A tribute to Pierre Lelong

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It was not long before the events of May 1968 that I first met Pierre Lelong, at the seminar he organized jointly with François Norguet at the Institute Henri Poincaré. I was then an advanced student preparing my thesis under the supervision of André Martineau who had advised me to attend it so that I could observe "the state of the art", in the field of holomorphic functions of several complex variables. At that time, there were no individual computers nor web. Papers were typewritten with carbon copies. Phone and television were still quite luxuries. So seminars took a more important place to interchange ideas and results between mathematicians than nowadays. They were fewer and the audience of P. Lelong and F. Norguet seminar was quite impressive for the beginner I was, because of the quality and the number of its members. For instance, Henri Cartan regularly attended it. As early as october1968, I began to study P. Lelong's works, especially those on the zeros of entire functions in $\mathbb{C}^{\mathbf{n}}$. I discovered in them, on one hand, the detailed study of the properties of plurisubharmonic functions family ([Le 45])(that is, the restriction of the function to every complex line is subharmonic) which included altogether convex functions and $\log|f|$ functions where f is an holomorphic function. On the other hand, as early as 1953, P. Lelong ([Le 57]) had proved the current of integration on a complex analytic set was well defined in spite of the singularities of the analytic set. It was closed and positive ([Le 69]) (in the sense that all the measures naturally associated to that current by the complex structure were positive) and was used as a model for the new concept of closed positive current he introduced. For every closed positive current, P. Lelong defined the density of the current in a point which coincided with the usual notion of multiplicity in the case of the current of integration on an analytic set. In the same way as that of Laurent Schwartz distributions or G. de Rahm currents, it gave the possibility to deal with the analytic sets of Complex Geometry with analytic methods totally compatible with the algebraic method of sheaf theory and of local algebra and especially well adapted to the study of metric and quantitative properties of analytic sets. The most important part of my works is immediately connected with the concepts P. Lelong has introduced. In my inaugural lecture at Colloque Européen en l'honneur de Pierre Lelong, in September 1997 ([Sk 00]), I have already widely explained the notable impact of these concepts on the developments of the Complex Analysis in several variables and on the Algebraic Geometry on the field of complex numbers from 1940 to 1997, in relation with Lars Hörmander ([Hör 65]), ([Hör 66]) and Enrico Bombiéri ([Bo70]) L^2 estimates for the $\bar{\partial}$ operator in 1965, then with Ohsawa-Takegoshi ([Ohs 88]) extension theorem and that of coherence from A.M Nadel ([Nad 89]). All these results are themselves based on the notion of plurisubharmonic function as well as on a long mathematical traditions of Partial Differential Equations and of Differential Geometry. To sum up, mathematicians could from that time use an extremely effective machinery : it was possible to associate to every closed positive current by the mean of a convenient integral kernel, a plurisubharmonic function then an analytic set. In this collective tribute, other mathematicians analyze this aspect of things. I would like to bring to light others which are perhaps less known. Pierre Lelong had built as early as 1956 ([Le 64]), the equivalent in $\mathbb{C}^{\mathbf{n}}$ of the canonical Weierstrass product (that is an holomorphic function F of minimal growth vanishing on a given zeros set) as a plurisubharmonic potential $\log|F|$ solving in $\mathbb{C}^{\mathbf{n}}$, in a very modern and inventive way, in the spirit of Hodge theory, the so called today Lelong-Poincaré equation : $\frac{i}{\pi}\partial\bar{\partial}\log|F| = [X]$ where $[X]$ is the current of integration on the hypersurface X. In my thesis in 1972 ([Sk 72]), I took benefit from all these methods dealing with potential theory and L^2 estimates to extend this P. Lelong's work about hypersurfaces of $\mathbb{C}^{\mathbf{n}}$ to any analytic sets. Then, in 1975, going back again to the solving of the same equation, Gennadi Henkin and myself ([Ru 80]), were independentely successful in characterizing the zeros of Nevanlinna class functions in strictly pseudoconvex domains of $\mathbb{C}^{\mathbf{n}}$ by the Blaschke condition. By the same way, we more generally have solved the equation $\frac{i}{\pi}\partial\bar{\partial}V = T$ where T is a closed positive current. P. Lelong's views have taken a prominent part in making possible for us to reveal the asymmetric behaviour of the current T with respect to the tangential or normal complex directions at the boundary of the domain, that was a compulsery condition to solve the equation with the expected estimates. The asymmetric behavior of a closed positive current

