Publications results for "Items authored by Demailly, Jean-Pierre"

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§ **1. Citations ↪**
MR4223969 (2021)
Demailly, Jean-Pierre (F-GREN-IF)
Hermitian–Yang–Mills approach to the conjecture of Griffiths on the positivity of ample vector bundles. (Russian)
Mat. Sb. 212 (2021), no. 3, 39–53.

Abstract. Given a vector bundle of arbitrary rank with ample determinant line bundle on a projective manifold, we propose a new elliptic system of differential equations of Hermitian-Yang-Mills type for the curvature tensor. The system is designed so that solutions provide Hermitian metrics with positive curvature in the sense of Griffiths – and even in the dual Nakano sense. As a consequence, if an existence result could be obtained for every ample vector bundle, the Griffiths conjecture on the equivalence between ampleness and positivity of vector bundles would be settled.

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§ **2. Citations ↪**
MR4081359 (2020)
Demailly, Jean-Pierre; Dinh, Tien-Cuong; Hai, Le Mau; Hiep, Pham Hoang; Khoai, Ha Huy; Ma, Xiaonan; Marinescu, George; Peternell, Thomas; Sibony, Nessim.
Special issue: Nevanlinna theory and complex geometry
Acta Math. Vietnam. 45 (2020), no. 1, 1–2.

Preface

Le Van Thiem received his doctorate in 1945 from Göttingen and he got his Doctorat d'État at the École Normale Supérieure de Paris in 1949.

In 1949, Le Van Thiem returned to Vietnam in the middle of a fierce resistance war for independence of Vietnam. He had a great contribution in building the University in the headquarters of the Resistance, and became a teacher of the first Vietnamese mathematicians. For the next dozen years, Vietnamese mathematicians were either his students or students of his students. We can say that Le Van Thiem is the founder of Mathematics in Vietnam.

In the first stage of his career, Le Van Thiem made a great contribution in the inverse problem of the value distribution theory of meromorphic functions (Nevanlinna theory). Later, he turned to applied mathematics, contributing to solving problems raised in Vietnamese practice such as oriented explosion, ground water under irrigation schemes, and the problem of calculating petroleum reserves.

Le Van Thiem himself is a part of the history of Mathematics in Vietnam. His name is given to a street in Hanoi and some schools throughout the country.

This issue of Acta Mathematica Vietnamica is dedicated to the proceedings of the Conference "Nevanlinna theory and Complex Geometry" in Honor of Le Van Thiem's Centenary (Hanoi, 26/02/2018 – 02/03/2018). We thank the Institute of Mathematics of Vietnam and its staff for their help to make the Conference possible.

Selected Papers of Le Van Thiem

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1. Thiem, L.-V.: Beitrag sum Typenproblem der Riemannschen Flachen. Comment. Math. Helv. 20, 270–287 (1947)
2. Thiem, L.-V.: Uber das Umkehrproblem der Werterteilungslehre. Comment. Math. Helv. 23, 26–49 (1949)
3. Thiem, L.-V.: Le degré de ramification d'une surface de Riemann et la croissance de la caractéristique de la fonction uniformisante. C. R. Acad. Sc. Paris 228, 1192–1195 (1949)
4. Thiem, L.-V.: Un problème de type généralisé. C. R. Acad. Sc. Paris 228, 1270–1272 (1949)
5. Thiem, L.-V.: Sur un problème d'inversion dans la théorie des fonctions méromorphes. Ann. Sci. Ecole Normale Sup. 67, 51–98 (1950)

6. Thiem, L.-V.: Sur un problème d'infiltration à travers un sol à deux couchés. Acta Sci. Vietnam., Sectio Sci. Math. et Phys. 1, 3–9 (1964)

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§ 3. Citations ↪
MR4068832 (2020)
Demailly, Jean-Pierre. Recent results on the Kobayashi and Green-Griffiths-Lang conjectures. Jpn. J. Math. 15 (2020), no. 1, 1–120.
32Q45 (32H30, 32L10, 53C55, 14J40)

Abstract: The study of entire holomorphic curves contained in projective algebraic varieties is intimately related to fascinating questions of geometry and number theory – especially through the concepts of curvature and positivity which are central themes in Kodaira's contributions to mathematics. The aim of these lectures is to present recent results concerning the geometric side of the problem. The Green-Griffiths-Lang conjecture stipulates that for every projective variety X of general type over $\mathbb{C}$, there exists a proper algebraic subvariety Y of X containing all entire curves $f : \mathbb{C} \to X$. Using the formalism of directed varieties and jet bundles, we show that this assertion holds true in case X satisfies a strong general type condition that is related to a certain jet-semi-stability property of the tangent bundle T_X. It is possible to exploit similar techniques to investigate a famous conjecture of Shoshichi Kobayashi (1970), according to which a generic algebraic hypersurface of dimension nn and of sufficiently large degree $d \geq d_n$ in the complex projective space $\mathbb{P}^{n+1}$ is hyperbolic: in the early 2000's, Yum-Tong Siu proposed a strategy that led in 2015 to a proof based on a clever use of slanted vector fields on jet spaces, combined with Nevanlinna theory arguments. In 2016, the conjecture has been settled in a different way by Damian Brotbek, making a more direct use of Wronskian differential operators and associated multiplier ideals; shortly afterwards, Ya Deng showed how the proof could be modified to yield an explicit value of dndn. We give here a short proof based on a substantial simplification of their ideas, producing a bound very similar to Deng's original estimate, namely $d_n = \lfloor \frac{1}{3}(en)^{2n+2} \rfloor$.

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§ 4. Citations ↪
MR4065066 (2020)
Campana, Frédéric; Demailly, Jean-Pierre; Peternell, Thomas.
The algebraic dimension of compact complex threefolds with vanishing second Betti number.
Compos. Math. 156 (2020), no. 4, 679–696.

Abstract. We study compact complex three-dimensional manifolds with vanishing second Betti number. In particular, we show that a compact complex manifold homeomorphic to the six-dimensional sphere does carry any non-constant meromorphic function.

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§ 5. Citations ↪
MR3936300 (2019)
Demailly, Jean-Pierre; Rahmati, Mohammad Reza.
Morse cohomology estimates for jet differential operators.
Boll. Unione Mat. Ital. 12 (2019), no. 1-2, 145–164.

Abstract. We provide detailed holomorphic Morse estimates for the cohomology of sheaves of jet differentials and their dual sheaves. These estimates apply on arbitrary directed varieties, and a special attention has been given to the analysis of the singular situation. As a consequence, we obtain existence results for global jet differentials and global differential operators under positivity conditions for the canonical or anticanonical sheaf of the directed structure.

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§ 6. Citations ↪
MR3923220 (2018)
Demailly, Jean-Pierre (F-GREN-IF)
Extension of holomorphic functions and cohomology classes from non reduced analytic subvarieties.
Geometric complex analysis, 97–113, Springer Proc. Math. Stat., 246, Springer, Singapore, 2018.

Abstract. The goal of this survey is to describe some recent results concerning the L^2 extension of holomorphic sections or cohomology classes with values in vector bundles satisfying weak semi-positivity properties. The results presented here are generalized versions of the Ohsawa–Takegoshi extension theorem, and borrow many techniques from the long series of papers by T. Ohsawa. The recent achievement that we want to point out is that the surjectivity property holds true for restriction morphisms to non necessarily reduced subvarieties, provided these are defined as zero varieties of multiplier ideal sheaves. The new idea involved to approach the existence problem is to make use of L^2 approximation in the Bochner-Kodaira technique. The extension results hold under curvature conditions that look pretty optimal. However, a major unsolved problem is to obtain natural (and hopefully best possible) L^2 estimates for the extension in the case of non reduced subvarieties—the case when Y has singularities or several irreducible components is also a substantial issue.

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§ 7. Citations ↪
MR3824567 (2018) Pending
Demailly, Jean-Pierre (F-GREN-IF)
Fano manifolds with nef tangent bundles are weakly almost Kähler-Einstein. (English summary)
Asian J. Math. 22 (2018), no. 2, 285–290.
32Q20 (14J45 14M17 32Q10 32U40)

Abstract. The goal of this short note is to point out that every Fano manifold with a nef tangent bundle possesses an almost Kähler–Einstein metric, in a weak sense. The technique relies on a regularization theorem for closed positive $(1,1)$-currents. We also discuss related semistability questions and Chern inequalities.

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§ 8. Citations ↪
From References: 0
From Reviews: 0
MR3794567 (2017) Reviewed
Demailly, Jean-Pierre(F-GREN-IFR)
Precise error estimate of the Brent-McMillan algorithm for Euler's constant. (English summary)
Mosc. J. Comb. Number Theory 7 (2017), no. 4, 3–38.
33C10 (11Y60)

The Brent-McMillan algorithm is the fastest known method for high-precision computation of Euler's constant γ. This relies on the result

$$\gamma = \frac{S_0(2x)}{I_0(2x)} - \log x - \frac{K_0(2x)}{I_0(2x)},$$

where $I_0(x$, $K_0(x)$ denote the modified Bessel functions and $S_0(x) = \sum_{n=1}^{\infty} H_n (x/2)^{2n}/(n!)^2$, with $H_n = 1+1/2+\cdots+1/n$ being the partial sum of the harmonic series. For large x the final term $K_0(2x)/I_0(2x)$ is exponentially small like e^{-8x}.

The algorithm makes use of the asymptotic expansion

$$I_0(2x)K_0(2x) \sim \frac{1}{4x}\sum_{k=0}^{\infty} \frac{((2k)!)^4}{(k!)^4(16x)^{2k}} \quad (x \to +\infty).$$

The procedure involves the optimal truncation of this series at its least term corresponding to $k = 2x$, when x is an integer. By expressing the product $I_0(2x)K_0(2x)$ as a double integral, the author considers the remainder term $\Delta(x)$ in the above optimally truncated expansion. The leading asymptotic form, together with an error bound, is shown to be

$$\Delta(x) = -e^{-4x}\left(\frac{5}{24\sqrt{\pi}x^{3/2}} + \epsilon(x)\right), \quad |\epsilon(x)| < \frac{0.863}{x^2}.$$

It then follows that the error in the term $K_0(2x)/I_0(2x)$ is $\Delta(x)/I_0(2x)^2 \sim -5\sqrt{2\pi}e^{-8x}/(12x^{1/2})$ as $x \to +\infty$.

Reviewed by Richard B. Paris

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§ 9. Citations ↪
From References: 0
From Reviews: 0
MR3713044 (2017) Reviewed
Demailly, Jean-Pierre(F-GREN-IFR); Gaussier, Hervé(F-GREN-IFR)
Algebraic embeddings of smooth almost complex structures. (English summary) J. Eur. Math. Soc. (JEMS) 19 (2017), no. 11, 3391–3419.
32Q60 (32G05 32Q40)

In this article, the problem of embedding an almost complex manifold into a complex algebraic variety is considered. More precisely, let Z be a complex projective (hence algebraic) manifold and let $\mathcal{D} \subset T_Z$ be a (not necessarily integrable) algebraic distribution, i.e., a subbundle of the tangent bundle. In particular, an algebraic distribution is invariant under the almost complex structure induced by the complex structure of Z. Recall that a foliation is a distribution that is closed under the Lie bracket of T_Z.

The basic question, which goes back to Bogomolov, is whether one can always realize an integrable almost complex structure on a compact (differentiable) manifold X by an embedding $X \hookrightarrow Z$ into a complex projective manifold Z such that it is transverse to an algebraic foliation $\mathcal{D} \subset T_X$. Here, transverse means that $\mathcal{D}_x \oplus T_{Z,x} = T_{X,x}$ for every $x \in X$; in particular, the rank of $\mathcal{D}$ equals the codimension of X. To realize the almost complex structure on X by such an embedding means that it be induced from the almost complex structure on $T_Z/\mathcal{D}$ by pullback. The authors provide the following universal solution to a weaker problem where $\mathcal{D}$ is allowed to be a (not necessarily integrable) distribution.

Theorem 1.2. *For all integers $n \geq 1$ and $k \geq 4n$, there exists a complex affine algebraic manifold $Z_{n,k}$ of dimension $2k + 2(k^2 + n(k-n))$ possessing a real structure (i.e. an anti-holomorphic algebraic involution) and an algebraic distribution $\mathcal{D}_{n,k} \subset T_{Z_{n,k}}$ of codimension n for which every compact n-dimensional almost complex manifold (X, J) admits an embedding $f : X \hookrightarrow Z^{\mathbb{R}}_{n,k}$ transverse to $\mathcal{D}_{n,k}$ and contained in the real part of $Z_{n,k}$, such that the almost complex structure on X is the one obtained by pullback.*

Regarding $\dim Z_{n,k}$, the authors show that the growth order $N = O(n^2)$ is optimal.

Turning to symplectic almost complex manifolds, it is natural to ask whether an embedding into a projective manifold transverse to a distribution can be chosen such that the symplectic structure is also induced from this embedding. This is also motivated by work of D. Tischler [J. Differential Geometry **12** (1977), no. 2, 229–235; MR0488108] on symplectic embeddings of symplectic differentiable manifolds into complex projective space with the Fubini-Study form. For this purpose, the notion of a transverse Kähler structure on a complex manifold $(Z, \mathcal{D})$ equipped with a holomorphic distribution is introduced. A transverse Kähler structure is a $(1,1)$-form β whose radical is contained in $\mathcal{D}$. Thus, a transverse Kähler structure induces a Kähler form on any complex submanifold that is transverse to $\mathcal{D}$.

It is proven in Theorem 1.5 that the aforementioned embedding problem for compact symplectic almost complex manifolds has a universal solution similar to Theorem 1.2. Towards the end of the paper, the authors also provide a result in the direction of Bogomolov's original question (where "distribution" is replaced by "foliation") under an additional assumption that holomorphic foliations can be approximated by Nash algebraic foliations on certain subsets of $\mathbb{C}^N$.

Reviewed by Christian Lehn

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§ 10. Citations ↪
From References: 0
From Reviews: 0
MR3666028 (2017) Reviewed
Demailly, Jean-Pierre(F-GREN-IFR) Variational approach for complex Monge-Ampère equations and geometric applications.
Séminaire Bourbaki. Vol. 2015/2016. Exposés 1104–1119. Astérisque No. 390 (2017), Exp. No. 1112, 245–275. ISBN: 978-2-85629-855-8 32W20 (32Q25 53C55)

The complex Monge-Ampère equations on compact Kähler manifolds can be solved by using different methods. One of them is a variational method that is independent of Yau's theorem. The aim of this paper is to present, starting from the beginning, the main ideas involved in this approach. The technique used by R. J. Berman et al. [Publ. Math. Inst. Hautes Études Sci. 117 (2013), 179–245; MR3090260] is based on the study of certain functionals (the Ding-Tian and Mabuchi functionals) defined on the space of Kähler metrics, and their geodesic convexity proved by X. Chen [J. Differential Geom. 56 (2000), no. 2, 189–234; MR1863016] and Berman and B. Berndtsson [J. Amer. Math. Soc. 30 (2017), no. 4, 1165–1196; MR3671939]. Recent applications include the existence and uniqueness of Kähler-Einstein metrics on $\mathbb{Q}$-Fano varieties with log terminal singularities, given by Berman et al. in ["Kähler-Einstein metrics and the Kähler-Ricci flow on log Fano varieties", J. Reine Angew. Math., posted September 14, 2016, doi:10.1515/crelle-2016-0033], and a new proof by Berman, S. Boucksom and M. Jonsson ["A variational approach to the Yau-Tian-Donaldson conjecture", preprint, arXiv:1509.04561] of a uniform version of the Yau-Tian-Donaldson conjecture. This provides a simpler way to the existence theorem for Kähler-Einstein metrics due to Chen, S. K. Donaldson and S. Sun [J. Amer. Math. Soc. 28 (2015), no. 1, 183–197; MR3264766].

For the collection containing this paper see MR3666019.

Reviewed by Rafał Czyż

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17. R. J. Berman, S. Boucksom & M. Jonsson – *A variational approach to the Yau-Tian-Donaldson conjecture*, preprint arXiv:1509.04561. MR2318616

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Cao, JunYan (F-PARIS6-IMJ); Demailly, Jean-Pierre (F-GREN-IFR); Matsumura, Shin-ichi (J-TOHOE)
A general extension theorem for cohomology classes on non reduced analytic subspaces. (English summary) Sci. China Math. 60 (2017), no. 6, 949–962
32L10, 32E05

In this paper the authors generalise the celebrated L^2-extension theorem of T. Ohsawa and K. Takegoshi [Math. Z. 195 (1987), no. 2, 197–204; MR0892051]. Let X be a Kähler complex manifold that is holomorphically convex, but not necessarily compact. Consider as well a holomorphic line bundle $E \to X$ defined over the ambient manifold and equipped with a possibly singular Hermitian metric h, so that this metric $h = e^{-\varphi}$ and its curvature current $\Theta_{E,h} = i\partial\overline{\partial}\varphi$ can be locally expressed in terms of a quasi-plurisubharmonic function φ. In this case, φ is the sum of a plurisubharmonic function and a smooth one. The main objective of the authors is to extend sections that are defined on a (not necessarily reduced) complex subspace $Y \subset X$, under the simple assumption that the structure sheaf $\mathcal{O}_Y = \mathcal{O}_X/\mathcal{I}(e^{-\psi})$ is given by the multiplier ideal sheaf of a quasi-plurisubharmonic function ψ with neat analytic singularities on X. Thus, suppose there exists a positive continuous function $\delta > 0$ on X such that the inequality

$$\Theta_{E,h} + (1 + \alpha\delta)i\partial\overline{\partial}\psi \geq 0, \quad \text{for all } \alpha \in [0, 1],$$

holds in the sense of currents. The authors prove that the morphism induced by the natural inclusion $\mathcal{I}(he^{-\psi}) \to \mathcal{I}(h)$,

$$H^q(X, K_X \otimes E \otimes \mathcal{I}(he^{-\psi})) \to H^q(X, K_X \otimes E \otimes \mathcal{I}(h)), \tag{1}$$

is injective for every index $q \geq 0$. The term $K_X = \Lambda^n T_X^*$ denotes by the canonical bundle of an n-dimensional complex manifold X. In other words, it is shown that the morphism induced by the natural sheaf surjection $\mathcal{I}(h) \to \mathcal{I}(h)/\mathcal{I}(he^{-\psi})$,

$$H^q(X, K_X \otimes E \otimes \mathcal{I}(h)) \to H^q(X, K_X \otimes E \otimes \mathcal{I}(h)/\mathcal{I}(he^{-\psi})), \tag{2}$$

is indeed surjective for every index $q \geq 0$. In particular, if h is smooth, one automatically has that $\mathcal{I}(h) = \mathcal{O}_X$, and so $\mathcal{I}(h)/\mathcal{I}(he^{-\psi})$ coincides with $\mathcal{O}_Y = \mathcal{O}_X/\mathcal{I}(e^{-\psi})$ for the zero subvariety Y of the ideal sheaf $\mathcal{I}(e^{-\psi})$. Hence, in the case when $q = 0$, the surjectivity statement above can be interpreted as an extension theorem for holomorphic sections, with respect to the restriction morphism

$$H^0(X, K_X \otimes E) \to H^0(Y, (K_X \otimes E)|_Y).$$

Reviewed by Eduardo S. Zeron

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MR3752612 (2016)
Demailly, Jean-Pierre
Analyse numérique et équations différentielles. (French) [[Numerical analysis and differential equations]] Fourth edition.

Grenoble Sciences. EDP Sciences, Les Ulis, 2016. vii+368 pp. ISBN: 978-2-7598-1926-3

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§ 13. Citations ↪
From References: 0
From Reviews: 0
MR3525916 (2016) Reviewed
Demailly, Jean-Pierre (F-GREN-F)
Extension of holomorphic functions defined on non reduced analytic subvarieties. (English summary) The legacy of Bernhard Riemann after one hundred and fifty years. Vol. I, 191–222, Adv. Lect. Math. (ALM), 35.1, Int. Press, Somerville, MA, 2016.
32D15 (14F05 32C35)

The problem of being able to extend sections of holomorphic bundles from subvarieties Y to the ambient variety X is quite important in algebraic geometry. Usually such results also produce an extension with controlled L^2 norm if the norm of the original section is controlled.

The Ohsawa-Takegoshi (OT) L^2 extension theorem is such a result. Usually, however, the OT theorem requires Y to be the zero locus of a section of a vector bundle. In this paper, the first theorem (Theorem 2.8) extends this to the case where Y is the (reduced) variety of a multiplier ideal sheaf. Theorem 2.12 generalises the above result to the case where Y can have "multiplicity" (some sort of a jet extension result). Theorem 2.13 speaks of the situation where the metric is singular. Finally, Theorem 2.14 is (not an L^2 result but) a qualitative extension result with semi-positivity assumptions (a sort of strengthening of Nadel's vanishing theorem).

Reviewed by Vamsi Pritham Pingali

Papers in this collection include the following:

For Vol. II see [MR3495160].

Lizhen Ji and Shing-Tung Yau, *What one should know about Riemann but may not know?*, 1–55. MR3525911

Michael F. Atiyah, *Riemann's influence in geometry, analysis and number theory*, 57–67. MR3525912

M. V. Berry, *Riemann's saddle-point method and the Riemann-Siegel formula*, 69–78. MR3525913

Ching-Li Chai, *The period matrices and theta functions of Riemann*, 79–106. MR3525914

Brian Conrey, *Riemann's hypothesis*, 107–190. MR3525915

Jean-Pierre Demailly, *Extension of holomorphic functions defined on non reduced analytic subvarieties*, 191–222. MR3525916

F. T. Farrell, *Bundles with extra geometric or dynamic structure*, 223–250. MR3525917

James Glimm, Dan Marchesin and Bradley Plohr, *The theory of shock waves: from Riemann through today*, 251–273. MR3525918

David Harbater, *Riemann's existence theorem*, 275–286. MR3525919

Lizhen Ji, *The historical roots of the concept of Riemann surfaces*, 287–305. MR3525920

Lizhen Ji, *The story of Riemann's moduli space*, 307–357. MR3525921

Jürgen Jost, *Riemann and the modern concept of space*, 359–377. MR3559991

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§ 14. Citations ↪
MR3587462 (2015)
Demailly, Jean-Pierre (FGRENF)
On the cohomology of pseudoeffective line bundles.
Complex geometry and dynamics, 51–99, Abel Symp., 10, Springer, Cham, 2015.

Abstract. The goal of this survey is to present various results concerning the cohomology of pseudoeffective line bundles on compact Kähler manifolds, and related properties of their multiplier ideal sheaves. In case the curvature is strictly positive, the prototype is the well known Nadel vanishing theorem, which is itself a generalized analytic version of the fundamental Kawamata-Viehweg vanishing theorem of algebraic geometry. We are interested here in the case where the curvature is merely semipositive in the sense of currents, and the base manifold is not necessarily projective. In this situation, one can still obtain

interesting information on cohomology, e.g. a Hard Lefschetz theorem with pseudoeffective coefficients, in the form of a surjectivity statement for the Lefschetz map. More recently, Junyan Cao, in his PhD thesis defended in Grenoble, obtained a general Kähler vanishing theorem that depends on the concept of numerical dimension of a given pseudoeffective line bundle. The proof of these results depends in a crucial way on a general approximation result for closed $(1,1)$-currents, based on the use of Bergman kernels, and the related intersection theory of currents. Another important ingredient is the recent proof by Guan and Zhou of the strong openness conjecture. As an application, we discuss a structure theorem for compact Kähler threefolds without nontrivial subvarieties, following a joint work with F. Campana and M. Verbitsky. We hope that these notes will serve as a useful guide to the more detailed and more technical papers in the literature; in some cases, we provide here substantially simplified proofs and unifying viewpoints.

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§ 15. Citations ↪
From References: 0
From Reviews: 0
MR3446751 (2015) Reviewed
Demailly, Jean-Pierre (FGRENF)
Structure theorems for compact Kähler manifolds with nef anticanonical bundles. (English summary) Complex analysis and geometry, 119–133, Springer Proc. Math. Stat., 144, Springer, Tokyo, 2015.
32Q15 (53C55)

Summary: "This survey presents various results concerning the geometry of compact Kähler manifolds with numerically effective first Chern class: structure of the Albanese morphism of such manifolds, relations tying semipositivity of the Ricci curvature with rational connectedness, positivity properties of the Harder-Narasimhan filtration of the tangent bundle."

For the collection containing this paper see MR3446743.

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§ 16. Citations ↪
From References: 0
From Reviews: 0
MR3445519 (2015) Reviewed
Demailly, Jean-Pierre (F-GREN-F)
Towards the Green-Griffiths-Lang conjecture. (English summary) Analysis and geometry, 141–159, Springer Proc. Math. Stat., 127, Springer, Cham, 2015.
32J25 (14Cxx 32Q45)

The paper under review studies the celebrated Green-Griffiths-Lang conjecture (GGL conjecture for short), in the general context of directed manifolds, introduced by the author in his seminal work [in Algebraic geometry—Santa Cruz 1995, 285–360, Proc. Sympos. Pure Math., 62, Part 2, Amer. Math. Soc., Providence, RI, 1997; MR1492539].

A directed manifold (X,V) is a pair, where X is a complex manifold and $V\subset T_X$ is a holomorphic vector subbundle of the holomorphic tangent space.

In this paper, the author introduces a more general notion of directed manifold, allowing V to possibly have singularities.

In this framework, one is interested only in entire curves $f:\mathbb{C}\to X$ whose derivatives are pointwise tangent to V. Whenever X is compact, one thus says that (X,V) is Kobayashi hyperbolic if there are no nonconstant entire curves with the above property, and that (X,V) satisfies the (generalized) GGL conjecture if the Zariski closure of the union of the images of all such entire curves is not the whole manifold. Observe that the original GGL conjecture states that the Zariski closure in question should not be the whole manifold whenever $V=T_X$ and X is of general type; i.e., K_X is big.

First of all, the author introduces a notion of being of general type for a compact directed manifold (X,V). This is quite a subtle point, since in order to be consistent with the natural generalization of the GGL conjecture in this singular context, one needs to take into account in a nontrivial way the singularities of V. Moreover, such a notion reduces to being $\det V^*$ big if V is an actual holomorphic vector subbundle of T_X.

Next, the author defines a stronger notion of being of general type, which he calls "strongly general type". This definition being quite technical, we refer to the original paper for a precise statement.

Finally, the author defines what he calls an “algebraically jet-hyperbolic” directed manifold. Roughly speaking, this means that the directed structure induced on every irreducible algebraic subvariety (has a desingularization which) is strongly of general type.

With these definitions in mind, we can state the two main results of the paper.

(1) Every projective directed manifold strongly of general type satisfies the generalized GGL conjecture.

(2) If a projective directed manifold is algebraically jet-hyperbolic, then it is Kobayashi hyperbolic.

Perhaps one flaw of the property of being strongly of general type is that it seems to us quite difficult in practice to establish whether or not a given compact directed manifold satisfies such a condition. Nevertheless, in order to overcome this issue, the author suggests at the very end of the paper that the notion of being strongly of general type should be related to some sort of stability condition, which he describes and calls “jet-stability”. Without any doubt, it would be desirable to have a better understanding of this jet-stability in the near future.

For the collection containing this paper see MR3379815.

Reviewed by Simone Diverio

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§ **17. Citations ↪**
From References: 0
From Reviews: 1
MR3380444 (2015) Reviewed
Campana, F. (F-LOR-IEC); Demailly, J.-P. (F-GREN-F); Peternell, Th. (D-BAYR-IM)
Rationally connected manifolds and semipositivity of the Ricci curvature. (English summary) Recent advances in algebraic geometry, 71–91,
London Math. Soc. Lecture Note Ser., 417, Cambridge Univ. Press, Cambridge, 2015.
53C55 (14M22)

The celebrated Beauville-Bogomolov decomposition theorem [A. Beauville, J. Differential Geom. **18** (1983), no. 4, 755–782 (1984); MR0730926] states (in one of its forms) that given a compact Kähler manifold X with zero real first Chern class, the universal cover of X splits holomorphically (and isometrically, once a Ricci-flat Kähler metric is chosen) into the product of a flat factor and simply connected compact Kähler manifolds with special unitary or compact symplectic holonomy (i.e., Calabi-Yau or hyper-Kähler manifolds).

The main result of the paper under review is a decomposition theorem in the same spirit as the one above for compact Kähler manifolds with semi-positive first Chern class, or equivalently, with a Kähler metric whose Ricci curvature is non-negative. In this more general setting, a new type of factor may appear in the decomposition of the universal cover, namely, simply connected compact Kähler manifolds with merely unitary holonomy. In order to have a nice characterization of these factors, which are indeed shown to be rationally connected, the authors prove a new characterization of rationally connected compact Kähler manifolds, in the spirit of Mumford’s conjecture.

The characterization is as follows. Let X be a compact Kähler manifold. Then, X is rationally connected if and only if for every invertible subsheaf $\mathcal{L} \subseteq \Omega^p_X$, $1 \leq p \leq n$, (resp. $\mathcal{L} \subseteq \mathcal{O}_X((T^*_X)^{\otimes m})$, $m \geq 1$), $\mathcal{L}$ is not pseudoeffective; moreover, in this case X is projective and the above is equivalent to the fact that for any ample line bundle A on X there exists a constant C such that

$$H^0(X, (T^*_X)^{\otimes m} \otimes A^{\otimes k}) = \{0\}, \quad \text{for all } m \geq Ck$$

(recall that Mumford’s conjecture is exactly the last statement without the auxiliary ample line bundle).

The proof of the above criterion relies upon a “generalized holonomy principle” which is stated for holomorphic Hermitian vector bundles on compact Hermitian manifolds, with semi-positive mean curvature (i.e. the trace of the Chern curvature with respect to the fixed Hermitian metric on the manifold). This generalized holonomy principle states, among other things, that if the restricted holonomy of the Hermitian vector bundle in question is unitary, then no invertible subsheaf of any tensor power of the dual of the vector bundle can be pseudoeffective.

The paper ends with an appendix by the second-named author, in which he gives a related version of the generalized holonomy principle over flag varieties.

For the collection containing this paper see MR3380552.

Reviewed by Simone Diverio

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§ **18. Citations ↪**
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Demailly, Jean-Pierre; van der Geer, Gerard; Hacon, Christopher; Kawamata, Yujiro; Kobayashi, Toshiyuki; Miyaoka, Yoichi; Schmid, Wilfried
Foreword [In commemoration of Professor Kunihiko Kodaira's centennial birthday, March 16, 2015].
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§ **19. Citations ↪**
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From Reviews: 0
MR3238108 (2014) Reviewed
Campana, Frédéric (F-NANC-IE); Demailly, Jean-Pierre (F-GREN-F); Verbitsky, Misha (RS-HSE-ALG)
Compact Kähler 3-manifolds without nontrivial subvarieties. (English summary)
Algebr. Geom. 1 (2014), no. 2, 131–139.
32J17 (32J25 32J27)

In the paper, the authors give an interesting characterization of a compact Kähler threefold which has no nontrivial complex subvarieties, namely that such a compact Kähler threefold should be a torus. This result yields an essential step toward the classification of compact Kähler threefolds. The authors also prove the following result: if X is a normal compact Kähler threefold such that (1) X has only terminal singularities, (2) the canonical bundle of X is nef and (3) X has no effective divisor, then X is a cyclic quotient of a simple non-projective torus.

