

Proof of the Kobayashi conjecture on the hyperbolicity of very general hypersurfaces

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Abstract. The Green-Griffiths-Lang conjecture stipulates that for every projective variety X of general type over \mathbb{C} , there exists a proper algebraic subvariety of X containing all non constant entire curves $f : \mathbb{C} \rightarrow X$. Using the formalism of directed varieties, we prove here that this assertion holds true in case X satisfies a strong general type condition that is related to a certain jet-semistability property of the tangent bundle T_X . We then use this fact to confirm a long-standing conjecture of Kobayashi (1970), according to which a very general algebraic hypersurface of dimension n and degree at least $2n + 2$ in the complex projective space \mathbb{P}^{n+1} is hyperbolic.

0. Introduction

The goal of this paper, among other results, is to prove the long standing conjecture of Kobayashi [Kob70, Kob78], according to which a very general algebraic hypersurface of dimension n and degree $d \geq 2n + 2$ in complex projective space \mathbb{P}^{n+1} is Kobayashi hyperbolic. It is expected that the bound can be improved to $2n + 1$ for $n \geq 2$, and such a bound would be optimal by Zaidenberg [Zai87], but we cannot yet prove this. Siu [Siu02, Siu04, Siu12] has introduced a more direct and more computational approach that yields the same conclusion for $d \geq d_n$, with a very large bound d_n instead of $2n + 2$. However, thanks to famous results of Clemens [Cle86], Ein [Ein88, Ein91] and Voisin [Voi96, Voi98], it was known that the bound $2n + 2$ would be a consequence of the Green-Griffiths-Lang conjecture on entire curve loci, cf. [GG79] and [Lan86]. Our technique consists in studying a generalized form of the GGL conjecture, and proving a special case that is strong enough to imply the Kobayashi conjecture, using e.g. [Voi96]. For this purpose, as was already observed in [Dem97], it is useful to work in the category of directed projective varieties, and to take into account the singularities that may appear in the directed structures, at all steps of the proof.

Since the basic problems we deal with are birationally invariant, the varieties under consideration can always be replaced by nonsingular models. A directed projective manifold is a pair (X, V) where X is a projective manifold equipped with an analytic linear subspace $V \subset T_X$, i.e. a closed irreducible complex analytic subset V of the total space of T_X , such that each fiber $V_x = V \cap T_{X,x}$ is a complex vector space [If X is not irreducible, V should rather be assumed to be irreducible merely over each component of X , but we will hereafter assume that our varieties are irreducible]. A morphism $\Phi : (X, V) \rightarrow (Y, W)$ in the category

of directed manifolds is an analytic map $\Phi : X \rightarrow Y$ such that $\Phi_*V \subset W$. We refer to the case $V = T_X$ as being the *absolute case*, and to the case $V = T_{X/S} = \text{Ker } d\pi$ for a fibration $\pi : X \rightarrow S$, as being the *relative case*; V may also be taken to be the tangent space to the leaves of a singular analytic foliation on X , or maybe even a non integrable linear subspace of T_X .

We are especially interested in *entire curves* that are tangent to V , namely non constant holomorphic morphisms $f : (\mathbb{C}, T_{\mathbb{C}}) \rightarrow (X, V)$ of directed manifolds. In the absolute case, these are just arbitrary entire curves $f : \mathbb{C} \rightarrow X$. The Green-Griffiths-Lang conjecture, in its strong form, stipulates

0.1. GGL conjecture. *Let X be a projective variety of general type. Then there exists a proper algebraic variety $Y \subsetneq X$ such that every entire curve $f : \mathbb{C} \rightarrow X$ satisfies $f(\mathbb{C}) \subset Y$.*

[The weaker form would state that entire curves are algebraically degenerate, so that $f(\mathbb{C}) \subset Y_f \subsetneq X$ where Y_f might depend on f]. The smallest admissible algebraic set $Y \subset X$ is by definition the *entire curve locus* of X , defined as the Zariski closure

$$(0.2) \quad \text{ECL}(X) = \overline{\bigcup_f f(\mathbb{C})}^{\text{Zar}}.$$

If $X \subset \mathbb{P}_{\mathbb{C}}^N$ is defined over a number field \mathbb{K}_0 (i.e. by polynomial equations with coefficients in \mathbb{K}_0) and $Y = \text{ECL}(X)$, it is expected that for every number field $\mathbb{K} \supset \mathbb{K}_0$ the set of \mathbb{K} -points in $X(\mathbb{K}) \setminus Y$ is finite, and that this property characterizes $\text{ECL}(X)$ as the smallest algebraic subset Y of X that has the above property for all \mathbb{K} ([Lan86]). This conjectural arithmetical statement would be a vast generalization of the Mordell-Faltings theorem, and is one of the strong motivations to study the geometric GGL conjecture as a first step.

0.3. Problem (generalized GGL conjecture). *Let (X, V) be a projective directed manifold. Find geometric conditions on V ensuring that all entire curves $f : (\mathbb{C}, T_{\mathbb{C}}) \rightarrow (X, V)$ are contained in a proper algebraic subvariety $Y \subsetneq X$. Does this hold when (X, V) is of general type, in the sense that the canonical sheaf K_V is big?*

As above, we define the entire curve locus set of a pair (X, V) to be the smallest admissible algebraic set $Y \subset X$ in the above problem, i.e.

$$(0.4) \quad \text{ECL}(X, V) = \overline{\bigcup_{f:(\mathbb{C}, T_{\mathbb{C}}) \rightarrow (X, V)} f(\mathbb{C})}^{\text{Zar}}.$$

We say that (X, V) is *Brody hyperbolic* if $\text{ECL}(X, V) = \emptyset$; as is well-known, this is equivalent to Kobayashi hyperbolicity whenever X is compact.

In case V has no singularities, the *canonical sheaf* K_V is defined to be $(\det \mathcal{O}(V))^*$ where $\mathcal{O}(V)$ is the sheaf of holomorphic sections of V , but in general this naive definition would not work. Take for instance a generic pencil of elliptic curves $\lambda P(z) + \mu Q(z) = 0$ of degree 3 in $\mathbb{P}_{\mathbb{C}}^2$, and the linear space V consisting of the tangents to the fibers of the rational map $\mathbb{P}_{\mathbb{C}}^2 \dashrightarrow \mathbb{P}_{\mathbb{C}}^1$ defined by $z \mapsto Q(z)/P(z)$. Then V is given by

$$0 \longrightarrow \mathcal{O}(V) \longrightarrow \mathcal{O}(T_{\mathbb{P}^2}) \xrightarrow{PdQ - QdP} \mathcal{O}_{\mathbb{P}^2}(6) \otimes \mathcal{I}_S \longrightarrow 0$$

where $S = \text{Sing}(V)$ consists of the 9 points $\{P(z) = 0\} \cap \{Q(z) = 0\}$, and \mathcal{I}_S is the corresponding ideal sheaf of S . Since $\det \mathcal{O}(T_{\mathbb{P}^2}) = \mathcal{O}(3)$, we see that $(\det(\mathcal{O}(V)))^* = \mathcal{O}(3)$ is ample,

thus Problem 0.3 would not have a positive answer (all leaves are elliptic or singular rational curves and thus covered by entire curves). An even more “degenerate” example is obtained with a generic pencil of conics, in which case $(\det(\mathcal{O}(V)))^* = \mathcal{O}(1)$ and $\#S = 4$.

If we want to get a positive answer to Problem 0.3, it is therefore indispensable to give a definition of K_V that incorporates in a suitable way the singularities of V ; this will be done in Def. 1.1 (see also Prop. 1.2). The goal is then to give a positive answer to Problem 0.3 under some possibly more restrictive conditions for the pair (X, V) . These conditions will be expressed in terms of the tower of Semple jet bundles

$$(0.5) \quad (X_k, V_k) \rightarrow (X_{k-1}, V_{k-1}) \rightarrow \dots \rightarrow (X_1, V_1) \rightarrow (X_0, V_0) := (X, V)$$

which we define more precisely in Section 1, following [Dem95]. It is constructed inductively by setting $X_k = P(V_{k-1})$ (projective bundle of *lines* of V_{k-1}), and all V_k have the same rank $r = \text{rank } V$, so that $\dim X_k = n + k(r-1)$ where $n = \dim X$. If $\mathcal{O}_{X_k}(1)$ is the tautological line bundle over X_k associated with the projective structure and $\pi_{k,\ell} : X_k \rightarrow X_\ell$ is the natural projection from X_k to X_ℓ , $0 \leq \ell \leq k$, we define the k -stage Green-Griffiths locus of (X, V) to be

$$(0.6) \quad \text{GG}_k(X, V) = \overline{(X_k \setminus \Delta_k) \cap \bigcap_{m \in \mathbb{N}} \left(\text{base locus of } \mathcal{O}_{X_k}(m) \otimes \pi_{k,0}^* A^{-1} \right)}$$

where A is any ample line bundle on X and $\Delta_k = \bigcup_{2 \leq \ell \leq k} \pi_{k,\ell}^{-1}(D_\ell)$ is the union of “vertical divisors” (see section 1; the vertical divisors play no role and have to be removed in this context). Clearly, $\text{GG}_k(X, V)$ does not depend on the choice of A . The basic vanishing theorem for entire curves (cf. [GG79], [SY96] and [Dem95]) asserts that for every entire curve $f : (\mathbb{C}, T_{\mathbb{C}}) \rightarrow (X, V)$, then its k -jet $f_{[k]} : (\mathbb{C}, T_{\mathbb{C}}) \rightarrow (X_k, V_k)$ satisfies

$$(0.7) \quad f_{[k]}(\mathbb{C}) \subset \text{GG}_k(X, V), \quad \text{hence} \quad f(\mathbb{C}) \subset \pi_{k,0}(\text{GG}_k(X, V)).$$

It is therefore natural to define the global Green-Griffiths locus of (X, V) to be

$$(0.8) \quad \text{GG}(X, V) = \bigcap_{k \in \mathbb{N}} \pi_{k,0}(\text{GG}_k(X, V)).$$

By (0.7) we infer that

$$(0.9) \quad \text{ECL}(X, V) \subset \text{GG}(X, V).$$

The main result of [Dem11] (Theorem 2.37 and Cor. 3.4) implies the following useful information:

0.10. Theorem. *Assume that (X, V) is of “general type”, i.e. that the canonical sheaf K_V is big on X . Then there exists an integer k_0 such that $\text{GG}_k(X, V)$ is a proper algebraic subset of X_k for $k \geq k_0$ [though $\pi_{k,0}(\text{GG}_k(X, V))$ might still be equal to X for all k].*

In fact, if F is an invertible sheaf on X such that $K_V \otimes F$ is big, the probabilistic estimates of [Dem11, Cor. 2.38 and Cor. 3.4] produce sections of

$$(0.11) \quad \mathcal{O}_{X_k}(m) \otimes \pi_{k,0}^* \mathcal{O} \left(\frac{m}{kr} \left(1 + \frac{1}{2} + \dots + \frac{1}{k} \right) F \right)$$

for $m \gg k \gg 1$. The (long and involved) proof uses a curvature computation and singular holomorphic Morse inequalities to show that the line bundles involved in (0.11) are big on

X_k for $k \gg 1$. One applies this to $F = A^{-1}$ with A ample on X to produce sections and conclude that $\text{GG}_k(X, V) \subsetneq X_k$.

