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FANO MANIFOLDS WITH NEF TANGENT BUNDLES ARE WEAKLY ALMOST KÄHLER-EINSTEIN*

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Dedicated to Professor Ngaiming Mok on the occasion of his sixtieth birthday

Abstract. The goal of this short note is to point out that every Fano manifold with a nef tangent bundle possesses an almost Kähler-Einstein metric, in a weak sense. The technique relies on a regularization theorem for closed positive $(1,1)$-currents. We also discuss related semistability questions and Chern inequalities.

Key words. Fano manifold, numerically effective vector bundle, rational homogeneous manifold, Campana-Peternell conjecture, Kähler-Einstein metric, closed positive current, regularization of currents, Schauder fixed point theorem.

Mathematics Subject Classification. 14J45, 14M17, 32C30, 32Q10, 32Q20.

1. Introduction. Recall that a holomorphic vector bundle E on a projective manifold X is said to be numerically effective (nef) if the line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ is nef on $Y = \mathbb{P}(E)$. Clearly, every homogeneous projective manifold X has its tangent bundle T_X generated by sections, so T_X is nef; morever, by [DPS94], every compact Kähler manifold with T_X nef admits a finite étale cover $\tilde{X}$ that is a locally trivial fibration over its Albanese torus, and the fibers are themselves Fano manifolds F with T_F nef. Hence, the classification problem is essentially reduced to the case when X is a Fano manifold with T_X nef. In this direction, Campana and Peternell [CP91] conjectured in 1991 that Fano manifolds with nef tangent bundles are rational homogeneous manifolds G/P, namely quotients of linear algebraic groups by parabolic subgroups. Although this can be checked up to dimension 3 by inspecting the classification of Fano 3-folds by Manin-Iskovskii and Mukai, very little is known in higher dimensions. It would be tempting to use the theory of VMRT's developed by J.M. Hwang and N. Mok ([Hwa01], [Mok08]), since the expected homogeneity property should be reflected in the geometry of rational curves. Even then, the difficulties to be solved remain formidable; see e.g. [Mok02] and also [MOSWW] for a recent account of the problem.

On the other hand, every rational homogeneous manifold $X = G/P$ carries a Kähler metric that is invariant by a compact real form $G^{\mathbb{R}}$ of G (cf. [AP86]), and the corresponding Ricci curvature form (i.e. the curvature of $-K_{G/P}$) is then a Kähler-Einstein metric. A stronger condition than nefness of T_X is the existence of a Kähler metric on X whose holomorphic bisectional curvature is nonnegative. N. Mok [Mok88] characterized those manifolds, they are exactly the products of hermitian symmetric spaces of compact type by flat compact complex tori $\mathbb{C}^q/\Lambda$ and projective spaces $\mathbb{P}^{n_j}$ with a Kähler metric of nonnegative holomorphic bisectional curvature. However, hermitian symmetric spaces of compact type are a smaller class than rational homogeneous manifolds (for instance, they do not include complete flag manifolds), thus one cannot expect these Kähler-Einstein metrics to have a nonnegative bisectional curvature in general.

It is nevertheless a natural related question to investigate whether every Fano manifold X with nef tangent bundle actually possesses a Kähler-Einstein metric. Our

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main observation is the following (much) weaker statement, whose proof is based on regularization techniques for closed positive currents [Dem92], [Dem99]. We denote here by $\mathrm{PC}^{1,1}(X)$ the cone of positive currents of bidegree $(1,1)$ on X and put $n = \dim_{\mathbb{C}} X$. Though it is a convex set of infinite dimension, the intersection $\mathrm{PC}^{1,1}(X) \cap c_1(X)$ is compact and metrizable for the weak topology (see § 2), and therefore it carries a unique uniform structure that can be defined by any compatible metric.

