5.3. Universal family of complete intersections. Let us consider the universal family of complete intersections of dimension n, codimension c and type $(d_1,\ldots,d_c)$ in complex projective $\mathbb{P}^{n+c}=P(E)$, where $E\simeq\mathbb{C}^{n+c+1}$ is a complex vector space. We can view it as a smooth family $\pi=\mathrm{pr}_1:\mathcal{X}\to S$ where S is a Zariski open set in

$$\overline{S}=\prod_{1\le j\le c}\mathrm{Sym}^{d_j}E^*\simeq\prod\mathbb{C}^{N_j}=\mathbb{C}^N,\qquad N_j=\binom{d_j+n+c}{n+c},$$

and $\mathcal{X}\subset S\times P(E)$ is the incidence variety defined by

$$\text{(5.3.1)}\qquad \begin{cases} t=(t_1,\ldots,t_c)\in S,\quad z\in E\simeq\mathbb{C}^{n+c+1},\quad t_j\simeq(t_{j,\alpha})\in\mathrm{Sym}^{d_j}E^*,\\ P_j(t,z):=t_j\cdot z^{d_j}=\sum_{|\alpha|=d_j}t_{j,\alpha}z^\alpha,\quad 1\le j\le c,\quad \alpha=(\alpha_\ell)\in\mathbb{N}^{n+c+1},\\ \mathcal{X}=\big\{(t,[z])\in S\times P(E)\,;\ P_j(t,z)=0,\ 1\le j\le c\big\}.\end{cases}$$

We denote by $\mathrm{pr}_1:\mathcal{X}\to S$ and $\mathrm{pr}_2:\mathcal{X}\to P(E)\simeq\mathbb{P}^{n+c}$ the natural projections. Here S is the set of coefficients $t\in\mathbb{C}^N$ that define a nonsingular subvariety $X_t=\mathrm{pr}_1^{-1}(t)$ of codimension c in $\mathbb{P}^{n+c}$ (or rather $\{t\}\times\mathbb{P}^{n+c}$). Notice that there is a natural action of $\mathrm{GL}(E)=\mathrm{GL}(n+c+1,\mathbb{C})$ on $\mathcal{X}$ defined by

$$\text{(5.3.2)}\qquad g\cdot((t_j),[z])=((t_j\circ g^{-1}),[g\cdot z]),\quad g\in\mathrm{GL}(E),\ t_j\in\mathrm{Sym}^{d_j}E^*,$$

which simply consists of transforming the equations via an arbitrary linear change of coordinates. We use the following famous result proved by Claire Voisin [Voi96, Corollary 1.3] (with the substitution of notation $k\mapsto c$, $n\mapsto n+c$, $l\mapsto N+n-q$ in our setting).

5.4. Proposition ([Voi96]). *Over any affine Zariski open set $S^0\subset S$, the twisted tangent bundle $T_{\mathcal{X}}\otimes\mathrm{pr}_2^*\mathcal{O}_{\mathbb{P}^{n+c}}(1)$ is generated by sections. Moreover, the vector bundle $\Lambda^qT^*_{\mathcal{X}}$ is generated by sections for $\sum d_j\ge 2n+c+N+1-q$, and it is relatively very ample with respect to the projection $\mathcal{X}\to S$ for $\sum d_j>2n+c+N+1-q$.*

Proof. Since the argument can be made very simple and very short, we give it here for the sake of completeness. If $(\varepsilon_j)_{0\le j\le n+c}$ denotes the canonical basis of $\mathbb{Z}^{n+c+1}$, we get sections of the tangent bundle of cone$(\mathcal{X})$ over $\mathcal{X}$ in $S\times E\simeq S\times\mathbb{C}^{n+c+1}$ by taking the explicit vector fields

$$\text{(5.4.1)}\qquad \xi_{\ell,m}:=z_m\frac{\partial}{\partial z_\ell}-\sum_{1\le j\le c,\,|\alpha|=d_j}\alpha_\ell t_{j,\alpha}\frac{\partial}{\partial t_{j,\alpha-\varepsilon_\ell+\varepsilon_m}},\quad 0\le\ell,m\le n+c,\ \alpha\in\mathbb{N}^{n+c+1},$$

$$\text{(5.4.2)}\qquad \eta_{j,\alpha,\ell,m}:=z_m\frac{\partial}{\partial t_{j,\alpha}}-z_\ell\frac{\partial}{\partial t_{j,\alpha-\varepsilon_\ell+\varepsilon_m}},\quad |\alpha|=d_j,\ \alpha_\ell>0,\ 0\le\ell\ne m\le n+c,$$

