PROOF OF THE KOBAYASHI CONJECTURE ON THE HYPERBOLICITY OF VERY GENERAL HYPERSURFACES

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Abstract. The Green-Griffiths-Lang conjecture stipulates that for every projective variety X of general type over $\mathbb{C}$, there exists a proper algebraic subvariety of X containing all non constant entire curves $f : \mathbb{C} \to X$. Using the formalism of directed varieties, we prove here that this assertion holds true in case X satisfies a strong general type condition that is related to a certain jet-semistability property of the tangent bundle T_X. We then use this fact to confirm a long-standing conjecture of Kobayashi (1970), according to which a very general algebraic hypersurface of dimension n and degree at least $2n + 2$ in the complex projective space $\mathbb{P}^{n+1}$ is hyperbolic.

dedicated to the memory of Salah Baouendi

0. Introduction

The goal of this paper, among other results, is to prove the long standing conjecture of Kobayashi [Kob70, Kob78], according to which a very general algebraic hypersurface of dimension n and degree $d \geq 2n+2$ in complex projective space $\mathbb{P}^{n+1}$ is Kobayashi hyperbolic. It is expected that the bound can be improved to $2n + 1$ for $n \geq 2$, and such a bound would be optimal by Zaidenberg [Zai87], but we cannot yet prove this. Siu [Siu02, Siu04, Siu12] has introduced a more explicit but more computationally involved approach that yields the same conclusion for $d \geq d_n$, with a very large bound d_n instead of $2n+2$. However, thanks to famous results of Clemens [Cle86], Ein [Ein88, Ein91] and Voisin [Voi96, Voi98], it was known that the bound $2n + 2$ would be a consequence of the Green-Griffiths-Lang conjecture on entire curve loci, cf. [GG79] and [Lan86]. Our technique consists in studying a generalized form of the GGL conjecture, and proving a special case that is strong enough to imply the Kobayashi conjecture, using e.g. [Voi96]. For this purpose, as was already observed in [Dem97], it is useful to work in the category of directed projective varieties, and to take into account the singularities that may appear in the directed structures, at all steps of the proof.

Since the basic problems we deal with are birationally invariant, the varieties under consideration can always be replaced by nonsingular models. A directed projective manifold is a pair (X, V) where X is a projective manifold equipped with an analytic linear subspace $V \subset T_X$, i.e. a closed irreducible complex analytic subset V of the total space of T_X, such that each fiber $V_x = V \cap T_{X,x}$ is a complex vector space [If X is not irreducible, V should rather be assumed to be irreducible merely over each component of X, but we will hereafter assume that our varieties are irreducible]. A morphism $\Phi : (X, V) \to (Y, W)$ in the category of directed manifolds is an analytic map $\Phi : X \to Y$ such that $\Phi_* V \subset W$. We refer to the case $V = T_X$ as being the *absolute case*, and to the case $V = T_{X/S} = \operatorname{Ker} d\pi$ for a fibration $\pi : X \to S$, as being the *relative case*; V may also be taken to be the tangent space to the

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leaves of a singular analytic foliation on X, or maybe even a non integrable linear subspace of T_X.

We are especially interested in *entire curves* that are tangent to V, namely non constant holomorphic morphisms $f:(\mathbb{C},T_{\mathbb{C}})\to(X,V)$ of directed manifolds. In the absolute case, these are just arbitrary entire curves $f:\mathbb{C}\to X$. The Green-Griffiths-Lang conjecture, in its strong form, stipulates

0.1. GGL conjecture. *Let X be a projective variety of general type. Then there exists a proper algebraic variety $Y\subsetneq X$ such that every entire curve $f:\mathbb{C}\to X$ satisfies $f(\mathbb{C})\subset Y$.*

[The weaker form would state that entire curves are algebraically degenerate, so that $f(\mathbb{C})\subset Y_f\subsetneq X$ where Y_f might depend on f]. The smallest admissible algebraic set $Y\subset X$ is by definition the *entire curve locus* of X, defined as the Zariski closure

$$\mathrm{ECL}(X)=\overline{\bigcup_f f(\mathbb{C})}^{\mathrm{Zar}}. \tag{0.2}$$

If $X\subset\mathbb{P}^N_{\mathbb{C}}$ is defined over a number field $\mathbb{K}_0$ (i.e. by polynomial equations with equations with coefficients in $\mathbb{K}_0$) and $Y=\mathrm{ECL}(X)$, it is expected that for every number field $\mathbb{K}\supset\mathbb{K}_0$ the set of $\mathbb{K}$-points in $X(\mathbb{K})\smallsetminus Y$ is finite, and that this property characterizes $\mathrm{ECL}(X)$ as the smallest algebraic subset Y of X that has the above property for all $\mathbb{K}$ ([Lan86]). This conjectural arithmetical statement would be a vast generalization of the Mordell-Faltings theorem, and is one of the strong motivations to study the geometric GGL conjecture as a first step.

0.3. Problem (generalized GGL conjecture). *Let (X,V) be a projective directed manifold. Find geometric conditions on V ensuring that all entire curves $f:(\mathbb{C},T_{\mathbb{C}})\to(X,V)$ are contained in a proper algebraic subvariety $Y\subsetneq X$. Does this hold when (X,V) is of general type, in the sense that the canonical sheaf K_V is big ?*

As above, we define the entire curve locus set of a pair (X,V) to be the smallest admissible algebraic set $Y\subset X$ in the above problem, i.e.

$$\mathrm{ECL}(X,V)=\overline{\bigcup_{f:(\mathbb{C},T_{\mathbb{C}})\to(X,V)} f(\mathbb{C})}^{\mathrm{Zar}}. \tag{0.4}$$

We say that (X,V) is *Brody hyperbolic* if $\mathrm{ECL}(X,V)=\emptyset$; as is well-known, this is equivalent to Kobayashi hyperbolicity whenever X is compact.

In case V has no singularities, the *canonical sheaf* K_V is defined to be $(\det\mathcal{O}(V))^*$ where $\mathcal{O}(V)$ is the sheaf of holomorphic sections of V, but in general this naive definition would not work. Take for instance a generic pencil of elliptic curves $\lambda P(z)+\mu Q(z)=0$ of degree 3 in $\mathbb{P}^2_{\mathbb{C}}$, and the linear space V consisting of the tangents to the fibers of the rational map $\mathbb{P}^2_{\mathbb{C}}\dashrightarrow\mathbb{P}^1_{\mathbb{C}}$ defined by $z\mapsto Q(z)/P(z)$. Then V is given by

$$0\longrightarrow\mathcal{O}(V)\longrightarrow\mathcal{O}(T_{\mathbb{P}^2_{\mathbb{C}}})\xrightarrow{PdQ-QdP}\mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(6)\otimes\mathcal{J}_S\longrightarrow 0$$

where $S=\mathrm{Sing}(V)$ consists of the 9 points $\{P(z)=0\}\cap\{Q(z)=0\}$, and $\mathcal{J}_S$ is the corresponding ideal sheaf of S. Since $\det\mathcal{O}(T_{\mathbb{P}^2})=\mathcal{O}(3)$, we see that $(\det(\mathcal{O}(V))^*=\mathcal{O}(3)$ is ample, thus Problem 0.3 would not have a positive answer (all leaves are elliptic or singular rational curves and thus covered by entire curves). An even more "degenerate" example is obtained with a generic pencil of conics, in which case $(\det(\mathcal{O}(V))^*=\mathcal{O}(1)$ and $\#S=4$.

If we want to get a positive answer to Problem 0.3, it is therefore indispensable to give a definition of K_V that incorporates in a suitable way the singularities of V ; this will be done in Def. 1.1 (see also Prop. 1.2). The goal is then to give a positive answer to Problem 0.3 under some possibly more restrictive conditions for the pair (X, V). These conditions will be expressed in terms of the tower of Semple jet bundles

$$(X_k, V_k) \to (X_{k-1}, V_{k-1}) \to \ldots \to (X_1, V_1) \to (X_0, V_0) := (X, V) \tag{0.5}$$

which we define more precisely in Section 1, following [Dem95]. It is constructed inductively by setting $X_k = P(V_{k-1})$ (projective bundle of *lines* of V_{k-1}), and all V_k have the same rank $r = \operatorname{rank} V$, so that $\dim X_k = n + k(r-1)$ where $n = \dim X$. If $\mathcal{O}_{X_k}(1)$ is the tautological line bundle over X_k associated with the projective structure and $\pi_{k,\ell} : X_k \to X_\ell$ is the natural projection from X_k to X_ℓ, $0 \le \ell \le k$, we define the k-stage Green-Griffiths locus of (X, V) to be

$$\mathrm{GG}_k(X, V) = \overline{(X_k \smallsetminus \Delta_k) \cap \bigcap_{m \in \mathbb{N}} \big(\text{base locus of } \mathcal{O}_{X_k}(m) \otimes \pi_{k,0}^* A^{-1}\big)} \tag{0.6}$$

where A is any ample line bundle on X and $\Delta_k = \bigcup_{2 \le \ell \le k} \pi_{k,\ell}^{-1}(D_\ell)$ is the union of "vertical divisors" (see section 1; the vertical divisors play no role and have to be removed in this context). Clearly, $\mathrm{GG}_k(X, V)$ does not depend on the choice of A. The basic vanishing theorem for entire curves (cf. [GG79], [SY96] and [Dem95]) asserts that for every entire curve $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$, then its k-jet $f_{[k]} : (\mathbb{C}, T_{\mathbb{C}}) \to (X_k, V_k)$ satisfies

$$f_{[k]}(\mathbb{C}) \subset \mathrm{GG}_k(X, V), \quad \text{hence} \quad f(\mathbb{C}) \subset \pi_{k,0}\left(\mathrm{GG}_k(X, V)\right). \tag{0.7}$$

(For this, one uses the fact that $f_{[k]}(\mathbb{C})$ is not contained in any component of Δ_k, cf. [Dem95]). It is therefore natural to define the global Green-Griffiths locus of (X, V) to be

$$\mathrm{GG}(X, V) = \bigcap_{k \in \mathbb{N}} \pi_{k,0}\left(\mathrm{GG}_k(X, V)\right). \tag{0.8}$$

By (0.7) we infer that

$$\mathrm{ECL}(X, V) \subset \mathrm{GG}(X, V). \tag{0.9}$$

The main result of [Dem11] (Theorem 2.37 and Cor. 3.4) implies the following useful information:

0.10. Theorem. *Assume that (X, V) is of "general type", i.e. that the canonical sheaf K_V is big on X. Then there exists an integer k_0 such that $\mathrm{GG}_k(X, V)$ is a proper algebraic subset of X_k for $k \ge k_0$ [though $\pi_{k,0}(\mathrm{GG}_k(X, V))$ might still be equal to X for all k].*

In fact, if F is an invertible sheaf on X such that $K_V \otimes F$ is big, the probabilistic estimates of [Dem11, Cor. 2.38 and Cor. 3.4] produce sections of

$$\mathcal{O}_{X_k}(m) \otimes \pi_{k,0}^* \mathcal{O}\Big(\frac{m}{kr}\Big(1 + \frac{1}{2} + \ldots + \frac{1}{k}\Big) F\Big) \tag{0.11}$$

for $m \gg k \gg 1$. The (long and involved) proof uses a curvature computation and singular holomorphic Morse inequalities to show that the line bundles involved in (0.11) are big on X_k for $k \gg 1$. One applies this to $F = A^{-1}$ with A ample on X to produce sections and conclude that $\mathrm{GG}_k(X, V) \subsetneq X_k$.

Thanks to (0.9), the GGL conjecture is satisfied whenever $\mathrm{GG}(X, V) \subsetneq X$. By [DMR10], this happens for instance in the absolute case when X is a generic hypersurface of degree $d \ge 2^{n^5}$ in $\mathbb{P}^{n+1}$ (see also [Pau08], e.g. for better bounds in low dimensions). However, as already mentioned in [Lan86], very simple examples show that one can have $\mathrm{GG}(X, V) = X$

even when (X,V) is of general type, and this already occurs in the absolute case as soon as $\dim X \geq 2$. A typical example is a product of directed manifolds

$$(X,V) = (X',V') \times (X'',V''), \qquad V = \mathrm{pr}'^* V' \oplus \mathrm{pr}''^* V''. \tag{0.12}$$

The absolute case $V = T_X$, $V' = T_{X'}$, $V'' = T_{X''}$ on a product of curves is the simplest instance. It is then easy to check that $\mathrm{GG}(X,V) = X$, cf. (3.2). Diverio and Rousseau [DR13] have given many more such examples, including the case of indecomposable varieties (X, T_X), e.g. Hilbert modular surfaces, or more generally compact quotients of bounded symmetric domains of rank ≥ 2. The problem here is the failure of some sort of stability condition that is introduced in Section 3. This leads to a somewhat technical concept of more manageable directed pairs (X,V) that we call *strongly of general type*, see Def. 3.1. Our main result can be stated

0.13. Theorem (partial solution to the generalized GGL conjecture). *Let (X,V) be a directed pair that is strongly of general type. Then the Green-Griffiths-Lang conjecture holds true for (X,V), namely $\mathrm{ECL}(X,V)$ is a proper algebraic subvariety of X.*

The proof proceeds through a complicated induction on $n = \dim X$ and $k = \mathrm{rank}\, V$, which is the main reason why we have to introduce directed varieties, even in the absolute case. An interesting feature of this result is that the conclusion on $\mathrm{ECL}(X,V)$ is reached without having to know anything about the Green-Griffiths locus $\mathrm{GG}(X,V)$, even a posteriori. Nevetherless, this is not yet enough to confirm the GGL conjecture. Our hope is that pairs (X,V) that are of general type without being strongly of general type – and thus exhibit some sort of "jet-instability" – can be investigated by different methods, e.g. by the diophantine approximation techniques of McQuillan [McQ98]. However, Theorem 0.13 is strong enough to imply the Kobayashi conjecture on generic hyperbolicity, thanks to the following concept of algebraic jet-hyperbolicity.

0.14. Definition. *A directed variety (X,V) will be said to be algebraically jet-hyperbolic if the induced directed variety structure (Z,W) on every irreducible algebraic variety Z of X such that $\mathrm{rank}\, W \geq 1$ has a desingularization that is strongly of general type* [*see Sections* 2 *and* 4 *for the definition of induced directed structures and further details*]. *We also say that a projective manifold X is algebraically jet-hyperbolic if (X,T_X) is.*

In this context, Theorem 0.13 yields the following connection between algebraic jet-hyperbolicity and the analytic concept of Kobayashi hyperbolicity.

0.15. Theorem. *Let (X,V) be a directed variety structure on a projective manifold X. Assume that (X,V) is algebraically jet-hyperbolic. Then (X,V) is Kobayashi hyperbolic.*

This strong link is useful to deal with generic hyperbolicity, i.e. the hyperbolicity of very general fibers in a deformation $\pi : \mathcal{X} \to S$ (by "very general fiber", we mean here a fiber $X_t = \pi^{-1}(t)$ where t is taken in a complement $S \smallsetminus \bigcup S_\nu$ of a countable union of algebraic subsets $S_\nu \subsetneq S$). In Section 5, we apply the above results to analyze the "relative" Semple tower of $(\mathcal{X}, T_{\mathcal{X}/S})$ versus the "absolute" one derived from $(\mathcal{X}, T_{\mathcal{X}})$, and obtain in this way:

0.16. Theorem. *Let $\pi : \mathcal{X} \to S$ be a deformation of complex projective nonsingular varieties $X_t = \pi^{-1}(t)$ over a smooth irreducible quasi-projective base S. Let $n = \dim X_t$ be the relative dimension and let $N = \dim S$. Assume that for all $q = N+1, \ldots, N+n$, the exterior power $\Lambda^q T^*_{\mathcal{X}}$ is a relatively ample vector bundle over S. Then the very general fiber X_t is algebraically jet-hyperbolic, and thus Kobayashi hyperbolic.*

In the special case of the universal family of complete intersections of codimension c and type $(d_1,\ldots,d_c)$ in complex projective $\mathbb{P}^{n+c}$, Ein [Ein88, 91] and Voisin [Voi96] have exploited in a crucial way the existence of global twisted vector fields, e.g. sections of $T_{\mathcal{X}}\otimes\mathcal{O}(1)$, on the total space $\mathcal{X}$ over S. This idea combined with Theorem 0.16 yields the following result.

0.17. Corollary (confirmation of the Kobayashi conjecture). *If $\sum d_j\ge 2n+c+1$, the very general complete intersection of type $(d_1,\ldots,d_c)$ in complex projective space $\mathbb{P}^{n+c}$ is Kobayashi hyperbolic.*

The border case $\sum d_j=2n+c$ is potentially also accessible by the techniques developed here, but further calculations would be needed to check the possible degenerations of the morphisms induced by twisted vector fields. I would like to thank Simone Diverio and Erwan Rousseau for very stimulating discussions on these questions. I am grateful to Mihai Păun for an invitation at KIAS (Seoul) in August 2014, during which further very fruitful exchanges took place, and for his extremely careful reading of earlier drafts of the manuscript.

1. Semple jet bundles and associated canonical sheaves

Let (X,V) be a directed projective manifold and $r=\operatorname{rank}V$, that is, the dimension of generic fibers. Then V is actually a holomorphic subbundle of T_X on the complement $X\smallsetminus\operatorname{Sing}(V)$ of a certain minimal analytic set $\operatorname{Sing}(V)\subsetneq X$ of codimension ≥ 2, called hereafter the singular set of V. If $\mu:\widehat{X}\to X$ is a proper modification (a composition of blow-ups with smooth centers, say), we get a directed manifold $(\widehat{X},\widehat{V})$ by taking $\widehat{V}$ to be the closure of $\mu_*^{-1}(V')$, where $V'=V_{|X'}$ is the restriction of V over a Zariski open set $X'\subset X\smallsetminus\operatorname{Sing}(V)$ such that $\mu:\mu^{-1}(X')\to X'$ is a biholomorphism. We will be interested in taking modifications realized by iterated blow-ups of certain nonsingular subvarieties of the singular set $\operatorname{Sing}(V)$, so as to eventually "improve" the singularities of V; outside of $\operatorname{Sing}(V)$ the effect of blowing-up will be irrelevant, as one can see easily. Following [Dem11], the canonical sheaf K_V is defined as follows.

