Summer School in Mathematics
June 18th – July 6th, 2012
FOLIATIONS, PSEUDOHOLOMORPHIC CURVES, APPLICATIONS

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“Kobayashi pseudo-metrics, entire curves and hyperbolicity of algebraic varieties”

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Introduction. The goal of these notes is to study complex varieties, mainly compact or projective algebraic ones, through a few geometric questions related to hyperbolicity in the sense of Kobayashi : a complex space X is said to be hyperbolic essentially if analytic disks $f : \mathbb{D} \to X$ through a given point form a normal family. If X is not hyperbolic, a basic question is to analyze entire holomorphic curves $f : \mathbb{C} \to X$, and especially to understand the locus $Y \subset X$ where these curves can be located. A tantalizing conjecture by Green-Griffiths and Lang says that one can take Y to be a proper algebraic subvariety of X whenever X is a projective variety of general type. It is also expected that very generic algebraic hypersurfaces X of high degree in complex projective space $\mathbb{P}^{n+1}$ are Kobayashi hyperbolic, i.e. without any entire holomorphic curves $f : \mathbb{C} \to X$. A convenient framework for this study is the category of "directed manifolds", that is, the category of pairs (X, V) where X is a complex manifold and V a holomorphic subbundle of T_X, possibly with singularities – this includes for instance the case of holomorphic foliations. If X is compact, the pair (X, V) is hyperbolic if and only if there are no nonconstant entire holomorphic curves $f : \mathbb{C} \to X$ tangent to V, as a consequence of Brody's criterion. We describe here the construction of certain jet bundles J_kX, $J_k(X, V)$, and corresponding projectivized k-jet bundles P_kV. These bundles, which were introduced in various contexts (Semple in 1954, Green-Griffiths in 1978) allow us to analyze hyperbolicity in terms of certain negativity properties of the curvature. For instance, $\pi_k : P_kV \to X$ is a tower of projective bundles over X and carries a canonical line bundle $\mathcal{O}_{P_kV}(1)$; the hyperbolicity of X is then conjecturally equivalent to the existence of suitable singular hermitian metrics of negative curvature on $\mathcal{O}_{P_kV}(-1)$ for k large enough. The direct images $(\pi_k)_\star\mathcal{O}_{P_kV}(m)$ can be viewed as bundles of algebraic differential operators of order k and degree m, acting on germs of curves and invariant under reparametrization.

Following an approach initiated by Green and Griffiths, we establish a basic Ahlfors-Schwarz lemma in the situation when $\mathcal{O}_{P_kV}(-1)$ has a (possibly singular) metric of negative curvature, and we infer that every nonconstant entire curve $f : \mathbb{C} \to V$ tangent to V must be contained in the base locus of the metric. Another fundamental tool is a vanishing theorem asserting that entire curves must be solutions of the algebraic differential equations provided by global sections of jet bundles, whenever their coefficients vanish on a given ample divisor. These results can in turn be used to prove various geometric statements such as the Bloch theorem, which asserts that the Zariski closure of an entire curve in a complex torus is a translate of a subtorus. Another important consequence is a partial answer to the Green-Griffiths-Lang conjecture : there exists a global algebraic differential operator P (in fact many such operators P_j) such that every entire curve $f : \mathbb{C} \to X$ drawn in a projective

variety of general type must satisfy the equations $P_j(f; f', \ldots, f^{(k)}) = 0$. The main idea is to make curvature calculations and use holomorphic Morse inequalities to show the existence of global sections for the relevant jet bundles. Using this, a differentiation technique of Siu based on "slanted" vector fields on jet bundles, implies that the Green-Griffiths conjecture holds true for generic hypersurfaces of projective space of sufficiently high degree.

Key words: Kobayashi hyperbolic variety, directed manifold, genus of a curve, jet bundle, jet differential, jet metric, Chern connection and curvature, negativity of jet curvature, variety of general type.

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Contents

§0. Preliminaries of complex differential geometry

§0.A. Dolbeault cohomology and sheaf cohomology

Let X be a $\mathbb{C}$-analytic manifold of dimension n. We denote by $\Lambda^{p,q}T_X^\star$ the bundle of differential forms of bidegree (p,q) on X, i.e., differential forms which can be written as

$$u = \sum_{|I|=p,\,|J|=q} u_{I,J}dz_I \wedge d\overline{z}_J.$$

Here $(z_1,\ldots,z_n)$ denote local holomorphic coordinates, $I=(i_1,\ldots,i_p)$, $J=(j_1,\ldots,j_q)$ are multiindices (increasing sequences of integers in the range $[1,\ldots,n]$, of lengths $|I|=p$, $|J|=q$), and

$$dz_I := dz_{i_1}\wedge\ldots\wedge dz_{i_p},\qquad d\overline{z}_J := d\overline{z}_{j_1}\wedge\ldots\wedge d\overline{z}_{j_q}.$$

Let $\mathcal{E}^{p,q}$ be the sheaf of germs of complex valued differential (p,q)-forms with C^∞ coefficients. Recall that the exterior derivative d splits as $d=\partial+\overline{\partial}$ where

$$\begin{aligned}\partial u &= \sum_{|I|=p,\,|J|=q,1\leqslant k\leqslant n} \frac{\partial u_{I,J}}{\partial z_k} dz_k\wedge dz_I\wedge d\overline{z}_J,\\ \overline{\partial} u &= \sum_{|I|=p,\,|J|=q,1\leqslant k\leqslant n} \frac{\partial u_{I,J}}{\partial \overline{z}_k} d\overline{z}_k\wedge dz_I\wedge d\overline{z}_J\end{aligned}$$

are of type $(p+1,q)$, $(p,q+1)$ respectively. (Another frequently used alternative notation is $d=d'+d''$, where $d'=\partial$, $d''=\overline{\partial}$). The well-known Dolbeault-Grothendieck lemma asserts that any $\overline{\partial}$-closed form of type (p,q) with $q>0$ is locally $\overline{\partial}$-exact (this is the analogue for $\overline{\partial}$ of the usual Poincaré lemma for d, see e.g. [Hör66]). In other words, the complex of sheaves $(\mathcal{E}^{p,\bullet},\overline{\partial})$ is exact in degree $q>0$; in degree $q=0$, $\operatorname{Ker}\overline{\partial}$ is the sheaf Ω_X^p of germs of holomorphic forms of degree p on X.

More generally, if F is a holomorphic vector bundle of rank r over X, there is a natural $\overline{\partial}$ operator acting on the space $C^\infty(X,\Lambda^{p,q}T_X^\star\otimes F)$ of smooth (p,q)-forms with values in F; if $s=\sum_{1\leqslant\lambda\leqslant r}s_\lambda e_\lambda$ is a (p,q)-form expressed in terms of a local holomorphic frame of F, we simply define $\overline{\partial}s := \sum\overline{\partial}s_\lambda\otimes e_\lambda$, observing that the holomorphic transition matrices involved in changes of holomorphic frames do not affect the computation of $\overline{\partial}$. It is then clear that the Dolbeault-Grothendieck lemma still holds for F-valued forms. For every integer $p=0,1,\ldots,n$, the *Dolbeault Cohomology* groups $H^{p,q}(X,F)$ are defined to be the cohomology groups of the complex of global (p,q) forms (graded by q):

$$H^{p,q}(X,F) = H^q\big(C^\infty(X,\Lambda^{p,\bullet}T_X^\star\otimes F)\big). \tag{0.1}$$

Now, let us recall the following fundamental result from sheaf theory (De Rham-Weil isomorphism theorem): let $(\mathcal{L}^\bullet,d)$ be a resolution of a sheaf $\mathcal{A}$ by acyclic sheaves, i.e. a complex of sheaves $(\mathcal{L}^\bullet,\delta)$ such that there is an exact sequence of sheaves

$$0\longrightarrow\mathcal{A}\xrightarrow{\ j\ }\mathcal{L}^0\xrightarrow{\ \delta^0\ }\mathcal{L}^1\longrightarrow\cdots\longrightarrow\mathcal{L}^q\xrightarrow{\ \delta^q\ }\mathcal{L}^{q+1}\longrightarrow\cdots\ ,$$

and $H^s(X,\mathcal{L}^q)=0$ for all $q\geqslant 0$ and $s\geqslant 1$. Then there is a functorial isomorphism

$$H^q\big(\Gamma(X,\mathcal{L}^\bullet)\big)\longrightarrow H^q(X,\mathcal{A}). \tag{0.2}$$

We apply this to the following situation: let $\mathcal{E}(F)^{p,q}$ be the sheaf of germs of C^∞ sections of $\Lambda^{p,q}T^\star_X \otimes F$. Then $(\mathcal{E}(F)^{p,\bullet}, \overline{\partial})$ is a resolution of the locally free $\mathcal{O}_X$-module $\Omega^p_X \otimes \mathcal{O}(F)$ (Dolbeault-Grothendieck lemma), and the sheaves $\mathcal{E}(F)^{p,q}$ are acyclic as modules over the soft sheaf of rings $\mathcal{C}^\infty$. Hence by (0.2) we get

0.3. Dolbeault isomorphism theorem (1953). *For every holomorphic vector bundle F on X, there is a canonical isomorphism*

$$H^{p,q}(X, F) \simeq H^q(X, \Omega^p_X \otimes \mathcal{O}(F)). \qquad \square$$

If X is projective algebraic and F is an algebraic vector bundle, Serre's GAGA theorem [Ser56] shows that the algebraic sheaf cohomology group $H^q(X, \Omega^p_X \otimes \mathcal{O}(F))$ computed with algebraic sections over Zariski open sets is actually isomorphic to the analytic cohomology group. These results are the most basic tools to attack algebraic problems via analytic methods. Another important tool is the theory of plurisubharmonic functions and positive currents introduced by K. Oka and P. Lelong in the decades 1940-1960.

§0.B. Plurisubharmonic Functions

Plurisubharmonic functions have been introduced independently by Lelong and Oka in the study of holomorphic convexity. We refer to [Lel67, 69] for more details.

0.4. Definition. *A function $u : \Omega \longrightarrow [-\infty, +\infty[$ defined on an open subset $\Omega \subset \mathbb{C}^n$ is said to be plurisubharmonic (psh for short) if*

a) *u is upper semicontinuous* ;

b) *for every complex line $L \subset \mathbb{C}^n$, $u_{\restriction \Omega \cap L}$ is subharmonic on $\Omega \cap L$, that is, for all $a \in \Omega$ and $\xi \in \mathbb{C}^n$ with $|\xi| < d(a, \complement\Omega)$, the function u satisfies the mean value inequality*

$$u(a) \leqslant \frac{1}{2\pi} \int_0^{2\pi} u(a + e^{i\theta}\,\xi)\, d\theta.$$

The set of psh functions on Ω is denoted by $\mathrm{Psh}(\Omega)$.

We list below the most basic properties of psh functions. They all follow easily from the definition.

0.5. Basic properties.

a) Every function $u \in \mathrm{Psh}(\Omega)$ is subharmonic, namely it satisfies the mean value inequality on euclidean balls or spheres:

$$u(a) \leqslant \frac{1}{\pi^n r^{2n}/n!} \int_{B(a,r)} u(z)\, d\lambda(z)$$

for every $a \in \Omega$ and $r < d(a, \complement\Omega)$. Either $u \equiv -\infty$ or $u \in L^1_{\mathrm{loc}}$ on every connected component of Ω.

b) For any decreasing sequence of psh functions $u_k \in \mathrm{Psh}(\Omega)$, the limit $u = \lim u_k$ is psh on Ω.

c) Let $u \in \mathrm{Psh}(\Omega)$ be such that $u \not\equiv -\infty$ on every connected component of Ω. If (ρ_ε) is a family of smoothing kernels, then $u \star \rho_\varepsilon$ is C^∞ and psh on

$$\Omega_\varepsilon = \big\{x \in \Omega\,;\, d(x, \complement\Omega) > \varepsilon\big\},$$

the family $(u \star \rho_\varepsilon)$ is increasing in ε and $\lim_{\varepsilon\to 0} u \star \rho_\varepsilon = u$.

d) Let $u_1, \ldots, u_p \in \mathrm{Psh}(\Omega)$ and $\chi : \mathbb{R}^p \longrightarrow \mathbb{R}$ be a convex function such that $\chi(t_1, \ldots, t_p)$ is increasing in each t_j. Then $\chi(u_1, \ldots, u_p)$ is psh on Ω. In particular $u_1 + \cdots + u_p$, $\max\{u_1, \ldots, u_p\}$, $\log(e^{u_1} + \cdots + e^{u_p})$ are psh on Ω. □

0.6. Lemma. *A function $u \in C^2(\Omega, \mathbb{R})$ is psh on Ω if and only if the hermitian form $Hu(a)(\xi) = \sum_{1\leqslant j,k\leqslant n} \partial^2 u/\partial z_j\partial\overline{z}_k(a)\, \xi_j\overline{\xi}_k$ is semipositive at every point $a \in \Omega$.*

Proof. This is an easy consequence of the following standard formula

$$\frac{1}{2\pi}\int_0^{2\pi} u(a + e^{i\theta}\,\xi)\,d\theta - u(a) = \frac{2}{\pi}\int_0^1 \frac{dt}{t}\int_{|\zeta|<t} Hu(a+\zeta\xi)(\xi)\,d\lambda(\zeta),$$

where $d\lambda$ is the Lebesgue measure on $\mathbb{C}$. Lemma 0.6 is a strong evidence that plurisubharmonicity is the natural complex analogue of linear convexity. □

For non smooth functions, a similar characterization of plurisubharmonicity can be obtained by means of a regularization process.

0.7. Theorem. *If $u \in \mathrm{Psh}(\Omega)$, $u \not\equiv -\infty$ on every connected component of Ω, then for all $\xi \in \mathbb{C}^n$*

$$Hu(\xi) = \sum_{1\leqslant j,k\leqslant n} \frac{\partial^2 u}{\partial z_j\partial\overline{z}_k}\,\xi_j\overline{\xi}_k \in \mathcal{D}'(\Omega)$$

is a positive measure. Conversely, if $v \in \mathcal{D}'(\Omega)$ is such that $Hv(\xi)$ is a positive measure for every $\xi \in \mathbb{C}^n$, there exists a unique function $u \in \mathrm{Psh}(\Omega)$ which is locally integrable on Ω and such that v is the distribution associated to u. □

In order to get a better geometric insight of this notion, we assume more generally that u is a function on a complex n-dimensional manifold X. If $\Phi : X \to Y$ is a holomorphic mapping and if $v \in C^2(Y, \mathbb{R})$, we have the commutation relation $\partial\overline{\partial}(v \circ \Phi) = \Phi^\star(\partial\overline{\partial} v)$, hence

$$H(v \circ \Phi)(a, \xi) = Hv\big(\Phi(a), \Phi'(a)\cdot\xi\big).$$

In particular Hu, viewed as a hermitian form on T_X, does not depend on the choice of coordinates $(z_1, \ldots, z_n)$. Therefore, the notion of psh function makes sense on any complex manifold. More generally, we have

0.8. Proposition. *If $\Phi : X \longrightarrow Y$ is a holomorphic map and $v \in \mathrm{Psh}(Y)$, then $v \circ \Phi \in \mathrm{Psh}(X)$.* □

0.9. Example. It is a standard fact that $\log|z|$ is psh (i.e. subharmonic) on $\mathbb{C}$. Thus $\log|f| \in \mathrm{Psh}(X)$ for every holomorphic function $f \in H^0(X, \mathcal{O}_X)$. More generally

$$\log\big(|f_1|^{\alpha_1} + \cdots + |f_q|^{\alpha_q}\big) \in \mathrm{Psh}(X)$$

for every $f_j \in H^0(X, \mathcal{O}_X)$ and $\alpha_j \geqslant 0$ (apply Property 0.5 d with $u_j = \alpha_j \log|f_j|$). We will be especially interested in the singularities obtained at points of the zero variety $f_1 = \ldots = f_q = 0$, when the α_j are rational numbers. □

0.10. Definition. *A psh function $u \in \mathrm{Psh}(X)$ will be said to have analytic singularities if u can be written locally as*

$$u = \frac{\alpha}{2} \log\big(|f_1|^2 + \cdots + |f_N|^2\big) + v,$$

where $\alpha \in \mathbb{R}_+$, v is a locally bounded function and the f_j are holomorphic functions. If X is algebraic, we say that u has algebraic singularities if u can be written as above on sufficiently small Zariski open sets, with $\alpha \in \mathbb{Q}_+$ and f_j algebraic.

We then introduce the ideal $\mathcal{J} = \mathcal{J}(u/\alpha)$ of germs of holomorphic functions h such that $|h| \leqslant Ce^{u/\alpha}$ for some constant C, i.e.

$$|h| \leqslant C\big(|f_1| + \cdots + |f_N|\big).$$

This is a globally defined ideal sheaf on X, locally equal to the integral closure $\overline{\mathcal{I}}$ of the ideal sheaf $\mathcal{I} = (f_1, \ldots, f_N)$, thus $\mathcal{J}$ is coherent on X. If $(g_1, \ldots, g_{N'})$ are local generators of $\mathcal{J}$, we still have

$$u = \frac{\alpha}{2} \log\big(|g_1|^2 + \cdots + |g_{N'}|^2\big) + O(1).$$

If X is projective algebraic and u has analytic singularities with $\alpha \in \mathbb{Q}_+$, then u automatically has algebraic singularities. From an algebraic point of view, the singularities of u are in 1:1 correspondence with the "algebraic data" $(\mathcal{J}, \alpha)$.

§0.C. Positive Currents

The reader can consult [Fed69] for a more thorough treatment of current theory. Let us first recall a few basic definitions. A *current* of degree q on an oriented differentiable manifold M is simply a differential q-form T with distribution coefficients. The space of currents of degree q over M will be denoted by $\mathcal{D}'^q(M)$. Alternatively, a current of degree q can be seen as an element T in the dual space $\mathcal{D}'_p(M) := \big(\mathcal{D}^p(M)\big)'$ of the space $\mathcal{D}^p(M)$ of smooth differential forms of degree $p = \dim M - q$ with compact support; the duality pairing is given by

$$\langle T, \alpha \rangle = \int_M T \wedge \alpha, \qquad \alpha \in \mathcal{D}^p(M). \tag{0.11}$$

A basic example is the *current of integration* $[S]$ over a compact oriented submanifold S of M :

$$\langle [S], \alpha \rangle = \int_S \alpha, \qquad \deg\alpha = p = \dim_{\mathbb{R}} S. \tag{0.12}$$

Then $[S]$ is a current with measure coefficients, and Stokes' formula shows that $d[S] = (-1)^{q-1}[\partial S]$, in particular $d[S] = 0$ if S has no boundary. Because of this example, the integer p is said to be the dimension of T when $T \in \mathcal{D}'_p(M)$. The current T is said to be *closed* if $dT = 0$.

On a complex manifold X, we have similar notions of bidegree and bidimension; as in the real case, we denote by

$$\mathcal{D}'^{p,q}(X) = \mathcal{D}'_{n-p,n-q}(X), \qquad n = \dim X,$$

the space of currents of bidegree (p,q) and bidimension $(n-p,n-q)$ on X. According to [Lel57], a current T of bidimension (p,p) is said to be (*weakly*) *positive* if for every choice of smooth $(1,0)$-forms $\alpha_1, \ldots, \alpha_p$ on X the distribution

$$T \wedge i\alpha_1 \wedge \overline{\alpha}_1 \wedge \ldots \wedge i\alpha_p \wedge \overline{\alpha}_p \quad \text{is a positive measure.} \tag{0.13}$$

0.14. Exercise. If T is positive, show that the coefficients $T_{I,J}$ of T are complex measures, and that, up to constants, they are dominated by the trace measure

$$\sigma_T = T \wedge \frac{1}{p!}\beta^p = 2^{-p}\sum T_{I,I}, \qquad \beta = \frac{i}{2}\partial\overline{\partial}|z|^2 = \frac{i}{2}\sum_{1\leqslant j\leqslant n} dz_j \wedge d\overline{z}_j,$$

which is a positive measure.

Hint. Observe that $\sum T_{I,I}$ is invariant by unitary changes of coordinates and that the (p,p)-forms $i\alpha_1 \wedge \overline{\alpha}_1 \wedge \ldots \wedge i\alpha_p \wedge \overline{\alpha}_p$ generate $\Lambda^{p,p}T^\star_{\mathbb{C}^n}$ as a $\mathbb{C}$-vector space. □

A current $T = i\sum_{1\leqslant j,k\leqslant n} T_{jk} dz_j \wedge dz_k$ of bidegree $(1,1)$ is easily seen to be positive if and only if the complex measure $\sum \lambda_j\overline{\lambda}_k T_{jk}$ is a positive measure for every n-tuple $(\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$.