which all yield zero derivative when applied to any of the polynomials $P_j(t,z)$. In fact the vector fields (5.4.1) are just the Killing vector fields induced by the action of $\mathrm{GL}(E)$ on cone$(\mathcal{X})$. The natural $\mathbb{C}^*$ action defined by $\lambda\cdot(t,z)=(t,\lambda z)$ has an associated Euler vector field $\varepsilon=\sum_{0\le\ell\le n+c}z_\ell\,\partial/\partial z_\ell$. By taking the quotient with the rank 1 subbundle $\mathcal{O}_{\mathcal{X}}\cdot\varepsilon$, the $\xi_{\ell,m}$'s actually define sections of $T_{\mathcal{X}}$ (homogeneity degree 0 in z), while the $\eta_{j,\alpha,\ell,m}$'s define sections of $T_{\mathcal{X}}\otimes\mathrm{pr}_2^*\mathcal{O}_{\mathbb{P}^{n+c}}(1)$ (homogeneity degree 1 in z). We claim that the vector fields

$$(t,[z])\mapsto\xi_{\ell,m}z_p\mod\mathcal{O}_{\mathcal{X}}\cdot\varepsilon,\quad (t,[z])\mapsto\eta_{j,\alpha,\ell,m}\mod\mathcal{O}_{\mathcal{X}}\cdot\varepsilon$$

generate $T_{\mathcal{X}}\otimes\mathrm{pr}_2^*\mathcal{O}_{\mathbb{P}^{n+c}}(1)$ at every point. In fact, as the $\xi_{\ell,m}$ already provide all "vertical" z-directions, we need only check that is is enough to add one (non tangent) "horizontal" vector field $\partial/\partial t_{j,\alpha_0^j}$ for each $j=1,\ldots,c$ to generate the whole ambient tangent space $T_{\mathbb{P}^{n+c}\times\mathbb{C}^N,x}$, since the claim then follows by a trivial (co)dimension argument. At a point $(t,[z])$ where $z_0\neq 0$ (say), we take $\alpha_0^j=(d_j,0,\ldots,0)\in\mathbb{N}^{n+c+1}$. Together with $\partial/\partial t_{j,\alpha_0^j}$, the vector fields

$$z_0^{-1}\eta_{j,\alpha,\ell,0}:=\frac{\partial}{\partial t_{j,\alpha}}-z_0^{-1}z_\ell\frac{\partial}{\partial t_{j,\alpha-\varepsilon_\ell+\varepsilon_0}},\quad \alpha_\ell>0,\quad \ell\neq 0 \tag{5.4.3}$$

then generate $T_{\mathbb{C}^{N_j}}$ by a simple triangular matrix argument (increase the value of the 0-th component of α and decrease the α_ℓ's, $\ell\neq 0$, until $\alpha=\alpha_0^j$). Let $\mathcal{O}_S(-1)$ be the tautological line bundle on S (coming from the tautological line bundle $\mathcal{O}_{P(\overline{S})}(-1)$ on $P(\overline{S})$). Since $K_{S\times\mathbb{P}^{n+c}}=\mathrm{pr}_1^*K_S\otimes\mathrm{pr}_2^*\mathcal{O}_{\mathbb{P}^{n+c}}(-n-c-1)$ and $\mathcal{X}$ is defined by sections of the line bundles $\mathrm{pr}_1^*\mathcal{O}_S(-1)\otimes\mathrm{pr}_2^*\mathcal{O}_{\mathbb{P}^{n+c}}(d_j)$, the adjunction formula gives

$$K_{\mathcal{X}}=\Lambda^{N+n}T^*_{\mathcal{X}}=\mathcal{L}_S\otimes\mathrm{pr}_2^*\mathcal{O}_{\mathbb{P}^{n+c}}\big(\textstyle\sum d_j-n-c-1\big).$$

where $\mathcal{L}_S$ is the line bundle $\mathrm{pr}_1^*(\det T^*_S\otimes\mathcal{O}_S(-c))$ (this bundle plays no role in the sequel since it can be made trivial by restricting S to a suitable affine chart). Therefore

$$\begin{aligned}\Lambda^q T^*_{\mathcal{X}}&=K_{\mathcal{X}}\otimes\Lambda^{N+n-q}T_{\mathcal{X}}\\&=\mathcal{L}_S\otimes\mathrm{pr}_2^*\mathcal{O}_{\mathbb{P}^{n+c}}\big(\textstyle\sum d_j-n-c-1\big)\otimes\Lambda^{N+n-q}T_{\mathcal{X}}\\&=\mathcal{L}_S\otimes\mathrm{pr}_2^*\mathcal{O}_{\mathbb{P}^{n+c}}\big(\textstyle\sum d_j-2n-c-N-1+q\big)\otimes\Lambda^{N+n-q}\big(T_{\mathcal{X}}\otimes\mathrm{pr}_2^*\mathcal{O}(1)\big).\end{aligned}\tag{5.4.4}$$

As $T_{\mathcal{X}}\otimes\mathrm{pr}_2^*\mathcal{O}(1)$ is generated by sections, Prop. 5.4 follows immediately. □