1.1. Definition. *For any directed pair (X,V) with X nonsingular, we define K_V to be the rank 1 analytic sheaf such that*

$$K_V(U)=\textit{sheaf of locally bounded sections of }\mathcal{O}_X(\Lambda^rV'^*)(U\cap X')$$

where $r=\mathrm{rank}(V)$, $X'=X\smallsetminus\operatorname{Sing}(V)$, $V'=V_{|X'}$, and "bounded" means bounded with respect to a smooth hermitian metric h on T_X.

For $r=0$, one can set $K_V=\mathcal{O}_X$, but this case is trivial: clearly $\operatorname{ECL}(X,V)=\emptyset$. The above definition of K_V may look like an analytic one, but it can easily be turned into an equivalent algebraic definition:

1.2. Proposition. *Consider the natural morphism $\mathcal{O}(\Lambda^rT_X^*)\to\mathcal{O}(\Lambda^rV^*)$ where $r=\operatorname{rank}V$ $[\mathcal{O}(\Lambda^rV^*)$ being defined here as the quotient of $\mathcal{O}(\Lambda^rT_X^*)$ by r-forms that have zero restrictions to $\mathcal{O}(\Lambda^rV^*)$ on $X\smallsetminus\operatorname{Sing}(V)\,]$. The bidual $\mathcal{L}_V=\mathcal{O}_X(\Lambda^rV^*)^{**}$ is an invertible sheaf, and our natural morphism can be written*

$$\mathcal{O}(\Lambda^rT_X^*)\to\mathcal{O}(\Lambda^rV^*)=\mathcal{L}_V\otimes\mathcal{J}_V\subset\mathcal{L}_V \tag{1.2.1}$$

where $\mathcal{J}_V$ is a certain ideal sheaf of $\mathcal{O}_X$ whose zero set is contained in $\operatorname{Sing}(V)$ and the arrow on the left is surjective by definition. Then

$$K_V=\mathcal{L}_V\otimes\overline{\mathcal{J}}_V \tag{1.2.2}$$

where $\overline{\mathcal{J}}_V$ is the integral closure of $\mathcal{J}_V$ in $\mathcal{O}_X$. In particular, K_V is always a coherent sheaf.

Proof. Let (u_k) be a set of generators of $\mathcal{O}(\Lambda^r V^*)$ obtained (say) as the images of a basis $(dz_I)_{|I|=r}$ of $\Lambda^r T_X^*$ in some local coordinates near a point $x \in X$. Write $u_k = g_k \ell$ where ℓ is a local generator of $\mathcal{L}_V$ at x. Then $\mathcal{J}_V = (g_k)$ by definition. The boundedness condition expressed in Def. 1.1 means that we take sections of the form $f\ell$ where f is a holomorphic function on $U \cap X'$ (and U a neighborhood of x), such that

$$|f| \le C \sum |g_k| \tag{1.2.3}$$

for some constant $C > 0$. But then f extends holomorphically to U into a function that lies in the integral closure $\overline{\mathcal{J}}_V$, and the latter is actually characterized analytically by condition (1.2.3). This proves Prop. 1.2. □

By blowing-up $\mathcal{J}_V$ and taking a desingularization $\widehat{X}$, one can always find a *log-resolution* of $\mathcal{J}_V$ (or K_V), i.e. a modification $\mu : \widehat{X} \to X$ such that $\mu^*\mathcal{J}_V \subset \mathcal{O}_{\widehat{X}}$ is an invertible ideal sheaf (hence integrally closed); it follows that $\mu^*\overline{\mathcal{J}}_V = \mu^*\mathcal{J}_V$ and $\mu^* K_V = \mu^*\mathcal{L}_V \otimes \mu^*\mathcal{J}_V$ are invertible sheaves on $\widehat{X}$. Notice that for any modification $\mu' : (X', V') \to (X, V)$, there is always a well defined natural morphism

$$\mu'^* K_V \to K_{V'} \tag{1.3}$$

(though it need not be an isomorphism, and $K_{V'}$ is possibly non invertible even when μ' is taken to be a log-resolution of K_V). Indeed $(\mu')_* = d\mu' : V' \to \mu^* V$ is continuous with respect to ambient hermitian metrics on X and X', and going to the duals reverses the arrows while preserving boundedness with respect to the metrics. If $\mu'' : X'' \to X'$ provides a simultaneous log-resolution of $K_{V'}$ and $\mu'^* K_V$, we get a non trivial morphism of invertible sheaves

$$(\mu' \circ \mu'')^* K_V = \mu''^* \mu'^* K_V \longrightarrow \mu''^* K_{V'}, \tag{1.4}$$

hence the bigness of $\mu'^* K_V$ with imply that of $\mu''^* K_{V'}$. This is a general principle that we would like to refer to as the "monotonicity principle" for canonical sheaves: one always get more sections by going to a higher level through a (holomorphic) modification.

1.5. Definition. *We say that the rank* 1 *sheaf K_V is "big" if the invertible sheaf $\mu^* K_V$ is big in the usual sense for any log resolution $\mu : \widehat{X} \to X$ of K_V. Finally, we say that (X, V) is of general type if there exists a modification $\mu' : (X', V') \to (X, V)$ such that $K_{V'}$ is big ; any higher blow-up $\mu'' : (X'', V'') \to (X', V')$ then also yields a big canonical sheaf by* (1.3).

Clearly, "general type" is a birationally (or bimeromorphically) invariant concept, by the very definition. When $\dim X = n$ and $V \subset T_X$ is a subbundle of rank $r \ge 1$, one constructs a tower of "Semple k-jet bundles" $\pi_{k,k-1} : (X_k, V_k) \to (X_{k-1}, V_{k-1})$ that are $\mathbb{P}^{r-1}$-bundles, with $\dim X_k = n + k(r-1)$ and $\mathrm{rank}(V_k) = r$. For this, we take $(X_0, V_0) = (X, V)$, and for every $k \ge 1$, we set inductively $X_k := P(V_{k-1})$ and

$$V_k := (\pi_{k,k-1})_*^{-1} \mathcal{O}_{X_k}(-1) \subset T_{X_k},$$

where $\mathcal{O}_{X_k}(1)$ is the tautological line bundle on X_k, $\pi_{k,k-1} : X_k = P(V_{k-1}) \to X_{k-1}$ the natural projection and $(\pi_{k,k-1})_* = d\pi_{k,k-1} : T_{X_k} \to \pi_{k,k-1}^* T_{X_{k-1}}$ its differential (cf. [Dem95]). In other terms, we have exact sequences

$$0 \longrightarrow T_{X_k/X_{k-1}} \longrightarrow V_k \xrightarrow{(\pi_{k,k-1})_*} \mathcal{O}_{X_k}(-1) \longrightarrow 0, \tag{1.6}$$

$$0 \longrightarrow \mathcal{O}_{X_k} \longrightarrow (\pi_{k,k-1})^* V_{k-1} \otimes \mathcal{O}_{X_k}(1) \longrightarrow T_{X_k/X_{k-1}} \longrightarrow 0, \tag{1.7}$$

where the last line is the Euler exact sequence associated with the relative tangent bundle of $P(V_{k-1}) \to X_{k-1}$. Notice that we by definition of the tautological line bundle we have

$$\mathcal{O}_{X_k}(-1) \subset \pi_{k,k-1}^* V_{k-1} \subset \pi_{k,k-1}^* T_{X_{k-1}},$$

and also $\mathrm{rank}(V_k) = r$. Let us recall also that for $k \geq 2$, there are "vertical divisors" $D_k = P(T_{X_{k-1}/X_{k-2}}) \subset P(V_{k-1}) = X_k$, and that D_k is the zero divisor of the section of $\mathcal{O}_{X_k}(1) \otimes \pi_{k,k-1}^* \mathcal{O}_{X_{k-1}}(-1)$ induced by the second arrow of the first exact sequence (1.6), when k is replaced by $k-1$. This yields in particular

$$\mathcal{O}_{X_k}(1) = \pi_{k,k-1}^* \mathcal{O}_{X_{k-1}}(1) \otimes \mathcal{O}(D_k). \tag{1.8}$$

By composing the projections we get for all pairs of indices $0 \leq j \leq k$ natural morphisms

$$\pi_{k,j} : X_k \to X_j, \quad (\pi_{k,j})_* = (d\pi_{k,j})_{|V_k} : V_k \to (\pi_{k,j})^* V_j,$$

and for every k-tuple $\mathbf{a} = (a_1, \ldots, a_k) \in \mathbb{Z}^k$ we define

$$\mathcal{O}_{X_k}(\mathbf{a}) = \bigotimes_{1 \leq j \leq k} \pi_{k,j}^* \mathcal{O}_{X_j}(a_j), \quad \pi_{k,j} : X_k \to X_j.$$

We extend this definition to all weights $\mathbf{a} \in \mathbb{Q}^k$ to get a $\mathbb{Q}$-line bundle in $\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. Now, Formula (1.8) yields

$$\mathcal{O}_{X_k}(\mathbf{a}) = \mathcal{O}_{X_k}(m) \otimes \mathcal{O}(-\mathbf{b} \cdot D) \quad \text{where } m = |\mathbf{a}| = \textstyle\sum a_j,\ \mathbf{b} = (0, b_2, \ldots, b_k) \tag{1.9}$$

and $b_j = a_1 + \ldots + a_{j-1}$, $2 \leq j \leq k$.

When $\mathrm{Sing}(V) \neq \emptyset$, one can always define X_k and V_k to be the respective closures of X'_k, V'_k associated with $X' = X \smallsetminus \mathrm{Sing}(V)$ and $V' = V_{|X'}$, where the closure is taken in the nonsingular "absolute" Semple tower (X^a_k, V^a_k) obtained from $(X^a_0, V^a_0) = (X, T_X)$. We leave the reader check the following easy (but important) observation.

1.10. Fonctoriality. *If $\Phi : (X, V) \to (Y, W)$ is a morphism of directed varieties such that $\Phi_* : T_X \to \Phi^* T_Y$ is injective (i.e. Φ is an immersion), then there is a corresponding natural morphism $\Phi_{[k]} : (X_k, V_k) \to (Y_k, W_k)$ at the level of Semple bundles. If one merely assumes that the differential $\Phi_* : V \to \Phi^* W$ is non zero, there is still a well defined meromorphic map $\Phi_{[k]} : (X_k, V_k) \dashrightarrow (Y_k, W_k)$ for all $k \geq 0$.*

In case V is singular, the k-th Semple bundle X_k will also be singular, but we can still replace (X_k, V_k) by a suitable modification $(\widehat{X}_k, \widehat{V}_k)$ if we want to work with a nonsingular model $\widehat{X}_k$ of X_k. The exceptional set of $\widehat{X}_k$ over X_k can be chosen to lie above $\mathrm{Sing}(V) \subset X$, and proceeding inductively with respect to k, we can also arrange the modifications in such a way that we get a tower structure $(\widehat{X}_{k+1}, \widehat{V}_{k+1}) \to (\widehat{X}_k, \widehat{V}_k)$; however, in general, it will not be possible to achieve that $\widehat{V}_k$ is a subbundle of $T_{\widehat{X}_k}$.

It is not true that $K_{\widehat{V}_k}$ is big in case (X, V) is of general type (especially since the fibers of $X_k \to X$ are towers of $\mathbb{P}^{r-1}$ bundles, and the canonical bundles of projective spaces are always negative !). However, a twisted version holds true, that can be seen as another instance of the "monotonicity principle" when going to higher stages in the Semple tower.

1.11. Lemma. *If (X, V) is of general type, then there is a modification $(\widehat{X}, \widehat{V})$ such that all pairs $(\widehat{X}_k, \widehat{V}_k)$ of the associated Semple tower have a twisted canonical bundle $K_{\widehat{V}_k} \otimes \mathcal{O}_{\widehat{X}_k}(p)$ that is still big when one multiplies $K_{\widehat{V}_k}$ by a suitable $\mathbb{Q}$-line bundle $\mathcal{O}_{\widehat{X}_k}(p)$, $p \in \mathbb{Q}_+$.*

Proof. First assume that V has no singularities. The exact sequences (1.6) and (1.7) provide

$$K_{V_k} := \det V_k^* = \det(T^*_{X_k/X_{k-1}}) \otimes \mathcal{O}_{X_k}(1) = \pi_{k,k-1}^* K_{V_{k-1}} \otimes \mathcal{O}_{X_k}(-(r-1))$$

where $r = \mathrm{rank}(V)$. Inductively we get

$$K_{V_k} = \pi_{k,0}^* K_V \otimes \mathcal{O}_{X_k}(-(r-1)\mathbf{1}), \qquad \mathbf{1} = (1, ..., 1) \in \mathbb{N}^k. \tag{1.11.1}$$

We know by [Dem95] that $\mathcal{O}_{X_k}(\mathbf{c})$ is relatively ample over X when we take the special weight $\mathbf{c} = (2\,3^{k-2}, ..., 2\,3^{k-j-1}, ..., 6, 2, 1)$, hence

$$K_{V_k} \otimes \mathcal{O}_{X_k}((r-1)\mathbf{1} + \varepsilon\mathbf{c}) = \pi_{k,0}^* K_V \otimes \mathcal{O}_{X_k}(\varepsilon\mathbf{c})$$

is big over X_k for any sufficiently small positive rational number $\varepsilon \in \mathbb{Q}_+^*$. Thanks to Formula (1.9), we can in fact replace the weight $(r-1)\mathbf{1} + \varepsilon\mathbf{c}$ by its total degree $p = (r-1)k + \varepsilon|\mathbf{c}| \in \mathbb{Q}_+$. The general case of a singular linear space follows by considering suitable "sufficiently high" modifications $\widehat{X}$ of X, the related directed structure $\widehat{V}$ on $\widehat{X}$, and embedding $(\widehat{X}_k, \widehat{V}_k)$ in the absolute Semple tower $(\widehat{X}_k^a, \widehat{V}_k^a)$ of $\widehat{X}$. We still have a well defined morphism of rank 1 sheaves

$$\pi_{k,0}^* K_{\widehat{V}} \otimes \mathcal{O}_{\widehat{X}_k}(-(r-1)\mathbf{1}) \to K_{\widehat{V}_k} \tag{1.11.2}$$

because the multiplier ideal sheaves involved at each stage behave according to the monotonicity principle applied to the projections $\pi_{k,k-1}^a : \widehat{X}_k^a \to \widehat{X}_{k-1}^a$ and their differentials $(\pi_{k,k-1}^a)_*$, which yield well-defined transposed morphisms from the $(k-1)$-st stage to the k-th stage at the level of exterior differential forms. Our contention follows. □

2. Induced directed structure on a subvariety of a jet space

Let Z be an irreducible algebraic subset of some k-jet bundle X_k over X, $k \geq 0$. We define the linear subspace $W \subset T_Z \subset T_{X_k|Z}$ to be the closure

$$W := \overline{T_{Z'} \cap V_k} \tag{2.1}$$

taken on a suitable Zariski open set $Z' \subset Z_{\mathrm{reg}}$ where the intersection $T_{Z'} \cap V_k$ has constant rank and is a subbundle of $T_{Z'}$. Alternatively, we could also take W to be the closure of $T_{Z'} \cap V_k$ in the k-th stage (X_k^a, V_k^a) of the absolute Semple tower. We say that (Z, W) is the *induced* directed variety structure. In the sequel, we always consider such a subvariety Z of X_k as a directed pair (Z, W) by taking the induced structure described above. Let us first quote the following easy observation.

2.2. Observation. *For $k \geq 1$, let $Z \subsetneq X_k$ be an irreducible algebraic subset that projects onto X_{k-1}, i.e. $\pi_{k,k-1}(Z) = X_{k-1}$. Then the induced directed variety $(Z, W) \subset (X_k, V_k)$, satisfies*

$$1 \leq \mathrm{rank}\, W < r := \mathrm{rank}(V_k).$$

Proof. Take a Zariski open subset $Z' \subset Z_{\mathrm{reg}}$ such that $W' = T_{Z'} \cap V_k$ is a vector bundle over Z'. Since $X_k \to X_{k-1}$ is a $\mathbb{P}^{r-1}$-bundle, Z has codimension at most $r-1$ in X_k. Therefore $\mathrm{rank}\, W \geq \mathrm{rank}\, V_k - (r-1) \geq 1$. On the other hand, if we had $\mathrm{rank}\, W = \mathrm{rank}\, V_k$ generically, then $T_{Z'}$ would contain $V_{k|Z'}$, in particular it would contain all vertical directions $T_{X_k/X_{k-1}} \subset V_k$ that are tangent to the fibers of $X_k \to X_{k-1}$. By taking the flow along vertical vector fields, we would conclude that Z' is a union of fibers of $X_k \to X_{k-1}$ up to an algebraic set of smaller dimension, but this is excluded since Z projects onto X_{k-1} and $Z \subsetneq X_k$. □

2.3. Definition. *For $k \geq 1$, let $Z \subset X_k$ be an irreducible algebraic subset of X_k that projects onto X_{k-1}. We assume moreover that $Z \not\subset D_k = P(T_{X_{k-1}/X_{k-2}})$ (and put here $D_1 = \emptyset$ in what follows to avoid to have to single out the case $k = 1$). In this situation we say that (Z, W) is of general type modulo $X_k \to X$ if there exists $p \in \mathbb{Q}_+$ such that $K_W \otimes \mathcal{O}_{X_k}(p)_{|Z}$*

is big over Z, possibly after replacing Z by a suitable nonsingular model $\widehat{Z}$ (and pulling-back W and $\mathcal{O}_{X_k}(p)_{|Z}$ to the nonsingular variety $\widehat{Z}$).

The main result of [Dem11] mentioned in the introduction as Theorem 0.10 implies the following important "induction step".