0.15. Example. If u is a (not identically $-\infty$) psh function on X, we can associate with u a (closed) positive current $T = i\partial\overline{\partial}u$ of bidegree $(1,1)$. Conversely, every closed positive current of bidegree $(1,1)$ can be written under this form on any open subset $\Omega \subset X$ such that $H^2_{DR}(\Omega, \mathbb{R}) = H^1(\Omega, \mathcal{O}) = 0$, e.g. on small coordinate balls (exercise to the reader). □

It is not difficult to show that a product $T_1 \wedge \ldots \wedge T_q$ of positive currents of bidegree $(1,1)$ is positive whenever the product is well defined (this is certainly the case if all T_j but one at most are smooth; much finer conditions will be discussed in Section 2).

We now discuss another very important example of closed positive current. In fact, with every closed analytic set $A \subset X$ of pure dimension p is associated a current of integration

$$\langle [A], \alpha\rangle = \int_{A_{\rm reg}} \alpha, \qquad \alpha \in \mathcal{D}^{p,p}(X), \tag{0.16}$$

obtained by integrating over the regular points of A. In order to show that (0.16) is a correct definition of a current on X, one must show that $A_{\rm reg}$ has locally finite area in a neighborhood of $A_{\rm sing}$. This result, due to [Lel57] is shown as follows. Suppose that 0 is a singular point of A. By the local parametrization theorem for analytic sets, there is a linear change of coordinates on $\mathbb{C}^n$ such that all projections

$$\pi_I : (z_1, \ldots, z_n) \mapsto (z_{i_1}, \ldots, z_{i_p})$$

define a finite ramified covering of the intersection $A \cap \Delta$ with a small polydisk Δ in $\mathbb{C}^n$ onto a small polydisk Δ_I in $\mathbb{C}^p$. Let n_I be the sheet number. Then the p-dimensional area of

$A \cap \Delta$ is bounded above by the sum of the areas of its projections counted with multiplicities, i.e.

$$\mathrm{Area}(A \cap \Delta) \leqslant \sum n_I \mathrm{Vol}(\Delta_I).$$

The fact that $[A]$ is positive is also easy. In fact

$$i\alpha_1 \wedge \overline{\alpha}_1 \wedge \ldots \wedge i\alpha_p \wedge \overline{\alpha}_p = |\det(\alpha_{jk})|^2\, iw_1 \wedge \overline{w}_1 \wedge \ldots \wedge iw_p \wedge \overline{w}_p$$

if $\alpha_j = \sum \alpha_{jk} dw_k$ in terms of local coordinates $(w_1, \ldots, w_p)$ on A_{reg}. This shows that all such forms are $\geqslant 0$ in the canonical orientation defined by $iw_1 \wedge \overline{w}_1 \wedge \ldots \wedge iw_p \wedge \overline{w}_p$. More importantly, Lelong [Lel57] has shown that $[A]$ is d-closed in X, even at points of A_{sing}. This last result can be seen today as a consequence of the Skoda-El Mir extension theorem. For this we need the following definition: a *complete pluripolar* set is a set E such that there is an open covering (Ω_j) of X and psh functions u_j on Ω_j with $E \cap \Omega_j = u_j^{-1}(-\infty)$. Any (closed) analytic set is of course complete pluripolar (take u_j as in Example 0.9).

0.17. Theorem (Skoda [Sko82], El Mir [EM84], Sibony [Sib85])**.** *Let E be a closed complete pluripolar set in X, and let T be a closed positive current on $X \smallsetminus E$ such that the coefficients $T_{I,J}$ of T are measures with locally finite mass near E. Then the trivial extension $\widetilde{T}$ obtained by extending the measures $T_{I,J}$ by 0 on E is still closed on X.* □

The proof proceeds by rather direct mass estimates and will be omitted here. Lelong's result $d[A] = 0$ is obtained by applying the Skoda-El Mir theorem to $T = [A_{\mathrm{reg}}]$ on $X \smallsetminus A_{\mathrm{sing}}$.

0.18. Corollary. *Let T be a closed positive current on X and let E be a complete pluripolar set. Then $\mathbb{1}_E T$ and $\mathbb{1}_{X \smallsetminus E} T$ are closed positive currents. In fact, $\widetilde{T} = \mathbb{1}_{X \smallsetminus E} T$ is the trivial extension of $T_{\restriction X \smallsetminus E}$ to X, and $\mathbb{1}_E T = T - \widetilde{T}$.* □

As mentioned above, any current $T = i\partial\overline{\partial} u$ associated with a psh function u is a closed positive $(1,1)$-current. In the special case $u = \log|f|$ where $f \in H^0(X, \mathcal{O}_X)$ is a non zero holomorphic function, we have the important

0.19. Lelong-Poincaré equation. *Let $f \in H^0(X, \mathcal{O}_X)$ be a non zero holomorphic function, $Z_f = \sum m_j Z_j$, $m_j \in \mathbb{N}$, the zero divisor of f and $[Z_f] = \sum m_j [Z_j]$ the associated current of integration. Then*

$$\frac{i}{\pi} \partial\overline{\partial} \log|f| = [Z_f].$$

Proof (*sketch*). It is clear that $i\partial\overline{\partial}\log|f| = 0$ in a neighborhood of every point $x \notin \mathrm{Supp}(Z_f) = \bigcup Z_j$, so it is enough to check the equation in a neighborhood of every point of $\mathrm{Supp}(Z_f)$. Let A be the set of singular points of $\mathrm{Supp}(Z_f)$, i.e. the union of the pairwise intersections $Z_j \cap Z_k$ and of the singular loci $Z_{j,\mathrm{sing}}$; we thus have $\dim A \leqslant n-2$. In a neighborhood of any point $x \in \mathrm{Supp}(Z_f) \smallsetminus A$ there are local coordinates $(z_1, \ldots, z_n)$ such that $f(z) = z_1^{m_j}$ where m_j is the multiplicity of f along the component Z_j which contains x and $z_1 = 0$ is an equation for Z_j near x. Hence

$$\frac{i}{\pi} \partial\overline{\partial} \log|f| = m_j \frac{i}{\pi} \partial\overline{\partial} \log|z_1| = m_j [Z_j]$$

in a neighborhood of x, as desired (the identity comes from the standard formula $\frac{i}{\pi}\partial\overline{\partial}\log|z| =$ Dirac measure δ_0 in $\mathbb{C}$). This shows that the equation holds on $X \smallsetminus A$.

Hence the difference $\frac{i}{\pi}\partial\overline{\partial}\log|f| - [Z_f]$ is a closed current of degree 2 with measure coefficients, whose support is contained in A. By Exercise 0.20, this current must be 0, because A has too small dimension to carry its support (A is stratified by submanifolds of real codimension $\geqslant 4$). $\square$

0.20. Exercise. Let T be a current of degree q on a real manifold M, such that both T and dT have measure coefficients ("normal current"). Suppose that $\mathrm{Supp}\, T$ is contained in a real submanifold A with $\mathrm{codim}_{\mathbb{R}} A > q$. Show that $T = 0$.

Hint: Let $m = \dim_{\mathbb{R}} M$ and let $(x_1, \ldots, x_m)$ be a coordinate system in a neighborhood Ω of a point $a \in A$ such that $A \cap \Omega = \{x_1 = \ldots = x_k = 0\}$, $k > q$. Observe that $x_j T = x_j dT = 0$ for $1 \leqslant j \leqslant k$, thanks to the hypothesis on supports and on the normality of T, hence $dx_j \wedge T = d(x_j T) - x_j dT = 0$, $1 \leqslant j \leqslant k$. Infer from this that all coefficients in $T = \sum_{|I|=q} T_I dx_I$ vanish. $\square$

§0.D. Hermitian Vector Bundles, Connections and Curvature

The goal of this section is to recall the most basic definitions of hemitian differential geometry related to the concepts of connection, curvature and first Chern class of a line bundle.

Let F be a complex vector bundle of rank r over a smooth differentiable manifold M. A *connection* D on F is a linear differential operator of order 1

$$D : C^\infty(M, \Lambda^q T^\star_M \otimes F) \to C^\infty(M, \Lambda^{q+1} T^\star_M \otimes F)$$

such that

$$D(f \wedge u) = df \wedge u + (-1)^{\deg f} f \wedge Du \tag{0.21}$$

for all forms $f \in C^\infty(M, \Lambda^p T^\star_M)$, $u \in C^\infty(X, \Lambda^q T^\star_M \otimes F)$. On an open set $\Omega \subset M$ where F admits a trivialization $\theta : F_{|\Omega} \xrightarrow{\simeq} \Omega \times \mathbb{C}^r$, a connection D can be written

$$Du \simeq_\theta du + \Gamma \wedge u$$

where $\Gamma \in C^\infty(\Omega, \Lambda^1 T^\star_M \otimes \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^r))$ is an arbitrary matrix of 1-forms and d acts componentwise. It is then easy to check that

$$D^2 u \simeq_\theta (d\Gamma + \Gamma \wedge \Gamma) \wedge u \quad \text{on } \Omega.$$

Since D^2 is a globally defined operator, there is a global 2-form

$$R_D \in C^\infty(M, \Lambda^2 T^\star_M \otimes \mathrm{Hom}_{\mathbb{C}}(F, F))$$

such that $D^2 u = R_D \wedge u$ for every form u with values in F. Locally, R_D is given by

$$R_D \simeq_\theta d\Gamma + \Gamma \wedge \Gamma \tag{0.22}$$

where Γ is the connection matrix.

Assume now that F is endowed with a C^∞ hermitian metric h along the fibers and that the isomorphism $F_{|\Omega} \simeq \Omega \times \mathbb{C}^r$ is given by a C^∞ frame (e_λ). We then have a canonical sesquilinear pairing $\{\bullet,\bullet\} = \{\bullet,\bullet\}_h$

$$\begin{aligned}(0.23)\qquad C^\infty(M,\Lambda^p T^\star_M\otimes F)\times C^\infty(M,\Lambda^q T^\star_M\otimes F) &\longrightarrow C^\infty(M,\Lambda^{p+q}T^\star_M\otimes\mathbb{C})\\ (u,v)&\longmapsto \{u,v\}_h\end{aligned}$$

given by

$$\{u,v\}_h=\sum_{\lambda,\mu} u_\lambda\wedge\overline{v}_\mu\langle e_\lambda,e_\mu\rangle_h,\qquad u=\sum u_\lambda\otimes e_\lambda,\quad v=\sum v_\mu\otimes e_\mu.$$

We will frequently omit the subscript h when no confusion can arise. The connection D is said to be *hermitian* (with respect to h) if it satisfies the additional property

$$d\{u,v\}=\{Du,v\}+(-1)^{\deg\, u}\{u,Dv\}.$$

Assuming that (e_λ) is orthonormal, one easily checks that D is hermitian if and only if $\Gamma^\star=-\Gamma$, i.e. Γ is hermitian skew symmetric. In this case $R^\star_D=-R_D$ [observe that $(\Gamma\wedge\Gamma)^*=-\Gamma^*\wedge\Gamma^*$ and more generally $(A\wedge B)^*=-B^*\wedge A^*$ for products of matrices of 1-forms, since reversing the order of the product of 1-forms changes the sign]. Therefore the 2-form $\Theta_D:=\frac{i}{2\pi}R_D=\frac{i}{2\pi}D^2$ takes values in hermitian symmetric tensors $\mathrm{Herm}(F,F)$, i.e.

$$\Theta_D\in C^\infty(M,\Lambda^2T^\star_M\otimes_{\mathbb{R}}\mathrm{Herm}(F,F))$$

where $\mathrm{Herm}(F,F)\subset\mathrm{Hom}(F,F)$ is the real subspace of hermitian endomorphisms. (The reason for introducing the additional factor 2π will appear below).

0.24. Special case. For a bundle F of rank 1, the connection form Γ of a hermitian connection D can be seen as a 1-form with purely imaginary coefficients (i.e. $\Gamma=iA$, A real). Then we have $R_D=d\Gamma=idA$, therefore $\Theta_F=\frac{i}{2\pi}R_D=-\frac{1}{2\pi}dA$ is a *d-closed and real* 2-form. The (*real*) *first Chern class* of F is defined to be the cohomology class

$$c_1(F)_{\mathbb{R}}=\{\Theta_D\}\in H^2_{\mathrm{DR}}(M,\mathbb{R}).$$

This cohomology class is actually independent of the connection D taken on F: any other connection D_1 differs by a global 1-form, i.e. $D_1u=Du+B\wedge u$, so that $\Theta_{D_1}=\Theta_D-\frac{i}{2\pi}dB$. It is well-known that $c_1(F)_{\mathbb{R}}$ is the image in $H^2(M,\mathbb{R})$ of an integral class $c_1(F)\in H^2(M,\mathbb{Z})$; by using the exponential exact sequence

$$0\to\mathbb{Z}\to\mathcal{E}\to\mathcal{E}^\star\to 0,$$

$c_1(F)$ can be defined in Čech cohomology theory as the image by the coboundary map $H^1(M,\mathcal{E}^\star)\to H^2(M,\mathbb{Z})$ of the cocycle $\{g_{jk}\}\in H^1(M,\mathcal{E}^\star)$ defining F; see e.g. [GrH78] for details. This is the essential reason for the introduction of a factor $\frac{i}{2\pi}$ in the definition of Θ_D. □

We now concentrate ourselves on the complex analytic case. If $M=X$ is a complex manifold X, every connection D on a complex C^∞ vector bundle F can be splitted in a unique way as a sum of a $(1,0)$ and of a $(0,1)$-connection, $D = D' + D''$. In a local trivialization θ given by a C^∞ frame, one can write

$$D'u \simeq_\theta d'u + \Gamma' \wedge u, \tag{0.25$'$}$$

$$D''u \simeq_\theta d''u + \Gamma'' \wedge u, \tag{0.25$''$}$$

with $\Gamma = \Gamma' + \Gamma''$. The connection is hermitian if and only if $\Gamma' = -(\Gamma'')^\star$ in any orthonormal frame. Thus there exists a unique hermitian connection D corresponding to a prescribed $(0,1)$ part D''.

Assume now that the hermitian bundle (F,h) itself has a *holomorphic* structure. The unique hermitian connection D_h for which D''_h is the $\overline{\partial}$ operator defined in §0.A is called the *Chern connection* of F. In a local holomorphic frame (e_λ) of $E_{|\Omega}$, the metric h is then given by a hermitian matrix $H = (h_{\lambda\mu})$, $h_{\lambda\mu} = \langle e_\lambda, e_\mu\rangle$. We have

$$\{u,v\} = \sum_{\lambda,\mu} h_{\lambda\mu} u_\lambda \wedge \overline{v}_\mu = u^\dagger \wedge H\overline{v},$$

where $u^\dagger$ is the transposed matrix of u. Easy computations yield

$$\begin{aligned} d\{u,v\} &= (du)^\dagger \wedge H\overline{v} + (-1)^{\deg u} u^\dagger \wedge (dH \wedge \overline{v} + H\overline{dv}) \\ &= \big(du + \overline{H}^{\,-1} d'\overline{H} \wedge u\big)^\dagger \wedge H\overline{v} + (-1)^{\deg u} u^\dagger \wedge \overline{(dv + \overline{H}^{\,-1} d'\overline{H} \wedge v)} \end{aligned}$$

using the fact that $dH = d'H + \overline{d'\overline{H}}$ and $\overline{H}^\dagger = H$. Therefore the Chern connection D_h coincides with the hermitian connection defined by

$$\begin{cases} D_h u \simeq_\theta du + \overline{H}^{\,-1} d'\overline{H} \wedge u, \\ D'_h \simeq_\theta d' + \overline{H}^{\,-1} d'\overline{H} \wedge \bullet = \overline{H}^{\,-1} d'(\overline{H}\bullet), \qquad D''_h = d''. \end{cases} \tag{0.26}$$

It is clear from the above relations (0.26) that $D'^2_h = D''^2_h = 0$. Consequently D^2_h is given by to $D^2_h = D'_h D''_h + D''_h D'_h$, and the curvature tensor R_{D_h} is of type $(1,1)$. Since $d'd'' + d''d' = 0$, we get

$$\begin{aligned} (D'_h D''_h + D''_h D'_h) u &\simeq_\theta \overline{H}^{\,-1} d'\overline{H} \wedge d''u + d''(\overline{H}^{\,-1} d'\overline{H} \wedge u) \\ &= d''(\overline{H}^{\,-1} d'\overline{H}) \wedge u. \end{aligned}$$

By the above calculation R_{D_h} is given by the matrix of $(1,1)$-forms

$$R_{D_h} \simeq_\theta d''(\overline{H}^{\,-1} d'\overline{H}) = \overline{H}^{\,-1} d''d'\overline{H} - \overline{H}^{\,-1} d''\overline{H} \wedge \overline{H}^{\,-1} d'\overline{H}$$

Since $H = \overline{H}^\dagger$ is hermitian symmetric and transposition reverses products, we find again in this setting that R_{D_h} is hermitian skew symmetric

$$R^*_{D_h} \simeq_\theta \overline{H}^{\,-1} \overline{R}^\dagger_F \overline{H} = -R_{D_h}.$$

0.27. Definition and proposition. *The Chern curvature tensor of (F,h) is defined to be $\Theta_{F,h} := \Theta_{D_h} = \frac{i}{2\pi} R_{D_h}$ where D_h is the Chern connection. It is such that*

$$\Theta_{F,h} \in C^\infty\big(X, \Lambda^{1,1}_{\mathbb{R}} T^\star_X \otimes_{\mathbb{R}} \mathrm{Herm}(F,F)\big) \subset C^\infty\big(X, \Lambda^{1,1} T^\star_X \otimes_{\mathbb{C}} \mathrm{Hom}(F,F)\big).$$

If $\theta : F_{\restriction\Omega} \to \Omega \times \mathbb{C}^r$ is a holomorphic trivialization and if H is the hermitian matrix representing the metric along the fibers of $F_{\restriction\Omega}$, then

$$\Theta_{F,h} \simeq_\theta \frac{i}{2\pi} d''(\overline{H}^{\,-1} d'\overline{H}) \quad \text{on } \Omega. \qquad \square$$

[*We will frequently omit the subscript h and write simply $D_h = D$, $\Theta_{F,h} = \Theta_F$ when no confusion can arise*].

The next proposition shows that the Chern curvature tensor is the obstruction to the existence of orthonormal holomorphic frames: a holomorphic frame can be made "almost orthonormal" only up to curvature terms of order 2 in a neighborhood of any point.

0.28. Proposition. *For every point $x_0 \in X$ and every holomorphic coordinate system $(z_j)_{1\leqslant j\leqslant n}$ at x_0, there exists a holomorphic frame $(e_\lambda)_{1\leqslant\lambda\leqslant r}$ of F in a neighborhood of x_0 such that*

$$\langle e_\lambda(z), e_\mu(z)\rangle = \delta_{\lambda\mu} - \sum_{1\leqslant j,k\leqslant n} c_{jk\lambda\mu}\, z_j \overline{z}_k + O(|z|^3)$$

where $(c_{jk\lambda\mu})$ are the coefficients of the Chern curvature tensor $\Theta_F(x_0)$, namely

$$\Theta_F(x_0) = \frac{i}{2\pi} \sum_{1\leqslant j,k\leqslant n,\, 1\leqslant\lambda,\mu\leqslant r} c_{jk\lambda\mu} dz_j \wedge d\overline{z}_k \otimes e_\lambda^\star \otimes e_\mu.$$

Such a frame (e_λ) is called a normal coordinate frame at x_0.