THEOREM 1.1. *Let X be a Fano manifold such that the tangent bundle T_X is nef. Then*

(i) *there exists a family of smoothing operators $(J_\varepsilon)_{\varepsilon\in]0,1]}$ that map every closed positive current $T \in \mathrm{PC}^{1,1}(X) \cap c_1(X)$ to a smooth closed positive definite $(1,1)$-form*

$$\alpha_\varepsilon = J_\varepsilon(T) \in c_1(X),$$

in such a way that $J_\varepsilon(T)$ converges weakly to T as $\varepsilon \to 0$ (uniformly for all T), and $T \mapsto J_\varepsilon(T)$ is continuous with respect to the weak topology of currents and the strong topology of C^∞ convergence on smooth $(1,1)$-forms;

(ii) *for every $\varepsilon \in]0,1]$, there exists a Kähler metric ω_ε on X such that $\mathrm{Ricci}(\omega_\varepsilon) = J_\varepsilon(\omega_\varepsilon)$, in other words ω_ε is "weakly almost Kähler-Einstein".*

The construction of J_ε that we have is unfortunately not very explicit, and we do not even know if J_ε can be constructed as a natural linear (say convolution-like) operator. The proof of (ii) is based on the use of the Schauder fixed point theorem. Let us denote by

$$0 < \rho_{1,\varepsilon}(x) \le \ldots \le \rho_{n,\varepsilon}(x)$$

the eigenvalues of $\mathrm{Ricci}(\omega_\varepsilon)$ with respect to ω_ε at each point $x \in X$, and $\zeta_{1,\varepsilon}, \ldots, \zeta_{n,\varepsilon} \in T_{X,x}$ a corresponding orthonormal family of eigenvectors with respect to ω_ε. We know that

$$\int_X \sum_{j=1}^{n} \rho_{j,\varepsilon}\, \omega_\varepsilon^n = n \int_X \mathrm{Ricci}(\omega_\varepsilon) \wedge \omega_\varepsilon^{n-1} = n\, c_1(X)^n, \tag{$*$}$$

in particular the left-hand side is bounded. Also, as $\mathrm{Ricci}(\omega_\varepsilon) - \omega_\varepsilon = J_\varepsilon(\omega_\varepsilon) - \omega_\varepsilon$ converges weakly to 0, we know that

$$\int_X \sum_{j=1}^{n} (\rho_{j,\varepsilon} - 1)\zeta_{1,\varepsilon}^* \wedge \overline{\zeta_{1,\varepsilon}^*} \wedge u$$

converges to 0 for every smooth $(n-1,n-1)$-form u on X, hence the eigenvalues $\rho_{j,\varepsilon}$ "converge weakly to 1" in the sense that $\sum_{j=1}^{n} (\rho_{j,\varepsilon} - 1)\zeta_{1,\varepsilon}^* \wedge \overline{\zeta_{1,\varepsilon}^*}$ converges weakly to 0 in the space of smooth $(1,1)$-forms. Therefore, it does not seem too unrealistic to expect that the family (ω_ε) is well behaved in the following sense.

DEFINITION 1.2. *We say the family (ω_ε) of weak almost Kähler-Einstein metrics is* well behaved *in L^p norm if*

$$\int_X (\rho_{n,\varepsilon} - \rho_{1,\varepsilon})^p\, \omega_\varepsilon^n \tag{$**^{p}$

converges to 0 as ε tends to 0.

Standard curvature inequalities then yield the following result.

THEOREM 1.3. *Assume that X is Fano with T_X nef, and possesses a family of weak almost Kähler-Einstein metrics that is well behaved in L^p norm. Then*

(i) *if $p \geq 1$, T_X is $c_1(X)$-semistable;*

(ii) *if $p \geq 2$, the Guggenheimer-Yau-Bogomolov-Miyaoka Chern class inequality*

$$\big[nc_1(X)^2 - (2n+2)c_2(X)\big] \cdot c_1(X)^{n-2} \leq 0$$

is satisfied.

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2. Proofs.