Reviewed by Atsushi Moriwaki

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§ **20. Citations ↪**
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MR3222365 (2014) Indexed
Skoda, Henri (F-PARIS6-IMJ); Demailly, Jean-Pierre (F-GREN-FM); Siu, Yum-Tong (1-HRV)
In memory of Pierre Lelong. Henri Skoda, coordinating editor.
Notices Amer. Math. Soc. 61 (2014), no. 6, 586–595.
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§ 21. Citations ↪
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MR3191972 (2014) Reviewed
Demailly, Jean-Pierre; Dinew, Sławomir; Guedj, Vincent; Phạm, Hoàng Hiệp; Kołodziej, Sławomir; Zeriahi, Ahmed;
Hölder continuous solutions to Monge–Ampère equations.
J. Eur. Math. Soc. (JEMS) 16 (2014), no. 4, 619–647.
32W20 (32Q15 32U05 32U15 32U40 35B65 35J96 53C55)

Let (X, ω) be a compact n-dimensional Kähler manifold. The authors study the following complex Monge-Ampère equation on X:

$$\mathrm{MA}(u) := \frac{1}{\int_X \omega^n}(\omega + dd^c u)^n = f\omega^n, \quad \text{where} \quad f \in L^p, \ p > 1.$$

The authors show that the solution to the above Monge-Ampère equation is Hölder continuous with exponent α arbitrarily close to $2/(1+nq)$, where q denotes the conjugate exponent of p. Furthermore, they obtain an analogue of this result when the cohomology class is semi-positive. They also obtain a uniform Hölder continuity result in the case of uniformly bounded geometries if the L_p norms of the right-hand sides are uniformly bounded. Next the authors obtain some properties of the range of Hölder continuous ω-plurisubharmonic functions under the complex Monge-Ampère operator, $\mathrm{MAH}(X, \omega) = \mathrm{MA}(X, \omega) \cup \mathrm{Hölder}(X, \mathbb{R})$. Theorem 1. If $\mu \in \mathrm{MAH}(X, \omega)$ and $0 < f \in L^p(\mu)$ with $p > 1$ and $\int_X f d\mu = 1$, then $f\mu \in \mathrm{MAH}(X, \omega)$. In particular, the set $\mathrm{MAH}(X, \omega)$ is convex. They further characterize the probability measures in $\mathrm{MAH}(X, \omega)$ which have finitely many isolated singularities of radial or toric type in terms of an integrability condition. Theorem 2. Let μ be a probability measure with finitely many isolated singularities of radial or toric type. Then μ belongs to $\mathrm{MAH}(X,\omega)$ if and only if the following strong integrability condition holds:

$$\exp(-\varepsilon\,\mathrm{PSH}(X,\omega)) \subset L^1(\mu) \quad \text{for some} \quad \varepsilon > 0.$$

Note that the strong integrability condition completely characterizes $\mathrm{MAH}(X,\omega)$ in dimension 1. It is not known whether the strong integrability condition characterizes $\mathrm{MAH}(X,\omega)$ when $n > 1$. The paper is very well written and gives insight into the subject.

Reviewed by Muhammed Ali Alan

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§ 22. Citations ↪
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MR3179606 (2014) Reviewed
Demailly, Jean-Pierre; Phạm, Hoàng Hiệp; A sharp lower bound for the log canonical threshold.
Acta Math. 212 (2014), no. 1, 1–9.
32W20 (32C99 32U05 32U25)

The purpose of this short article is to prove a sharp lower bound for the log canonical threshold of a plurisubharmonic function φ with an isolated singularity in an open subset of $\mathbb{C}^n$. This threshold is defined as the supremum of constants $c > 0$ such that $e^{-2c\varphi}$ is integrable in a neighborhood of the singular point, say 0. The authors relate $c(\varphi)$ to the intermediate multiplicity numbers $e_j(\varphi)$ defined as the Lelong numbers of $(dd^c\varphi)^j$ at 0 (in particular, $e_0(\varphi) = 1$ and $e_1(\varphi)$ is the Lelong number of the psh function φ). The main result of the article is the sharp inequality

$$c(\varphi) \geq \sum_{j=0}^{n-1} \frac{e_j(\varphi)}{e_{j+1}(\varphi)}.$$

This improves an old result of H. Skoda, $c(\varphi) \geq 1/e_1(\varphi)$ [see Bull. Soc. Math. France 100 (1972), 353–408; MR0352517 (50 #5004)], as well as previous lower estimates [P. Ahag et al., Adv. Math. 222 (2009), no. 6, 2036–2058; MR2562773 (2010h:32042)] and $c(\varphi) \geq n/e_n(\varphi)^{1/n}$ [see T. de Fernex, L. Ein and M. Mustaţă, J. Algebraic Geom. 13 (2004), no. 3, 603–615; MR2047683 (2005b:14008)], which have been important in applications to birational geometry in recent years [see A. V. Pukhlikov, Izv. Ross. Akad. Nauk Ser. Mat. 66 (2002), no. 6, 159–186; MR1970356 (2004a:14012b); I. Cheltsov, Uspekhi Mat. Nauk 60 (2005), no. 5(365), 71–160; MR2195677 (2007d:14028)]. The proof consists of a reduction to the toric case.

Reviewed by Vincent Guedj

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§ 23. Citations ↪
From References: 2
From Reviews: 0
MR3089070 (2013) Reviewed
Demailly, Jean-Pierre (F-GREN-F)
Applications of pluripotential theory to algebraic geometry. (English summary)
Pluripotential theory, 143–263, Lecture Notes in Math., 2075, Springer, Heidelberg, 2013.
32U40 (14F43 32U05 32W20)

This nice survey article is based on lectures by the author in 2011 and explains how transcendental techniques, particularly those coming from pluripotential theory, can be used as a powerful tool for the study of many problems in algebraic geometry.

The article is divided into four sections. In the first, the holomorphic Morse inequalities are studied, in particular their connections with Monge-Ampère operators and intersection theory.

The second section is focused on regularization of positive closed currents using Bergman kernel techniques, ultimately arising from the Ohsawa-Takegoshi theorem. These regularization results are used to study natural convex cones defined by suitable positivity conditions, in the cohomology space of a compact Kähler manifold, as well as their algebraic counterparts in the Néron-Severi space in the case of a smooth complex projective variety. The author discusses many important topics here: approximate Zariski decompositions, the mobile intersection product introduced by Boucksom, the dual of the pseudoeffective cone, and more. He uses these techniques in Section 3 to study the asymptotic cohomology functionals introduced by Küronya, and some natural transcendental analogues.

Finally, Section 4 is devoted to the Green-Griffiths-Lang conjecture on the structure of entire curves in a projective manifold X (or, more generally, a directed projective manifold (X, V)) satisfying a suitable "general type" condition.

For the entire collection see MR3089067.

Reviewed by Mattias Jonsson

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§ 24. Citations ↪
From References: 2
From Reviews: 0
MR3089067 (2013) Reviewed
Patrizio, Giorgio(I-FRNZ); Błocki, Zbigniew(PL-JAGL); Berteloot, François(F-TOUL3-IM); Demailly, Jean-Pierre(F-GREN-FM)
Pluripotential theory.

Lectures from the Centro Internazionale Matematico Estivo (CIME) Session held in Cetraro, 2011. Edited by Filippo Bracci and John Erik Fornæss. Lecture Notes in Mathematics, 2075. Fondazione CIME/CIME Foundation Subseries. Springer, Heidelberg; Fondazione C.I.M.E., Florence, 2013. x+319 pp. ISBN: 978-3-642-36420-4; 978-3-642-36421-1 32-06 (32Uxx)

Contents: François Berteloot, Bifurcation currents in holomorphic families of rational maps (1–93) MR3089068; Zbigniew Błocki, The complex Monge-Ampère equation in Kähler geometry (95–141) MR3089069; Jean-Pierre Demailly, Applications of pluripotential theory to algebraic geometry (143–263) MR3089070; G. Patrizio and A. Spiro [Andrea F. Spiro], Pluripotential theory and Monge-Ampère foliations (265–319) MR3089071.

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§ 25. Citations ↪
From References: 2
From Reviews: 0
MR3070567 (2013) Reviewed
Demailly, Jean-Pierre (F-GREN-NDM); Hacon, Christopher D.(1-UT-NDM); Păun, Mihai (F-NANC-NDM)
Extension theorems, non-vanishing and the existence of good minimal models. (English summary)
Acta Math. 210 (2013), no. 2, 203-259. (Reviewer: Vladimir Lazić) 14E30
PDF Clipboard Journal Article Make Link
Publication Year 2013

Let (X, Δ) be a log canonical pair in characteristic zero. One of the goals of the Minimal Model Program is to show, when the divisor $K_X + \Delta$ is pseudoeffective, that the pair (X, Δ) has a suitable birational model (called a good minimal model) on which the proper transform of $K_X + \Delta$ becomes semiample. In particular, conjecturally some multiple of $K_X + \Delta$ has sections, and this problem is known as nonvanishing. Even when one knows nonvanishing, it is a very difficult problem to show that a good minimal model exists. An essential ingredient in recent advances in the Minimal Model Program, related to both of the above-mentioned issues, is to find a result which enables one to extend sections from a suitable (prime) divisor on X, and then use induction on the dimension.

The paper under review finds such an extension result under some additional hypotheses. The main result of the paper is too technical to state here; however, the main application is in the case when the adjoint divisor $K_X + \Delta$ is nef, and it has the following simple form.

Corollary 1.8. Let $(X, S + B)$ be a plt pair, where S is a prime divisor and the coefficients of B lie in the interval $(0, 1)$. Assume that $K_X + S + B$ is nef, and that there exists an effective $\mathbb{Q}$-divisor

$D \sim_{\mathbb{Q}} K_X + S + B$ with $S \subseteq \operatorname{Supp}(D) \subseteq \operatorname{Supp}(S + B)$. Then the restriction map

$$H^0(X, \mathcal{O}_X(m(K_X + S + B))) \to H^0(S, \mathcal{O}_S(m(K_X + S + B)))$$

is surjective for all sufficiently divisible positive integers m.

The proof is analytic, and it is a skillful use of an extension of the classical Ohsawa-Takegoshi theorem. The assumptions on the supports of D and $S + B$ are natural from the point of view of the Minimal Model Program. Further, previous results of a similar form assumed certain positivity of the divisor B, and this is the first extension result which works for any B, albeit at the expense of assuming that some multiple of $K_X + S + B$ has sections. However, the assumption that the pair is plt is very difficult to achieve in practice.

Section 8 of the paper contains very interesting applications of the main result of [C. D. Hacon, J. McKernan and C. Xu, "ACC for log canonical thresholds”, preprint, arXiv:1208.4150, Ann. of Math. (2), to appear], which are of independent interest.

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§ 26. Citations ↪
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Gaz. Math. No. 135 (2013),63-66. 01A70
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§ 27. Citations ↪
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Boucksom, Sébastien (F-PARIS7-IM); Demailly, Jean-Pierre (F-GREN-F); Păun, Mihai (F-STRAS); Peternell, Thomas(D-BAYR-IM)
The pseudo-effective cone of a compact Kähler manifold and varieties of negative Kodaira dimension. (English summary)
J. Algebraic Geom. 22 (2013), no. 2, 201–248. (Reviewer: Thomas Eckl) 32L05
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This is a paper whose results were reproduced in a monograph [R. K. Lazarsfeld, Positivity in algebraic geometry. II, Ergeb. Math. Grenzgeb. (3), 49, Springer, Berlin, 2004; MR2095472 (2005k:14001b)], re-proven in a new language [S. Boucksom, C. Favre and M. Jonsson, J. Algebraic Geom. 18 (2009), no. 2, 279–308; MR2475816 (2009m:14005)] and applied in several other papers [for example F. Campana and T. Peternell, Bull. Soc. Math. France 139 (2011), no. 1, 41–74; MR2815027 (2012e:14031); C. D. Hacon and J. McKernan, Duke Math. J. 138 (2007), no. 1, 119–136; MR2309156 (2008f:14030)] before it was officially published. Still, its publication is not only justified by setting the historic records right, but also by the fact that the paper is written in a setting different from its successors and contains discussions published nowhere else, on questions that follow naturally from its main result:

Theorem 1. On a non-uniruled projective complex manifold X the canonical bundle K_X is pseudoeffective.

This statement should be seen as a first fundamental step towards the Abundance Conjecture in arbitrary dimensions:

Conjecture 1. A projective complex manifold X has Kodaira dimension $\kappa(X) = -\infty$ if and only if X is uniruled.

The theorem is a consequence of a uniruledness criterion of Y. Miyaoka and S. Mori [Ann. of Math. (2) 124 (1986), no. 1, 65–69; MR0847952 (87k:14046)] and a general fact on pseudoeffective line bundles:

Theorem 2. A line bundle L on an n-dimensional projective complex manifold X is pseudoeffective if and only if $L \cdot C \geq 0$ for all strongly moving curves C that are curves $C = \mu(\widetilde{A}_1 \cap \cdots \cap \widetilde{A}_{n-1})$ where $\mu : \widetilde{X} \to X$ is a modification of X and the $\widetilde{A}_i$ are sufficiently general very ample divisors on $\widetilde{X}$.

This theorem is one of several duality statements on convex cones in the spaces of curves and divisors on projective complex manifolds, or more generally Kähler manifolds:

• $\overline{NE}(X)$, the closed convex cone generated by irreducible curves, is dual to the convex cone $\overline{K}_{NS}(X)$ generated by nef divisors (a classical result by Kleiman [see R. Hartshorne, Ample subvarieties of algebraic varieties, Lecture Notes in Mathematics, Vol. 156, Springer, Berlin, 1970; MR0282977 (44 #211)]).

• On an n-dimensional Kähler manifold X, the closed convex cone $\overline{K}(X)$ generated by Kähler forms on X is dual to the convex cone N generated by classes $[Y] \cap \omega^{p-1}$, where $Y \subset X$ ranges over p-dimensional analytic subsets of X, $p = 1, 2, \ldots, n$, and ω ranges over the Kähler forms (a generalization of the Nakai-Moishezon criterion, shown by J.-P. Demailly and M. Păun [Ann. of Math. (2) 159 (2004), no. 3, 1247–1274; MR2113021 (2005i:32020)]).

• Theorem 2 shows that the convex cone $SME(X)$ generated by all strongly movable curves is dual to the closed convex cone $\mathrm{Eff}(X)$ generated by all effective divisors on X. Since $\overline{ME}(X)$, the closed convex cone generated by all irreducible curves moving in a family covering X, is contained in $\mathrm{Eff}(X)^\vee = \overline{SME}(X)$, it is also true that $\overline{ME}(X) = \overline{SME}(X)$.

• The authors conjecture that Theorem 2 also holds in the Kähler setting:

Conjecture 2. On an n-dimensional Kähler manifold the pseudoeffective cone $\mathcal{E}(X)$ of classes represented by positive $(1,1)$-currents is dual to the closed convex cone $\mathcal{M}(X)$ generated by classes of currents of the form $\mu_*(\widetilde{\omega}_1 \wedge \cdots \wedge \widetilde{\omega}_{n-1})$, where $\mu : \widetilde{X} \to X$ is a modification of X and the $\widetilde{\omega}_i$ are Kähler forms on $\widetilde{X}$.

In the paper this is shown for compact hyper-Kähler manifolds but remains wide open in the general case (see also the remarks below).

The main tool in the proof of Theorem 2 is Fujita's approximative Zariski decomposition of a big line bundle L on a projective complex manifold X: For arbitrary $\epsilon > 0$ there exist a modification $\mu_\epsilon : \widetilde{X}_\epsilon \to X$ and a decomposition $\mu_\epsilon^* L = E_\epsilon + D_\epsilon$ into an effective divisor E_ϵ and a big and nef divisor D_ϵ such that the volumes of L and D_ϵ differ at most by ϵ (proven by T. Fujita in [Kodai Math. J. 17 (1994), no. 1, 1–3; MR1262949 (95c:14053)] and by Demailly, L. Ein and Lazarsfeld in [Michigan Math. J. 48 (2000), 137–156; MR1786484 (2002a:14016)]). A Kähler version of an approximative Zariski decomposition follows from a result of Demailly in [J. Algebraic Geom. 1 (1992), no. 3, 361–409; MR1158622 (93e:32015)].

The main idea in the proof of Theorem 2 is that the fixed part E_ϵ and the moving part D_ϵ of such an approximative Zariski decomposition are almost orthogonal, with an estimate

$$D_\epsilon^{n-1} \cdot E_\epsilon \leq C \cdot \epsilon,$$

where C is a constant independent of ϵ. In turn, the proof of this orthogonality estimate on projective complex manifolds X relies on the inequality

$$\mathrm{Vol}(A - B) \geq A^n - n \cdot A^{n-1} \cdot B \qquad (*)$$

for two nef divisors A, B on X. There are simple proofs of $(*)$ on projective complex manifolds, but they also follow from the holomorphic Morse inequalities [see J.-P. Demailly, in School on Vanishing Theorems and Effective Results in Algebraic Geometry (Trieste, 2000), 1–148, ICTP Lect. Notes, 6, Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2001; MR1919457 (2003f:32020)]. On a Kähler manifold, $(*)$, with the optimal coefficient n, would follow from a conjectural transcendental version of these Morse inequalities. In the paper a version of $(*)$ with a coefficient quadratic in n is shown.

The approximative Zariski decomposition is also used to define a "movable intersection product" on the cone of pseudoeffective classes, with values $\langle \alpha_1 \cdots \alpha_k \rangle$ in the cone of non-negative (k,k)-classes. In particular, for $k = 1$ this product yields a "divisorial Zariski decomposition" of a pseudoeffective class α on a Kähler manifold X,

$$\alpha = \langle \alpha \rangle + N(\alpha)$$

where $N(\alpha)$ is a finite sum of effective $\mathbb{R}$-divisors from a countable set of irreducible divisors determined by X.

Furthermore, a numerical dimension of a pseudoeffective class α may be defined as

$$\mathrm{nd}(\alpha) = \max\{p \in \mathbb{N} : \langle \alpha^p \rangle \neq 0\},$$

and a generalized Abundance Conjecture can be stated:

Conjecture 3. On a projective complex manifold, or even a Kähler manifold X,

$$\kappa(X) = \mathrm{nd}(X) := \mathrm{nd}(c_1(X)).$$

This development of a movable intersection product closely follows Boucksom's thesis. A completely algebraic approach was taken by Boucksom, Favre and Jonsson [op. cit.], leading to the same results.

Next, the paper deals with natural extensions of pseudoeffectivity of line bundles to holomorphic vector bundles E on projective complex manifolds X. The vector bundle E is called almost nef if the restrictions $E|_C$ to curves $C \subset X$ not contained in a countable union of algebraic subsets of X are always nef. E is called pseudoeffective if $\mathcal{O}_{\mathbb{P}(E)}(1)$ is a pseudoeffective line bundle on $\mathbb{P}(E)$, and the non-nef locus of $\mathcal{O}_{\mathbb{P}(E)}(1)$ is not projected onto X. The non-nef locus $L_{\mathrm{nonnef}}(\alpha)$ of a pseudo-effective class α is constructed as the union of all logarithmic singularities that currents in perturbed classes $\alpha + \delta\omega$ have in common, where $\delta > 0$ and ω is a Kähler form. Note that $L_{\mathrm{nonnef}}(\alpha)$ contains all irreducible algebraic curves C such that $\alpha \cdot C < 0$, but may be bigger: an example on $\mathbb{P}^2$ blown up in several points is provided.

If E is almost nef then $\mathcal{O}_{\mathbb{P}(E)}(1)$ is pseudoeffective, by Theorem 2. But contrary to a claim of Demailly, T. Peternell and M. H. Schneider in [Internat. J. Math. 12 (2001), no. 6, 689–741; MR1875649 (2003a:32032)] it is not certain whether an almost nef vector bundle is also pseudoeffective. In the paper this implication is at least shown for vector bundles of rank at most 3 with vanishing first Chern class and then applied to the tangent bundle T_X of a K3 surface X: neither is T_X almost nef, nor is $\mathcal{O}_{\mathbb{P}(T_X)}(1)$ pseudoeffective. The last statement was also shown by N. Nakayama [Zariski-decomposition and abundance, MSJ Mem., 14, Math. Soc. Japan, Tokyo, 2004; MR2104208 (2005h:14015)].

Finally, the authors prove the Abundance Conjecture for projective 4-folds under the additional assumption that K_X is numerically trivial when restricted to curves in a large family:

Theorem 3. If the canonical bundle K_X on a projective 4-fold X is pseudoeffective and if there is a good covering family (C_t) of curves on X such that $K_X \cdot C_t = 0$ then $\kappa(X) \geq 0$.

A good covering family (C_t) of (generically irreducible) curves on X is either non-connecting or strongly connecting, that is, either two general points on X cannot be connected by a chain of curves C_t, or they can be connected by such a chain, furthermore avoiding any given codim ≥ 2 algebraic subset. Strong connectedness is needed to show that $C_t \cdot L = 0$ implies $\mathrm{nd}(L) = 0$ for a pseudo-effective divisor L. In that case $\kappa(L) \geq 0$ follows from the results of Campana and Peternell [op. cit.]. An example is constructed where $C_t \cdot L = 0$ does not imply $\mathrm{nd}(L) = 0$ if the family (C_t) is only connected; however this can (but need not) occur for $L = K_X$ only on very special X.

If $C_t \cdot K_X > 0$ for all strongly connected families (C_t) of curves, X should be of general type. The authors are at least able to show that $\kappa(X) \leq 0$ or $= \dim X$.

If C_t is a non-connecting family of curves then the proof of Theorem 3 applies the $C_{n,m}$-conjectures on the reduction map associated to the family (C_t) (see the results of Campana [Bull. Soc. Math. France 122 (1994), no. 2, 255–284; MR1273904 (95f:32036); Ann. Inst. Fourier (Grenoble) 54 (2004), no. 3, 631–665; MR2097417 (2006c:14014)]) and the log MMP to the basis of this reduction map. Both applications restrict the theorem to dimension 4. These methods resemble and sometimes use results from F. Ambro in [Math. Ann. 330 (2004), no. 2, 309–322; MR2089428 (2005h:14036)].

Reviewed by Thomas Eckl

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§ 28. Citations ↪
From References: 0
From Reviews: 1
MR3014194 (2012) Reviewed
Demailly, Jean-Pierre(F-GREN-F)
Henri Cartan et les fonctions holomorphes de plusieurs variables. (French) [Henri Cartan and multivariate holomorphic functions] Henri Cartan & André Weil, mathématiciens du XX^{e} siècle,
99-168, Ed. Éc. Polytech., Palaiseau, 2012. 32-03 (32A10 32B10 32C35 32H99)
The paper is a review of some fundamental results on holomorphic functions of several variables, and of the role of Henri Cartan in the development of this theory, in particular in the theory of coherent sheaves which he developed and which stands now as one of the most fundamental tools in complex geometry and in algebraic geometry. The exposition is concise but self-contained and the stress is on the essential facts. A few historical remarks are useful for understanding the motivations behind the ideas. All this makes the text much more attractive than many other texts written on the subject. The bibliographical references organized in sections are also very useful.
For the entire collection see MR3059568.

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§ 29. Citations ↪

From References: 0
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MR3058660 (2012) Reviewed
Demailly, Jean-Pierre (F-GREN-F)
Hyperbolic algebraic varieties and holomorphic differential equations. Acta Math. Vietnam. 37(2012),no. 4, 441-512.
32F45 (14Jxx 32H30 32L10 53C55)
PDF Clipboard Journal Article Make Link
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In this survey on Kobayashi hyperbolicity of complex varieties, the author updates his earlier notes [in *Algebraic geometry—Santa Cruz 1995*, 285–360, Proc. Sympos. Pure Math., 62, Part 2, Amer. Math. Soc., Providence, RI, 1997; MR1492539 (99b:32037)]. First, the author reviews the general techniques used in the study of entire curves in complex varieties in the general framework of directed manifolds. In comparison with the previous article, the author not only considers holomorphic subbundles of the tangent bundle but also saturated coherent subsheaves. This allows one to work in particular with singular holomorphic foliations. An interesting generalized Green-Griffiths-Lang conjecture is stated, claiming the algebraic degeneracy of entire curves in projective manifolds tangent to distributions of general type. Then the author concentrates on one of the main new tools used to attack these problems: the holomorphic Morse inequalities. He gives a detailed account of his recent result [Pure Appl. Math. Q. **7** (2011), no. 4, Special Issue: In memory of Eckart Viehweg, 1165–1207; MR2918158] on the existence of many algebraic differential equations satisfied by entire curves on directed projective varieties of general type. Finally, the author gives some applications in the setting of projective hypersurfaces of large degree, recovering a result of S. Diverio, J. Merker and E. Rousseau [Invent. Math. **180** (2010), no. 1, 161–223; MR2593279 (2011e:14079)] with a better bound on the minimal degree ensuring that the Green-Griffiths-Lang conjecture is satisfied in this case.

Reviewed by Erwan Rousseau

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This list reflects references listed in the original paper as accurately as possible with no attempt to correct error.

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§ 30. Citations ↪
From References: 1
From Reviews: 0
MR2978333 (2012) Reviewed
Demailly, Jean-Pierre (F-GREN-FM)
Analytic methods in algebraic geometry.
Surveys of Modern Mathematics, 1. International Press, Somerville, MA; Higher Education Press, Beijing, 2012. viii+231 pp. ISBN: 978-1-57146-234-3 (Reviewer: Valentino Tosatti) 32-02 (14C30 14F18 32J25 32Q15 32U40)
PDF Clipboard Series Book Make Link
Publication Year 2012 Review Published 2013-09-25

This book is a compilation and expansion of lecture notes that the author has written in recent years, most notably the Park City 2008 notes [in Analytic and algebraic geometry, 295-370, IAS/Park City Math. Ser., 17, Amer. Math. Soc., Providence, RI, 2010; MR2743818 (2012g:32001)], which have a considerable overlap with this book. This is arguably the first written textbook on applications of analysis to problems in algebraic geometry; it is written by one of the leaders and developers of this field, and is aimed at advanced graduate students and other researchers who have already a working knowledge of the basic concepts of complex geometry.

This book is divided into twenty chapters. The first four chapters contain mostly introductory material about complex manifolds, Dolbeault cohomology, closed positive currents and Lelong numbers, Monge-Ampère operators, Hermitian vector bundles, the Bochner technique and vanishing theorems.

Chapter 5, "L^2 estimates and existence theorems", gives a quick introduction to the Hörmander L^2 estimates for $\overline{\partial}$ and to multiplier ideal sheaves, including Nadel's vanishing theorem.

Chapter 6, "Numerically effective and pseudo-effective line bundles", introduces several different notions of positivity for line bundles, their corresponding positive cones in cohomology, and proves the Kawamata-

Viehweg vanishing theorem. It also covers analytic Zariski decompositions and Siu's uniform global generation theorem.

Chapter 7, "A simple algebraic approach to Fujita's conjecture", discusses an algebraic approach of Siu towards Fujita's conjecture.

Chapter 8, "Holomorphic Morse inequalities", describes (without proof) these inequalities, and then discusses their algebraic versions, asymptotic cohomology and a conjectural transcendental version of these inequalities.

Chapter 9, "Effective version of Matsusaka's big theorem", contains an exposition of Siu's proof of this theorem, with some simplifications and improvements due to the author.

Chapter 10, "Positivity concepts for vector bundles", discusses the relationship between Griffiths and Nakano positivity for vector bundles.

Chapter 11, "Skoda's L^2 estimates for surjective bundle morphisms", proves Skoda's division theorem, and the Briançon-Skoda theorem as an application.

Chapter 12, "The Ohsawa-Takegoshi L^2 extension theorem", proves this celebrated result, and rederives the Skoda division theorem from it.

Chapter 13, "Approximation of closed positive currents by analytic cycles", contains an exposition of the author's celebrated regularization procedure for closed positive currents. There is also a proof of the fact that a compact complex manifold is bimeromorphic to Kähler iff it supports a Kähler current. The author also introduces the complex singularity exponents of plurisubharmonic functions, relates them to log canonical thresholds, gives a sketch of proof of their semicontinuity, and finally discusses the relation between the Hodge conjecture and the approximation of (p, p) currents by algebraic cycles.

Chapter 14, "Subadditivity of multiplier ideals and Fujita's approximate Zariski decomposition", is an exposition of results due to Demailly-Ein- Lazarsfeld on this topic.

Chapter 15, "Hard Lefschetz theorem with multiplier ideal sheaves", proves a version of the Hard Lefschetz theorem for pseudo-effective line bundles.

Chapter 16, "Invariance of plurigenera of projective varieties", gives Păun's simplified proof of Siu's celebrated result.

Chapter 17, "Numerical characterization of the Kähler cone", is an exposition of this groundbreaking result of Demailly-Păun.

Chapter 18, "Structure of the pseudo-effective cone and mobile intersection theory", discusses results of Boucksom-Demailly-Păun-Peternell characterizing the dual of the pseudoeffective cone, and thus shows that a projective manifold is uniruled iff its canonical bundle is not pseudo- effective.

Chapter 19, "Super-canonical metrics and abundance", deals with recent results of Berman-Demailly on these topics, and ends with a discussion of Tsuji's approach towards the abundance conjecture.

Chapter 20, "Siu's analytic approach and Păun's non vanishing theorem", contains a very short introduction to Siu's analytic proof of the finite generation of the canonical ring, and of Păun's improved non-vanishing theorem.

To summarize, this book is essentially a panoramic view of Demailly's work on analytic geometry over the past 30 years, with an emphasis on very recent results and developments. The writing style is quite condensed, which makes it a demanding but extremely rewarding read for students interested in entering this exciting field of research, and also a very useful reference book for more advanced readers. The great variety of results covered (some of which appear here for the first time in book form) and the numerous open questions and conjectures that are scattered throughout this book make it an especially useful resource.

Reviewed by Valentino Tosatti

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§ 31. Citations ↪
From References: 1
From Reviews: 1
MR2884031 (2012m:32035) Reviewed
Berman, Robert(S-CHAL); Demailly, Jean-Pierre(F-GREN-F)
Regularity of plurisubharmonic upper envelopes in big cohomology classes. (English summary) Perspectives in analysis, geometry, and topology, 39-66,

Progr. Math., 296, Birkhäuser/Springer, New York, 2012.
32U05 (32Q15 32U40 32W20)

The goal of this article is to establish a fundamental regularity result for upper envelopes of quasi-plurisubharmonic (q-psh) functions.

Assume that X is a compact complex manifold in the Fujiki class $\mathcal{C}$ (i.e. bimeromorphic to a Kähler manifold) and let $\{\alpha\} \in H^{1,1}(X, \mathbb{R})$ be a big cohomology class, i.e. a class that can be represented by a positive current which dominates a Hermitian form (it follows from the work of the second author and M. Păun [Ann. of Math. (2) 159 (2004), no. 3, 1247-1274; MR2113021 (2005i:32020)] that such classes exist precisely on these manifolds).

Let α be a smooth closed $(1,1)$-form in $\{\alpha\}$. A function $\psi = X \to \mathbb{R} \cup \{-\infty\}$ is said to be α-plurisubharmonic if it is q-psh (i.e. locally given as the sum of a smooth and a plurisubharmonic function) and such that $\alpha + dd^c\psi \geq 0$ in the sense of currents. Let $\mathrm{PSH}(X, \alpha)$ denote the set of all α-psh functions and set

$$V_\alpha(x) := \sup\{\psi(x) \,|\, \psi \in \mathrm{PSH}(X, \alpha),\ \psi \leq 0\}.$$

The authors show that V_α has locally bounded mixed complex derivatives $\partial^2 V_\alpha / \partial z_i \partial \overline{z_j}$ in the ample locus of α (a Zariski open subset) and derive several consequences.

The main technical tool is a regularization result of the second author [in Contributions to complex analysis and analytic geometry, 105-126, Aspects Math., E26, Friedr. Vieweg, Braunschweig, 1994; MR1319346 (96k:32012)]. A slightly stronger result was obtained by the first author in [Amer. J. Math. 131 (2009), no. 5, 1485-1524; MR2559862 (2010g:32030)] when the cohomology class $\{\alpha\}$ is the first Chern class of a big line bundle. Here, however, the proof is somewhat simpler and will without any doubt find several other applications.

For the entire collection see MR2867634 (2012h:00045).

Reviewed by Vincent Guedj

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§ 32. Citations ↪
From References: 2
From Reviews: 0
MR2918158 (2011) Reviewed
Demailly, Jean-Pierre(F-GREN) Holomorphic Morse inequalities and the Green-Griffiths-Lang conjecture. (English, French summary)
Pure Appl. Math. Q. 7(2011),no. 4, Special Issue: In memory of Eckart Viehweg, 1165-1207. 32Q45 (14C30 32L20)
PDF Clipboard Journal Article Make Link
Publication Year 2011 Review Published 2013-10-10

Consider an irreducible projective complex n-dimensional variety X of general type. According to the Green-Griffiths-Lang conjecture, the locus of rational, elliptic or more generally entire curves is expected to be contained in a strict Zariski closed subset of X.