Proof. Let (h_λ) be a holomorphic frame of F. After replacing (h_λ) by suitable linear combinations with constant coefficients, we may assume that $\big(h_\lambda(x_0)\big)$ is an orthonormal basis of F_{x_0}. Then the inner products $\langle h_\lambda, h_\mu\rangle$ have an expansion

$$\langle h_\lambda(z), h_\mu(z)\rangle = \delta_{\lambda\mu} + \sum_j (a_{j\lambda\mu}\, z_j + a'_{j\lambda\mu}\, \overline{z}_j) + O(|z|^2)$$

for some complex coefficients $a_{j\lambda\mu}$, $a'_{j\lambda\mu}$ such that $a'_{j\lambda\mu} = \overline{a}_{j\mu\lambda}$. Set first

$$g_\lambda(z) = h_\lambda(z) - \sum_{j,\mu} a_{j\lambda\mu}\, z_j\, h_\mu(z).$$

Then there are coefficients $a_{jk\lambda\mu}$, $a'_{jk\lambda\mu}$, $a''_{jk\lambda\mu}$ such that

$$\begin{aligned}\langle g_\lambda(z), g_\mu(z)\rangle &= \delta_{\lambda\mu} + O(|z|^2)\\ &= \delta_{\lambda\mu} + \sum_{j,k} \big(a_{jk\lambda\mu}\, z_j\overline{z}_k + a'_{jk\lambda\mu}\, z_j z_k + a''_{jk\lambda\mu}\overline{z}_j\overline{z}_k\big) + O(|z|^3).\end{aligned}$$

The holomorphic frame (e_λ) we are looking for is

$$e_\lambda(z) = g_\lambda(z) - \sum_{j,k,\mu} a'_{jk\lambda\mu}\, z_j z_k\, g_\mu(z).$$

Since $a''_{jk\lambda\mu} = \overline{a}'_{jk\mu\lambda}$, we easily find

$$\begin{aligned}
\langle e_\lambda(z), e_\mu(z)\rangle &= \delta_{\lambda\mu} + \sum_{j,k} a_{jk\lambda\mu}\, z_j \overline{z}_k + O(|z|^3),\\
d'\langle e_\lambda, e_\mu\rangle = \{D'e_\lambda, e_\mu\} &= \sum_{j,k} a_{jk\lambda\mu}\, \overline{z}_k\, dz_j + O(|z|^2),\\
\Theta_F \cdot e_\lambda = D''(D'e_\lambda) &= \sum_{j,k,\mu} a_{jk\lambda\mu}\, d\overline{z}_k \wedge dz_j \otimes e_\mu + O(|z|),
\end{aligned}$$

therefore $c_{jk\lambda\mu} = -a_{jk\lambda\mu}$. □

According to (0.27), one can associate canonically with the curvature tensor of F a hermitian form on $T_X \otimes F$ defined by

$$\widetilde{\Theta}_F(\xi \otimes v) = \sum_{1\leqslant j,k\leqslant n,\, 1\leqslant\lambda,\mu\leqslant r} c_{jk\lambda\mu}\xi_j\overline{\xi}_k v_\lambda \overline{v}_\mu, \qquad \xi \in T_X, \quad v \in F. \tag{0.29}$$

This leads in a natural way to positivity concepts, following definitions introduced by Kodaira [Kod53], Nakano [Nak55] and Griffiths [Gri69].

0.30. Definition. *The hermitian vector bundle F is said to be*

a) *positive in the sense of Nakano if $\widetilde{\Theta}_F(\tau) > 0$ for all non zero tensors $\tau = \sum \tau_{j\lambda}\partial/\partial z_j \otimes e_\lambda \in T_X \otimes F$.*

b) *positive in the sense of Griffiths if $\widetilde{\Theta}_F(\xi \otimes v) > 0$ for all non zero decomposable tensors $\xi \otimes v \in T_X \otimes F$;*

Corresponding semipositivity concepts are defined by relaxing the strict inequalities.

0.31. Special case of rank 1 bundles. Assume that F is a line bundle. The hermitian matrix $H = (h_{11})$ associated to a trivialization $\theta : F_{\restriction\Omega} \simeq \Omega \times \mathbb{C}$ is simply a positive function which we find convenient to denote by $e^{-\varphi}$, $\varphi \in C^\infty(\Omega, \mathbb{R})$. In this case the curvature form $R_{F,h}$ can be identified to the $(1,1)$-form $\partial\overline{\partial}\varphi$, and thus we get a real $(1,1)$-form

$$\Theta_{F,h} = \frac{i}{2\pi}\partial\overline{\partial}\varphi.$$

Hence F is semipositive (in either the Nakano or Griffiths sense) if and only if φ is psh, resp. positive if and only if φ is *strictly psh*. In this setting, the Lelong-Poincaré equation can be generalized as follows: let $\sigma \in H^0(X, F)$ be a non zero holomorphic section. Then

$$\frac{i}{2\pi}\partial\overline{\partial}\log\|\sigma\|_h = [Z_\sigma] - \frac{i}{2\pi}\Theta_{F,h}. \tag{0.32}$$

Formula (0.32) is immediate if we write $\|\sigma\|_h^2 = |\theta(\sigma)|^2 e^{-\varphi}$ and if we apply (0.19) to the holomorphic function $f = \theta(\sigma)$. As we shall see later, it is very important for the applications to consider also singular hermitian metrics.

0.33. Definition. *A singular (hermitian) metric h on a line bundle F is a metric h which is given in any trivialization $\theta : F_{\restriction\Omega} \xrightarrow{\simeq} \Omega \times \mathbb{C}$ by*

$$\|\xi\|_h^2 = |\theta(\xi)|^2\, e^{-\varphi(x)}, \qquad x \in \Omega,\ \xi \in F_x$$

where φ is an arbitrary measurable function in $L^1_{\rm loc}(\Omega)$, called the weight of the metric with respect to the trivialization θ.

If $\theta' : F_{\restriction\Omega'} \longrightarrow \Omega' \times \mathbb{C}$ is another trivialization, φ' the associated weight and $g \in \mathcal{O}^\star(\Omega \cap \Omega')$ the transition function, then $\theta'(\xi) = g(x)\,\theta(\xi)$ for $\xi \in F_x$, and so $\varphi' = \varphi + \log|g|^2$ on $\Omega \cap \Omega'$. The curvature form of F is then given formally by the closed $(1,1)$-current $\Theta_{F,h} = \frac{i}{2\pi}\partial\overline{\partial}\varphi$ on Ω ; our assumption $\varphi \in L^1_{\rm loc}(\Omega)$ guarantees that $\Theta_{F,h}$ exists in the sense of distribution theory. As in the smooth case, $\Theta_{F,h}$ is globally defined on X and independent of the choice of trivializations, and its De Rham cohomology class is the image of the first Chern class $c_1(F) \in H^2(X,\mathbb{Z})$ in $H^2_{DR}(X,\mathbb{R})$. Before going further, we discuss two basic examples.

0.34. Example. Let $D = \sum \alpha_j D_j$ be a divisor with coefficients $\alpha_j \in \mathbb{Z}$ and let $F = \mathcal{O}(D)$ be the associated invertible sheaf of meromorphic functions u such that $\mathrm{div}(u) + D \geqslant 0$; the corresponding line bundle can be equipped with the singular metric defined by $\|u\| = |u|$. If g_j is a generator of the ideal of D_j on an open set $\Omega \subset X$ then $\theta(u) = u\prod g_j^{\alpha_j}$ defines a trivialization of $\mathcal{O}(D)$ over Ω, thus our singular metric is associated to the weight $\varphi = \sum \alpha_j \log|g_j|^2$. By the Lelong-Poincaré equation, we find

$$\Theta_{\mathcal{O}(D)} = \frac{i}{2\pi}\partial\overline{\partial}\varphi = [D],$$

where $[D] = \sum \alpha_j [D_j]$ denotes the current of integration over D. □

0.35. Example. Assume that $\sigma_1, \ldots, \sigma_N$ are non zero holomorphic sections of F. Then we can define a natural (possibly singular) hermitian metric h^* on $F^\star$ by

$$\|\xi^\star\|^2_{h^*} = \sum_{1 \leqslant j \leqslant n} \big|\xi^\star.\sigma_j(x)\big|^2 \quad \text{for } \xi^\star \in F^\star_x.$$

The dual metric h on F is given by

$$\|\xi\|^2_h = \frac{|\theta(\xi)|^2}{|\theta(\sigma_1(x))|^2 + \ldots + |\theta(\sigma_N(x))|^2} \tag{0.35 a}$$

with respect to any trivialization θ. The associated weight function is thus given by $\varphi(x) = \log\big(\sum_{1\leqslant j\leqslant N} |\theta(\sigma_j(x))|^2\big)$. In this case φ is a psh function, thus $\Theta_{F,h}$ is a closed positive current, given explicity by

$$\Theta_{F,h} = \frac{i}{2\pi}\partial\overline{\partial}\varphi = \frac{i}{2\pi}\partial\overline{\partial}\log\Big(\sum_{1\leqslant j\leqslant N} |\theta(\sigma_j(x))|^2\Big). \tag{0.35 b}$$

Let us denote by Σ the linear system defined by $\sigma_1, \ldots, \sigma_N$ and by $B_\Sigma = \bigcap \sigma_j^{-1}(0)$ its base locus. We have a meromorphic map

$$\Phi_\Sigma : X \smallsetminus B_\Sigma \to \mathbb{P}^{N-1}, \qquad x \mapsto (\sigma_1(x) : \sigma_2(x) : \ldots : \sigma_N(x)).$$

Then Θ_F is equal to the pull-back by Φ_Σ over $X \smallsetminus B_\Sigma$ of the so called *Fubini-Study metric* on $\mathbb{P}^{N-1}$:

$$\omega_{\rm FS} = \frac{i}{2\pi}\partial\overline{\partial}\log(|z_1|^2 + \ldots + |z_N|^2) \tag{0.35 c}$$

□

0.36. Ample and very ample line bundles. *A holomorphic line bundle F over a compact complex manifold X is said to be*

a) *very ample if the map $\Phi_{|F|} : X \to \mathbb{P}^{N-1}$ associated to the complete linear system $|F| = P(H^0(X,F))$ is a regular embedding (by this we mean in particular that the base locus is empty, i.e. $B_{|F|} = \emptyset$).*

b) *ample if some multiple mF, $m > 0$, is very ample.*

Here we use an additive notation for $\mathrm{Pic}(X) = H^1(X, \mathcal{O}^\star)$, hence the symbol mF denotes the line bundle $F^{\otimes m}$. By Example 0.35, every ample line bundle F has a smooth hermitian metric with positive definite curvature form; indeed, if the linear system $|mF|$ gives an embedding in projective space, then we get a smooth hermitian metric on $F^{\otimes m}$, and the m-th root yields a metric on F such that $\Theta_F = \frac{1}{m}\Phi^\star_{|mF|}\omega_{\mathrm{FS}}$. Conversely, the Kodaira embedding theorem [Kod54] tells us that every positive line bundle F is ample.

§1. Hyperbolicity concepts and directed manifolds

We first recall a few basic facts concerning the concept of hyperbolicity, according to S. Kobayashi [Kob70, Kob76]. Let X be a complex n-dimensional manifold. We denote by $f : \Delta \to X$ an arbitrary holomorphic map from the unit disk $\Delta \subset \mathbb{C}$ to X. The *Kobayashi-Royden infinitesimal pseudometric* on X is the Finsler pseudometric on the tangent bundle T_X defined by

$$\mathbf{k}_X(\xi) = \inf\big\{\lambda > 0\,;\, \exists f : \Delta \to X,\ f(0) = x,\ \lambda f'(0) = \xi\big\}, \qquad x \in X,\ \xi \in T_{X,x}$$

(see H. Royden [Roy71], [Roy74]). In the terminology of Kobayashi [Kob75], a *Finsler metric* (resp. *pseudometric*) on a vector bundle E is a homogeneous positive (resp. nonnegative) positive function N on the total space E, that is,

$$N(\lambda\xi) = |\lambda|\,N(\xi) \qquad \text{for all } \lambda \in \mathbb{C} \text{ and } \xi \in E,$$

but in general N is not assumed to be subbadditive (i.e. convex) on the fibers of E.

A Finsler (pseudo-)metric on E is thus nothing but a hermitian (semi-)norm on the tautological line bundle $\mathcal{O}_{P(E)}(-1)$ of lines of E over the projectivized bundle $Y = P(E)$. The *Kobayashi pseudodistance* $d_K(x,y)$ is the geodesic pseudodistance obtained by integrating the Kobayashi-Royden infinitesimal metric. The manifold X is said to be *hyperbolic* (in the sense of Kobayashi) if d_K is actually a distance, namely if $d_K(x,y) > 0$ for all pairs of distinct points (x,y) in X. In this context, we have the following well-known results of Brody [Bro78].

1.1. Brody reparametrization lemma. *Let ω be a hermitian metric on X and let $f : \Delta \to X$ be a holomorphic map. For every $\varepsilon > 0$, there exists a radius $R \geqslant (1-\varepsilon)\|f'(0)\|_\omega$ and a homographic transformation ψ of the disk $D(0,R)$ onto $(1-\varepsilon)\Delta$ such that*

$$\|(f\circ\psi)'(0)\|_\omega = 1, \qquad \|(f\circ\psi)'(t)\|_\omega \leqslant \frac{1}{1-|t|^2/R^2} \quad \textit{for every } t \in D(0,R).$$

In particular, if X is compact, given any sequence of holomorphic mappings $f_\nu : \Delta \to X$ such that $\lim \|f'_\nu(0)\|_\omega = +\infty$, one can find a sequence of homographic transformations $\psi_\nu : D(0,R_\nu) \to (1-1/\nu)\Delta$ with $\lim R_\nu = +\infty$, such that, after passing possibly to a *subsequence,* $(f_\nu \circ \psi_\nu)$ *converges uniformly on every compact subset of* $\mathbb{C}$ *towards a non constant holomorphic map* $g : \mathbb{C} \to X$ *with* $\|g'(0)\|_\omega = 1$ *and* $\sup_{t\in\mathbb{C}} \|g'(t)\|_\omega \leqslant 1$.

Proof. The first assertion of Brody's lemma is obtained by selecting $t_0 \in \Delta$ such that $(1-|t|^2)\|f'((1-\varepsilon)t)\|_\omega$ reaches its maximum for $t = t_0$. The reason for this choice is that $(1-|t|^2)\|f'((1-\varepsilon)t)\|_\omega$ is the norm of the differential $f'((1-\varepsilon)t) : T_\Delta \to T_X$ with respect to the Poincaré metric $|dt|^2/(1-|t|^2)^2$ on T_Δ, which is conformally invariant under $\mathrm{Aut}(\Delta)$. One then adjusts R and ψ so that $\psi(0) = (1-\varepsilon)t_0$ and $|\psi'(0)|\,\|f'(\psi(0))\|_\omega = 1$. As $|\psi'(0)| = \frac{1-\varepsilon}{R}(1-|t_0|^2)$, the only possible choice for R is

$$R = (1-\varepsilon)(1-|t_0|^2)\|f'(\psi(0))\|_\omega \geqslant (1-\varepsilon)\|f'(0)\|_\omega.$$

The inequality for $(f\circ\psi)'$ follows from the fact that the Poincaré norm is maximum at the origin, where it is equal to 1 by the choice of R. □

1.2. Corollary (Brody's theorem)**.** *A compact complex manifold* X *is hyperbolic if and only if there are no non constant entire holomorphic maps* $g : \mathbb{C} \to X$.

Proof. The arguments are rather standard and will be developped in more detail in the proof of Prop. 1.5 below. □

Now, more generally, let (X, V) be a complex manifold equipped with a *holomorphic subbundle* $V \subset T_X$. We will refer to such a pair as being a *complex directed manifold.* A morphism $\Phi : (X, V) \to (Y, W)$ in the category of complex directed manifolds is a holomorphic map such that $\Phi_\star(V) \subset W$. Our philosophy is that directed manifolds are also useful to study the "absolute case", i.e. the case $V = T_X$, because there are fonctorial constructions which work better in the category of directed manifolds (see e.g. § 4, 5, 6). We think of directed manifolds as a kind of "relative situation", covering e.g. the case when V is the relative tangent sheaf to a smooth map $X \to S$. We want to stress here that no assumption need be made on the Lie bracket tensor $[\ ,\ ] : V \times V \to T_X/V$, and the rank $r = \mathrm{rank}\, V$ may be an arbitrary integer in the range $1 \leqslant r \leqslant n := \dim_{\mathbb{C}} X$. For the sake of generality, one might also wish to allow singularities in the subbundle V: for this, one can take V to be given by an arbitrary coherent subsheaf $\mathcal{V} \subset \mathcal{O}(T_X)$ such that $\mathcal{O}(T_X)/\mathcal{V}$ has no torsion; then V is a subbundle outside an analytic subset of codimension at least 2 (it is however somewhat safer to view $\mathcal{V}^\star$ as given by a quotient sheaf morphism $\Omega^1_X \to \mathcal{V}^\star$ and let $V^\star$ be the associated linear space, see Remark 3.10 below). For the sake of simplicity, we will assume most of the time that V is actually a subbundle of T_X. In this situation, we generalize the notion of hyperbolicity as follows.

1.3. Definition. *Let* (X, V) *be a complex directed manifold.*

i) *The Kobayashi-Royden infinitesimal metric of* (X, V) *is the Finsler metric on* V *defined for any* $x \in X$ *and* $\xi \in V_x$ *by*

$$\mathbf{k}_{(X,V)}(\xi) = \inf\big\{\lambda > 0\,;\, \exists f : \Delta \to X,\ f(0) = x,\ \lambda f'(0) = \xi,\ f'(\Delta) \subset V\big\}.$$

Here $\Delta \subset \mathbb{C}$ *is the unit disk and the map* f *is an arbitrary holomorphic map which is tangent to* V*, i.e., such that* $f'(t) \in V_{f(t)}$ *for all* $t \in \Delta$*. We say that* (X, V) *is infinitesimally hyperbolic if* $\mathbf{k}_{(X,V)}$ *is positive definite on every fiber* V_x *and satisfies a uniform lower bound* $\mathbf{k}_{(X,V)}(\xi) \geqslant \varepsilon\|\xi\|_\omega$ *in terms of any smooth hermitian metric* ω *on* X*, when* x *describes a compact subset of* X.

ii) *More generally, the Kobayashi-Eisenman infinitesimal pseudometric of* (X,V) *is the pseudometric defined on all decomposable* p*-vectors* $\xi = \xi_1 \wedge \cdots \wedge \xi_p \in \Lambda^p V_x$, $1 \leqslant p \leqslant r = \operatorname{rank} V$, *by*

$$\mathbf{e}^p_{(X,V)}(\xi) = \inf\big\{\lambda > 0\,;\, \exists f : \mathbb{B}_p \to X,\ f(0) = x,\ \lambda f_\star(\tau_0) = \xi,\ f_\star(T_{\mathbb{B}_p}) \subset V\big\}$$

where $\mathbb{B}_p$ *is the unit ball in* $\mathbb{C}^p$ *and* $\tau_0 = \partial/\partial t_1 \wedge \cdots \wedge \partial/\partial t_p$ *is the unit* p*-vector of* $\mathbb{C}^p$ *at the origin. We say that* (X,V) *is infinitesimally* p*-measure hyperbolic if* $\mathbf{e}^p_{(X,V)}$ *is positive definite on every fiber* $\Lambda^p V_x$ *and satisfies a locally uniform lower bound in terms of any smooth metric.*

If $\Phi : (X,V) \to (Y,W)$ is a morphism of directed manifolds, it is immediate to check that we have the monotonicity property

$$(1.4) \qquad \mathbf{k}_{(Y,W)}(\Phi_\star\xi) \leqslant \mathbf{k}_{(X,V)}(\xi), \qquad \forall \xi \in V,$$

$$(1.4^p) \qquad \mathbf{e}^p_{(Y,W)}(\Phi_\star\xi) \leqslant \mathbf{e}^p_{(X,V)}(\xi), \qquad \forall \xi = \xi_1 \wedge \cdots \wedge \xi_p \in \Lambda^p V.$$

The following proposition shows that virtually all reasonable definitions of the hyperbolicity property are equivalent if X is compact (in particular, the additional assumption that there is locally uniform lower bound for $\mathbf{k}_{(X,V)}$ is not needed). We merely say in that case that (X,V) is *hyperbolic.*

1.5. Proposition. *For an arbitrary directed manifold* (X,V), *the Kobayashi-Royden infinitesimal metric* $\mathbf{k}_{(X,V)}$ *is upper semicontinuous on the total space of* V. *If* X *is compact,* (X,V) *is infinitesimally hyperbolic if and only if there are no non constant entire curves* $g : \mathbb{C} \to X$ *tangent to* V. *In that case,* $\mathbf{k}_{(X,V)}$ *is a continuous* (*and positive definite*) *Finsler metric on* V.