Existence of regularization operators in 1.1 (i). Let us fix once for all a Kähler metric $\beta \in c_1(X)$. It is well known that $K = \mathrm{PC}^{1,1}(X) \cap c_1(X)$ is compact for the weak topology of currents; this comes from the fact that currents $T \in K$ have a uniformly bounded mass

$$\int_X T \wedge \beta^{n-1} = c_1(X)^n.$$

We also know from [Dem92], [Dem99] that every such current T admits a family of regularizations $T_\varepsilon \in C^\infty(X) \cap c_1(X)$ converging weakly to T, such that $T_\varepsilon \geq -\varepsilon\beta$. Here, the fact that we can achieve an arbitrary small lower bound uses in an essential way the assumption that T_X is nef. After replacing T_ε by $T'_\varepsilon = (1-\varepsilon)T_\varepsilon + \varepsilon\beta \geq \varepsilon^2\beta$, we can assume that T_ε is a Kähler form. An important issue is that we want to produce a continuous operator $J_\varepsilon(T) = T_\varepsilon$ defined on the weakly compact set K. This is clearly the case when the regularization is produced by convolution, as is done [Dem94], but in that case one needs a more demanding condition on T_X, e.g. that T_X possesses a hermitian metric with nonnegative bisectional curvature (i.e. that T_X is Griffiths semipositive). However, the existence of a continuous global operator J_ε is an easy consequence of the convexity of K. In fact, the weak topology of K is induced by the L^2 topology on the space of potentials φ when writing $T = \beta + dd^c\varphi$ (and taking the quotient by constants) – it is also induced by the L^p topology for any $p \in [1, \infty[$, but L^2 has the advantage of being a Hilbert space topology. By compactness, for every $\delta \in]0, 1]$ we can then find finitely many currents $(T_j)_{1 \leq j \leq N(\delta)}$ such that the Hilbert balls $B(T_j, \delta)$ cover K. Let K_δ be the convex hull of the family $(T_j)_{1 \leq j \leq N(\delta)}$ and let $p_\delta : K \to K_\delta$ be induced by the (nonlinear) Hilbert projection $L^2/\mathbb{R} \to K_\delta$. Since K_δ is finite dimensional and is a finite union of simplices, we can take regularizations $T_{j,\varepsilon}$ of the vertices T_j and construct a piecewise linear operator $\tilde{J}_{\delta,\varepsilon} : K_\delta \to K \cap C^\infty(X)^+$ to Kähler forms, simply by taking linear combinations of the $T_{j,\varepsilon}$'s. Then $\tilde{J}_{\delta,\varepsilon} \circ p_\delta(T)$ is the continuous regularization operator we need. The uniform weak convergence to T is guaranteed if we take $J_\varepsilon := \tilde{J}_{\delta(\varepsilon),\varepsilon}$ with $\varepsilon \ll \delta(\varepsilon) \to 0$, e.g. with a step function $\varepsilon \mapsto \delta(\varepsilon)$ that converges slowly to 0 compared to ε. These operators have the drawback of being non explicit and a priori non linear.

QUESTION 2.1. *Is it possible to construct a* linear *regularization operator J_ε with the same properties as above, e.g. by means of a convolution process ?*

In fact, such a "linear" regularization operator J_ε is constructed in [Dem94] by putting $T = \beta + dd^c\varphi$ and $J_\varepsilon(T) = \beta + dd^c\varphi_\varepsilon$ where

$$\varphi_\varepsilon(z) = \int_{\zeta\in T_{X,z}} \varphi(\mathrm{exph}_z(\zeta))\,\chi(|\zeta|_h^2/\varepsilon^2)\,dV_h(\zeta)$$

for a suitable hermitian metrics h on T_X, taking exph to be the holomorphic part of the exponential map associated with the Chern connection ∇_h. However, the positivity of $J_\varepsilon(T)$ is in general not preserved, unless one assumes that (T_X, h) is (say) Griffiths semipositive. As the relation between nefness and Griffiths semipositivity is not yet elucidated, one would perhaps need to extend the above formula to the case of *Finsler metrics*. This is eventually possible by applying some of the techniques of [Dem99] to produce suitable dual Finsler metrics on T_X (notice that any assumption that $T_X \otimes L$ is ample for some $\mathbb{Q}$-line bundle L translates into the existence of a strictly plurisubharmonic Finsler metric on the total space of $(T_X \otimes L)^* \smallsetminus \{0\}$, so one needs to dualize).