2.4. Proposition. *Let (X,V) be a directed pair where X is projective algebraic. Take an irreducible algebraic subset $Z \not\subset D_k$ of the associated k-jet Semple bundle X_k that projects onto X_{k-1}, $k \geq 1$, and assume that the induced directed space $(Z,W) \subset (X_k,V_k)$ is of general type modulo $X_k \to X$. Then there exists a divisor $\Sigma \subset Z_\ell$ in a sufficiently high stage of the Semple tower (Z_ℓ, W_ℓ) associated with (Z,W), such that every non constant holomorphic map $f : \mathbb{C} \to X$ tangent to V that satisfies $f_{[k]}(\mathbb{C}) \subset Z$ also satisfies $f_{[k+\ell]}(\mathbb{C}) \subset \Sigma$.*

Proof. Let $E \subset Z$ be a divisor containing $Z_{\rm sing} \cup (Z \cap \pi_{k,0}^{-1}(\mathrm{Sing}(V)))$, chosen so that on the nonsingular Zariski open set $Z' = Z \smallsetminus E$ all linear spaces $T_{Z'}$, $V_{k|Z'}$ and $W' = T_{Z'} \cap V_k$ are subbundles of $T_{X_k|Z'}$, the first two having a transverse intersection on Z'. By taking closures over Z' in the absolute Semple tower of X, we get (singular) directed pairs $(Z_\ell, W_\ell) \subset (X_{k+\ell}, V_{k+\ell})$, which we eventually resolve into $(\widehat{Z}_\ell, \widehat{W}_\ell) \subset (\widehat{X}_{k+\ell}, \widehat{V}_{k+\ell})$ over nonsingular bases. By construction, locally bounded sections of $\mathcal{O}_{\widehat{X}_{k+\ell}}(m)$ restrict to locally bounded sections of $\mathcal{O}_{\widehat{Z}_\ell}(m)$ over $\widehat{Z}_\ell$.

Since Theorem 0.10 and the related estimate (0.11) are universal in the category of directed varieties, we can apply them by replacing X with $\widehat{Z} \subset \widehat{X}_k$, the order k by a new index ℓ, and F by

$$F_k = \mu^*\Big(\big(\mathcal{O}_{X_k}(p) \otimes \pi_{k,0}^*\mathcal{O}_X(-\varepsilon A)\big)_{|Z}\Big)$$

where $\mu : \widehat{Z} \to Z$ is the desingularization, $p \in \mathbb{Q}_+$ is chosen such that $K_W \otimes \mathcal{O}_{x_k}(p)_{|Z}$ is big, A is an ample bundle on X and $\varepsilon \in \mathbb{Q}_+^*$is small enough. The assumptions show that $K_{\widehat{W}} \otimes F_k$ is big on $\widehat{Z}$, therefore, by applying our theorem and taking $m \gg \ell \gg 1$, we get in fine a large number of (metric bounded) sections of

$$\begin{aligned}\mathcal{O}_{\widehat{Z}_\ell}(m) &\otimes \widehat{\pi}_{k+\ell,k}^*\mathcal{O}\Big(\frac{m}{\ell r'}\Big(1+\frac{1}{2}+\ldots+\frac{1}{\ell}\Big)F_k\Big)\\ &= \mathcal{O}_{\widehat{X}_{k+\ell}}(m\mathbf{a}') \otimes \widehat{\pi}_{k+\ell,0}^*\mathcal{O}\Big(-\frac{m\varepsilon}{kr}\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)A\Big)_{|\widehat{Z}_\ell}\end{aligned}$$

where $\mathbf{a}' \in \mathbb{Q}_+^{k+\ell}$ is a positive weight (of the form $(0,\ldots,\lambda,\ldots,0,1)$ with some non zero component $\lambda \in \mathbb{Q}_+$ at index k). These sections descend to metric bounded sections of

$$\mathcal{O}_{X_{k+\ell}}((1+\lambda)m) \otimes \widehat{\pi}_{k+\ell,0}^*\mathcal{O}\Big(-\frac{m\varepsilon}{kr}\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)A\Big)_{|Z_\ell}.$$

Since A is ample on X, we can apply the fundamental vanishing theorem (see e.g. [Dem97] or [Dem11], Statement 8.15), or rather an "embedded" version for curves satisfying $f_{[k]}(\mathbb{C}) \subset Z$, proved exactly by the same arguments. The vanishing theorem implies that the divisor Σ of any such section satisfies the conclusions of Proposition 2.4, possibly modulo exceptional divisors of $\widehat{Z} \to Z$; to take care of these, it is enough to add to Σ the inverse image of the divisor $E = Z \smallsetminus Z'$ initially selected. □

3. Strong general type condition for directed manifolds

Our main result is the following partial solution to the Green-Griffiths-Lang conjecture, providing a sufficient algebraic condition for the analytic conclusion to hold true. We first give an ad hoc definition.

3.1. Definition. *Let (X,V) be a directed pair where X is projective algebraic. We say that that (X,V) is "strongly of general type" if it is of general type and for every irreducible algebraic set $Z \subsetneq X_k$, $Z \not\subset D_k$, that projects onto X_{k-1}, $k \geq 1$, the induced directed structure $(Z,W) \subset (X_k,V_k)$ is of general type modulo $X_k \to X$.*

3.2. Example. The situation of a product $(X,V) = (X',V') \times (X'',V'')$ described in (0.12) shows that (X,V) can be of general type without being strongly of general type. In fact, if (X',V') and (X'',V'') are of general type, then $K_V = \mathrm{pr}'^* K_{V'} \otimes \mathrm{pr}''^* K_{V''}$ is big, so (X,V) is again of general type. However

$$Z = P(\mathrm{pr}'^* V') = X'_1 \times X'' \subset X_1$$

has a directed structure $W = \mathrm{pr}'^* V'_1$ which does not possess a big canonical bundle over Z, since the restriction of K_W to any fiber $\{x'\} \times X''$ is trivial. The higher stages (Z_k,W_k) of the Semple tower of (Z,W) are given by $Z_k = X'_{k+1} \times X''$ and $W_k = \mathrm{pr}'^* V'_{k+1}$, so it is easy to see that $\mathrm{GG}_k(X,V)$ contains Z_{k-1}. Since Z_k projects onto X, we have here $\mathrm{GG}(X,V) = X$ (see [DR13] for more sophisticated indecomposable examples).

3.3. Remark. It follows from Definition 2.3 that $(Z,W) \subset (X_k,V_k)$ is automatically of general type modulo $X_k \to X$ if $\mathcal{O}_{X_k}(1)_{|Z}$ is big. Notice further that

$$\mathcal{O}_{X_k}(1+\varepsilon)_{|Z} = \big(\mathcal{O}_{X_k}(\varepsilon) \otimes \pi^*_{k,k-1}\mathcal{O}_{X_{k-1}}(1) \otimes \mathcal{O}(D_k)\big)_{|Z}$$

where $\mathcal{O}(D_k)_{|Z}$ is effective and $\mathcal{O}_{X_k}(1)$ is relatively ample with respect to the projection $X_k \to X_{k-1}$. Therefore the bigness of $\mathcal{O}_{X_{k-1}}(1)$ on X_{k-1} also implies that every directed subvariety $(Z,W) \subset (X_k,V_k)$ is of general type modulo $X_k \to X$. If (X,V) is of general type, we know by the main result of [Dem11] that $\mathcal{O}_{X_k}(1)$ is big for $k \geq k_0$ large enough, and actually the precise estimates obtained therein give explicit bounds for such a k_0. The above observations show that we need to check the condition of Definition 3.1 only for $Z \subset X_k$, $k \leq k_0$. Moreover, at least in the case where V, Z, and $W = T_Z \cap V_k$ are nonsingular, we have

$$K_W \simeq K_Z \otimes \det(T_Z/W) \simeq K_Z \otimes \det(T_{X_k}/V_k)_{|Z} \simeq K_{Z/X_{k-1}} \otimes \mathcal{O}_{X_k}(1)_{|Z}.$$

Thus we see that, in some sense, it is only needed to check the bigness of K_W modulo $X_k \to X$ for "rather special subvarieties" $Z \subset X_k$ over X_{k-1}, such that $K_{Z/X_{k-1}}$ is not relatively big over X_{k-1}. □

3.4. Hypersurface case. Assume that $Z \neq D_k$ is an irreducible hypersurface of X_k that projects onto X_{k-1}. To simplify things further, also assume that V is nonsingular. Since the Semple jet-bundles X_k form a tower of $\mathbb{P}^{r-1}$-bundles, their Picard groups satisfy $\mathrm{Pic}(X_k) \simeq \mathrm{Pic}(X) \oplus \mathbb{Z}^k$ and we have $\mathcal{O}_{X_k}(Z) \simeq \mathcal{O}_{X_k}(\mathbf{a}) \otimes \pi^*_{k,0}B$ for some $\mathbf{a} \in \mathbb{Z}^k$ and $B \in \mathrm{Pic}(X)$, where $a_k = d > 0$ is the relative degree of the hypersurface over X_{k-1}. Let $\sigma \in H^0(X_k, \mathcal{O}_{X_k}(Z))$ be the section defining Z in X_k. The induced directed variety (Z,W) has $\operatorname{rank} W = r-1 = \operatorname{rank} V - 1$ and formula (1.12) yields $K_{V_k} = \mathcal{O}_{X_k}(-(r-1)\mathbf{1}) \otimes \pi^*_{k,0}(K_V)$.

We claim that

$$K_W \supset \big(K_{V_k} \otimes \mathcal{O}_{X_k}(Z)\big)_{|Z} \otimes \mathcal{J}_S = \big(\mathcal{O}_{X_k}(\mathbf{a} - (r-1)\mathbf{1}) \otimes \pi_{k,0}^*(B \otimes K_V)\big)_{|Z} \otimes \mathcal{J}_S \tag{3.4.1}$$

where $S \subsetneq Z$ is the set (containing $Z_{\rm sing}$) where σ and $d\sigma_{|V_k}$ both vanish, and $\mathcal{J}_S$ is the ideal locally generated by the coefficients of $d\sigma_{|V_k}$ along $Z = \sigma^{-1}(0)$. In fact, the intersection $W = T_Z \cap V_k$ is transverse on $Z \smallsetminus S$; then (3.4.1) can be seen by looking at the morphism

$$V_{k|Z} \xrightarrow{d\sigma_{|V_k}} \mathcal{O}_{X_k}(Z)_{|Z},$$

and observing that the contraction by $K_{V_k} = \Lambda^r V_k^*$ provides a metric bounded section of the canonical sheaf K_W. In order to investigate the positivity properties of K_W, one has to show that B cannot be too negative, and in addition to control the singularity set S. The second point is a priori very challenging, but we get useful information for the first point by observing that σ provides a morphism $\pi_{k,0}^*\mathcal{O}_X(-B) \to \mathcal{O}_{X_k}(\mathbf{a})$, hence a nontrivial morphism

$$\mathcal{O}_X(-B) \to E_{\mathbf{a}} := (\pi_{k,0})_*\mathcal{O}_{X_k}(\mathbf{a})$$

By [Dem95, Section 12], there exists a filtration on $E_{\mathbf{a}}$ such that the graded pieces are irreducible representations of $\mathrm{GL}(V)$ contained in $(V^*)^{\otimes \ell}$, $\ell \le |\mathbf{a}|$. Therefore we get a nontrivial morphism

$$\mathcal{O}_X(-B) \to (V^*)^{\otimes \ell}, \qquad \ell \le |\mathbf{a}|. \tag{3.4.2}$$

If we know about certain (semi-)stability properties of V, this can be used to control the negativity of B. □

We further need the following useful concept that generalizes entire curve loci.

3.5. Definition. *If Z is an algebraic set contained in some stage X_k of the Semple tower of (X, V), we define its "induced entire curve locus" $\mathrm{IEL}_{X,V}(Z) \subset Z$ to be the Zariski closure of the union $\bigcup f_{[k]}(\mathbb{C})$ of all jets of entire curves $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$ such that $f_{[k]}(\mathbb{C}) \subset Z$.*

We have of course $\mathrm{IEL}_{X,V}(\mathrm{IEL}_{X,V}(Z)) = \mathrm{IEL}_{X,V}(Z)$ by definition. It is not hard to check that modulo certain "vertical divisors" of X_k, the $\mathrm{IEL}_{X,V}(Z)$ locus is essentially the same as the entire curve locus $\mathrm{ECL}(Z, W)$ of the induced directed variety, but we will not use this fact here. Since $\mathrm{IEL}_{X,V}(X) = \mathrm{ECL}(X, V)$, proving the Green-Griffiths-Lang property amounts to showing that $\mathrm{IEL}_{X,V}(X) \subsetneq X$ in the stage $k = 0$ of the tower.

3.6. Theorem. *Let (X, V) be a directed pair of general type. Assume that there is an integer $k_0 \ge 0$ such that for every $k > k_0$ and every irreducible algebraic set $Z \subsetneq X_k$, $Z \not\subset D_k$, that projects onto X_{k-1}, the induced directed structure $(Z, W) \subset (X_k, V_k)$ is of general type modulo $X_k \to X$. Then $\mathrm{IEL}_{X,V}(X_{k_0}) \subsetneq X_{k_0}$.*

Proof. We argue here by contradiction, assuming that $\mathrm{IEL}_{X,V}(X_{k_0}) = X_{k_0}$. The main argument consists of producing inductively an increasing sequence of integers

$$k_0 < k_1 < \ldots < k_j < \ldots$$

and directed varieties $(Z^j, W^j) \subset (X_{k_j}, V_{k_j})$ satisfying the following properties :

(3.6.1) $(Z^0, W^0) = (X_{k_0}, V_{k_0})$;

(3.6.2) for all $j \ge 0$, $\mathrm{IEL}_{X,V}(Z^j) = Z^j$;

(3.6.3) Z^j is an irreducible algebraic variety such that $Z^j \subsetneq X_{k_j}$ for $j \ge 1$, Z^j is not contained in the vertical divisor $D_{k_j} = P(T_{X_{k_j-1}/X_{k_j-2}})$ of X_{k_j}, and (Z^j, W^j) is of general type modulo $X_{k_j} \to X$ (i.e. some nonsingular model is) ;

(3.6.4) for all $j \geq 0$, the directed variety (Z^{j+1}, W^{j+1}) is contained in some stage (of order $\ell_j = k_{j+1} - k_j$) of the Semple tower of (Z^j, W^j), namely

$$(Z^{j+1}, W^{j+1}) \subset (Z^j_{\ell_j}, W^j_{\ell_j}) \subset (X_{k_{j+1}}, V_{k_{j+1}})$$

and

$$W^{j+1} = \overline{T_{Z^{j+1\,\prime}} \cap W^j_{\ell_j}} = \overline{T_{Z^{j+1\,\prime}} \cap V_{k_j}}$$

is the induced directed structure.

(3.6.5) for all $j \geq 0$, we have $Z^{j+1} \subsetneq Z^j_{\ell_j}$ but $\pi_{k_{j+1},k_{j+1}-1}(Z^{j+1}) = Z^j_{\ell_j - 1}$.

For $j = 0$, we have nothing to do by our hypotheses. Assume that (Z^j, W^j) has been constructed. By Proposition 2.4, we get an algebraic subset $\Sigma \subsetneq Z^j_\ell$ in some stage of the Semple tower (Z^j_ℓ) of Z^j such that every entire curve $f : (\mathbb{C}, T_\mathbb{C}) \to (X, V)$ satisfying $f_{[k_j]}(\mathbb{C}) \subset Z^j$ also satisfies $f_{[k_j+\ell]}(\mathbb{C}) \subset \Sigma$. By definition, this implies the first inclusion in the sequence

$$Z^j = \mathrm{IEL}_{X,V}(Z^j) \subset \pi_{k_j+\ell,k_j}(\mathrm{IEL}_{X,V}(\Sigma)) \subset \pi_{k_j+\ell,k_j}(\Sigma) \subset Z^j$$

(the other ones being obvious), so we have in fact an equality throughout. Let (S_α) be the irreducible components of $\mathrm{IEL}_{X,V}(\Sigma)$. We have $\mathrm{IEL}_{X,V}(S_\alpha) = S_\alpha$ and one of the components S_α must already satisfy $\pi_{k_j+\ell,k_j}(S_\alpha) = Z^j = Z^j_0$. We take $\ell_j \in [1, \ell]$ to be the smallest order such that $Z^{j+1} := \pi_{k_j+\ell,k_j+\ell_j}(S_\alpha) \subsetneq Z^j_{\ell_j}$, and set $k_{j+1} = k_j + \ell_j > k_j$. By definition of ℓ_j, we have $\pi_{k_{j+1},k_{j+1}-1}(Z^{j+1}) = Z^j_{\ell_j-1}$, otherwise ℓ_j would not be minimal. The fact that $\mathrm{IEL}_{X,V}(S_\alpha) = S_\alpha$ immediately implies $\mathrm{IEL}_{X,V}(Z^{j+1}) = Z^{j+1}$. Also Z^{j+1} cannot be contained in the vertical divisor $D_{k_{j+1}}$. In fact no irreducible algebraic set Z such that $\mathrm{IEL}_{X,V}(Z) = Z$ can be contained in a vertical divisor D_k, because $\pi_{k,k-2}(D_k)$ corresponds to stationary jets in X_{k-2} ; as every non constant curve f has non stationary points, its k-jet $f_{[k]}$ cannot be entirely contained in D_k. Finally, the induced directed structure (Z^{j+1}, W^{j+1}) must be of general type modulo $X_{k_{j+1}} \to X$, by the assumption of the theorem and the fact that $k_{j+1} > k_0$. The inductive procedure is therefore complete.