Proof. The proof is almost identical to the standard proof for $\mathbf{k}_X$, so we only give a brief outline of the ideas. In order to prove the upper semicontinuity, let $\xi_0 \in V_{x_0}$ and $\varepsilon > 0$ be given. Then there is a curve $f : \Delta \to X$ tangent to V such that $f(0) = x_0$ and $\lambda f'(0) = \xi_0$ with $0 < \lambda < \mathbf{k}_X(\xi_0) + \varepsilon$. Take $\lambda = 1$ for simplicity, and replace ξ_0 by $\lambda^{-1}\xi_0$. We may assume that f is a proper embedding, otherwise we replace (X,V) by $(X',V') = (X \times \Delta, \mathrm{pr}_1^\star V \oplus \mathrm{pr}_2^\star T_\Delta)$, f by $f \times \mathrm{Id}_\Delta$, ξ_0 by $\xi_0 \oplus 1$, and use a monotonicity argument for the projection $\mathrm{pr}_1 : X' \to X$. If f is an embedding, then $f(\Delta)$ is a Stein submanifold of X, and thus $f(\Delta)$ has a Stein neighborhood Ω. As Ω is Stein, there exists a section $\theta \in H^0(\Omega, \mathcal{O}(V))$ extending $f' \in H^0(f(\Delta), \mathcal{O}(V))$. The map f can be viewed as the solution of the differential equation $f' = \theta(f)$ with initial value $f(0) = x_0$. Take a small perturbation $g' = \theta_\eta(g)$ with initial value $g(0) = x$, where $\theta_\eta = \theta + \sum \eta_j s_j$ and $s_1, \ldots, s_N$ are finitely many sections of $H^0(\Omega, \mathcal{O}(V))$ which generate V in a neighborhood of x_0. We can achieve that $g'(0) = \theta_\eta(x)$ is equal to any prescribed vector $\xi \in V_x$ close to $\xi_0 = \theta(x_0)$, and the solution g exists on $(1-\varepsilon)\Delta$ if the perturbation is small enough. We conclude that $\mathbf{k}_{(X,V)}$ is upper semicontinuous by considering $t \mapsto g((1-\varepsilon)t)$.

If there exists a non constant entire curve $g : \mathbb{C} \to X$ tangent to V, it is clear that $\mathbf{k}_{(X,V)}(g'(t)) \equiv 0$, hence (X,V) cannot be hyperbolic. Conversely, if X is compact and if there are no non constant entire curves $g : \mathbb{C} \to X$ tangent to V, Brody's lemma implies that there is an absolute bound $\|f'(0)\|_\omega \leqslant C$ for all holomorphic maps $f : \Delta \to X$ tangent to V; hence $\mathbf{k}_{(X,V)}(\xi) \geqslant C^{-1}\|\xi\|_\omega$ and (X,V) is infinitesimally hyperbolic. By reparametrizing f with an arbitrary automorphism of Δ, we find $\|f'(t)\|_\omega \leqslant C/(1-|t|^2)$. The space of maps $f : \Delta \to X$ tangent to V is therefore compact for the topology of uniform convergence on compact subsets of Δ, thanks to Ascoli's theorem. We easily infer from this that $\mathbf{k}_{(X,V)}$ is lower semicontinuous on V. □

Another easy observation is that the concept of p-measure hyperbolicity gets weaker and weaker as p increases :

1.6. Proposition. *If (X,V) is p-measure hyperbolic, then it is $(p+1)$-measure hyperbolic for all $p \in \{1,\ldots,r-1\}$.*

Proof. Asserting that (X,V) is p-measure hyperbolic means that for all maps $f:\mathbb{B}_p \to X$ tangent to V with $f(0)=x$, there is a uniform upper bound $\|\Lambda^p f_*(0)\|_\omega \leqslant A$ for $\Lambda^p f_*(0) : \Lambda^p T_{\mathbb{B}_p} \to \Lambda^p V$ with respect to a given hermitian metric ω on X. Consider $g:\mathbb{B}_{p+1}\to X$ tangent to V with $g(0)=x$ fixed. Let us restrict g to all p-dimensional balls $\mathbb{B}_{p+1}\cap H$ where H is a hyperplane in $\mathbb{B}^{p+1}$. Applying this to $f = g_{\restriction \mathbb{B}_{p+1}\cap H}$ and H arbitrary, one gets a bound for $\|(\Lambda^p g_*(0))_{\restriction H}\|_\omega$ and therefore a bound for $\|\Lambda^p g_*(0)\|_\omega$. However, there are orthonormal bases of $\mathbb{C}^{p+1}$ and $V\simeq \mathbb{C}^r$ such that $u:=g_*(0):\mathbb{C}^{p+1}\to V$ has a diagonal matrix with diagonal entries $\lambda_j\in\mathbb{R}_+$ (the λ_j's are the square roots of the eigenvalues of the hermitian form $\tau\mapsto\|u(\tau)\|_\omega^2$). Then

$$\|\Lambda^k u\|_\omega^2 = \sum_{i_1<\ldots<i_k} (\lambda_{i_1}\ldots\lambda_{i_k})^2, \qquad \text{especially}\quad \|\Lambda^{p+1}u\|_\omega^2 = (\lambda_1\ldots\lambda_{p+1})^2, \quad \text{hence}$$

$$\|\Lambda^{p+1}u\|_\omega^{2p} = \prod_{j=1}^{p+1}(\lambda_1\ldots\widehat{\lambda_j}\ldots\lambda_{p+1})^2 \leqslant \|\Lambda^p u\|_\omega^{2(p+1)}, \qquad \text{i.e.}\quad \|\Lambda^{p+1}u\|_\omega \leqslant \|\Lambda^p u\|_\omega^{1+1/p}.$$

This implies our claim. □

We conclude this section by showing that hyperbolicity is an open property.

1.7. Proposition. *Let $(\mathcal{X},\mathcal{V})\to S$ be a holomorphic family of compact directed manifolds (by this, we mean a proper holomorphic map $\mathcal{X}\to S$ together with a holomorphic subbundle $\mathcal{V}\subset T_{\mathcal{X}/S}$ of the relative tangent bundle, defining a deformation $(X_t,V_t)_{t\in S}$ of the fibers). Then the set of $t\in S$ such that the fiber (X_t,V_t) is hyperbolic is open in S with respect to the euclidean topology.*

Proof. Take a sequence of non hyperbolic fibers (X_{t_ν},V_{t_ν}) with $t_\nu\to t$ and fix a hermitian metric ω on $\mathcal{X}$. By Brody's lemma, there is a sequence of entire holomorphic maps $g_\nu:\mathbb{C}\to X_{t_\nu}$ tangent to V_{t_ν}, such that $\|g'_\nu(0)\|_\omega=1$ and $\|g'_\nu\|\leqslant 1$. Ascoli's theorem shows that there is a subsequence of (g_ν) converging uniformly to a limit $g:\mathbb{C}\to X_t$, tangent to V_t, with $\|g'(0)\|_\omega=1$. Hence (X_t,V_t) is not hyperbolic, and the collection of non hyperbolic fibers is closed in S. □

Let us mention here an impressive result proved by Marco Brunella [Bru03, Bru05, Bru06] concerning the behavior of the Kobayashi metric on foliated varieties.

1.8. Theorem (Brunella)**.** *Let X be a compact Kähler manifold equipped with a (possibly singular) holomorphic foliation which is not a foliation by rational curves. Then the canonical bundle $K_{\mathcal{F}}$ of the foliation is pseudoeffective (i.e. the curvature of $K_{\mathcal{F}}$ is $\geqslant 0$ in the sense of currents).*

The proof is obtained by putting on $K_{\mathcal{F}}$ precisely the metric induced by the Kobayashi metric on the leaves whenever they are generically hyperbolic (i.e. covered by the unit disk). The case of parabolic leaves (covered by $\mathbb{C}$) has to be treated separately.

§2. Hyperbolicity and bounds for the genera of curves

In the case of projective algebraic varieties, hyperbolicity is expected to be related to other properties of a more algebraic nature. Theorem 2.1 below is a first step in this direction.

2.1. Theorem. *Let (X, V) be a compact complex directed manifold and let $\sum \omega_{jk} dz_j \otimes d\overline{z}_k$ be a hermitian metric on X, with associated positive $(1,1)$-form $\omega = \frac{i}{2} \sum \omega_{jk} dz_j \wedge d\overline{z}_k$. Consider the following three properties, which may or not be satisfied by (X, V) :*

i) *(X, V) is hyperbolic.*

ii) *There exists $\varepsilon > 0$ such that every compact irreducible curve $C \subset X$ tangent to V satisfies*

$$-\chi(\overline{C}) = 2g(\overline{C}) - 2 \geqslant \varepsilon \deg_\omega(C)$$

where $g(\overline{C})$ is the genus of the normalization $\overline{C}$ of C, $\chi(\overline{C})$ its Euler characteristic and $\deg_\omega(C) = \int_C \omega$. (This property is of course independent of ω.)

iii) *There does not exist any non constant holomorphic map $\Phi : Z \to X$ from an abelian variety Z to X such that $\Phi_\star(T_Z) \subset V$.*

Then i) $\Rightarrow$ ii) $\Rightarrow$ iii)*.*

Proof. i) $\Rightarrow$ ii). If (X, V) is hyperbolic, there is a constant $\varepsilon_0 > 0$ such that $\mathbf{k}_{(X,V)}(\xi) \geqslant \varepsilon_0 \|\xi\|_\omega$ for all $\xi \in V$. Now, let $C \subset X$ be a compact irreducible curve tangent to V and let $\nu : \overline{C} \to C$ be its normalization. As (X, V) is hyperbolic, $\overline{C}$ cannot be a rational or elliptic curve, hence $\overline{C}$ admits the disk as its universal covering $\rho : \Delta \to \overline{C}$.

The Kobayashi-Royden metric $\mathbf{k}_\Delta$ is the Finsler metric $|dz|/(1 - |z|^2)$ associated with the Poincaré metric $|dz|^2/(1 - |z|^2)^2$ on Δ, and $\mathbf{k}_{\overline{C}}$ is such that $\rho^\star \mathbf{k}_{\overline{C}} = \mathbf{k}_\Delta$. In other words, the metric $\mathbf{k}_{\overline{C}}$ is induced by the unique hermitian metric on $\overline{C}$ of constant Gaussian curvature -4. If $\sigma_\Delta = \frac{i}{2} dz \wedge d\overline{z}/(1 - |z|^2)^2$ and $\sigma_{\overline{C}}$ are the corresponding area measures, the Gauss-Bonnet formula (integral of the curvature $= 2\pi\,\chi(\overline{C})$) yields

$$\int_{\overline{C}} d\sigma_{\overline{C}} = -\frac{1}{4} \int_{\overline{C}} \operatorname{curv}(\mathbf{k}_{\overline{C}}) = -\frac{\pi}{2} \chi(\overline{C})$$

On the other hand, if $j : C \to X$ is the inclusion, the monotonicity property (1.4) applied to the holomorphic map $j \circ \nu : \overline{C} \to X$ shows that

$$\mathbf{k}_{\overline{C}}(t) \geqslant \mathbf{k}_{(X,V)}\big((j \circ \nu)_\star t\big) \geqslant \varepsilon_0 \big\|(j \circ \nu)_\star t\big\|_\omega, \qquad \forall t \in T_{\overline{C}}.$$

From this, we infer $d\sigma_{\overline{C}} \geqslant \varepsilon_0^2 (j \circ \nu)^\star \omega$, thus

$$-\frac{\pi}{2} \chi(\overline{C}) = \int_{\overline{C}} d\sigma_{\overline{C}} \geqslant \varepsilon_0^2 \int_{\overline{C}} (j \circ \nu)^\star \omega = \varepsilon_0^2 \int_C \omega.$$

Property ii) follows with $\varepsilon = 2\varepsilon_0^2/\pi$.

ii) $\Rightarrow$ iii). First observe that ii) excludes the existence of elliptic and rational curves tangent to V. Assume that there is a non constant holomorphic map $\Phi : Z \to X$ from an abelian variety Z to X such that $\Phi_\star(T_Z) \subset V$. We must have $\dim \Phi(Z) \geqslant 2$, otherwise $\Phi(Z)$ would be a curve covered by images of holomorphic maps $\mathbb{C} \to \Phi(Z)$, and so $\Phi(Z)$ would be elliptic or rational, contradiction. Select a sufficiently general curve Γ in Z (e.g., a curve obtained as an intersection of very generic divisors in a given very ample linear system $|L|$ in Z). Then all isogenies $u_m : Z \to Z$, $s \mapsto ms$ map Γ in a $1:1$ way to curves $u_m(\Gamma) \subset Z$, except maybe for finitely many double points of $u_m(\Gamma)$ (if $\dim Z = 2$). It follows that the normalization of $u_m(\Gamma)$ is isomorphic to Γ. If Γ is general enough, similar arguments show that the images

$$C_m := \Phi(u_m(\Gamma)) \subset X$$

are also generically $1:1$ images of Γ, thus $\overline{C}_m \simeq \Gamma$ and $g(\overline{C}_m) = g(\Gamma)$. We would like to show that C_m has degree $\geqslant \mathrm{Const}\, m^2$. This is indeed rather easy to check if ω is Kähler, but the general case is slightly more involved. We write

$$\int_{C_m} \omega = \int_\Gamma (\Phi \circ u_m)^\star \omega = \int_Z [\Gamma] \wedge u_m^\star(\Phi^\star\omega),$$

where Γ denotes the current of integration over Γ. Let us replace Γ by an arbitrary translate $\Gamma + s$, $s \in Z$, and accordingly, replace C_m by $C_{m,s} = \Phi \circ u_m(\Gamma + s)$. For $s \in Z$ in a Zariski open set, $C_{m,s}$ is again a generically $1:1$ image of $\Gamma + s$. Let us take the average of the last integral identity with respect to the unitary Haar measure $d\mu$ on Z. We find

$$\int_{s\in Z} \left(\int_{C_{m,s}} \omega\right) d\mu(s) = \int_Z \left(\int_{s\in Z} [\Gamma + s]\, d\mu(s)\right) \wedge u_m^\star(\Phi^\star\omega).$$

Now, $\gamma := \int_{s\in Z} [\Gamma + s]\, d\mu(s)$ is a translation invariant positive definite form of type $(p-1, p-1)$ on Z, where $p = \dim Z$, and γ represents the same cohomology class as $[\Gamma]$, i.e. $\gamma \equiv c_1(L)^{p-1}$. Because of the invariance by translation, γ has constant coefficients and so $(u_m)_\star\gamma = m^2\gamma$. Therefore we get

$$\int_{s\in Z} d\mu(s) \int_{C_{m,s}} \omega = m^2 \int_Z \gamma \wedge \Phi^\star\omega.$$

In the integral, we can exclude the algebraic set of values z such that $C_{m,s}$ is not a generically $1:1$ image of $\Gamma + s$, since this set has measure zero. For each m, our integral identity implies that there exists an element $s_m \in Z$ such that $g(\overline{C}_{m,s_m}) = g(\Gamma)$ and

$$\deg_\omega(C_{m,s_m}) = \int_{C_{m,s_m}} \omega \geqslant m^2 \int_Z \gamma \wedge \Phi^\star\omega.$$

As $\int_Z \gamma \wedge \Phi^\star\omega > 0$, the curves C_{m,s_m} have bounded genus and their degree is growing quadratically with m, contradiction to property ii). □

2.2. Definition. *We say that a projective directed manifold (X, V) is "algebraically hyperbolic" if it satisfies property* 2.1 ii)*, namely, if there exists $\varepsilon > 0$ such that every algebraic curve $C \subset X$ tangent to V satisfies*

$$2g(\overline{C}) - 2 \geqslant \varepsilon \deg_\omega(C).$$

A nice feature of algebraic hyperbolicity is that it satisfies an algebraic analogue of the openness property.

2.3. Proposition. *Let $(\mathcal{X}, \mathcal{V}) \to S$ be an algebraic family of projective algebraic directed manifolds* (*given by a projective morphism $\mathcal{X} \to S$*)*. Then the set of $t \in S$ such that the fiber*

(X_t, V_t) is algebraically hyperbolic is open with respect to the "countable Zariski topology" of S (by definition, this is the topology for which closed sets are countable unions of algebraic sets).

Proof. After replacing S by a Zariski open subset, we may assume that the total space $\mathcal{X}$ itself is quasi-projective. Let ω be the Kähler metric on $\mathcal{X}$ obtained by pulling back the Fubini-Study metric via an embedding in a projective space. If integers $d > 0$, $g \geqslant 0$ are fixed, the set $A_{d,g}$ of $t \in S$ such that X_t contains an algebraic 1-cycle $C = \sum m_j C_j$ tangent to V_t with $\deg_\omega(C) = d$ and $g(\overline{C}) = \sum m_j\, g(\overline{C}_j) \leqslant g$ is a closed algebraic subset of S (this follows from the existence of a relative cycle space of curves of given degree, and from the fact that the geometric genus is Zariski lower semicontinuous). Now, the set of non algebraically hyperbolic fibers is by definition

$$\bigcap_{k>0} \; \bigcup_{2g-2<d/k} A_{d,g}.$$

This concludes the proof (of course, one has to know that the countable Zariski topology is actually a topology, namely that the class of countable unions of algebraic sets is stable under arbitrary intersections; this can be easily checked by an induction on dimension). □

2.4. Remark. More explicit versions of the openness property have been dealt with in the literature. H. Clemens ([Cle86] and [CKL88]) has shown that on a very generic surface of degree $d \geqslant 5$ in $\mathbb{P}^3$, the curves of type (d, k) are of genus $g > kd(d-5)/2$ (recall that a very generic surface $X \subset \mathbb{P}^3$ of degree $\geqslant 4$ has Picard group generated by $\mathcal{O}_X(1)$ thanks to the Noether-Lefschetz theorem, thus any curve on the surface is a complete intersection with another hypersurface of degree k; such a curve is said to be of type (d, k); genericity is taken here in the sense of the countable Zariski topology). Improving on this result of Clemens, Geng Xu [Xu94] has shown that every curve contained in a very generic surface of degree $d \geqslant 5$ satisfies the sharp bound $g \geqslant d(d-3)/2 - 2$. This actually shows that a very generic surface of degree $d \geqslant 6$ is algebraically hyperbolic. Although a very generic quintic surface has no rational or elliptic curves, it seems to be unknown whether a (very) generic quintic surface is algebraically hyperbolic in the sense of Definition 2.2.

2.5. Remark. It would be interesting to know whether algebraic hyperbolicity is open with respect to the euclidean topology; still more interesting would be to know whether Kobayashi hyperbolicity is open for the countable Zariski topology (of course, both properties would follow immediately if one knew that algebraic hyperbolicity and Kobayashi hyperbolicity coincide, but they seem otherwise highly non trivial to establish). The latter openness property has raised an important amount of work around the following more particular question: is a (very) generic hypersurface $X \subset \mathbb{P}^{n+1}$ of degree d large enough (say $d \geqslant 2n+1$) Kobayashi hyperbolic? Again, "very generic" is to be taken here in the sense of the countable Zariski topology. Brody-Green [BrGr77] and Nadel [Nad89] produced examples of hyperbolic surfaces in $\mathbb{P}^3$ for all degrees $d \geqslant 50$, and Masuda-Noguchi [MaNo93] recently gave examples of such hypersurfaces in $\mathbb{P}^n$ for arbitrary $n \geqslant 2$, of degree $d \geqslant d_0(n)$ large enough. The question of studying the hyperbolicity of complements $\mathbb{P}^n \smallsetminus D$ of generic divisors is in principle closely related to this; in fact if $D = \{P(z_0, \ldots, z_n) = 0\}$ is a smooth generic divisor of degree d, one may look at the hypersurface

$$X = \big\{z_{n+1}^d = P(z_0, \ldots, z_n)\big\} \subset \mathbb{P}^{n+1}$$

which is a cyclic $d\!:\!1$ covering of $\mathbb{P}^n$. Since any holomorphic map $f : \mathbb{C} \to \mathbb{P}^n \smallsetminus D$ can be lifted to X, it is clear that the hyperbolicity of X would imply the hyperbolicity of $\mathbb{P}^n \smallsetminus D$.

The hyperbolicity of complements of divisors in $\mathbb{P}^n$ has been investigated by many authors. □

In the "absolute case" $V = T_X$, it seems reasonable to expect that properties 2.1 i), ii) are equivalent, i.e. that Kobayashi and algebraic hyperbolicity coincide. However, it was observed by Serge Cantat [Can00] that property 2.1 (iii) is not sufficient to imply the hyperbolicity of X, at least when X is a general complex surface: a general (non algebraic) K3 surface is known to have no elliptic curves and does not admit either any surjective map from an abelian variety; however such a surface is not Kobayashi hyperbolic. We are uncertain about the sufficiency of 2.1 (iii) when X is assumed to be projective.