If we want the relative ampleness of $\Lambda^qT^*_{\mathcal{X}}$ to hold for $q>N$, we need $\sum d_j\geq 2n+c+1$. In order to analyze the much more delicate border case $\sum d_j=2n+c$, we will have to introduce further considerations. First, there is an exact sequence

$$0\to T^h_{\mathcal{X}}\to T_{\mathcal{X}}\to\mathrm{pr}_2^*T_{\mathbb{P}^{n+c}}\to 0,\tag{5.4.5}$$

and via the differential $(\mathrm{pr}_1)_*$ of the first projection, the "horizontal subbundle" $T^h_{\mathcal{X}}$ can be seen as a subbundle of the trivial bundle $\mathrm{pr}_1^*T_S$ on $\mathcal{X}$. The Killing vector fields $\xi_{\ell,m}$ (cf. (5.4.1)) are global sections of $T_{\mathcal{X}}$, and their images in $\mathrm{pr}_2^*T_{\mathbb{P}^{n+c}}$ generate that bundle. On the other hand, the $\eta_{j,\alpha,\ell,m}$'s defined in (5.4.2) are global sections of $T^h_{\mathcal{X}}\otimes\mathrm{pr}_2^*\mathcal{O}_{\mathbb{P}^{n+c}}(1)$ and they generate $T^h_{\mathcal{X}}\otimes\mathrm{pr}_2^*\mathcal{O}_{\mathbb{P}^{n+c}}(1)$ as well. A combination of Theorem 5.1 with Proposition 5.4 implies

5.5. Corollary (solution of the Kobayashi conjecture). *For all $n,c\geq 1$ and d_j such that $\sum d_j\geq 2n+c+1$, the very general complete intersection of type $(d_1,\ldots,d_c)$ in complex projective space $\mathbb{P}^{n+c}$ is Kobayashi hyperbolic. The same conclusion holds for $\sum d_j\geq 2n+c$ and $n\geq 2$.*

Proof. When $\sum d_j\geq 2n+c+1$, our previous discussion shows that $\Lambda^qT^*_{\mathcal{X}}$ is very ample for $q\geq N+1$, so we conclude directly by Theorem 5.1. In the border case $\sum d_j=2n+c$, these bundles are still very ample with the potential exception of $\Lambda^{N+1}T^*_{\mathcal{X}}$ that is merely generated by sections. According to Remark 5.2, we try to find a suitable ample subbundle $\mathcal{A}\subset\Lambda^{N+1}T^*_{\mathcal{X}}$, the pull-back of which is not contained in the kernel $\mathcal{R}$ of the associated morphism (5.1.7); $\mathcal{R}$ has codimension 1 since $\operatorname{rank}K_{\widetilde{\mathcal{W}}}=1$. By (5.4.4), we simply have in this situation

$$\Lambda^{N+1}T^*_{\mathcal{X}}\simeq\mathcal{L}_S\otimes\Lambda^{n-1}\big(T_{\mathcal{X}}\otimes\mathrm{pr}_2^*\mathcal{O}(1)\big).\tag{5.5.1}$$

In other words, one can take exterior products of $(n-1)$ $\mathcal{O}(1)$-twisted vector fields and contract them with a (then nowhere vanishing) twisted $(N+n)$-form on $\mathcal{X}$, to produce sections of $\Lambda^{N+1}T^*_{\mathcal{X}}$. We reach our goal if we can take $\mathcal{A}$ to be generated by contractions with at least one vector field $\xi_{\ell,m}$ (i.e. not all of the type $\eta_{j,\alpha,\ell,m}$), because in that case one of the sections lies in $T_{\mathcal{X}}$, and we have a gain of one factor $\mathcal{O}(1)$, i.e. $\mathcal{A}\otimes \mathrm{pr}_2^*\,\mathcal{O}(-1)$ is still generated by sections. This argument fails if the only possibility to get a section outside of $\mathcal{R}$ is to take all factors to be vector fields of the type $\eta_{j,\alpha,\ell,m}$. In order to analyze this situation, we need the following general observation which is a consequence of the fonctorial properties of Semple towers.