By Observation 2.2, we have

$$\mathrm{rank}\, W^j < \mathrm{rank}\, W^{j-1} < \ldots < \mathrm{rank}\, W^1 < \mathrm{rank}\, W^0 = \mathrm{rank}\, V.$$

After a sufficient number of iterations we reach $\mathrm{rank}\, W^j = 1$. In this situation the Semple tower of Z^j is trivial, $K_{W^j} = W^{j\,*} \otimes \overline{\mathcal{J}}_{W^j}$ is big, and Proposition 2.4 produces a divisor $\Sigma \subsetneq Z^j_\ell = Z^j$ containing all jets of entire curves with $f_{[k_j]}(\mathbb{C}) \subset Z^j$. This contradicts the fact that $\mathrm{IEL}_{X,V}(Z^j) = Z^j$. We have reached a contradiction, and Theorem 3.6 is thus proved. □

3.7. Remark. As it proceeds by contradiction, the proof is unfortunately non constructive – especially it does not give any information on the degree of the locus $Y \subsetneq X_{k_0}$ whose existence is asserted. On the other hand, and this is a bit surprising, the conclusion is obtained even though the conditions to be checked do not involve cutting down the dimensions of the base loci of jet differentials; in fact, the contradiction is obtained even though the integers k_j may increase and $\dim Z^j$ may become very large.

The special case $k_0 = 0$ of Theorem 3.6 yields the following

3.8. Partial solution to the generalized GGL conjecture. *Let (X, V) be a directed pair that is strongly of general type. Then the Green-Griffiths-Lang conjecture holds true for (X, V), namely* $\mathrm{ECL}(X, V) \subsetneq X$, *in other words there exists a proper algebraic variety*

$Y \subsetneq X$ such that every non constant holomorphic curve $f : \mathbb{C} \to X$ tangent to V satisfies $f(\mathbb{C}) \subset Y$.

3.9. Remark. The condition that (X, V) is strongly of general type seems to be related to some sort of stability condition. We are unsure what is the most appropriate definition, but here is one that makes sense. Fix an ample divisor A on X. For every irreducible subvariety $Z \subset X_k$ that projects onto X_{k-1} for $k \geq 1$, and $Z = X = X_0$ for $k = 0$, we define the slope $\mu_A(Z, W)$ of the corresponding directed variety (Z, W) to be

$$\mu_A(Z, W) = \frac{\inf \lambda}{\operatorname{rank} W},$$

where λ runs over all rational numbers such that there exists $m \in \mathbb{Q}_+$ for which

$$K_W \otimes \big(\mathcal{O}_{X_k}(m) \otimes \pi_{k,0}^* \mathcal{O}(\lambda A)\big)_{|Z} \quad \text{is big on } Z$$

(again, we assume here that $Z \not\subset D_k$ for $k \geq 2$). Notice that (X, V) is of general type if and only if $\mu_A(X, V) < 0$, and that $\mu_A(Z, W) = -\infty$ if $\mathcal{O}_{X_k}(1)_{|A}$ is big. Also, the proof of Lemma 1.11 shows that

$$\mu_A(X_k, V_k) \leq \mu_A(X_{k-1}, V_{k-1}) \leq \ldots \leq \mu_A(X, V) \quad \text{for all } k$$

(with $\mu_A(X_k, V_k) = -\infty$ for $k \geq k_0 \gg 1$ if (X, V) is of general type). We say that (X, V) is *A-jet-stable* (resp. *A-jet-semi-stable*) if $\mu_A(Z, W) < \mu_A(X, V)$ (resp. $\mu_A(Z, W) \leq \mu_A(X, V)$) for all $Z \subsetneq X_k$ as above. It is then clear that if (X, V) is of general type and A-jet-semi-stable, then it is strongly of general type in the sense of Definition 3.1. It would be useful to have a better understanding of this condition of stability (or any other one that would have better properties). □

3.10. Example: case of surfaces. Assume that X is a minimal complex surface of general type and $V = T_X$ (absolute case). Then K_X is nef and big and the Chern classes of X satisfy $c_1 \leq 0$ ($-c_1$ is big and nef) and $c_2 \geq 0$. The Semple jet-bundles X_k form here a tower of $\mathbb{P}^1$-bundles and $\dim X_k = k + 2$. Since $\det V^* = K_X$ is big, the strong general type assumption of 3.6 and 3.8 need only be checked for irreducible hypersurfaces $Z \subset X_k$ distinct from D_k that project onto X_{k-1}, of relative degree m. The projection $\pi_{k,k-1} : Z \to X_{k-1}$ is a ramified $m : 1$ cover. Putting $\mathcal{O}_{X_k}(Z) \simeq \mathcal{O}_{X_k}(\mathbf{a}) \otimes \pi_{k,0}(B)$, $B \in \operatorname{Pic}(X)$, we can apply (3.4.1) to get an inclusion

$$K_W \supset \big(\mathcal{O}_{X_k}(\mathbf{a} - \mathbf{1}) \otimes \pi_{k,0}^*(B \otimes K_X)\big)_{|Z} \otimes \mathcal{J}_S, \qquad \mathbf{a} \in \mathbb{Z}^k,\ a_k = m.$$

Let us assume $k = 1$ and $S = \emptyset$ to make things even simpler, and let us perform numerical calculations in the cohomology ring

$$H^\bullet(X_1, \mathbb{Z}) = H^\bullet(X)[u]/(u^2 + c_1 u + c_2), \qquad u = c_1(O_{X_1}(1))$$

(cf. [DEG00, Section 2] for similar calculations and more details). We have

$$Z \equiv mu + b \quad \text{where} \quad b = c_1(B) \quad \text{and} \quad K_W \equiv (m-1)u + b - c_1.$$

We are allowed here to add to K_W an arbitrary multiple $\mathcal{O}_{X_1}(p)$, $p \geq 0$, which we rather write $p = mt + 1 - m$, $t \geq 1 - 1/m$. An evaluation of the Euler-Poincaré characteristic of $K_W + \mathcal{O}_{X_1}(p)_{|Z}$ requires computing the intersection number

$$\begin{aligned}\big(K_W + \mathcal{O}_{X_1}(p)_{|Z}\big)^2 \cdot Z &= \big(mt\, u + b - c_1\big)^2 (mu + b) \\ &= m^2 t^2 \big(m(c_1^2 - c_2) - bc_1\big) + 2mt(b - mc_1)(b - c_1) + m(b - c_1)^2,\end{aligned}$$

taking into account that $u^3 \cdot X_1 = c_1^2 - c_2$. In case $S \neq \emptyset$, there is an additional (negative) contribution from the ideal $\mathcal{J}_S$ which is $O(t)$ since S is at most a curve. In any case, for $t \gg 1$, the leading term in the expansion is $m^2t^2(m(c_1^2 - c_2) - bc_1)$ and the other terms are negligible with respect to t^2, including the one coming from S. We know that T_X is semistable with respect to $c_1(K_X) = -c_1 \geq 0$. Multiplication by the section σ yields a morphism $\pi_{1,0}^*\mathcal{O}_X(-B) \to \mathcal{O}_{X_1}(m)$, hence by direct image, a morphism $\mathcal{O}_X(-B) \to S^m T_X^*$. Evaluating slopes against K_X (a big nef class), the semistability condition implies $bc_1 \leq \frac{m}{2}c_1^2$, and our leading term is bigger that $m^3t^2(\frac{1}{2}c_1^2 - c_2)$. We get a positive anwer in the well-known case where $c_1^2 > 2c_2$, corresponding to T_X being almost ample. Analyzing positivity for the full range of values (k, m, t) and of singular sets S seems an unsurmountable task at this point; in general, calculations made in [DEG00] and [McQ99] indicate that the Chern class and semistability conditions become less demanding for higher order jets (e.g. $c_1^2 > c_2$ is enough for $Z \subset X_2$, and $c_1^2 > \frac{9}{13}c_2$ suffices for $Z \subset X_3$). When rank $V = 1$, major gains come from the use of Ahlfors currents in combination with McQuillan's tautological inequalities [McQ98]. We therefore hope for a substantial strengthening of the above sufficient conditions, and a better understanding of the stability issues, possibly in combination with a use of Ahlfors currents and tautological inequalities. In the case of surfaces, an application of Theorem 3.6 for $k_0 = 1$ and an analysis of the behaviour of rank 1 (multi-)foliations on the surface X (with the crucial use of [McQ98]) was the main argument used in [DEG00] to prove the hyperbolicity of very general surfaces of degree $d \geq 21$ in $\mathbb{P}^3$. For these surfaces, one has $c_1^2 < c_2$ and $c_1^2/c_2 \to 1$ as $d \to +\infty$. Applying Theorem 3.6 for higher values $k_0 \geq 2$ might allow to enlarge the range of tractable surfaces, if the behavior of rank 1 (multi)-foliations on X_{k_0-1} can be analyzed independently.

4. Algebraic jet-hyperbolicity implies Kobayashi hyperbolicity

Let (X, V) be a directed variety, where X is an irreducible projective variety; the concept still makes sense when X is singular, by embedding (X, V) in a projective space $(\mathbb{P}^N, T_{\mathbb{P}^N})$ and taking the linear space V to be an irreducible algebraic subset of $T_{\mathbb{P}^n}$ that is contained in T_X at regular points of X.

For any irreducible algebraic subvariety $Z \subset X$, we get as in section 2 a directed variety structure $(Z, W) \subset (X, V)$ by taking $W = \overline{T_{Z'} \cap V}$ on a sufficiently small Zariski open set $Z' \subset Z_{\mathrm{reg}}$ where the intersection has minimal rank. Notice that when $W = 0$ there cannot exist entire curves $f : (\mathbb{C}, T_{\mathbb{C}}) \to (Z, W)$, except possibly those which lie in the algebraic set $Z \smallsetminus Z'$, hence this case is easy to deal with by induction on dimension. Otherwise, we can resolve singularities of Z to get a directed variety $(\widehat{Z}, \widehat{W})$ where $\widehat{Z}$ is nonsingular and $\operatorname{rank} \widehat{W} \geq 1$.

4.1. Definition. *Let (X, V) be a directed variety. We say that*

(a) (X, V) *is algebraically jet-hyperbolic if for every irreducible algebraic subvariety $Z_0 \subset X$, the induced directed structure (Z_0, W_0) either satisfies $W_0 = 0$, or has a desingularization $(\widehat{Z}_0, \widehat{W}_0)$, $\operatorname{rank} \widehat{W}_0 \geq 1$, that is strongly of general type.*

(b) (X, V) *is algebraically fully jet-hyperbolic if for every $k \geq 0$ and every irreducible algebraic subvariety $Z \subset X_k$ that is not contained in the union Δ_k of vertical divisors, the induced directed structure (Z, W) either satisfies $W = 0$, or is of general type modulo $X_k \to X$, i.e. has a desingularization $(\widehat{Z}, \widehat{W})$, $\mu : \widehat{Z} \to Z$, such that some twisted canonical sheaf $K_{\widehat{W}} \otimes \mu^*(\mathcal{O}_{X_k}(\mathbf{a})_{|Z})$, $\mathbf{a} \in \mathbb{N}^k$, is big.*

It is clear that hypothesis 4.1 (b) is stronger than 4.1 (a). In fact, in 4.1 (a), one first takes an induced directed subvariety $(Z_0, W_0) \subset (X, V)$, its Semple tower (Z_k, W_k), and the question is to check whether every induced subvariety $(Z, W) \subset (Z_k, W_k) \subset (X_k, V_k)$ such that $\pi_{k,k-1}(Z) = Z_{k-1}$ is of general type modulo $Z_k \to Z_0$. On the other hand, for property 4.1 (b), we have to check right away all induced structures $(Z, W) \subset (X_k, V_k)$, whatever are their projections $\pi_{k,\ell}(Z)$. It is unclear to us whether the resulting concepts are really different. Thanks to Theorem 3.8, a very easy induction on the dimension of X implies

4.2. Theorem. *Let (X, V) be an irreducible projective directed variety that is algebraically jet-hyperbolic in the sense of the above definition. Then (X, V) is Brody (or Kobayashi) hyperbolic, i.e.* $\mathrm{ECL}(X, V) = \emptyset$.

Proof. By Theorem 3.8, we have $Y := \mathrm{ECL}(X, V) \subsetneq X$. If $Y \neq \emptyset$, apply induction on dimension to each of the irreducible components X'_j of Y and to the induced directed structures (X'_j, V'_j) to get $\mathrm{ECL}(X, V) \subset \bigcup \mathrm{ECL}(X'_j, V'_j) \subsetneq \bigcup X'_j = Y$, a contradiction. □

5. Proof of the Kobayashi conjecture on generic hyperbolicity

We start with a general situation, and then restrict ourselves to the special case of complete intersections in projective space. Consider a smooth deformation $\pi : \mathcal{X} \to S$ of complex projective manifolds, i.e. a proper algebraic submersion over a quasi-projective algebraic manifold S such that the fibers are nonsingular. By a "very general fiber", we mean here a fiber $X_t = \pi^{-1}(t)$ over a point t taken in the complement $S \smallsetminus \bigcup S_\nu$ of a countable union of algebraic subsets $S_\nu \subsetneq S$. We are only interested in the very general fiber and can therefore restrict ourselves to the case where S is affine after replacing S with a suitable Zariski open subset $S^0 \subset S$. Ample vector bundles over the total space $\mathcal{X}$ are then the same as vector bundles that are relatively ample over S, as one can see immediately by the direct image theorem and the fact that every locally free sheaf on an affine variety is very ample.

5.1. Theorem. *Let $\pi : \mathcal{X} \to S$ be a deformation of complex projective nonsingular varieties $X_t = \pi^{-1}(t)$ over a smooth quasi-projective irreducible base S. Let $n = \dim X_t$ be the relative dimension and let $N = \dim S$. Assume that for all $q = N+1, \ldots, N+n$, the exterior power $\Lambda^q T^*_{\mathcal{X}}$ is a relatively ample vector bundle over S. Then the very general fiber X_t is algebraically jet-hyperbolic, and thus Kobayashi hyperbolic.*

Proof. By taking the relative directed structure $\mathcal{V} = T_{\mathcal{X}/S} = \mathrm{Ker}(d\pi : T_{\mathcal{X}} \to \pi^* T_S)$ on $\mathcal{X}$, one constructs a "relative" Semple tower $(\mathcal{X}_k, \mathcal{V}_k)$ over $\mathcal{X}$. It specializes to the absolute Semple tower $X_{t,k}$ of $X_t = \pi^{-1}(t) \subset \mathcal{X}$ when one takes $X_{t,k} = \pi_{k,0}^{-1}(X_t) \subset \mathcal{X}_k$ via the natural projection $\pi_{k,0} : \mathcal{X}_k \to \mathcal{X}_0 = \mathcal{X}$. By construction $\mathcal{V}_0 = \mathcal{V} = T_{\mathcal{X}/S}$ and all $\mathcal{V}_k$ have rank n. Let $(\mathcal{X}^a_k, \mathcal{V}^a_k)$ be the absolute Semple tower of $\mathcal{X}$, so that $\mathcal{X}^a_0 = \mathcal{X}$ and $\mathcal{V}^a_0 = T_{\mathcal{X}}$, and let $\widetilde{\mathcal{V}}_k$ be the restriction of the vector bundle $\mathcal{V}^a_k$ to $\mathcal{X}_k \subset \mathcal{X}^a_k$, so that $\mathrm{rank}\, \widetilde{\mathcal{V}}_k = \mathrm{rank}\, \mathcal{V}^a_k = N + n$. For every $k \geq 0$, we claim that there is an exact sequence of vector bundles

$$(5.1.1) \qquad 0 \to \mathcal{V}_k \to \widetilde{\mathcal{V}}_k \to \mathcal{S}_k \to 0, \qquad \mathcal{S}_k \simeq (\pi \circ \pi_{k,0})^* T_S \otimes \mathcal{O}_{\mathcal{X}_k}(\mathbf{1}) \quad \text{over } \mathcal{X}_k,$$

where $\mathbf{1} = (1, \ldots, 1) \in \mathbb{N}^k$, $\mathrm{rank}\, \mathcal{V}_k = n$, and $\mathrm{rank}\, \widetilde{\mathcal{V}}_k = \mathrm{rank}\, \mathcal{V}^a_k = N + n = \dim \mathcal{X}$. Since $\widetilde{\mathcal{V}}_0 = \mathcal{V}^a_0 = T_{\mathcal{X}}$ and $\mathcal{V}_0 = T_{\mathcal{X}/S}$, this is true by definition for $k = 0$, with $\mathcal{S}_0 = \pi^* T_S$ and $\mathcal{O}_{\mathcal{X}_0}(\mathbf{1}) = \mathcal{O}_{\mathcal{X}}$. In general, there is a well defined injection of bundles $\mathcal{V}_k \to \widetilde{\mathcal{V}}_k$, the quotient is of rank N, and we simply put $\mathcal{S}_k = \widetilde{\mathcal{V}}_k / \mathcal{V}_k$ by definition. The relative (resp. absolute)

Semple tower of $\pi_{k,\ell} : \mathcal{X}_k \to \mathcal{X}_\ell$ (resp. $\pi^a_{k,\ell} : \mathcal{X}^a_k \to \mathcal{X}^a_\ell$) yields exact sequences

$$(5.1.2)\qquad 0 \longrightarrow \mathcal{G}_k \longrightarrow \mathcal{V}_k \stackrel{(\pi_{k,k-1})_*}{\longrightarrow} \mathcal{O}_{\mathcal{X}_k}(-1) \longrightarrow 0, \qquad \mathcal{G}_k := T_{\mathcal{X}_k/\mathcal{X}_{k-1}},$$

$$(5.1.3)\qquad 0 \longrightarrow \mathcal{O}_{\mathcal{X}_k} \longrightarrow (\pi_{k,k-1})^*\mathcal{V}_{k-1} \otimes \mathcal{O}_{\mathcal{X}_k}(1) \longrightarrow \mathcal{G}_k \longrightarrow 0,$$

$$(5.1.2^a)\qquad 0 \longrightarrow \mathcal{G}^a_k \longrightarrow \mathcal{V}^a_k \stackrel{(\pi^a_{k,k-1})_*}{\longrightarrow} \mathcal{O}_{\mathcal{X}^a_k}(-1) \longrightarrow 0, \qquad \mathcal{G}^a_k := T_{\mathcal{X}^a_k/\mathcal{X}^a_{k-1}},$$

$$(5.1.3^a)\qquad 0 \longrightarrow \mathcal{O}_{\mathcal{X}^a_k} \longrightarrow (\pi^a_{k,k-1})^*\mathcal{V}^a_{k-1} \otimes \mathcal{O}_{\mathcal{X}^a_k}(1) \longrightarrow \mathcal{G}^a_k \longrightarrow 0.$$