The current strategy towards this result, originating in the works of A. Bloch [J. Math. Pures Appl. (9) 5 (1926), 19-66; JFM 52.0373.04], M. L. Green and P. A. Griffiths [in The Chern Symposium 1979 (Proc. Internat. Sympos., Berkeley, Calif., 1979), 41-74, Springer, New York, 1980; MR0609557 (82h:32026)], J.-P. Demailly [in Algebraic geometry–Santa Cruz 1995, 285-360, Proc. Sympos. Pure Math., 62, Part 2, Amer. Math. Soc., Providence, RI, 1997; MR1492539 (99b:32037)] and Y. T. Siu and S.-K. Yeung [Invent. Math. 124 (1996), no. 1-3, 573-618; MR1369429 (97e:32028)], is to first find a differential equation fulfilled by all the entire curves in X and then, as a second step, to derive from this differential equation an algebraic equation fulfilled by all such curves. Some special cases of the first issue have been worked out by Green and Griffiths [op. cit.] for surfaces, and by S. Diverio [Math. Ann. 344 (2009), no. 2, 293-315; MR2495771 (2010c:32047)], J. Merker ["Complex projective hypersurfaces of general type: toward a conjecture of Green and Griffiths", preprint, arXiv: 1005.0405] and G. Bérczi [in Contributions to algebraic geometry, 141-167, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2012; MR2976941] for high degree hypersurfaces in projective spaces. The whole program was successfully worked out in the case of generic hypersurfaces of high degree in projective spaces by Siu [in The legacy of Niels Henrik Abel, 543-566, Springer, Berlin, 2004; MR2077584 (2005h:32061)] and Diverio, Merker and E. Rousseau [Invent. Math. 180 (2010), no. 1, 161-223; MR2593279 (2011e:14079)].

The main achievement of the present work is to settle the first issue in full generality. More precisely, for an irreducible projective complex variety X and a big subbundle V of the tangent bundle of X, up to taking high enough order, the number of differential equations vanishing on an ample divisor which are fulfilled by all entire curves tangent to V has maximal growth with respect to the degree. The proof here goes through precise curvature estimates for high jet bundles. This provides new tools for working on jet spaces.

The author also offers some potential strategies for the second issue, which, in view of an example of Diverio and Rousseau showing that the exceptional set and the Green-Griffiths locus do not always coincide, may require some additional non-rigidity properties.

Reviewed by Christophe Mourougane

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§ 33. Citations ↪
From References: 4
From Reviews: 0
MR2858170 (2012j:32021) Reviewed

Demailly, Jean-Pierre(F-GREN-F)
A converse to the Andreotti-Grauert theorem. (English, French summary)
Ann. Fac. Sci. Toulouse Math. (6) 20 (2011), Fascicule Spécial, 123-135.
32L20 (32L10 32W20)

Let X be a compact complex manifold, and L a line bundle over X. The Andreotti-Grauert vanishing theorem states that if for some integer q and some $u \in c_1(L)$ the form $u(z)$ has at least $n-q+1$ positive eigenvalues everywhere then $H^j(X, L^{\otimes k}) = 0$ for $j \geq q$ and $k \gg 1$. The holomorphic Morse inequalities [in J.-P. Demailly, Ann. Inst. Fourier (Grenoble) 35 (1985), no. 4, 189-229; MR0812325 (87d:58147)] establish an upper bound for the q-th asymptotic cohomology functional defined as

$$\widehat{h}^q(X, L) := \limsup_{k \to +\infty} \frac{n!}{k^n} h^q(X, L^{\otimes k})$$

and for the q-th asymptotic holomorphic Morse sum of L defined as

$$\widehat{h}^{\leq q}(X, L) := \limsup_{k \to +\infty} \frac{n!}{k^n} \sum_{0 \leq j \leq q} (-1)^{q-j} h^j(X, L^{\otimes k}),$$

of the type $\widehat{h}^q(X, L), \widehat{h}^{\leq q}(X, L) \leq \inf_{u \in c_1(L)} \int_D (-1)^q u^n$, where $u \in c_1(L)$ and, in the first case, D is the open set of X made of points where u has signature $(n-q, q)$ while, in the second case, it is the open set of points of X where u has signature $(n-j, j)$ with $0 \leq j \leq q$.

In the paper under review, the author asks for the following converse of the Andreotti-Grauert vanishing theorem. If it is known that the cohomology groups are asymptotically small in a certain degree q, is it true that there exists a Hermitian metric on L with suitable curvature, i.e., with almost no q-index points? This question is clearly related to having equality in the previous holomorphic Morse inequalities. In fact, the author proves that, if X is projective, then $\mathrm{Vol}(X, L) := \widehat{h}^0(X, L) = \inf_{u \in c_1(L)} \int_D (-1)^q u^n$.

In the case of higher cohomology groups, the author proves that if X is a complex projective surface, then the limsups involved in the definition of $\widehat{h}^q(X, L)$ and $\widehat{h}^{\leq q}(X, L)$ are in fact limits and equalities occur in the holomorphic Morse inequalities.

Reviewed by Filippo Bracci

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MR2743818 (2012g:32001) Reviewed
Demailly, Jean-Pierre(F-GREN-F)
Structure theorems for projective and Kähler varieties. Analytic and algebraic geometry, 295-370,
IAS/Park City Math. Ser., 17, Amer. Math. Soc., Providence, RI, 2010.
32-02 (14F18 14J40 32C30 32J25 32Q15 32U40)

This is a thorough survey of analytic approaches to problems in complex algebraic geometry, exemplified by the results of Păun, Boucksom, Tsuji, Campana and the author on positivity.

The paper is organised in eight sections. Section 1, 'Numerically effective and pseudo-effective $(1,1)$ classes', gives the basic definitions of such concepts as nefness and other types of positivity, and multiplier ideal sheaves. It then covers analytic Zariski decomposition; vanishing; Siu's uniform global generation theorem; and Hard Lefschetz in its multiplier ideal sheaf version. Section 2 is entitled 'Holomorphic Morse inequalities': it states these results, asymptotic estimates for $h^q(X, E\otimes \mathcal{O}(kL))$, first in terms of curvature integrals and then algebraically. Section 3 is called 'Approximation of closed positive $(1,1)$-currents by divisors' and goes into rather more analytic detail than we find elsewhere in the paper. One reward is an essentially self-contained proof that a compact complex manifold is of class $\mathcal{C}$ (that is, bimeromorphic to a Kähler manifold) if and only if it admits a Kähler current. Another is a technical result that essentially means that the cone of closed positive currents is a completion of the cone of effective $\mathbb{Q}$-divisors. This, and the fact that the cone of currents is locally compact, is what makes the study of currents useful for the study of asymptotic behaviour of linear systems. This section also contains a sketch of Tian's α invariant and the associated criterion for the existence of Kähler-Einstein metrics.

Section 4, 'Subadditivity of multiplier ideals and Fujita's approximate Zariski decomposition', deals with those topics and gives a geometric interpretation of the volume of a line bundle as the growth of the moving self-intersection of the linear system $|kL|$. Section 5 is called 'Numerical characterization of the Kähler cone': it raises some hard questions about the duality of positive cones in $H^{p,p}$ and $H^{n-p,n-p}$ but considers only the cases $p=1$ and $n-p=1$, the others being at present out of reach. In those cases there are rather satisfactory results, including the theorem that if X is projective then the Kähler cone is the same as the cone of numerically positive real $(1,1)$-classes: this was a completely new result when it was proved by the author and Păun in 2004, even though it is a special case of their more general result for Kähler manifolds. A consequence is the invariance of the Kähler cone under very general deformations.

Section 6, 'Structure of the pseudo-effective cone and mobile intersection theory', deals with the results of Boucksom and compares various positive cones in $H^{n-k,n-k}(X)$ (mostly real, mostly $k=1$), interpreting some of them in terms of mobility of linear systems, Zariski decomposition, etc., and ending with a generalised version of the abundance conjecture (also apparently out of reach at present). This leads into Section 7, 'Super-canonical metrics and abundance', which outlines the recent work of the author and Boucksom on these topics and also outlines Tsuji's ideas about the positivity of relative canonical divisors and the invariance of plurigenera. In addition, it contains a conjectural approach to abundance also suggested by Tsuji. The final Section 8, 'Siu's analytic approach and Păun's non vanishing theorem', covers some recent developments very briefly– although Siu's approach (to the results of Birkar, Cascini, Hacon and McKernan) was announced in 2006, the details are not yet fully worked out.

The paper is thorough and covers a wide range of results. A few misprints have crept in: the mathematical ones are mostly easily corrected, but a more serious error, for a paper likely to be given to research students as background, is that the bibliography is badly incomplete. Many important papers are cited in the text but not listed. The experts will identify them easily, and may find their students asking them to do so.

For the entire collection see MR2742533 (2011i:32002).

Reviewed by G. K. Sankaran

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§ 35. Citations ↪
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MR2667492 (2012) Indexed
Demailly, Jean-Pierre (F-GREN); Kobayashi, Shoshichi (1-CA); Narasimhan, Raghavan (1-CHI); Siu, Yum-Tong(1-HRV)
Cartan and complex analytic geometry.
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There will be no review of this item.

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MR2684780 (2012a:32017) Reviewed
Demailly, Jean-Pierre(F-GREN-F)
Holomorphic Morse inequalities and asymptotic cohomology groups: a tribute to Bernhard Riemann. (English, French summary)
Milan J. Math. 78 (2010), no. 1, 265-277.
32L10 (32C35 32W20)

This expository paper presents several interesting results on the asymptotic q-cohomology functions for tensor powers of line bundles in connection with the author's previous work on holomorphic Morse inequalities [Ann. Inst. Fourier (Grenoble) 35 (1985), no. 4, 189-229; MR0812325 (87d:58147)]. The asymptotic q-cohomology function is an upper semi-continuous, positively homogeneous function defined on the real Néron-Severi subspace $\mathrm{NS}_{\mathbf{R}}(X)$ of the Bott-Chern cohomology group $H^{1,1}_{\mathrm{BC}}(X, \mathbf{R})$ of a compact complex manifold. It is shown that these functions are locally Lipschitz continuous on the divisorial Néron-Severi subspaces $\mathrm{DNS}_{\mathbf{R}}(X)$, and they are a natural generalization of the notion of volume of a line bundle. This paper also contains several open questions relating these functions to certain Monge-Ampère integrals.

Reviewed by Siqi Fu

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MR2647006 (2012e:32039) Reviewed
Demailly, Jean-Pierre(F-GREN-F); Pali, Nefton(F-PARIS11)
Degenerate complex Monge-Ampère equations over compact Kähler manifolds. (English summary)
Internat. J. Math. 21 (2010), no. 3, 357-405.
32Q20 (32J27 32W20)

The authors obtain the existence and uniqueness of the solutions of a quite general type of degenerate complex Monge-Ampère equations and investigate their regularity. They use some techniques developed by Yau, Bedford-Taylor (BT) and Tsuji in order to obtain some very general and sharp results on the above problems. Theorem 1.2: Let X be a compact Kähler manifold of complex dimension n and let χ be a $(1,1)$-cohomology class admitting a smooth closed semi-positive $(1,1)$-form ω such that $\int_X \omega^n > 0$. (A) For any $L\log^{n+\varepsilon} L$-density $v \geq 0$, $\varepsilon > 0$, such that $\int_X v = \int_X \chi^n$, there exists a unique closed positive current $T \in \mathrm{BT}_\chi$ such that $T^n = v$. Moreover, this current possesses bounded local potentials over X and continuous local potentials outside a complex analytic set $\Sigma_\chi \subset X$. This set depends only on the class χ and can be taken to be empty if the class χ is Kähler. (B) In the special case of a density $v \geq 0$ possessing complex analytic singularities the current T is also smooth outside the complex analytic subset $\Sigma_\chi \cup Z(v) \subset X$, where $Z(v)$ is the set of zeros and poles of v. Theorem 1.3: Let X be a smooth complex projective variety of general type. If the canonical bundle is nef, then there exists a unique closed positive current $\omega_E \in \mathrm{BT}^{\log}_{2\pi c_1(K_\chi)}$ solution of the Einstein equation $\mathrm{Ric}(\omega_E) = -\omega_E$. This current possesses bounded local potentials over X and defines a smooth Kähler metric outside a complex analytic subset Σ which is empty if and only if the canonical bundle is ample. Theorem 1.4: Let X be a smooth variety of general type and let $SB \subset \Sigma$ be respectively the stable and augmented stable base locus of the canonical bundle K_X. Then there exists a closed positive current $\omega_E \in 2\pi c_1(K_X)$ over X with locally bounded potentials over $X \smallsetminus SB$, and a solution of the Einstein equation $\mathrm{Ric}(\omega_E) = -\omega_E$ over $X \smallsetminus SB$ which restricts to a smooth (nondegenerate) Kähler-Einstein metric over $X \smallsetminus \Sigma$. The L^∞-estimate used by the authors allows them to solve the conjecture of Tian: Let (X, ω_X) be a polarized compact connected Kähler manifold of complex dimension n, (Y, ω_Y) be a compact irreducible Kähler space of complex dimension $m \leq n$, $\pi\colon X \to Y$ be a surjective holomorphic map and $0 \leq f \in L \log^{n+\varepsilon} L(X, \omega_X^n)$, for some $\varepsilon > 0$ such that $1 = \int_X f\omega_X^n$. Set $K_t := \{\pi^*\omega_Y + t\omega_X\}^n > 0$ for $t \in (0,1)$. Then the solution of the complex Monge-Ampère equations $(\pi^*\omega_Y + t\omega_X + i\partial\overline{\partial}\psi_t)^n = K_t f \omega_X^n$ satisfy the uniform L^∞-estimate $\mathrm{Osc}(\psi)_t := \sup_X \psi_t - \inf_X \psi_t \leq C < +\infty$ for all $t \in (0,1)$.

Reviewed by V. Oproiu

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§ 38. Citations ↪
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MR2742678 (2011j:32055) Reviewed
Demailly, Jean-Pierre(F-GREN-M2)
Estimates on Monge-Ampère operators derived from a local algebra inequality. (English, French summary) Complex analysis and digital geometry, 131-143,
Acta Univ. Upsaliensis Skr. Uppsala Univ. C Organ. Hist., 86, Uppsala Universitet, Uppsala, 2009.
32W20

The author proves an integrability theorem for plurisubharmonic (psh) functions. Let K be a compact subset of a domain Ω in $\mathbb{C}^n$, and u a psh function in Ω which off K takes values between $-A$ and 0. Suppose

$$\int_\Omega (dd^c u)^n \leq M < n^n \quad \left(d^c = \frac{1}{2\pi}(\overline{\partial} - \partial)\right).$$

Then

$$\int_K e^{-2u}\, dV \leq C \quad (dV \text{ the Lebesgue measure})$$

with C depending only on Ω, K, A, M. This is no longer true if $M = n^n$ since for $u(z) = n \log |z|$ the function e^{-2u} is not integrable and the Monge-Ampère mass of u is exactly n^n. The proof is carried out by reduction to an inequality

$$\mathrm{lc}(I) \geq n e(I)^{-1/n},$$

relating log-canonical thresholds $\mathrm{lc}(I)$ and the Hilbert-Samuel multiplicity $e(I)$ of an ideal I of germs of holomorphic functions near zero with zero variety $V(I)$ equal to $\{0\}$. This inequality is due to A. Corti [in Explicit birational geometry of 3-folds, 259-312, London Math. Soc. Lecture Note Ser., 281, Cambridge Univ. Press, Cambridge, 2000; MR1798984 (2001k:14041)] for $n = 2$, and to T. de Fernex, L. M. H. Ein and M. Mustaţă [J. Algebraic Geom. 13 (2004), no. 3, 603-615; MR2047683 (2005b:14008)] in the general case. The reduction uses an approximation (due to the author and based on the Ohsawa-Takegoshi theorem) of general u by psh functions of the form $\log \sum |g_j|$, g_j holomorphic, and the semicontinuity theorem for complex singularity exponents of psh functions from [J.-P. Demailly and J. Kollár, Ann. Sci. École Norm. Sup. (4) 34 (2001), no. 4, 525-556; MR1852009 (2002e:32032)]. The author also indicates that by having an independent, "analytic" proof of the integrability statement one could use the arguments of his proof to show the inequality of Corti, de Fernex, Ein and Mustaţă. The analytic proof, relying on pluripotential theory, was later given in [P. Ahag et al., Adv. Math. 222 (2009), no. 6, 2036-2058; MR2562773 (2010h:32042)].

For further information pertaining to this item see [A. Zériahi, in Complex analysis and digital geometry, 144-146, Acta Univ. Upsaliensis Skr. Uppsala Univ. C Organ. Hist., 86, Uppsala Universitet, Uppsala, 2009; MR2742763].

For the entire collection see MR2742180 (2011f:32001).

Reviewed by Sławomir Kołodziej

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§ 39. Citations ↪
From References: 0
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MR2393263 (2009b:32039) Reviewed
Demailly, Jean-Pierre (F-GREN-F); Hwang, Jun-Muk (KR-AIST-SM); Peternell, Thomas (D-BAYR-IM)

Compact manifolds covered by a torus. (English summary)
J. Geom. Anal. 18 (2008), no. 2, 324-340.
32Q57 (14K99 32L05 32Q15)

Suppose that A is a complex torus and $f\colon A \to X$ is a surjective map to a complex manifold. What can be said about the structure of X? O. Debarre [C. R. Acad. Sci. Paris Sér. I Math. 309 (1989), no. 2, 119-122; MR1004953 (90f:14027)] showed that X is $\mathbb{P}^n$ if A is a simple abelian n-fold and f is not an isogeny. Hwang and N. Mok [Math. Z. 238 (2001), no. 1, 89-100; MR1860736 (2002h:14024)] extended this to general abelian varieties, showing that X is an iterated projective space bundle over an étale quotient Y of an abelian variety.

Here A is allowed to be any complex torus (not necessarily algebraic), and it is shown that X is Kähler, and that there is an étale map $X' \to X$ such that X' is a product $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k} \times B$ of projective spaces and its Albanese torus B. Moreover, X is an étale quotient of X', so X is a $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$-bundle over an étale quotient of B.

The fact that X is Kähler is proved by using the easy fact that f is equidimensional (it factors as $A \to A/S \to X$, where S is a subtorus and $A/S \to X$ is finite). It is then a general fact, due to J. Varouchas, that if Y is a Kähler manifold and $g\colon Y \to Z$ is a proper surjective holomorphic map to a complex manifold Z, then Z is Kähler. Varouchas [Math. Ann. 283 (1989), no. 1, 13-52; MR0973802 (89m:32021)] proved this for complex spaces, with only weak conditions on X, and the authors here give a quick proof for the case of g finite, which is all that they need. In an appendix they also give a short proof of Varouchas's result under an extra assumption on the singularities of Z.

Back to the main stream of the paper and the bundle structure of X. The map $f\colon A \to X$ has ramification divisor $R = f^*\mathcal{O}_X(-K_X)$, so the anticanonical morphism of X induces a morphism from A whose image is a quotient torus V. The trick is to replace X by an étale cover $\tilde{X}$ of largest possible irregularity. Then $\tilde{X}$ is covered by a complex torus $\tilde{A}$, which is isogenous to the product of $\mathrm{Alb}(\tilde{X})$ and a torus $\tilde{V}$ covering the anticanonical image of $\tilde{X}$. Under these circumstances the fibres of the Albanese map of $\tilde{X}$ are products of projective spaces. These methods, and some of the intermediate results, come from the techniques of Demailly, Peternell and M. H. Schneider [J. Algebraic Geom. 3 (1994), no. 2, 295-345; MR1257325 (95f:32037)], though the context there is rather different because here the tangent bundle of X is not a priori nef.

Reviewed by G. K. Sankaran

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§ 40. Citations ↪
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MR2344027 (2008) Indexed
Demailly, Jean-Pierre; Kosarew, Siegmund; Malgrange, Bernard(F-GREN-F)
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Gaz. Math. No. 113 (2007), 35-38.
01A70 (01A60 46-03)

There will be no review of this item.

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§ 41. Citations ↪
From References: 5
From Reviews: 1
MR2334190 (2008k:32057) Reviewed
Demailly, Jean-Pierre(F-GREN-F)
Kähler manifolds and transcendental techniques in algebraic geometry. (English summary)
International Congress of Mathematicians. Vol. I, 153-186, Eur. Math. Soc., Zürich, 2007.
32J25 (14C30 32L20 32Q15)

Written from the perspective of one of the leading contemporary authorities in the field, this survey traces the development of transcendental algebraic geometry from the early decades of the last century through to the present, with ample indication of its future directions. It begins with the work of Hodge, and the famous conjecture concerning the role of algebraic cycles with rational coefficients as generators of cohomology for projective manifolds. The author points to the impetus it has given the development of Kähler geometry as a more general framework for the study of complex varieties, this in spite of the fact that the conjecture is well-known to be false for Kähler manifolds in general. A brief summary is given of the L^2-existence theory of the Cauchy-Riemann equation for (p, q)-forms with coefficients in a Hermitian-holomorphic vector bundle, associated prominently with the names of Bochner, Kodaira, Kohn, Andreotti-Vesentini, Hörmander and Skoda among many others, as well as some of the theory's most striking consequences, such as the vanishing theorems of Kodaira, Nakano, Kawamata-Viehweg and Nadel, the extension theorem of Ohsawa and Takegoshi, and the embedding theorem of Kodaira. The significance of this last result for the recent development of transcendental algebraic geometry rests largely on the equivalence it establishes between the existence of very ample line bundles and that of Kähler metrics representing classes in integer-valued cohomology. With the introduction of the Kähler and pseudo-effective cones in $H^{1,1}(X, \mathbf{R})$ for a compact Kähler manifold X, there is a shift in emphasis towards the study of closed positive currents which characterizes the contemporary theory. The intersection of the Kähler cone with the Néron-Severi lattice is generated by ample divisors, while the "numerically effective" , or "nef", divisors generate its closure. The term "big divisor" is used to refer to the generators of the interior of the Néron-Severi part of the pseudo-effective cone. Major contributions to the theory of approximation of currents and their Zariski decomposition have been made by the author and S. Boucksom, a notable corollary being the fact that a compact complex manifold is seen to be bimeromorphically equivalent to a Kähler manifold if and only if it carries a Kähler current. Further collaboration between the author and M. Păun has led to a natural generalization of the Nakai-Moishezon criterion which yields a numerical characterization of the Kähler cone. The deformation theory of Kähler structures also begins with a result of Kodaira, to the effect that every Kähler surface is a deformation-limit of algebraic surfaces. By contrast, a series of decisive counterexamples to such a property in higher dimension have been constructed by C. Voisin. Another theorem of Kodaira and Spencer refers to the local deformation-stability of Kähler structures. A global stability theorem for Kähler deformations due to the author and Păun is foreshadowed in dimension two by independent work of Buchdahl and Lamari. These results support the conjecture that all non-Kähler fibres in the deformation space of a Kähler manifold are parametrized by a finite (or perhaps countable) union of analytic substrata of the base space. The last two sections survey work of the author and his collaborators on the duality theory of closed positive currents and their positive cones. In particular, the concepts of "volume" and "numerical dimension" of classes (the latter being conjectured to be equivalent, in the case of the Chern class of the canonical bundle, to the Kodaira dimension) are outlined as tools for the characterization of ampleness. One striking consequence is that any projective manifold which is not uniruled has a pseudo-effective canonical divisor. A synopsis of the current state of the minimal models program in higher dimension and its relationship to Siu's work on deformation-invariance of plurigenera is also covered in conclusion.

For the entire collection see MR2334180 (2008b:00006).

Reviewed by Adam Gregory Harris

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§ 42. Citations ↪
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From Reviews: 0
MR2142242 (2006d:32033) Reviewed
Demailly, Jean-Pierre(F-GREN-F); Eckl, Thomas(D-KOLN); Peternell, Thomas(D-BAYR-IM)
Line bundles on complex tori and a conjecture of Kodaira.
Comment. Math. Helv. 80 (2005), no. 2, 229-242.
32Q15 (32G20 32J27)

A compact Kähler manifold is called almost algebraic if it can be approximated by smooth projective varieties. K. Kodaira proved in [Ann. of Math. (2) 78 (1963), 1-40; MR0184257 (32 #1730)] that every Kähler surface is almost algebraic. The statement that this should be true also in higher dimensions is known as the Kodaira conjecture. Recently, C. Voisin ["On the homotopy types of Kähler manifolds and the birational Kodaira problem", preprint, arxiv.org/abs/math/0410040] and K. Oguiso ["Automorphisms of hyperkähler manifolds in the view of topological entropy", preprint, arxiv.org/abs/math/0407476] constructed counterexamples by constructing rigid non-algebraic Kähler threefolds. The present paper, which was completed before the counterexamples appeared, gives some observations concerning the Kodaira conjecture. A certain blow-up of a $\mathbb{P}_1^3$-bundle over a 3-dimensional complex torus with Picard number ≥ 3 is shown to be rigid. It turns out, however, that these complex tori are algebraic. Some interesting generalizations are also considered.

Reviewed by H. Lange

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§ 43. Citations ↪
From References: 2
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MR2112584 (2006b:14011) Reviewed
Demailly, Jean-Pierre(F-GREN-IF)
On the geometry of positive cones of projective and Kähler varieties. (English summary)
The Fano Conference, 395-422, Univ. Torino, Turin, 2004.
14C25 (32J27)

Some of the more fundamental problems and results in complex geometry revolve around such questions as when a complex manifold would be projective or Kähler, or how much of its geometry could be determined by divisors and curves. All projective manifolds are Kähler, and a famous theorem of Kodaira proves that a Kähler manifold (X, ω) is projective precisely when the class $[\omega] \in H^{1,1}(X) \subset H^2(X, \mathbb{R})$ moreover represents a class in $H^2(X, \mathbb{Z})$. Kodaira had also conjectured that a compact complex surface admits a Kähler metric if and only if the first Betti number is even. A closely related question concerns the ampleness of holomorphic line bundles L on X. When X is projective, the Nakai-Moishezon criterion establishes that ampleness of L is equivalent to having a strictly positive integral for the p-th exterior power of the Chern class of L over any algebraic subset of dimension p for $1 \le p \le n = \dim(X)$. Mori's theory of complex three-manifolds brought new techniques to bear on the projective context via the geometry of cones of divisors and curves lying within their respective cohomology groups. For example, a conjecture of Fano asserts that a projective X is "uniruled" by rational curves precisely when the Chern class of the canonical line bundle lies outside the closure of the cone of effective divisors. The article under review is a survey of relatively recent achievements of the author and his collaborators, S. Boucksom, M. Păun and T. Peternell, in further unifying and extending the theory surrounding these questions. Central to their programme are the powerful techniques associated with positive currents of type $(1,1)$ on compact Kähler manifolds, and the interplay between the open convex cone of Kähler forms and the enveloping closed convex cone of positive $(1,1)$-currents (the "pseudo-effective" cone). While some basic familiarity with Kähler geometry and the theory of currents is assumed, the author's exposition is designed to be informative to the non-specialist. Among the results surveyed, some highlights are a generalization of the Nakai-Moishezon criterion and its application to the characterisation of Kähler currents on compact complex manifolds, as well as a theory of Poincaré duality between cones of positive currents of type $(1,1)$ and $(n-1, n-1)$, which leads in particular to a proof of Fano's conjecture.

For the entire collection see MR2112562 (2005g:14003).

Reviewed by Adam Gregory Harris

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§ 44. Citations ↪
From References: 41
From Reviews: 7
MR2113021 (2005i:32020) Reviewed
Demailly, Jean-Pierre(F-GREN); Păun, Mihai(F-STRAS)
Numerical characterization of the Kähler cone of a compact Kähler manifold. (English summary)
Ann. of Math. (2) 159 (2004), no. 3, 1247-1274.
32J27 (32Q15)

This article gives a beautiful solution of a long-standing basic problem in Kähler geometry and, as such, can be viewed as a classic. It is likely to have a lasting impact on the field.

The problem was to generalize the classical Nakai-Moishezon criterion of ampleness to a numerical characterization of the Kähler cone of a compact Kähler manifold.

Let us first recall the statement of the Nakai-Moishezon theorem. Let k be a field and X be a projective scheme over k. Let L be a Cartier divisor on X. Then L is ample iff for every positive dimensional reduced closed subscheme $Z \subset X$, $L^{\dim Z} \cdot Z > 0$.

If $k = \mathbf{C}$ and X is smooth, we can reformulate this using Kodaira's theorem that ample divisors L on X are characterized by the existence of a smooth Hermitian metric of positive curvature or, in an equivalent fashion, by the fact that the first Chern class $c_1(L)$, as an element of the vector space $H^{1,1}(X)$ of degree 2 de Rham real cohomology classes represented by closed $(1,1)$-forms, has a Kähler representative. The open convex cone in $H^{1,1}(X)$ consisting of classes with a Kähler representative is called the Kähler cone and will be denoted by $\mathcal{K}(X)$.

Thus, we get the following statement: Let X be a complex projective manifold. Let $\mathrm{NS}(X) \subset H^{1,1}(X)$ be the subset of $H^{1,1}(X)$ consisting of classes with integral periods. A class $\omega \in \mathrm{NS}(X)$ lies in $\mathcal{K}(X)$ iff $\int_Z \omega^{\dim Z} > 0$ for every positive-dimensional closed analytic subset Z of X.

We will denote by $\mathcal{P}(X)$ the set of classes ω cut out by the conditions that $\int_Z \omega^{\dim Z} > 0$ for every positive-dimensional closed analytic subset Z of X.

It was widely believed that a similar result holds for general real $(1,1)$ classes, namely that $\mathcal{K}(X) = \mathcal{P}(X)$.

The article under review confirms this conjecture in the more general case of compact Kähler manifolds. Here the statement should be modified to the effect that $\mathcal{K}(X)$ is a connected component of $\mathcal{P}(X)$.

The proof consists in a reduction to the nef case and a subtle application of Yau's fundamental work on the solution of the inhomogeneous complex Monge-Ampère equation in which the volume form acquires a singularity.

Reviewed by Philippe P. Eyssidieux

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§ 45. Citations ↪
From References: 1
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MR2039988 (2005f:32033) Reviewed
Demailly, Jean-Pierre(F-GREN-F); Peternell, Thomas(D-BAYR-IM)
A Kawamata-Viehweg vanishing theorem on compact Kähler manifolds. (English summary) Surveys in differential geometry, Vol. VIII (Boston, MA, 2002), 139-169,
Surv. Differ. Geom., VIII, Int. Press, Somerville, MA, 2003.
32L20 (32J27)

In this article the authors prove a partial generalization of the Kawamata-Viehweg vanishing theorem for a normal compact Kähler space X of dimension n: if L is a nef divisor with $L^2 \neq 0$, then $H^q(X, \mathcal{O}_X(K_X + L)) = 0$ for $q \geq n-1$.

As an application of this vanishing result, the authors derive the following corollary: Let X be a $\mathbf{Q}$-Gorenstein minimal Kähler threefold. Then $\kappa(X) \geq 0$. Recall that $\mathbf{Q}$-Gorenstein means that mK_X is Cartier for some $m > 0$ and a minimal Kähler threefold is a Kähler threefold with only terminal singularities such that K_X is nef.

The main importance of this result is that it applies to simple Kähler threefolds that are not Kummer. The interesting twist involved in this case is that it is expected that a simple minimal Kähler threefold is always Kummer. Now it may appear that because of this the above case is not that interesting, but the situation is the exact opposite. The above result reduces the problem of non-existence of non-Kummer simple minimal Kähler threefolds to the case of Kodaira dimension zero.

Furthermore, besides the main application mentioned above, vanishing theorems have been proven extremely useful in general, so one expects that this theorem will be helpful in many different ways.

For the entire collection see MR2039983 (2004j:00036).

Reviewed by Sándor J. Kovács

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§ 46. Citations ↪
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From Reviews: 0
MR2015548 (2004m:32040) Reviewed
Demailly, Jean-Pierre(F-GREN-F); Peternell, Thomas(D-BAYR-IM)
A Kawamata-Viehweg vanishing theorem on compact Kähler manifolds. (English summary)
J. Differential Geom. 63 (2003), no. 2, 231-277.
32L20 (32J27)

The purpose of this work is to extend to the compact Kähler case some important results in the Mori theory of projective complex varieties.