5.5.2. Observation. *If ζ is a local holomorphic vector field on an open set $U\subset\mathcal{X}$, its flow defines local biholomorphisms of $\mathcal{X}$ which admit unique liftings to the absolute Semple bundles $\mathcal{X}^a_k$. In particular any such vector field ζ has an intrinsically defined lifting $\zeta^{(k)}$ over $\pi_{k,0}^{-1}(U)\subset\mathcal{X}^a_k$. Moreover, if $\zeta_1,\ldots,\zeta_{N'}$ generate $T_{\mathcal{X}}$ over U, $N'=\dim\mathcal{X}=N+n$, then*

$$\zeta_1^{(k)},\ldots,\zeta_{N'}^{(k)}\ \text{generate}\ \mathcal{V}^a_k\ \text{over}\ \pi_{k,0}^{-1}(U)\cap(\mathcal{X}_k\smallsetminus\Delta_k)$$

($X\smallsetminus\Delta_k$ being the Zariski open set of points of X_k associated with regular jets). $\square$

Notice that by fonctoriality, our morphism (5.1.7) commutes with the action of flows of vector fields. Therefore, the failure described above means that we can find $n-1$ vector fields among the $\eta_{j,\alpha,\ell,m}$, say $\eta_1,\ \ldots\ ,\ \eta_{n-1}$ to make notation shorter, so that

$$\widetilde{\mathcal{W}}\oplus\mathrm{Span}(\eta_1^{(k)},\ldots,\ \eta_{n-1}^{(k)})=\mathcal{V}^a_k$$

generically, but the replacement of some $\eta_j^{(k)}$ (say $\eta_{n-1}^{(k)}$) by any $\xi_{\ell,m}^{(k)}$ no longer generates $\mathcal{V}^a_k$. In that case we conclude that

$$\mathrm{Span}(\xi_{\ell,m}^{(k)})_{0\le\ell,m\le n+c}\subset\widetilde{\mathcal{W}}\oplus\mathrm{Span}(\eta_1^{(k)},\ldots,\ \eta_{n-2}^{(k)})\quad\text{[a hyperplane in }\mathcal{V}^a_k].\tag{5.5.3}$$

In fact, we can change the direction of $\mathrm{Span}(\eta_1^{(k)},\ldots,\ \eta_{n-2}^{(k)})$ by adding linear combinations of the η_j's, and by taking the intersection of all hyperplanes involved in the right hand side of (5.5.3), we conclude that the choice of an ample $\mathcal{A}$ possibly fails only in case

$$\mathrm{Span}(\xi_{\ell,m}^{(k)})_{0\le\ell,m\le n+c}\subset\widetilde{\mathcal{W}}\subset T_{\mathcal{Z}}\tag{5.5.4}$$

at a generic point of $\mathcal{Z}$. Condition (5.5.4) implies (and is equivalent to the fact) that $\mathcal{Z}$ is invariant by the action of $\mathrm{GL}(E)$ on $\mathcal{X}_k$. Also, since the vertical vector fields do not contribute in this situation, we see that (5.1.7) induces by restriction via (5.5.1) a surjective morphism

$$\mathcal{L}_S\otimes(\pi_{k,0})^*\Lambda^{n-1}(T^h_{\mathcal{X}}\otimes\mathcal{O}(1))\to K_{\widetilde{W}}\otimes\mathcal{O}_{X_k}(a)_{|\widetilde{Z}}.\tag{5.5.5}$$

This situation is solved by the next lemma, which shows that the image of the morphism (5.5.5) is still a big rank 1 sheaf, as needed. $\square$

5.5.6. Lemma. *Let $\mathcal{Z}\subset\mathcal{X}_k$ be an irreducible algebraic variety that dominates $\mathcal{X}_{k-1}$ (if $k>0$) resp. S (if $k=0$). Assume that $\mathcal{Z}$ is invariant by $\mathrm{GL}(E)$ and that the directed structure $(\mathcal{Z},\widetilde{\mathcal{W}})$ induced by the absolute Semple bundle $(\mathcal{X}^a_k,\mathcal{V}^a_k)$ has $\mathrm{rank}\,\mathcal{W}=q:=N+1$. For $n\ge2$ and $\sum d_j=2n+c$, consider the associated morphism* (5.1.7)

$$\theta:(\pi_{k,0})^*\Lambda^{q'}T^*_{\mathcal{X}\,|\mathcal{Z}}\to K_{\widetilde{\mathcal{W}}}\otimes\mathcal{O}_{\mathcal{X}_k}(\mathbf{a})_{|\mathcal{Z}},\qquad q'\ge q.$$

Then

(i) *either one has $q'>q=N+1$, in which case $\Lambda^{q'}T^*_{\mathcal{X}}$ is relatively ample,*

(ii) *or* $q' = q = N+1$, *in which case generic points of* $\mathcal{Z}$ *are separated by global sections of* $H^0(\mathcal{Z}, K_{\widetilde{\mathcal{W}}} \otimes \mathcal{O}_{\mathcal{X}_k}(\mathbf{a})_{|\mathcal{Z}})$ *obtained as images of sections of* $\Lambda^q T^*_{\mathcal{X}} \simeq \Lambda^{n-1}(T_{\mathcal{X}} \otimes \mathcal{O}(1))$ *arising from* $(n-1)$ *exterior products of twisted vector fields* $\eta_{j,\alpha,\ell,m}$.