Thanks to (0.9), the GGL conjecture is satisfied whenever $\text{GG}(X, V) \subsetneq X$. By [DMR10], this happens for instance in the absolute case when X is a generic hypersurface of degree $d \geq 2^{n^5}$ in \mathbb{P}^{n+1} (see also [Pau08], e.g. for better bounds in low dimensions). However, as already mentioned in [Lan86], very simple examples show that one can have $\text{GG}(X, V) = X$ even when (X, V) is of general type, and this already occurs in the absolute case as soon as $\dim X \geq 2$. A typical example is a product of directed manifolds

$$(0.12) \quad (X, V) = (X', V') \times (X'', V''), \quad V = \text{pr}'^* V' \oplus \text{pr}''^* V''.$$

The absolute case $V = T_X$, $V' = T_{X'}$, $V'' = T_{X''}$ on a product of curves is the simplest instance. It is then easy to check that $\text{GG}(X, V) = X$, cf. (3.2). Diverio and Rousseau [DR13] have given many more such examples, including the case of indecomposable varieties (X, T_X) , e.g. Hilbert modular surfaces, or more generally compact quotients of bounded symmetric domains of rank ≥ 2 . The problem here is the failure of some sort of stability condition that is introduced in Section 3. This leads to a somewhat technical concept of more manageable directed pairs (X, V) that we call *strongly of general type*, see Def. 3.1. Our main result can be stated

0.13. Theorem (partial solution to the generalized GGL conjecture). *Let (X, V) be a directed pair that is strongly of general type. Then the Green-Griffiths-Lang conjecture holds true for (X, V) , namely $\text{ECL}(X, V)$ is a proper algebraic subvariety of X .*

The proof proceeds through a complicated induction on $n = \dim X$ and $k = \text{rank } V$, which is the main reason why we have to introduce directed varieties, even in the absolute case. An interesting feature of this result is that the conclusion on $\text{ECL}(X, V)$ is reached without having to know anything about the Green-Griffiths locus $\text{GG}(X, V)$, even a posteriori. Nevertheless, this is not yet enough to confirm the GGL conjecture. Our hope is that pairs (X, V) that are of general type without being strongly of general type – and thus exhibit some sort of “jet-instability” – can be investigated by different methods, e.g. by the diophantine approximation techniques of McQuillan [McQ98]. However, Theorem 0.13 is strong enough to imply the Kobayashi conjecture on generic hyperbolicity, thanks to the following concept of algebraic jet-hyperbolicity.

0.14. Definition. *A directed variety (X, V) will be said to be algebraically jet-hyperbolic if the induced directed variety structure (Z, W) on every irreducible algebraic variety Z of X such that $\text{rank } W \geq 1$ has a desingularization that is strongly of general type [see section 2 for the definition of induced directed structures and further details]. We also say that a projective manifold X is algebraically jet-hyperbolic if (X, T_X) is.*

In this context, Theorem 0.13 yields the following connection between algebraic jet-hyperbolicity and the analytic concept of Kobayashi hyperbolicity.

0.15. Theorem. *Let (X, V) be a directed variety structure on a projective manifold X . Assume that (X, V) is algebraically jet-hyperbolic. Then (X, V) is Kobayashi hyperbolic.*

This strong link appears to be very useful to deal with generic hyperbolicity, i.e. the hyperbolicity of very general fibers in a deformation $\pi : \mathcal{X} \rightarrow S$ (by “very general fiber”, we mean here a fiber $X_t = \pi^{-1}(t)$ where t is taken in a complement $S \setminus \bigcup S_\nu$ of a countable union of algebraic subsets $S_\nu \subsetneq S$).

0.16. Theorem. *Let $\pi : \mathcal{X} \rightarrow S$ be a deformation of complex projective nonsingular varieties $X_t = \pi^{-1}(t)$ over a smooth irreducible quasi-projective base S . Let $n = \dim X_t$ be the relative dimension and let $N = \dim S$. Assume that for all $q = N + 1, \dots, N + n$, the exterior power $\Lambda^q T_{\mathcal{X}}^*$ is a relatively ample vector bundle over S . Then the very general fiber X_t is algebraically jet-hyperbolic, and thus Kobayashi hyperbolic.*

In the special case of the universal family of complete intersections of codimension s and type (d_1, \dots, d_s) in complex projective \mathbb{P}^{n+s} , a combination of Theorem 0.16 with the results of Voisin [Voi96] implies

0.17. Corollary (confirmation of the Kobayashi conjecture). *If $\sum d_j \geq 2n + s + 1$, the very general complete intersection of type (d_1, \dots, d_s) in complex projective space \mathbb{P}^{n+s} is Kobayashi hyperbolic.*

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1. Simple jet bundles and associated canonical sheaves

Let (X, V) be a directed projective manifold and $r = \text{rank } V$, that is, the dimension of generic fibers. Then V is actually a holomorphic subbundle of T_X on the complement $X \setminus V_{\text{sing}}$ of a certain minimal analytic set $V_{\text{sing}} \subsetneq X$ of codimension ≥ 2 , called hereafter the singular set of V . If $\mu : \widehat{X} \rightarrow X$ is a proper modification (a composition of blow-ups with smooth centers, say), we get a directed manifold $(\widehat{X}, \widehat{V})$ by taking \widehat{V} to be the closure of $\mu_*^{-1}(V')$, where $V' = V|_{X'}$ is the restriction of V over a Zariski open set $X' \subset X \setminus V_{\text{sing}}$ such that $\mu : \mu^{-1}(X') \rightarrow X'$ is a biholomorphism. We will be interested in taking modifications realized by iterated blow-ups of certain nonsingular subvarieties of the singular set V_{sing} , so as to eventually “improve” the singularities of V ; outside of V_{sing} the effect of blowing-up will be irrelevant, as one can see easily. Following [Dem11], the canonical sheaf K_V is defined as follows.

1.1. Definition. *For any directed pair (X, V) with X non singular, we define K_V to be the rank 1 analytic sheaf such that*

$$K_V(U) = \text{sheaf of locally bounded sections of } \mathcal{O}_X(\Lambda^r V'^*)(U \cap X')$$

where $r = \text{rank}(V)$, $X' = X \setminus V_{\text{sing}}$, $V' = V|_{X'}$, and “bounded” means bounded with respect to a smooth hermitian metric h on T_X .

For $r = 0$, one can set $K_V = \mathcal{O}_X$, but this case is essentially trivial: clearly $\text{ECL}(X, V) = \emptyset$. This definition may look like an analytic one, but it can easily be turned into an equivalent algebraic definition:

1.2. Proposition. *Consider the natural morphism $\mathcal{O}(\Lambda^r T_X^*) \rightarrow \mathcal{O}(\Lambda^r V^*)$ where $r = \text{rank } V$ [$\mathcal{O}(\Lambda^r V^*)$ being defined here as the quotient of $\mathcal{O}(\Lambda^r T_X^*)$ by r -forms that have zero restrictions to $\mathcal{O}(\Lambda^r V^*)$ on $X \setminus V_{\text{sing}}$]. The bidual $\mathcal{L}_V = \mathcal{O}_X(\Lambda^r V^*)^{**}$ is an invertible sheaf, and our natural morphism can be written*

$$(1.2') \quad \mathcal{O}(\Lambda^r T_X^*) \rightarrow \mathcal{O}(\Lambda^r V^*) = \mathcal{L}_V \otimes \mathcal{J}_V \subset \mathcal{L}_V$$

where \mathcal{J}_V is a certain ideal sheaf of \mathcal{O}_X whose zero set is contained in V_{sing} and the arrow on the left is surjective by definition. Then

$$(1.2'') \quad K_V = \mathcal{L}_V \otimes \bar{\mathcal{J}}_V$$

where $\bar{\mathcal{J}}_V$ is the integral closure of \mathcal{J}_V in \mathcal{O}_X . In particular, K_V is always a coherent sheaf.