By restricting the absolute ones to $\mathcal{X}_k \subset \mathcal{X}^a_k$ and denoting $\widetilde{\mathcal{G}}_k := \mathcal{G}^a_{k|\mathcal{X}_k}$, we get exact sequences

$$(5.1.2^\sim)\qquad 0 \longrightarrow \widetilde{\mathcal{G}}_k \longrightarrow \widetilde{\mathcal{V}}_k \stackrel{(\pi_{k,k-1})_*}{\longrightarrow} \mathcal{O}_{\mathcal{X}_k}(-1) \longrightarrow 0,$$

$$(5.1.3^\sim)\qquad 0 \longrightarrow \mathcal{O}_{\mathcal{X}_k} \longrightarrow (\pi_{k,k-1})^*\widetilde{\mathcal{V}}_{k-1} \otimes \mathcal{O}_{\mathcal{X}_k}(1) \longrightarrow \widetilde{\mathcal{G}}_k \longrightarrow 0$$

There is an inclusion morphism of (5.1.i) into (5.1.$i^\sim$), $i = 2, 3$, and by taking cokernels, we see that

$$\mathcal{S}_k := \widetilde{\mathcal{V}}_k/\mathcal{V}_k \underset{(*)}{=} \widetilde{\mathcal{G}}_k/\mathcal{G}_k \underset{(**)}{=} (\pi_{k,k-1})^*\mathcal{S}_{k-1} \otimes \mathcal{O}_{\mathcal{X}_k}(1)$$

where $(*)$ comes from (5.1.$2^\sim$) and $(**)$ from (5.1.$3^\sim$). This induction formula for $\mathcal{S}_k$ completes the proof of (5.1.1). If we take the dual exact sequences, we get

$$(5.1.2^*)\qquad 0 \longrightarrow \mathcal{O}_{\mathcal{X}_k}(1) \longrightarrow \widetilde{\mathcal{V}}^*_k \longrightarrow \widetilde{\mathcal{G}}^*_k \longrightarrow 0,$$

$$(5.1.3^*)\qquad 0 \longrightarrow \widetilde{\mathcal{G}}^*_k \longrightarrow (\pi_{k,k-1})^*\widetilde{\mathcal{V}}^*_{k-1} \otimes \mathcal{O}_{\mathcal{X}_k}(-1) \longrightarrow \mathcal{O}_{\mathcal{X}_k} \longrightarrow 0,$$

and the q-th (resp. q'-th) exterior power of these yield

$$(5.1.4)\qquad 0 \longrightarrow \Lambda^{q-1}\widetilde{\mathcal{G}}^*_k \otimes \mathcal{O}_{\mathcal{X}_k}(1) \longrightarrow \Lambda^q\widetilde{\mathcal{V}}^*_k \longrightarrow \Lambda^q\widetilde{\mathcal{G}}^*_k \longrightarrow 0,$$

$$(5.1.5)\qquad 0 \longrightarrow \Lambda^{q'}\widetilde{\mathcal{G}}^*_k \longrightarrow (\pi_{k,k-1})^*\Lambda^{q'}\widetilde{\mathcal{V}}^*_{k-1} \otimes \mathcal{O}_{\mathcal{X}_k}(-q') \longrightarrow \Lambda^{q'-1}\widetilde{\mathcal{G}}^*_k \longrightarrow 0.$$

In a next step, we will need local vector fields and their liftings to the absolute and relative Semple towers to justify certain delicate arguments about multiplier ideals.

5.1.6. Lemma.

(a) *Every local holomorphic vector field ζ on an open set $U \subset \mathcal{X}$ has a natural lifting $\zeta^{(k)}$ to the open set $(\pi^a_{k,0})^{-1}(U)$ in the total space of the absolute Semple bundle $\mathcal{X}^a_k$.*

(b) *In particular, if one assumes that ζ is in $H^0(U, T_{\mathcal{X}/S}) \subset H^0(U, T_{\mathcal{X}})$, the lifted flow leaves invariant the relative Semple tower $\mathcal{X}_k \subset \mathcal{X}^a_k$ in $\pi^{-1}_{k,0}(U)$, thus $\zeta^{(k)}_{|\mathcal{X}_k}$ is tangent to $\mathcal{V}_k$.*

(c) *Every local holomorphic vector field $\tau \in H^0(\Omega, T_S)$ on S has a (nonunique) lifting $\widetilde{\tau} \in H^0(U, T_{\mathcal{X}})$ on a neighborhood U of every point $x \in \pi^{-1}(\Omega)$. Once $\widetilde{\tau}$ is chosen, there is a (unique) lifting $\widetilde{\tau}^{(k)}$ of $\widetilde{\tau}$ to the open set $(\pi^a_{k,0})^{-1}(U)$ in the total space of the absolute Semple bundle $\mathcal{X}^a_k$, and the flow of $\widetilde{\tau}^{(k)}$ induces a local biholomorphism $(\mathcal{X}_k, \mathcal{V}_k) \to (\mathcal{X}_k, \mathcal{V}_k)$ of the relative Semple tower. These local flows preserve the exact sequences (5.1.1 – 5.1.5); moreover $\widetilde{\tau}^{(k)}_{|\mathcal{X}_k}$ is tangent to $\mathcal{V}^a_k$ (but is not tangent to $\mathcal{V}_k$).*

Proof. (a) Every local holomorphic vector field ζ on $\mathcal{X}$ generates a flow of local biholomorphisms on open subsets of $\mathcal{X}$, and we can apply the fonctoriality property 1.10 to lift it to a flow on $(\mathcal{X}^a_k, \mathcal{V}^a_k)$. The differentiation of the lifted flow gives back what we define to be the lifted vector field $\zeta^{(k)}$ on $\mathcal{X}^a_k$.

(b) If ζ lies in $\mathcal{V}_0 = T_{\mathcal{X}/S}$, the lifted flow acts similarly on the relative tower $(\mathcal{X}_k, \mathcal{V}_k)$ since the fibers $X_{t,k}$ over $X_t = \pi^{-1}(t)$ are preserved. Therefore $\zeta^{(k)}_{|\mathcal{X}_k}$ is tangent to $\mathcal{V}_k$, since the relative Semple tower is the absolute Semple tower of the fibers X_t.

(c) Since $\mathcal{X} \to S$ is a holomorphic submersion, $\mathcal{X}$ is locally holomorphically trivialized as a product $S \times X_t$; if τ is a local vector field on S, it can thus be lifted (possibly not uniquely) as a local vector field $\widetilde{\tau}$ on $\mathcal{X}$. The resulting flow of $\widetilde{\tau}$ on $\mathcal{X}$ commutes with the flow of σ on S, i.e. acts by (local) biholomorphisms $(\mathcal{X}, T_{\mathcal{X}/S}) \to (\mathcal{X}, T_{\mathcal{X}/S})$ of directed varieties. From there we conclude, again by fonctoriality, that the lifting of the flow to $\mathcal{X}^a_k$ preserves $(\mathcal{X}_k, \mathcal{V}_k) \subset (\mathcal{X}^a_k, \mathcal{V}^a_k)$, in particular the exact sequences are preserved. Moreover the differential $\widetilde{\tau}^{(k)}$ is tangent to $\mathcal{V}^a_k$ (but, already for $k = 0$, it is not tangent to $\mathcal{V}_k$). □

Now, let $\mathcal{Z} \subset \mathcal{X}_k$ be an irreducible algebraic subvariety of $\mathcal{X}_k$ that is not contained in the union Δ_k of vertical divisors and projects surjectively onto S, and let $(\mathcal{Z}, \mathcal{W}) \subset (\mathcal{X}_k, \mathcal{V}_k)$ be the induced directed structure, Our ultimate goal is to show that the generic fibers (Z_t, W_t) are of general type modulo $X_{t,k} \to X_t$. We assume $r := \operatorname{rank} \mathcal{W} \geq 1$, otherwise there is nothing to do. We first need to extend $\mathcal{Z}$ within the absolute Semple tower $\mathcal{X}^a_k$, "horizontally" with respect to the projection $\mathcal{X}^a_k \to S$. Later, we will have to care that the extension $\mathcal{Z}^a$ is made in an "equisingular way". This procedure is what replaces here the use of oblique vector fields (introduced by Siu [Siu02, 04], and employed later by [Pau08] and [DMR10]). The main advantage is that "equisingular horizontal extensions" do not introduce additional poles.

5.1.7. Lemma. *There exists an irreducible algebraic variety $\mathcal{Z}^a \subset \mathcal{X}^a_k$ satisfying the following properties with respect to the projections $\mathcal{Z}_\ell = \pi_{k,\ell}(\mathcal{Z})$ and $\mathcal{Z}^a_\ell = \pi_{k,\ell}(\mathcal{Z}^a)$, $\ell = 0, 1, \ldots, k$:*

(a) *$\mathcal{Z}$ is one of the irreducible components of $\mathcal{Z}^a \cap \mathcal{X}_k$ and likewise, $\mathcal{Z}_\ell$ is one of the irreducible components of $\mathcal{Z}^a_\ell \cap \mathcal{X}_\ell$ for all ℓ;*

(b) *the intersection $\mathcal{Z}^a_\ell \cap \mathcal{X}_\ell$ is smooth and transverse at the generic point of $\mathcal{Z}_\ell$;*

(c) *the codimension of $\mathcal{Z}^a_\ell$ in $\mathcal{X}^a_\ell$ is equal to the codimension p_ℓ of $\mathcal{Z}_\ell$ in $\mathcal{X}_\ell$ for all ℓ;*

(d) *if r_ℓ is the (generic) rank of $\mathcal{W}_\ell = T_{\mathcal{Z}_\ell} \cap \mathcal{V}_\ell$, then the rank of $\mathcal{W}^a_\ell = T_{\mathcal{Z}^a_\ell} \cap \mathcal{V}^a_\ell$ is $N + r_\ell$;*

(e) *we have $r_\ell = n - p_\ell$, $p_0 \leq p_1 \leq \ldots \leq p_k$ and $r_0 \geq r_1 \geq \ldots \geq r_k = r \geq 1$.*

Proof. For $\ell = 0$, no extension is needed as $\mathcal{X}^a_0 = \mathcal{X}_0 = \mathcal{X}$, we simply take $\mathcal{Z}^a_0 = \mathcal{Z}_0 = \pi_{k,0}(\mathcal{Z})$ (and thus we are done if $k = 0$). For $k \geq 1$, we construct $\mathcal{Z}^a_\ell$ inductively, assuming that $\mathcal{Z}^a_{\ell-1}$ has already been constructed, $\ell \geq 1$. Since $\mathcal{Z}_{\ell-1} = \pi_{k,k-1}(\mathcal{Z}_\ell)$ and $\pi_{\ell,\ell-1} : \mathcal{X}_\ell \to \mathcal{X}_{\ell-1}$ is a fibration, it is clear that $p_{\ell-1} \leq p_\ell$. Also, $\mathcal{Z}_\ell$ is contained in $P(\mathcal{W}_{\ell-1}) \subset P(\mathcal{V}_{\ell-1}) = \mathcal{X}_\ell$. The codimension of $P(\mathcal{W}_{\ell-1})$ in $\mathcal{X}_\ell$ is $p_{\ell-1} = n - r_{\ell-1}$, hence the codimension of $\mathcal{Z}_\ell$ in $P(\mathcal{W}_{\ell-1})$ is $p_\ell - p_{\ell-1}$. Now, $\mathcal{Z}_\ell$ is the zero locus of a family of sections s_j of some very ample line bundle $\mathcal{O}_{\mathcal{X}_\ell}(\mathbf{m}_\ell)$ on $\mathcal{X}_\ell$, $\mathbf{m}_\ell \in \mathbb{N}^\ell$, such that the differentials ds_j are independent at all nonsingular points of $\mathcal{Z}_\ell$. If we take $\mathbf{m}_\ell$ large enough, those sections s_j extend as sections s^a_j of $\mathcal{O}_{\mathcal{X}^a_\ell}(\mathbf{m}_\ell)$, and we can pick $p_\ell - p_{\ell-1}$ linear combinations σ_j of them so that $P(\mathcal{W}_{\ell-1}) \cap \{\sigma_j = 0\}$ and $P(\mathcal{W}^a_{\ell-1}) \cap \{\sigma^a_j = 0\}$ are generically transverse intersections of pure dimension, the dimension of $P(\mathcal{W}_{\ell-1}) \cap \{\sigma_j = 0\}$ being equal to that of $\mathcal{Z}_\ell$. We take $\mathcal{Z}^a_\ell$ to be the irreducible component of $P(\mathcal{W}^a_{\ell-1}) \cap \{s^a_j = 0\}$ that contains $\mathcal{Z}_\ell$ (in general, the intersection will not be irreducible, as its degree may be very large). Properties (a), (b), (c) are then satisfied by construction, and $\mathcal{W}_\ell$ (resp. $\mathcal{W}^a_\ell$) is obtained from the lifting of $\mathcal{W}_{\ell-1}$ to $P(\mathcal{W}_{\ell-1})$ (resp. of $\mathcal{W}^a_{\ell-1}$ to $P(\mathcal{W}^a_{\ell-1})$) by cutting the corresponding lifted directed structure by the generically independent linear equations $d\sigma_j = 0$ (resp. $d\sigma^a_j = 0$). Therefore we see by induction that the rank of $\mathcal{W}_\ell$

(resp. $\mathcal{W}^a_\ell$) is

$$r_\ell = r_{\ell-1} - (p_\ell - p_{\ell-1}) = n - p_\ell, \qquad \text{resp.} \quad N + r_{\ell-1} - (p_\ell - p_{\ell-1}) = N + r_\ell.$$

Properties (d), (e) follow, and Lemma 5.1.7 is proved. □

To simplify notation, we let $(\mathcal{Z}^a, \mathcal{W}^a) = (\mathcal{Z}^a_k, \mathcal{W}^a_k)$ be the induced directed structure at the top level k (the lower levels will no longer be needed in the sequel), i.e. $\mathcal{W}^a = T_{\mathcal{Z}^a} \cap \mathcal{V}^a_k$ at the generic point. We take $\widetilde{\mathcal{W}}$ to be the irreducible component of $\mathcal{W}^a_{|\mathcal{Z}}$ that contains $\mathcal{W}$ (recall that $\mathcal{Z} \subset \mathcal{Z}^a \cap \mathcal{X}_k$ have the same dimension, but the intersection need not be irreducible). We claim that there is an exact sequence

$$(5.1.8) \qquad 0 \longrightarrow \mathcal{W} \longrightarrow \widetilde{\mathcal{W}} \longrightarrow (\pi \circ \pi_{k,0})^* T_S \otimes \mathcal{O}_{\mathcal{X}_k}(\mathbf{1}) \longrightarrow 0$$

at a generic point of $\mathcal{Z}$. In fact, at a generic point, the proof of Lemma 5.1.7 shows that

$$\mathcal{W} = \mathcal{V}_k \cap \{d\sigma_j = 0\}, \quad \mathcal{W}^a = \mathcal{V}^a_k \cap \{d\sigma^a_j = 0\}, \quad \widetilde{\mathcal{W}} = \widetilde{\mathcal{V}}_k \cap \{d\sigma^a_j = 0\}$$

have the same codimension in $\mathcal{V}_k$, $\mathcal{V}^a_k$, $\widetilde{\mathcal{V}}_k$ respectively, hence we have $\widetilde{\mathcal{W}}/\mathcal{W} \simeq \widetilde{\mathcal{V}}_k/\mathcal{V}_k \simeq \mathcal{S}_k$, and the conclusion follows by restricting the exact sequence (5.1.1). We get by definition a non trivial morphism over $\mathcal{Z}$ induced by the natural inclusion $\widetilde{\mathcal{W}} \subset \widetilde{\mathcal{V}}_k$

$$\Lambda^q \widetilde{\mathcal{V}}^*_{k\,|\mathcal{Z}} \longrightarrow K_{\widetilde{\mathcal{W}}}, \qquad q = N + r = \operatorname{rank} \widetilde{\mathcal{W}}.$$

One should notice that $\widetilde{\mathcal{V}}_k$ and $\widetilde{\mathcal{G}}_k$ are genuine vector bundles without singularities, hence the above morphism actually has its image *contained in* $K_{\widetilde{\mathcal{W}}} = K_{\mathcal{W}^a|\mathcal{Z}}$ even when one takes into account the relevant multiplier ideal sheaf that defines $K_{\mathcal{W}^a}$ ("monotonicity principle"); however, there will be more delicate singularity issues later on. We conclude by (5.1.4) that either we have a non trivial morphism

$$\Lambda^{q-1} \widetilde{\mathcal{G}}^*_k \otimes \mathcal{O}_{\mathcal{X}_k}(1)_{|\mathcal{Z}} \longrightarrow K_{\widetilde{\mathcal{W}}}$$

or (if the above vanishes) a non trivial morphism

$$\Lambda^q \widetilde{\mathcal{G}}^*_{k\,|\mathcal{Z}} \longrightarrow K_{\widetilde{\mathcal{W}}}.$$

By (5.1.5) with $q' = q$ or $q' = q + 1$, we infer that we have a non trivial morphism

$$(\pi_{k,k-1})^* \Lambda^q \widetilde{\mathcal{V}}^*_{k-1} \otimes \mathcal{O}_{\mathcal{X}_k}(-q+1)_{|\mathcal{Z}} \longrightarrow K_{\widetilde{\mathcal{W}}}$$

or a non trivial morphism

$$(\pi_{k,k-1})^* \Lambda^{q+1} \widetilde{\mathcal{V}}^*_{k-1} \otimes \mathcal{O}_{\mathcal{X}_k}(-q-1)_{|\mathcal{Z}} \longrightarrow K_{\widetilde{\mathcal{W}}}.$$

Proceeding inductively with the lower stages and getting down to $\widetilde{\mathcal{V}}_0 = T_{\mathcal{X}}$, we conclude that there exists an integer $q' \geq q = \operatorname{rank} \widetilde{\mathcal{W}}$, a weight $\mathbf{a} = (a_1, \ldots, a_k) \in \mathbb{N}^k$, $a_j \geq q - 1$, and a non trivial morphism

$$(5.1.9) \qquad (\pi_{k,0})^* \Lambda^{q'} T^*_{\mathcal{X}\,|\mathcal{Z}} \to K_{\widetilde{\mathcal{W}}} \otimes \mathcal{O}_{\mathcal{X}_k}(\mathbf{a})_{|\mathcal{Z}}.$$

Assume that $q \geq N + 1$ (i.e. $\operatorname{rank} \mathcal{W} \geq 1$). By our assumption (assuming S affine here), $\Lambda^{q'} T^*_{\mathcal{X}}$ is ample over $\mathcal{X}$, thus, by twisting with a certain relatively ample line bundle $\mathcal{O}_{\mathcal{X}_k}(\varepsilon \mathbf{c})$ with respect to $\pi_{k,0}$, we see that $(\pi_{k,0})^* \Lambda^{q'} T^*_{\mathcal{X}} \otimes \mathcal{O}_{\mathcal{X}_k}(\varepsilon \mathbf{c})$ is ample over $\mathcal{X}_k$ for $0 < \varepsilon \ll 1$. From this, we infer that there exists a weight $\mathbf{b} \in \mathbb{Q}^k_+$, $b_j > q - 1$, such that $K_{\widetilde{\mathcal{W}}} \otimes \mathcal{O}_{\mathcal{X}_k}(\mathbf{b})_{|\mathcal{Z}}$ is big over $\mathcal{Z}$. In fact, these arguments could have been given instead for $(\mathcal{Z}^a, \mathcal{W}^a)$, and we would have obtained in the same manner the existence of a non trivial morphism

$$(5.1.9^a) \qquad (\pi^a_{k,0})^* \Lambda^{q'} T^*_{\mathcal{X}\,|\mathcal{Z}^a} \to K_{\mathcal{W}^a} \otimes \mathcal{O}_{\mathcal{X}^a_k}(\mathbf{a})_{|\mathcal{Z}},$$

in other words $(\mathcal{Z}^a,\mathcal{W}^a)$ is of general type modulo $\mathcal{X}_k^a\to\mathcal{X}$. The reasoning is of course much easier for $k=0$, in that case we simply take $\mathcal{Z}^a=\mathcal{Z}$, $\mathcal{W}^a=T_{\mathcal{Z}}$, $q'=q$, and $\Lambda^qT^*_{\mathcal{X}}$ is ample on $\mathcal{X}$, hence on $\mathcal{Z}$.