Proof. We only have to consider case (ii) $q' = q = N+1$. First assume $k = 0$. Then $\mathcal{Z} \to S$ is a family of curves $Z_t \subset \mathbb{P}^{n+c+1}$ over some Zariski open set $S^0 \subset S \subset \mathbb{C}^N$ with $\mathcal{Z} = \overline{\bigcup Z_t} \subset \mathcal{X}$ and $\widetilde{\mathcal{W}} = T_{\mathcal{Z}}$. We already know that the sections described above generate $K_{\widetilde{\mathcal{W}}} \otimes \mathcal{O}_{\mathcal{X}_k}(\mathbf{a})_{|\mathcal{Z}}$ at nonsingular points, so it suffices to prove that we can find a nonzero section in the image that vanishes at a generic point. We pick a point $x^0 \in Z_t$ such that x^0 is smooth on Z_t, $\mathcal{Z}$, X_t and $\mathcal{X}$ (this is true generically). By a suitable choice of a basis $(e_j)_{0 \le j \le n+c}$ of $E \simeq \mathbb{C}^{n+c+1}$, we may assume that $x^0 = (t, p^0)$ where $p^0 = [e_0]$ is identified with the origin in the affine chart $\mathbb{C}^{N+c} = \{z_0 = 1\} \subset \mathbb{P}^{n+c}$ equipped with coordinates $z_1, \ldots, z_{n+c}$, that T_{Z_t, x^0} is generated by $e_1 = \partial/\partial z_1$ in $T_{\mathbb{P}^{n+c}}$, and finally that $T_{X_t, x^0} = \mathrm{Span}(\partial/\partial z_1, \ldots, \partial/\partial z_n)$. By (5.4.3), the horizontal subspace of $T_{\mathcal{X}, x^0}$ defined as the kernel of $(\mathrm{pr}_2)_* : T_{\mathcal{X}} \to T_{\mathbb{P}^{n+c}}$ contains the $(N-c)$ vectors

$$\eta_{j,\alpha,\ell,0}(x^0) = \frac{\partial}{\partial t_{j,\alpha}}, \quad 1 \le j \le c, \quad \alpha \in \mathbb{N}^{n+c+1}, \alpha \ne \alpha_0^j = (d_j, 0, \ldots, 0)$$

(the value at x^0 does not depend on ℓ, for any such α we agree to take e.g. the largest $\ell > 0$ such that $\alpha_\ell > 0$). Beyond these horizontal vectors and the vertical ones $\partial/\partial z_1, \ldots, \partial/\partial z_n$ in $T_{X_t, x^0} \subset T_{\mathcal{X}, z}$, there are also c "oblique" (still possibly horizontal!) tangent vectors

$$\left(\frac{\partial}{\partial t_{j,\alpha_0^j}}, \zeta_j \right) \in T_{\mathcal{X}, x^0}, \qquad \alpha_0^j = (d_j, 0, \ldots, 0), \quad \zeta_j \in T_{\mathbb{P}^{n+c}, p^0}, \quad 1 \le j \le c.$$

By our choices, $(t_{j,\alpha}, z_1)$ are local coordinates of $\mathcal{Z}$ near x^0, $(t_{j,\alpha}, z_1, \ldots, z_n)$ are local coordinates of $\mathcal{X}$, and the varieties can be locally expressed in $\mathbb{C}^N \times \mathbb{P}^{n+c}$ as certain graphs

$$\mathcal{Z}: \quad z_\ell = g_\ell(t_{j,\alpha}, z_1), \qquad 2 \le \ell \le n+c, \tag{5.5.7}$$

$$\mathcal{X}: \quad z_\ell = h_\ell(t_{j,\alpha}, z_1, \ldots, z_n), \qquad n+1 \le \ell \le n+c. \tag{5.5.8}$$

Accordingly, $K_{\widetilde{\mathcal{W}}} = K_{\mathcal{Z}}$, resp. $K_{\mathcal{X}}$, are locally generated near x^0 by the top degree forms

$$\omega_1 = \bigwedge_{j,\alpha} dt_{j,\alpha} \wedge dz_1, \quad \text{resp.} \quad \omega_{1\ldots n} = \bigwedge_{j,\alpha} dt_{j,\alpha} \wedge dz_1 \wedge \ldots \wedge dz_n$$

of respective degrees $N+1$, $N+n$. By our assumption that $\mathcal{Z}$ is $\mathrm{GL}(E)$ invariant, we know that there is a certain section of $\Lambda^{n-1}(T_{\mathcal{X}} \otimes \mathcal{O}(1))$ of the form

$$u = \eta_{j_1,\alpha_1,\ell_1,0} \wedge \ldots \wedge \eta_{j_{n-1},\alpha_{n-1},\ell_{n-1},0}, \qquad \alpha_s \ne (d_{j_s}, 0, \ldots, 0),$$

that yields after contraction a section θu of $K_{\widetilde{\mathcal{W}}} \otimes \mathcal{O}_{\mathcal{X}_k}(\mathbf{a})_{|\mathcal{Z}}$ such that $\theta u(x^0) \ne 0$. In terms of the graph description (5.5.7), this means that

$$\det \left(\frac{\partial g_\ell}{\partial t_{j_s,\alpha_s}}(x^0) \right)_{2 \le \ell \le n,\, 1 \le s \le n-1} \ne 0. \tag{5.5.9}$$