The authors first obtain a Kawamata-Viehweg type vanishing theorem for the cohomology group $H^{n,n-1}(X,L)$ of a nef line bundle L with $L^2 \neq 0$ on a normal compact Kähler space X of dimension n. This is done by first showing the vanishing of the map $H^{n-1}(X, K_X \otimes L \otimes \mathcal{I}) \to H^{n-1}(X, K_X \otimes L)$ for a well-chosen Nadel ideal sheaf and then showing that $\mathcal{O}(-L+D)$ has no nonzero holomorphic section on the divisorial part D of the variety of $\mathcal{I}$.

Then the authors obtain an abundance result on the existence of a pluricanonical section on a minimal Kähler 3-fold with terminal singularities. In order to apply the Riemann-Roch formula, they first prove a stability type inequality between Chern numbers and then a bound on $h^2(X, mK_X)$ thanks to the first part.

Reviewed by Christophe Mourougane

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Bertin, José(F-GREN); Demailly, Jean-Pierre(F-GREN); Illusie, Luc(F-PARIS11); Peters, Chris(F-GREN)
Introduction to Hodge theory. (English summary)
Translated from the 1996 French original by James Lewis and Peters. SMF/AMS Texts and Monographs, 8. American Mathematical Society, Providence, RI; Société Mathématique de France, Paris, 2002. x+232 pp. ISBN: 0-8218-2040-0
14C30 (14D07 14J32 32G20 32J25)

The French original has been reviewed [Introduction à la théorie de Hodge, Soc. Math. France, Paris, 1996; MR1409818 (97e:14011)].

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§ 48. Citations ↪
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MR1922099 (2003f:32029) Reviewed
Demailly, Jean-Pierre(F-GREN-F)
On the Frobenius integrability of certain holomorphic p-forms. (English summary) Complex geometry (Göttingen, 2000), 93-98, Springer, Berlin, 2002.
32Q15 (32J27)

Let X be a compact Kähler manifold, L a line bundle on X and $\theta \in H^0(X, \Omega_X^p \otimes L^{-1})$ a holomorphic p-form with values in L^{-1}. Finally, let S_θ be the subsheaf of germs of vector fields ξ in the tangent sheaf T_X such that the contraction $i_\xi \theta$ vanishes. In this paper, the author asks if the sheaf S_θ is integrable, i.e. if S_θ is closed under the Lie bracket $[S_\theta, S_\theta] \subset S_\theta$. He shows that S_θ is integrable if L is pseudo-effective.

For an application of this result, assume that X is a contact manifold, i.e. a manifold of odd complex dimension $\dim X = 2n+1$ that carries a form $\theta \in H^0(X, \Omega_X^1 \otimes L^{-1})$ such that $\theta \wedge (d\theta)^n \in H^0(X, K_X \otimes L^{-(n+1)})$ does not have any zeros– it is an elementary computation to see that the expression $\theta \wedge (d\theta)^n$ is well-defined even if θ is an L^{-1}-valued form. It was long conjectured that contact manifolds with $b_2(X) = 1$ must be Fano. Demailly's result immediately gives a positive answer to this conjecture. Together with the results of [Invent. Math. 142 (2000), no. 1, 1-15; MR1784795 (2002a:14047)], this implies that a projective contact manifold with $b_2(X) > 2$ is always isomorphic to the projectivized hyperplane bundle $X = \mathbf{P}(T_Y^*)$ associated with a projective manifold Y.

This paper is amazingly short and very well written. After the statement of the result, a brief review of contact manifolds and a list of known results, the actual proof takes a little less than two pages.

For the entire collection see MR1922091 (2003b:14001).

Reviewed by Stefan Kebekus

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§ **49. Citations ↪**
From References: 29
From Reviews: 2
MR1919457 (2003f:32020) Reviewed
Demailly, Jean-Pierre(F-GREN-F)
Multiplier ideal sheaves and analytic methods in algebraic geometry. School on Vanishing Theorems and Effective Results in Algebraic Geometry (Trieste, 2000), 1-148, ICTP Lect. Notes, 6, Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2001.
32J25 (32L10 32Q15)

This is an extended version of the CIME lectures by the same author [in Transcendental methods in algebraic geometry (Cetraro, 1994), 1-97, Lecture Notes in Math., 1646, Springer, Berlin, 1996; MR1603616 (99k:32051)].

The main changes have been made to present Siu's result on deformation invariance of plurigenera of varieties of general type. Along the way, a complete picture of the theory of multiplier ideal sheaves is drawn from the point of view of analytic geometry: Nadel vanishing theorem, subadditivity properties, global generation properties, Hard Lefschetz type theorem, semicontinuity properties. This theory hence provides new formulations of classical theorems such as the Skoda division theorem, the Ohsawa-Takegoshi-Manivel extension theorem, and the approximation of plurisubharmonic functions.

These notes written for non-specialists collect material published elsewhere. They serve as a nice display of the modern analytic toolbox in algebraic geometry.

For the entire collection see MR1919456 (2003b:14002).

Reviewed by Christophe Mourougane

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§ **50. Citations ↪**
From References: 29
From Reviews: 1
MR1875649 (2003a:32032) Reviewed
Demailly, Jean-Pierre(F-GREN); Peternell, Thomas(D-BAYR); Schneider, Michael (D-BAYR)
Pseudo-effective line bundles on compact Kähler manifolds. (English summary)
Internat. J. Math. 12 (2001), no. 6, 689-741.

32J27 (32Q15 32Q57)

This work is intended first to study the classification theory of compact Kähler manifolds with canonical bundle having weak positivity or negativity properties. The usual features of Kodaira dimension, Albanese map and fundamental group are studied in detail. (The case of projective 3-folds is carried further thanks to Mori theory.)

Those results are obtained from a nice generalization of the Hard Lefschetz theorem for the cohomology with values in a pseudo-effective line bundle. Although very technical, the heart of the work is a regularization process for quasi-pluri-subharmonic functions, for it enables one to deal with singular metrics via the Bochner technique.

Pseudoeffective line bundles and vector bundles are also studied on their own and especially with regard to their nefness properties.

Reviewed by Christophe Mourougane

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MR1861061 (2002k:32033) Reviewed
Campana, Frédéric(F-NANCS); Demailly, Jean-Pierre(F-GREN-F)
Cohomologie L^2 sur les revêtements d'une variété complexe compacte. (French) [**L^2-cohomology on the coverings of a compact complex manifold**]
Ark. Mat. 39 (2001), no. 2, 263-282.
32J25 (32C35)

Let X be a complex analytic space and $\tilde{X} \to X$ an unramified covering space (possibly with infinite fibers). Any coherent analytic sheaf F on X lifts to a coherent analytic sheaf on $\tilde{X}$. The article under review defines L^2 cohomology groups on $\tilde{X}$ with values in $\tilde{F}$ and gives a proof of its expected properties: cohomology exact sequences, Leray spectral sequences, Serre duality, vanishing theorems, finiteness of Γ-dimension and a variant of Atiyah's L^2 index theorem for Galois coverings [M. F. Atiyah, in Colloque "Analyse et Topologie" en l'Honneur de Henri Cartan (Orsay, 1974), 43-72. Astérisque, 32-33, Soc. Math. France, Paris, 1976; MR0420729 (54 #8741)]. When X is a compact manifold and F is free these L^2 cohomology groups can be computed by the L^2 Dolbeault complex. Similar results with a slightly different perspective have been independently developed by the reviewer [Math. Ann. 317 (2000), no. 3, 527-566 MR1776117].

Reviewed by Philippe P. Eyssidieux

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MR1852009 (2002e:32032) Reviewed
Demailly, Jean-Pierre(F-GREN-F); Kollár, János(1-PRIN)
Semi-continuity of complex singularity exponents and Kähler-Einstein metrics on Fano orbifolds. (English, French summary)
Ann. Sci. École Norm. Sup. (4) 34 (2001), no. 4, 525-556.
32Q20 (32U05)

Summary: “We introduce complex singularity exponents of plurisubharmonic functions and prove a general semicontinuity result for them. This concept contains as a special case several similar concepts which have been considered, e.g., by Arnol'd and Varchenko, mostly for the study of hypersurface singularities. The plurisubharmonic version is somehow based on a reduction to the algebraic case,

but it also takes into account more quantitative information of great interest for complex analysis and complex Einstein metrics on certain Fano orbifolds, following Nadel's original ideals (but with a drastic simplification in the technique, once the semicontinuity result is taken for granted). In this way, three new examples of rigid Kähler-Einstein del Pezzo surfaces with quotient singularities are obtained."

Reviewed by Lin Weng

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MR1786484 (2002a:14016) Reviewed
Demailly, Jean-Pierre(F-GREN-FM); Ein, Lawrence(1-ILCC); Lazarsfeld, Robert(1-MI)
A subadditivity property of multiplier ideals.
Dedicated to William Fulton on the occasion of his 60th birthday.
Michigan Math. J. 48 (2000), 137-156.
14E99 (14J17)

Given an effective fractional coefficient divisor D on a smooth projective complex algebraic variety X, the multiplier ideal is an ideal sheaf on the variety which reflects subtle features of the singularities of the divisor. For example, the closed subset defined by (the radical of) this ideal is precisely the locus of points where the pair (X, D) fails to be log-terminal. Thus the multiplier ideal endows the non-log terminal locus with a scheme structure that somehow quantifies the singularities of the pair. Multiplier ideals are playing an increasingly important role in higher-dimensional birational geometry, in part because of their strong vanishing properties.

This paper establishes an interesting relationship for the multiplier ideals associated to two different fractional coefficient divisors, D and E. Specifically, it is shown that the multiplier ideal of the sum $D + E$ is contained in the product of the multiplier ideal of D and the multiplier ideal of E. Proofs are presented both in the analytic framework, yielding the corresponding "subadditivity property" for multiplier ideals associated to plurisubharmonic functions, and in the algebraic framework, yielding the same subadditivity property also for multiplier ideals of ideals.

As an application, the authors re-prove a result of Fujita which gives a sort of numerical form of an approximate Zariski decomposition for big divisors on a smooth projective variety. Further applications of the "subadditivity" property appeared shortly thereafter in [L. M. H. Ein, R. K. Lazarsfeld and K. E. Smith, Invent. Math. 144 (2001), no. 2, 241-252 MR1826369]. The subadditivity property seems destined to become a fundamental result in the subject.

For the entire collection see MR1778979 (2001d:00057).

Reviewed by Karen E. Smith

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MR1782659 (2001m:32041) Reviewed
Demailly, Jean-Pierre(F-GREN-F)
On the Ohsawa-Takegoshi-Manivel L^2 extension theorem. (English, French summary) Complex analysis and geometry (Paris, 1997), 47-82,
Progr. Math., 188, Birkhäuser, Basel, 2000.
32L10 (32D15 32J25 32U05)

This paper provides a rather deep insight into the current status of L^2 extension techniques for sections [resp. $(0,q)$-forms] of vector bundles over complex analytic submanifolds. The fundamental extension theorem of T. Ohsawa and K. Takegoshi [Math. Z. 195 (1987), no. 2, 197-204; MR0892051 (88g:32029)], refined in many ways by Ohsawa, and in a more geometric setting by L. Manivel [Math. Z. 212 (1993), no. 1, 107-122; MR1200166 (94e:32050)], is proven here in its most general form. Unfortunately, a gap in the proof of Manivel is pointed out, regarding the regularity of the extension in the case of $(0,q)$-forms when $q > 0$. It thus reappears here as a conjecture, which is discussed in detail, but without being settled.

This theorem can yield powerful constructions that have been used in transcendental algebraic geometry. First, any psh function on a pseudoconvex open set in $\mathbb{C}^n$ can be approximated accurately with functions of the form $c \log |f|$ where f is a holomorphic function; this can be applied for instance to approximate the curvature current of a singular metric by divisors with multiplicities controlled by the Lelong numbers of the current. Other implications are detailed, among them a Briançon-Skoda theorem for multiplier ideal sheaves, and an analytical proof of Fujita's approximate Zariski decomposition for big line bundles.

For the entire collection see MR1782699 (2001c:32002).

Reviewed by Thierry Bouche

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§ 55. Citations ↪
From References: 10
From Reviews: 1
MR1772670 (2001m:32042) Reviewed
Demailly, Jean-Pierre(F-GREN-F)
Méthodes L^2 et résultats effectifs en géométrie algébrique. (French. French summary) [L^2-methods and effective results in algebraic geometry]
Séminaire Bourbaki, Vol. 1998/99.

Astérisque No. 266 (2000), Exp. No. 852, 3, 59-90.
32L10 (14C20 32J25)

This paper surveys the recent work that has been done by Demailly, Siu and Nadel among others about effective results in algebraic geometry obtained through Hörmander's L^2-methods for the $\overline{\partial}$ equation with singular metrics. A first section provides good insight into the basic tools, which are defined, and the results, whose proofs are outlined: singular metrics of holomorphic line bundles over complex analytic manifolds, the Bochner-Kodaira-Nakano identity for the antiholomorphic Laplace-Beltrami operator (in the case where the metric is smooth), L^2 estimates with singular metrics, Nadel's multiplier ideal sheaves and the corresponding vanishing theorem. Two important applications of these techniques are then described in detail: the Fujita conjecture [see Y. T. Siu, in Modern methods in complex analysis (Princeton, NJ, 1992), 291-318, Ann. of Math. Stud., 137, Princeton Univ. Press, Princeton, NJ, 1995; MR1369144 (98f:32032)] and Siu's theorem about the invariance of plurigenera under deformation [see Y. T. Siu, Invent. Math. 134 (1998), no. 3, 661-673; MR1660941 (99i:32035)].

For the entire collection see MR1772667 (2001b:00028).

Reviewed by Thierry Bouche

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Demailly, Jean-Pierre(F-GREN-F); El Goul, Jawher(F-TOUL3)
Hyperbolicity of generic surfaces of high degree in projective 3-space. (English summary)
Amer. J. Math. 122 (2000), no. 3, 515-546.
32Q45 (14J29 14J70)

S. Kobayashi [Hyperbolic manifolds and holomorphic mappings, Dekker, New York, 1970; MR0277770 (43 #3503)] conjectured that a generic hypersurface of dimension n in the projective space $\mathbf{P}^{n+1}$ is hyperbolic, i.e., every holomorphic map from the affine complex line $\mathbf{C}$ into such a hypersurface is constant. In the paper under review the authors verify the above conjecture for a very generic surface in $\mathbf{P}^3$ of degree $d \geq 21$ (i.e., away from a possible countable union of subvarieties in the moduli space of surfaces of degree d). More precisely, the surfaces for which the claim holds are of general type, have Picard number 1 and their Chern classes satisfy certain inequalities. The methods and techniques developed and used in the paper might be of independent interest but they are far too elaborate to be discussed here.. For the purpose of this review we outline briefly the key ideas of the proof which goes as follows. Using the Riemann-Roch theorem one produces a branched covering Z of X living in the projectivized tangent bundle of X. If $f\colon \mathbf{C} \longrightarrow X$ is a non-constant holomorphic map, then its first differential extends to a holomorphic map whose image is contained in a leaf of an algebraic foliation on Z. By a recent argument of M. McQuillan [Inst. Hautes Études Sci. Publ. Math. No. 87 (1998), 121-174; MR1659270 (99m:32028)], the resulting curve must be algebraically degenerate, i.e., contained in a proper algebraic subvariety of Z. In order to apply this result one is in fact forced to consider 2-jets. Then the closure of the image of f is either a rational or an elliptic curve. On the other hand, by a result of H. Clemens [Ann. Sci. École Norm. Sup. (4) 19 (1986), no. 4, 629-636; MR0875091 (88c:14037)], a generic surface of degree at least 7 in the projective space contains no rational or elliptic curves which implies that f is in fact constant.

Similar results in a more general context (implying, in particular, the Kobayashi conjecture for generic surfaces of degree at least 36 in $\mathbf{P}^3$) were obtained recently in [M. McQuillan, Geom. Funct. Anal. 9 (1999), no. 2, 370-392; MR1692470 (2000f:32035)].

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§ 57. Citations ↪
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Demailly, Jean-Pierre(F-GREN-F)

Pseudoconvex-concave duality and regularization of currents. (English summary) Several complex variables (Berkeley, CA, 1995-1996), 233-271,
Math. Sci. Res. Inst. Publ., 37, Cambridge Univ. Press, Cambridge, 1999.
32U40 (32C30 32C37 32F10 53C60)

The goal of this paper is to investigate some duality properties connecting pseudoconvexity and pseudoconcavity in a certain perspective to obtain a geometric version of the Serre duality theorem. These duality properties are related to several geometric problems, such as the conjecture of Hartshorne asserting that the complement of a q-codimensional algebraic subvariety Y with ample normal bundle N_Y in a projective algebraic variety X is q-convex in the sense of Andreotti-Grauert. M. Schneider proved the conjecture in the case that the normal bundle is positive in the sense of Griffiths. Using Sommese's result, the author proves the conjecture in the case that N_Y^* has a strictly convex plurisubharmonic Finsler metric.

Let X be a complex manifold of dimension n and E a holomorphic vector bundle of rank r. Demailly treats the problem of approximation of closed positive $(1,1)$-currents and the attenuation of their singularities. In general a closed positive current T cannot be approximated in the weak topology by smooth closed positive currents. J.-P. Demailly [Ann. Sci. École Norm. Sup. (4) 15 (1982), no. 3, 457-511; MR0690650 (85d:32057); J. Algebraic Geom. 1 (1992), no. 3, 361-409; MR1158622 (93e:32015); in Contributions to complex analysis and analytic geometry, 105-126, Vieweg, Braunschweig, 1994; MR1319346 (96k:32012)] proved that this approximation is possible if we allow the regularization T_ε to have a small negative part. The main point is to control the negative part in terms of the global geometry of the ambient geometry X. It turns out that more or less optimal bounds can be described in terms of the convexity of a Finsler metric on the tangent bundle T_X. The author gives an easy proof based on the use of symmetric products of Finsler metrics.

For the entire collection see MR1748597 (2000k:32002).

Reviewed by Mongi Blel

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§ **58. Citations ↪**
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MR1622747 (99e:32047) Reviewed
Campana, Frédéric(F-NANC); Demailly, Jean-Pierre(F-GREN-F); Peternell, Thomas (D-BAYR-IM)
The algebraic dimension of compact complex threefolds with vanishing second Betti number. (English summary)
Compositio Math. 112 (1998), no. 1, 77-91.
32J17

A compact complex threefold with vanishing second Betti number cannot be algebraic or Kähler. Then the natural question is: What possibilities are there for the algebraic dimension of such manifolds? (Algebraic dimension is the transcendence degree of the field of meromorphic functions over **C**.)

The main result of this article is that if the algebraic dimension is positive, then the topological Euler characteristic is 0 and then either $b_1 = 0$ and $b_3 = 2$ or $b_1 = 1$ and $b_3 = 0$. An interesting corollary is that S^6 does not admit a complex structure with a non-constant meromorphic function. The authors deduce the main result as a straightforward consequence of a vanishing theorem for vector bundles twisted by generic elements of Pic^0. Examples of threefolds with positive algebraic dimension and vanishing second Betti number and topological Euler characteristics are also given showing that the result is optimal.

The authors also investigate more deeply threefolds with vanishing second Betti number and whose algebraic dimension is 1.

This is another interesting article from these distinguished authors presenting ideas of great interest to algebraic and complex analytic geometers alike.

Reviewed by Sándor J. Kovács

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This list reflects references listed in the original paper as accurately as possible with no attempt to correct error.

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§ 59. Citations ↪
From References: 1
From Reviews: 0
MR1492596 (99a:32033) Reviewed
Demailly, Jean-Pierre(F-GREN-F)
Variétés projectives hyperboliques et équations différentielles algébriques. (French) [Hyperbolic projective varieties and algebraic differential equations] Journée en l'Honneur de Henri Cartan, 3-17,
SMF Journ. Annu., 1997, Soc. Math. France, Paris, 1997.
32H20 (32J10 32L05)

This is a very well-written survey of some of the more recent developments in the theory of holomorphic curves in algebraic varieties, a holomorphic curve in an algebraic variety being a holomorphic mapping from the complex plane to the variety. The author's survey concentrates in particular on the recent work of Y. T. Siu and S.-K. Yeung [Amer. J. Math. 119 (1997), no. 5, 1139-1172; MR1473072 (98h:32044)]. An extensive bibliography is also provided. Although probably best suited for those readers already familiar with the language of complex differential geometry, and in particular the language used when working with Hermitian vector bundles and meromorphic connections, the survey is for the most part a very accessible introduction to some of the latest developments in the field and assumes little prior knowledge of Nevanlinna theory, algebraic geometry, or the other techniques commonplace in the study of holomorphic curves.

The reviewer's translation of the author's first paragraph reads as follows: "The goal of this text is to offer an introduction, which is as elementary as possible, to an important result concerning the geometry of the images of holomorphic curves in complex algebraic varieties. This result finds its origin in the fundamental work of A. Bloch [J. Math. Pures Appl. (9) 5 (1926), 19-66; JFM 52.0373.05] and in the thesis of H. Cartan [Ann. Sci. École Norm. Sup. 45 (1928), 255-346; JFM 54.0357.06]. The proof that we give here is a very recent contribution by Siu and Yeung [op. cit.]. It proceeds in a relatively simple manner with help from classical estimates in Nevanlinna theory, like the lemma on the logarithmic derivative, and by making use of differential operators such as Wronskians, all ideas whose germs were already sown in Henri Cartan's thesis [op. cit.]."

More specifically, the author explains techniques for showing that a holomorphic curve in an algebraic variety is algebraically degenerate, meaning that its image is contained in a proper algebraic subvariety. A fundamental conjecture along these lines is the conjecture of Green and Griffiths stating that a holomorphic curve in a variety of general type must be algebraically degenerate. The survey is centered around the following fundamental vanishing theorem. If $f\colon \mathbf{C} \to X$ is a holomorphic curve in a projective variety X, if L is a positive line bundle on X, and if P is an algebraic differential operator on X with values in L^{-1}, then P applied to f is zero. For hypersurfaces in projective space, this theorem can be applied to Wronskian-like differential operators coming from explicitly constructed meromorphic connections, as in the work of A. M. Nadel [Duke Math. J. 58 (1989), no. 3, 749-771; MR1016444 (91a:32036)]. This results in specific examples of general type projective varieties in which every holomorphic curve is algebraically degenerate. This method also proves that in some of these varieties, the image of every holomorphic curve must be constant; such varieties are called hyperbolic.

For the entire collection see MR1492594 (98h:00041).

Reviewed by William A. Cherry

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§ 60. Citations ↪
From References: 0
From Reviews: 0
MR1492594 (98h:00041) Reviewed
Hirzebruch, Friedrich; Demailly, Jean-Pierre(F-GREN-F); Lannes, Jean
Journée en l'Honneur de Henri Cartan. (French) [Conference in Honor of Henri Cartan]
SMF Journée Annuelle [SMF Annual Conference], 1997. Société Mathématique de France, Paris, 1997. iv+27 pp.
00B30

Contents: F. Hirzebruch, Learning complex analysis in Muenster-Paris, Zuerich and Princeton from 1945 to 1953 (1-2); Jean-Pierre Demailly, Variétés projectives hyperboliques et équations différentielles algébriques [Hyperbolic projective varieties and algebraic differential equations] (3-17); Jean Lannes, Divers aspects des opérations de Steenrod [Various aspects of the Steenrod operations] (18-27).

Most of the papers are being reviewed individually.

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§ 61. Citations ↪
From References: 51
From Reviews: 15
MR1492539 (99b:32037) Reviewed
Demailly, Jean-Pierre(F-GREN-F)
Algebraic criteria for Kobayashi hyperbolic projective varieties and jet differentials. (English summary) Algebraic geometry– Santa Cruz 1995, 285-360,
Proc. Sympos. Pure Math., 62, Part 2, Amer. Math. Soc., Providence, RI, 1997.
32H20 (14J40 32L10)

The article under review is an expanded version of five lectures delivered at the Santa Cruz AMS Summer School on Algebraic Geometry. It proposes an important framework for solving several geometry questions related to hyperbolicity in the sense of Kobayashi. This framework was initiated by M. Green and P. Griffiths [in The Chern Symposium 1979 (Proc. Internat. Sympos., Berkeley, Calif., 1979), 41-74, Springer, New York, 1980; MR0609557 (82h:32026)]. Aiming, among other things, to fix a gap in Green-Griffiths' proof of the pointwise version of the Ahlfors-Schwarz lemma for jet differentials, Demailly introduces the concept of "directed manifold" and an associated tower of projective bundles over X (called Semple jet bundles). The Ahlfors-Schwarz lemma is then established in this setting, and the proof of Bloch's theorem is recovered following the approach of Green and Griffiths. Several important new results are also obtained in this paper. Moreover, the author believes that the Semple bundle construction should be an efficient tool to calculate the case locus; therefore several important open problems in the theory of complex hyperbolicity hopefully could be settled under this framework. It should be noted that, since the appearance of this article, Demailly has proved jointly with J. El Goul that every generic surface in $\mathbf{P}^3$ of degree greater than or equal to 42 is Kobayashi hyperbolic. It has been conjectured by Kobayashi that every generic surface in $\mathbf{P}^3$ of degree greater than or equal to 5 is Kobayashi hyperbolic.

This paper is a quite important contribution to the theory of complex hyperbolicity. The paper is self-contained and the exposition is excellent. It is highly recommended to the experts in this field, as well as to anyone who desires a general overview of this subject.

We will now try to outline this article. A complex directed manifold is a pair (X, V) where X is a complex manifold and V is a holomorphic subbundle of T_X; here T_X is the tangent bundle of X. To study the complex hyperbolicity of (X, V), a well-known major technique is the so-called "negative curvature method". The method is based on the following observation: by the Ahlfors-Schwarz lemma, the existence of a Hermitian metric on the line bundle $\mathcal{O}_{\mathbf{P}(V)}(-1)$ over $\mathbf{P}(V)$ (i.e. a Finsler metric on V) with negative curvature implies that (X, V) is hyperbolic. Let us recall here how to construct such a metric. Assume that V^* is "very big" in the following sense: there exist an ample line bundle L and a sufficiently large integer m such that the global sections $H^0(X, S^m V^* \otimes L^{-1})$ generate all fibers over

$X \setminus Y$, for some analytic subset $Y \subset X$. Let $\sigma_1, \cdots, \sigma_N$ be such global sections, and define

$$N(\xi) = \Big(\sum_{1 \le j \le N} |\sigma_j(x) \cdot \xi^m|^2 \Big)^{1/2m}, \quad \xi \in V_x^*;$$

N then gives rise to such a metric. Therefore we have the following result: Let (X, V) be a directed complex manifold. Assume that V^* is "very big". Then every entire curve $f\colon \mathbf{C} \to X$ tangent to V satisfies $f(\mathbf{C}) \subset Y$, where Y is the subset of X defined above. In particular, if V^* is ample, then (X, V) is hyperbolic.

The heart of the article consists of Chapters 4 to 7. They are devoted to extending the above result to k-jet differentials. The idea is based on the important fact, first observed by Green and Griffiths, that the Ahlfors-Schwarz lemma still works for k-jet differentials, and thus k-jet negativity also implies hyperbolicity. Unfortunately, there is a slight technical gap in Green and Griffiths' approach in the step proving the pointwise Ahlfors-Schwarz lemma for jet differentials. In his paper, Demailly fills the gap in the case of invariant jet differentials, and also extends the result to the more general situation of directed manifolds. (Note: Another solution has been provided later by Y. T. Siu and S.-K. Yeung by means of Nevanlinna's second main theorem [see Amer. J. Math. 119 (1997), no. 5, 1139-1172; MR1473072 (98h:32044)].)

To do this, Demailly introduces a canonical tower of projective bundles (also called Semple jet bundles). Given a complex directed manifold (X, V), a new complex directed manifold $(\widetilde{X}, \widetilde{V})$ is produced as follows. Let $\widetilde{X} = \mathbf{P}(V)$ be the projectivized bundle of lines of V, and let $\widetilde{V} \subset T_{\widetilde{X}}$ be the subbundle of $T_{\widetilde{X}}$ defined as follows: for every point $(x, [v]) \in \widetilde{X}$ associated with a vector $v \in V_x \setminus \{0\}$,

$$\widetilde{V}_{(x,[v])} = \big\{ \xi \in T_{\widetilde{X},(x,[v])} \colon \pi_* \xi \in \mathbf{C} \big\}, \quad V_x \subset T_{X,x},$$

where $\pi\colon \widetilde{X} = \mathbf{P}(V) \to X$ is the natural projection. The projectivized k-jet bundle $\mathbf{P}_k V = X_k$ (or Semple k-jet bundle) and the associated subbundle $V_k \subset T_{X_k}$ are defined inductively by $(X_0, V_0) = (X, V)$, $(X_k, V_k) = (\widetilde{X}_{k-1}, \widetilde{V}_{k-1})$. Every non-constant tangent trajectory $f\colon \Delta_R \to X$ of (X, V) lifts to a well-defined and unique tangent trajectory $f_{[k]}\colon \Delta_R \to X_k$ of (X_k, V_k).

The author shows that the Ahlfors-Schwarz lemma works at each level of the tower of projective bundles. That is: If (X, V) has a k-jet metric h_k on the line bundle $\mathcal{O}_{\mathbf{P}_k V}(-1)$ (i.e. a Finsler metric on the vector bundle V_{k-1} over $\mathbf{P}_{k-1} V$), with negative jet curvature, then every entire curve $f\colon \mathbf{C} \to X$ tangent to V satisfies $f_{[k]}(\mathbf{C}) \subset \Sigma_{h_k}$, where Σ_{h_k} is the singularity set of the metric h_k.

To produce such metrics h_k, one uses global sections of $H^0(\mathbf{P}_k V, \mathcal{O}_{\mathbf{P}_k V}(m) \otimes \pi_{0,k}^* L^{-1})$, where L is an ample line bundle on X. The author also shows that the direct images $(\pi_{0,k})_* \mathcal{O}_{\mathbf{P}_k V}(m)$ can be viewed as bundles of algebraic differential operators of order k and degree m, acting on germs of curves and invariant under reparametrization. This bundle is denoted by $E_{k,m}(V^*)$. Therefore $H^0(\mathbf{P}_k V, \mathcal{O}_{\mathbf{P}_k V}(m) \otimes \pi_{0,k} L^{-1}) \simeq H^0(X, E_{k,m}(V^*) \otimes L^{-1})$.

The above discussion leads to the following result: Assume that there exist integers $k, m > 0$ and an ample line bundle L on X such that $H^0(X, E_{k,m}(V^*) \otimes L^{-1})$ has nonzero sections $\sigma_0, \cdots, \sigma_N$. Let $Z \subset \mathbf{P}_k V$ be the base locus of these sections. Then every entire curve $f\colon \mathbf{C} \to X$ tangent to V satisfies $f_{[k]}(\mathbf{C}) \subset Z$. In other words, for every global parametrization invariant polynomial differential operator P with values in L^{-1}, every entire curve f as above must satisfy the algebraic differential equation $P(f) = 0$.

The dimension

$$h^0(X, E_{k,m}(V^*) \otimes L^{-1}) = \dim H^0(X, E_{k,m}(V^*) \otimes L^{-1})$$

can be computed by using the Riemann-Roch theorem and a vanishing theorem due to Bogomolov. In particular, in the surface case, the Riemann-Roch theorem yields the following (see Chapter 13, Corollary 13.9): If X is an algebraic surface of general type and L an ample line bundle over X, then

$$h^0(X, E_{2,m} T^* X \otimes \mathcal{O}(-L)) \ge \frac{m^4}{648}(13c_1^2 - 9c_2) - O(m^3).$$

In particular, every smooth surface $X \subset \mathbf{P}^3$ of degree $d \ge 15$ admits a nontrivial section, and every entire function $f\colon \mathbf{C} \to X$ must satisfy the corresponding algebraic differential equations.