§3. The Ahlfors-Schwarz lemma for metrics of negative curvature

One of the most basic ideas is that hyperbolicity should somehow be related with suitable negativity properties of the curvature. For instance, it is a standard fact already observed in Kobayashi [Kob70] that the negativity of T_X (or the ampleness of $T^\star_X$) implies the hyperbolicity of X. There are many ways of improving or generalizing this result. We present here a few simple examples of such generalizations. If (V, h) is a holomorphic vector bundle equipped with a smooth hermitian metric, we denote by $\nabla_h = \nabla'_h + \nabla''_h$ the associated Chern connection and by $\Theta_{V,h} = \frac{i}{2\pi}\nabla_h^2$ its Chern curvature tensor.

3.1. Proposition. *Let (X, V) be a compact directed manifold. Assume that $V^\star$ is ample. Then (X, V) is hyperbolic.*

Proof (from an original idea of [Kob75]). Recall that a vector bundle E is said to be ample if $S^m E$ has enough global sections $\sigma_1, \ldots, \sigma_N$ so as to generate 1-jets of sections at any point, when m is large. One obtains a Finsler metric N on $E^\star$ by putting

$$N(\xi) = \Big(\sum_{1\leqslant j\leqslant N} |\sigma_j(x)\cdot\xi^m|^2\Big)^{1/2m}, \qquad \xi \in E^\star_x,$$

and N is then a strictly plurisubharmonic function on the total space of $E^\star$ minus the zero section (in other words, the line bundle $\mathcal{O}_{P(E^\star)}(1)$ has a metric of positive curvature). By the ampleness assumption on $V^\star$, we thus have a Finsler metric N on V which is strictly plurisubharmonic outside the zero section. By Brody's lemma, if (X, V) is not hyperbolic, there is a non constant entire curve $g : \mathbb{C} \to X$ tangent to V such that $\sup_{\mathbb{C}} \|g'\|_\omega \leqslant 1$ for some given hermitian metric ω on X. Then $N(g')$ is a bounded subharmonic function on $\mathbb{C}$ which is strictly subharmonic on $\{g' \neq 0\}$. This is a contradiction, for any bounded subharmonic function on $\mathbb{C}$ must be constant. □

This result can be generalized a little bit further by means of the Ahlfors-Schwarz lemma (see e.g. [Lang87]).

3.2. Ahlfors-Schwarz lemma. *Let $\gamma(t) = \gamma_0(t)\, i\, dt\wedge d\bar t$ be a hermitian metric on Δ_R where $\log\gamma_0$ is a subharmonic function such that $i\,\partial\bar\partial\log\gamma_0(t) \geqslant A\,\gamma(t)$ in the sense of currents, for some positive constant A. Then γ can be compared with the Poincaré metric of Δ_R as follows:*

$$\gamma(t) \leqslant \frac{2}{A}\frac{R^{-2}|dt|^2}{(1-|t|^2/R^2)^2}.$$

More generally, let $\gamma = i\sum\gamma_{jk}dt_j\wedge d\bar t_k$ be an almost everywhere positive hermitian form on the ball $B(0,R) \subset \mathbb{C}^p$, such that $-\mathrm{Ricci}(\gamma) := i\,\partial\bar\partial\log\det\gamma \geqslant A\gamma$ in the sense of currents, *for some constant $A>0$ (this means in particular that* $\det\gamma=\det(\gamma_{jk})$ *is such that* $\log\det\gamma$ *is plurisubharmonic). Then*

$$\det(\gamma)\leqslant\Big(\frac{p+1}{AR^2}\Big)^p\frac{1}{(1-|t|^2/R^2)^{p+1}}.$$

Proof. It is of course sufficient to deal with the more general case of a ball in $\mathbb{C}^p$. First assume that γ is smooth and positive definite on $\overline{B}(0,R)$. Take a point $t_0\in B(0,R)$ at which $(1-|t|^2/R^2)^{p+1}\det(\gamma(t))$ is maximum. The logarithmic $i\,\partial\overline\partial$-derivative of this function at t_0 must be $\leqslant 0$, hence

$$i\,\partial\overline\partial\log\det\gamma(t)_{t=t_0}-(p+1)\,i\,\partial\overline\partial\log(1-|t|^2/R^2)^{-1}_{t=t_0}\leqslant 0.$$

The hypothesis on the Ricci curvature implies

$$A^p\,\gamma(t_0)^p\leqslant\big(i\,\partial\overline\partial\log\det\gamma(t)_{t=t_0}\big)^p\leqslant(p+1)^p\,\big(i\,\partial\overline\partial\log(1-|t|^2/R^2)^{-1}_{t=t_0}\big)^p.$$

An easy computation shows that the determinant of $i\,\partial\overline\partial\log(1-|t|^2/R^2)^{-1}$ is equal to $R^{-2p}(1-|t|^2/R^2)^{-p-1}$. From this, we conclude that

$$(1-|t|^2/R^2)^{p+1}\det\gamma(t)\leqslant(1-|t_0|^2/R^2)^{p+1}\det\gamma(t_0)\leqslant\Big(\frac{p+1}{AR^2}\Big)^p.$$

If γ is not smooth, we use a regularization argument. Namely, we shrink R a little bit and look at the maximum of the function

$$u(t)=(1-|t|^2/R^2)^{p+1}\exp\big(\rho_\varepsilon\star\log\det\gamma(t)\big)$$

where (ρ_ε) is a family of regularizing kernels. The argument goes through because

$$i\,\partial\overline\partial(\rho_\varepsilon\star\log\det\gamma)\geqslant A\,\rho_\varepsilon\star\gamma$$

and $\log\det(\rho_\varepsilon\star\gamma)\geqslant\rho_\varepsilon\star\log\det\gamma$ by concavity of the log det function. □

3.3. Proposition. *Let (X,V) be a compact directed manifold. Assume that $V^\star$ is "very big" in the following sense: there exists an ample line bundle L and a sufficiently large integer m such that the global sections in $H^0(X,S^mV^\star\otimes L^{-1})$ generate all fibers over $X\smallsetminus Y$, for some analytic subset $Y\subsetneq X$. Then all entire curves $f:\mathbb{C}\to X$ tangent to V satisfy $f(\mathbb{C})\subset Y$ [under our assumptions, X is a projective algebraic manifold and Y is an algebraic subvariety, thus it is legitimate to say that the entire curves are "algebraically degenerate"].*

Proof. Let $\sigma_1,\ldots,\sigma_N\in H^0(X,S^mV^\star\otimes L^{-1})$ be a basis of sections generating $S^mV^\star\otimes L^{-1}$ over $X\smallsetminus Y$. If $f:\mathbb{C}\to X$ is tangent to V, we define a semipositive hermitian form $\gamma(t)=\gamma_0(t)\,|dt|^2$ on $\mathbb{C}$ by putting

$$\gamma_0(t)=\sum\|\sigma_j(f(t))\cdot f'(t)^m\|^{2/m}_{L^{-1}}$$

where $\|\ \|_L$ denotes a hermitian metric with positive curvature on L. If $f(\mathbb{C})\not\subset Y$, the form γ is not identically 0 and we then find

$$i\,\partial\overline\partial\log\gamma_0\geqslant\frac{2\pi}{m}f^\star\Theta_L$$

where Θ_L is the curvature form. The positivity assumption combined with an obvious homogeneity argument yield

$$\frac{2\pi}{m} f^\star\Theta_L \geqslant \varepsilon \|f'(t)\|^2_\omega\,|dt|^2 \geqslant \varepsilon'\,\gamma(t)$$

for any given hermitian metric ω on X. Now, for any t_0 with $\gamma_0(t_0) > 0$, the Ahlfors-Schwarz lemma shows that f can only exist on a disk $D(t_0, R)$ such that $\gamma_0(t_0) \leqslant \frac{2}{\varepsilon'}R^{-2}$, contradiction. □

There are similar results for p-measure hyperbolicity, e.g.

3.4. Proposition. *Let (X, V) be a compact directed manifold. Assume that $\Lambda^p V^\star$ is ample. Then (X, V) is infinitesimally p-measure hyperbolic. More generally, assume that $\Lambda^p V^\star$ is very big with base locus contained in $Y \subsetneq X$ (see* 3.3*). Then $\mathbf{e}^p$ is non degenerate over $X \smallsetminus Y$.*

Proof. By the ampleness assumption, there is a smooth Finsler metric N on $\Lambda^p V$ which is strictly plurisubharmonic outside the zero section. We select also a hermitian metric ω on X. For any holomorphic map $f : \mathbb{B}_p \to X$ we define a semipositive hermitian metric $\widetilde{\gamma}$ on $\mathbb{B}_p$ by putting $\widetilde{\gamma} = f^\star\omega$. Since ω need not have any good curvature estimate, we introduce the function $\delta(t) = N_{f(t)}(\Lambda^p f'(t) \cdot \tau_0)$, where $\tau_0 = \partial/\partial t_1 \wedge \cdots \wedge \partial/\partial t_p$, and select a metric $\gamma = \lambda\widetilde{\gamma}$ conformal to $\widetilde{\gamma}$ such that $\det\gamma = \delta$. Then λ^p is equal to the ratio $N/\Lambda^p\omega$ on the element $\Lambda^p f'(t) \cdot \tau_0 \in \Lambda^p V_{f(t)}$. Since X is compact, it is clear that the conformal factor λ is bounded by an absolute constant independent of f. From the curvature assumption we then get

$$i\,\partial\overline{\partial}\log\det\gamma = i\,\partial\overline{\partial}\log\delta \geqslant (f, \Lambda^p f')^\star(i\,\partial\overline{\partial}\log N) \geqslant \varepsilon f^\star\omega \geqslant \varepsilon'\,\gamma.$$

By the Ahlfors-Schwarz lemma we infer that $\det\gamma(0) \leqslant C$ for some constant C, i.e., $N_{f(0)}(\Lambda^p f'(0) \cdot \tau_0) \leqslant C'$. This means that the Kobayashi-Eisenman pseudometric $\mathbf{e}^p_{(X,V)}$ is positive definite everywhere and uniformly bounded from below. In the case $\Lambda^p V^\star$ is very big with base locus Y, we use essentially the same arguments, but we then only have N being positive definite on $X \smallsetminus Y$. □

3.5. Corollary ([Gri71], KobO71]). *If X is a projective variety of general type, the Kobayashi-Eisenmann volume form $\mathbf{e}^n$, $n = \dim X$, can degenerate only along a proper algebraic set $Y \subsetneq X$.*

The converse of Corollary 3.5 is expected to be true, namely, the generic non degeneracy of $\mathbf{e}^n$ should imply that X is of general type, but this is only known for surfaces (see [GrGr80] and [MoMu82]):

3.6. Conjecture (Green-Griffiths [GrGr80]). *A projective algebraic variety X is measure hyperbolic (i.e. $\mathbf{e}^n$ degenerates only along a proper algebraic subvariety) if and only if X is of general type.*

An essential step in the proof of the necessity of having general type subvarieties would be to show that manifolds of Kodaira dimension 0 (say, Calabi-Yau manifolds and holomorphic symplectic manifolds, all of which have $c_1(X) = 0$) are not measure hyperbolic, e.g. by exhibiting enough families of curves $C_{s,\ell}$ covering X such that $(2g(\overline{C}_{s,\ell}) - 2)/\deg(C_{s,\ell}) \to 0$. Another related conjecture which we will investigate at the end of these notes is

3.7. Conjecture (Green-Griffiths [GrGr80]). *If X is a variety of general type, there exists a proper algebraic set $Y \subsetneq X$ such that every entire holomorphic curve $f : \mathbb{C} \to X$ is contained in Y.*

The most outstanding result in the direction of Conjecture 3.7 is the proof of the Bloch theorem, as proposed by Bloch [Blo26] and Ochiai [Och77]. The Bloch theorem is the special case of 3.7 when the irregularity of X satisfies $q = h^0(X, \Omega^1_X) > \dim X$. Various solutions have then been obtained in fundamental papers of Noguchi [Nog77, 81, 84], Kawamata [Kaw80] and Green-Griffiths [GrGr80], by means of different techniques. See section § 9 for a proof based on jet bundle techniques.

3.8. Conjecture ([Lang86, 87]). *A projective algebraic variety X is hyperbolic if and only if all its algebraic subvarieties* (*including X itself*) *are of general type.*

The relation between these conjectures is as follows.

3.9. Proposition. *Conjecture* 3.7 *implies the "if" part of conjecture* 3.8*, and Conjecture* 3.6 *implies the "only if" part of Conjecture* 3.8*, hence* (3.6 *and* 3.7) $\Rightarrow$ (3.8)*.*

Proof. In fact if Conjecture 3.7 holds and every subariety Y of X is of general type, then it is easy to infer that every entire curve $f : \mathbb{C} \to X$ has to be constant by induction on $\dim X$, because in fact f maps $\mathbb{C}$ to a certain subvariety $Y \subsetneq X$. Therefore X is hyperbolic.

Conversely, if Conjecture 3.6 holds and X has a certain subvariety Y which is not of general type, then Y is not measure hyperbolic. However it is easy to see that hyperbolicity implies measure hyperbolicity, since it is enough to bound a differential in every direction to bound its determinant. Therefore Y is not hyperbolic and so X itself is not hyperbolic either. □

§4. Projectivization of a directed manifold

The basic idea is to introduce a fonctorial process which produces a new complex directed manifold $(\widetilde{X}, \widetilde{V})$ from a given one (X, V). The new structure $(\widetilde{X}, \widetilde{V})$ plays the role of a space of 1-jets over X. We let

$$\widetilde{X} = P(V), \qquad \widetilde{V} \subset T_{\widetilde{X}}$$

be the projectivized bundle of lines of V, together with a subbundle $\widetilde{V}$ of $T_{\widetilde{X}}$ defined as follows: for every point $(x, [v]) \in \widetilde{X}$ associated with a vector $v \in V_x \smallsetminus \{0\}$,

$$\widetilde{V}_{(x,[v])} = \big\{\xi \in T_{\widetilde{X},\,(x,[v])}\,;\ \pi_\star\xi \in \mathbb{C}v\big\}, \qquad \mathbb{C}v \subset V_x \subset T_{X,x}, \tag{4.1}$$

where $\pi : \widetilde{X} = P(V) \to X$ is the natural projection and $\pi_\star : T_{\widetilde{X}} \to \pi^\star T_X$ is its differential. On $\widetilde{X} = P(V)$ we have a tautological line bundle $\mathcal{O}_{\widetilde{X}}(-1) \subset \pi^\star V$ such that $\mathcal{O}_{\widetilde{X}}(-1)_{(x,[v])} = \mathbb{C}v$. The bundle $\widetilde{V}$ is characterized by the two exact sequences

$$0 \longrightarrow T_{\widetilde{X}/X} \longrightarrow \widetilde{V} \xrightarrow{\ \pi_\star\ } \mathcal{O}_{\widetilde{X}}(-1) \longrightarrow 0, \tag{4.2}$$

$$0 \longrightarrow \mathcal{O}_{\widetilde{X}} \longrightarrow \pi^\star V \otimes \mathcal{O}_{\widetilde{X}}(1) \longrightarrow T_{\widetilde{X}/X} \longrightarrow 0, \tag{4.2'}$$

where $T_{\widetilde{X}/X}$ denotes the relative tangent bundle of the fibration $\pi : \widetilde{X} \to X$. The first sequence is a direct consequence of the definition of $\widetilde{V}$, whereas the second is a relative version of the Euler exact sequence describing the tangent bundle of the fibers $P(V_x)$. From these exact sequences we infer

$$\dim \widetilde{X} = n + r - 1, \qquad \operatorname{rank} \widetilde{V} = \operatorname{rank} V = r, \tag{4.3}$$

and by taking determinants we find $\det(T_{\widetilde{X}/X}) = \pi^\star \det V \otimes \mathcal{O}_{\widetilde{X}}(r)$, thus

$$\det \widetilde{V} = \pi^\star \det V \otimes \mathcal{O}_{\widetilde{X}}(r-1). \tag{4.4}$$

By definition, $\pi : (\widetilde{X}, \widetilde{V}) \to (X, V)$ is a morphism of complex directed manifolds. Clearly, our construction is fonctorial, i.e., for every morphism of directed manifolds $\Phi : (X, V) \to (Y, W)$, there is a commutative diagram

$$\begin{tikzcd} (\widetilde{X}, \widetilde{V}) \arrow[r, "\pi"] \arrow[d, dashed, "\widetilde{\Phi}"'] & (X, V) \arrow[d, "\Phi"] \\ (\widetilde{Y}, \widetilde{W}) \arrow[r, "\pi"] & (Y, W) \end{tikzcd} \tag{4.5}$$

where the left vertical arrow is the meromorphic map $P(V) \dashrightarrow P(W)$ induced by the differential $\Phi_\star : V \to \Phi^\star W$ ($\widetilde{\Phi}$ is actually holomorphic if $\Phi_\star : V \to \Phi^\star W$ is injective).

Now, suppose that we are given a holomorphic curve $f : \Delta_R \to X$ parametrized by the disk Δ_R of centre 0 and radius R in the complex plane, and that f is a *tangent trajectory* of the directed manifold, i.e., $f'(t) \in V_{f(t)}$ for every $t \in \Delta_R$. If f is non constant, there is a well defined and unique tangent line $[f'(t)]$ for every t, even at stationary points, and the map

$$\widetilde{f} : \Delta_R \to \widetilde{X}, \qquad t \mapsto \widetilde{f}(t) := (f(t), [f'(t)]) \tag{4.6}$$

is holomorphic (at a stationary point t_0, we just write $f'(t) = (t - t_0)^s u(t)$ with $s \in \mathbb{N}^\star$ and $u(t_0) \neq 0$, and we define the tangent line at t_0 to be $[u(t_0)]$, hence $\widetilde{f}(t) = (f(t), [u(t)])$ near t_0 ; even for $t = t_0$, we still denote $[f'(t_0)] = [u(t_0)]$ for simplicity of notation). By definition $f'(t) \in \mathcal{O}_{\widetilde{X}}(-1)_{\widetilde{f}(t)} = \mathbb{C}\, u(t)$, hence the derivative f' defines a section

$$f' : T_{\Delta_R} \to \widetilde{f}^\star \mathcal{O}_{\widetilde{X}}(-1). \tag{4.7}$$

Moreover $\pi \circ \widetilde{f} = f$, therefore

$$\pi_\star \widetilde{f}'(t) = f'(t) \in \mathbb{C} u(t) \Longrightarrow \widetilde{f}'(t) \in \widetilde{V}_{(f(t), u(t))} = \widetilde{V}_{\widetilde{f}(t)}$$

and we see that $\widetilde{f}$ is a tangent trajectory of $(\widetilde{X}, \widetilde{V})$. We say that $\widetilde{f}$ is the *canonical lifting* of f to $\widetilde{X}$. Conversely, if $g : \Delta_R \to \widetilde{X}$ is a tangent trajectory of $(\widetilde{X}, \widetilde{V})$, then by definition of $\widetilde{V}$ we see that $f = \pi \circ g$ is a tangent trajectory of (X, V) and that $g = \widetilde{f}$ (unless g is contained in a vertical fiber $P(V_x)$, in which case f is constant).