As $x^0 \in \mathcal{X}$ and $\partial/\partial z_\ell \in T_{\mathcal{X}, x^0}$, $1 \le \ell \le n$, we get

$$P_j(x^0) = t_{j,\alpha_0^j} = 0 \quad \text{and} \quad \frac{\partial}{\partial z_\ell} P_j(x^0) = t_{j,\beta_\ell^j} = 0, \ \beta_\ell^j = (d_j - 1, 0, \ldots, 1, \ldots, 0) = (d_j - 1)\varepsilon_0 + \varepsilon_\ell.$$

On the curve Z_t, the tangency of $\partial/\partial z_1$ at x^0 shows that we can use z_1 as the local coordinate near x^0 and that $z_\ell = O(z_1^2)$ for $\ell > 1$. If $\ell_s > 1$, we find

$$\eta_{j_s,\alpha_s,\ell_s,0}(x^0) = \frac{\partial}{\partial t_{j_s,\alpha_s}} \quad \text{while} \quad \eta_{j_s,\alpha_s,\ell_s,1} = z_1\frac{\partial}{\partial t_{j_s,\alpha_s}} - z_{\ell_s}\frac{\partial}{\partial t_{j_s,\alpha_s-\varepsilon_{\ell_s}+\varepsilon_1}},\ z_{\ell_s} = O(z_1^2).$$

Therefore, if we replace $\eta_{j_s,\alpha_s,\ell_s,0}$ by $\eta_{j_s,\alpha_s,\ell_s,1}$ in the wedge product defining u, we get another product v such that θv vanishes exactly at order 1 at $x^0 \in Z_t$; this is the section we were looking for, and in that case conclusion (ii) is met. We obtain such sections unless all choices of α_s that satisfy (5.5.9) are of the form $(d_j - r, r, 0, \ldots, 0)$, i.e. all their components of index $p > 1$ vanish. In other words, we fail if $\partial g_\ell/\partial t_{j,\alpha}(x^0) = 0$ for all $\ell \in \{2, \ldots, n\}$ and all (j, α) such that $\alpha_p > 0$ for some $p > 1$ (as any nonzero vector of $\mathbb{C}^{n-1}$ can be completed into a basis by $(n-2)$ vectors extracted from $(n-1)$ linearly independent ones). Geometrically, this means that the corresponding horizontal vectors $\frac{\partial}{\partial t_{j,\alpha}}$, $\exists p > 1$, $\alpha_p > 0$ that are tangent to $\mathcal{X}$ are also tangent to $\mathcal{Z}$ (the horizontality implies $\partial g_\ell/\partial t_{j,\alpha}(x^0) = \partial h_\ell/\partial t_{j,\alpha}(x^0) = 0$ for $\ell \geq n{+}1$). More intrinsically, these vectors form a basis of the subspace $\mathcal{F}_\delta \subset \mathrm{pr}_1^* T_S$ of vectors $\tau \in \mathbb{C}^N = \prod \mathrm{Sym}^{d_j} E^*$ such that $\tau_j \cdot \zeta^{d_j} = 0$ for all ζ in the plane $\Pi_\delta = \mathrm{Span}(e_0, e_1) \subset E$ that defines the tangent line $\delta = T_{Z_t,x^0} \subset \mathbb{P}^{n+c}$, and our condition is that this horizontal subspace $\mathcal{F}_\delta$ should be tangent to $\mathcal{Z}$. The equality $t_j \cdot \zeta^{d_j} \equiv 0$ yields $d_j + 1$ independent conditions, thus $\dim \mathcal{F}_\delta = N - \sum(d_j + 1) = N - (2n + 2c)$. Since $\mathcal{Z}$ is invariant by the action of $\mathrm{GL}(E)$, $T_{\mathcal{Z},x^0}$ also contains the subspace $\mathcal{G}_{x^0}$ generated by the $(n + c + 1)^2$ Killing vector fields at point $x^0 \in \mathbb{C}^N \times \mathbb{P}^{n+c}$. For t generic (see below), we have $\dim \mathcal{G}_{x^0} = (n + c + 1)^2$ and as all vertical directions are attained, the horizontal directions form a subspace of dimension $\dim \mathcal{G}^h_{x^0} = (n{+}c{+}1)^2 - (n{+}c)$. When failure occurs, we have $\mathcal{F}_\delta + \mathcal{G}_{x^0} \subset T_{\mathcal{Z},x^0}$ or equivalently $\mathcal{F}_\delta + \mathcal{G}^h_{x^0} \subset T^h_{\mathcal{Z},x^0}$, hence