From this, we are going to conclude by a Hilbert scheme argument that (X_t,T_{X_t}) is algebraically jet-hyperbolic for very general $t\in S$. Otherwise, consider the collection of non vertical irreducible varieties $Z_t\times\{t\}\subset\mathcal{X}_k$ such that the induced directed structure (Z_t,W_t) is not of general type modulo $X_{t,k}\to X_t$, with $\operatorname{rank}W_t\ge 1$ and t running over S. If we fix k, the degree δ of Z_t with respect to some polarization and the weight $\mathbf{b}\in\mathbb{Q}_+^k$ such that $K_{W_t}\otimes\mathcal{O}_{\mathcal{X}_k}(\mathbf{b})_{|Z_t}$ is not big, we get a Zariski closed set $\mathcal{H}_{k,\delta,\mathbf{b}}$ in the Hilbert scheme of $\mathcal{X}_k$, and so is $\mathcal{H}_{k,\delta}=\bigcap_{\mathbf{b}}\mathcal{H}_{k,\delta,\mathbf{b}}$. We have a natural projection $p_{k,\delta}:\mathcal{H}_{k,\delta}\to S$. If $p_{k,\delta}$ were dominant, it would be possible to find a Zariski open set $S^0\subset S$, a finite unramified cover $\widehat{S}^0$ of S^0 and a branched section $\widehat{S}^0\to\mathcal{H}_{k,\delta}$ of $p_{k,\delta}$. This would give an algebraic family $Z_t\subset X_{t,k}$ for $t\in\widehat{S}^0$, such that the induced directed structure (Z_t,W_t) is not of general type modulo $X_{t,k}\to X_t$, with $\operatorname{rank}W_t\ge 1$. In order to avoid finite covers of the base, we apply a base change $\widehat{S}^0\to S$ and consider the resulting deformation $\widehat{\mathcal{X}}\to\widehat{S}^0$, which we still denote $\mathcal{X}\to S$ to simplify notation (so that we just have $\widehat{S}^0=S$ in the new setting). In this way, we obtain a directed subvariety $(\mathcal{Z},\mathcal{W})$ of $(\mathcal{X}_k,\mathcal{V}_k)$, and we extend it horizontally as a subvariety $(\mathcal{Z}^a,\mathcal{W}^a)\subset(\mathcal{X}_k^a,\mathcal{V}_k^a)$ satisfying the following properties:

(5.1.10) $\mathcal{Z}:=\overline{\bigcup_{t\in S}Z_t}\subset\mathcal{X}_k\subset\mathcal{X}_k^a,\quad \mathcal{W}:=\overline{T_{\mathcal{Z}_{\rm reg}}\cap\mathcal{V}_k}\subset\mathcal{V}_{k\,|\mathcal{Z}}$ restricts to W_t on Z_t, $t\in S$,

(5.1.11) $\mathcal{Z}$ is one of the irreducible components of $\mathcal{Z}^a\cap\mathcal{X}_k$,

(5.1.12) if one takes $\widetilde{\mathcal{W}}=\mathcal{W}^a_{|\mathcal{Z}}$ where $\mathcal{W}^a=\overline{T_{\mathcal{Z}^a_{\rm reg}}\cap\mathcal{V}_k^a}$, there is an exact sequence

$$0\to\mathcal{W}\to\widetilde{\mathcal{W}}\to(\pi\circ\pi_{k,0})^*T_S\otimes\mathcal{O}_{\mathcal{X}_k}(\mathbf{1})_{|\mathcal{Z}}\to 0\quad\text{generically on }\mathcal{Z}$$

(maybe after shrinking again S to a smaller Zariski open set). The next idea, which refines the technique used by [Ein88, 91] and [Voi96], is to consider an embedded desingularization of our spaces to take care of the singularities and their associated multiplier ideal sheaves.

5.1.13. Lemma. *Let $\mu:\widehat{\mathcal{Z}}\to\mathcal{Z}$ be a desingularization of $\mathcal{Z}$, and let $\widehat{Z}_t$ be the fiber over $t\in S$ of the projection $\pi\circ\pi_{k,0}\circ\mu:\widehat{\mathcal{Z}}\to S$. In fact, we take $\mu:\widehat{\mathcal{X}}_k^a\to\mathcal{X}_k^a$ to be a simultaneous embedded resolution of singularities of $\mathcal{Z}$ and $\mathcal{Z}^a$ inside the nonsingular space $\mathcal{X}_k^a$. Also, by composing if necessary with further blow-ups, we ask μ to resolve the indeterminacies of the meromorphic maps to Grassmannian bundles*

$$\varphi:\mathcal{Z}\dashrightarrow\operatorname{Gr}(\mathcal{V}_k,r),\qquad\varphi^a:\mathcal{Z}^a\dashrightarrow\operatorname{Gr}(\mathcal{V}_k^a,N+r)$$

associated with the linear spaces $\mathcal{W}\subset\mathcal{V}_k\subset T_{\mathcal{X}_k}$, $\mathcal{W}^a\subset\mathcal{V}_k^a\subset T_{\mathcal{X}_k^a}$ of respective ranks r and $N+r$; in other words, we want μ to provide holomorphic maps

$$\varphi\circ\mu:\widehat{\mathcal{Z}}\longrightarrow\mu^*(\operatorname{Gr}(\mathcal{V}_k,r)_{|\mathcal{Z}}),\qquad\varphi^a\circ\mu:\widehat{\mathcal{Z}}^a\longrightarrow\mu^*(\operatorname{Gr}(\mathcal{V}_k^a,N+r)_{|\mathcal{Z}^a}).$$

Under the above assumptions for μ, let $(\widehat{\mathcal{Z}},\widehat{\mathcal{W}})$, $(\widehat{\mathcal{Z}}^a,\widehat{\mathcal{W}}^a)$ be the pull-backs of $(\mathcal{Z},\mathcal{W})$, $(\mathcal{Z}^a,\mathcal{W}^a)$ by μ, and let $(\widehat{Z}_t,\widehat{W}_t)$ be the restriction of $(\widehat{\mathcal{Z}}^a,\widehat{\mathcal{W}}^a)$ to $\widehat{Z}_t$. Then, provided that the extension $\mathcal{Z}\rightsquigarrow\mathcal{Z}^a$ has been made in an "equisingular way", there is a well defined nontrivial morphism

$$(\mu^*K_{\widetilde{\mathcal{W}}})_{|\widehat{Z}_t}\longrightarrow K_{\widehat{W}_t}\otimes\mu^*\big((\pi\circ\pi_{k,0})^*K_S\otimes\mathcal{O}_{X_k}(-N\,\mathbf{1})\big)_{|\widehat{Z}_t}$$

on the generic smooth fiber $\widehat{Z}_t$, $t\in S$, taking into account the respective multiplier ideal sheaves of $K_{\widetilde{\mathcal{W}}}$ and $K_{\widehat{W}_t}$.

Proof. We first consider the much easier case $k = 0$ (which does not require to take an extension $\mathcal{Z} \rightsquigarrow \mathcal{Z}^a$). Then $\mu : \widehat{\mathcal{Z}} \to \mathcal{Z} \subset \mathcal{X} \xrightarrow{\pi} S$ is a fibration over S, we have $\widetilde{W} = T_{\mathcal{Z}} \subset T_{\mathcal{X}}$, and the pull-back of (5.1.12) by μ reduces to an exact sequence of sheaves

$$0 \to T_{\widehat{\mathcal{Z}}/S} \to T_{\widehat{\mathcal{Z}}} \to (\pi \circ \mu)^* T_S \to 0.$$

It restricts to an exact sequence of vector bundles on a neighborhood of a generic (smooth) fiber $\widehat{Z}_t$, and $T_{\widehat{\mathcal{Z}}/S\,|\widehat{Z}_t} = T_{\widehat{Z}_t} = \widehat{W}_t$ by definition. We then get a composition of morphisms

$$(5.1.14_0) \qquad (\mu^* K_{\widetilde{\mathcal{W}}})_{|\widehat{Z}_t} \longrightarrow K_{\widehat{\mathcal{Z}}\,|\widehat{Z}_t} \xrightarrow{\simeq} K_{\widehat{Z}_t} \otimes (\pi \circ \mu)^* K_S,$$

which is just the morphism whose existence is asserted in the Lemma. The first arrow is well defined everywhere since by definition (incorporating multiplier ideals) the canonical sheaf $K_{\widetilde{\mathcal{W}}}$ is obtained by restricting smooth sections of the appropriate exterior power $\Lambda^q T^*_{\mathcal{X}}$ to $\mathcal{Z}$, an operation followed by pulling-back via a morphism $i_{\mathcal{Z}} \circ \mu : \widehat{\mathcal{Z}} \to \mathcal{Z} \subset \mathcal{X}$ between nonsingular varieties. This completes the case $k = 0$.

For $k \geq 1$, the situation is more involved: in particular the induced linear structure $\widehat{\mathcal{W}}$ on $\widehat{\mathcal{Z}}$ will probably remain singular, and $\widehat{W}_t$ may be singular as well on $\widehat{Z}_t$ even when $\widehat{Z}_t$ is nonsingular. In any case, (5.1.12) gives an isomorphism

$$(K_{\widetilde{\mathcal{W}}})_{|Z_t} \longrightarrow K_{\mathcal{W}} \otimes (\pi \circ \pi_{k,0})^* K_S \otimes \mathcal{O}_{X_k}(-N\,\mathbf{1})_{|Z_t}$$

at the generic point of Z_t ($t \in S$ being itself generic). By pulling-back via μ, we get an isomorphism

$$(5.1.14_k) \qquad (\mu^* K_{\widetilde{\mathcal{W}}})_{|\widehat{Z}_t} \longrightarrow K_{\widehat{W}_t} \otimes \mu^*\big((\pi \circ \pi_{k,0})^* K_S \otimes \mathcal{O}_{X_k}(-N\,\mathbf{1})\big)_{|\widehat{Z}_t}$$

at the generic point of $\widehat{Z}_t$. It is obtained from a fibration between non singular varieties

$$\pi \circ \pi^a_{k,0} \circ \mu : \widehat{\mathcal{Z}} \subset \widehat{\mathcal{X}}^a_k \to S$$

and we only consider what happens when $\widehat{Z}_t$ is a nonsingular fiber. We still have to show that ($5.1.14_k$) is everywhere defined and factorizes through the corresponding multiplier ideals. The hypothesis that μ resolves the indeterminacies of the Grassmannian structure maps of $\mathcal{W}$ and $\widetilde{\mathcal{W}}$ implies that we get an exact sequence of *vector bundles*

$$(5.1.15) \quad 0 \to \mu^*\mathcal{W} \to \mu^*\widetilde{\mathcal{W}} \to \mathcal{F} \to 0, \quad \mathcal{F} \subset \mu^*\mathcal{S}_{k\,|\widehat{\mathcal{Z}}}, \quad \mathcal{S}_k = (\pi \circ \pi_{k,0})^* T_S \otimes \mathcal{O}_{\mathcal{X}_k}(\mathbf{1})\big)_{|\mathcal{Z}}$$

over $\widehat{\mathcal{Z}}$ [here $\mu^*\mathcal{W}$, $\mu^*\widetilde{\mathcal{W}}$ mean pull-backs of *linear spaces*, not pull-backs of sheaves, namely, one takes the closure of fibers over the open set of regular points]. In fact, we have a map $\psi = (\varphi, \varphi^a{}_{|\mathcal{Z}})$ into the flag bundle $p : F_{r,N+r}(\mathcal{V}_k, \widetilde{\mathcal{V}}_k) \to \mathcal{X}_k$ of subspaces $L \subset \mathcal{V}_k$, $L' \subset \widetilde{\mathcal{V}}_k$ of respective dimensions $(r, N + r)$ with $L \subset L'$. Let $\mathcal{L}$ and $\mathcal{L}'$ be the tautological vector bundles of rank r and $N + r$ on this flag bundle. By construction $\psi \circ \mu$ is holomorphic, while $\mathcal{W} = \psi^*\mathcal{L}$, $\widetilde{\mathcal{W}} = \psi^*\mathcal{L}'$ and $\widetilde{\mathcal{V}}_k/\mathcal{V}_k \simeq \mathcal{S}_k$. Therefore $\mu^*\mathcal{W} = (\psi \circ \mu)^*\mathcal{L}$ and $\mu^*\widetilde{\mathcal{W}} = (\psi \circ \mu)^*\mathcal{L}'$ are genuine vector bundles, $\mu^*\mathcal{W}$ is a subbundle of $\mu^*\widetilde{\mathcal{W}}$ and

$$\mu^*\widetilde{\mathcal{W}}/\mu^*\mathcal{W} = (\psi \circ \mu)^*(\mathcal{L}'/\mathcal{L})$$

is of rank N. On the flag bundle, there is also a natural morphism

$$\mathcal{L}'/\mathcal{L} \to p^*\widetilde{\mathcal{V}}_k/p^*\mathcal{V}_k = p^*\mathcal{S}_k$$

between rank N bundles, whose determinant vanishes along a certain intrinsically defined divisor Δ, so that $\det(\mathcal{L}'/\mathcal{L}) = p^* \det \mathcal{S}_k \otimes \mathcal{O}(-\Delta)$. In the end, we get

$$\det \mathcal{F} = \mu^*\Lambda^N \mathcal{S}_k \otimes (\psi \circ \mu)^*\mathcal{O}(-\Delta).$$

Now, let u be a local section of $K_{\widetilde{\mathcal{W}}}$. By definition u is the restriction of a local holomorphic section u' of $\Lambda^{N+r}T^*_{\mathcal{X}^a}$ near a point $x \in \mathcal{Z}$. In order to construct its image by (5.1.14$_k$), we pick a local generator $\tau_1 \wedge \ldots \wedge \tau_N$ of $\Lambda^N T_S$ near $s = \pi \circ \pi_{k,0}(x)$. By Lemma 5.1.6 (c), the local vector fields τ_j can be lifted as vector fields $\widetilde{\tau}_j$ on $\mathcal{X}$ near $\pi_{k,0}(x)$, and then as vector fields $\widetilde{\tau}_j^{(k)}$ tangent to $\mathcal{V}^a_k \subset T_{\mathcal{X}^a_k}$ near x. The image of the wedge product $\widetilde{\tau}_1^{(k)} \wedge \ldots \wedge \widetilde{\tau}_N^{(k)}{}_{|\mathcal{X}_k}$ in $\Lambda^N \mathcal{S}_k$ via (5.1.1) is a local generator θ of

$$\Lambda^N \mathcal{S}_k = (\pi \circ \pi_{k,0})^* K_S^{-1} \otimes \mathcal{O}_{\mathcal{X}_k}(N\,\mathbf{1}),$$

and we want to emphasize here that the construction of the τ_j's, $\widetilde{\tau}_j$'s, and θ is made entirely on nonsingular spaces. Let δ be a local section of $(\psi \circ \mu)^*\mathcal{O}(-\Delta)$ on $\widehat{\mathcal{Z}}$ whose zero divisor is $(\psi \circ \mu)^*\Delta$. By what we have done, $\delta\,\mu^*\theta$ is a local generator of $\det \mathcal{F}$, and thanks to (5.1.15), $\delta\,\mu^*\theta$ has a local holomorphic lifting ξ with values in $\mathcal{O}(\mu^*\Lambda^N\widetilde{\mathcal{W}}) \subset \mathcal{O}(\mu^*\Lambda^N T_{\mathcal{X}^a_k})$. We contract μ^*u' with ξ to get a local section $\mu^*u' \cdot \xi$ of $\Lambda^r T^*_{\mathcal{X}^a_k}$, and define in this way a morphism

$$\mu^* u \longmapsto (\mu^* u' \cdot \xi) \otimes (\delta\,\mu^*\theta)^{-1}{}_{|\widehat{Z}_t} \tag{5.1.16}$$

with values in

$$K_{\widehat{W}_t} \otimes \mu^*\big((\pi \circ \pi_{k,0})^* K_S \otimes \mathcal{O}_{X_k}(-N\,\mathbf{1})\big) \otimes \mathcal{O}((\psi \circ \mu)^*\Delta)_{|\widehat{Z}_t}. \tag{5.1.17}$$