However, it seems very difficult to conclude that f satisfies an algebraic equation. The author suggests in Chapter 13 that the Riemann-Roch calculations might be helpful to locate the base locus, thus to conclude the algebraic degeneracy.

Another important part of this article is Chapter 2 and Chapter 9, where Demailly shows that Kobayashi hyperbolicity is related to other properties of a more algebraic nature. A projective directed manifold (X, V) is called algebraically hyperbolic if there exists $\varepsilon > 0$ such that every algebraic curve $C \subset X$ tangent to V satisfies $2\,g(\overline{C}) - 2 \geq \varepsilon \deg_\omega(C)$ ($\overline{C}$ is the normalization of C). The main result of Chapter 2 is that if (X, V) is hyperbolic, then (X, V) is algebraically hyperbolic. Chapter 9 extends this result to k-jet metrics and shows that the negativity of k-jet curvature implies strong restrictions of an algebraic nature on curve genera and their singularity indices.

Chapter 11 recalls the "meromorphic connection" method introduced by Siu [Y. T. Siu, Duke Math. J. 55 (1987), no. 1, 213-251; MR0883671 (89a:32030); A. M. Nadel, Duke Math. J. 58 (1989), no. 3, 749-771; MR1016444 (91a:32036)]. Using this method, the author reports on a joint work with J. El Goul, where examples of hyperbolic surfaces in $\mathbf{P}^3$ are produced for any degree ≥ 11.

For the entire collection see MR1492532 (98h:14003).

Reviewed by Min Ru

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§ 62. Citations ↪
From References: 5
From Reviews: 0
MR1462789 (98j:32027) Reviewed
Demailly, Jean-Pierre(F-GREN)
Variétés hyperboliques et équations différentielles algébriques. (French) [Hyperbolic manifolds and algebraic differential equations]
Gaz. Math. No. 73 (1997), 3-23.
32H20 (14J99 32H30)

Let X be a projective algebraic variety and $f\colon \mathbf{C} \to X$ be a nonconstant entire curve. Then for every algebraic differential operator P with values in the dual L^* of a holomorphic line bundle L over X with positive curvature, one has $P(f', \cdots, f^{(k)}) \equiv 0$.

This theorem, which was stated by M. Green and P. Griffiths [in The Chern Symposium 1979 (Proc. Internat. Sympos., Berkeley, Calif., 1979), 41-74, Springer, New York, 1980; MR0609557 (82h:32026)], plays a key role if one wants to prove hyperbolicity of X. After Demailly [in Algebraic geometry– Santa Cruz 1995, 285-360, Proc. Sympos. Pure Math., 62, Part 2, Amer. Math. Soc., Providence, RI, 1997 MR1492539] had proved a slightly weaker version, Y. T. Siu and S.-K. Yeung [Amer. J. Math. 119 (1997), no. 5, 1139-1172; MR1473072 (98h:32044)] gave a convincing proof.

The paper under review is an extended version of the author's talk given in honor of Henri Cartan. The first part deals with another proof of the theorem, the idea of which was also apparently given by Siu. The main tool used in this proof is Nevanlinna theory. The presentation is self contained and very elegant. The second part of the paper follows Demailly and J. El Goul [C. R. Acad. Sci. Paris Sér. I Math. 324 (1997), no. 12, 1385-1390; MR1457092 (98j:32026); see the preceding review]. Here the author shows how to use the theorem in combination with a Wronskian differential operator and the author's concept of partial projective connections to obtain families of smooth hyperbolic surfaces $Y \subset \mathbf{P}^3$ of any degree $d \geq 11$. The hyperbolicity of the same families was also obtained by Siu and Yeung [op. cit.] with different methods.

Reviewed by Gerd Dethloff

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§ 63. Citations ↪
From References: 2
From Reviews: 1
MR1457092 (98j:32026) Reviewed
Demailly, Jean-Pierre(F-GREN-FM); El Goul, Jawher(F-GREN-FM)
Connexions méromorphes projectives partielles et variétés algébriques hyperboliques. (French. English, French summary) [Partial projective meromorphic connections and hyperbolic projective varieties]
C. R. Acad. Sci. Paris Sér. I Math. 324 (1997), no. 12, 1385-1390.
32H20 (32H30)

This paper presents one more exposition on the hyperbolicity of special hypersurfaces in $\mathrm{P}^3(\mathbf{C})$ and related topics [see A. M. Nadel, Duke Math. J. 58 (1989), no. 3, 749-771; MR1016444 (91a:32036); J. El Goul, Manuscripta Math. 90 (1996), no. 4, 521-532; MR1403721 (97k:32039)].

The main result of Nadel and the scheme of its proof are not modified in the paper under review. One can single out two new aspects of the exposition. The authors indicate a new possibility for proving the main auxiliary result, which following Nadel is usually named "the Siu degeneration theorem", and introduce a notion of partial projective connections.

As for possible development of the subject, it seems that a generalization of the Siu degeneration theorem for the case of singular projective varieties would give substantial progress in this area. Till now the classical second main theorem of H. Cartan [Mathematica (Cluj) 7 (1933), 5-31; Zbl 007.41503 (p. 12)] enables one to obtain more general and sharper results than that of Nadel-El Goul-Demailly. For the details see the review of the above-cited paper of El Goul.

Reviewed by Evgenii Nochka

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§ 64. Citations ↪
From References: 67
From Reviews: 3
MR1603616 (99k:32051) Reviewed
Demailly, Jean-Pierre(F-GREN-F)
L^2 vanishing theorems for positive line bundles and adjunction theory. Transcendental methods in algebraic geometry (Cetraro, 1994), 1-97,
Lecture Notes in Math., 1646, Springer, Berlin, 1996.
32L20 (14C20 14F17 32L10)

As explained by the author, “these notes are essentially written with the idea of serving as an analytic toolbox for algebraic geometry, with special attention paid to linear series and vanishing theorems for algebraic vector bundles”. This paper provides a nice review in particular on currents, multiplier ideal sheaves, and Seshadri constants. The main analytic ideas occurring in the author’s and Y. T. Siu’s results concerning Fujita’s conjecture on global generation of adjoint linear systems, and an effective version of Matsusaka’s big theorem on numerical criteria for very ampleness, are described in detail. We recommend that one also read ["Méthodes L^2 et résultats effectifs en géométrie algébrique”, in Séminaire Bourbaki (1998), Exp. No. 852; per revr.], where the author describes the use of the whole analytic toolbox in Siu’s proof of invariance of plurigenera in a family of general type. These two papers contain complete lists of references.

For the entire collection see MR1603612 (98h:32001).

Reviewed by Christophe Mourougane

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§ 65. Citations ↪
From References: 2
From Reviews: 1
MR1603612 (98h:32001) Reviewed
Demailly, J.-P.(F-GREN-F); Peternell, T.(D-BAYR-IM); Tian, G.(1-MIT); Tyurin, A. N.(RS-AOS)
Transcendental methods in algebraic geometry.
Lectures given at the 3rd C.I.M.E. Session held in Cetraro, July 4-12, 1994. Edited by F. Catanese and C. Ciliberto. Lecture Notes in Mathematics, 1646. Fondazione C.I.M.E.. [C.I.M.E. Foundation] Springer-Verlag, Berlin; Centro Internazionale Matematico Estivo (C.I.M.E.), Florence, 1996. viii+247 pp. ISBN: 3-540-62038-9
32-06 (32J25)

Contents: Jean-Pierre Demailly, L^2 vanishing theorems for positive line bundles and adjunction theory (1-97); Thomas Peternell, Manifolds of semi-positive curvature (98-142); Gang Tian, Kähler-Einstein metrics on algebraic manifolds (143-185); Andrei Tyurin [A. N. Tyurin], Six lectures on four-manifolds (186-246).

The papers are being reviewed individually.

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§ 66. Citations ↪
From References: 10
From Reviews: 0

MR1409819 (1997) Indexed
Demailly, Jean-Pierre(F-GREN-F)
Théorie de Hodge L^2 et théorèmes d'annulation. (French) [L^2 Hodge theory and vanishing theorems] Introduction à la théorie de Hodge, 3-111,
Panor. Synthèses, 3, Soc. Math. France, Paris, 1996.
32L20 (32J25)

This item will not be reviewed individually.

For the entire collection see MR1409818 (97e:14011).

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§ **67. Citations ↪**
From References: 8
From Reviews: 2
MR1409818 (97e:14011) Reviewed
Bertin, José(F-GREN-F); Demailly, Jean-Pierre(F-GREN-F); Illusie, Luc(F-PARIS11); Peters, Chris(F-GREN-F)
Introduction à la théorie de Hodge. (French. English, French summary) [Introduction to Hodge theory]
Panoramas et Synthèses [Panoramas and Syntheses], 3. Société Mathématique de France, Paris, 1996. vi+273 pp. ISBN: 2-85629-049-3
14C30 (14D07 14J32 14N10 32G20 32J25)

Since its introduction, Hodge theory and its variants (variations of Hodge structures, mixed Hodge structures, variations of mixed Hodge structures) have been a source of some of the deepest results in algebraic geoemetry. The Hodge index theorem, for example, was one of the first results relating analytic invariants (the dimensions $h^{p,q}$ of the space of harmonic (p, q)-forms) with algebraic-topological invariants of an algebraic variety X over $\mathbf{C}$; in a sense this is even a model for the much deeper Donaldson theory of four-manifolds. More recently the theory of variations of Hodge structures has served as the basis of the theory of Shimura varieties, and, similarly, mixed Hodge structures form a basis for a theory of mixed Shimura varieties, important for the theory of compactifications.

The book under review is a collection of three articles about Hodge theory and related developments, which are all aimed at non-experts and fulfill, in an extremely satisfactory manner, two functions. First, the basic methods used in the theories are discussed and developed in great detail; second, some newer developments are described, giving the reader a good overview of the more important applications. Furthermore, the style makes these articles a joy to work through, even for the mathematician not encountering these subjects for the first time. I now sketch the contents of the individual articles.

The first, by Demailly, concerns the L^2-theory of Hodge structures. Although Hodge theory was not originally introduced in this manner, the method is very appropriate today, especially concerning applications to non-compact or singular varieties. The L^2-condition is just right to carry over basic results to this context. This paper consists of two parts. In the first the general theory is derived. The author is very thorough in the development, and in particular, the entire theory of pseudo-elliptic operators (necessary for the finite-dimensionality of Hodge groups) is developed in the more general context. Similarly, the theory of Kähler manifolds is presented in this context. One section is devoted to the Hodge-Fröhlicher spectral sequence, which describes exactly how, for a general (not necessarily Kähler) compact complex manifold, the actual Hodge groups deviate from the accustomed (Kähler) symmetry $H^{p,q} = \overline{H^{q,p}}$.

In the second part of this paper, vanishing results and applications are discussed, again everything in the L^2-context. A very complete explanation of the notions of pseudoconvexity and positivity of vector bundles is given, together with the corresponding vanishing results. The Bochner method is described in detail, with several applications. It turns out that the L^2-methods allow a simplified proof of some very recent results of Y. T. Siu [Invent. Math. 124 (1996), no. 1-3, 563-571; MR1369428 (97a:32036)], culminating in an effective version of Matsusaka's big theorem (an effective bound m_0 so that, for positive L and all $m \geq m_0$, L^m is very ample).

The second article, by Illusie, is concerned with a very different aspect: applications of characteristic $p > 0$ methods to characteristic 0, in particular, the proof of degeneration of the Hodge spectral sequence and the vanishing theorem of Kodaira-Akizuki-Nakano of P. Deligne and Illusie [Invent. Math. 89 (1987), no. 2, 247-270; MR0894379 (88j:14029)], which applied these methods. Again the level of presentation is for non-specialists and contains a fair amount of rather inaccessible background material, written in a very lucid and understandable manner. One section discusses the notions of smoothness and liftability (essentially EGA material). The next introduces and describes in detail the principal characteristic p

methods: Frobenius and the Cartier isomorphism. The following section sketches the homological algebra needed to formulate the result: derived categories and spectral sequences. After the proof of the main theorem of Deligne and Ilusie [op. cit.], the "well-known" methods used to deduce from characteristic p the corresponding statements in characteristic 0 is described in detail. An additional section considers more recent results and open problems.

Finally, the third article, by Bertin and Peters, discusses variations of Hodge structures and its applications to mirror symmetry. This article also consists of two parts, the first with general theory, the second with more specific applications in mind. In particular, the discussion of the Gauss-Manin connection is done in great detail, as this is one of the fundamentals of the entire theory and in particular of the applications which are to follow. The local monodromy theorem is described, and a proof, due to W. Schmid, is sketched; also, other results of the fundamental work of his [Invent. Math. 22 (1973), 211-319; MR0382272 (52 #3157)] are described, like the limit mixed Hodge structure. Further topics include the local invariant cycle theorem and the status of vanishing cycles. This leads naturally to the complicated theory of Hodge modules due to Sato, which is then introduced. In the second part, after a brief introduction to mirror symmetry, the Picard-Fuchs equations of a family of Calabi-Yau manifolds is derived in great detail. In the last section, the authors then give a new interpretation of the mirror symmetry conjecture: For every $q \in \Delta^*$, the mixed structure on $H^+(M^*) \times \{q\}$ coincides with the mixed structure of Deligne on $H^3(M_q)$. (Notation: M^* is the generic member of the mirror family and H^+ denotes the even cohomology.)

Reviewed by Bruce Hunt

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§ 68. Citations ↪
From References: 11
From Reviews: 0
MR1389367 (97d:32039) Reviewed
Demailly, Jean-Pierre(F-GREN); Peternell, Thomas(D-BAYR-IM); Schneider, Michael (D-BAYR-IM)
Compact Kähler manifolds with Hermitian semipositive anticanonical bundle. (English summary)
Compositio Math. 101 (1996), no. 2, 217-224.
32J27 (53C55)

Summary: "This note states a structure theorem for compact Kähler manifolds with semipositive Ricci curvature: Any such manifold has a finite étale covering possessing a de Rham decomposition as a product of irreducible compact Kähler manifolds, each one being either Ricci flat (torus, symplectic or Calabi-Yau manifold) or Ricci semipositive without nontrivial holomorphic forms. Related questions and conjectures concerning the latter case are discussed."

Reviewed by Thierry Bouche

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§ 69. Citations ↪
From References: 15
From Reviews: 3
MR1369417 (97a:32035) Reviewed
Demailly, Jean-Pierre(F-GREN-F)
Effective bounds for very ample line bundles. (English summary)
Invent. Math. 124 (1996), no. 1-3, 243-261.
32L10 (14C20 32J25)

This paper is a continuation of the variations on the Fujita conjecture. Using carefully the same technique as Y. T. Siu [Invent. Math. 124 (1996), no. 1-3, 563-571; see the following review], it is shown that $2K_X + mL$ is very ample if $m \geq m_0 = 2 + \binom{n}{3n+1}$ for any ample line bundle L on X. This is an improvement in the bounds by a factor n. But the main interest of this paper is to present a detailed yet very clear exposition of the state of the art techniques involved. Although m_0 is not sharp, the methods already yield a few sharp results, such as the nefness of $K_X + (n+1)L$ if $\dim X = n$ (this result is due to Fujita himself). The paper also includes a proof of Y. T. Siu's effective big Matsusaka theorem [Ann. Inst. Fourier (Grenoble) 43 (1993), no. 5, 1387-1405; MR1275204 (95f:32035)], with corresponding refined bounds, so that it provides a panoramic view on effective very ampleness results.

Reviewed by Thierry Bouche

§ 70. Citations ↪
From References: 4
From Reviews: 0
MR1360502 (96k:14016) Reviewed
Demailly, Jean-Pierre(F-GREN-F); Peternell, Thomas(D-BAYR); Schneider, Michael (D-BAYR)
Holomorphic line bundles with partially vanishing cohomology. (English summary) Proceedings of the Hirzebruch 65 Conference on Algebraic Geometry (Ramat Gan, 1993), 165-198,
Israel Math. Conf. Proc., 9, Bar-Ilan Univ., Ramat Gan, 1996.
14F17 (32L10 32L20)

Let L be a holomorphic line bundle over an n-dimensional complex projective manifold X. Classical results of Andreotti-Grauert show that the existence of a Hermitian metric on L with a curvature form having $n-q$ positive eigenvalues at all points of X implies some important results on the vanishing for $j>q$ of the jth cohomology groups of coherent sheaves on X tensored with sufficiently high powers of L. In this article the authors reverse the question and ask when, given a line bundle L on X, the appropriate vanishing condition for cohomology implies the existence of a Hermitian metric on L with an appropriate number of positive eigenvalues. The authors introduce the technically useful notion of an ample q-flag. This is a flexible generalization of the notion of a sequence $Y_q\subset Y_{q+1}\subset\cdots\subset X$ of subvarieties with Y_j an ample Cartier divisor of Y_{j+1}, with the condition of being Cartier relaxed in a way that the concept becomes invariant under finite maps. It is shown that if L is q-flag positive, i.e., if there is an ample q-flag with L_{Y_q} ample, then there occurs the cohomology vanishing that one would expect if L had an Hermitian metric on L with a curvature form having $n-q$ positive eigenvalues at all points of X. An example is given which shows that the converse of this is not true when $n=3$, $q=2$. Some positive results are given when $q=1$. Finally, in the case when $L:=-K_X$ is nef and big, an interesting structure result is given relating appropriate vanishing of cohomology with the situation in which the maximal fiber dimension of the morphism associated to a high power of $-K_X$ is one.

For the entire collection see MR1360492 (96f:14002).

Reviewed by Andrew J. Sommese

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§ 71. Citations ↪
From References: 1
From Reviews: 1
MR1403982 (98e:32055) Reviewed
Demailly, Jean-Pierre(F-GREN-F)
L^2-methods and effective results in algebraic geometry. (English summary) Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994), 817-827, Birkhäuser, Basel, 1995.
32L20 (32L10)

Summary: "One important problem arising in algebraic geometry is the computation of effective bounds for the degree of embeddings in a projective space of given algebraic varieties. This problem is intimately related to the following question: Given a positive (or ample) line bundle L on a projective manifold X, can one compute explicitly an integer m_0 such that mL is very ample for $m\geq m_0$? It turns out that the answer is much easier to obtain in the case of adjoint line bundles $2(K_X+mL)$, for which universal values of m_0 exist. We indicate here how such bounds can be derived by a combination of powerful analytic methods; theory of positive currents and plurisubharmonic functions (Lelong), L^2 estimates for $\overline{\partial}$ (Andreotti-Vesentini, Hörmander, Bombieri, Skoda), Nadel vanishing theorem, Aubin-Calabi-Yau theorem, and holomorphic Morse inequalities."

For the entire collection see MR1403907 (97c:00049).

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§ 72. Citations ↪
From References: 0
From Reviews: 0
MR1351504 (96h:32040) Reviewed
Demailly, Jean-Pierre(F-GREN-F)

Compact complex manifolds whose tangent bundles satisfy numerical effectivity properties. Joint work with Thomas Peternell and Michael Schneider. Geometry and analysis (Bombay, 1992), 67-86, Tata Inst. Fund. Res., Bombay, 1995.
32J27 (14J45)

A line bundle L on a compact complex manifold is called numerically effective (nef) if $c_1(L) \cdot C \geq 0$ for all curves $C \subset X$. A bundle E on X is nef if $\mathcal{O}(1)$ on $\mathbf{P}(E)$ is nef. F. Campana and T. Peternell [Math. Ann. 289 (1991), no. 1, 169-187; MR1087244 (91m:14061)] considered the class of projective manifolds X with nef tangent bundles, and gave a complete classification in dimension 3.

For projective algebraic X, Seshadri's ampleness criterion implies that a line bundle L is nef iff for every $\epsilon > 0$ there is a Hermitian metric on L with curvature F such that $F \geq -\epsilon\omega$. This enables the definition of nef bundles on arbitrary compact Hermitian (X, ω) by defining a bundle E to be nef if there are a sequence of positive numbers ϵ_m decreasing to 0 and a sequence of metrics h_m on $S^m E$ with curvatures $F(h_m)$ such that $F(h_m) \geq -m\epsilon_m\omega \otimes \mathrm{Id}_{S^m E}$. Every homogeneous manifold has nef tangent bundle (Proposition 3.1).

The main theorem of the paper, which is close to a classification up to finite étale cover in the Kähler case, is as follows: If X is a compact Kähler manifold with nef tangent bundle and $\widetilde{X}$ is a finite étale cover of maximum irregularity $q = h^1(\widetilde{X}, \mathcal{O})$ then (i) $\pi_1(\widetilde{X}) \simeq \mathbf{Z}^q$; (ii) the Albanese map $\widetilde{X} \to A(\widetilde{X})$ is a smooth fibration over a q-dimensional torus with nef relative tangent bundle; and (iii) the fibres of the Albanese map are Fano manifolds with nef tangent bundles. It is conjectured (by Campana and Peternell) that a Fano manifold has nef tangent bundle if and only if it is homogeneous and rational.

For the entire collection see MR1351499 (96c:00026).

Reviewed by Nicholas Buchdahl

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§ **73. Citations ↪**
From References: 0
From Reviews: 0
MR1320382 (95k:32013) Reviewed
Demailly, Jean-Pierre(F-GREN-F)
Propriétés de semi-continuité de la cohomologie et de la dimension de Kodaira-Iitaka. (French. English, French summary) [Semicontinuity properties of cohomology and of the Kodaira-Iitaka dimension]
C. R. Acad. Sci. Paris Sér. I Math. 320 (1995), no. 3, 341-346.
32C35 (32G05 32L10)

Let $X \to S$ be a proper flat morphism of reduced complex spaces with fibres X_t, $t \in S$, and E be a locally free $\mathcal{O}_X$-module giving a family of sheaves $E_t \to X_t$. It is well known that the cohomology group dimensions $h^q(t) = \dim H^q(X_t, E_t)$ are upper semi-continuous functions of t, even in the analytic Zariski topology on S.

Using the proofs of Grauert's direct image theorem due to Forster-Knorr and Kiehl-Verdier, the author gives a short proof of a considerably stronger semicontinuity result. Namely, the alternating sum $h^q - h^{q-1} + \cdots + (-1)^q h^0$ is Zariski upper semi-continuous for each $q \geq 0$. He outlines a similar proof using harmonic forms and Hodge theory (for the Hausdorff topology).

When $\operatorname{rank} E = 1$, this result is applied to the Kodaira-Iitaka dimension $\kappa(E_t)$. An example is given to show that $\kappa(E_t)$ is in general neither upper nor lower semicontinuous, even in the Hausdorff topology. However, it is proved that if $\kappa(E_0) = k \geq 0$ for some $0 \in S$ and

$$(*) \qquad \limsup_{m\to\infty} m^{-k}(h^0(X_0, E_0^{\otimes m}) - h^1(X_0, E_0^{\otimes m})) > 0,$$

then $\kappa(E_t) \geq k$ for t in a Zariski neighbourhood of 0.

It follows that if $(X_t)_{t\in S}$ is a family of compact complex manifolds, X_0 has Kodaira dimension $\kappa(X_0) = k$ and $(*)$ holds with E_0 replaced by the canonical bundle of X_0, then $\kappa(X_t) \geq k$ for t in a Zariski neighbourhood of 0. For surfaces, it is known that $\kappa(X_t)$ is constant. In higher dimensions, this is an open problem. The paper concludes with a discussion of this and other related questions.

Current version of review. Go to earlier version.

Reviewed by Finnur Lárusson

§ **74. Citations** $\hookrightarrow$
From References: 2
From Reviews: 1
MR1313858 (96b:32012) Reviewed
Demailly, Jean-Pierre(F-GREN-F); Passare, Mikael(S-RIT-IM)
Courants résiduels et classe fondamentale. (French) [Residue currents and fundamental class]
Bull. Sci. Math. 119 (1995), no. 1, 85-94.
32C30 (32A27)

Let $f = (f_1, \cdots, f_p)\colon U \to \mathbf{C}^p$ be a holomorphic mapping. If $Y_j = f_j^{-1}\{0\}$ are the loci of the zeros of the components of f, one has

$$\overline{\partial}\partial \log |f_1|^2 \wedge \cdots \wedge \overline{\partial}\partial \log |f_p|^2 = 2\pi i[Y_1] \wedge \cdots \wedge 2\pi i[Y_p] = (2\pi i)^p[Y],$$

where $[Y]$ is the current corresponding to the intersection $Y_1 \cap \cdots \cap Y_p$. One has the factorization

$$\overline{\partial}\partial \log |f_1|^2 \wedge \cdots \wedge \overline{\partial}\partial \log |f_p|^2 = df_1 \wedge \cdots \wedge df_p + \overline{\partial}(1/f_1) \wedge \cdots \wedge \overline{\partial}(1/f_p).$$

The current $R_Y = \overline{\partial}(1/f_1) \wedge \cdots \wedge \overline{\partial}(1/f_p)$ is called a residue current [cf. N. R. Coleff and M. E. Herrera, Les courants résiduels associés à une forme méromorphe, Lecture Notes in Math., 633, Springer, Berlin, 1978; MR0492769 (80j:32016)].

In this paper the authors show that the above factorization may be extended to the case in which Y is the complex subspace associated with a coherent ideal $\mathcal{I}$ on a complex manifold X, provided that $\mathcal{I}$ is locally a complete intersection. They prove essentially that the residue current R_Y may be identified in an intrinsic way with a canonical element of the infinitesimal cohomology of order 1 with support in Y and with values in the sheaf of sections of the determinant of the conormal bundle of Y, following the general formalism for residues introduced by Grothendieck. In fact, the residue current is well defined even if the Y_j are not in the situation of a complete intersection [cf. M. Passare, J. Reine Angew. Math. 392 (1988), 37-56; MR0965056 (90d:32020)]. The cohomological meaning of the residue current remains an open problem in the case in which one does not have a complete intersection.

Reviewed by Mongi Blel

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§ **75. Citations** $\hookrightarrow$
From References: 7
From Reviews: 3
MR1319346 (96k:32012) Reviewed
Demailly, Jean-Pierre(F-GREN-F)
Regularization of closed positive currents of type $(1,1)$ by the flow of a Chern connection.
Contributions to complex analysis and analytic geometry, 105-126,
Aspects Math., E26, Vieweg, Braunschweig, 1994.
32C30 (32J25)

Let X be a compact n-dimensional complex manifold and let T be a closed positive current of bidegree $(1,1)$ on X. In general, T cannot be approximated by closed positive currents of class C^∞: a necessary condition for this is that the cohomology class $\{T\}$ be numerically effective in the sense that $\int_Y \{T\}^p \geq 0$ for every p-dimensional subvariety $Y \subset X$. The author proves that it is always possible to approximate a closed positive current T of type $(1,1)$ by closed real currents admitting a small negative part, and that this negative part can be estimated in terms of the Lelong numbers of T and the geometry of X. Let α be a smooth closed $(1,1)$-form representing the same $\partial\overline{\partial}$-cohomology class as T and let ψ be a quasi-psh function on X such that $T = \alpha + (i/\pi)\partial\overline{\partial}\psi$ (a function is said to be quasi-psh if it is locally the sum of a psh function and a smooth function). Such a decomposition exists even when X is non-Kähler. If ψ_ε is an approximation of ψ, then $T_\varepsilon = \alpha + (i/\pi)\partial\overline{\partial}\psi_\varepsilon$ is an approximation of T. The author proves the following: Theorem. Let T be a closed almost positive $(1,1)$-current and let α be a smooth real $(1,1)$-form in the same $\partial\overline{\partial}$-cohomology class as T, i.e. $T = \alpha + (i/\pi)\partial\overline{\partial}\psi$, where ψ is an almost psh function. Let γ be a continuous real $(1,1)$-form such that $T \geq \gamma$. Suppose that the tangent bundle T_X is equipped with a smooth Hermitian metric ω such that the Chern curvature form $\Theta(T_X)$ satisfies $(\Theta(T_X) + u \otimes \mathrm{Id}_{T_X})(\theta \otimes \xi, \theta \otimes \xi) \geq 0$ for all $\theta, \xi \in T_X$ with $\langle \theta, \xi \rangle = 0$, for some continuous nonnegative

$(1,1)$-form u on X. Then there is a family of closed almost positive $(1,1)$-currents $T_\varepsilon = \alpha + (i/\pi)\partial\overline{\partial}\psi_\varepsilon$, $\varepsilon \in (0,\varepsilon_0)$, such that ψ_ε is smooth over X, increases with ε, and converges to ψ as ε tends to 0 (in particular, T_ε is smooth and converges weakly to T on X), and such that (i) $T_\varepsilon \geq \gamma - \lambda_\varepsilon u - \delta_\varepsilon\omega$, where (ii) $\lambda_\varepsilon(x)$ is an increasing family of continuous functions on X such that $\lim_{\varepsilon\to 0}\lambda_\varepsilon(x) = \nu(T,x)$ (Lelong number of T at x) at every point, and (iii) δ_ε is an increasing family of positive constants such that $\lim_{\varepsilon\to 0}\delta_\varepsilon = 0$.

For the proof, the author uses a smoothing procedure involving a convolution kernel constructed by means of the exponential map associated to the Chern connection on T_X. From a calculation of the Taylor expansion of the exponential map at order 3, a precise estimate of the complex Hessian of the regularized function is derived. Kiselman's singularity attenuation technique is then applied in combination with the above theorem to obtain a family of approximating currents $T_{c,\varepsilon}$ which are smooth in the complement $X \smallsetminus E_c(T)$ of the Lelong sublevel set

$$E_c(T) = \{x \in X\colon\ \nu(T,x) \geq c\}$$

and have Lelong numbers $\nu(T_{c,\varepsilon},x) = \nu(T,x) - c$ along $E_c(T)$. It should be observed that similar results have been proved by the author in a related paper [J. Algebraic Geom. 1 (1992), no. 3, 361-409; MR1158622 (93e:32015)], under slightly different curvature assumptions. Some geometric applications of the smoothing theorem to the study of compact complex manifolds with partially semipositive curvature are given.

For the entire collection see MR1319341 (95j:32001).

Reviewed by Mongi Blel

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§ 76. Citations ↪
From References: 18
From Reviews: 1
MR1302317 (95i:32022) Reviewed
Demailly, Jean-Pierre (F-GREN-F); Lempert, László (1-PURD); Shiffman, Bernard (1-JHOP)
Algebraic approximations of holomorphic maps from Stein domains to projective manifolds.
Duke Math. J. 76 (1994), no. 2, 333-363.
32E30 (32H02 32H15 32H20)

This paper studies the problem of approximation of holomorphic maps by algebraic maps. The authors show that algebraic approximation is always possible in the case of holomorphic maps to quasiprojective manifolds and of locally free sheaves. In particular, they obtain that any holomorphic map from a Runge domain Ω in an affine algebraic variety S into a quasiprojective algebraic manifold X can be approximated by Nash algebraic maps uniformly on every relatively compact domain $\Omega_0 \subset\subset \Omega$. As an application, they describe how both the Kobayashi-Royden pseudometric and the Kobayashi pseudodistance on projective algebraic manifolds can be given in terms of algebraic curves.

Reviewed by Min Ru

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§ 77. Citations ↪
From References: 65
From Reviews: 12
MR1257325 (95f:32037) Reviewed
Demailly, Jean-Pierre(F-GREN-F); Peternell, Thomas(D-BAYR); Schneider, Michael (D-BAYR)
Compact complex manifolds with numerically effective tangent bundles.
J. Algebraic Geom. 3 (1994), no. 2, 295-345.
32J27 (14J45 32L07)

A vector bundle E on a projective variety is said to be numerically effective (nef) if the tautological line bundle $\mathcal{O}_E(1)$ on the associated projective bundle $\mathbf{P}(E)$ is numerically effective. This notion is extended to vector bundles over compact complex manifolds as follows. Let X be a compact complex manifold and let E be a vector bundle on X. Fix a Hermitian metric ω on $\mathbf{P}(E)$. Then E is nef if, for every positive number ϵ, we can find a Hermitian metric h_ϵ on $\mathcal{O}_E(1)$ (or a Hermitian Finsler metric on E) such that its curvature Θ_{h_ϵ} satisfies $\Theta_{h_\epsilon} \geq -\epsilon\omega$. This definition does not depend on the choice of the Hermitian metric ω.