(5.5.10) $$\dim(\mathcal{F}_\delta + \mathcal{G}_{x^0}) \leq \dim T_{\mathcal{Z},x^0} = N + 1,$$

(5.5.10^h) $$\dim(\mathcal{F}_\delta + \mathcal{G}^h_{x^0}) \leq \dim T^h_{\mathcal{Z},x^0} = N + 1 - (n + c).$$

Since $\dim \mathcal{F}_\delta = N - (2n + 2c)$ and $\dim \mathcal{G}_{x^0} = (n + c + 1)^2$ is rather large, we are going to see that this situation is highly non generic, at least if $n \geq 3$.

5.5.11. Lemma. *Let $F_{1,2}(E)$ be the flag manifold of pairs $([z], \delta)$ such that $[z] \in \delta \subset P(E)$ where δ is a projective line, associated to a $(1,2)$-flag $\mathbb{C}z \subset \Pi_\delta \subset E$, $\dim \Pi_\delta = 2$. For $r = 1, \ldots, n$, consider the algebraic variety M_r of triples $(t, [z], \delta) \in S \times F_{1,2}(E)$ satisfying the conditions*

(a) $(t, [z]) \in \mathcal{X}$*, i.e.* $t_j \cdot z^{d_j} = 0$ *for* $1 \leq j \leq c$ *;*

(b) $[z] \in \delta \subset T_{X_t}$ *and*

(c) $\dim(\mathcal{F}_\delta + \mathcal{G}_{(t,[z])}) \leq N + r$*, where* $\mathcal{F}_\delta = \{\tau \in T_S\,;\ \tau_j \cdot \zeta^{d_j} = 0,\ \forall \zeta \in \Pi_\delta\}$ *and* $\mathcal{G}_p$ *is the tangent space to the orbit of a point* $p = (t, [z]) \in \mathcal{X}$ *under* $\mathrm{GL}(E)$.

Then

$$\mathrm{codim}_S\, \mathrm{pr}_1(M_r) \geq 2c - 1 + (n - r - 2)(n + c - 1),$$

in particular $\mathrm{codim}_S\, \mathrm{pr}_1(M_1) > 0$ *as soon as* $n \geq 3$.

Proof. The variety M_r admits an equivariant fibration over $F_{1,2}(E)$ with respect to the action of $\mathrm{GL}(E)$, and the action is transitive on $F_{1,2}(E)$, hence

(5.5.12) $$\dim M_r \leq \dim F_{1,2}(E) + \dim(\text{fiber}) = 2n + 2c - 1 + \dim(\text{fiber}).$$

Now, by differentiating the action $g \mapsto g \cdot (t, [z]) = (\zeta \mapsto t_j \cdot g^{-1}(\zeta)^{d_j}, [gz])$ at $g = \mathrm{Id}_E$, we see that $\mathcal{G}_{(t,[z])}$ is the image of the differential

$$\mathfrak{gl}(E) \ni u \longmapsto \gamma(u) := \big(\tau_j : \zeta \mapsto -d_j\, t_j \cdot (\zeta^{d_j-1} u\zeta), \pi_*(uz)\big) \tag{5.5.13}$$

where π_* is the differential of $\pi : E \smallsetminus \{0\} \to P(E)$ and the product $\zeta^{d_j-1}u\zeta$ is computed in the symmetric algebra $\mathrm{Sym}^\bullet E$. The horizontal part $\mathcal{G}^h_{(t,[z])}$ is obtained by taking only those $u \in \mathfrak{gl}(E)$ such that $uz \,/\!/\, z$, in which case $\pi_*(uz) = 0$. We make more explicit calculations assuming $z = e_0 = (1, 0, \ldots, 0)$ and $\Pi_\delta = \mathrm{Span}(e_0, e_1)$ where $(e_j)_{0 \le j \le n+c}$ is a basis of E. Then $\mathcal{F}_\delta$ consists of elements $\tau = (\tau_j)$ such that all components $\tau_{j,\alpha}$ vanish for α of the form $(\alpha_1, \alpha_2, 0, \ldots, 0)$, $\alpha_1 + \alpha_2 = d_j$. Condition (a) is equivalent to $t_{j,\alpha} = 0$ for $\alpha = \alpha^j_0 = (d_j, 0, \ldots, 0)$ and (b) is equivalent to $t_{j,\alpha} = 0$ for $\alpha = \beta^j_1 = (d_j - 1, 1, 0, \ldots, 0)$, which already count for $2c$ linear conditions on t. Since $\dim \mathcal{F}_\delta = N - (2n+2c)$, condition (c), which is equivalent to $\dim(\mathcal{F}_\delta + \mathcal{G}^h_{x^0}) \le N + 1 - (n + c)$, is realized if $\mathcal{G}^h_{x^0}$ recovers via (5.5.13) at most $n + c + 1$ directions from the missing $(2n + 2c)$ directions $\tau_{j,\alpha}$, $\alpha = (\alpha_0, \alpha_1, 0, \ldots, 0)$, $\alpha_1 + \alpha_2 = d_j$. Denote by $u_{\ell,m} \in \mathfrak{gl}(E)$ the endomorphism such that $u_{\ell,m}(e_m) = e_\ell$ and all other entries of the matrix are zero. Let $\gamma_1(u) = \mathrm{pr}_1\, \gamma(u)$ in (5.5.13), and let $\gamma_1(u)_{j,\alpha}$ be the coefficient of the monomial ζ^α in the component in $\mathrm{Sym}^{d_j} E^*$. The action restricted to $\lambda \mapsto g = \mathrm{Id} + \lambda u_{\ell,m}$ has the effect of substituting $\zeta_\ell + \lambda\zeta_m$ to ζ_ℓ, hence we find by taking the $(d/d\lambda)_{|\lambda=0}$ derivative the simple formula