Observe that $(\mu^*u' \cdot \theta')_{|\widehat{Z}_t}$ is the restriction of a section of the ambient r-form bundle

$$\mu^*\Lambda^r T^*_{\mathcal{X}^a_k} \subset \Lambda^r T^*_{\widehat{\mathcal{X}}^a_k},$$

hence it does restrict to a section of $K_{\widehat{W}_t}$ on $\widehat{Z}_t$ when the latter is equipped with its corresponding multiplier ideal sheaf. A priori, (5.1.16) seems to be dependent on the choice of our liftings $\widetilde{\tau}_j$, u' and ξ, but as it coincides with the intrinsically defined morphism (5.1.14$_k$) at the generic point of $\mathcal{Z}$, it must be uniquely defined. We further pass to the integral closures on both sides of (5.1.16) to reach what we defined to be the multiplier ideals, according to Prop. 1.2. The only potential trouble to complete the proof of Lemma 5.1.13 is the presence of the divisor $(\psi \circ \mu)^*\Delta$ in (5.1.17). This is overcome, at least in the special case we need, by the next lemma. □

5.1.18. Lemma. *Assume that $(\mathcal{Z}, \mathcal{W})$ is induced by the k-th stage $(\mathcal{Z}_k, \mathcal{W}_k)$ of a directed subvariety $(\mathcal{Z}_0, \mathcal{W}_0) \subset (\mathcal{X}, T_{\mathcal{X}/S})$ such that $\pi_{k,k-1}(\mathcal{Z}) = \mathcal{Z}_{k-1}$. If $N \geq (k+1)(n-1)$, one can arrange the choice of $\mathcal{Z}^a \subset \mathcal{X}^a_k$ in Lemma 5.1.7 and of the desingularization μ in Lemma 5.1.13 in such a way that the generic fiber $\widehat{Z}_t$ does not meet the support of $(\psi \circ \mu)^*\Delta$ in (5.1.17) (hence the projection of $(\psi \circ \mu)^{-1}(\Delta) \subset \widehat{\mathcal{Z}} \subset \widehat{\mathcal{X}}^a_k$ on S will be contained in a proper algebraic subvariety of S). We then say that the extension $\mathcal{Z} \rightsquigarrow \mathcal{Z}^a$ is equisingular.*

Proof. One can first take a desingularization $\mu_0 : \widehat{\mathcal{Z}}_0 \to \mathcal{Z}_0$ of $\mathcal{Z}_0$, and find a Zariski open set $S' \subset S$ over which $\pi \circ \mu_0 : \widehat{\mathcal{Z}}_0 \to S$ is a submersion. We replace the family $\mathcal{X} \to S$ by its restriction $\widehat{\mathcal{Z}}'_0 := (\pi \circ \mu_0)^{-1}(S') \to S'$ and observe that it is enough to extend $\mathcal{Z}$ into $\mathcal{Z}^a$ within the relative and absolute Semple towers of $\widehat{\mathcal{Z}}'_0 \to S'$ (an arbitrary extension over $\widehat{\mathcal{X}}^a_{k-1} \supset \widehat{\mathcal{Z}}^a_{k-1}$ will then do). Therefore, it is enough to prove Lemma 5.1.18 when $\mathcal{Z}_0 = \mathcal{X}$ and $\pi_{k,k-1}(\mathcal{Z}) = \mathcal{X}_{k-1}$ (after a replacement of our ambient space $\mathcal{X}$ by $\mathcal{X}' = \widehat{\mathcal{Z}}'_0$ of smaller dimension $n' \leq n$). We set $r = \operatorname{rank} \mathcal{W}$ and notice that

$$\dim \mathcal{Z} = \dim \mathcal{X}_{k-1} + \operatorname{rank} T_{\mathcal{Z}/\mathcal{X}_{k-1}} = \dim \mathcal{X}_{k-1} + r - 1.$$

On the other hand, we have to take

$$\dim \mathcal{Z}^a = \dim \mathcal{X}^a_{k-1} + N + r - 1 < \dim \mathcal{X}^a_k = \dim \mathcal{X}^a_{k-1} + N + n - 1,$$

hence the codimension of $\mathcal{Z}$ in $\mathcal{Z}^a$ is extremely large, while the codimension of $\mathcal{Z}^a$ in $\mathcal{X}_k^a$ is small, equal to $n-r$ (this makes the result not so surprising).

For the sake of explaining the argument in simpler terms, first assume that $\mathcal{Z}$ and $\mathcal{W}$ are nonsingular. We produce $\mathcal{Z}^a \subset \mathcal{X}_k^a$ as a complete intersection $\{\sigma_j = 0\}$ of codimension $n-r$ in $\mathcal{X}_k^a$. We claim that we can take $\mathcal{Z}^a$ to be nonsingular along any fixed smooth fiber Z_t, $t = t_0$, if $N \geq (k+1)(n-1)$. In fact, if $\mathcal{I}_\mathcal{Z}$ is the ideal sheaf of $\mathcal{Z}$ in $\mathcal{X}_k^a$, we can take the sections $\sigma_j \in H^0(\mathcal{X}_k^a, \mathcal{I}_\mathcal{Z} \otimes A)$ in a sufficiently ample line bundle A on $\mathcal{X}_k^a$, chosen so that global sections σ vanishing along $\mathcal{Z}$ still generate all possible 1-jets at any point of Z_t, i.e. their differentials $d\sigma$ generate the conormal bundle $\mathcal{N}^*_{\mathcal{Z}|Z_t} = (T_{\mathcal{X}_k^a}/T_\mathcal{Z})^*_{|Z_t} = (\mathcal{I}_\mathcal{Z}/\mathcal{I}_\mathcal{Z}^2)_{|Z_t}$; it is enough to take $A \gg 0$ so that $(\mathcal{N}^*_\mathcal{Z} \otimes A)_{|Z_t}$ is generated by sections and the groups $H^1(\mathcal{X}_k^a, \mathcal{I}_\mathcal{Z}^2 \otimes A)$, $H^1(\mathcal{X}_k^a, \mathcal{I}_{Z_t} \otimes \mathcal{N}^*_\mathcal{Z} \otimes A)$ vanish, thanks to the exact sequences

$$H^0(\mathcal{X}_k^a, \mathcal{I}_\mathcal{Z} \otimes A) \to H^0(\mathcal{X}_k^a, \mathcal{I}_\mathcal{Z}/\mathcal{I}_\mathcal{Z}^2 \otimes A) \to H^1(\mathcal{X}_k^a, \mathcal{I}_\mathcal{Z}^2 \otimes A) = 0,$$
$$H^0(\mathcal{X}_k^a, \mathcal{N}^*_\mathcal{Z} \otimes A) \to H^0(Z_t, (\mathcal{N}^*_\mathcal{Z} \otimes A)_{|Z_t}) \to H^1(\mathcal{X}_k^a, \mathcal{I}_{Z_t} \otimes \mathcal{N}^*_\mathcal{Z} \otimes A) = 0.$$

The conormal bundle $\mathcal{N}^*_\mathcal{Z}$ projects surjectively onto the vector bundle $(\widetilde{\mathcal{V}}_k/\mathcal{W})^*$ of rank $N+n-r$. What we want is that the differentials $d\sigma_1, \ldots, d\sigma_r$ be pointwise linearly independent as sections of $(\widetilde{\mathcal{V}}_k/\mathcal{W})^*$ along Z_t ; we then get transverse intersections

$$\mathcal{W} = \mathcal{V}_k \cap \{d\sigma_j = 0\}, \qquad \widetilde{\mathcal{W}} = \widetilde{\mathcal{V}}_k \cap \{d\sigma_j = 0\}.$$

Now, $\dim Z_t = n + (k-1)(n-1) + (r-1) \leq (k+1)(n-1)$, and by a well known argument, our condition on the independence of $n-r$ sections is true for generic sections in a spanned vector bundle of rank $\geq \dim Z_t + (n-r)$, i.e. $N \geq \dim Z_t$. Then $\widetilde{\mathcal{W}}$ will be nonsingular, and $\widetilde{\mathcal{W}}/\mathcal{W} \simeq \widetilde{\mathcal{V}}_k/\mathcal{V}_k = \mathcal{S}_k$ along Z_t, as desired.

When $\mathcal{Z}$ and $\mathcal{W}$ are singular, we take an embedded desingularization $\mu : \widehat{\mathcal{Z}} \to \mathcal{Z} \subset \mathcal{X}_k^a$ in such a way that $\mu^*\mathcal{W}$ becomes a nonsingular subbundle of $\mu^*\mathcal{V}_k$ (for this, we resolve the indeterminacies of the meromorphic map φ to the relevant Grassmannian bundle, as already explained in Lemma 5.1.13). We work on a fixed nonsingular fiber $\widehat{Z}_t$. What we want is that the pull-backs $\mu^*d\sigma_j$, $1 \leq j \leq n-r$ still cut out a nonsingular rank N vector subbundle in the rank $N+n-r$ bundle $(\mu^*\widetilde{\mathcal{V}}_k/\mu^*\mathcal{W})^*$, in restriction to the fiber $\widehat{Z}_t$. This subbundle will become our $\mu^*\widetilde{\mathcal{W}}$ [and as it will be already nonsingular, there will be no need to resolve further the indeterminacies of $\mu^*\widetilde{\mathcal{W}}$ along the given fiber $\widehat{Z}_t$]. Moreover $(\psi \circ \mu)^{-1}(\Delta)$ will not meet $\widehat{Z}_t$ by construction. The embedded resolution of $\mathcal{Z}$ in $\widehat{\mathcal{X}}_k^a$ yields $\mu^*\mathcal{I}_\mathcal{Z} = \mathcal{I}_{\widehat{\mathcal{Z}}} \cdot \mathcal{O}(-\sum m_j E_j)$ where $\widehat{\mathcal{Z}}$ is the strict transform of $\mathcal{Z}$ and the E_j's are the exceptional divisors, which we can assume to be normal crossing and intersecting $\widehat{\mathcal{Z}}$ transversally. The pull-back $\mu^*\sigma$ of any section $\sigma \in H^0(\mathcal{X}_k^a, A \otimes \mathcal{I}_\mathcal{Z})$ can be seen as a section

$$\mu^*\sigma \in H^0(\widehat{\mathcal{X}}_k^a, \mathcal{I}_{\widehat{\mathcal{Z}}} \otimes \mu^*A \otimes \mathcal{O}(-\textstyle\sum m_j E_j)) \subset H^0(\widehat{\mathcal{X}}_k^a, \mu^*A \otimes \mathcal{O}(-\textstyle\sum m_j E_j)).$$

By taking $A \gg 0$, we can achieve that the differentials $d(\mu^*\sigma)_{|\widehat{Z}_t}$, viewed as sections of

$$H^0(\widehat{Z}_t, \mathcal{N}^*_{\widehat{\mathcal{Z}}} \otimes \mu^*A \otimes \mathcal{O}(-\textstyle\sum m_j E_j)_{|\widehat{Z}_t})$$

still generate the normal bundle of $\widehat{\mathcal{Z}}$ along $\widehat{Z}_t \subset \mu^{-1}(Z_t)$. For this, imitating what we did in the nonsingular case, we rely on the fact that $\mathcal{I}_\mathcal{Z}/\mathcal{I}_\mathcal{Z}^2 \otimes A$ is generated by global sections on $\mathcal{Z}$ for $A \gg 0$, and on the vanishing properties

$$H^1(\widehat{\mathcal{X}}_k^a, \mu^*(\mathcal{I}_\mathcal{Z}^2 \otimes A)) = 0, \qquad H^1(\widehat{\mathcal{X}}_k^a, \mu^*(\mathcal{I}_{Z_t} \otimes (\mathcal{I}_\mathcal{Z}/\mathcal{I}_\mathcal{Z}^2) \otimes A)) = 0$$

obtained from the Leray spectral sequence and from $R^q\mu_*\mathcal{O}_{\widehat{\mathcal{X}}_k^a} = 0$ for $q > 0$. The same dimension argument as in the nonsingular case then allows us to choose $\sigma_1, \ldots, \sigma_{n-r}$ so that $\mu^*\widetilde{\mathcal{W}}/\mu^*\mathcal{W} \simeq \mu^*(\widetilde{\mathcal{V}}_k/\mathcal{V}_k) = \mu^*\mathcal{S}_k$ along $\widehat{Z}_t$, and we are done. □

End of proof of Theorem 5.1. If we combine (5.1.9) and Lemmas 5.1.13, 5.1.18 (thus, under the assumption $N \geq (k+1)(n-1)$), we get a restriction morphism

$$(\pi_{k,0} \circ \mu)^* \Lambda^{q'} T^*_{\mathcal{X}|\widehat{Z}_t} \longrightarrow K_{\widehat{W}_t} \otimes \mu^* \big((\pi \circ \pi_{k,0})^* K_S \otimes \mathcal{O}_{X_k}(\mathbf{a} - N\,\mathbf{1})\big)_{|\widehat{Z}_t}. \tag{5.1.19}$$

Since $\Lambda^{q'} T^*_{\mathcal{X}}$ is relatively ample over S and the factor $(\pi \circ \pi_{k,0})^* K_{S\,|Z_t}$ is trivial, we see after multiplication by an additional small relatively ample term $\mathcal{O}_{X_k}(\varepsilon \mathbf{c})$ that there is a weight $\mathbf{b} \in \mathbb{N}^k$ such that $K_{\widehat{W}_t} \otimes \mu^* \mathcal{O}_{X_k}(\mathbf{b})_{|\widehat{Z}_t}$ is big, hence (Z_t, W_t) is of general type modulo $X_{k,t} \to X_t$ for generic t (it is helpful here to know that $a_j > q - 1 \geq N$). This is a contradiction, hence $p_{k,\delta}$ is not dominant and $S_{k,\delta} = \overline{p_{k,\delta}(S)} \subsetneq S$ (here we reproject down to S in case there were a finite cover $\widehat{S}^0 \to S$). If $N = \dim S$ is too small, we can artificially increase the dimension of S by replacing S with $S \times \mathbb{C}^m$, $m \gg 1$, and put $X_{(t,t')} = X_t$, $\mathcal{Z}_{(t,t')} = \mathcal{Z}_t$, $(t,t') \in S \times \mathbb{C}^m$. The projection of the extended $p_{k,\delta}$'s to $S \times \mathbb{C}^m$ is then non dominant, but since the property of $\mathcal{Z}_{(t,t')}$ to be strongly of general type does not depend on t', we conclude that the projection to S itself is non dominant; in fact, the argument amounts to add many additional deformation parameters that are used to "distort" $\mathcal{Z}^a$ along the fibers $\widehat{Z}_t$, and so achieve the required equisingularity property of $\mathcal{Z}^a$ in Lemma 5.1.7. We conclude that X_t is algebraically jet-hyperbolic for $t \in S \smallsetminus \bigcup_{k,\delta} S_{k,\delta}$ and Theorem 5.1 is proved. □

5.2. Remarks. In fine, the main argument of the proof is the existence of a non trivial morphism given by (5.1.9). If for all relevant subvarieties $\mathcal{Z} \subset \mathcal{X}_k$ one can find an ample subbundle $\mathcal{A} \subset \Lambda^{q'} T_{\mathcal{X}}$ such that the composition

$$(\pi_{k,0})^* \mathcal{A}_{|\mathcal{Z}} \longrightarrow (\pi_{k,0})^* \Lambda^{q'} T^*_{\mathcal{X}\,|\mathcal{Z}} \longrightarrow K_{\widetilde{\mathcal{W}}} \otimes \mathcal{O}_{\mathcal{X}_k}(\mathbf{a})_{|\mathcal{Z}}, \quad \mathbf{a} \in \mathbb{Q}_+^k$$

is non zero, then the conclusion still holds. This may allow to weaken the hypotheses on the positivity of $\Lambda^{q'} T_{\mathcal{X}}$.