The main result of the paper is a structure theorem for Kähler manifolds with nef tangent bundles. Main Theorem: Let X be a compact Kähler manifold with nef tangent bundle. Then there exists an étale finite cover $\tilde{X}$ such that the Albanese mapping $\alpha\colon \tilde{X} \to \mathrm{Alb}(\tilde{X})$ is a surjective, smooth morphism, every fibre of which is a Fano manifold with nef tangent bundle.

The result states that X is essentially constructed by a torus and Fano manifolds. The torus part defines a flat quotient E of T_X, the tangent bundle of X (or, more precisely, of $T_{\tilde{X}}$). Since a Fano manifold is always simply connected, the main theorem implies that the fundamental group of X, a compact Kähler manifold with nef tangent bundle, is an extension of a finite group by a free abelian group of even rank $\mathbf{Z}^{2q}$.

One of the essential steps toward the main theorem is to show the following. Proposition: Let X be an n-dimensional compact Kähler manifold with T_X nef. If $c_1(X)^n = 0$, then (1) ${}_\chi(\mathcal{O}_X) = 0$, (2) there is a nowhere vanishing p-form u for suitable odd p, and, (3) by lifting to a suitable étale cover $\tilde{X}$, the irregularity $q(\tilde{X})$ becomes positive.

The proof of this part depends on a result of J. Tits on subgroups of linear groups [J. Algebra 20 (1972), 250-270; MR0286898 (44 #4105)] and on the authors' result to the effect that $\pi_1(X)$ has subexponential growth [Compositio Math. 89 (1993), no. 2, 217-240; MR1255695 (95b:32044)].

Choosing $\tilde{X}$ such that its irregularity attains the maximum, we get a subsheaf $E^* \subset \Omega^1_{\tilde{X}} = (T_X)^*$ generated by global sections. It is easy to show that E^* is a subbundle with trivial Chern classes, and results of Uhlenbeck-Yau and S. Kobayashi tell us that it has a filtration with Hermitian-flat graded pieces. Taking the duals, we get a quotient E of T_X which defines the torus part of $\tilde{X}$, or the image of the Albanese map. The smoothness of the fibres of the Albanese fibration is proved via the theory of Mori contractions.

As by-products of the proof, the authors obtain (a) the projectivity of Moishezon manifolds with nef tangent bundles and (b) the classification of nonalgebraic compact 3-folds with nef tangent bundles (up to finite étale coverings) into nonalgebraic tori and some $\mathbf{P}^1$-bundles over nonalgebraic two-dimensional tori.

Reviewed by Yoichi Miyaoka

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§ **78. Citations ↪**
From References: 19
From Reviews: 8
MR1255695 (95b:32044) Reviewed
Demailly, Jean-Pierre(F-GREN-F); Peternell, Thomas(D-BAYR); Schneider, Michael (D-BAYR)
Kähler manifolds with numerically effective Ricci class.
Compositio Math. 89 (1993), no. 2, 217-240.
32J27 (14J40 32C17 53C55)

The purpose of this paper is to contribute to the solution of the following conjectures: Let X be a compact Kähler manifold with numerically effective (nef) anticanonical bundle $-K_X$; then: Conjecture 1: The fundamental group $\pi_1(X)$ of X has polynomial growth. Conjecture 2: The Albanese map $\alpha\colon X \to \mathrm{Alb}(X)$ is surjective. Section 1 is devoted to proving the following theorem, which is the main contribution to Conjecture 1. Theorem 1: Let X be a compact Kähler manifold with nef anticanonical bundle; then $\pi_1(X)$ has subexponential growth. The main tools used in order to prove Theorem 1 are the solution of the Calabi conjecture and volume bounds for geodesic balls due to Bishop and Gage. It should be mentioned that from the proof of Theorem 1 it follows that Conjecture 1 holds in the case $-K_X$ is Hermitian semipositive (Theorem 2).

In Section 2 the following theorem concerning Conjecture 2 is proved. Theorem 3: Let X be an n-dimensional compact Kähler manifold such that $-K_X$ is nef. Then the Albanese map α is surjective as soon as $\dim \alpha(X)$ is $0, 1$ or n, and, if X is projective, also for $n-1$; moreover, if X is projective and if the generic fibre F of α has $-K_F$ big, then α is surjective. Finally, Section 3 is devoted to the study of the structure of projective 3-folds with nef anticanonical bundles; in particular Conjecture 2 is proved in dimension 3 with purely algebraic methods, except in one very special case.

Reviewed by Antonella Nannicini

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§ **79. Citations ↪**

From References: 93
From Reviews: 11
MR1211880 (94k:32009) Reviewed
Demailly, Jean-Pierre(F-GREN-F)
Monge-Ampère operators, Lelong numbers and intersection theory. Complex analysis and geometry, 115-193,
Univ. Ser. Math., Plenum, New York, 1993.
32C30 (32F07)

This article surveys the theory of Lelong numbers and their applications for studying intersection theory of analytic cycles. For this point of view, we define a plurisubharmonic (psh) function on a complex manifold X to be an upper semicontinuous function u for which $dd^c u$ is a positive closed current of bidegree $(1,1)$. If u is a locally bounded psh function on X and T is a positive, closed current of bidimension (p,p), the wedge product $dd^c u \wedge T := dd^c(uT)$ defines a closed positive current; by induction, if $1 \leq q \leq p$ and $u_1, \cdots, u_q$ are locally bounded psh functions on X then

$$dd^c u_1 \wedge \cdots \wedge dd^c u_q \wedge T := dd^c(u_1 dd^c u_2 \wedge \cdots \wedge dd^c u_q \wedge T)$$

is a closed positive current. If $u_1^k, \cdots, u_q^k$ are decreasing sequences of psh functions converging pointwise to $u_1, \cdots, u_q$ then, following E. Bedford and B. A. Taylor [Acta Math. 149 (1982), no. 1-2, 1-40; MR0674165 (84d:32024)], it is shown in Section 2 that

$$(1) \qquad u_1^k dd^c u_2^k \wedge \cdots \wedge dd^c u_q^k \wedge T \to u_1 dd^c u_2 \wedge \cdots \wedge dd^c u_q \wedge T$$

"and"

$$(2) \qquad dd^c u_1^k \wedge \cdots \wedge dd^c u_q^k \wedge T \to dd^c u_1 \wedge \cdots \wedge dd^c u_q \wedge T$$

weakly.

Section 3 discusses the definition of $dd^c u_1 \wedge \cdots \wedge dd^c u_q \wedge T$ for certain psh u_i and closed positive currents T even if some of the u_i are not necessarily bounded below. Define the unbounded locus $L(u)$ of u to be the set of points $x \in X$ such that u is unbounded in every neighborhood of x. Modifying the arguments of the previous section, the following result is proved: Theorem 1. Let $u_1, \cdots, u_q$ be psh functions on X. The currents $u_1 dd^c u_2 \wedge \cdots \wedge dd^c u_q \wedge T$ and $dd^c u_1 \wedge \cdots \wedge dd^c u_q \wedge T$ are well-defined and have locally finite mass in X provided $q \leq p =$ dimension of T and

$$\mathcal{H}_{2p-2m+1}(L(u_{j_1}) \cap \cdots \cap L(u_{j_m}) \cap \operatorname{Supp} T) = 0$$

for all choices of indices $j_1 < \cdots < j_m$ in $\{1, \cdots, q\}$. Here, $\mathcal{H}_s(E)$ denotes the s-dimensional Hausdorff measure of E. In addition, it is shown that the analogues of the monotone convergence theorems in Section 2 remain valid.

In Section 4, the definition of generalized Lelong numbers is given. For a Stein manifold X, let T be a positive, closed current of bidimension (p,p) and let $\phi\colon X \to [-\infty, +\infty)$ be a semiexhaustive weight function on $\operatorname{Supp} T$, i.e., ϕ is continuous and psh on X and there exists R such that $B(R) \cap \operatorname{Supp} T \subset\subset X$, where $B(R) = \{x \in X\colon \phi(x) < R\}$. It follows that $\{\phi = -\infty\} \cap \operatorname{Supp} T$ is compact and $T \wedge (dd^c\phi)^p$ is well-defined. For each $r \in (-\infty, R)$, set $\nu(T, \phi, r) = \int_{B(r)} T \wedge (dd^c\phi)^p$ and define the generalized Lelong number of T with respect to ϕ as

$$(3) \qquad \nu(T, \phi) = \int_{\{\phi=-\infty\}} T \wedge (dd^c\phi)^p = \lim_{r\to-\infty} \nu(T, \phi, r).$$

For $X = \mathbf{C}^n$ and $\phi(z) = \log|z|$, this agrees with the ordinary Lelong number of T at 0,

$$\nu(T, 0) = \lim_{r\to 0} \frac{\sigma_T(B(r))}{\pi^p r^{2p}/p!},$$

where $\sigma_T = T \wedge (dd^c|z|^2)^p$ is the trace measure of T and $B(r)$ is the (Euclidean) ball of radius r centered at 0.

In Section 5, a Lelong-Jensen type formula is proved. Suppose $B(R) \subset\subset X$. Let $\mu_r = (dd^c[\max\{\phi, r\}])^n - \mathbf{1}_{X-B(r)}(dd^c\phi)^n$, $r < R$. In the case $X = \mathbf{C}^n$ and $\phi(z) = \log|z|$, this is just a normalized surface area measure on the sphere of radius e^r. For any psh function V on X, V is μ_r-integrable for each $r < R$ and

$$\mu_r(V) - \int_{B(r)} V(dd^c\phi)^n = \int_{-\infty}^{r} \nu(dd^c V, \phi, t)\, dt.$$

This reduces to the classical Jensen formula if $n = 1$ ($X = \mathbf{C}$) and $V = \log|f|$. If $(dd^c\phi)^n = 0$ on $X - \{\phi = -\infty\}$, one obtains the formula $\nu(dd^cV, \phi) = \lim_{r\to-\infty} \mu_r(V)/r$. An interesting case occurs in $\mathbf{C}^n$ if one takes $\phi(z) = \log\max|z_j|^{\lambda_j}$, where $\lambda_j > 0$. It can be shown that $(dd^c\phi)^n = \lambda_1\cdots\lambda_n\delta_0$ and $\mu_r = \lambda_1\cdots\lambda_n(2\pi)^{-n}d\theta_1\cdots d\theta_n$ on the distinguished boundary $\{z\colon |z_j| = e^{r/|\lambda_j|}\}$ of the polydisk $B(r)$. The Lelong number $\nu(dd^cV, \phi)$ is then

$$\lim_{r\to-\infty} \frac{\lambda_1\cdots\lambda_n}{r} \int_{\theta_j\in[0,2\pi]} V(e^{r/\lambda_1+i\theta_1}, \cdots, e^{r/\lambda_n+i\theta_n}) \frac{d\theta_1\cdots d\theta_n}{(2\pi)^n},$$

which is the directional Lelong number of dd^cV at 0 with coefficients $\lambda := (\lambda_1, \cdots, \lambda_n)$ introduced by Kiselman. In general, for any current T, define $\nu(T, x, \lambda) = \nu(T, \log\max|z_j - x_j|^{\lambda_j})$; in $\mathbf{C}^n$, taking $\phi(z) = \log|z - x|$, it follows that the usual Lelong numbers $\nu(T, x)$ agree with the Kiselman numbers $\nu(T, x, (1, \cdots, 1))$.

In Section 6, a new proof is given of Thie's theorem: If A is an analytic set of pure dimension p and $[A]$ is the current of integration over A, then for each $x \in A$, $\nu([A], x)$ is the multiplicity of A at x. Also, a result of Siu's on stability of Lelong numbers for closed, positive currents under slicing along linear subspaces is presented. In Section 7, a generalization of Siu's upper semicontinuity theorem is proved [J.-P. Demailly, Acta Math. 159 (1987), no. 3-4, 153-169; MR0908144 (89b:32019)]. Section 8 describes the behavior of Lelong numbers under proper morphisms. As a concrete application, it is shown that for T a closed, positive current of bidimension (p, p) and for $0 < \lambda_1 \leq \cdots \leq \lambda_n$, the directional Lelong numbers of Kiselman satisfy $\lambda_1\cdots\lambda_p\nu(T, x) \leq \nu(T, x, \lambda) \leq \lambda_{n-p+1}\cdots\lambda_n\nu(T, x)$. A type of Schwarz lemma relating growth of zeros of entire functions f in $\mathbf{C}^n$ which involves Lelong numbers of the current of integration $[Z_f]$ of the zero variety of f is proved in Section 9. This yields a theorem of Bombieri on algebraic values of meromorphic maps satisfying algebraic differential equations. Finally, in Section 10, a self-intersection inequality for closed positive currents of bidegree (1,1) is given. The motivation behind this inequality is the following. The wedge product of smooth differential forms defines a ring structure on de Rham cohomology, and, for two currents Θ_1, Θ_2 on X, there is a well-defined intersection class $\{\Theta_1\}\cdot\{\Theta_2\}$ in the cohomology ring, even if $\Theta_1 \wedge \Theta_2$ is not defined pointwise as a current. But the wedge product of closed, positive currents cannot always be defined in a reasonable way, and, moreover, such currents cannot necessarily be approximated in the weak topology by smooth closed, positive currents. Indeed, for T a closed, positive current, a necessary condition for such an approximation to be possible is that $\{T\}^p \cdot \{Y\} \geq 0$ for every p-dimensional subvariety $Y \subset X$. The author showed [J. Algebraic Geom. 1 (1992), no. 3, 361-409; MR1158622 (93e:32015)] that T can be approximated by closed, real currents with small negative part governed by the curvature of X. Then, by regularizing and taking weak limits, one can compute self-intersections.

For the entire collection see MR1211876 (93j:32001).

Reviewed by Norman Levenberg

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§ 80. Citations ↪
From References: 0
From Reviews: 0
MR1207864 (94f:32060) Reviewed
Demailly, Jean-Pierre(F-GREN)
Holomorphic Morse inequalities on q-convex manifolds. Several complex variables (Stockholm, 1987/1988), 245-257,
Math. Notes, 38, Princeton Univ. Press, Princeton, NJ, 1993.
32L10 (32F10 32L15)

This note is a report on a paper by T. Bouche [Ann. Sci. École Norm. Sup. (4) 22 (1989), no. 4, 501-513; MR1026747 (91a:32041)] in which holomorphic Morse inequalities for strongly q-convex manifolds were obtained, extending the results of the author [in Séminaire d'analyse P. Lelong-P. Dolbeault-H. Skoda, années 1983/1984, 88-97, Lecture Notes in Math., 1198, Springer, Berlin, 1986; MR0874763 (88f:32069)]. The author carefully describes the main ideas and techniques used by Bouche and illustrates some interesting applications.

For the entire collection see MR1207850 (93j:32002).

Reviewed by Antonella Nannicini

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§ 81. Citations ↪
From References: 49
From Reviews: 10
MR1205448 (94d:14007) Reviewed
Demailly, Jean-Pierre(F-GREN)
A numerical criterion for very ample line bundles. (English summary)
J. Differential Geom. 37 (1993), no. 2, 323-374.
14C20 (14E25 32J25 32L10)

Let X be a smooth projective variety of dimension n defined over the field of complex numbers. A divisor (or, equivalently, a line bundle) L on X is ample if, by definition, some positive multiple of L is very ample, that is, it is isomorphic to the restriction of $\mathcal{O}(1)$ for some embedding $X \subset \mathbf{P}^N$. The notion of very ampleness has a definite geometric meaning while, as it follows from well-known Kleiman and Nakai criteria, ampleness has a very numerical character. Therefore, it is natural to ask for a numerical criterion asserting very ampleness of an ample divisor. One observes easily that for curves the answer depends not only on the degree of the divisor L but also on the geometry of X; in particular, it depends on the genus of the curve X. A uniform answer for curves is as follows: if L is an ample divisor on a curve X then $K_X + 3L$ is very ample, where K_X denotes the canonical divisor of X. For surfaces, a result of I. Reider implies very ampleness of $K_X + 4L$. On the other hand, the theory of extremal rays of S. Mori implies ampleness of $K_X + (n+2)L$ for an ample divisor L on a smooth n-fold X; T. Fujita conjectured that $K_X + (n+2)L$ is actually very ample.

The main result of the paper under review is a significant step towards the conjecture of Fujita. Namely, the main theorem of the paper gives numerical conditions for an ample (or, more generally, big and nef) divisor L to imply spannedness or very ampleness (or, more generally, separation of s-jets of sections) of the adjoint divisor $K_X + L$. From the theorem it follows that $2K_X + mL$ is very ample if only $m \geq 12n^n$. This in turn yields an effective version of T. Matsusaka's big theorem [cf. Amer. J. Math. 94 (1972), 1027-1077; MR0337960 (49 #2729); Y. T. Siu, "An effective Matsusaka big theorem", Preprint, 1993; per revr.], and combined with results of Catanese and of Green and Morrison implies an effective bound on the number of irreducible families of n-dimensional smooth polarised varieties (X, L) depending only on the intersection numbers L^n and $K_X \cdot L^{n-1}$ [J. Kollár, Math. Ann. 296 (1993), no. 4, 595-605]. Although the main result of the paper and its applications are formulated in terms of algebraic geometry, the proof of the theorem is analytic and it applies Hörmander L^2 estimates for the operator $\overline{\partial}$, the Aubin-Calabi-Yau theorem and the theory of positive currents and Lelong numbers.

Reviewer's remarks: Among most recent developments following the paper under review there are the above-mentioned papers of Siu and Kollár as well as a paper of L. M. H. Ein and R. K. Lazarsfeld [J. Amer. Math. Soc. 6 (1993), no. 4, 875-903; MR1207013 (94c:14016)]. The Kollár and Ein-Lazarsfeld papers involve algebraic settings (Kollár works with varieties having Kawamata log terminal singularities) but, in the end, they depend on the transcendental Kodaira-Kawamata-Viehweg vanishing.

Reviewed by Jarosław A. Wiśniewski

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§ 82. Citations ↪
From References: 75
From Reviews: 12
MR1178721 (93g:32044) Reviewed
Demailly, Jean-Pierre(F-GREN-F)
Singular Hermitian metrics on positive line bundles. (English summary) Complex algebraic varieties (Bayreuth, 1990), 87-104,
Lecture Notes in Math., 1507, Springer, Berlin, 1992.
32L10 (14J60 32L05)

In this paper, the author introduces the notion of singular Hermitian metric on a holomorphic line bundle on a complex manifold. A singular Hermitian metric on a holomorphic line bundle L on a complex manifold M is a product of the form $h_0 e^{-\phi}$, where h_0 is a smooth Hermitian metric on L and ϕ is a locally L^1-function. The beauty of the definition is that we can take the curvature of a singular Hermitian metric in the sense of a closed current. This extended definition of Hermitian metrics enables us to characterize the line bundle to be pseudoeffective, big or nef in terms of the positivity properties of the curvature of the singular metrics. For the applications of these metrics, the author shows the relation between the Seshadri constant and the singularity of singular Hermitian metrics. This is related to the global generation of nef line bundles. Finally, he uses the singular metrics to prove a new asymptotic estimate for the dimension of the cohomology groups with values in high multiples $\mathcal{O}(kL)$ of a big line bundle L.

For the entire collection see MR1178715 (93d:14006).

Reviewed by Hajime Tsuji

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§ 83. Citations ↪
From References: 7
From Reviews: 0
MR1175540 (93f:32012) Reviewed
Demailly, Jean-Pierre(F-GREN-F)
Courants positifs et théorie de l'intersection. (French) [Positive currents and intersection theory]
Gaz. Math. No. 53 (1992), 131-159.
32C30 (32-01)

This article is a brief exposition of intersection theory from the point of view of positive currents.

Let Y be a complex analytic manifold of dimension n. To any subvariety X of Y, we can associate a current denoted $[X]$ defined by $\langle [X], \phi\rangle = \int_X \phi$. After recalling all the requisite definitions, the author gives in the first part of this article the relation between the intersection number of two analytic subvarieties A and B of Y ($\dim A + \dim B = n$) and the product $[A] \wedge [B]$. The main tools are two theorems of P. Lelong, for which short proofs are given; as an application, Example 5.2 calculates the intersection number of the curves Γ_1 and Γ_2 of $\mathbf{C}^2$ defined, respectively, by the equation $z^2 = w^3$ and $z^3 = w^5$, where (z, w) are the coordinate functions on $\mathbf{C}^2$.

The second part (the last two sections) deals with the "Lelong numbers": to any positive d-closed (p, p)-current T and any $y \in Y$ is associated a number $\nu(T, y) \in \mathbf{R}_+$ called the "Lelong number of T at y", which is related to the "regularity" of T at the point $y \in Y$. Its definition is recalled and generalized in Section 6. The last section deals with more recent results: by using a theorem of Siu, the author states interesting estimations for the degree of the irreducible components of $E_c(T) = \{y \in Y : \nu(T, y) \geq c\}$.

This article uses mainly elementary results of analytic geometry in the proofs. It is an excellent introduction to the subject.

Reviewed by Salomon Ofman

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§ 84. Citations ↪
From References: 90
From Reviews: 8
MR1158622 (93e:32015) Reviewed
Demailly, Jean-Pierre(F-GREN)
Regularization of closed positive currents and intersection theory.
J. Algebraic Geom. 1 (1992), no. 3, 361-409.
32C30 (32C17 32J25)

Let X be a complex manifold of dimension n and let T be a closed current of bidegree $(1, 1)$ on X. T is said to be almost positive if there exists a smooth form v of bidegree $(1, 1)$ such that $T + v \geq 0$. If T is closed and almost positive, let (T, x) denote the Lelong number of T at x. For every $c > 0$, let $E_c(T) = \{x \in X\colon\ \nu(T, x) \geq c\}$. By Siu's theorem $E_c(T)$ is an analytic subset of X. The main result of the paper under review is an "approximation-regularization" theorem for closed almost positive currents T. It says that T can be approximated, for every $c > 0$, by T_c which are smooth outside $E_c(T)$, and such that $\nu(T_c, x) = (\nu(T, x) - c)_+$ at every point $x \in X$. The T_c can be chosen so that they satisfy certain estimates from below. Although the precise statement of the theorem is too long to be described here, it includes the good old regularization theorem of Richberg as a special case, where $T = \partial\overline{\partial}\psi$ for finite and continuous ψ. The proof of the result is based on a combination of three different types of L^2-techniques. Among other things, the reader will be delighted to find an elegant proof of a local approximation theorem for plurisubharmonic functions by logarithms of holomorphic functions. Interesting applications are also given. Some of them are described below. Let $H^{p,q}_{\partial\overline{\partial}}(X) = \{d\text{-closed } (p, q)\text{-forms}\}/\{\overline{\partial}\text{-exact } (p, q)\text{-forms}\}$. A cohomology class $\{\alpha\} \in H^{1,1}_{\partial\overline{\partial}}(X)$ is said to be pseudo-effective (psef) if it can be represented by a closed positive $(1, 1)$-current, and nef if for some fixed Hermitian metric ω on X and for every $\epsilon > 0$ there is a smooth form $\alpha_\epsilon \in \{\alpha\}$ such that $\alpha_\epsilon \geq -\epsilon\omega$.

Denote respectively by $H^{1,1}_{\rm psef}(X)$ and $H^{1,1}_{\rm nef}(X)$ the cones of pseudo-effective and nef cohomology classes. Then, for compact X, the main result of this article implies that $H^{1,1}_{\rm nef}(X) = H^{1,1}_{\rm psef}(X)$ if the tangent bundle TX of X is nef. Here one says that a vector bundle E is nef if $C_1(\mathcal{O}_E(1))$ is nef over the projectivized bundle $P(E^*)$ of hyperplanes of E. A related new result is that X is Kähler if TX is nef and X is in the Fujiki class. A self-intersection inequality proved in an earlier work of the author's is extended here to an arbitrary closed positive $(1,1)$-current T on a Kähler manifold X. It may be worthwhile to note that this work was strongly motivated by the question of classifying compact Kähler varieties with nef tangent bundle, which is of course of current research interest.

Reviewed by Takeo Ohsawa

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§ 85. Citations ↪
From References: 4
From Reviews: 0
MR1222208 (94j:32025) Reviewed
Demailly, J.-P.(F-GREN-F)
Transcendental proof of a generalized Kawamata-Viehweg vanishing theorem. (English summary) Geometrical and algebraical aspects in several complex variables (Cetraro, 1989), 81-94,
Sem. Conf., 8, EditEl, Rende, 1991.
32L20 (14F17 32L10)

Let L be a numerically effective line bundle on a projective algebraic manifold X of dimension n. The main purpose of this paper is to show a vanishing theorem for the cohomology group $H^q(X, \Omega^n(L \otimes D))$ for the effective $\mathbf{Q}$-divisor D which may have nonnormal crossings, under a certain natural integrability condition for D. As a corollary, the Kawamata-Viehweg vanishing theorem follows, i.e. if L is as above and $\bigwedge^k c_1(L) \neq 0$, then $H^q(X, \Omega^n(L)) = 0$ for $q > n-k$. The author's proof is analytic in the sense that his method is based on a vanishing theorem of L^2 cohomology for $\overline{\partial}$ on complete Kähler manifolds for line bundles provided with a singular metric which yields a positive curvature in the sense of currents. This theorem is induced from an L^2 estimate for $\overline{\partial}$ by the Bochner-Kodaira-Nakano curvature inequality and a smooth procedure for plurisubharmonic functions on Kähler manifolds by the author. To show the vanishing theorem the problem is reduced to the case that L is ample by standard slicing arguments and a trick of Kawamata in algebraic geometry. Finally, the theorem is shown by using a vanishing theorem on compact Kähler manifolds which follows from the vanishing theorem of L^2 cohomology.

For the entire collection see MR1222200 (93m:32002).

Reviewed by Kensho Takegoshi

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§ 86. Citations ↪
From References: 18
From Reviews: 0
MR1128538 (93b:32048) Reviewed
Demailly, Jean-Pierre(F-GREN)
Holomorphic Morse inequalities. Several complex variables and complex geometry, Part 2 (Santa Cruz, CA, 1989), 93-114,
Proc. Sympos. Pure Math., 52, Part 2, Amer. Math. Soc., Providence, RI, 1991.
32L10 (58G11 58G25)

A simple heat equation proof of the author's holomorphic Morse inequalities is presented. For this purpose, the asymptotic eigenvalue distribution of $\Box_k = (1/k)D_k^*D_k - V$ is studied. Here D_k is the associated connection on $E^k \otimes F$, where E, F are complex vector bundles over a smooth manifold M equipped with Hermitian connections, and $V = V \otimes \mathrm{id}_{E^k}$ is a Hermitian endomorphism of F. The asymptotic estimates of the heat kernel $e^{-t\Box_k}$ are purely local and the explicit form of the heat kernel in the case of connections with constant curvature, assuming M has a flat metric, is obtained by using Mehler's formula. From these facts, the asymptotic estimate of the heat kernel of $\Box_k$ is expressed in terms of $c(E)$, the curvature form of E (Theorem 3.1). To obtain holomorphic Morse inequalities from this estimate, $(2/k)\Delta''_k = (1/k)\nabla_k^*\nabla_k - V + (1/k)\Theta$ is shown. Here Δ''_k and ∇_k are the Dolbeault Laplacian and Chern connection on $E^k \otimes F \otimes \bigwedge^{0,q} T^*X$, X a compact complex manifold, E and F are holomorphic Hermitian bundles over X of ranks 1 and r, Θ a Hermitian form independent of k; the eigenvalues of V are easily counted (reviews of complex geometry and Hodge theory are given in Section 2). Thus the asymptotic formula of the heat kernel $e^{-(2t/k)}\Delta''_k$ in bidegree $(0,q)$ is obtained from Theorem 3.1 (Theorem 4.4). Then, according to Witten's idea, a finite-dimensional subcomplex of the Dolbeault complex on $E^k \otimes F$ with the same cohomology is constructed by using the eigenspaces of $(1/k)\Delta''_k$. This allows one to use linear algebraic considerations, and combining these considerations and Theorem 4.4, the strong holomorphic Morse inequality follows. We have dim $H^0(X, E^k) \geq O(k^n)$ if $ic(E) \geq 0$ and $ic(E) > 0$ at at least one point by this inequality. Hence we have an alternative proof of Siu's theorem (the Grauert-Riemenschneider conjecture, Section 5). In Section 6, a generalization of the holomorphic Morse inequality to q-convex manifolds and its application to an a priori estimate for the Monge-Ampère operator are given (Theorem 6.1 and Corollary 6.6, the author states, were obtained by Bouche and Siu). The case rank $E \geq 2$ is discussed in Section 7 [cf. E. Getzler, C. R. Acad. Sci. Paris Sér. I Math. 304 (1987), no. 16, 475-478; MR0894572 (88j:32040)]. Related open problems are discussed in Section 8, the last section.

For the entire collection see MR1128530 (92d:32002).

Reviewed by Akira Asada

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§ 87. Citations ↪
From References: 6
From Reviews: 3
MR1084013 (92f:32017) Reviewed
Blel, Mongi (TN-TUNISM); Demailly, Jean-Pierre (F-GREN-F); Mouzali, Mokhtar (F-GREN-F)
Sur l'existence du cône tangent à un courant positif fermé. (French) [Existence of the tangent cone of a closed positive current]
Ark. Mat. 28 (1990), no. 2, 231-248.
32C30

Summary: "Let T be a closed positive current in a neighbourhood of 0 in $\mathbf{C}^n$. We show here that T admits a tangent cone (limit of the family of its homotheties) when the projective mass $\nu_T(r)$ converges to $\nu_T(0)$ rapidly enough for the function $(\nu_T(r)-\nu_T(0))/r$ to be locally integrable at 0. This sufficient condition is optimal: we build $(1,1)$ currents without tangent cone such that the integral at $r=0$ of $(\nu_T(r)-\nu_T(0))/r$ has as small a divergence as one likes. When T is given by integration on an analytic set, we show that $\nu_T(r)-\nu_T(0)=O(r^\varepsilon)$ and that this recovers the Thie-King theorem on the existence of the tangent cone."

Reviewed by Daniel Barlet

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§ 88. Citations ↪
From References: 45
From Reviews: 2
MR1055992 (91e:32014) Reviewed
Demailly, Jean-Pierre(F-GREN-F)
Cohomology of q-convex spaces in top degrees.
Math. Z. 204 (1990), no. 2, 283-295.
32F10 (32L10)

Highly elegant proofs of the following three theorems are given. Theorem 1: Let Y be an analytic subvariety in a complex space X. If Y is strongly q-complete, then Y has a fundamental family of strongly q-complete neighbourhoods in X. Theorem 2: Let X be a complex space such that all irreducible components have dimension $\leq n$. Then: (a) X is always strongly $(n+1)$-complete. (b) If X has no compact irreducible component of dimension n, then X is strongly n-complete. (c) If X has only finitely many irreducible components of dimension n, then X is strongly n-convex. Theorem 3: Let (M,ω) be an n-dimensional Kähler manifold. Suppose that M is absolutely q-convex, i.e. admits a smooth plurisubharmonic exhaustion function that is strongly q-convex on $M \setminus K$ for some compact set K in M. Set $\Omega^r = \mathcal{O}(\Lambda^r T^*M)$. Then the de Rham cohomology groups with arbitrary [resp. compact] support have decompositions $H^k(M,\mathbf{C}) \simeq \bigoplus H^s(M,\Omega^r)$, $H^r(M,\Omega^s) \simeq H^s(M,\Omega^r)$, $k \geq n+q$, $H^k_c(M,\mathbf{C}) \simeq \bigoplus H^s_c(M,\Omega^r)$, $H^r_c(M,\Omega^s) \simeq H^s_c(M,\Omega^r)$, $k \leq n-q$, and these groups are finite-dimensional. Moreover, there is a Lefschetz isomorphism $\omega^{n-r-s} \wedge \cdot : H^s_c(M,\Omega^r) \to H^{n-r}(M,\Omega^{n-s})$, $r+s \leq n-q$.