$$\gamma_1(u_{\ell,m})_{j,\alpha-\varepsilon_\ell+\varepsilon_m} = -\alpha_\ell t_{j,\alpha} \iff \gamma_1(u_{\ell,m})_{j,\alpha} = -(\alpha_\ell + 1 - \delta_{\ell,m}) t_{j,\alpha+\varepsilon_\ell-\varepsilon_m}. \tag{5.5.14}$$

Restricting the right hand side of (5.5.14) to indices α of the form $\alpha = (\alpha_0, \alpha_1, 0, \ldots, 0)$, we see that $\gamma_1(u_{\ell,m})_{j,\alpha} = 0$ for $m > 1$, thus we only have to consider the images of $u_{0,0}$ and $u_{\ell,1}$, $0 \le \ell \le n + c$, namely $n + c + 2$ vectors. We get

$$\gamma_1(u_{0,0})_{j,\alpha} = -\alpha_0 t_{j,\alpha}, \qquad \gamma_1(u_{\ell,1})_{j,\alpha} = -(\alpha_\ell + 1 - \delta_{\ell,1}) t_{j,\alpha+\varepsilon_\ell-\varepsilon_1}.$$

to pairs (ℓ, m) such that $m = 0 \Rightarrow \ell = 0$, we get a matrix with $(n + c)(n + c + 1)$ columns and $\sum d_j = 2n + c$ rows, whose entries are the $*t_{j,\alpha+\varepsilon_\ell-\varepsilon_m}$ (notice that the entry is always zero if $\alpha_1 = 0$). The degeneration condition (c) occurs if the rank of this matrix does not exceed $n + c + r$.

The element $u_{0,0}$ yields one missing direction unless $t_{j,\alpha} = 0$ for $\alpha_0 > 0$.

We can already recover 3 directions by taking

$$u_1(e_0) = e_0,\ u_1(e_1) = 0, \qquad u_2(e_0) = 0,\ u_1(e_1) = e_0, \qquad u_3(e_0) = 0,\ u_1(e_1) = e_0$$

. We can further recover every direction $\tau_{j,\alpha}$ with $\alpha_2 > 0$ by taking $u(e_0) = 0$, $u(e_1) = e_\ell$, $\ell > 1$, and $u(e_j) = 0$, $j > 1$ in (5.5.13), as soon as we have $t_{j,\beta} \ne 0$ for

$$\beta = (\alpha_1, \alpha_2 - 1, 0, \ldots, 1, \ldots, 0),$$

the component 1 being in the ℓ-th position. Failure to recover $\tau_{j,\alpha}$ occurs if these $(n + c - 1)$ components $t_{j,\beta}$ vanish. However, in order to recover several directions at the same time, we have to use independent elements $u \in \mathfrak{gl}(E)$, namely elements corresponding to different values of ℓ. Hence we lose one dimension for the second direction to recover, $1 + 2 = 3$ dimensions for the third direction, ... , $p(p - 1)/2$ dimensions for the p-th direction. Since we can recover in total $\sum d_j = 2n + c$ directions $\tau_{j,\alpha}$, the degeneration occurs when we have at least $(n - r)$ directions not recovered, a situation that occurs only when t is in a finite union of subspaces of codimension $2c + 1 + (n - r)(n + c - 1) - (n - r)(n - r - 1)/2$. From this, we infer by (5.5.12) that

$$\begin{aligned}\dim M_r &\le (2n+2c-1) + \big(N - 2c - (n-r)(n+c-1) + (n-r)(n-r-1)/2\big)\\ &\le N - \big(2c - 1 + (n-r-2)(n+c-1)\big).\end{aligned}$$

Lemma 5.5.11 is proved. □

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