Proof. Let (u_k) be a set of generators of $\mathcal{O}(\Lambda^r V^*)$ obtained (say) as the images of a basis $(dz_I)_{|I|=r}$ of $\Lambda^r T_X^*$ in some local coordinates near a point $x \in X$. Write $u_k = g_k \ell$ where ℓ is a local generator of \mathcal{L}_V at x . Then $\mathcal{J}_V = (g_k)$ by definition. The boundedness condition expressed in Def. 1.1 means that we take sections of the form $f\ell$ where f is a holomorphic function on $U \cap X'$ (and U a neighborhood of x), such that

$$(1.3) \quad |f| \leq C \sum |g_k|$$

for some constant $C > 0$. But then f extends holomorphically to U into a function that lies in the integral closure $\bar{\mathcal{J}}_V$, and the latter is actually characterized analytically by condition (1.3). This proves Prop. 1.2. \square

By blowing-up \mathcal{J}_V and taking a desingularization \widehat{X} , one can always find a *log-resolution* of \mathcal{J}_V (or K_V), i.e. a modification $\mu : \widehat{X} \rightarrow X$ such that $\mu^* \mathcal{J}_V \subset \mathcal{O}_{\widehat{X}}$ is an invertible ideal sheaf (hence integrally closed); it follows that $\mu^* \bar{\mathcal{J}}_V = \mu^* \mathcal{J}_V$ and $\mu^* K_V = \mu^* \mathcal{L}_V \otimes \mu^* \mathcal{J}_V$ are invertible sheaves on \widehat{X} . Notice that for any modification $\mu' : (X', V') \rightarrow (X, V)$, there is always a well defined natural morphism

$$(1.4) \quad \mu'^* K_V \rightarrow K_{V'}$$

(though it need not be an isomorphism, and $K_{V'}$ is possibly non invertible even when μ' is taken to be a log-resolution of K_V). Indeed $(\mu')_* = d\mu' : V' \rightarrow \mu^* V$ is continuous with respect to ambient hermitian metrics on X and X' , and going to the duals reverses the arrows while preserving boundedness with respect to the metrics. If $\mu'' : X'' \rightarrow X'$ provides a simultaneous log-resolution of $K_{V'}$ and $\mu'^* K_V$, we get a non trivial morphism of invertible sheaves

$$(1.5) \quad (\mu' \circ \mu'')^* K_V = \mu''^* \mu'^* K_V \longrightarrow \mu''^* K_{V'},$$

hence the bigness of $\mu'^* K_V$ with imply that of $\mu''^* K_{V'}$.

1.6. Definition. We say that the rank 1 sheaf K_V is “big” if the invertible sheaf $\mu^* K_V$ is big in the usual sense for any log resolution $\mu : \widehat{X} \rightarrow X$ of K_V . Finally, we say that (X, V) is of general type if there exists a modification $\mu' : (X', V') \rightarrow (X, V)$ such that $K_{V'}$ is big; any higher blow-up $\mu'' : (X'', V'') \rightarrow (X', V')$ then also yields a big canonical sheaf by (1.4).

Clearly, “general type” is a birationally (or bimeromorphically) invariant concept, by the very definition. When $\dim X = n$ and $V \subset T_X$ is a subbundle of rank r , one constructs a tower of “Simple k -jet bundles” $\pi_{k,k-1} : (X_k, V_k) \rightarrow (X_{k-1}, V_{k-1})$ that are \mathbb{P}^{r-1} -bundles, with $\dim X_k = n + k(r-1)$ and $\text{rank}(V_k) = r$. For this, we take $(X_0, V_0) = (X, V)$, and for every $k \geq 1$, we set inductively $X_k := P(V_{k-1})$ and

$$V_k := (\pi_{k,k-1})_*^{-1} \mathcal{O}_{X_k}(-1) \subset T_{X_k},$$

where $\mathcal{O}_{X_k}(1)$ is the tautological line bundle on X_k , $\pi_{k,k-1} : X_k = P(V_{k-1}) \rightarrow X_{k-1}$ the natural projection and $(\pi_{k,k-1})_* = d\pi_{k,k-1} : T_{X_k} \rightarrow \pi_{k,k-1}^* T_{X_{k-1}}$ its differential (cf. [Dem95]). In other terms, we have exact sequences

$$(1.7) \quad 0 \longrightarrow T_{X_k/X_{k-1}} \longrightarrow V_k \xrightarrow{(\pi_{k,k-1})_*} \mathcal{O}_{X_k}(-1) \longrightarrow 0,$$

$$(1.8) \quad 0 \longrightarrow \mathcal{O}_{X_k} \longrightarrow (\pi_{k,k-1})^* V_{k-1} \otimes \mathcal{O}_{X_k}(1) \longrightarrow T_{X_k/X_{k-1}} \longrightarrow 0,$$

where the last line is the Euler exact sequence associated with the relative tangent bundle of $P(V_{k-1}) \rightarrow X_{k-1}$. Notice that we by definition of the tautological line bundle we have

$$\mathcal{O}_{X_k}(-1) \subset \pi_{k,k-1}^* V_{k-1} \subset \pi_{k,k-1}^* T_{X_{k-1}},$$

and also $\text{rank}(V_k) = r$. Let us recall also that for $k \geq 2$, there are ‘‘vertical divisors’’ $D_k = P(T_{X_{k-1}/X_{k-2}}) \subset P(V_{k-1}) = X_k$, and that D_k is the zero divisor of the section of $\mathcal{O}_{X_k}(1) \otimes \pi_{k,k-1}^* \mathcal{O}_{X_{k-1}}(-1)$ induced by the second arrow of the first exact sequence (1.7), when k is replaced by $k-1$. This yields in particular

$$(1.9) \quad \mathcal{O}_{X_k}(1) = \pi_{k,k-1}^* \mathcal{O}_{X_{k-1}}(1) \otimes \mathcal{O}(D_k).$$

By composing the projections we get for all pairs of indices $0 \leq j \leq k$ natural morphisms

$$\pi_{k,j} : X_k \rightarrow X_j, \quad (\pi_{k,j})_* = (d\pi_{k,j})|_{V_k} : V_k \rightarrow (\pi_{k,j})^* V_j,$$

and for every k -tuple $\mathbf{a} = (a_1, \dots, a_k) \in \mathbb{Z}^k$ we define

$$\mathcal{O}_{X_k}(\mathbf{a}) = \bigotimes_{1 \leq j \leq k} \pi_{k,j}^* \mathcal{O}_{X_j}(a_j), \quad \pi_{k,j} : X_k \rightarrow X_j.$$

We extend this definition to all weights $\mathbf{a} \in \mathbb{Q}^k$ to get a \mathbb{Q} -line bundle in $\text{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. Now, Formula (1.9) yields

$$(1.10) \quad \mathcal{O}_{X_k}(\mathbf{a}) = \mathcal{O}_{X_k}(m) \otimes \mathcal{O}(-\mathbf{b} \cdot D) \quad \text{where } m = |\mathbf{a}| = \sum a_j, \mathbf{b} = (0, b_2, \dots, b_k)$$

and $b_j = a_1 + \dots + a_{j-1}$, $2 \leq j \leq k$.

When $V_{\text{sing}} \neq \emptyset$, one can always define X_k and V_k to be the respective closures of X'_k , V'_k associated with $X' = X \setminus V_{\text{sing}}$ and $V' = V|_{X'}$, where the closure is taken in the nonsingular ‘‘absolute’’ Semple tower $(\mathcal{X}_k, \mathcal{A}_k)$ obtained from $(\mathcal{X}_0, \mathcal{A}_0) = (X, T_X)$. We can then replace (X_k, V_k) by a suitable modification $(\widehat{X}_k, \widehat{V}_k)$ if we want to work with a non singular model \widehat{X}_k of X_k . The exceptional set of \widehat{X}_k over X_k can be chosen to lie above $V_{\text{sing}} \subset X$, and proceeding inductively with respect to k , we can also arrange the modifications in such a way that we get a tower structure $(\widehat{X}_{k+1}, \widehat{V}_{k+1}) \rightarrow (\widehat{X}_k, \widehat{V}_k)$; however, in general, it will not be possible to achieve that \widehat{V}_k is a subbundle of $T_{\widehat{X}_k}$.

It is not true that $K_{\widehat{V}_k}$ is big in case (X, V) is of general type (especially since the fibers of $X_k \rightarrow X$ are towers of \mathbb{P}^{r-1} bundles, and the canonical bundles of projective spaces are always negative!). However, a ‘‘twisted version’’ holds true.

1.11. Lemma. *If (X, V) is of general type, then all pairs (X_k, V_k) have a twisted canonical bundle $K_{V_k} \otimes \mathcal{O}_{X_k}(p)$ that is still big when one multiplies K_{V_k} by a suitable \mathbb{Q} -line bundle $\mathcal{O}_{X_k}(p)$, $p \in \mathbb{Q}_+$.*

Proof. First assume that V has no singularities. The exact sequences (1.7) and (1.8) provide

$$K_{V_k} := \det V_k^* = \det(T_{X_k/X_{k-1}}^*) \otimes \mathcal{O}_{X_k}(1) = \pi_{k,k-1}^* K_{V_{k-1}} \otimes \mathcal{O}_{X_k}(-(r-1))$$

where $r = \text{rank}(V)$. Inductively we get

$$(1.12) \quad K_{V_k} = \pi_{k,0}^* K_V \otimes \mathcal{O}_{X_k}(-(r-1)\mathbf{1}), \quad \mathbf{1} = (1, \dots, 1) \in \mathbb{N}^k.$$

We know by [Dem95] that $\mathcal{O}_{X_k}(\mathbf{c})$ is relatively ample over X when we take the special weight $\mathbf{c} = (2 \cdot 3^{k-2}, \dots, 2 \cdot 3^{k-j-1}, \dots, 6, 2, 1)$, hence

$$K_{V_k} \otimes \mathcal{O}_{X_k}((r-1)\mathbf{1} + \varepsilon\mathbf{c}) = \pi_{k,0}^* K_V \otimes \mathcal{O}_{X_k}(\varepsilon\mathbf{c})$$

is big over X_k for any sufficiently small positive rational number $\varepsilon \in \mathbb{Q}_+^*$. Thanks to Formula (1.10), we can in fact replace the weight $(r-1)\mathbf{1} + \varepsilon\mathbf{c}$ by its total degree $p = (r-1)k + \varepsilon|\mathbf{c}| \in \mathbb{Q}_+$. The general case of a singular linear space follows by considering suitable modifications of X (notice that the condition for a section of \widehat{V}_k^* to be bounded with respect to the dual of a metric on \mathcal{X}_k is less restrictive than saying that it is bounded with respect to the dual of the pull-back of a metric taken downstairs on a birational model \mathcal{X} of X , so we eventually get even more sections than those coming from downstairs in the above formula). \square

2. Induced directed structure on a subvariety of a jet space

Let Z be an irreducible algebraic subset of some k -jet bundle X_k over X , $k \geq 0$. We define the linear subspace $W \subset T_Z \subset T_{X_k|Z}$ to be the closure

$$(2.1) \quad W := \overline{T_{Z'} \cap V_k}$$

taken on a suitable Zariski open set $Z' \subset Z_{\text{reg}}$ where the intersection $T_{Z'} \cap V_k$ has constant rank and is a subbundle of $T_{Z'}$. Alternatively, we could also take W to be the closure of $T_{Z'} \cap V_k$ in the k -th stage $(\mathcal{X}_k, \mathcal{A}_k)$ of the absolute Semple tower. We say that (Z, W) is the *induced* directed variety structure. In the sequel, we always consider such a subvariety Z of X_k as a directed pair (Z, W) by taking the induced structure described above. Let us first quote the following easy observation.