Reviewed by Takeo Ohsawa

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§ **89. Citations ↪**
From References: 2
From Reviews: 0
MR1004954 (90e:32032) Reviewed
Demailly, Jean-Pierre(F-GREN-F)
Une généralisation du théorème d'annulation de Kawamata-Viehweg. (French. English summary) [A generalization of the Kawamata-Viehweg vanishing theorem]
C. R. Acad. Sci. Paris Sér. I Math. 309 (1989), no. 2, 123-126.
32L20 (14F05)

The author gives a short and elegant proof, using analytic methods via Bochner-Kodaira-Nakano type inequalities, of the vanishing theorem due to Kawamata and Viehweg: $H^q(X, \mathcal{L}^{-1}) = 0$ for $q < \dim X$ if X is a projective smooth complex manifold and L is a numerically effective invertible sheaf of maximal Kodaira dimension.

In fact, it is sufficient to assume that $\mathcal{L}^N(-D)$ is numerically effective whenever $N^{-1} \cdot D$ satisfies a certain integrability condition. This condition is the standard one if D is a normal crossing divisor: all the multiplicities of the components of D should be smaller than N. Unfortunately, the author does not explain whether his vanishing theorem is stronger than the one obtained by reduction to the normal crossing case.

By well-known methods of cutting down, the author gives the version of the vanishing theorem in the case when $\mathcal{L}^N(-D)$ is not a maximal Kodaira dimension.

Reviewed by Hélène Esnault

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§ **90. Citations ↪**
From References: 11
From Reviews: 1
MR0982833 (90d:32033) Reviewed
Bedford, Eric(1-IN); Demailly, Jean-Pierre(F-GREN)
Two counterexamples concerning the pluri-complex Green function in $\mathbf{C}^n$.
Indiana Univ. Math. J. 37 (1988), no. 4, 865-867.
32F05 (31C10)

Let z be a fixed point of a domain D in $\mathbf{C}^n$. The pluricomplex Green function $u_z(w)$ on D with logarithmic pole at z is defined by the formula $u_z(w) := \sup\{v(w); v$ is plurisubharmonic on D, $v < 0$, and $v(w) \leq \log|w - z| + O(1)\}$.

The authors show that (1) there exists a bounded pseudoconvex domain D in $\mathbf{C}^2$ with $\mathcal{C}^2$ boundary such that, for some points z in D, u_z is not $\mathcal{C}^2$ in $D \smallsetminus \{z\}$; (2) there is a bounded, strongly pseudoconvex domain in $\mathbf{C}^2$ with real analytic boundary for which the Green function $u_z(w)$ is not symmetric. Hence, in general, the function $u_z(w)$ is not plurisubharmonic in z.

This gives negative answers to the corresponding questions raised by U. Cegrell [Capacities in complex analysis, Vieweg, Braunschweig, 1988; MR0964469 (89m:32001)].

Reviewer's remark: If $0 < r < 1$, then $D := \{(z_1, z_2) \in \mathbf{C}^2;\ |z_1^2 + z_2^2 - 1|^2 + y_1^2 + y_2^2 < r^2\}$ is bounded strongly pseudoconvex domain with real-analytic boundary such that for some points z in D the Green function $u_z(w)$ is not $\mathcal{C}^2$ in $D \smallsetminus \{z\}$.

Reviewed by J. Siciak

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§ **91. Citations ↪**
From References: 2
From Reviews: 0
MR0961475 (89k:32058) Reviewed
Demailly, Jean-Pierre(F-GREN-F)
Vanishing theorems for tensor powers of a positive vector bundle. Geometry and analysis on manifolds (Katata/Kyoto, 1987), 86-105,
Lecture Notes in Math., 1339, Springer, Berlin, 1988.
32L20

Let p, q be integers with $p + q \geq n + 1$. Further define $A(n, p, q) := n(n+1)(p+1)(q+1)/4(p+q-n+1)$ if $p < n$ and $A(n, p, q) := 0$ if $p = n$. Let E be a holomorphic vector bundle of rank r over a compact

n-dimensional complex manifold X and L also a holomorphic vector bundle over X. Assume that, in the sense of Griffiths, E is positive and L is semi-positive, or E is semi-positive and L is positive. Then the author proves that $H^{p,q}(X, S^k E \otimes (\det E)^l \otimes L) = 0$ if $l \geq 1 + A(n,p,q)$ and $H^{p,q}(X, E^k \otimes (\det E)^l \otimes L) = 0$ if $l \geq \min\{k, r-1\} + A(n,p,q)$. A partial result of the above is a generalization of the vanishing theorem by P. A. Griffiths [in Global analysis, 185-251, Univ. Tokyo Press, Tokyo, 1969; MR0258070 (41 #2717)]. Further, the author gives a counterexample to a question of A. J. Sommese [Math. Ann. 233 (1978), no. 3, 229-256; MR0466647 (57 #6524)].

For the entire collection see MR0961468 (89c:58003).

Reviewed by Hideaki Kazama

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§ 92. Citations ↪
From References: 11
From Reviews: 2
MR0918242 (89d:32066) Reviewed
Demailly, Jean-Pierre(F-GREN-F)
Vanishing theorems for tensor powers of an ample vector bundle.
Invent. Math. 91 (1988), no. 1, 203-220.
32L20 (32L10)

The following vanishing theorem is the main result of the paper. Let X be a projective manifold, E a holomorphic vector bundle of rank r on X, L a holomorphic line bundle on X. Assume that E is ample and L semiample or L ample and E semiample. (Semiampleness of E means that for $k \geq k_0$, $S^k E$ is generated by global sections.)

Then $H^{p,q}(X, E^{\otimes k} \otimes (\det E)^l \otimes L) = 0$ provided $p + q \geq n + 1$, $k \geq 1$, $l \geq n - p + r - 1$. This generalizes for instance the le Potier vanishing theorem (valid for the case E ample, L trivial, $l = 0$). In general the factor $\det E$ cannot be omitted.

Reviewed by Thomas Peternell

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§ 93. Citations ↪
From References: 0
From Reviews: 0
MR1047721 (91d:32043) Reviewed
Demailly, Jean-Pierre(F-GREN-F)
Sur les théorèmes d'annulation et de finitude de T. Ohsawa et O. Abdelkader. (French) [On the vanishing and finiteness theorems of T. Ohsawa and O. Abdelkader] Séminaire d'Analyse P. Lelong-P. Dolbeault-H. Skoda, Années 1985/1986, 48-58,
Lecture Notes in Math., 1295, Springer, Berlin, 1987.
32L20

Let X be a weakly 1-complete complex manifold, $\dim_{\mathbf{C}} X = n$. Let E be a Hermitian holomorphic line bundle on X. Let ψ be a plurisubharmonic exhaustion C^∞-function on X. The following three theorems are known: (A) If X is Kähler and the curvature form of E is semipositive, with rank $\geq s$ at each point of X, then $H^{p,q}(X, E) = 0$ for $p + q \geq 2n - s + 1$. (B) Let Y be a compact subset of X. Assume that the curvature form of E is semipositive with rank $\geq s$ at each point of $X \setminus Y$ and that X admits a Hermitian metric which is Kähler on $X \setminus Y$. Then $\dim H^{p,q}(X, E) < +\infty$ for $p + q \geq 2n - s + 1$. (C) Let X, Y, E fulfill the assumptions of (B). For each $c \in \mathbf{R}$, set $X_c = \{x \in X \colon \psi(x) < c\}$. If $X_d \supset Y$, then the restriction morphism $H^{p,q}(X, E) \to H^{p,q}(X_d, E)$ is an isomorphism for $p + q \geq 2n - s + 1$.

For Theorem (A) see papers by Abdelkader [C. R. Acad. Sci. Paris Sér. A 290 (1980), no. 2, 75-78; MR0563942 (81c:32054); "Théoremes de finitude pour la cohomologie d'une variété faiblement I-complète à valeurs dans un fibré en droites semi-positif", Thèse d'État, Univ. Paris, Paris, 1985; per bibl.]. For (B) and (C) see papers by Ohsawa [Publ. Res. Inst. Math. Sci.15 (1979), no. 3, 853-870; MR0566085 (81g:32024); ibid. 17 (1981), no. 1, 113-126; MR0613936 (82j:32058)].

In the present work, a new, very simple and clever proof of Theorems (A), (B) and (C) is given. Let $\mathcal{C}$ be the cone of all increasing convex C^∞-functions $\mathbf{R} \to \mathbf{R}$. For each $\chi \in \mathcal{C}$ denote by E_χ the vector bundle E endowed with the metric obtained from the metric of E by multiplication by $\exp(-\chi \circ \psi)$.

Under the assumptions of (A), the author is able to deduce from a non-Kähler Bochner-Kodaira-Nakano identity (see a paper by the author [in Séminaire d'analyse P. Lelong-P. Dolbeault-H. Skoda, années

1983/1984, 88-97, Lecture Notes in Math., 1198, Springer, Berlin, 1986; MR0874763 (88f:32069)]) the existence of a function $\chi_0 \in \mathcal{C}$ such that for each $\chi \in \chi_0 + \mathcal{C}$ one has $H^{p,q}(X, E_\chi) = 0$. Then, $H^{p,q}(X, E) = \lim\operatorname{ind}\{H^{p,q}(X, E_\chi) \colon \chi \in \chi_0 + \mathcal{C}\} = 0$. In the same way and by the use of some classical tools, the author shows that, under the assumptions of the finiteness theorem (B), a function $\chi_0 \in \mathcal{C}$ exists such that $\dim H^{p,q}(X, E_\chi) \le N$ for every $\chi \in \chi_0 + \mathcal{C}$, $\chi = \chi_0$ on $(-\infty, d]$. Here N is a fixed natural number. Then it is enough to recall the equality $H^{p,q}(X, E) = \lim\operatorname{ind}\{H^{p,q}(X, E_\chi)\}$. In proving (C), interesting new results about smooth real convex functions are also proved and the final result is completed with new L^2-estimates.

For the entire collection see MR1047717 (90m:32002).

Reviewed by Salvatore Coen

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§ 94. Citations ↪
From References: 1
From Reviews: 2
MR1047720 (91h:32025) Reviewed
Demailly, Jean-Pierre(F-GREN-F)
Une preuve simple de la conjecture de Grauert-Riemenschneider. (French. English summary) [A simple proof of the Grauert-Riemenschneider conjecture] Séminaire d'Analyse P. Lelong-P. Dolbeault-H. Skoda, Années 1985/1986, 24-47,
Lecture Notes in Math., 1295, Springer, Berlin, 1987.
32L10 (32L20 58G05)

Let Y be a compact irreducible analytic space. The conjecture of Grauert and Riemenschneider says that Y is a Moishezon space if and only if it has a desingularisation $\pi : X \to Y$ which admits a quasipositive holomorphic line bundle. The conjecture was proved by Y. T. Siu [J. Differential Geom. 19 (1984), no. 2, 431-452; MR0755233 (86c:32029)] and in a survey article [Siu, in Arbeitstagung Bonn 1984, 169-192, Lecture Notes in Math., 1111, Springer, Berlin, 1985; MR0797421 (87b:32055)].

This paper contains a simpler proof of the conjecture, based on an asymptotic upper bound for the dimension of the cohomology groups of the tensor powers of a holomorphic, Hermitian line bundle over a complex compact manifold, in terms of an integral of the bundle curvature form. The upper bound is obtained by interpreting the cohomology groups as harmonic-forms spaces, using the non-Kählerian Bochner-Kodaira-Nakano identity of P. Griffiths and a Rellich-type lemma. Furthermore, the asymptotic upper bound result allows the author to weaken the quasipositivity hypothesis in the conjecture to an integral condition which does not require the bundle to be pointwise semipositive.

For the entire collection see MR1047717 (90m:32002).

Reviewed by Andrei Baran

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§ 95. Citations ↪
From References: 2
From Reviews: 4
MR0932799 (89g:32023) Reviewed
Demailly, Jean-Pierre(F-GREN-F); Laurent-Thiébaut, Christine(F-PARIS6-G)
Formules intégrales pour les formes différentielles de type (p, q) dans les variétés de Stein. (French. English summary) [Integral formulas for differential forms of type (p, q) in Stein manifolds]
Ann. Sci. École Norm. Sup. (4) 20 (1987), no. 4, 579-598.
32E10 (32A25)

Since the late 1960s integral representation formulas for differential forms have played a new important role in complex analysis. It is a problem to generalize these formulas, which are well known for domains in $\mathbf{C}^n$, to more general complex manifolds. Several authors have succeeded in obtaining such a generalization for $(0, q)$-forms on Stein manifolds, whereas the case of arbitrary (p, q)-forms (with $p \neq 0$) has not yet been solved completely. In the work under review, a new, interesting approach is proposed to solve this problem: instead of the exterior differential operator, the authors use the Chern connection related to a Hermitian metric in the complex tangent bundle. In general, the formulas obtained in this way contain some "parasitic" terms coming from the (nonvanishing) curvature of the Chern connection. But there are interesting concrete situations where, by an appropriate choice of the metric, these terms can be removed. For applications of this new approach see papers by B. Berndtsson [Publ. Math. 32 (1988), no. 1, 7-41; MR0939766 (89f:32006)] and M. Andersson ["Weighted solution formulas for the $\overline{\partial}$-equation on a Stein manifold", Preprint, Univ. Göteborg, Göteborg, 1988; per revr.].

Reviewed by Jürgen Leiterer

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§ 96. Citations ↪
From References: 0
From Reviews: 0
MR0916343 (89c:32076) Reviewed
Demailly, Jean-Pierre(F-GREN-F)
Théorèmes d'annulation pour la cohomologie des puissances tensorielles d'un fibré vectoriel positif. (French. English summary) [Vanishing theorems for cohomology groups of tensor powers of a positive vector bundle]
C. R. Acad. Sci. Paris Sér. I Math. 305 (1987), no. 10, 419-422.
32L20 (32L15)

Summary: "Let E be a holomorphic vector bundle of rank r over a compact complex manifold X of dimension n. If E is positive in the sense of Griffiths, it is shown that the Dolbeault cohomology groups $H^{p,q}(X, E^{\otimes k} \otimes (\det E)^l)$ vanish for $p+q > n+1$ and $l > r+C(n,p,q)$. The proof rests on the well-known fact that any tensor power $E^{\otimes k}$ splits into irreducible representations of $\mathrm{Gl}(E)$, each component being canonically isomorphic to the space of sections of a positive homogeneous line bundle over a flag manifold of E. The vanishing property is then obtained by a combination of J. Le Potier's isomorphism theorem with a new curvature estimate for the bundle of X-relative differential forms on the flag manifold of E."

Detailed proofs of these results appear elsewhere [the author, in Geometry and analysis on manifolds (Katata/Kyoto, 1987), 86-105, Lecture Notes in Math., 1339, Springer, Berlin, 1988; Invent. Math. 91 (1988), no. 1, 203-220].

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§ 97. Citations ↪
From References: 17
From Reviews: 4
MR0908144 (89b:32019) Reviewed
Demailly, Jean-Pierre(F-GREN)
Nombres de Lelong généralisés, théorèmes d'intégralité et d'analyticité. (French) [Generalized Lelong numbers, integrality and analyticity theorems]
Acta Math. 159 (1987), no. 3-4, 153-169.
32C30

The author gives a generalised definition of the classical Lelong numbers on a Stein complex space with a closed positive current on it. This definition is also a generalisation of the Lelong numbers introduced recently by C. O. Kiselman ["Un nombre de Lelong raffiné", oral presentation, Journées Complexes du Sud de la France, May 1986; per bibl.]. Consequently the author is able to give very simple proofs of the classical results about the usual Lelong numbers together with a proof of P. Thie's theorem about the coincidence of Lelong numbers of an analytic set X at a point $x \in X$ with its algebraic multiplicity. Finally, a generalisation of Y. T. Siu's theorem [Invent. Math. 27 (1974), 53-156; MR0352516 (50 #5003); MR errata, EA 50] on the analyticity of sets associated to the Lelong numbers is proved, which has Kiselman's theorem [op. cit.] concerning directional Lelong numbers as a special case.

Reviewed by Adib A. Fadlalla

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§ 98. Citations ↪
From References: 69
From Reviews: 10
MR0881709 (88g:32034) Reviewed
Demailly, Jean-Pierre(F-GREN)
Mesures de Monge-Ampère et mesures pluriharmoniques. (French) [Monge-Ampère measures and pluriharmonic measures]
Math. Z. 194 (1987), no. 4, 519-564.
32F05 (31C10)

The author addresses the problem of developing an appropriate potential theory, for plurisubharmonic (psh) functions on a bounded domain Ω in a Stein manifold. A domain Ω is hyperconvex if there is a psh function $\varphi: \Omega \to [-\infty, 0)$ such that $\{\varphi < -\varepsilon\}$ is relatively compact for all $\varepsilon > 0$. The connection with potential theory is to show that for Ω hyperconvex, there are "pluricomplex" Green and Poisson kernels and to derive some of their properties. (The Green kernel discussed here has a singularity at a point $z \in \Omega$ and is not to be confused with the extremal psh function associated to a compact set [cf. J. Siciak , Extremal plurisubharmonic functions and capacities in $\mathbf{C}^n$, Sophia Univ., Dept. Math., Tokyo, 1982; Zbl 579:32025], which plays the role of Green function with logarithmic pole at infinity.)

First it is shown that if $\Omega \subset\subset \mathbf{C}^n$ is pseudoconvex and $\partial\Omega$ is Lipschitz, then there exists a smooth, psh exhaustion $\varphi: \Omega \to (-\infty, 0)$ such that $\varphi(z)$ is bounded above and below by a multiple of $(\log(1/\delta))^{-1}$, where $\delta(z)$ is the distance from z to $\partial\Omega$. This improves an earlier result of N. Kerzman and J.-P. Rosay [Math. Ann. 257 (1981), no. 2, 171-184; MR0634460 (83g:32019)].

The pluricomplex Green function with pole at $z \in \Omega$ is defined as the unique psh function u_z on Ω with the properties $u_z = 0$ on $\partial\Omega$, $(dd^c u_z)^n = 0$ on $\Omega - \{z\}$, and $u_z(w) = \log|z - w| + O(1)$. This function was studied by L. Lempert [Bull. Soc. Math. France 109 (1981), no. 4, 427-474; MR0660145 (84d:32036)] for Ω strictly convex. In this case, u_z arises from the Kobayashi extremal disks for the point z. For the more general case of hyperconvex domains, the author follows the approach of M. Klimek [Bull. Soc. Math. France 113 (1985), no. 2, 231-240; MR0820321 (87d:32032)] to show that $u_z(\zeta)$ exists and is continuous for $(z, \zeta) \in \Omega \times \overline{\Omega}$.

This gives rise to an invariant distance function defined by $\delta_\Omega(x, y) = \limsup_{\delta\to\partial\Omega} |\mathrm{Log}(u_x(\zeta)/u_y(\zeta))|$. The distance δ_Ω is compatible with the Euclidean topology on Ω, and in many cases Ω is complete with respect to δ_Ω. But δ_Ω is not always comparable with the Carathéodory and Kobayashi metrics.

For a continuous psh exhaustion φ of Ω, there is a family of measures $\mu_{\varphi,r}$ for which a generalized Jensen-Lelong formula holds. That is, if $B(r) = \{\varphi < r\}$, then

$$\int V\, d\mu_{\varphi,r} = \int_{B(r)} V (dd^c\varphi)^n + \int_{B(r)} (r - \varphi) dd^c V \wedge (dd^c\varphi)^{n-1}$$

holds for any psh V on Ω. If φ is smooth, then $\mu_{\varphi,r} = (dd^c\varphi)^{n-1} \wedge d^c\varphi \big|_{\partial B(r)}$. For $z \in \Omega$ and $\varphi = u_z/2\pi$, it is shown that $\lim_{r\to 0} \mu_{\varphi,r}$ exists, and this is called the pluriharmonic measure μ_z, i.e., μ_z is a measure supported on $\partial\Omega$, and $\int p\mu_z = p(z)$ for p pluriharmonic on $\overline{\Omega}$. In a natural sense, the measure μ_z is the balayage of the singularity of u_z to $\partial\Omega$. It is shown that if $\partial\Omega$ is of class C^2, then μ_z is carried by the strongly pseudoconvex points of $\partial\Omega$.

In the last section of the paper, the author applies these ideas to the analogous situation in the convex case. That is, he considers convex $K \subset \mathbf{R}^n$, which is envisioned as the logarithmic image of a Reinhardt domain. In this case, he shows that each $x \in K$ is the barycenter of a measure supported on the extreme points of K (the Choquet representation).

Reviewed by Eric Bedford

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§ 99. Citations ↪
From References: 0
From Reviews: 0
MR1046067 (91a:32020) Reviewed
Demailly, Jean-Pierre
Fonction de Green pluricomplexe et mesures pluriharmoniques. (French) [Pluricomplex Green functions and pluriharmonic measures] Séminaire de Théorie Spectrale et Géométrie, No. 4, Année 1985-1986, 131-143, Univ. Grenoble I, Saint-Martin-d'Hères, 1986.
32F05 (31C10)

This is a condensed version of a paper that has appeared elsewhere [Math. Z. 194 (1987), no. 4, 519-564; MR0881709 (88g:32034)].

For the entire collection see MR1046058 (90m:58220).

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§ 100. Citations ↪
From References: 1
From Reviews: 0

MR0874764 (88c:32025) Reviewed
Demailly, Jean-Pierre(F-GREN-F)
Un exemple de fibré holomorphe non de Stein à fibre $\mathbf{C}^2$ au-dessus du disque ou du plan. (French. English summary) [An example of a non-Stein holomorphic fiber bundle with fiber $\mathbf{C}^2$ over the disc or the plane] Séminaire d'analyse P. Lelong-P. Dolbeault-H. Skoda, années 1983/1984, 98-104,
Lecture Notes in Math., 1198, Springer, Berlin, 1986.
32E10 (32L15)

The author gives a simple counterexample to the Serre problem. The first counterexample was given by H. Skoda [Invent. Math. 43 (1977), no. 2, 97-107; MR0508091 (58 #22657)]. By an improvement of Skoda's construction, the author obtained a counterexample whose base is simply connected [ibid. 48 (1978), no. 3, 293-302; MR0508989 (81m:32036)]. In this paper, the author constructs a non-Stein holomorphic fiber bundle over the disc or the plane with fiber $\mathbf{C}^2$ and with structural automorphisms of exponential type, which is a simplification of the above counterexamples.

For the entire collection see MR0874757 (87m:32004).

Reviewed by Yukitaka Abe

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§ 101. Citations ↪
From References: 7
From Reviews: 3
MR0874763 (88f:32069) Reviewed
Demailly, Jean-Pierre(F-GREN-F)
Sur l'identité de Bochner-Kodaira-Nakano en géométrie hermitienne. (French. English summary) [On the Bochner-Kodaira-Nakano identity in Hermitian geometry] Séminaire d'analyse P. Lelong-P. Dolbeault-H. Skoda, années 1983/1984, 88-97,
Lecture Notes in Math., 1198, Springer, Berlin, 1986.
32L15 (32L10 53C55)

Let (E,π,X) be a holomorphic Hermitian vector bundle over a complex Hermitian manifold X ($\pi\colon E\to X$), equipped with a smooth ($\mathcal{C}^\infty$) Hermitian metric ω. By $D=D'+D''$ is denoted the canonical connection of E, δ' and δ'' being the formal adjoints of D' and D'', respectively, and $\Delta'=[D',\delta']$, $\Delta''=[D'',\delta'']$ the Laplace-Beltrami operators, respectively the holomorphic and the anti-holomorphic one. The proof of the Bochner-Calabi-Kodaira-Nakano (BCKN) identity $\Delta''=\Delta'+[ic(E),\Lambda]$ is based on the known commutation relation $[L,\delta'']=iD'$, where $c(E)=D^2$ is the associated curvature form of D, and Λ is the adjoint operator of L ($u\mapsto Lu:=\omega\wedge u$, u being a differential form). A new commutation relation based on the operator $\tau=[\Lambda,d'\omega]$, ω being in general a non-Kähler metric, is developed in this paper. As suggested by J. Le Potier in his Thèse, and essentially simplified here, the operator τ leads to the general commutation relation $[L,\delta'']=i(D'+\tau)$. From this relation, with the help of elementary methods (say with an intensive use of the Jacobi identity), the following generalized BCKN formula is deduced: $\Delta''=\Delta'+[ic(E),\Lambda]+T_\omega$, where $\Delta'_\tau=[D'+\tau,\delta'+\tau^*]$ and $T_\omega=[\Lambda,[\Lambda,(i/2)d'd''\omega]]-[d'\omega,(d'\omega)^*]$.

Let us remark that T_ω is an operator of degree 0, as is τ, which depends only on the curvature of ω. The new interesting operator Δ'_τ is a selfadjoint positive operator for which it is true that $\Delta_\tau:=[D+\tau,\delta+\tau^*]=\Delta'_\tau+\Delta''$. This relation replaces the classical relation $\Delta=\Delta'+\Delta''$ of the Kählerian case.

One application of the generalized BCKN formula, related to the existence of D'' on general Hermitian manifolds, is given.

For the entire collection see MR0874757 (87m:32004).

Reviewed by S. Dimiev

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§ 102. Citations ↪
From References: 0
From Reviews: 0
MR0874578 (88b:32042) Reviewed
Demailly, J.-P.

Mesures de Monge-Ampère et mesures pluriharmoniques. (French. English summary) [Monge-Ampère measures and pluriharmonic measures] Séminaire sur les équations aux dérivées partielles, 1985-1986, Exp. No. XIX, 15 pp., École Polytech., Palaiseau, 1986.
32F05 (32H15)

This is a short preliminary version of a recent paper by the author [Math. Z. 194 (1987), no. 4, 519-564].

For the entire collection see MR0874559 (87j:58003).

Reviewed by M. Klimek

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§ **103. Citations ↪**
From References: 34
From Reviews: 5
MR0813252 (87g:32030) Reviewed
Demailly, Jean-Pierre
Mesures de Monge-Ampère et caractérisation géométrique des variétés algébriques affines. (French. English summary) [Monge-Ampère measures and geometric characterization of affine algebraic varieties]
Mém. Soc. Math. France (N.S.) No. 19 (1985), 124 pp.
32H35 (32C10 32F05)

The main result of this big work consists in obtaining the following geometric characterisation of an affine algebraic manifold: Let X be a complex analytic manifold of dimension n. Then X is biholomorphically equivalent to the affine algebraic manifold $X_{\rm alg}$ if and only if the cohomology spaces $H^{2q}(X,\mathbf{R})$ have finite dimension and there exists a strictly plurisubharmonic exhaustion function φ of class C^∞ on X such that $\operatorname{vol}(X)=\int_X(dd^c\varphi)^n<+\infty$ and the Ricci curvature of the metric $\beta=dd^c(e^\varphi)$ has the following estimate from below:

$$\operatorname{Ricci}(\beta)\geq -\tfrac{1}{2}dd^c\psi,\ \text{ where } \psi\in C^0(X,\mathbf{R}),\ \psi\leq A\varphi+B,$$

and A, B are positive constants.

This result can be considered as a very deep generalisation of the known work of W. Stoll (1980) and D. Burns (1982), which gives a characterisation in terms of plurisubharmonic exhaustion functions of analytic manifolds which are equivalent to $\mathbf{C}^n$.

The proof of the necessity of the given conditions for an algebraic submanifold X of $\mathbf{C}^N$ uses the exhaustion function $\varphi=\log(1+|z|^2)$ and an explicit calculation of the Ricci curvature for $\beta=dd^c(1+|z|^2)$ on X. For the proof of the sufficiency of the given conditions, the author constructs (using the L^2-estimates for $\overline{\partial}$ of L. Hörmander, S. Nakano and H. Skoda) the system of holomorphic functions $F=(f_1,\cdots,f_N)$ on X, which are φ-polynomials on X and give an imbedding of $X\smallsetminus S$ in $\mathbf{C}^N$, where S is some analytic subset of X. A holomorphic function f on X is called a φ-polynomial if $\delta(f)=\limsup_{r\to+\infty}(1/r)\mu_r(\log_+|f|)<\infty$, where $\mu_r(h)=\int_{\varphi(z)=r}h(dd^c\varphi)^{n-1}\wedge d^c\varphi$. In order to prove that $F(X\smallsetminus S)$ is a Zariski-open subset of some algebraic manifold $M\subset\mathbf{C}^N$, the author finds the following new algebraicity theorem.

Let $K_\varphi(X)$ be the field of meromorphic function on$X$ of the form f/g, where f and g are φ-polynomials. Then $0\leq\deg\operatorname{tr}_{\mathbf{C}}K_\varphi(X)\leq\dim_{\mathbf{C}}X$ and if $\deg\operatorname{tr}_{\mathbf{C}}K_\varphi(X)=\dim_{\mathbf{C}}X$ then $K_\varphi(X)$ has finite type. This result contains, in particular, the known theorem of Stoll (1964) about the characterisation of an algebraic subset of $\mathbf{C}^N$ in terms of the growth of its Euclidean volume. It is also interesting to compare the main result of this paper with recent results in the same direction of N. Mok, Y. T. Siu and S. T. Yau (1983, 1984).

Reviewed by G. M. Khenkin

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§ **104. Citations ↪**
From References: 27
From Reviews: 14
MR0812325 (87d:58147) Reviewed
Demailly, Jean-Pierre(F-GREN)
Champs magnétiques et inégalités de Morse pour la d''-cohomologie. (French. English summary) [Magnetic fields and Morse inequalities for the d''-cohomology]

Ann. Inst. Fourier (Grenoble) 35 (1985), no. 4, 189-229.
58G25 (32J25 58E05)

A new viewpoint on the Morse inequalities was recently introduced by E. Witten [J. Differential Geom. 17 (1982), no. 4, 661-692; MR0683171 (84b:58111)], relying on the behaviour of the spectrum of the Laplacian associated to the modified de Rham complex obtained by taking the exterior differential $e^{-kf}de^{kf}$ for large k, and Morse function f. The paper under review applies the same ideas to the operator $k^{-1}(\nabla^*\nabla - kV)$ on a compact complex manifold X, where ∇ is the covariant derivative of a connection on a holomorphic line bundle E and V is a C^∞ function. (This operator is the "Schrödinger operator" associated to the electric field kV and the magnetic field kB, where B is the curvature of ∇ – hence the title.)

The analogue of the Morse inequalities obtained in this way consists of an asymptotic estimate for the dimension of the sheaf cohomology groups $H^q(X, E^k \otimes F)$ for large k. The basic inequality is

$$\dim H^q(X, E^k \otimes F) \le (\operatorname{rk} F)\frac{k^n}{n!}\int_{X(q)} (-1)^q \left(\frac{ic(E)}{2\pi}\right)^n + o(k^n),$$

where the curvature form $c(E)^n$ is integrated over the open set of points $X(q)$ for which $ic(E)$ has exactly q negative eigenvalues and $(n-q)$ positive ones. By analogy with the Morse inequalities, stronger inequalities are obtained for alternating sums, culminating in the (known) asymptotic form of the Riemann-Roch formula for the full Euler characteristic of the complex.

An interesting application concerns a characterization ofMoishezon manifolds as complex manifolds which admit the existence of a holomorphic line bundle E which is positive in a weak way: the integral of $(ic(E))^n$ over the set of points for which $ic(E)$ has one or fewer negative eigenvalues must be positive.

Reviewed by N. J. Hitchin

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§ 105. Citations ↪
From References: 0
From Reviews: 0
MR0803576 (86m:11105) Reviewed
Demailly, Jean-Pierre(F-GREN-F)
Sur le calcul numérique de la constante d'Euler. (French) [The numerical computation of the Euler constant]
Gaz. Math. No. 27 (1985), 113-126.
11Y60 (65D20 68Q25)

The author gives an account of various methods used to calculate Euler's constant, and estimates the number of operations that each requires to obtain d decimal places. Euler's own methods required $O(d^3(\log d)^{-2})$ and $O(d^2 \log d)$ operations: modern methods are all $O(d^2)$, the most economical being those of R. P. Brent and E. M. McMillan [Math. Comp. 34 (1980), no. 149, 305-312; MR0551307 (82g:10002)]. By evaluating series by nested polynomials the author achieves smaller multiples of d^2 than Brent and McMillan did, though he does not appear to have implemented his methods.

On the first line of p. 122, the equation should be $S_0(x) + H_N(1 - I_0(x)) = -x^2/1^2$ (etc.).