Also, one can in fact get the stronger conclusion that the very general fiber X_t is algebraically fully jet-hyperbolic in the sense of Definition 4.1 (b). The only change is that we need a more sophisticated version of Lemma 5.1.18; instead of just desingularizing $\mathcal{Z}_0 \subset \mathcal{X}$ and $\mathcal{Z} \subset \mathcal{X}_k$, we perform embedded desingularizations step by step for all intermediate projections $\mathcal{Z}_\ell = \pi_{k,\ell}(\mathcal{Z})$, in case the ranks of the associated directed structures $\mathcal{W}_\ell$ gradually change. The arguments are essentially the same and are left to the reader. □

5.3. Universal family of complete intersections. Let us consider the universal family of complete intersections of dimension n, codimension c and type $(d_1, \ldots, d_c)$ in complex projective $\mathbb{P}^{n+c} = P(E)$, where $E \simeq \mathbb{C}^{n+c+1}$ is a complex vector space. We can view it as a smooth family $\pi = \mathrm{pr}_1 : \mathcal{X} \to S$ where S is a Zariski open set in

$$\overline{S} = \prod_{1 \leq j \leq c} \mathrm{Sym}^{d_j} E^* \simeq \prod \mathbb{C}^{N_j} = \mathbb{C}^N, \qquad N_j = \binom{d_j + n + c}{n + c},$$

and $\mathcal{X} \subset S \times P(E)$ is the incidence variety defined by

$$\begin{cases} t = (t_1, \ldots, t_c) \in S, \quad z \in E \simeq \mathbb{C}^{n+c+1}, \quad t_j \simeq (t_{j,\alpha}) \in \mathrm{Sym}^{d_j} E^*, \\ P_j(t,z) := t_j \cdot z^{d_j} = \displaystyle\sum_{|\alpha| = d_j} t_{j,\alpha} z^\alpha, \quad 1 \leq j \leq c, \quad \alpha = (\alpha_\ell) \in \mathbb{N}^{n+c+1}, \\ \mathcal{X} = \big\{(t,[z]) \in S \times P(E)\,;\ P_j(t,z) = 0,\ 1 \leq j \leq c\big\}. \end{cases} \tag{5.3.1}$$

We denote by $\mathrm{pr}_1 : \mathcal{X} \to S$ and $\mathrm{pr}_2 : \mathcal{X} \to P(E) \simeq \mathbb{P}^{n+c}$ the natural projections. Here S is the set of coefficients $t \in \mathbb{C}^N$ that define a nonsingular subvariety $X_t = \mathrm{pr}_1^{-1}(t)$ of codimension c in $\mathbb{P}^{n+c}$ (or rather $\{t\} \times \mathbb{P}^{n+c}$). Notice that there is a natural action of $\mathrm{GL}(E) = \mathrm{GL}(n+c+1, \mathbb{C})$ on $\mathcal{X}$ defined by

$$g \cdot ((t_j), [z]) = ((t_j \circ g^{-1}), [g \cdot z]), \quad g \in \mathrm{GL}(E),\ t_j \in \mathrm{Sym}^{d_j} E^*, \tag{5.3.2}$$

which simply consists of transforming the equations via an arbitrary linear change of coordinates. We use the following famous result proved by Claire Voisin [Voi96, Corollary 1.3] (with the substitution of notation $k \mapsto c$, $n \mapsto n+c$, $l \mapsto N+n-q$ in our setting).

5.4. Proposition ([Voi96]). *Over any affine Zariski open set $S^0 \subset S$, the twisted tangent bundle $T_{\mathcal{X}} \otimes \mathrm{pr}_2^* \mathcal{O}_{\mathbb{P}^{n+c}}(1)$ is generated by sections. Moreover, the vector bundle $\Lambda^q T^*_{\mathcal{X}}$ is generated by sections for $\sum d_j \geq 2n+c+N+1-q$, and it is relatively very ample with respect to the projection $\mathcal{X} \to S$ for $\sum d_j > 2n+c+N+1-q$.*

Proof. Since the argument can be made very simple and very short, we give it here for the sake of completeness. If $(\varepsilon_j)_{0 \leq j \leq n+c}$ denotes the canonical basis of $\mathbb{Z}^{n+c+1}$, we get sections of the tangent bundle of cone$(\mathcal{X})$ over $\mathcal{X}$ in $S \times E \simeq S \times \mathbb{C}^{n+c+1}$ by taking the explicit vector fields

$$\xi_{\ell,m} := z_m \frac{\partial}{\partial z_\ell} - \sum_{1 \leq j \leq c,\, |\alpha| = d_j} \alpha_\ell t_{j,\alpha} \frac{\partial}{\partial t_{j,\alpha-\varepsilon_\ell+\varepsilon_m}}, \quad 0 \leq \ell, m \leq n+c,\ \alpha \in \mathbb{N}^{n+c+1}, \tag{5.4.1}$$

$$\eta_{j,\alpha,\ell,m} := z_m \frac{\partial}{\partial t_{j,\alpha}} - z_\ell \frac{\partial}{\partial t_{j,\alpha-\varepsilon_\ell+\varepsilon_m}}, \quad |\alpha| = d_j,\ \alpha_\ell > 0,\ 0 \leq \ell \neq m \leq n+c, \tag{5.4.2}$$

which all yield zero derivative when applied to any of the polynomials $P_j(t,z)$. In fact the vector fields (5.4.1) are just the Killing vector fields induced by the action of $\mathrm{GL}(E)$ on cone$(\mathcal{X})$. The natural $\mathbb{C}^*$ action defined by $\lambda \cdot (t,z) = (t, \lambda z)$ has an associated Euler vector field $\varepsilon = \sum_{0 \leq \ell \leq n+c} z_\ell\, \partial/\partial z_\ell$. By taking the quotient with the rank 1 subbundle $\mathcal{O}_{\mathcal{X}} \cdot \varepsilon$, the $\xi_{\ell,m}$'s actually define sections of $T_{\mathcal{X}}$ (homogeneity degree 0 in z), while the $\eta_{j,\alpha,\ell,m}$'s define sections of $T_{\mathcal{X}} \otimes \mathrm{pr}_2^* \mathcal{O}_{\mathbb{P}^{n+c}}(1)$ (homogeneity degree 1 in z). We claim that the vector fields

$$(t,[z]) \mapsto \xi_{\ell,m} z_p \mod \mathcal{O}_{\mathcal{X}} \cdot \varepsilon, \quad (t,[z]) \mapsto \eta_{j,\alpha,\ell,m} \mod \mathcal{O}_{\mathcal{X}} \cdot \varepsilon$$

generate $T_{\mathcal{X}} \otimes \mathrm{pr}_2^* \mathcal{O}_{\mathbb{P}^{n+c}}(1)$ at every point. In fact, as the $\xi_{\ell,m}$ already provide all "vertical" z-directions, we need only check that is is enough to add one (non tangent) "horizontal" vector field $\partial/\partial t_{j,\alpha_0^j}$ for each $j = 1, \ldots, c$ to generate the whole ambient tangent space $T_{\mathbb{P}^{n+c} \times \mathbb{C}^N, x}$, since the claim then follows by a trivial (co)dimension argument. At a point $(t,[z])$ where $z_0 \neq 0$ (say), we take $\alpha_0^j = (d_j, 0, \ldots, 0) \in \mathbb{N}^{n+c+1}$. Together with $\partial/\partial t_{j,\alpha_0^j}$, the vector fields

$$z_0^{-1} \eta_{j,\alpha,\ell,0} := \frac{\partial}{\partial t_{j,\alpha}} - z_0^{-1} z_\ell \frac{\partial}{\partial t_{j,\alpha-\varepsilon_\ell+\varepsilon_0}}, \quad \alpha_\ell > 0, \quad \ell \neq 0 \tag{5.4.3}$$

then generate $T_{\mathbb{C}^{N_j}}$ by a simple triangular matrix argument (increase the value of the 0-th component of α and decrease the α_ℓ's, $\ell \neq 0$, until $\alpha = \alpha_0^j$). Let $\mathcal{O}_S(-1)$ be the tautological line bundle on S (coming from the tautological line bundle $\mathcal{O}_{P(\overline{S})}(-1)$ on $P(\overline{S})$). Since $K_{S \times \mathbb{P}^{n+c}} = \mathrm{pr}_1^* K_S \otimes \mathrm{pr}_2^* \mathcal{O}_{\mathbb{P}^{n+c}}(-n-c-1)$ and $\mathcal{X}$ is defined by sections of the line bundles $\mathrm{pr}_1^* \mathcal{O}_S(-1) \otimes \mathrm{pr}_2^* \mathcal{O}_{\mathbb{P}^{n+c}}(d_j)$, the adjunction formula gives

$$K_{\mathcal{X}} = \Lambda^{N+n} T^*_{\mathcal{X}} = \mathcal{L}_S \otimes \mathrm{pr}_2^* \mathcal{O}_{\mathbb{P}^{n+c}}\big(\textstyle\sum d_j - n - c - 1\big).$$

where $\mathcal{L}_S$ is the line bundle $\mathrm{pr}_1^*(\det T_S^* \otimes \mathcal{O}_S(-c))$ (this bundle plays no role in the sequel since it can be made trivial by restricting S to a suitable affine chart). Therefore

$$\begin{aligned}(5.4.4)\quad \Lambda^q T_{\mathcal{X}}^* &= K_{\mathcal{X}} \otimes \Lambda^{N+n-q} T_{\mathcal{X}}\\ &= \mathcal{L}_S \otimes \mathrm{pr}_2^* \mathcal{O}_{\mathbb{P}^{n+c}}\big(\textstyle\sum d_j - n - c - 1\big) \otimes \Lambda^{N+n-q} T_{\mathcal{X}}\\ &= \mathcal{L}_S \otimes \mathrm{pr}_2^* \mathcal{O}_{\mathbb{P}^{n+c}}\big(\textstyle\sum d_j - 2n - c - N - 1 + q\big) \otimes \Lambda^{N+n-q}\big(T_{\mathcal{X}} \otimes \mathrm{pr}_2^* \mathcal{O}(1)\big).\end{aligned}$$

As $T_{\mathcal{X}} \otimes \mathrm{pr}_2^* \mathcal{O}(1)$ is generated by sections, Prop. 5.4 follows immediately. □

If we want the relative ampleness of $\Lambda^q T_{\mathcal{X}}^*$ to hold for $q > N$, we need $\sum d_j \geq 2n + c + 1$. Theorem 5.1 then implies:

5.5. Corollary (solution of the Kobayashi conjecture). *For all $n, c \geq 1$ and d_j such that $\sum d_j \geq 2n + c + 1$, the very general complete intersection of type $(d_1, \ldots, d_c)$ in complex projective space $\mathbb{P}^{n+c}$ is algebraically jet-hyperbolic, and thus Kobayashi hyperbolic.*

The simplest non trivial situation is the surface case $n = 2$ in codimension $c = 1$. We then obtain the Kobayashi hyperbolicity of a very general surface $X \subset \mathbb{P}^3$ of degree $d \geq 6$. The result seems to be new even in this case, although Duval [Duv04] has shown by elementary means the existence of a hyperbolic sextic (from this, it already follows that there is a family of hyperbolic sextics over an open set of parameters in Hausdorff topology). Geng Xu [Xu95] has shown that a very general quintic surface X does not contain curves of genus $g \leq 2$, but as far as we know, this is not enough to conclude that X is Kobayashi hyperbolic.

5.6. Remark. It would be good to know if Kobayashi hyperbolicity is a Zariski open condition, in particular, whether one can replace "very general" by "general" in Cor. 5.5. This would require further investigations, but such a result might be accessible by taking into account Remark 3.3, which shows that the "bad sets" Z to consider are somehow bounded.

5.7. Remark. In the case $n \geq 2$ and $\sum d_j = 2n + c$ (and especially in the "border case" $d = 2n+1$ of hypersurfaces), it follows from Prop. 5.4 due to [Voi96] that $(\Lambda^q T_{\mathcal{X}}^*)_{|X_t}$ is generated by sections for $q = N+1$ and very ample for $q \geq N+2$. It would then be natural to look at the degeneration sets occurring for all appropriate subvarieties $\mathcal{Z}$ in the various stages of the relative Semple tower. The arguments used by Claire Voisin ([Voi96], Theorem 1.6 and its proof) indicate a possibility to analyze the situation, but certain remaining degeneracies seem to require intricate Wronskian and flag manifold arguments.

5.8. Case of complements. Our techniques also apply to study the Kobayashi hyperbolicity of complements $\mathbb{P}^n \smallsetminus X$, when X is an algebraic hypersurface of degree d in $\mathbb{P}^n$. In fact, if $X = \{P(z_0, \ldots, z_n) = 0\}$, one can introduce the hypersurface

$$Y = \{z_{n+1}^d - P(z_0, \ldots, z_n) = 0\} \subset \mathbb{P}^{n+1}.$$

It is trivial to show that the Kobayashi hyperbolicity of X implies the Kobayashi hyperbolicity of X, since the natural projection

$$\rho : Y \to \mathbb{P}^n, \quad (z_0, \ldots, z_{n+1}) \mapsto (z_0, \ldots, z_n)$$

defines an unramified $d:1$ cover from $Y \smallsetminus \rho^{-1}(X)$ onto $\mathbb{P}^n \smallsetminus X$. We have a universal family $\mathcal{Y} \to S$ by looking at the parameter space given by coefficients of P. This is just a subfamily of the universal family of degree d hypersurfaces, and we only have to check that Prop. 5.4 still applies when we have no dependence on the variable z_{n+1} except for the monomial z_{n+1}^d.

Here the group acting on the ambient projective space $\mathbb{P}^{n+1}$ is taken to be

$$\mathrm{GL}(n+1,\mathbb{C})\times\mathbb{C}^*\subset\mathrm{GL}(n+2,\mathbb{C}),$$

and one can see that the last Killing vector field $z_{n+1}\partial/\partial z_{n+1}+(\ldots)$ introduces some degenerations on $z_{n+1}=0$ – and only there. We easily conclude by our techniques that $\mathrm{ECL}(X)\subset X\cap\{z_{n+1}=0\}$ for P very general of degree $d\geq 2n+2$, but since we also have $\mathrm{ECL}(Y)=\emptyset$, we conclude that $\mathbb{P}^n\smallsetminus X$ is Kobayashi hyperbolic for X very general of degree $d\geq 2n+2$. Zaidenberg [Zai87] has shown that this conclusion fails for $d=2n$. One could hope to improve the bound to $d\geq 2n+1$ by introducing logarithmic Semple jet bundles, as suggested by Dethloff and Lu [DL01], and apply the idea suggested in Remark 5.7.

References

[Cle86] Clemens, H.: *Curves in generic hypersurfaces.* Ann. Sci. École Norm. Sup. **19** (1986), 629–636.

[Dem95] Demailly, J.-P.: *Algebraic criteria for Kobayashi hyperbolic projective varieties and jet differentials.* AMS Summer School on Algebraic Geometry, Santa Cruz 1995, Proc. Symposia in Pure Math., ed. by J. Kollár and R. Lazarsfeld, Amer. Math. Soc., Providence, RI (1997), 285-360.

[Dem97] Demailly, J.-P.: *Variétés hyperboliques et équations différentielles algébriques.* Gaz. Math. **73** (juillet 1997) 3–23, and `http://www-fourier.ujf-grenoble.fr/~demailly/manuscripts/cartan_augm.pdf`.

[DEG00] Demailly, J.-P., El Goul, J.: *Hyperbolicity of generic surfaces of high degree in projective 3-space.* Amer. J. Math. **122** (2000) 515–546.

[Dem11] Demailly, J.-P.: *Holomorphic Morse inequalities and the Green-Griffiths-Lang conjecture.* November 2010, arXiv: math.AG/1011.3636, dedicated to the memory of Eckart Viehweg; Pure and Applied Mathematics Quarterly **7** (2011) 1165–1208.

[DL01] Dethloff, G., Lu, S.S.Y.: *Logarithmic jet bundles and applications.* Osaka J. Math. **38** (2001) 185–237.

[DMR10] Diverio, S., Merker, J., Rousseau, E.: *Effective algebraic degeneracy.* Invent. Math. **180** (2010) 161–223.

[DR13] Diverio, S., and Rousseau, E.: *The exceptional set and the Green-Griffiths locus do not always coincide.* arXiv: math.AG/1302.4756 (v2).

[Duv04] Duval, J.: *Une sextique hyperbolique dans* $\mathbb{P}^3(\mathbb{C})$. Math. Ann. **330** (2004) 473–476.

[Ein88] Ein, L.: *Subvarieties of generic complete intersections.* Invent. Math. **94** (1988), 163–169.

[Ein91] Ein, L.: *Subvarieties of generic complete intersections, II.* Math. Ann. **289** (1991), 465–471.

[GG79] Green, M., Griffiths, P.: *Two applications of algebraic geometry to entire holomorphic mappings.* The Chern Symposium 1979, Proc. Internal. Sympos. Berkeley, CA, 1979, Springer-Verlag, New York (1980), 41–74.

[Kob70] Kobayashi, S. *Hyperbolic manifolds and holomorphic mappings.* Volume 2 of Pure and Applied Mathematics. Marcel Dekker Inc., New York, 1970.

[Kob78] Kobayashi, S. *Hyperbolic complex spaces.* Volume 318 of Grundlehren der Mathematischen Wissenschaften, Springer-Verlag, Berlin, 1998.

[Lan86] Lang, S.: *Hyperbolic and Diophantine analysis.* Bull. Amer. Math. Soc. **14** (1986) 159–205.

[McQ98] McQuillan, M.: *Diophantine approximation and foliations.* Inst. Hautes Études Sci. Publ. Math. **87** (1998) 121–174.

[McQ99] McQuillan, M.: *Holomorphic curves on hyperplane sections of 3-folds.* Geom. Funct. Anal. **9** (1999) 370–392.

[Pau08] Păun, M.: *Vector fields on the total space of hypersurfaces in the projective space and hyperbolicity.* Math. Ann. **340** (2008) 875–892.

[Siu02] Siu, Y.T.: *Some recent transcendental techniques in algebraic and complex geometry.* In: Proceedings of the International Congress of Mathematicians, Vol. I (Beijing, 2002), Higher Ed. Press, Beijing, 2002, 439–448.

[Siu04] Siu, Y.T.: *Hyperbolicity in complex geometry.* In: The legacy of Niels Henrik Abel, Springer, Berlin, 2004, 543–566.

[Siu12] Siu, Y.T: *Hyperbolicity of generic high-degree hypersurfaces in complex projective spaces.* arXiv: math.CV, 1209.2723v3, 87 pages.

[SY96] Siu, Y.T., Yeung, S.K.: *Hyperbolicity of the complement of a generic smooth curve of high degree in the complex projective plane.* Invent. Math. **124** (1996), 573–618.

[Voi96] Voisin, C.: *On a conjecture of Clemens on rational curves on hypersurfaces.* J. Differential Geometry **44** (1996) 200–213.

[Voi98] Voisin, C.: *A correction on "A conjecture of Clemens on rational curves on hypersurfaces".* J. Differential Geometry **49** (1998) 601–611.

[Xu94] Xu Geng: *Subvarieties of general hypersurfaces in projective space.* J. Differential Geom. **39** (1) (1994) 139–172.

[Zai87] Zaidenberg, M.: *The complement to a general hypersurface of degree* $2n$ *in* $\mathbb{CP}^n$ *is not hyperbolic.* (Russian) Sibirsk. Mat. Zh. **28** (3) (1987) 91–100.

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