2.2. Observation. *For $k \geq 1$, let $Z \subsetneq X_k$ be an irreducible algebraic subset that projects onto X_{k-1} , i.e. $\pi_{k,k-1}(Z) = X_{k-1}$. Then the induced directed variety $(Z, W) \subset (X_k, V_k)$, satisfies*

$$1 \leq \text{rank } W < r := \text{rank}(V_k).$$

Proof. Take a Zariski open subset $Z' \subset Z_{\text{reg}}$ such that $W' = T_{Z'} \cap V_k$ is a vector bundle over Z' . Since $X_k \rightarrow X_{k-1}$ is a \mathbb{P}^{r-1} -bundle, Z has codimension at most $r-1$ in X_k . Therefore $\text{rank } W \geq \text{rank } V_k - (r-1) \geq 1$. On the other hand, if we had $\text{rank } W = \text{rank } V_k$ generically, then $T_{Z'}$ would contain $V_{k|Z'}$, in particular it would contain all vertical directions $T_{X_k/X_{k-1}} \subset V_k$ that are tangent to the fibers of $X_k \rightarrow X_{k-1}$. By taking the flow along vertical vector fields, we would conclude that Z' is a union of fibers of $X_k \rightarrow X_{k-1}$ up to an algebraic set of smaller dimension, but this is excluded since Z projects onto X_{k-1} and $Z \subsetneq X_k$. \square

2.3. Definition. For $k \geq 1$, let $Z \subset X_k$ be an irreducible algebraic subset of X_k that projects onto X_{k-1} . We assume moreover that $Z \not\subset D_k = P(T_{X_{k-1}/X_{k-2}})$ (and put here $D_1 = \emptyset$ in what follows to avoid to have to single out the case $k = 1$). In this situation we say that (Z, W) is of general type modulo $X_k \rightarrow X$ if there exists $p \in \mathbb{Q}_+$ such that $K_W \otimes \mathcal{O}_{X_k}(p)|_Z$ is big over Z , possibly after replacing Z by a suitable nonsingular model \widehat{Z} (and pulling-back W and $\mathcal{O}_{X_k}(p)|_Z$ to the non singular variety \widehat{Z}).

The main result of [Dem11] mentioned in the introduction as Theorem 0.10 implies the following important “induction step”.

2.4. Proposition. Let (X, V) be a directed pair where X is projective algebraic. Take an irreducible algebraic subset $Z \not\subset D_k$ of the associated k -jet Semple bundle X_k that projects onto X_{k-1} , $k \geq 1$, and assume that the induced directed space $(Z, W) \subset (X_k, V_k)$ is of general type modulo $X_k \rightarrow X$. Then there exists a divisor $\Sigma \subset Z_\ell$ in a sufficiently high stage of the Semple tower (Z_ℓ, W_ℓ) associated with (Z, W) , such that every non constant holomorphic map $f : \mathbb{C} \rightarrow X$ tangent to V that satisfies $f_{[k]}(\mathbb{C}) \subset Z$ also satisfies $f_{[k+\ell]}(\mathbb{C}) \subset \Sigma$.

Proof. Let $E \subset Z$ be a divisor containing $Z_{\text{sing}} \cup (Z \cap \pi_{k,0}^{-1}(V_{\text{sing}}))$, chosen so that on the nonsingular Zariski open set $Z' = Z \setminus E$ all linear spaces $T_{Z'}$, $V_{k|Z'}$ and $W' = T_{Z'} \cap V_k$ are subbundles of $T_{X_k|Z'}$, the first two having a transverse intersection on Z' . By taking closures over Z' in the absolute Semple tower of X , we get (singular) directed pairs $(Z_\ell, W_\ell) \subset (X_{k+\ell}, V_{k+\ell})$, which we eventually resolve into $(\widehat{Z}_\ell, \widehat{W}_\ell) \subset (\widehat{X}_{k+\ell}, \widehat{V}_{k+\ell})$ over nonsingular bases. By construction, locally bounded sections of $\mathcal{O}_{\widehat{X}_{k+\ell}}(m)$ restrict to locally bounded sections of $\mathcal{O}_{\widehat{Z}_\ell}(m)$ over \widehat{Z}_ℓ .

Since Theorem 0.10 and the related estimate (0.11) are universal in the category of directed varieties, we can apply them by replacing X with $\widehat{Z} \subset \widehat{X}_k$, the order k by a new index ℓ , and F by

$$F_k = \mu^* \left((\mathcal{O}_{X_k}(p) \otimes \pi_{k,0}^* \mathcal{O}_X(-\varepsilon A))|_Z \right)$$

where $\mu : \widehat{Z} \rightarrow Z$ is the desingularization, $p \in \mathbb{Q}_+$ is chosen such that $K_W \otimes \mathcal{O}_{x_k}(p)|_Z$ is big, A is an ample bundle on X and $\varepsilon \in \mathbb{Q}_+^*$ is small enough. The assumptions show that $K_{\widehat{W}} \otimes F_k$ is big on \widehat{Z} , therefore, by applying our theorem and taking $m \gg \ell \gg 1$, we get in fine a large number of (metric bounded) sections of

$$\begin{aligned} & \mathcal{O}_{\widehat{Z}_\ell}(m) \otimes \widehat{\pi}_{k+\ell,k}^* \mathcal{O} \left(\frac{m}{\ell r'} \left(1 + \frac{1}{2} + \dots + \frac{1}{\ell} \right) F_k \right) \\ & = \mathcal{O}_{\widehat{X}_{k+\ell}}(m \mathbf{a}') \otimes \widehat{\pi}_{k+\ell,0}^* \mathcal{O} \left(- \frac{m\varepsilon}{kr} \left(1 + \frac{1}{2} + \dots + \frac{1}{k} \right) A \right)|_{\widehat{Z}_\ell} \end{aligned}$$

where $\mathbf{a}' \in \mathbb{Q}_+^{k+\ell}$ is a positive weight (of the form $(0, \dots, \lambda, \dots, 0, 1)$ with some non zero component $\lambda \in \mathbb{Q}_+$ at index k). These sections descend to metric bounded sections of

$$\mathcal{O}_{X_{k+\ell}}((1+\lambda)m) \otimes \widehat{\pi}_{k+\ell,0}^* \mathcal{O} \left(- \frac{m\varepsilon}{kr} \left(1 + \frac{1}{2} + \dots + \frac{1}{k} \right) A \right)|_{Z_\ell}.$$

Since A is ample on X , we can apply the fundamental vanishing theorem (see e.g. [Dem97] or [Dem11], Statement 8.15), or rather an “embedded” version for curves satisfying $f_{[k]}(\mathbb{C}) \subset Z$, proved exactly by the same arguments. The vanishing theorem implies that the divisor Σ of any such section satisfies the conclusions of Proposition 2.4, possibly modulo exceptional divisors of $\widehat{Z} \rightarrow Z$; to take care of these, it is enough to add to Σ the inverse image of the divisor $E = Z \setminus Z'$ initially selected. \square

3. Strong general type condition for directed manifolds

Our main result is the following partial solution to the Green-Griffiths-Lang conjecture, providing a sufficient algebraic condition for the analytic conclusion to hold true. We first give an ad hoc definition.

3.1. Definition. *Let (X, V) be a directed pair where X is projective algebraic. We say that (X, V) is “strongly of general type” if it is of general type and for every irreducible algebraic set $Z \subsetneq X_k$, $Z \not\subset D_k$, that projects onto X_{k-1} , $k \geq 1$, the induced directed structure $(Z, W) \subset (X_k, V_k)$ is of general type modulo $X_k \rightarrow X$.*

3.2. Example. The situation of a product $(X, V) = (X', V') \times (X'', V'')$ described in (0.12) shows that (X, V) can be of general type without being strongly of general type. In fact, if (X', V') and (X'', V'') are of general type, then $K_V = \text{pr}'^* K_{V'} \otimes \text{pr}''^* K_{V''}$ is big, so (X, V) is again of general type. However

$$Z = P(\text{pr}'^* V') = X'_1 \times X'' \subset X_1$$

has a directed structure $W = \text{pr}'^* V'_1$ which does not possess a big canonical bundle over Z , since the restriction of K_W to any fiber $\{x'\} \times X''$ is trivial. The higher stages (Z_k, W_k) of the Semple tower of (Z, W) are given by $Z_k = X'_{k+1} \times X''$ and $W_k = \text{pr}'^* V'_{k+1}$, so it is easy to see that $\text{GG}_k(X, V)$ contains Z_{k-1} . Since Z_k projects onto X , we have here $\text{GG}(X, V) = X$ (see [DR13] for more sophisticated indecomposable examples).