For any point $x_0 \in X$, there are local coordinates $(z_1, \ldots, z_n)$ on a neighborhood Ω of x_0 such that the fibers $(V_z)_{z \in \Omega}$ can be defined by linear equations

$$V_z = \Big\{ \xi = \sum_{1 \leqslant j \leqslant n} \xi_j \frac{\partial}{\partial z_j}\ ;\ \xi_j = \sum_{1 \leqslant k \leqslant r} a_{jk}(z) \xi_k \text{ for } j = r+1, \ldots, n \Big\}, \tag{4.8}$$

where (a_{jk}) is a holomorphic $(n-r)\times r$ matrix. It follows that a vector $\xi \in V_z$ is completely determined by its first r components $(\xi_1,\ldots,\xi_r)$, and the affine chart $\xi_j \neq 0$ of $P(V)_{\restriction\Omega}$ can be described by the coordinate system

$$\Big(z_1,\ldots,z_n;\frac{\xi_1}{\xi_j},\ldots,\frac{\xi_{j-1}}{\xi_j},\frac{\xi_{j+1}}{\xi_j},\ldots,\frac{\xi_r}{\xi_j}\Big). \tag{4.9}$$

Let $f\simeq(f_1,\ldots,f_n)$ be the components of f in the coordinates $(z_1,\ldots,z_n)$ (we suppose here R so small that $f(\Delta_R)\subset\Omega$). It should be observed that f is uniquely determined by its initial value x and by the first r components $(f_1,\ldots,f_r)$. Indeed, as $f'(t)\in V_{f(t)}$, we can recover the other components by integrating the system of ordinary differential equations

$$f'_j(t)=\sum_{1\leqslant k\leqslant r}a_{jk}(f(t))f'_k(t),\qquad j>r, \tag{4.10}$$

on a neighborhood of 0, with initial data $f(0)=x$. We denote by $m=m(f,t_0)$ the *multiplicity* of f at any point $t_0\in\Delta_R$, that is, $m(f,t_0)$ is the smallest integer $m\in\mathbb{N}^\star$ such that $f_j^{(m)}(t_0)\neq 0$ for some j. By (4.10), we can always suppose $j\in\{1,\ldots,r\}$, for example $f_r^{(m)}(t_0)\neq 0$. Then $f'(t)=(t-t_0)^{m-1}u(t)$ with $u_r(t_0)\neq 0$, and the lifting $\widetilde f$ is described in the coordinates of the affine chart $\xi_r\neq 0$ of $P(V)_{\restriction\Omega}$ by

$$\widetilde f\simeq\Big(f_1,\ldots,f_n;\frac{f'_1}{f'_r},\ldots,\frac{f'_{r-1}}{f'_r}\Big). \tag{4.11}$$

We end this section with a few curvature computations. Assume that V is equipped with a smooth hermitian metric h. Denote by $\nabla_h=\nabla'_h+\nabla''_h$ the associated Chern connection and by $\Theta_{V,h}=\frac{i}{2\pi}\nabla_h^2$ its Chern curvature tensor. For every point $x_0\in X$, there exists a "normalized" holomorphic frame $(e_\lambda)_{1\leqslant\lambda\leqslant r}$ on a neighborhood of x_0, such that

$$\langle e_\lambda,e_\mu\rangle_h=\delta_{\lambda\mu}-\sum_{1\leqslant j,k\leqslant n}c_{jk\lambda\mu}z_j\overline z_k+O(|z|^3), \tag{4.12}$$

with respect to any holomorphic coordinate system $(z_1,\ldots,z_n)$ centered at x_0. A computation of $d'\langle e_\lambda,e_\mu\rangle_h=\langle\nabla'_he_\lambda,e_\mu\rangle_h$ and $\nabla_h^2e_\lambda=d''\nabla'_he_\lambda$ then gives

$$\begin{aligned}\nabla'_he_\lambda&=-\sum_{j,k,\mu}c_{jk\lambda\mu}\overline z_k\,dz_j\otimes e_\mu+O(|z|^2),\\ \Theta_{V,h}(x_0)&=\frac{i}{2\pi}\sum_{j,k,\lambda,\mu}c_{jk\lambda\mu}dz_j\wedge d\overline z_k\otimes e_\lambda^\star\otimes e_\mu.\end{aligned} \tag{4.13}$$

The above curvature tensor can also be viewed as a hermitian form on $T_X\otimes V$. In fact, one associates with $\Theta_{V,h}$ the hermitian form $\langle\Theta_{V,h}\rangle$ on $T_X\otimes V$ defined for all $(\zeta,v)\in T_X\times_X V$ by

$$\langle\Theta_{V,h}\rangle(\zeta\otimes v)=\sum_{1\leqslant j,k\leqslant n,\,1\leqslant\lambda,\mu\leqslant r}c_{jk\lambda\mu}\zeta_j\overline\zeta_kv_\lambda\overline v_\mu. \tag{4.14}$$

Let h_1 be the hermitian metric on the tautological line bundle $\mathcal{O}_{P(V)}(-1)\subset\pi^\star V$ induced by the metric h of V. We compute the curvature $(1,1)$-form $\Theta_{h_1}(\mathcal{O}_{P(V)}(-1))$ at an arbitrary point $(x_0,[v_0])\in P(V)$, in terms of $\Theta_{V,h}$. For simplicity, we suppose that the frame

$(e_\lambda)_{1\leqslant\lambda\leqslant r}$ has been chosen in such a way that $[e_r(x_0)] = [v_0] \in P(V)$ and $|v_0|_h = 1$. We get holomorphic local coordinates $(z_1, \ldots, z_n\,;\, \xi_1, \ldots, \xi_{r-1})$ on a neighborhood of $(x_0, [v_0])$ in $P(V)$ by assigning

$$(z_1, \ldots, z_n\,;\, \xi_1, \ldots, \xi_{r-1}) \longmapsto (z, [\xi_1 e_1(z) + \cdots + \xi_{r-1}e_{r-1}(z) + e_r(z)]) \in P(V).$$

Then the function

$$\eta(z, \xi) = \xi_1 e_1(z) + \cdots + \xi_{r-1}e_{r-1}(z) + e_r(z)$$

defines a holomorphic section of $\mathcal{O}_{P(V)}(-1)$ in a neighborhood of $(x_0, [v_0])$. By using the expansion (4.12) for h, we find

$$\begin{aligned}
|\eta|^2_{h_1} = |\eta|^2_h &= 1 + |\xi|^2 - \sum_{1\leqslant j,k\leqslant n} c_{jkrr} z_j \overline{z}_k + O((|z| + |\xi|)^3),\\
\Theta_{h_1}(\mathcal{O}_{P(V)}(-1))_{(x_0,[v_0])} &= -\frac{i}{2\pi}\partial\overline{\partial}\log|\eta|^2_{h_1}\\
&= \frac{i}{2\pi}\Big(\sum_{1\leqslant j,k\leqslant n} c_{jkrr}\, dz_j \wedge d\overline{z}_k - \sum_{1\leqslant\lambda\leqslant r-1} d\xi_\lambda \wedge d\overline{\xi}_\lambda\Big).
\end{aligned}\tag{4.15}$$

Now, the connection ∇_h on V defines on $\widetilde{X} = P(V)$ a C^∞ decomposition

$$T_{\widetilde{X}} = {}^H T_{\widetilde{X}} \oplus {}^V T_{\widetilde{X}}, \qquad {}^H T_{\widetilde{X},(x,[v])} \simeq T_{X,x}, \quad {}^V T_{\widetilde{X},(x,[v])} \simeq T_{P(V_x),[v]},$$

in horizontal and vertical components. With respect to this decomposition, (4.15) can be rewritten as

$$\langle\Theta_{h_1}(\mathcal{O}_{P(V)}(-1))\rangle_{(x_0,[v_0])}(\tau) = \langle\Theta_{V,h}\rangle_{x_0}({}^H\tau \otimes v_0) - |{}^V\tau|^2_{\mathrm{FS}}\tag{4.16}$$

where $|\ \ |_{\mathrm{FS}}$ is the Fubini-Study metric along the fibers $T_{P(V_x)}$. By definition of $\widetilde{V}$, we have $\widetilde{V}_{(x,[v])} \subset V_x \oplus T_{P(V_x),[v]}$ with respect to the decomposition. By this observation, if we equip $P(V)$ with the Fubini-Study metric rescaled by $\rho^2 > 0$, the metric h on V induces a canonical hermitian metric $\widetilde{h}_\rho$ on $\widetilde{V}$ such that

$$|w|^2_{\widetilde{h}_\rho} = |{}^H w|^2_h + \rho^2 |{}^V w|^2_h \qquad \text{for } w \in \widetilde{V}_{(x_0,[v_0])},$$

where ${}^H w \in \mathbb{C}v_0 \subset V_{x_0}$ and ${}^V w \in T_{P(V_{x_0}),[v_0]}$ is viewed as an element of $v_0^\perp \subset V_{x_0}$. A computation (left to the reader) gives the formula

$$\begin{aligned}
\langle\Theta_{\widetilde{h}_\rho}(\widetilde{V})\rangle_{(x_0,[v_0])}(\tau \otimes w) &= \langle\Theta_{V,h}\rangle_{x_0}({}^H\tau \otimes v_0)\ (|{}^H w|^2_h - \rho^2|{}^V w|^2_h)\\
&\quad + \rho^2\langle\Theta_{V,h}\rangle_{x_0}({}^H\tau \otimes {}^V w)\\
&\quad + \rho^2\big(|\langle{}^V\tau, {}^V w\rangle_h|^2 + |{}^V\tau|^2_h|{}^V w|^2_h\big) - |{}^V\tau|^2_h|{}^H w|^2_h\\
&\quad + O(\rho)|\tau|^2_\omega|w|^2_{\widetilde{h}_\rho}, \qquad \tau \in T_{\widetilde{X}},\ w \in \widetilde{V},
\end{aligned}\tag{4.17}$$

where $|\tau|^2_\omega$ is computed from a fixed hermitian metric ω on T_X.

§5. Jets of curves and Semple jet bundles

Let X be a complex n-dimensional manifold. Following ideas of Green-Griffiths [GrGr80], we let $J_k \to X$ be the bundle of k-jets of germs of parametrized curves in X, that is, the set of equivalence classes of holomorphic maps $f : (\mathbb{C}, 0) \to (X, x)$, with the equivalence relation $f \sim g$ if and only if all derivatives $f^{(j)}(0) = g^{(j)}(0)$ coincide for $0 \leqslant j \leqslant k$, when computed in some local coordinate system of X near x. The projection map $J_k \to X$ is simply $f \mapsto f(0)$. If $(z_1, \ldots, z_n)$ are local holomorphic coordinates on an open set $\Omega \subset X$, the elements f of any fiber $J_{k,x}$, $x \in \Omega$, can be seen as $\mathbb{C}^n$-valued maps

$$f = (f_1, \ldots, f_n) : (\mathbb{C}, 0) \to \Omega \subset \mathbb{C}^n,$$

and they are completetely determined by their Taylor expansion of order k at $t = 0$

$$f(t) = x + t\, f'(0) + \frac{t^2}{2!} f''(0) + \cdots + \frac{t^k}{k!} f^{(k)}(0) + O(t^{k+1}).$$

In these coordinates, the fiber $J_{k,x}$ can thus be identified with the set of k-tuples of vectors $(f'(0), \ldots, f^{(k)}(0)) \in (\mathbb{C}^n)^k$. It follows that J_k is a holomorphic fiber bundle with typical fiber $(\mathbb{C}^n)^k$ over X (however, J_k is not a vector bundle for $k \geqslant 2$, because of the nonlinearity of coordinate changes; see formula (6.2) in § 6).

According to the philosophy developed throughout this paper, we describe the concept of jet bundle in the general situation of complex directed manifolds. If X is equipped with a holomorphic subbundle $V \subset T_X$, we associate to V a k-jet bundle $J_k V$ as follows.

5.1. Definition. *Let (X, V) be a complex directed manifold. We define $J_k V \to X$ to be the bundle of k-jets of curves $f : (\mathbb{C}, 0) \to X$ which are tangent to V, i.e., such that $f'(t) \in V_{f(t)}$ for all t in a neighborhood of 0, together with the projection map $f \mapsto f(0)$ onto X.*

It is easy to check that $J_k V$ is actually a subbundle of J_k. In fact, by using (4.8) and (4.10), we see that the fibers $J_k V_x$ are parametrized by

$$\big((f_1'(0), \ldots, f_r'(0)); (f_1''(0), \ldots, f_r''(0)); \ldots; (f_1^{(k)}(0), \ldots, f_r^{(k)}(0))\big) \in (\mathbb{C}^r)^k$$

for all $x \in \Omega$, hence $J_k V$ is a locally trivial $(\mathbb{C}^r)^k$-subbundle of J_k. $\square$

We now describe a convenient process for constructing "projectivized jet bundles", which will later appear as natural quotients of our jet bundles $J_k V$ (or rather, as suitable desingularized compactifications of the quotients). Such spaces have already been considered since a long time, at least in the special case $X = \mathbb{P}^2$, $V = T_{\mathbb{P}^2}$ (see Gherardelli [Ghe41], Semple [Sem54]), and they have been mostly used as a tool for establishing enumerative formulas dealing with the order of contact of plane curves (see [Coll88], [CoKe94]); the article [ASS92] is also concerned with such generalizations of jet bundles*.

We define inductively the *projectivized k-jet bundle* $P_k V = X_k$ (or *Semple k-jet bundle*) and the associated subbundle $V_k \subset T_{X_k}$ by

$$(5.2) \qquad (X_0, V_0) = (X, V), \qquad (X_k, V_k) = (\widetilde{X}_{k-1}, \widetilde{V}_{k-1}).$$

* Very recently, a preprint [LaTh96] by Laksov and Thorup has also appeared, dealing in depth with algebraic-theoretic properties of jet differentials. The formalism of "higher order" differentials has been part of the mathematical folklore during the 18th and 19th centuries (without too much concern, in those times, on the existence of precise definitions !). During the 20th century, this formalism almost disappeared, before getting revived in several ways. See e.g. the interested article by P.A. Meyer [Mey89], which was originally motivated by applications to probability theory.

In other words, $(P_kV, V_k) = (X_k, V_k)$ is obtained from (X, V) by iterating k-times the lifting construction $(X, V) \mapsto (\widetilde{X}, \widetilde{V})$ described in § 4. By (4.2–4.7), we find

$$\dim P_kV = n + k(r-1), \qquad \operatorname{rank} V_k = r, \tag{5.3}$$

together with exact sequences

$$0 \longrightarrow T_{P_kV/P_{k-1}V} \longrightarrow V_k \xrightarrow{(\pi_k)_\star} \mathcal{O}_{P_kV}(-1) \longrightarrow 0, \tag{5.4}$$

$$0 \longrightarrow \mathcal{O}_{P_kV} \longrightarrow \pi_k^\star V_{k-1} \otimes \mathcal{O}_{P_kV}(1) \longrightarrow T_{P_kV/P_{k-1}V} \longrightarrow 0. \tag{5.4'}$$

where π_k is the natural projection $\pi_k : P_kV \to P_{k-1}V$ and $(\pi_k)_\star$ its differential. Formula (4.4) yields

$$\det V_k = \pi_k^\star \det V_{k-1} \otimes \mathcal{O}_{P_kV}(r-1). \tag{5.5}$$

Every non constant tangent trajectory $f : \Delta_R \to X$ of (X, V) lifts to a well defined and unique tangent trajectory $f_{[k]} : \Delta_R \to P_kV$ of (P_kV, V_k). Moreover, the derivative $f'_{[k-1]}$ gives rise to a section

$$f'_{[k-1]} : T_{\Delta_R} \to f_{[k]}^\star \mathcal{O}_{P_kV}(-1). \tag{5.6}$$

In coordinates, one can compute $f_{[k]}$ in terms of its components in the various affine charts (4.9) occurring at each step: we get inductively

$$f_{[k]} = (F_1, \ldots, F_N), \qquad f_{[k+1]} = \Big(F_1, \ldots, F_N, \frac{F'_{s_1}}{F'_{s_r}}, \ldots, \frac{F'_{s_{r-1}}}{F'_{s_r}}\Big) \tag{5.7}$$

where $N = n + k(r-1)$ and $\{s_1, \ldots, s_r\} \subset \{1, \ldots, N\}$. If $k \geqslant 1$, $\{s_1, \ldots, s_r\}$ contains the last $r-1$ indices of $\{1, \ldots, N\}$ corresponding to the "vertical" components of the projection $P_kV \to P_{k-1}V$, and in general, s_r is an index such that $m(F_{s_r}, 0) = m(f_{[k]}, 0)$, that is, F_{s_r} has the smallest vanishing order among all components F_s (s_r may be vertical or not, and the choice of $\{s_1, \ldots, s_r\}$ need not be unique).

By definition, there is a canonical injection $\mathcal{O}_{P_kV}(-1) \hookrightarrow \pi_k^\star V_{k-1}$, and a composition with the projection $(\pi_{k-1})_\star$ (analogue for order $k-1$ of the arrow $(\pi_k)_\star$ in sequence (5.4)) yields for all $k \geqslant 2$ a canonical line bundle morphism

$$\mathcal{O}_{P_kV}(-1) \hookrightarrow \pi_k^\star V_{k-1} \xrightarrow{(\pi_k)^\star(\pi_{k-1})_\star} \pi_k^\star \mathcal{O}_{P_{k-1}V}(-1), \tag{5.8}$$

which admits precisely $D_k = P(T_{P_{k-1}V/P_{k-2}V}) \subset P(V_{k-1}) = P_kV$ as its zero divisor (clearly, D_k is a hyperplane subbundle of P_kV). Hence we find

$$\mathcal{O}_{P_kV}(1) = \pi_k^\star \mathcal{O}_{P_{k-1}V}(1) \otimes \mathcal{O}(D_k). \tag{5.9}$$

Now, we consider the composition of projections

$$\pi_{j,k} = \pi_{j+1} \circ \cdots \circ \pi_{k-1} \circ \pi_k : P_kV \longrightarrow P_jV. \tag{5.10}$$

Then $\pi_{0,k} : P_kV \to X = P_0V$ is a locally trivial holomorphic fiber bundle over X, and the fibers $P_kV_x = \pi_{0,k}^{-1}(x)$ are k-stage towers of $\mathbb{P}^{r-1}$-bundles. Since we have (in both directions) morphisms $(\mathbb{C}^r, T_{\mathbb{C}^r}) \leftrightarrow (X,V)$ of directed manifolds which are bijective on the level of bundle morphisms, the fibers are all isomorphic to a "universal" nonsingular projective algebraic variety of dimension $k(r-1)$ which we will denote by $\mathbb{R}_{r,k}$; it is not hard to see that $\mathbb{R}_{r,k}$ is rational (as will indeed follow from the proof of Theorem 6.8 below). The following Proposition will help us to understand a little bit more about the geometric structure of P_kV. As usual, we define the *multiplicity* $m(f,t_0)$ of a curve $f:\Delta_R\to X$ at a point $t\in\Delta_R$ to be the smallest integer $s\in\mathbb{N}^\star$ such that $f^{(s)}(t_0)\neq 0$, i.e., the largest s such that $\delta(f(t),f(t_0)) = O(|t-t_0|^s)$ for any hermitian or riemannian geodesic distance δ on X. As $f_{[k-1]}=\pi_k\circ f_{[k]}$, it is clear that the sequence $m(f_{[k]},t)$ is non increasing with k.

5.11. Proposition. *Let $f:(\mathbb{C},0)\to X$ be a non constant germ of curve tangent to V. Then for all $j\geqslant 2$ we have $m(f_{[j-2]},0)\geqslant m(f_{[j-1]},0)$ and the inequality is strict if and only if $f_{[j]}(0)\in D_j$. Conversely, if $w\in P_kV$ is an arbitrary element and $m_0\geqslant m_1\geqslant\cdots\geqslant m_{k-1}\geqslant 1$ is a sequence of integers with the property that*

$$\forall j\in\{2,\ldots,k\},\qquad m_{j-2}>m_{j-1}\quad \textit{if and only if } \pi_{j,k}(w)\in D_j,$$

there exists a germ of curve $f:(\mathbb{C},0)\to X$ tangent to V such that $f_{[k]}(0)=w$ and $m(f_{[j]},0)=m_j$ for all $j\in\{0,\ldots,k-1\}$.