Reviewed by H. J. Godwin

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§ 106. Citations ↪
From References: 1
From Reviews: 0
MR0800173 (87f:32037) Reviewed
Demailly, Jean-Pierre(F-GREN-F)
Propagation des singularités des courants positifs fermés. (French. English summary) [Propagation of singularities of closed positive currents]
Ark. Mat. 23 (1985), no. 1, 35-52.
32D20 (32C30)

Let T be a closed positive current of bidegree (q,q) on a bounded Runge open subset Ω of $\mathbf{C}^n$; $\sigma = (\frac{i}{2}\partial\overline{\partial}|z|^2)^{n-q} \wedge T$ themeasure trace of T; $\sigma(z,r) = \sigma(\{\zeta \in \mathbf{C}^n\colon |\zeta - z| < r\}$;$\Omega_r = \{\zeta \in \mathbf{C}^n\colon d(\zeta, \mathbf{C}^n \setminus \Omega) > r\})$. Theorem: If T has a sufficientlylow density (i.e. $\sup_{z\in K}\int_0^\varepsilon (\sigma(z,r)/r)\,dr < \infty$, if $q = 1$, and$\sup_{z\in K}\int_0^{\varepsilon}((\sigma(z,r))^{1/2}/r^n)\,dr<\infty$, if $1<q<n$ for every $K\subset\Omega_\varepsilon$ and every $\varepsilon>0$) then there exists a global extension of T to $\mathbf{C}^n$ without propagation of singularities (i.e., for every $\delta>\eta>0$ there exists a closed positive current Θ on $\mathbf{C}^n$, which equals T on Ω_δ and equals some C^∞-form outside $\overline{\Omega}_\eta$). Conversely, using results of Y. T. Siu, N. Skoda, H. El Mir, J. P. Demailly, K. Diederich and J. E. Fornæss , examples are given which show that the above sufficient conditions are optimal.

Reviewed by G. M. Khenkin

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§ 107. Citations ↪
From References: 7
From Reviews: 0
MR0799607 (86k:32028) Reviewed
Demailly, Jean-Pierre(F-GREN-F)
Champs magnétiques et inégalités de Morse pour la d''-cohomologie. (French. English summary) [Magnetic fields and Morse inequalities for d''-cohomology]
C. R. Acad. Sci. Paris Sér. I Math. 301 (1985), no. 4, 119-122.
32L10 (58G10)

Author summary: "Using E. Witten's method, we prove asymptotic Morse inequalities for the d''-cohomology of tensor powers of a Hermitian line bundle over a compact complex manifold: the dimension H^q is bounded above by an integral of the $(1,1)$-curvature form, extended to the set of points of index q. The proof rests upon a spectral theorem which describes the asymptotic distribution of the spectrum of the Schrödinger operator associated to a large magnetic field. As an application, we find new geometric characterizations of Moishezon spaces, which improve Y. T. Siu's recent solution of the Grauert-Riemenschneider conjecture."

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§ 108. Citations ↪
From References: 0
From Reviews: 0
MR0773103 (86k:32013) Reviewed
Demailly, Jean-Pierre(F-GREN-F)
Sur la propagation des singularités des courants positifs fermés. (French. English summary) [On the propagation of singularities of closed positive currents] Complex analysis (Toulouse, 1983), 53-64,
Lecture Notes in Math., 1094, Springer, Berlin, 1984.
32D15 (32C30 32F05)

Let T be a closed positive current on a bounded Runge open subset Ω of $\mathbf{C}^n$. The author gives sufficient conditions in terms of the density of T in order that there exists a closed positive current Θ on $\mathbf{C}^n$ such that $\Theta=T$ on $\Omega_\varepsilon=\{z\in\Omega\colon d(z,\partial\Omega)>\varepsilon\}$. When T is of bidegree $(1,1)$, the sufficient condition may be written:

$$\sup_{z\in\Omega_\delta\setminus\Omega_\varepsilon}\int_0^{\delta/2}\frac{\sigma_T(z,r)}{r^{2n-1}}\,dr<\infty$$

for some $\delta<\varepsilon$, where $\sigma_T(z,r)$ is the mass of the ball $B(z,r)$, and it is shown to be the best possible; the counterexample is constructed by a summation process on the fibers of an analytic morphism and use of the El Mir-Skoda theorem.

Similar, but not so sharp, results are given for currents T of bidegree (q,q), $q\geq 2$. The author constructs a function V such that $i\partial\bar{\partial}V=\chi T+$ remaining terms, where χ is a truncation function, using suitable kernels. Moreover, the extension Θ is C^∞ outside $\overline{\Omega}$.

For the entire collection see MR0773096 (85i:32002).

Reviewed by Aline Bonami

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§ 109. Citations ↪
From References: 1
From Reviews: 0
MR0768075 (86b:43013) Reviewed
Demailly, Jean-Pierre

Sur les transformées de Fourier de fonctions continues et le théorème de de Leeuw-Katznelson-Kahane. (French. English summary) [On Fourier transforms of continuous functions and a theorem of de Leeuw, Katznelson and Kahane]
C. R. Acad. Sci. Paris Sér. I Math. 299 (1984), no. 10, 435-438.
43A25

Author's summary: "Given any locally compact abelian group G and any function $\varphi \in L^2(\hat{G})$, we prove the existence of a function $f \in L^2(G)$, continuous and vanishing at infinity, such that $|\hat{f}| \geq |\varphi|$ a.e. on $\hat{G}$."

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§ **110. Citations ↪**
From References: 0
From Reviews: 0
MR0774974 (86j:32041) Reviewed
Demailly, Jean-Pierre(F-PARIS6-G); Gaveau, Bernard
Majoration statistique de la courbure d'une variété analytique. (French) [Statistical upper bound for the curvature of an analytic manifold] P. Lelong-P. Dolbeault-H. Skoda analysis seminar, 1981/1983, 96-124,
Lecture Notes in Math., 1028, Springer, Berlin, 1983.
32F15 (53C20)

Let Ω be a strongly pseudoconvex domain in $\mathbf{C}^n$ equipped with the induced flat metric from $\mathbf{C}^n$. Let $F\colon \Omega \to \mathbf{C}^p$ be a holomorphic map, with $F = (F_1, \cdots, F_p)$. Let $X_t = F^{-1}(t)$, $t \in \mathbf{C}^p$; then for almost all t, X_t is a nonsingular subvariety of codimension p in Ω. One is interested in the growth properties of the Ricci curvature of the induced metric on X_t relative to the growth of $|F|$. To state the main theorems of this paper, let δ denote the distance from the boundary of Ω and let ω be the canonical Kähler form $\sum_i dz_i \wedge d\bar{z}_i$ on Ω. Let R denote the negative of the Ricci form of the induced metric on X_t; then R is positive semidefinite on X_t. If each $F_1, \cdots, F_p$ is bounded, then, for each integer $q = 0, 1, \cdots, n-p$, one has

$$\int_{\mathbf{C}^p} d\lambda(t) \int_{X_t} \delta^{p+q} [\log(1+1/\delta)]^{-q} R^q \wedge \omega^{n-p-q} < \infty,$$

where $d\lambda$ denotes the Lebesgue measure on $\mathbf{C}^p$. If on the other hand each $F_1, \cdots, F_p$ has polynomial growth in the sense that for each i there is a positive constant C_i satisfying $\log |F_i| \leq C_i \log(1+1/\delta)$, then

$$\int_{\mathbf{C}^p} (1+|t|^2)^{-p-\varepsilon} d\lambda(t) \int_{X_t} \frac{\delta^{p+q} R^q \wedge \omega^{n-p-q}}{[\log(1+1/(1-\varepsilon)\delta)]^{-p-q-\varepsilon}} < \infty,$$

where ε is any positive constant. Three relevant comments have to be made in connection with these two assertions. One is that both are special cases of a general theorem showing the finiteness of iterated integrals of the above type when each F_i is in a weighted Nevanlinna class; the technical definition of the latter will not be attempted here. A second remark is that both are statements to the effect that the inner integral $\int_{X_t}$ behaves well on the average. However, one cannot expect a statement on each individual $\int_{X_t}$ for the following reason. If $q = 0$, the inner integral of the first inequality is just the volume of X_t and the well-known example of M. Cornalba and B. Shiffman [Ann. of Math. (2) 96 (1972), 402-406; MR0311937 (47 #499)] shows that in fact the volume of X_t does not behave well at all. Finally, the proof of this inequality is a series of careful integrations by parts plus a good estimate of the potential H of R, $R = \partial\bar{\partial} \log H$.

The last section of the paper shows that when $p = 1$, the total curvature Γ, which is defined to be the determinant of R, satisfies a Monge-Ampère equation

$$(i\partial\bar{\partial} \log \Gamma)^{n-1} = (-1)^{n-1}(n+1)^{n-1} \Gamma \omega^{n-1},$$

the equality holding on each X_t outside the zero set of Γ. This is a direct calculation but is nevertheless interesting.

For the entire collection see MR0774968 (85k:32003).

Reviewed by Hung-Hsi Wu

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§ **111. Citations ↪**

From References: 0
From Reviews: 0
MR0774973 (1986) Indexed
Demailly, Jean-Pierre
Constructibilité des faisceaux de solutions des systèmes différentiels holonômes d'après Masaki Kashiwara. (French) [Constructibility of sheaves of solutions of holonomic differential systems after Masaki Kashiwara] P. Lelong-P. Dolbeault-H. Skoda analysis seminar, 1981/1983, 83-95,
Lecture Notes in Math., 1028, Springer, Berlin, 1983.
58G07 (32C38)

This item will not be reviewed individually.

For the entire collection see MR0774968 (85k:32003).

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§ **112. Citations** ↪
From References: 0
From Reviews: 0
MR0748313 (86f:32009) Reviewed
Demailly, J.-P.(F-PARIS6)
Sur la structure des courants positifs fermés. (French) [The structure of closed positive currents] Conference on complex analysis, Nancy 82 (Nancy, 1982), 52-62,
Inst. Élie Cartan, 8, Univ. Nancy, Nancy, 1983.
32C30

The first part of the paper is a survey of a previous paper by the author [Ann. Inst. Fourier Grenoble 32 (1982), no. 2, 37-66; MR0662440 (84k:32011)]: generalized Lelong numbers of a closed, positive current relative to a logarithmically plurisubharmonic weight, invariance properties of these numbers with respect to proper analytic morphisms, arithmetical applications. In the second part, a negative answer is given to a question of P. Lelong [Pierre Lelong seminar (analysis) 1971-1972 (French), 112-131, Lecture Notes in Math., 332, Springer, Berlin, 1973; MR0412474 (54 #600)] and R. Harvey [Several complex variables, Part 1 (Williamstown, Mass., 1975), 309-382, Proc. Sympos. Pure Math., XXX, Amer. Math. Soc., Providence, R.I., 1977; MR0447619 (56 #5929)]. Namely, an example of an extremal, positive, closed current of bidimension $(1,1)$ on $\mathbf{C}\mathrm{P}^2$ and $\mathbf{C}^2$ is constructed which is not an analytic cycle. Then, by H. Skoda's extension theorem [Invent. Math. 66 (1982), no. 3, 361-376; MR0662596 (84k:32020)], this example is extended to $\mathbf{C}\mathrm{P}^n$ and $\mathbf{C}^n$ for each bidimension p with $0 < p < n$. Further, in the second part, a connection with the Hodge conjecture via the Kreĭn-Mil'man theorem is discussed [see the author, ibid. 69 (1982), no. 3, 347-374; MR0679762 (84f:32007)].

For the entire collection see MR0748309 (85d:32002).

Reviewed by Jürgen Leiterer

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§ **113. Citations** ↪
From References: 4
From Reviews: 1
MR0662440 (84k:32011) Reviewed
Demailly, Jean-Pierre
Sur les nombres de Lelong associés à l'image directe d'un courant positif fermé. (French. English summary) [On the Lelong numbers associated with the direct image of a closed positive current]
Ann. Inst. Fourier (Grenoble) 32 (1982), no. 2, ix, 37-66.
32C30

Generalized Lelong numbers with respect to a logarithmically plurisubharmonic weight are defined for closed positive currents. The invariance properties of these numbers with respect to analytic morphisms give precise bounds for Lelong numbers of direct images.

Reviewed by Jürgen Leiterer

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§ **114. Citations** ↪

From References: 10
From Reviews: 8
MR0679762 (84f:32007) Reviewed
Demailly, Jean-Pierre
Courants positifs extrêmaux et conjecture de Hodge. (French) [Extremal positive currents and the Hodge conjecture]
Invent. Math. 69 (1982), no. 3, 347-374.
32C30 (14C30)

On a complex manifold X, the convex cone $\mathrm{SPC}^p(X)$ of strongly positive closed (p,p)-currents is defined (for $0 \le p \le n = \dim_{\mathbf{C}} X$), and it is known that the current $[Z]$ corresponding to integration on an irreducible p-dimensional subvariety Z of X is an element of the set $E^p(X)$ of extremal elements of $\mathrm{SPC}^p(X)$. It has been conjectured [cf., e.g., P. Lelong, Séminaire Pierre Lelong (Analyse), Année 1971-1972, 112-131, Lecture Notes in Math., 332, Springer, Berlin, 1973; MR0412474 (54 #600)] that, for a Stein manifold X, every $T \in E^p(X)$ is a scalar multiple of such a $[Z]$. In this paper the author shows that the conjecture is false for $X = \mathbf{C}^n$ or $\mathbf{P}^n$ and $1 \le p \le n-1$. In fact, he proves that, if $C_d \subset \mathbf{P}^2$ is the curve $Z_0^d + Z_1^d + Z_2^d = 0$, then $((1/d)[C_d])$ converges (in the weak topology on currents) to a counterexample, and he passes to the case of $\mathbf{C}^n$ and $\mathbf{P}^n$ by using an extension theorem for closed positive currents due to H. Skoda [Invent. Math. 66 (1982), 361-376].

In the rest of the paper, the author indicates why the above conjecture is too optimistic (even for Stein or projective manifolds). When X is projective, let $\mathrm{SPC}^p_{\mathbf{Z}}(X)$ consist of those $T \in \mathrm{SPC}^p(X)$ whose cohomology class belongs to the $\mathbf{R}$-span of $(\mathrm{Image}\, H^{2q}(X,\mathbf{Z}) \cap H^{q,q}(X,\mathbf{C}))$, $q = n-p$. Then, the conjecture that the convex cone generated by irreducible p-dimensional subvarieties of X is dense in $\mathrm{SPC}^p_{\mathbf{Z}}(X)$ already implies the Hodge conjecture that $H^{q,q}(X,\mathbf{C}) \cap H^{2q}(X,\mathbf{Q})$ is generated by algebraic p-cycles, while the original conjecture (density in all of $\mathrm{SPC}^p(X)$) implies a stronger condition which is obviously false in general. The author proves the new conjecture (and its analogue when X is Stein) for $p = n-1$.

Reviewed by R. R. Simha

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§ 115. Citations ↪
From References: 5
From Reviews: 0
MR0662130 (84d:32014) Reviewed
Demailly, J.-P.
Formules de Jensen en plusieurs variables et applications arithmétiques. (French. English summary) [Jensen formulas in several variables and arithmetic applications]
Bull. Soc. Math. France 110 (1982), no. 1, 75-102.
32C30 (10F35)

Let F be an entire function on $\mathbf{C}^n$ and $P_1, \cdots, P_N$ be polynomials of degree δ, the maximal homogeneous parts $Q_1, \cdots, Q_N$ of which have a unique common zero at $0 \in \mathbf{C}^N$. Let $\varphi(z) = \sum_{j=1}^N |P_j(z)|^2$, $\beta = i\partial\overline{\partial}\varphi$, $T = (i/\pi)\partial\overline{\partial}\mathrm{Log}\,|F|$ and $|F|_r = \sup_{|z|\le r} |F(z)|$.

Using an appropriate generalization of the Poisson-Jensen formula, the author proves the following new variant of the Schwarz lemma in $\mathbf{C}^n$: There exists a constant $C \in (0,1]$, depending only on $P_1, \cdots, P_N$, such that for all $R \ge r \ge 1$ we have

$$\int_{r^{2\delta}}^{CR^{2\delta}} \frac{dt}{t^n} \int_{\varphi(z)<t} T \wedge \beta^{n-1} \le (2\delta)^n \pi^{n-1} \mathrm{Log} \frac{|F|_R}{|F|_r}.$$

This inequality permits the author to give a new proof of E. Bombieri's theorem on algebraic values of meromorphic maps without using L^2-estimates for the $\overline{\partial}$-operator.

Some new results concerning zero sets of polynomials in $\mathbf{C}^n$ are also given.

Reviewed by G. M. Khenkin

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§ 116. Citations ↪
From References: 80
From Reviews: 5

MR0690650 (85d:32057) Reviewed
Demailly, Jean-Pierre
Estimations L^2 pour l'opérateur $\bar{\partial}$ d'un fibré vectoriel holomorphe semi-positif au-dessus d'une variété kählérienne complète. (French) [L^2-estimates for the $\bar{\partial}$-operator of a semi-positive holomorphic vector bundle over a complete Kähler manifold]
Ann. Sci. École Norm. Sup. (4) 15 (1982), no. 3, 457-511.
32L15 (32L20)

Author's review: Let E be a Hermitian vector bundle of rank r over an n-dimensional Kähler manifold X. The bundle E is said to be s-positive if its curvature tensor $K(E)$ identified with a Hermitian form on $TX \otimes E$ takes positive values on tensors of rank $\leq s$ and $\neq 0$. For example, if E is Griffiths positive (i.e. 1-positive) of rank $r \geq 2$, we show that $E^* \otimes (\det E)^s$ is s-positive and that $E \otimes \det E$ is Nakano positive (i.e. n-positive). In connection with these results, we prove the following vanishing theorem: If E is s-positive and X is weakly pseudoconvex, then $H^q(X, \bigwedge^n T^*X \otimes E) = 0$ for $q \geq \sup(1, n-S+1)$. Given a surjective morphism $E \to Q \to 0$ of Hermitian bundles, we also obtain curvature conditions which imply the surjectivity of the map $H^q(X, E \otimes L) \to H^q(X, Q \otimes L)$, $0 \leq q < n$, where L is a line bundle. All these results are proved in quantitative versions using L^2 estimates and plurisubharmonic weights. In order to get rid of continuity assumptions for weights or exhaustion on X, a smoothing method is developed for psh functions involving the exponential map $TX \to X$. In particular, if X has an upper semicontinuous exhaustive psh function, then it can be endowed with a complete Kähler metric.

Reviewed by Autorreferat (Zbl 507:32021)

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§ **117. Citations ↪**
From References: 3
From Reviews: 1
MR0658880 (83j:32019) Reviewed
Demailly, J.-P.
Scindage holomorphe d'un morphisme de fibrés vectoriels semi-positifs avec estimations L^2. (French) [Holomorphic splitting of a morphism of semipositive vector bundles with L^2 estimates] Seminar Pierre Lelong-Henri Skoda (Analysis), 1980/1981, and Colloquium at Wimereux, May 1981, pp. 77-107,
Lecture Notes in Math., 919, Springer, Berlin-New York, 1982.
32D15 (32L05)

Let $0 \to S \to E \overset{g}{\to} Q \to 0$ be an exact sequence of holomorphic Hermitian vector bundles over the weakly pseudoconvex complex Kählerian manifold X and let M be a line bundle over X. Using a result of H. Skoda [Ann. Sci. École Norm. Sup. (4) 11 (1978), no. 4, 577-611; MR0533068 (80j:32047)] the author gives geometric conditions in terms of the curvature of the bundles which are sufficient to find for every $f \in \Gamma(X, \mathrm{Hom}(Q, Q \otimes M))$ a holomorphic preimage $h \in \Gamma(X, \mathrm{Hom}(Q, E \otimes M))$ with L^2-estimates. In the natural situation of the exact sequence $0 \to TX \to T\Omega|_X \to NX \to 0$, where $\Omega \subset \mathbf{C}^n$ is a pseudoconvex domain and X is a closed submanifold in Ω, the above results are applied to prove extension theorems for holomorphic functions on X with precise estimates. Similar results have been obtained before by B. Jennane [Séminaire Pierre Lelong-Henri Skoda (Analyse), Année 1976/77 (French), pp. 126-133, Lecture Notes in Math., 694, Springer, Berlin, 1978; MR0522474 (80m:32016)], C. A. Bernstein and B. A. Taylor [J. Analyse Math. 38 (1980), 188-254; MR0600786 (82h:32002)] and T. Yoshioka [Proc. Japan Acad. Ser. A Math. Sci. 57 (1981), no. 3, 181-184; MR0618087 (82f:32029)].

For the entire collection see MR0658876 (83d:32001).

Reviewed by Peter Pflug

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§ **118. Citations ↪**
From References: 0
From Reviews: 0
MR0658879 (83j:32032) Reviewed
Demailly, J.-P.
Relations entre les différentes notions de fibrés et de courants positifs. (French) [Relations between the different notions of positive vector bundles and currents] Seminar Pierre Lelong-Henri Skoda (Analysis), 1980/1981, and Colloquium at Wimereux, May 1981, pp. 56-76,

Lecture Notes in Math., 919, Springer, Berlin-New York, 1982.
32L20 (32L05)

The author discusses relations among various positivity concepts for Hermitian forms Θ on vector spaces of the form $T \otimes E$. The main result states: If Θ is semipositive in the sense of P. A. Griffiths, i.e., $\Theta(x,x) \geq 0$ for the decomposable vectors $x \in T \otimes E$, then for a scalar product φ on E the form $\Theta + \mathrm{Tr}_E\, \Theta \otimes \varphi$ is strictly semipositive, i.e., for all $x \in T \otimes E$ one has: $(\Theta + \mathrm{Tr}_E\, \Theta \otimes \varphi)(x,x) = \sum |x_i^*(x)|^2$, where x_i^* are finitely many decomposable linear forms on $T \otimes E$. In particular, it follows that $(\Theta + \mathrm{Tr}_E\, \Theta \otimes \varphi)(x,x) \geq 0$, i.e., this form is semipositive in the sense of S. Nakano.

This result is then applied to vector bundles and the positivity concepts valid there [cf. the author and H. Skoda, Séminaire Pierre Lelong- Henri Skoda (Analyse), Années 1978/79 (French), pp. 304-309, Lecture Notes in Math., 822, Springer, Berlin, 1980; MR0599033 (82h:32028)].

In the last section the author compares a weakly positive (p,p)-form $\alpha \in \Lambda^{p,p}\, \mathrm{Hom}_{\mathbf{R}}(T, \mathbf{C})$ (i.e., a form α for which, with every (k,k)-form $\beta = \sum_j \varepsilon_k \cdot \beta_j \wedge \overline{\beta}_j$, where the β_j's are decomposable $(k,0)$-forms, the form $\alpha \wedge \beta$ is a positive (n,n)-form) with forms of the type $L^{p-1}\Lambda^{p-1}\alpha$. Here, with the existing scalar product on T the operators L, Λ are modeled after the usual Kähler operators. For example,

$$C(n,p) \cdot \frac{1}{p!^2} \cdot L^{p-1}\Lambda^{p-1}\alpha - \alpha$$

is a sum of forms $\varepsilon_p \cdot \alpha_j \wedge \overline{\alpha}_j$ with decomposable $(p,0)$-forms α_j and exactly definable constants $C(n,p)$. An essential tool in the proofs is a summation formula for the q-th roots of unity.

For the entire collection see MR0658876 (83d:32001).

Reviewed by Peter Pflug

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§ 119. Citations ↪
From References: 5
From Reviews: 3
MR0599033 (82h:32028) Reviewed
Demailly, J.-P.; Skoda, H.
Relations entre les notions de positivités de P. A. Griffiths et de S. Nakano pour les fibrés vectoriels. (French) Séminaire Pierre Lelong-Henri Skoda (Analyse). Années 1978/79 (French), pp. 304-309,
Lecture Notes in Math., 822, Springer, Berlin, 1980.
32L20 (32J25)

La différence entre les deux notions en question est que celle de Nakano se teste sur tous les tenseurs alors que celle de Griffiths se teste sur les tenseurs décomposables. Les auteurs démontrent que si E est un fibré semi-positif au sens de Nakano, alors $E \otimes \det E$ est semi-positif au sens de Griffiths. Cet énoncé, dont la démonstration très simple relève de l'algèbre multilinéaire, est suffisamment précis par exemple pour réduire le théorème d'annulation de Griffiths à celui de Nakano. Il permet aussi, à partir des résultats de Skoda [Ann. Sci. École Norm. Sup. (4) 11 (1978), no. 4, 577-611; MR0533068 (80j:32047)] concernant la notion de Nakano, d'obtenir des énoncés concernant celle de Griffiths, sensiblement plus fins que ceux obtenus directement par Skoda [see MR0599032 (82h:32027) above].

For the entire collection see MR0599013 (81j:32003).

Reviewed by A. Hirschowitz

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§ 120. Citations ↪
From References: 2
From Reviews: 0
MR0597024 (82f:32007) Reviewed
Demailly, Jean-Pierre
Construction d'hypersurfaces irréductibles avec lieu singulier donné dans $\mathbf{C}^n$. (French)
Ann. Inst. Fourier (Grenoble) 30 (1980), no. 3, 219-236.
32A15 (32C25)

The author obtains the following interesting result. Theorem: If $S = \{f_1 = \cdots = f_k = 0\} \subset \mathbf{C}^n$ is an analytic subvariety of codimension ≥ 2, then there exist entire functions $g_1, \cdots, g_k$ of slow growth such that the variety $X = \{\sum f_j g_j = 0\}$ is irreducible, and the singular set of X is contained in S. (If f_j is replaced by f_j^2, then the singular set of X may be taken to be exactly S.)

This theorem then is used to construct two noteworthy examples. The first is an irreducible algebraic curve in $\mathbf{C}^2$ of order zero such that the number of singular points in the ball of radius R is larger than any preassigned function $\psi(R)$. The reason for giving the example is that an irreducible algebraic curve of degree n can have at most $\frac{1}{2}(n-1)(n-2)$ double points. Thus this example is related to an earlier example of M. Cornalba and B. Shiffman [Ann. of Math. (2) 96 (1972), 402-406; MR0311937 (47 #499)].

Next, the author considers the Fourier transforms of functions in $\mathcal{D}(\mathbf{R}^n)$ and $\mathcal{E}'(\mathbf{R}^n)$, $n \geq 2$. The elements of $\widehat{\mathcal{D}(\mathbf{R}^n)}$ and $\widehat{\mathcal{E}'(\mathbf{R}^n)}$ are entire functions on $\mathbf{C}^n$ with certain growth conditions. By an application of the theorem above, there exists a function $V = \sum u_j * v_j$, where $u_j, v_j \in \mathcal{D}(\mathbf{R}^n)$, and V is irreducible in $\mathcal{E}'(\mathbf{R}^n)$. As a consequence, it follows that $\mathcal{D}(\mathbf{R}^n) * \mathcal{D}(\mathbf{R}^n) \neq \mathcal{D}(\mathbf{R}^n)$ for $n \geq 2$, a result which was obtained for $n \geq 3$ by L. A. Rubel, W. A. Squires and B. A. Taylor [ibid. (2) 108 (1978), no. 3, 553-567; MR0512433 (80d:32003)] and for $n = 2$ by J. Dixmier and P. Malliavin [Bull. Sci. Math. (2) 102 (1978), no. 4, 307-330; MR0517765 (80f:22005)].

Reviewed by Eric Bedford

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§ 121. Citations ↪
From References: 1
From Reviews: 0
MR0533896 (81c:32049) Reviewed
Demailly, Jean-Pierre
Fonctions holomorphes à croissance polynomiale sur la surface d'équation $e^x + e^y = 1$. (French. English summary)
Bull. Sci. Math. (2) 103 (1979), no. 2, 179-191.
32H30 (32A10)

Let $S \subset \mathbf{C}^2$ be the surface $e^x + e^y = 1$, $(x, y) \in \mathbf{C}^2$. The author proves that if $f(x, y)\colon S \to \mathbf{C}$ is a holomorphic function on S with polynomial growth, then $f(x, y)$ is the restriction of a polynomial on $\mathbf{C}^2$. As a result he deduces that if $f\colon S \to \mathbf{P}^1(\mathbf{C})$ is a meromorphic function on S with finite fibres, then f is constant. In particular, if $f\colon S \to \mathbf{C}$ is a bounded holomorphic function then f is constant. ($\mathbf{P}^1(\mathbf{C})$ is the complex projective "plane".)

Reviewed by Adib A. Fadlalla

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§ 122. Citations ↪
From References: 0
From Reviews: 0
MR0522015 (80e:32002) Reviewed
Demailly, Jean-Pierre
Fonctions holomorphes bornées ou à croissance polynomiale sur la courbe $e^x + e^y = 1$. (French. English summary)
C. R. Acad. Sci. Paris Sér. A-B 288 (1979), no. 1, A39-A40.
32A10 (32H99)

Author's summary: "We prove a very precise extension theorem for holomorphic functions on the curve $e^x + e^y = 1$, and deduce from it that bounded holomorphic functions are constant, or more generally, that every holomorphic function with polynomial growth extends to a polynomial in $\mathbf{C}^2$. This result immediately applies to meromorphic functions, and can also be used to study some hypersurfaces of $\mathbf{C}^n$."

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§ 123. Citations ↪
From References: 9
From Reviews: 1
MR0508989 (81m:32036) Reviewed
Demailly, Jean-Pierre
Un exemple de fibré holomorphe non de Stein à fibre $\mathbf{C}^2$ ayant pour base le disque ou le plan. (French)
Invent. Math. 48 (1978), no. 3, 293-302.

32L05 (32E10)

The author provides yet another example of a holomorphic fiber space with Stein base, Stein fiber and non-Stein total space in the mainstream originated by H. Skoda's brilliant counterexample to Serre's problem [same journal 43 (1977), no. 2, 97-107; MR0508091 (58 #22657)]. Here the base is any nonempty connected open subset of $\mathbf{C}$ with $\mathbf{C}^2$ as fiber and with transition automorphisms of exponential type, while in Skoda's example the base is multiply connected and the transition automorphisms are locally constant with exponential growth. Moreover the holomorphic functions on such a bundle are constant on each fiber and its Dolbeault group $H^{0,1}$ is non-Hausdorff of infinite dimension.

Reviewed by Alessandro Silva

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§ 124. Citations ↪
From References: 4
From Reviews: 3
MR0522471 (80e:32008) Reviewed
Demailly, J.-P.
Différents exemples de fibrés holomorphes non de Stein. (French) Séminaire Pierre Lelong-Henri Skoda (Analyse), Année 1976/77, pp. 15-41,
Lecture Notes in Math., 694, Springer, Berlin, 1978.
32E10 (32L05)

L'auteur simplifie notablement l'exemple de H. Skoda [Invent. Math. 43 (1977), no. 2, 97-107; MR0508091 (58 #22657)] et construit un fibré de base $\mathbf{C}^*$ et de fibre $\mathbf{C}^2$ dont l'espace total n'est pas de Stein. L'automorphisme de transition est constant et polynomial (dans l'exemple de Skoda, les automorphismes de transition étaient à croissance exponentielle). L'exemple peut être décrit en une ligne comme quotient de $\mathbf{C}\times\mathbf{C}^2$ par le groupe cyclique engendré par α avec $\alpha(x,z_1,z_2)=(x+2i\pi,z_1^k-z_2,z_1)$, $k\geq 2$.

L'auteur montre plus précisément que ce fibré X est de Stein au-dessus de la couronne $\rho_1<|x|<\rho_2$ si et seulement si $\mathrm{Log}(\rho_2/\rho_1)\leq 2\pi^2/\mathrm{Log}\,k$; dans le cas contraire, $H^1(X,\mathcal{O}_X)$ est grossier. Toujours en utilisant l'inégalité de Lelong sur la croissance des fonctions plurisousharmoniques, l'auteur construit aussi un fibré de base le disque, de fibre $\mathbf{C}^2$ et dont l'espace total n'est pas de Stein (en particulier, ce fibré n'est pas trivial!).

For the entire collection see MR0522469 (80a:32001).

Reviewed by A. Hirschowitz