3.3. Remark. It follows from Definition 2.3 that $(Z, W) \subset (X_k, V_k)$ is automatically of general type modulo $X_k \rightarrow X$ if $\mathcal{O}_{X_k}(1)|_Z$ is big. Notice further that

$$\mathcal{O}_{X_k}(1 + \varepsilon)|_Z = (\mathcal{O}_{X_k}(\varepsilon) \otimes \pi_{k,k-1}^* \mathcal{O}_{X_{k-1}}(1) \otimes \mathcal{O}(D_k))|_Z$$

where $\mathcal{O}(D_k)|_Z$ is effective and $\mathcal{O}_{X_k}(1)$ is relatively ample with respect to the projection $X_k \rightarrow X_{k-1}$. Therefore the bigness of $\mathcal{O}_{X_{k-1}}(1)$ on X_{k-1} also implies that every directed subvariety $(Z, W) \subset (X_k, V_k)$ is of general type modulo $X_k \rightarrow X$. If (X, V) is of general type, we know by the main result of [Dem11] that $\mathcal{O}_{X_k}(1)$ is big for $k \geq k_0$ large enough, and actually the precise estimates obtained therein give explicit bounds for such a k_0 . The above observations show that we need to check the condition of Definition 3.1 only for $Z \subset X_k$, $k \leq k_0$. Moreover, at least in the case where V , Z , and $W = T_Z \cap V_k$ are nonsingular, we have

$$K_W \simeq K_Z \otimes \det(T_Z/W) \simeq K_Z \otimes \det(T_{X_k}/V_k)|_Z \simeq K_{Z/X_{k-1}} \otimes \mathcal{O}_{X_k}(1)|_Z.$$

Thus we see that, in some sense, it is only needed to check the bigness of K_W modulo $X_k \rightarrow X$ for “rather special subvarieties” $Z \subset X_k$ over X_{k-1} , such that $K_{Z/X_{k-1}}$ is not relatively big over X_{k-1} . \square

3.4. Hypersurface case. Assume that $Z \neq D_k$ is an irreducible hypersurface of X_k that projects onto X_{k-1} . To simplify things further, also assume that V is nonsingular. Since the Semple jet-bundles X_k form a tower of \mathbb{P}^{r-1} -bundles, their Picard groups satisfy $\text{Pic}(X_k) \simeq \text{Pic}(X) \oplus \mathbb{Z}^k$ and we have $\mathcal{O}_{X_k}(Z) \simeq \mathcal{O}_{X_k}(\mathbf{a}) \otimes \pi_{k,0}^* B$ for some $\mathbf{a} \in \mathbb{Z}^k$ and $B \in \text{Pic}(X)$, where $a_k = d > 0$ is the relative degree of the hypersurface over X_{k-1} . Let $\sigma \in H^0(X_k, \mathcal{O}_{X_k}(Z))$ be the section defining Z in X_k . The induced directed variety (Z, W)

has rank $W = r - 1 = \text{rank } V - 1$ and formula (1.12) yields $K_{V_k} = \mathcal{O}_{X_k}(-(r-1)\mathbf{1}) \otimes \pi_{k,0}^*(K_V)$. We claim that

$$(3.5) \quad K_W \supset (K_{V_k} \otimes \mathcal{O}_{X_k}(Z))|_Z \otimes \mathcal{I}_S = (\mathcal{O}_{X_k}(\mathbf{a} - (r-1)\mathbf{1}) \otimes \pi_{k,0}^*(B \otimes K_V))|_Z \otimes \mathcal{I}_S$$

where $S \subsetneq Z$ is the set (containing Z_{sing}) where σ and $d\sigma|_{V_k}$ both vanish, and \mathcal{I}_S is the ideal locally generated by the coefficients of $d\sigma|_{V_k}$ along $Z = \sigma^{-1}(0)$. In fact, the intersection $W = T_Z \cap V_k$ is transverse on $Z \setminus S$; then (3.5) can be seen by looking at the morphism

$$V_k|_Z \xrightarrow{d\sigma|_{V_k}} \mathcal{O}_{X_k}(Z)|_Z,$$

and observing that the contraction by $K_{V_k} = \Lambda^r V_k^*$ provides a metric bounded section of the canonical sheaf K_W . In order to investigate the positivity properties of K_W , one has to show that B cannot be too negative, and in addition to control the singularity set S . The second point is a priori very challenging, but we get useful information for the first point by observing that σ provides a morphism $\pi_{k,0}^* \mathcal{O}_X(-B) \rightarrow \mathcal{O}_{X_k}(\mathbf{a})$, hence a nontrivial morphism

$$\mathcal{O}_X(-B) \rightarrow E_{\mathbf{a}} := (\pi_{k,0})_* \mathcal{O}_{X_k}(\mathbf{a})$$

By [Dem95, Section 12], there exists a filtration on $E_{\mathbf{a}}$ such that the graded pieces are irreducible representations of $\text{GL}(V)$ contained in $(V^*)^{\otimes \ell}$, $\ell \leq |\mathbf{a}|$. Therefore we get a nontrivial morphism

$$(3.6) \quad \mathcal{O}_X(-B) \rightarrow (V^*)^{\otimes \ell}, \quad \ell \leq |\mathbf{a}|.$$

If we know about certain (semi-)stability properties of V , this can be used to control the negativity of B . \square

We further need the following useful concept that generalizes entire curve loci.

3.7. Definition. *If Z is an algebraic set contained in some stage X_k of the Semple tower of (X, V) , we define its “induced entire curve locus” $\text{IEL}(Z) \subset Z$ to be the Zariski closure of the union $\bigcup f_{[k]}(\mathbb{C})$ of all jets of entire curves $f : (\mathbb{C}, T_{\mathbb{C}}) \rightarrow (X, V)$ such that $f_{[k]}(\mathbb{C}) \subset Z$.*

We have of course $\text{IEL}(\text{IEL}(Z)) = \text{IEL}(Z)$ by definition. It is not hard to check that modulo certain “vertical divisors” of X_k , the $\text{IEL}(Z)$ locus is essentially the same as the entire curve locus $\text{ECL}(Z, W)$ of the induced directed variety, but we will not use this fact here. Since $\text{IEL}(X) = \text{ECL}(X, V)$, proving the Green-Griffiths-Lang property amounts to showing that $\text{IEL}(X) \subsetneq X$ in the stage $k = 0$ of the tower.

3.8. Theorem. *Let (X, V) be a directed pair of general type. Assume that there is an integer $k_0 \geq 0$ such that for every $k > k_0$ and every irreducible algebraic set $Z \subsetneq X_k$, $Z \not\subset D_k$, that projects onto X_{k-1} , the induced directed structure $(Z, W) \subset (X_k, V_k)$ is of general type modulo $X_k \rightarrow X$. Then $\text{IEL}(X_{k_0}) \subsetneq X_{k_0}$.*

Proof. We argue here by contradiction, assuming that $\text{IEL}(X_{k_0}) = X_{k_0}$. The main argument consists of producing inductively an increasing sequence of integers

$$k_0 < k_1 < \dots < k_j < \dots$$

and directed varieties $(Z^j, W^j) \subset (X_{k_j}, V_{k_j})$ satisfying the following properties :

- (a) $(Z^0, W^0) = (X_{k_0}, V_{k_0})$;
- (b) for all $j \geq 0$, $\text{IEL}(Z^j) = Z^j$;
- (c) Z^j is an irreducible algebraic variety such that $Z^j \subsetneq X_{k_j}$ for $j \geq 1$, Z^j is not contained in the vertical divisor $D_{k_j} = P(T_{X_{k_{j-1}}/X_{k_{j-2}}})$ of X_{k_j} , and (Z^j, W^j) is of general type modulo $X_{k_j} \rightarrow X$ (i.e. some nonsingular model is) ;
- (d) for all $j \geq 0$, the directed variety (Z^{j+1}, W^{j+1}) is contained in some stage (of order $\ell_j = k_{j+1} - k_j$) of the Semple tower of (Z^j, W^j) , namely

$$(Z^{j+1}, W^{j+1}) \subset (Z_{\ell_j}^j, W_{\ell_j}^j) \subset (X_{k_{j+1}}, V_{k_{j+1}})$$

and

$$W^{j+1} = \overline{T_{Z^{j+1}, \ell_j} \cap W_{\ell_j}^j} = \overline{T_{Z^{j+1}, \ell_j} \cap V_{k_j}}$$

is the induced directed structure.

- (e) for all $j \geq 0$, we have $Z^{j+1} \subsetneq Z_{\ell_j}^j$ but $\pi_{k_{j+1}, k_{j+1}-1}(Z^{j+1}) = Z_{\ell_j-1}^j$.

For $j = 0$, we have nothing to do by our hypotheses. Assume that (Z^j, W^j) has been constructed. By Proposition 2.4, we get an algebraic subset $\Sigma \subsetneq Z_{\ell}^j$ in some stage of the simple tower (Z_{ℓ}^j) of Z^j such that every entire curve $f : (\mathbb{C}, T_{\mathbb{C}}) \rightarrow (X, V)$ satisfying $f_{[k_j]}(\mathbb{C}) \subset Z^j$ also satisfies $f_{[k_j+\ell]}(\mathbb{C}) \subset \Sigma$. By definition, this implies the first inclusion in the sequence

$$Z^j = \text{IEL}(Z^j) \subset \pi_{k_j+\ell, k_j}(\text{IEL}(\Sigma)) \subset \pi_{k_j+\ell, k_j}(\Sigma) \subset Z^j$$

(the other ones being obvious), so we have in fact an equality throughout. Let (S_{α}) be the irreducible components of $\text{IEL}(\Sigma)$. We have $\text{IEL}(S_{\alpha}) = S_{\alpha}$ and one of the components S_{α} must already satisfy $\pi_{k_j+\ell, k_j}(S_{\alpha}) = Z^j = Z_0^j$. We take $\ell_j \in [1, \ell]$ to be the smallest order such that $Z^{j+1} := \pi_{k_j+\ell, k_j+\ell_j}(S_{\alpha}) \subsetneq Z_{\ell_j}^j$, and set $k_{j+1} = k_j + \ell_j > k_j$. By definition of ℓ_j , we have $\pi_{k_{j+1}, k_{j+1}-1}(Z^{j+1}) = Z_{\ell_j-1}^j$, otherwise ℓ_j would not be minimal. The fact that $\text{IEL}(S_{\alpha}) = S_{\alpha}$ immediately implies $\text{IEL}(Z^{j+1}) = Z^{j+1}$. Also Z^{j+1} cannot be contained in the vertical divisor $D_{k_{j+1}}$. In fact no irreducible algebraic set Z such that $\text{IEL}(Z) = Z$ can be contained in a vertical divisor D_k , because $\pi_{k, k-2}(D_k)$ corresponds to stationary jets in X_{k-2} ; as every non constant curve f has non stationary points, its k -jet $f_{[k]}$ cannot be entirely contained in D_k . Finally, the induced directed structure (Z^{j+1}, W^{j+1}) must be of general type modulo $X_{k_{j+1}} \rightarrow X$, by the assumption of the theorem and the fact that $k_{j+1} > k_0$. The inductive procedure is therefore complete.