remained as an essential argument in the numerous researches today on hard analysis about zeros of functions in Hardy classes on pseudoconvex domains. In fifties, sixties, P. Lelong's ideas seemed to be very useful only to control the asymptotic behaviour of holomorphic objects. P. Lelong had observed the Hadamard's inequality of convexity generalized to holomorphic functions of several variables had much deeper consequences than with one variable. It implies for instance that the asymptotic behavior at infinity of an entire function is remarkably stable along almost all complex lines. That has been the decisive argument which was used to build ([Sk 77]) examples of holomorphic fiber spaces, with Stein ($\mathbb{C}^2$) fiber, with Stein basis (an open subset of $\mathbb{C}$) and nevertheless not Stein and which gave a negative answer to a question of Jean-Pierre Serre which has aroused the curiosity of geometers for many years and also showed that P. Lelong's methods could be more effective for strictly geometrical problems than the more traditional sheaf theoretic methods.
By chance or destiny, somewhat probably with the help of Jean-Louis Verdier, Directeur des Études at École Normale Supérieure and Michel Hervé, Directeur adjoint of École Normale Supérieure, one of the first who attended my lessons for advanced studies in 1976 was Jean-Pierre Demailly, a young student who also raised on P. Lelong and L. Hörmander's methods. He immediately took up the torch by extending and making more flexible the notion of Lelong number ([Dem 87]) and connecting it with other important problems of that time as Hodge conjecture ([Dem 82]) and numerical vanishing theorems. From that time forward, P. Lelong's ideas are far from seeing their end. More and more mathematicians took interest in studying his works which had also a decisive impact on other fields, as algebraic Geometry with, for instance the deep J.P. Demailly's results on Fujita conjecture ([Dem 93]) and Y.T.Siu's one ([Siu 98]) on the plurigenera invariance. Let us quote too the emergence of closed positive currents in the holomorphic dynamic systems with several variables, following Nessim Sibony's works ([Sib 99]). I am grateful to Pierre Lelong for having in an unpretentious way, laid the bases of all these mathematical developments which I think, are far from end.
I woud like now throw light on his seminar which took an important part in scientific life. Many French or foreign researchers have been invited and could benefit from his audience and the broadcasting of their talks which have been published in Seminar Acts, at Springer's, between 1957 and 1986, in Lecture Notes Series. Pierre Lelong, Pierre Dolbeault and myself have shared the managing of the Complex Analysis Seminar which went on as

far as today. Gennadi Henkin and Jean-Marie Trépreau have taken part in seminar managing. From October 2006, with the arrival of Olivier Bicquard and Tien Cong Dinh, it has become Geometry and Complex Analysis Seminar and has quite turned to Differential Geometry, dealing however with an important part of Complex Analysis. From times to times, Henri Cartan and Laurent Schwartz have attended the seminar, and more regularly Paul Malliavin and Michel Hervé. In organizing it, we could talk not only about Mathematics, Research, University, but also about the part involving Administration and State in Research field.
I have immediately seen in P. Lelong's speech and individual characteristics the mark of humanistic tradition, reenforced by his classical studies in secondary school, which gives more importance to man than to ideology and technics. It was in that way P. Lelong has undertaken to serve the State. He has been training for public service for a long time, by attending, in the thirties, the Institut Politique de Paris, in addition to his mathematical activities. In the sixties, he became a scientific consultant of the President of French Republic, General de Gaulle and so he gave a contribution to the effort for the expansion of universities and planning for research. He has especially contributed to the expansion of computer science in France with the establishment of INRIA. I think his influence has been highly beneficial and we owe him much still today. He tried also, during the eighties to protect Institute Henri Poincaré. For, because of a legal vacuum, mathematicians could have been excluded from this institute. He did his best to obtain clear statutes for IHP, acting in mathematicians and theoretical physicians best interests. I wish to pay tribute to the memory of Pierre Lelong. One of the best mathematicians of the XXth century has left, whose influence will go on for a long time, but also an academic deeply wedded to humanistic and republican values, who has highly been at the service of the State.

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