Use of the Schauder fixed point theorem for 1.1 (ii). By Yau's theorem [Yau78], for every closed $(1,1)$-form $\rho \in c_1(X)$, there exists a unique Kähler metric $\gamma(\rho) \in c_1(X)$ such that $\mathrm{Ricci}(\gamma(\rho)) = \rho$. Moreover, by the regularity theory of nonlinear elliptic operators, the map $\rho \mapsto \gamma(\rho)$ is continuous in C^∞ topology. We consider the composition

$$\gamma\circ J_\varepsilon : K \mapsto K\cap C^\infty(X)^+ \to K\cap C^\infty(X)^+ \subset K, \qquad T\mapsto \rho = J_\varepsilon(T)\mapsto \gamma(\rho).$$

Since J_ε is continuous from the weak topology to the strong C^∞ topology, we infer that $\gamma\circ J_\varepsilon$ is continuous on K in the weak topology. Now, K is convex and weakly compact, therefore $\gamma\circ J_\varepsilon$ must have a fixed point $T = \omega_\varepsilon$ by the theorem of Schauder. By construction ω_ε must be a Kähler metric in $c_1(X)$, since $\gamma\circ J_\varepsilon$ maps K into the space of Kähler metrics contained in $c_1(X)$. This proves Theorem 1.1.

Semistability of T_X (1.3 (i)). Let $\mathcal{F} \subset \mathcal{O}(T_X)$ be a coherent subsheaf such that $\mathcal{O}(T_X)/\mathcal{F}$ is torsion free. We can view $\mathcal{F}$ as a holomorphic subbundle of T_X outside of an algebraic subset of codimension 2 in X. The Chern curvature tensor $\Theta_{T_X,\omega_\varepsilon} = \frac{i}{2\pi}\nabla^2$ satisfies the Hermite-Einstein condition

$$\Theta_{T_X,\omega_\varepsilon}\wedge\omega_\varepsilon^{n-1} = \frac{1}{n}\rho_\varepsilon\,\omega_\varepsilon^n,$$

where $\rho_\varepsilon \in C^\infty(X, \mathrm{End}(T_X))$ is the Ricci operator (with the same eigenvalues $\rho_{j,\varepsilon}$ as $\mathrm{Ricci}(\omega_\varepsilon)$). It is well known that the curvature of a subbundle is always bounded above by the restriction of the full curvature tensor, i.e. $\Theta_{\mathcal{F},\omega_\varepsilon} \le \Theta_{T_X,\omega_\varepsilon|\mathcal{F}}$ (say, in the sense of Griffiths positivity, viewing the curvature tensors as hermitian forms on $T_X\otimes\mathcal{F}$). By taking the trace with respect to ω_ε, we get

$$\Theta_{\mathcal{F},\omega_\varepsilon}\wedge\omega_\varepsilon^{n-1} \le \frac{1}{n}\rho_{\varepsilon|\mathcal{F}}\,\omega_\varepsilon^n.$$

By the minimax principle, if $r = \mathrm{rank}\,\mathcal{F}$, the eigenvalues of $\rho_{\varepsilon|\mathcal{F}}$ are bounded above by

$$\rho_{n-r+1,\varepsilon} \le \cdots \le \rho_{n,\varepsilon},$$

hence

$$\int_X c_1(\mathcal{F}) \wedge \omega_\varepsilon^{n-1} = \int_X \mathrm{tr}_{\mathcal{F}}\, \Theta_{\mathcal{F},\omega_\varepsilon} \wedge \omega_\varepsilon^{n-1} \le \frac{1}{n} \int_X \sum_{j=1}^{r} \rho_{n-r+j,\varepsilon}\, \omega_\varepsilon^n .$$

The left hand side is unchanged if we replace ω_ε by $\beta \in c_1(X)$, and our assumption $(**)^p$ for $p = 1$ (cf. Definition 1.2) implies

$$\int_X c_1(\mathcal{F}) \wedge \beta^{n-1} \le \lim_{\varepsilon\to 0} \frac{1}{n} \int_X \sum_{j=1}^{r} \rho_{n-r+j,\varepsilon}\, \omega_\varepsilon^n = \lim_{\varepsilon\to 0} \frac{r}{n^2} \int_X \sum_{j=1}^{n} \rho_{j,\varepsilon}\, \omega_\varepsilon^n = \frac{r}{n} \int_X c_1(X) \wedge \beta^{n-1}$$

by $(*)$. This means that T_X is $c_1(X)$-semistable.