By Observation 2.2, we have

$$\text{rank } W^j < \text{rank } W^{j-1} < \dots < \text{rank } W^1 < \text{rank } W^0 = \text{rank } V.$$

After a sufficient number of iterations we reach $\text{rank } W^j = 1$. In this situation the Semple tower of Z^j is trivial, $K_{W^j} = W^{j*} \otimes \bar{\mathcal{J}}_{W^j}$ is big, and Proposition 2.4 produces a divisor $\Sigma \subsetneq Z_{\ell}^j = Z^j$ containing all jets of entire curves with $f_{[k_j]}(\mathbb{C}) \subset Z^j$. This contradicts the fact that $\text{IEL}(Z^j) = Z^j$. We have reached a contradiction, and Theorem 3.8 is thus proved. \square

3.9. Remark. As it proceeds by contradiction, the proof is unfortunately non constructive – especially it does not give any information on the degree of the locus $Y \subsetneq X_{k_0}$ whose existence is asserted. On the other hand, and this is a bit surprising, the conclusion is obtained even though the conditions to be checked do not involve cutting down the dimensions of the base

loci of jet differentials; in fact, the contradiction is obtained even though the integers k_j may increase and $\dim Z^j$ may become very large.

The special case $k_0 = 0$ of Theorem 3.8 yields the following

3.10. Partial solution to the generalized GGL conjecture. *Let (X, V) be a directed pair that is strongly of general type. Then the Green-Griffiths-Lang conjecture holds true for (X, V) , namely $\text{ECL}(X, V) \subsetneq X$, in other words there exists a proper algebraic variety $Y \subsetneq X$ such that every non constant holomorphic curve $f : \mathbb{C} \rightarrow X$ tangent to V satisfies $f(\mathbb{C}) \subset Y$.*

3.11. Remark. The condition that (X, V) is strongly of general type seems to be related to some sort of stability condition. We are unsure what is the most appropriate definition, but here is one that makes sense. Fix an ample divisor A on X . For every irreducible subvariety $Z \subset X_k$ that projects onto X_{k-1} for $k \geq 1$, and $Z = X = X_0$ for $k = 0$, we define the slope $\mu_A(Z, W)$ of the corresponding directed variety (Z, W) to be

$$\mu_A(Z, W) = \frac{\inf \lambda}{\text{rank } W},$$

where λ runs over all rational numbers such that there exists $m \in \mathbb{Q}_+$ for which

$$K_W \otimes (\mathcal{O}_{X_k}(m) \otimes \pi_{k,0}^* \mathcal{O}(\lambda A))|_Z \text{ is big on } Z$$

(again, we assume here that $Z \not\subset D_k$ for $k \geq 2$). Notice that (X, V) is of general type if and only if $\mu_A(X, V) < 0$, and that $\mu_A(Z, W) = -\infty$ if $\mathcal{O}_{X_k}(1)|_A$ is big. Also, the proof of Lemma 1.11 shows that

$$\mu_A(X_k, V_k) \leq \mu_A(X_{k-1}, V_{k-1}) \leq \dots \leq \mu_A(X, V) \quad \text{for all } k$$

(with $\mu_A(X_k, V_k) = -\infty$ for $k \geq k_0 \gg 1$ if (X, V) is of general type). We say that (X, V) is *A-jet-stable* (resp. *A-jet-semi-stable*) if $\mu_A(Z, W) < \mu_A(X, V)$ (resp. $\mu_A(Z, W) \leq \mu_A(X, V)$) for all $Z \subsetneq X_k$ as above. It is then clear that if (X, V) is of general type and *A-jet-semi-stable*, then it is strongly of general type in the sense of Definition 3.1. It would be useful to have a better understanding of this condition of stability (or any other one that would have better properties). \square

3.12. Example: case of surfaces. Assume that X is a minimal complex surface of general type and $V = T_X$ (absolute case). Then K_X is nef and big and the Chern classes of X satisfy $c_1 \leq 0$ ($-c_1$ is big and nef) and $c_2 \geq 0$. The Semple jet-bundles X_k form here a tower of \mathbb{P}^1 -bundles and $\dim X_k = k + 2$. Since $\det V^* = K_X$ is big, the strong general type assumption of 3.8 and 3.10 need only be checked for irreducible hypersurfaces $Z \subset X_k$ distinct from D_k that project onto X_{k-1} , of relative degree m . The projection $\pi_{k,k-1} : Z \rightarrow X_{k-1}$ is a ramified $m : 1$ cover. Putting $\mathcal{O}_{X_k}(Z) \simeq \mathcal{O}_{X_k}(\mathbf{a}) \otimes \pi_{k,0}(B)$, $B \in \text{Pic}(X)$, we can apply (3.5) to get an inclusion

$$K_W \supset (\mathcal{O}_{X_k}(\mathbf{a} - \mathbf{1}) \otimes \pi_{k,0}^*(B \otimes K_X))|_Z \otimes \mathcal{J}_S, \quad \mathbf{a} \in \mathbb{Z}^k, \quad a_k = m.$$

Let us assume $k = 1$ and $S = \emptyset$ to make things even simpler, and let us perform numerical calculations in the cohomology ring

$$H^\bullet(X_1, \mathbb{Z}) = H^\bullet(X)[u]/(u^2 + c_1 u + c_2), \quad u = c_1(\mathcal{O}_{X_1}(1))$$

(cf. [DEG00, Section 2] for similar calculations and more details). We have

$$Z \equiv mu + b \quad \text{where} \quad b = c_1(B) \quad \text{and} \quad K_W \equiv (m-1)u + b - c_1.$$

We are allowed here to add to K_W an arbitrary multiple $\mathcal{O}_{X_1}(p)$, $p \geq 0$, which we rather write $p = mt + 1 - m$, $t \geq 1 - 1/m$. An evaluation of the Euler-Poincaré characteristic of $K_W + \mathcal{O}_{X_1}(p)|_Z$ requires computing the intersection number

$$\begin{aligned} (K_W + \mathcal{O}_{X_1}(p)|_Z)^2 \cdot Z &= (mtu + b - c_1)^2(mu + b) \\ &= m^2t^2(m(c_1^2 - c_2) - bc_1) + 2mt(b - mc_1)(b - c_1) + m(b - c_1)^2, \end{aligned}$$

taking into account that $u^3 \cdot X_1 = c_1^2 - c_2$. In case $S \neq \emptyset$, there is an additional (negative) contribution from the ideal \mathcal{J}_S which is $O(t)$ since S is at most a curve. In any case, for $t \gg 1$, the leading term in the expansion is $m^2t^2(m(c_1^2 - c_2) - bc_1)$ and the other terms are negligible with respect to t^2 , including the one coming from S . We know that T_X is semistable with respect to $c_1(K_X) = -c_1 \geq 0$. Multiplication by the section σ yields a morphism $\pi_{1,0}^* \mathcal{O}_X(-B) \rightarrow \mathcal{O}_{X_1}(m)$, hence by direct image, a morphism $\mathcal{O}_X(-B) \rightarrow S^m T_X^*$. Evaluating slopes against K_X (a big nef class), the semistability condition implies $bc_1 \leq \frac{m}{2}c_1^2$, and our leading term is bigger than $m^3t^2(\frac{1}{2}c_1^2 - c_2)$. We get a positive answer in the well-known case where $c_1^2 > 2c_2$, corresponding to T_X being almost ample. Analyzing positivity for the full range of values (k, m, t) and of singular sets S seems an unsurmountable task at this point; in general, calculations made in [DEG00] and [McQ99] indicate that the Chern class and semistability conditions become less demanding for higher order jets (e.g. $c_1^2 > c_2$ is enough for $Z \subset X_2$, and $c_1^2 > \frac{9}{13}c_2$ suffices for $Z \subset X_3$). When $\text{rank } V = 1$, major gains come from the use of Ahlfors currents in combination with McQuillan's tautological inequalities [McQ98]. We therefore hope for a substantial strengthening of the above sufficient conditions, and a better understanding of the stability issues, possibly in combination with a use of Ahlfors currents and tautological inequalities. In the case of surfaces, an application of Theorem 3.8 for $k_0 = 1$ and an analysis of the behaviour of rank 1 (multi-)foliations on the surface X (with the crucial use of [McQ98]) was the main argument used in [DEG00] to prove the hyperbolicity of very general surfaces of degree $d \geq 21$ in \mathbb{P}^3 . For these surfaces, one has $c_1^2 < c_2$ and $c_1^2/c_2 \rightarrow 1$ as $d \rightarrow +\infty$. Applying Theorem 3.8 for higher values $k_0 \geq 2$ might allow to enlarge the range of tractable surfaces, if the behavior of rank 1 (multi-)foliations on X_{k_0-1} can be analyzed independently.

4. Algebraic jet-hyperbolicity implies Kobayashi hyperbolicity

Let (X, V) be a directed variety, where X is an irreducible projective variety; if X is singular, this still makes sense by possibly considering (X, V) as embedded in $(\mathbb{P}^N, T_{\mathbb{P}^N})$ where V is closed and irreducible and contained in T_X at regular points.