Proof. i) Suppose first that f is given and put $m_j = m(f_{[j]},0)$. By definition, we have $f_{[j]} = (f_{[j-1]},[u_{j-1}])$ where $f'_{[j-1]}(t) = t^{m_{j-1}-1}u_{j-1}(t)\in V_{j-1}$, $u_{j-1}(0)\neq 0$. By composing with the differential of the projection $\pi_{j-1}:P_{j-1}V\to P_{j-2}V$, we find $f'_{[j-2]}(t)=t^{m_{j-1}-1}(\pi_{j-1})_\star u_{j-1}(t)$. Therefore

$$m_{j-2}=m_{j-1}+\mathrm{ord}_{t=0}(\pi_{j-1})_\star u_{j-1}(t),$$

and so $m_{j-2}>m_{j-1}$ if and only if $(\pi_{j-1})_\star u_{j-1}(0)=0$, that is, if and only if $u_{j-1}(0)\in T_{P_{j-1}V/P_{j-2}V}$, or equivalently $f_{[j]}(0)=(f_{[j-1]}(0),[u_{j-1}(0)])\in D_j$.

ii) Suppose now that $w\in P_kV$ and $m_0,\ldots,m_{k-1}$ are given. We denote by $w_{j+1}=(w_j,[\eta_j])$, $w_j\in P_jV$, $\eta_j\in V_j$, the projection of w to $P_{j+1}V$. Fix coordinates $(z_1,\ldots,z_n)$ on X centered at w_0 such that the r-th component $\eta_{0,r}$ of η_0 is non zero. We prove the existence of the germ f by induction on k, in the form of a Taylor expansion

$$f(t)=a_0+t\,a_1+\cdots+t^{d_k}a_{d_k}+O(t^{d_k+1}),\qquad d_k=m_0+m_1+\cdots+m_{k-1}.$$

If $k=1$ and $w=(w_0,[\eta_0])\in P_1V_x$, we simply take $f(t)=w_0+t^{m_0}\eta_0+O(t^{m_0+1})$. In general, the induction hypothesis applied to $P_kV=P_{k-1}(V_1)$ over $X_1=P_1V$ yields a curve $g:(\mathbb{C},0)\to X_1$ such that $g_{[k-1]}=w$ and $m(g_{[j]},0)=m_{j+1}$ for $0\leqslant j\leqslant k-2$. If $w_2\notin D_2$, then $[g'_{[1]}(0)]=[\eta_1]$ is not vertical, thus $f=\pi_1\circ g$ satisfies $m(f,0)=m(g,0)=m_1=m_0$ and we are done.

If $w_2\in D_2$, we express $g=(G_1,\ldots,G_n;G_{n+1},\ldots,G_{n+r-1})$ as a Taylor expansion of order $m_1+\cdots+m_{k-1}$ in the coordinates (4.9) of the affine chart $\xi_r\neq 0$. As $\eta_1=\lim_{t\to 0}g'(t)/t^{m_1-1}$ is vertical, we must have $m(G_s,0)>m_1$ for $1\leqslant j\leqslant n$. It follows from (5.7) that $G_1,\ldots,G_n$ are never involved in the calculation of the liftings $g_{[j]}$. We can therefore replace g by $f\simeq(f_1,\ldots,f_n)$ where $f_r(t)=t^{m_0}$ and $f_1,\ldots,f_{r-1}$ are obtained by integrating the equations $f'_j(t)/f'_r(t)=G_{n+j}(t)$, i.e., $f'_j(t)=m_0t^{m_0-1}G_{n+j}(t)$, while $f_{r+1},\ldots,f_n$ are obtained by integrating (4.10). We then get the desired Taylor expansion of order d_k for f. □

Since we can always take $m_{k-1} = 1$ without restriction, we get in particular:

5.12. Corollary. *Let $w \in P_k V$ be an arbitrary element. Then there is a germ of curve $f : (\mathbb{C}, 0) \to X$ such that $f_{[k]}(0) = w$ and $f'_{[k-1]}(0) \neq 0$ (thus the liftings $f_{[k-1]}$ and $f_{[k]}$ are regular germs of curve). Moreover, if $w_0 \in P_k V$ and w is taken in a sufficiently small neighborhood of w_0, then the germ $f = f_w$ can be taken to depend holomorphically on w.*

Proof. Only the holomorphic dependence of f_w with respect to w has to be guaranteed. If f_{w_0} is a solution for $w = w_0$, we observe that $(f_{w_0})'_{[k]}$ is a non vanishing section of V_k along the regular curve defined by $(f_{w_0})_{[k]}$ in $P_k V$. We can thus find a non vanishing section ξ of V_k on a neighborhood of w_0 in $P_k V$ such that $\xi = (f_{w_0})'_{[k]}$ along that curve. We define $t \mapsto F_w(t)$ to be the trajectory of ξ with initial point w, and we put $f_w = \pi_{0,k} \circ F_w$. Then f_w is the required family of germs. □

Now, we can take $f : (\mathbb{C}, 0) \to X$ to be regular at the origin (by this, we mean $f'(0) \neq 0$) if and only if $m_0 = m_1 = \cdots = m_{k-1} = 1$, which is possible by Proposition 5.11 if and only if $w \in P_k V$ is such that $\pi_{j,k}(w) \notin D_j$ for all $j \in \{2, \ldots, k\}$. For this reason, we define

$$(5.13) \qquad \begin{aligned} P_k V^{\mathrm{reg}} &= \bigcap_{2 \leqslant j \leqslant k} \pi_{j,k}^{-1}(P_j V \smallsetminus D_j), \\ P_k V^{\mathrm{sing}} &= \bigcup_{2 \leqslant j \leqslant k} \pi_{j,k}^{-1}(D_j) = P_k V \smallsetminus P_k V^{\mathrm{reg}}, \end{aligned}$$

in other words, $P_k V^{\mathrm{reg}}$ is the set of values $f_{[k]}(0)$ reached by all regular germs of curves f. One should take care however that there are singular germs which reach the same points $f_{[k]}(0) \in P_k V^{\mathrm{reg}}$, e.g., any s-sheeted covering $t \mapsto f(t^s)$. On the other hand, if $w \in P_k V^{\mathrm{sing}}$, we can reach w by a germ f with $m_0 = m(f, 0)$ as large as we want.

5.14. Corollary. *Let $w \in P_k V^{\mathrm{sing}}$ be given, and let $m_0 \in \mathbb{N}$ be an arbitrary integer larger than the number of components D_j such that $\pi_{j,k}(w) \in D_j$. Then there is a germ of curve $f : (\mathbb{C}, 0) \to X$ with multiplicity $m(f, 0) = m_0$ at the origin, such that $f_{[k]}(0) = w$ and $f'_{[k-1]}(0) \neq 0$.*

§6. Jet differentials

Following Green-Griffiths [GrGr80], we now introduce the concept of jet differential. This concept gives an intrinsic way of describing holomorphic differential equations that a germ of curve $f : (\mathbb{C}, 0) \to X$ may satisfy. In the sequel, we fix a directed manifold (X, V) and suppose implicitly that all germs f are tangent to V.

Let $\mathbb{G}_k$ be the group of germs of k-jets of biholomorphisms of $(\mathbb{C}, 0)$, that is, the group of germs of biholomorphic maps

$$t \mapsto \varphi(t) = a_1 t + a_2 t^2 + \cdots + a_k t^k, \qquad a_1 \in \mathbb{C}^\star,\ a_j \in \mathbb{C},\ j \geqslant 2,$$

in which the composition law is taken modulo terms t^j of degree $j > k$. Then $\mathbb{G}_k$ is a k-dimensional nilpotent complex Lie group, which admits a natural fiberwise right action on $J_k V$. The action consists of reparametrizing k-jets of maps $f : (\mathbb{C}, 0) \to X$ by a biholomorphic change of parameter $\varphi : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$, that is, $(f, \varphi) \mapsto f \circ \varphi$. There is an exact sequence of groups

$$1 \to \mathbb{G}'_k \to \mathbb{G}_k \to \mathbb{C}^\star \to 1$$

where $\mathbb{G}_k \to \mathbb{C}^\star$ is the obvious morphism $\varphi \mapsto \varphi'(0)$, and $\mathbb{G}'_k = [\mathbb{G}_k, \mathbb{G}_k]$ is the group of k-jets of biholomorphisms tangent to the identity. Moreover, the subgroup $\mathbb{H} \simeq \mathbb{C}^\star$ of homotheties $\varphi(t) = \lambda t$ is a (non normal) subgroup of $\mathbb{G}_k$, and we have a semidirect decomposition $\mathbb{G}_k = \mathbb{G}'_k \ltimes \mathbb{H}$. The corresponding action on k-jets is described in coordinates by

$$\lambda \cdot (f', f'', \ldots, f^{(k)}) = (\lambda f', \lambda^2 f'', \ldots, \lambda^k f^{(k)}).$$

Following [GrGr80], we introduce the vector bundle $E^{\mathrm{GG}}_{k,m} V^\star \to X$ whose fibers are complex valued polynomials $Q(f', f'', \ldots, f^{(k)})$ on the fibers of $J_k V$, of weighted degree m with respect to the $\mathbb{C}^\star$ action defined by H, that is, such that

$$Q(\lambda f', \lambda^2 f'', \ldots, \lambda^k f^{(k)}) = \lambda^m Q(f', f'', \ldots, f^{(k)}) \tag{6.1}$$

for all $\lambda \in \mathbb{C}^\star$ and $(f', f'', \ldots, f^{(k)}) \in J_k V$. Here we view $(f', f'', \ldots, f^{(k)})$ as indeterminates with components

$$\big((f'_1, \ldots, f'_r); (f''_1, \ldots, f''_r); \ldots; (f^{(k)}_1, \ldots, f^{(k)}_r)\big) \in (\mathbb{C}^r)^k.$$

Notice that the concept of polynomial on the fibers of $J_k V$ makes sense, for all coordinate changes $z \mapsto w = \Psi(z)$ on X induce polynomial transition automorphisms on the fibers of $J_k V$, given by a formula

$$(\Psi \circ f)^{(j)} = \Psi'(f) \cdot f^{(j)} + \sum_{s=2}^{s=j} \sum_{j_1+j_2+\cdots+j_s=j} c_{j_1\ldots j_s} \Psi^{(s)}(f) \cdot (f^{(j_1)}, \ldots, f^{(j_s)}) \tag{6.2}$$

with suitable integer constants $c_{j_1\ldots j_s}$ (this is easily checked by induction on s). In the "absolute case" $V = T_X$, we simply write $E^{\mathrm{GG}}_{k,m} T^\star_X = E^{\mathrm{GG}}_{k,m}$. If $V \subset W \subset T_X$ are holomorphic subbundles, there are natural inclusions

$$J_k V \subset J_k W \subset J_k, \qquad P_k V \subset P_k W \subset P_k.$$

The restriction morphisms induce surjective arrows

$$E^{\mathrm{GG}}_{k,m} \to E^{\mathrm{GG}}_{k,m} W^\star \to E^{\mathrm{GG}}_{k,m} V^\star,$$

in particular $E^{\mathrm{GG}}_{k,m} V^\star$ can be seen as a quotient of $E^{\mathrm{GG}}_{k,m}$. (The notation $V^\star$ is used here to make the contravariance property implicit from the notation).

If $Q \in E^{\mathrm{GG}}_{k,m} V^\star$ is decomposed into multihomogeneous components of multidegree $(\ell_1, \ell_2, \ldots, \ell_k)$ in $f', f'', \ldots, f^{(k)}$ (the decomposition is of course coordinate dependent), these multidegrees must satisfy the relation

$$\ell_1 + 2\ell_2 + \cdots + k\ell_k = m.$$

The bundle $E^{\mathrm{GG}}_{k,m} V^\star$ will be called the *bundle of jet differentials of order k and weighted degree m*. It is clear from (6.2) that a coordinate change $f \mapsto \Psi \circ f$ transforms every monomial $(f^{(\bullet)})^\ell = (f')^{\ell_1} (f'')^{\ell_2} \cdots (f^{(k)})^{\ell_k}$ of partial weighted degree $|\ell|_s := \ell_1 + 2\ell_2 + \cdots + s\ell_s$, $1 \leqslant s \leqslant k$, into a polynomial $((\Psi \circ f)^{(\bullet)})^\ell$ in $(f', f'', \ldots, f^{(k)})$ which has the same partial weighted degree of order s if $\ell_{s+1} = \cdots = \ell_k = 0$, and a larger or equal partial degree of order s otherwise. Hence, for each $s = 1, \ldots, k$, we get a well defined (i.e., coordinate invariant) decreasing filtration $F_s^\bullet$ on $E_{k,m}^{\mathrm{GG}} V^\star$ as follows:

$$F_s^p(E_{k,m}^{\mathrm{GG}} V^\star) = \left\{ \begin{matrix} Q(f', f'', \ldots, f^{(k)}) \in E_{k,m}^{\mathrm{GG}} V^\star \text{ involving} \\ \text{only monomials } (f^{(\bullet)})^\ell \text{ with } |\ell|_s \geqslant p \end{matrix} \right\}, \qquad \forall p \in \mathbb{N}. \tag{6.3}$$

The graded terms $\mathrm{Gr}_{k-1}^p(E_{k,m}^{\mathrm{GG}} V^\star)$ associated with the filtration $F_{k-1}^p(E_{k,m}^{\mathrm{GG}} V^\star)$ are precisely the homogeneous polynomials $Q(f', \ldots, f^{(k)})$ whose monomials $(f^\bullet)^\ell$ all have partial weighted degree $|\ell|_{k-1} = p$ (hence their degree ℓ_k in $f^{(k)}$ is such that $m - p = k\ell_k$, and $\mathrm{Gr}_{k-1}^p(E_{k,m}^{\mathrm{GG}} V^\star) = 0$ unless $k|m-p$). The transition automorphisms of the graded bundle are induced by coordinate changes $f \mapsto \Psi \circ f$, and they are described by substituting the arguments of $Q(f', \ldots, f^{(k)})$ according to formula (6.2), namely $f^{(j)} \mapsto (\Psi \circ f)^{(j)}$ for $j < k$, and $f^{(k)} \mapsto \Psi'(f) \circ f^{(k)}$ for $j = k$ (when $j = k$, the other terms fall in the next stage F_{k-1}^{p+1} of the filtration). Therefore $f^{(k)}$ behaves as an element of $V \subset T_X$ under coordinate changes. We thus find

$$G_{k-1}^{m-k\ell_k}(E_{k,m}^{\mathrm{GG}} V^\star) = E_{k-1,m-k\ell_k}^{\mathrm{GG}} V^\star \otimes S^{\ell_k} V^\star. \tag{6.4}$$

Combining all filtrations $F_s^\bullet$ together, we find inductively a filtration $F^\bullet$ on $E_{k,m}^{\mathrm{GG}} V^\star$ such that the graded terms are

$$\mathrm{Gr}^\ell(E_{k,m}^{\mathrm{GG}} V^\star) = S^{\ell_1} V^\star \otimes S^{\ell_2} V^\star \otimes \cdots \otimes S^{\ell_k} V^\star, \qquad \ell \in \mathbb{N}^k, \quad |\ell|_k = m. \tag{6.5}$$

The bundles $E_{k,m}^{\mathrm{GG}} V^\star$ have other interesting properties. In fact,

$$E_{k,\bullet}^{\mathrm{GG}} V^\star := \bigoplus_{m \geqslant 0} E_{k,m}^{\mathrm{GG}} V^\star$$

is in a natural way a bundle of graded algebras (the product is obtained simply by taking the product of polynomials). There are natural inclusions $E_{k,\bullet}^{\mathrm{GG}} V^\star \subset E_{k+1,\bullet}^{\mathrm{GG}} V^\star$ of algebras, hence $E_{\infty,\bullet}^{\mathrm{GG}} V^\star = \bigcup_{k \geqslant 0} E_{k,\bullet}^{\mathrm{GG}} V^\star$ is also an algebra. Moreover, the sheaf of holomorphic sections $\mathcal{O}(E_{\infty,\bullet}^{\mathrm{GG}} V^\star)$ admits a canonical derivation ∇ given by a collection of $\mathbb{C}$-linear maps

$$\nabla : \mathcal{O}(E_{k,m}^{\mathrm{GG}} V^\star) \to \mathcal{O}(E_{k+1,m+1}^{\mathrm{GG}} V^\star), \tag{6.6}$$

constructed in the following way. A holomorphic section of $E_{k,m}^{\mathrm{GG}} V^\star$ on a coordinate open set $\Omega \subset X$ can be seen as a differential operator on the space of germs $f : (\mathbb{C}, 0) \to \Omega$ of the form

$$Q(f) = \sum_{|\alpha_1| + 2|\alpha_2| + \cdots + k|\alpha_k| = m} a_{\alpha_1 \ldots \alpha_k}(f)\, (f')^{\alpha_1} (f'')^{\alpha_2} \cdots (f^{(k)})^{\alpha_k}$$

in which the coefficients $a_{\alpha_1 \ldots \alpha_k}$ are holomorphic functions on Ω. Then ∇Q is given by the formal derivative $(\nabla Q)(f)(t) = d(Q(f))/dt$ with respect to the 1-dimensional parameter t in $f(t)$. For example, in dimension 2, if $Q \in H^0(\Omega, \mathcal{O}(E_{2,4}^{\mathrm{GG}}))$ is the section of weighted degree 4

$$Q(f) = a(f_1, f_2)\, f_1'^3 f_2' + b(f_1, f_2)\, f_1''^2,$$

we find that $\nabla Q \in H^0(\Omega, \mathcal{O}(E_{3,5}^{\mathrm{GG}}))$ is given by

$$\begin{aligned}(\nabla Q)(f) &= \frac{\partial a}{\partial z_1}(f_1, f_2)\, f_1'^4 f_2' + \frac{\partial a}{\partial z_2}(f_1, f_2)\, f_1'^3 f_2'^2 + \frac{\partial b}{\partial z_1}(f_1, f_2)\, f_1' f_1''^2 \\ &\quad + \frac{\partial b}{\partial z_2}(f_1, f_2)\, f_2' f_1''^2 + a(f_1, f_2)\, \big(3 f_1'^2 f_1'' f_2' + f_1'^3 f_2''\big) + b(f_1, f_2)\, 2 f_1'' f_1'''.\end{aligned}$$

Associated with the graded algebra bundle $E^{\mathrm{GG}}_{k,\bullet}V^\star$, we have an analytic fiber bundle $\mathrm{Proj}(E^{\mathrm{GG}}_{k,\bullet}V^\star) = J_kV^{\mathrm{nc}}/\mathbb{C}^\star$ over X, which has weighted projective spaces $\mathbb{P}(1^{[r]}, 2^{[r]}, \ldots, k^{[r]})$ as fibers (these weighted projective spaces are singular for $k > 1$, but they only have quotient singularities, see [Dol81] ; here J_kV^{nc} denotes the set of non constant jets of order k ; we refer e.g. to Hartshorne's book [Har77] for a definition of the Proj fonctor). However, we are not really interested in the bundles $J_kV^{\mathrm{nc}}/\mathbb{C}^\star$ themselves, but rather on their quotients $J_kV^{\mathrm{nc}}/\mathbb{G}_k$ (would such nice complex space quotients exist!). We will see that the Semple bundle P_kV constructed in § 5 plays the role of such a quotient. First we introduce a canonical bundle subalgebra of $E^{\mathrm{GG}}_{k,\bullet}V^\star$.

6.7. Definition. *We introduce a subbundle $E_{k,m}V^\star \subset E^{\mathrm{GG}}_{k,m}V^\star$, called the bundle of invariant jet differentials of order k and degree m, defined as follows: $E_{k,m}V^\star$ is the set of polynomial differential operators $Q(f', f'', \ldots, f^{(k)})$ which are invariant under arbitrary changes of parametrization, i.e., for every $\varphi \in \mathbb{G}_k$*

$$Q\big((f\circ\varphi)', (f\circ\varphi)'', \ldots, (f\circ\varphi)^{(k)}\big) = \varphi'(0)^m Q(f', f'', \ldots, f^{(k)}).$$

Alternatively, $E_{k,m}V^\star = (E^{\mathrm{GG}}_{k,m}V^\star)^{\mathbb{G}'_k}$ is the set of invariants of $E^{\mathrm{GG}}_{k,m}V^\star$ under the action of $\mathbb{G}'_k$. Clearly, $E_{\infty,\bullet}V^\star = \bigcup_{k\geqslant 0}\bigoplus_{m\geqslant 0} E_{k,m}V^\star$ is a subalgebra of $E^{\mathrm{GG}}_{k,m}V^\star$ (observe however that this algebra is not invariant under the derivation ∇, since e.g. $f''_j = \nabla f_j$ is not an invariant polynomial). In addition to this, there are natural induced filtrations $F^p_s(E_{k,m}V^\star) = E_{k,m}V^\star \cap F^p_s(E^{\mathrm{GG}}_{k,m}V^\star)$ (all locally trivial over X). These induced filtrations will play an important role later on.