Chern class inequality (1.3 (ii)). Let us write

$$\Theta_{T_X,\omega_\varepsilon} = (\theta_{\alpha\beta})_{1\le\alpha,\beta\le n} = \Big(\sum_{j,k} \theta_{\alpha\beta jk} dz_j \wedge d\overline{z}_k \Big)_{1\le\alpha,\beta\le n}$$

as an $n \times n$ matrix of $(1,1)$-forms with respect to an orthonormal frame of T_X that diagonalizes the Ricci operator ρ_ε (with respect to the metric ω_ε). A standard calculation yields

$$\begin{aligned} &2\Big[n\, c_1(X)_h^2 - (2n+2)\, c_2(X)\Big] \wedge \frac{\omega_\varepsilon^{n-2}}{(n-2)!} \\ &= \Bigg(- \sum_{\alpha\ne\beta, j\ne k} |\theta_{\alpha\alpha jk} - \theta_{\beta\beta jk}|^2 - (n+1) \sum_{\alpha,\beta,j,k,\ \text{pairwise} \ne} |\theta_{\alpha\beta jk}|^2 \\ &\quad - (4n+8) \sum_{\alpha\ne j<k\ne\alpha} |\theta_{\alpha\alpha jk}|^2 - 4 \sum_{\alpha\ne j} |\theta_{\alpha\alpha\alpha j}|^2 \\ &\quad - \sum_{\alpha\ne\beta\ne j\ne\alpha} |\theta_{\alpha\alpha jj} - \theta_{\beta\beta jj}|^2 - \sum_{\alpha\ne\beta} |\theta_{\alpha\alpha\alpha\alpha} - 2\theta_{\alpha\alpha\beta\beta}|^2 \\ &\quad + n \sum_\alpha \rho_{\alpha,\varepsilon}^2 - \Big(\sum_\alpha \rho_{\alpha,\varepsilon}\Big)^2 \ \Bigg) \omega_\varepsilon^n \end{aligned}$$

where all terms in the summation are nonpositive except the ones involving the Ricci eigenvalues $\rho_{\alpha,\varepsilon}$. However, the assumption $(**)^p$ for $p \ge 2$ implies that the integral of the difference appearing in the last line converges to 0 as $\varepsilon \to 0$. Theorem 1.3 is proved.

Our results lead to the following interesting question.

Question 2.2. *Assuming that a "sufficiently good" family of regularization operators J_ε is used, can one infer that the resulting family (ω_ε) of fixed points such that* $\mathrm{Ricci}(\omega_\varepsilon) = J_\varepsilon(\omega_\varepsilon)$ *(or some subsequence) is well behaved in L^1, resp. L^2 norm?*

Remark 2.3. A strategy to attack the Campana-Peternell conjecture could be as follows. The first step would be to prove that $H^0(X, T_X) \ne 0$ (if $\dim X > 0$). Assume therefore $H^0(X, T_X) = 0$ and try to reach a contradiction. It is well known in that case that there are fewer obstructions to the existence of Kähler-Einstein metrics, for instance one has the Bando-Mabuchi uniqueness theorem [BM85] and (obviously) the vanishing of the classical Futaki invariant [Fut83]; hence we could expect our family of weakly almost Kähler-Einstein metrics to converge to a genuine Kähler-Einstein metric. But then the resulting known Chern class inequalities (the ones of

Theorem 1.3 (ii) combined with the Fulton-Lazarsfeld inequalities [FL83], [DPS94]) might help to contradict $H^0(X, T_X) = 0$, e.g. by the Riemann-Roch formula or by ad hoc nonvanishing theorems such as the generalized hard Lefschetz theorem [DPS01]. Now, the existence of holomorphic vector fields on X implies that X has a non trivial group of automorphisms $H = \mathrm{Aut}(X)$, and one could then try to apply induction on dimension on a suitable desingularization Y of $X /\!/ H$, if Y is not reduced to a point and one can achieve Y to have T_Y nef.

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