For every irreducible algebraic subvariety $Z \subset X$, we get as in section 2 a directed variety structure $(Z, W) \subset (X, V)$ by taking $W = \overline{T_{Z'} \cap V}$ on a sufficiently small Zariski open set $Z' \subset Z_{\text{reg}}$ where the intersection has minimal rank. Notice that when $W = 0$ there cannot exist entire curves $f : (\mathbb{C}, T_{\mathbb{C}}) \rightarrow (Z, W)$ except possibly those which lie in the algebraic set $Z \setminus Z'$, hence this case is easy to deal with by induction on dimension. Otherwise, we can resolve singularities of Z to get a directed variety $(\widehat{Z}, \widehat{W})$ where \widehat{Z} is non singular and $\text{rank } \widehat{W} \geq 1$.

4.1. Definition. *We say that (X, V) is algebraically jet-hyperbolic if for every irreducible algebraic subvariety $Z \subset X$, the induced directed structure (Z, W) either satisfies $W = 0$, or has a desingularization $(\widehat{Z}, \widehat{W})$, $\text{rank } \widehat{W} \geq 1$, that is strongly of general type.*

Thanks to Theorem 3.10, a very easy induction on the dimension of X implies

4.2. Theorem. *Let (X, V) be an irreducible projective directed variety that is algebraically jet-hyperbolic in the sense of the above definition. Then (X, V) is Brody (or Kobayashi) hyperbolic, i.e. $\text{ECL}(X, V) = \emptyset$.*

Proof. By Theorem 3.10, we have $Y := \text{ECL}(X, V) \subsetneq X$. If $Y \neq \emptyset$, apply induction on dimension to each of the irreducible components X'_j of Y and to the induced directed structures (X'_j, V'_j) to get $\text{ECL}(X, V) \subset \bigcup \text{ECL}(X'_j, V'_j) \subsetneq \bigcup X'_j = Y$, a contradiction. \square

5. Proof of the Kobayashi conjecture on generic hyperbolicity

We start with a general situation, and then restrict ourselves to the special case of complete intersections in projective space. Consider a smooth deformation $\pi : \mathcal{X} \rightarrow S$ of complex projective manifolds, i.e. a proper algebraic submersion over a quasi-projective algebraic manifold S such that the fibers are non singular. By a “very general fiber”, we mean here a fiber $X_t = \pi^{-1}(t)$ over a point t taken in the complement $S \setminus \bigcup S_\nu$ of a countable union of algebraic subsets $S_\nu \subsetneq S$. We are only interested in the very general fiber and can therefore restrict ourselves to the case where S is affine by taking a suitable Zariski open subset $S' \subset S$. Ample vector bundles over the total space \mathcal{X} are then the same as vector bundles that are relatively ample over S .

5.1. Theorem. *Let $\pi : \mathcal{X} \rightarrow S$ be a deformation of complex projective nonsingular varieties $X_t = \pi^{-1}(t)$ over a smooth quasi-projective irreducible base S . Let $n = \dim X_t$ be the relative dimension and let $N = \dim S$. Assume that for all $q = N + 1, \dots, N + n$, the exterior power $\Lambda^q T_{\mathcal{X}}^*$ is a relatively ample vector bundle over S . Then the very general fiber X_t is algebraically jet-hyperbolic, and thus Kobayashi hyperbolic.*

Proof. By taking the relative directed structure $\mathcal{V} = T_{\mathcal{X}/S}$ on \mathcal{X} , one constructs a relative Semple tower $(\mathcal{X}_k, \mathcal{V}_k)$ over \mathcal{X} . It specializes to the absolute Semple tower of X_t when one takes the restriction of \mathcal{X}_k to the inverse image of $X_t = \pi^{-1}(t) \subset \mathcal{X}$ by $\pi_{k,0} : \mathcal{X}_k \rightarrow \mathcal{X}_0 = \mathcal{X}$. By construction $\mathcal{V}_0 = \mathcal{V} = T_{\mathcal{X}/S}$ has rank n . Let $(\mathcal{X}'_k, \mathcal{V}'_k)$ be the absolute Semple tower of \mathcal{X} (so that $\mathcal{X}'_0 = \mathcal{X}$ and $\mathcal{V}'_0 = T_{\mathcal{X}}$), and let $\tilde{\mathcal{V}}_k$ be the restriction of the vector bundle \mathcal{V}'_k to $\mathcal{X}_k \subset \mathcal{X}'_k$. For every $k \geq 0$, we have an exact sequence of vector bundles

$$0 \rightarrow \mathcal{V}_k \rightarrow \tilde{\mathcal{V}}_k \rightarrow (\pi_{k,0} \circ \pi)^* T_S \rightarrow 0$$

over \mathcal{X}_k where $\text{rank } \mathcal{V}_k = n$, $\text{rank } \tilde{\mathcal{V}}_k = \text{rank } \mathcal{V}'_k = N + n = \dim \mathcal{X}$. Now, the absolute Semple tower of \mathcal{X} yields exact sequences

$$\begin{aligned} 0 &\longrightarrow T_{\mathcal{X}'_k/\mathcal{X}'_{k-1}} \longrightarrow \mathcal{V}'_k \xrightarrow{(\pi_{k,k-1})^*} \mathcal{O}_{\mathcal{X}'_k}(-1) \longrightarrow 0, \\ 0 &\longrightarrow \mathcal{O}_{\mathcal{X}'_k} \longrightarrow (\pi_{k,k-1})^* \mathcal{V}'_{k-1} \otimes \mathcal{O}_{\mathcal{X}'_k}(1) \longrightarrow T_{\mathcal{X}'_k/\mathcal{X}'_{k-1}} \longrightarrow 0. \end{aligned}$$

If we take the duals and restrict to $\mathcal{X}_k \subset \mathcal{X}'_k$, we get exact sequences

$$\begin{aligned} 0 &\longrightarrow \mathcal{O}_{\mathcal{X}_k}(1) \longrightarrow \tilde{\mathcal{V}}_k^* \longrightarrow \mathcal{H}_k^* \longrightarrow 0, \\ 0 &\longrightarrow \mathcal{H}_k^* \longrightarrow (\pi_{k,k-1})^* \tilde{\mathcal{V}}_{k-1}^* \otimes \mathcal{O}_{\mathcal{X}_k}(-1) \longrightarrow \mathcal{O}_{\mathcal{X}_k} \longrightarrow 0. \end{aligned}$$

where $\mathcal{H} = (T_{\mathcal{X}'_k/\mathcal{X}'_{k-1}})|_{\mathcal{X}_k}$. Now, take the q -th (resp. s -th) exterior power of these sequences; we find

$$(5.2') \quad 0 \longrightarrow \Lambda^{q-1}\mathcal{H}_k^* \otimes \mathcal{O}_{\mathcal{X}_k}(1) \longrightarrow \Lambda^q\tilde{\mathcal{V}}_k^* \longrightarrow \Lambda^q\mathcal{H}_k^* \longrightarrow 0,$$

$$(5.2'') \quad 0 \longrightarrow \Lambda^s\mathcal{H}_k^* \longrightarrow (\pi_{k,k-1})^*\Lambda^s\tilde{\mathcal{V}}_{k-1}^* \otimes \mathcal{O}_{\mathcal{X}_k}(-s) \longrightarrow \Lambda^{s-1}\mathcal{H}_k^* \longrightarrow 0.$$

If $\mathcal{Z} \subset \mathcal{X}_k$ is an irreducible algebraic subvariety of \mathcal{X}_k and $(\mathcal{Z}, \mathcal{W}) \subset (\mathcal{X}_k, \tilde{\mathcal{V}}_k)$ is the directed structure induced by $(\mathcal{X}'_k, \mathcal{V}'_k)$, we get by definition a non trivial morphism over \mathcal{Z} :

$$\Lambda^q\tilde{\mathcal{V}}_k^* \longrightarrow K_{\mathcal{W}}, \quad q = \text{rank } \mathcal{W}.$$

One should notice that $\tilde{\mathcal{V}}_k$ and \mathcal{H}_k are genuine vector bundles without singularities, hence the above morphism actually has its image *contained in* $K_{\mathcal{W}}$ even when one takes into account the relevant multiplier ideal sheaf. Now, we conclude by (5.2') that either we have a non trivial morphism

$$\mathcal{O}_{\mathcal{X}_k}(1) \otimes \Lambda^{q-1}\mathcal{H}_k^* \longrightarrow K_{\mathcal{W}}$$

or (if the above vanishes) a non trivial morphism

$$\Lambda^q\mathcal{H}_k^* \longrightarrow K_{\mathcal{W}}.$$

By (5.2'') with $s = q$ or $s = q + 1$, we infer that we have a non trivial morphism

$$(\pi_{k,k-1})^*\Lambda^q\tilde{\mathcal{V}}_{k-1}^* \otimes \mathcal{O}_{\mathcal{X}_k}(-q+1) \longrightarrow K_{\mathcal{W}}$$

or a non trivial morphism

$$(\pi_{k,k-1})^*\Lambda^{q+1}\tilde{\mathcal{V}}_{k-1}^* \otimes \mathcal{O}_{\mathcal{X}_k}(-q-1) \longrightarrow K_{\mathcal{W}}.$$

Proceeding inductively with the lower stages and getting down to $\tilde{\mathcal{V}}_0 = T_{\mathcal{X}}$, we conclude that there exists an integer $q' \geq q = \text{rank } \mathcal{W}$, a weight $\mathbf{a} = (a_1, \dots, a_k) \in \mathbb{N}^k$ and a non trivial morphism

$$(5.3) \quad (\pi_{k,0})^*\Lambda^{q'}T_{\mathcal{X}}^* \rightarrow K_{\mathcal{W}} \otimes \mathcal{O}_{\mathcal{X}_k}(\mathbf{a}).$$

Assume that $q > N$. By our assumption (assuming S affine here), $\Lambda^{q'}T_{\mathcal{X}}^*$ is ample over \mathcal{X} , thus, by twisting with a certain relatively ample line bundle $\mathcal{O}_{\mathcal{X}_k}(\varepsilon\mathbf{c})$ with respect to $\pi_{k,0}$, we see that $(\pi_{k,0})^*\Lambda^{q'}T_{\mathcal{X}}^* \otimes \mathcal{O}_{\mathcal{X}_k}(\varepsilon\mathbf{c})$ is ample over \mathcal{X}_k for $0 < \varepsilon \ll 1$. From this, we infer that there exists a weight $\mathbf{b} \in \mathbb{Q}_+$ such that $K_{\mathcal{W}} \otimes \mathcal{O}_{\mathcal{X}_k}(\mathbf{b})$ is big over \mathcal{Z} , in other words $(\mathcal{Z}, \mathcal{W})$ is of general type modulo $\mathcal{X}_k \rightarrow \mathcal{X}$. Notice that this is true especially for $k = 0$ (the argument is then obvious from the assumption that $\Lambda^q T_{\mathcal{X}}$ is ample over \mathcal{X}).

From this, one easily concludes by a Hilbert scheme argument that (X_t, T_{X_t}) is algebraically jet-hyperbolic for very general $t \in S$. Otherwise (by the fact that $k, \mathbf{b} \in \mathbb{Q}_+$, and that all dimensions and degrees run in countable sets), there would exist an algebraic family $Z_t \subset X_{t,k}$, $t \in S$, such that $\dim Z_t \geq 1$ and the induced directed structure (Z_t, W_t) is not of general type modulo $X_{t,k} \rightarrow X_t$. Equivalently, this family can be viewed as an algebraic directed subvariety $(\mathcal{Z}, \mathcal{W}) \subset (\mathcal{X}_k, \mathcal{V}_k)$ in the relative simple tower of $(\mathcal{X}, T_{\mathcal{X}/S})$. By construction, \mathcal{Z} projects onto S and has dimension $q > N = \dim S$. We have seen that $K_{\mathcal{W}} \otimes \mathcal{O}_{\mathcal{X}_k}(\mathbf{b})$ is big over \mathcal{X} for a suitable weight $\mathbf{b} \in \mathbb{Q}_+$. For a general $t \in S$, we have

$(K_W)_{|Z_t} = K_{W_t} \otimes (K_S)_{|Z_t} \simeq K_{W_t}$, therefore K_{W_t} is big modulo $X_{k,t} \rightarrow X_t$, a contradiction. Theorem 5.1 is proved. \square

Let us now consider the special case of the universal family of complete intersections of codimension ℓ and type (d_1, \dots, d_ℓ) in complex projective $\mathbb{P}^{n+\ell}$. We can view it as a smooth family $\mathcal{X} \rightarrow S$ where S is a Zariski open set in $\mathbb{C}^N = \prod \mathbb{C}^{N_j}$, $N_j = \binom{d_j+n+\ell}{n+\ell}$, and $\mathcal{X} \subset \mathbb{P}^{n+\ell} \times S$ is the incidence variety defined by

$$\mathcal{X} = \{([z], t) \in \mathbb{P}^{n+\ell} \times S; P_{t,j}(z) = 0, 1 \leq j \leq \ell\}.$$

Here S is the set of coefficients $t \in \mathbb{C}^N$ of polynomial ℓ -tuples $P_t = (P_{t,1}, \dots, P_{t,\ell})$ that define a nonsingular codimensional ℓ subvariety X_t . We use the following famous result proved by Claire Voisin [Voi96, Corollary 1.3] (with the substitution of notation $k \mapsto \ell$, $n \mapsto n + \ell$, $l \mapsto N + n - q$ in our setting).

5.4. Proposition ([Voi96]). *For $q > N + 2n + \ell + 1 - \sum d_j$, the vector bundle $\Lambda^q T_{\mathcal{X}}^*$ is relatively very ample with respect to the projection $\mathcal{X} \rightarrow S$.*

Since we want the relative ampleness of $\Lambda^q T_{\mathcal{X}}^*$ to hold for $q > N$, we need $\sum d_j \geq 2n + \ell + 1$. A combination of Theorem 5.1 with Proposition 5.4 implies

5.5. Corollary. *For $\sum d_j \geq 2n + \ell + 1$, the very general complete intersection of type (d_1, \dots, d_ℓ) in complex projective space $\mathbb{P}^{n+\ell}$ is Kobayashi hyperbolic.*

The simplest non trivial situation is the surface case $n = 2$ in codimension $\ell = 1$. We then obtain the Kobayashi hyperbolicity of a very general surface $X \subset \mathbb{P}^3$ of degree $d \geq 6$. The result seems to be new even in this case, although Duval [Duv04] has shown by elementary means the existence of a hyperbolic sextic (from this, it already follows that there is a family of hyperbolic sextics over an open set of parameters in Hausdorff topology).

5.6. Remark. It would be good to know if Kobayashi hyperbolicity is a Zariski open condition, in particular, whether one can replace “very general” by “general” in Cor. 5.5. This would require further investigations, but such a result might be accessible by taking into account Remark 3.3, which shows that the “bad sets” Z to consider are somehow bounded.

5.7. Remark. In the case $\sum d_j = 2n + \ell$ (and especially in the “border case” $d = 2n + 1$ of hypersurfaces), it follows from [Voi96] that $(\Lambda^q T_{\mathcal{X}})_{|X_t}$ is generated by sections for $q = N + 1$ and very ample for $q \geq N + 2$. It would then be natural to look at the degeneration set of the curvature, and decide whether it can accommodate a non trivial algebraic subvariety in one the stages of the Semple tower. If not, Corollary 5.5 could still hold when $\sum d_j = 2n + \ell$.

References

- [Cle86] Clemens, H.: *Curves in generic hypersurfaces*. Ann. Sci. École Norm. Sup. **19** (1986), 629–636.
- [Dem95] Demailly, J.-P.: *Algebraic criteria for Kobayashi hyperbolic projective varieties and jet differentials*. AMS Summer School on Algebraic Geometry, Santa Cruz 1995, Proc. Symposia in Pure Math., ed. by J. Kollár and R. Lazarsfeld, Amer. Math. Soc., Providence, RI (1997), 285–360.

- [Dem97] Demailly, J.-P.: *Variétés hyperboliques et équations différentielles algébriques*. Gaz. Math. **73** (juillet 1997) 3–23, and <http://www-fourier.ujf-grenoble.fr/~demailly/manuscripts/cartan.augm.pdf>.
- [DEG00] Demailly, J.-P., El Goul, J.: *Hyperbolicity of generic surfaces of high degree in projective 3-space*. Amer. J. Math. **122** (2000) 515–546.
- [Dem11] Demailly, J.-P.: *Holomorphic Morse inequalities and the Green-Griffiths-Lang conjecture*. November 2010, arXiv: math.AG/1011.3636, dedicated to the memory of Eckart Viehweg; Pure and Applied Mathematics Quarterly **7** (2011) 1165–1208.
- [DMR10] Diverio, S., Merker, J., Rousseau, E.: *Effective algebraic degeneracy*. Invent. Math. **180** (2010) 161–223.
- [DR13] Diverio, S., and Rousseau, E.: *The exceptional set and the Green-Griffiths locus do not always coincide*. arXiv: math.AG/1302.4756 (v2).
- [Duv04] Duval, J.: *Une sextique hyperbolique dans $\mathbb{P}^3(\mathbb{C})$* . Math. Ann. **330** (2004) 473–476.
- [Ein88] Ein, L.: *Subvarieties of generic complete intersections*. Invent. Math. **94** (1988), 163–169.
- [Ein91] Ein, L.: *Subvarieties of generic complete intersections, II*. Math. Ann. **289** (1991), 465–471.
- [GG79] Green, M., Griffiths, P.: *Two applications of algebraic geometry to entire holomorphic mappings*. The Chern Symposium 1979, Proc. Internal. Sympos. Berkeley, CA, 1979, Springer-Verlag, New York (1980), 41–74.
- [Kob70] Kobayashi, S. *Hyperbolic manifolds and holomorphic mappings*. Volume 2 of Pure and Applied Mathematics. Marcel Dekker Inc., New York, 1970.
- [Kob78] Kobayashi, S. *Hyperbolic complex spaces*. Volume 318 of Grundlehren der Mathematischen Wissenschaften, Springer-Verlag, Berlin, 1998.
- [Lan86] Lang, S.: *Hyperbolic and Diophantine analysis*. Bull. Amer. Math. Soc. **14** (1986) 159–205.
- [McQ98] McQuillan, M.: *Diophantine approximation and foliations*. Inst. Hautes Études Sci. Publ. Math. **87** (1998) 121–174.
- [McQ99] McQuillan, M.: *Holomorphic curves on hyperplane sections of 3-folds*. Geom. Funct. Anal. **9** (1999) 370–392.
- [Pau08] Păun, M.: *Vector fields on the total space of hypersurfaces in the projective space and hyperbolicity*. Math. Ann. **340** (2008) 875–892.
- [Siu02] Siu, Y.T.: *Some recent transcendental techniques in algebraic and complex geometry*. In: Proceedings of the International Congress of Mathematicians, Vol. I (Beijing, 2002), Higher Ed. Press, Beijing, 2002, 439–448.
- [Siu04] Siu, Y.T.: *Hyperbolicity in complex geometry*. In: The legacy of Niels Henrik Abel, Springer, Berlin, 2004, 543–566.
- [Siu12] Siu, Y.T.: *Hyperbolicity of generic high-degree hypersurfaces in complex projective spaces*. arXiv: math.CV, 1209.2723v3, 87 pages.
- [SY96] Siu, Y.T., Yeung, S.K.: *Hyperbolicity of the complement of a generic smooth curve of high degree in the complex projective plane*. Invent. Math. **124** (1996), 573–618.
- [Voi96] Voisin, C.: *On a conjecture of Clemens on rational curves on hypersurfaces*. J. Differential Geometry **44** (1996) 200–213.
- [Voi98] Voisin, C.: *A correction on “A conjecture of Clemens on rational curves on hypersurfaces”*. J. Differential Geometry **49** (1998) 601–611.
- [Xu94] Xu Geng: *Subvarieties of general hypersurfaces in projective space*. J. Differential Geom. **39** (1) (1994) 139–172.

[Zai87] Zaidenberg, M.: *The complement to a general hypersurface of degree $2n$ in $\mathbb{C}P^n$ is not hyperbolic.* (Russian) *Sibirsk. Mat. Zh.* **28** (3) (1987) 91–100.

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