6.8. Theorem. *Suppose that V has rank $r \geqslant 2$. Let $\pi_{0,k} : P_kV \longrightarrow X$ be the Semple jet bundles constructed in section 5, and let J_kV^{reg} be the bundle of regular k-jets of maps $f : (\mathbb{C}, 0) \to X$, that is, jets f such that $f'(0) \neq 0$.*

i) *The quotient $J_kV^{\mathrm{reg}}/\mathbb{G}_k$ has the structure of a locally trivial bundle over X, and there is a holomorphic embedding $J_kV^{\mathrm{reg}}/\mathbb{G}_k \hookrightarrow P_kV$ over X, which identifies $J_kV^{\mathrm{reg}}/\mathbb{G}_k$ with P_kV^{reg} (thus P_kV is a relative compactification of $J_kV^{\mathrm{reg}}/\mathbb{G}_k$ over X).*

ii) *The direct image sheaf*
$$(\pi_{0,k})_\star\mathcal{O}_{P_kV}(m) \simeq \mathcal{O}(E_{k,m}V^\star)$$
can be identified with the sheaf of holomorphic sections of $E_{k,m}V^\star$.

iii) *For every $m > 0$, the relative base locus of the linear system $|\mathcal{O}_{P_kV}(m)|$ is equal to the set P_kV^{sing} of singular k-jets. Moreover, $\mathcal{O}_{P_kV}(1)$ is relatively big over X.*

Proof. i) For $f \in J_kV^{\mathrm{reg}}$, the lifting $\widetilde{f}$ is obtained by taking the derivative $(f, [f'])$ without any cancellation of zeroes in f', hence we get a uniquely defined $(k-1)$-jet $\widetilde{f} : (\mathbb{C}, 0) \to \widetilde{X}$. Inductively, we get a well defined $(k-j)$-jet $f_{[j]}$ in P_jV, and the value $f_{[k]}(0)$ is independent of the choice of the representative f for the k-jet. As the lifting process commutes with reparametrization, i.e., $(f\circ\varphi)^\sim = \widetilde{f}\circ\varphi$ and more generally $(f\circ\varphi)_{[k]} = f_{[k]}\circ\varphi$, we conclude that there is a well defined set-theoretic map

$$J_kV^{\mathrm{reg}}/\mathbb{G}_k \to P_kV^{\mathrm{reg}}, \qquad f \bmod \mathbb{G}_k \mapsto f_{[k]}(0).$$

This map is better understood in coordinates as follows. Fix coordinates $(z_1, \ldots, z_n)$ near a point $x_0 \in X$, such that $V_{x_0} = \mathrm{Vect}(\partial/\partial z_1, \ldots, \partial/\partial z_r)$. Let $f = (f_1, \ldots, f_n)$ be a regular

k-jet tangent to V. Then there exists $i \in \{1, 2, \ldots, r\}$ such that $f_i'(0) \neq 0$, and there is a unique reparametrization $t = \varphi(\tau)$ such that $f \circ \varphi = g = (g_1, g_2, \ldots, g_n)$ with $g_i(\tau) = \tau$ (we just express the curve as a graph over the z_i-axis, by means of a change of parameter $\tau = f_i(t)$, i.e. $t = \varphi(\tau) = f_i^{-1}(\tau)$). Suppose $i = r$ for the simplicity of notation. The space P_kV is a k-stage tower of $\mathbb{P}^{r-1}$-bundles. In the corresponding inhomogeneous coordinates on these $\mathbb{P}^{r-1}$'s, the point $f_{[k]}(0)$ is given by the collection of derivatives

$$\big((g_1'(0), \ldots, g_{r-1}'(0)); (g_1''(0), \ldots, g_{r-1}''(0)); \ldots ; (g_1^{(k)}(0), \ldots, g_{r-1}^{(k)}(0))\big).$$

[Recall that the other components $(g_{r+1}, \ldots, g_n)$ can be recovered from $(g_1, \ldots, g_r)$ by integrating the differential system (4.10)]. Thus the map $J_kV^{\mathrm{reg}}/\mathbb{G}_k \to P_kV$ is a bijection onto P_kV^{reg}, and the fibers of these isomorphic bundles can be seen as unions of r affine charts $\simeq (\mathbb{C}^{r-1})^k$, associated with each choice of the axis z_i used to describe the curve as a graph. The change of parameter formula $\frac{d}{d\tau} = \frac{1}{f_r'(t)}\frac{d}{dt}$ expresses all derivatives $g_i^{(j)}(\tau) = d^j g_i/d\tau^j$ in terms of the derivatives $f_i^{(j)}(t) = d^j f_i/dt^j$

$$\begin{aligned}
&(g_1', \ldots, g_{r-1}') = \Big(\frac{f_1'}{f_r'}, \ldots, \frac{f_{r-1}'}{f_r'}\Big);\\
&(g_1'', \ldots, g_{r-1}'') = \Big(\frac{f_1''f_r' - f_r''f_1'}{f_r'^3}, \ldots, \frac{f_{r-1}''f_r' - f_r''f_{r-1}'}{f_r'^3}\Big);\ \ldots\ ;\\
&(g_1^{(k)}, \ldots, g_{r-1}^{(k)}) = \Big(\frac{f_1^{(k)}f_r' - f_r^{(k)}f_1'}{f_r'^{k+1}}, \ldots, \frac{f_{r-1}^{(k)}f_r' - f_r^{(k)}f_{r-1}'}{f_r'^{k+1}}\Big) + (\text{order} < k).
\end{aligned}\tag{6.9}$$

Also, it is easy to check that $f_r'^{2k-1}g_i^{(k)}$ is an invariant polynomial in f', f'', $\ldots$, $f^{(k)}$ of total degree $2k-1$, i.e., a section of $E_{k,2k-1}$.

ii) Since the bundles P_kV and $E_{k,m}V^\star$ are both locally trivial over X, it is sufficient to identify sections σ of $\mathcal{O}_{P_kV}(m)$ over a fiber $P_kV_x = \pi_{0,k}^{-1}(x)$ with the fiber $E_{k,m}V_x^\star$, at any point $x \in X$. Let $f \in J_kV_x^{\mathrm{reg}}$ be a regular k-jet at x. By (5.6), the derivative $f_{[k-1]}'(0)$ defines an element of the fiber of $\mathcal{O}_{P_kV}(-1)$ at $f_{[k]}(0) \in P_kV$. Hence we get a well defined complex valued operator

$$Q(f', f'', \ldots, f^{(k)}) = \sigma(f_{[k]}(0)) \cdot (f_{[k-1]}'(0))^m. \tag{6.10}$$

Clearly, Q is holomorphic on $J_kV_x^{\mathrm{reg}}$ (by the holomorphicity of σ), and the $\mathbb{G}_k$-invariance condition of Def. 6.7 is satisfied since $f_{[k]}(0)$ does not depend on reparametrization and $(f \circ \varphi)_{[k-1]}'(0) = f_{[k-1]}'(0)\varphi'(0)$. Now, $J_kV_x^{\mathrm{reg}}$ is the complement of a linear subspace of codimension n in J_kV_x, hence Q extends holomorphically to all of $J_kV_x \simeq (\mathbb{C}^r)^k$ by Riemann's extension theorem (here we use the hypothesis $r \geqslant 2$; if $r = 1$, the situation is anyway not interesting since $P_kV = X$ for all k). Thus Q admits an everywhere convergent power series

$$Q(f', f'', \ldots, f^{(k)}) = \sum_{\alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{N}^r} a_{\alpha_1 \ldots \alpha_k}\, (f')^{\alpha_1}(f'')^{\alpha_2} \cdots (f^{(k)})^{\alpha_k}.$$

The $\mathbb{G}_k$-invariance (6.7) implies in particular that Q must be multihomogeneous in the sense of (6.1), and thus Q must be a polynomial. We conclude that $Q \in E_{k,m}V_x^\star$, as desired.

Conversely, Corollary 5.12 implies that there is a holomorphic family of germs $f_w : (\mathbb{C}, 0) \to X$ such that $(f_w)_{[k]}(0) = w$ and $(f_w)_{[k-1]}'(0) \neq 0$, for all w in a neighborhood of any given point $w_0 \in P_k V_x$. Then every $Q \in E_{k,m}V_x^\star$ yields a holomorphic section σ of $\mathcal{O}_{P_kV}(m)$ over the fiber P_kV_x by putting

$$\sigma(w) = Q(f_w', f_w'', \ldots, f_w^{(k)})(0)\,\big((f_w)_{[k-1]}'(0)\big)^{-m}. \tag{6.11}$$

iii) By what we saw in i-ii), every section σ of $\mathcal{O}_{P_kV}(m)$ over the fiber P_kV_x is given by a polynomial $Q \in E_{k,m}V_x^\star$, and this polynomial can be expressed on the Zariski open chart $f_r' \neq 0$ of $P_kV_x^{\rm reg}$ as

$$Q(f', f'', \ldots, f^{(k)}) = f_r'^m \widehat{Q}(g', g'', \ldots, g^{(k)}), \tag{6.12}$$

where $\widehat{Q}$ is a polynomial and g is the reparametrization of f such that $g_r(\tau) = \tau$. In fact $\widehat{Q}$ is obtained from Q by substituting $f_r' = 1$ and $f_r^{(j)} = 0$ for $j \geqslant 2$, and conversely Q can be recovered easily from $\widehat{Q}$ by using the substitutions (6.9).

In this context, the jet differentials $f \mapsto f_1', \ldots, f \mapsto f_r'$ can be viewed as sections of $\mathcal{O}_{P_kV}(1)$ on a neighborhood of the fiber P_kV_x. Since these sections vanish exactly on $P_kV^{\rm sing}$, the relative base locus of $\mathcal{O}_{P_kV}(m)$ is contained in $P_kV^{\rm sing}$ for every $m > 0$. We see that $\mathcal{O}_{P_kV}(1)$ is big by considering the sections of $\mathcal{O}_{P_kV}(2k-1)$ associated with the polynomials $Q(f', \ldots, f^{(k)}) = f_r'^{2k-1} g_i^{(j)}$, $1 \leqslant i \leqslant r-1$, $1 \leqslant j \leqslant k$; indeed, these sections separate all points in the open chart $f_r' \neq 0$ of $P_kV_x^{\rm reg}$.

Now, we check that every section σ of $\mathcal{O}_{P_kV}(m)$ over P_kV_x must vanish on $P_kV_x^{\rm sing}$. Pick an arbitrary element $w \in P_kV^{\rm sing}$ and a germ of curve $f : (\mathbb{C}, 0) \to X$ such that $f_{[k]}(0) = w$, $f_{[k-1]}'(0) \neq 0$ and $s = m(f, 0) \gg 0$ (such an f exists by Corollary 5.14). There are local coordinates $(z_1, \ldots, z_n)$ on X such that $f(t) = (f_1(t), \ldots, f_n(t))$ where $f_r(t) = t^s$. Let Q, $\widehat{Q}$ be the polynomials associated with σ in these coordinates and let $(f')^{\alpha_1}(f'')^{\alpha_2}\cdots(f^{(k)})^{\alpha_k}$ be a monomial occurring in Q, with $\alpha_j \in \mathbb{N}^r$, $|\alpha_j| = \ell_j$, $\ell_1 + 2\ell_2 + \cdots + k\ell_k = m$. Putting $\tau = t^s$, the curve $t \mapsto f(t)$ becomes a Puiseux expansion $\tau \mapsto g(\tau) = (g_1(\tau), \ldots, g_{r-1}(\tau), \tau)$ in which g_i is a power series in $\tau^{1/s}$, starting with exponents of τ at least equal to 1. The derivative $g^{(j)}(\tau)$ may involve negative powers of τ, but the exponent is always $\geqslant 1 + \frac{1}{s} - j$. Hence the Puiseux expansion of $\widehat{Q}(g', g'', \ldots, g^{(k)})$ can only involve powers of τ of exponent $\geqslant -\max_\ell((1-\frac{1}{s})\ell_2 + \cdots + (k-1-\frac{1}{s})\ell_k)$. Finally $f_r'(t) = st^{s-1} = s\tau^{1-1/s}$, thus the lowest exponent of τ in $Q(f', \ldots, f^{(k)})$ is at least equal to

$$\begin{aligned}\Big(1-\frac{1}{s}\Big)m - \max_\ell\Big(\Big(1-\frac{1}{s}\Big)\ell_2 + \cdots + \Big(k-1-\frac{1}{s}\Big)\ell_k\Big)&\\ \geqslant \min_\ell \Big(1-\frac{1}{s}\Big)\ell_1 + \Big(1-\frac{1}{s}\Big)\ell_2 + \cdots + \Big(1-\frac{k-1}{s}\Big)\ell_k&\end{aligned}$$

where the minimum is taken over all monomials $(f')^{\alpha_1}(f'')^{\alpha_2}\cdots(f^{(k)})^{\alpha_k}$, $|\alpha_j| = \ell_j$, occurring in Q. Choosing $s \geqslant k$, we already find that the minimal exponent is positive, hence $Q(f', \ldots, f^{(k)})(0) = 0$ and $\sigma(w) = 0$ by (6.11). □

Theorem (6.8 iii) shows that $\mathcal{O}_{P_kV}(1)$ is never relatively ample over X for $k \geqslant 2$. In order to overcome this difficulty, we define for every $\boldsymbol{a} = (a_1, \ldots, a_k) \in \mathbb{Z}^k$ a line bundle $\mathcal{O}_{P_kV}(\boldsymbol{a})$ on P_kV such that

$$\mathcal{O}_{P_kV}(\boldsymbol{a}) = \pi_{1,k}^\star \mathcal{O}_{P_1V}(a_1) \otimes \pi_{2,k}^\star \mathcal{O}_{P_2V}(a_2) \otimes \cdots \otimes \mathcal{O}_{P_kV}(a_k). \tag{6.13}$$

By (5.9), we have $\pi_{j,k}^\star \mathcal{O}_{P_jV}(1) = \mathcal{O}_{P_kV}(1) \otimes \mathcal{O}_{P_kV}(-\pi_{j+1,k}^\star D_{j+1} - \cdots - D_k)$, thus by putting $D_j^\star = \pi_{j+1,k}^\star D_{j+1}$ for $1 \leqslant j \leqslant k-1$ and $D_k^\star = 0$, we find an identity

$$\begin{aligned} \text{(6.14)} \qquad & \mathcal{O}_{P_kV}(\boldsymbol{a}) = \mathcal{O}_{P_kV}(b_k) \otimes \mathcal{O}_{P_kV}(-\boldsymbol{b}\cdot D^\star), \qquad \text{where} \\ & \boldsymbol{b} = (b_1, \ldots, b_k) \in \mathbb{Z}^k, \quad b_j = a_1 + \cdots + a_j, \\ & \boldsymbol{b}\cdot D^\star = \sum_{1 \leqslant j \leqslant k-1} b_j\, \pi_{j+1,k}^\star D_{j+1}. \end{aligned}$$

In particular, if $\boldsymbol{b} \in \mathbb{N}^k$, i.e., $a_1 + \cdots + a_j \geqslant 0$, we get a morphism

$$\text{(6.15)} \qquad \mathcal{O}_{P_kV}(\boldsymbol{a}) = \mathcal{O}_{P_kV}(b_k) \otimes \mathcal{O}_{P_kV}(-\boldsymbol{b}\cdot D^\star) \to \mathcal{O}_{P_kV}(b_k).$$

6.16. Proposition. *Let $\boldsymbol{a} = (a_1, \ldots, a_k) \in \mathbb{Z}^k$ and $m = a_1 + \cdots + a_k$.*

i) *We have the direct image formula*

$$(\pi_{0,k})_\star \mathcal{O}_{P_kV}(\boldsymbol{a}) \simeq \mathcal{O}(\overline{F}^{\boldsymbol{a}} E_{k,m}V^\star) \subset \mathcal{O}(E_{k,m}V^\star)$$

where $\overline{F}^{\boldsymbol{a}} E_{k,m}V^\star$ is the subbundle of polynomials $Q(f', f'', \ldots, f^{(k)}) \in E_{k,m}V^\star$ involving only monomials $(f^{(\bullet)})^\ell$ such that

$$\ell_{s+1} + 2\ell_{s+2} + \cdots + (k-s)\ell_k \leqslant a_{s+1} + \cdots + a_k$$

for all $s = 0, \ldots, k-1$.

ii) *If $a_1 \geqslant 3a_2, \ldots, a_{k-2} \geqslant 3a_{k-1}$ and $a_{k-1} \geqslant 2a_k \geqslant 0$, the line bundle $\mathcal{O}_{P_kV}(\boldsymbol{a})$ is relatively nef over X.*

iii) *If $a_1 \geqslant 3a_2, \ldots, a_{k-2} \geqslant 3a_{k-1}$ and $a_{k-1} > 2a_k > 0$, the line bundle $\mathcal{O}_{P_kV}(\boldsymbol{a})$ is relatively ample over X.*

Proof. i) By (6.15), we find a sheaf injection

$$(\pi_{0,k})_\star \mathcal{O}_{P_kV}(\boldsymbol{a}) \hookrightarrow (\pi_{0,k})_\star \mathcal{O}_{P_kV}(m) = \mathcal{O}(E_{k,m}V^\star).$$

Given a section σ of $\mathcal{O}_{P_kV}(\boldsymbol{a})$ over a fiber P_kV_x, the associated polynomial

$$Q(f', f'', \ldots, f^{(k)}) \in E_{k,m}V_x^\star$$

is given by the identity

$$Q(f', f'', \ldots, f^{(k)}) = \sigma(f_{[k]}(0)) \cdot (f'(0))^{a_1} \cdot (f'_{[1]}(0))^{a_2} \cdots (f'_{[k-1]}(0))^{a_k}.$$

Indeed, we see this from (6.10) and from the fact that $f'_{[k-1]}(0)$ is mapped to $f'_{[j-1]}(0)$ by the projection morphism

$$(\pi_{j-1,k-1})_\star : \mathcal{O}_{P_kV}(-1) \to \pi_{j,k}^\star \mathcal{O}_{P_jV}(-1)$$

(cf. (5.8)), which is dual to the corresponding morphism (6.15). Now, we prove the inclusion $(\pi_{0,k})_\star \mathcal{O}_{P_kV}(\boldsymbol{a}) \subset \mathcal{O}(\overline{F}^{\boldsymbol{a}} E_{k,m}V^\star)$ by induction on k. For $s = 0$, the desired inequality comes from the weighted homogeneity condition, hence we may assume $s \geqslant 1$. Let f

run over all regular germs having their first derivative $f'(0)$ fixed. This means that σ is viewed as a section of $\pi^\star_{2,k}\mathcal{O}_{P_2V}(a_2)\otimes\cdots\otimes\mathcal{O}_{P_kV}(a_k)$ on the fibers of the projection $P_kV=P_{k-1}V_1\to X_1=P_1V$. Then we get a polynomial $Q_1\in E_{k-1,m-a_1}V_1^\star$ such that

$$Q_1(f'_{[1]},f''_{[1]},\ldots,f^{(k-1)}_{[1]})=Q(f',f'',\ldots,f^{(k)}).$$

In the affine chart $f'_r\neq 0$, the map $f_{[1]}$ is defined in coordinates by

$$f_{[1]}\simeq(f_1,\ldots,f_n\,;\,f'_1/f'_r,\ldots,f'_{r-1}/f'_r).$$

Its derivative $f'_{[1]}\in V_1$ can thus be described by $f'_{[1]}\simeq\big((f'_1/f'_r)',\ldots,(f'_{r-1}/f'_r)',f'_r\big)$, by taking $r-1$ vertical components and a horizontal one. All this becomes much simpler if we replace f by $g=f\circ f_r^{-1}$, since $g_r(t)=t$ and $g'_r(t)=1$. Then we get

$$\begin{aligned}(g',g'',\ldots,g^{(k)})&\simeq\big((g'_1,\ldots,g'_{r-1},1),(g''_1,\ldots,g''_{r-1},0),\ldots,(g^{(k)}_1,\ldots,g^{(k)}_{r-1},0)\big),\\(g'_{[1]},g''_{[1]},\ldots,g^{(k)}_{[1]})&\simeq\big((g''_1,\ldots,g''_{r-1},1),(g'''_1,\ldots,g'''_{r-1},0),\ldots,(g^{(k)}_1,\ldots,g^{(k)}_{r-1},0)\big)\end{aligned}$$

in the corresponding charts of J_kV and $J_{k-1}V_1$. The inequality (6.16 i) for the monomials $(g^{(\bullet)})^\ell$ of $Q(g',g'',\ldots,g^{(k)})$ follows clearly from the corresponding inequality on the monomials $(g^{(\bullet)}_{[1]})^\ell$ of Q_1, when (k,s) is replaced by $(k-1,s-1)$. Now, thanks to (6.9), we get $Q(f',f'',\ldots,f^{(k)})=(f'_r)^mQ(g',g'',\ldots,g^{(k)})$, and the desired inequality (6.16 i) for the monomials $(f^{(\bullet)})^\ell$ follows easily. In the opposite direction, if we are given a section $Q(f',f'',\ldots,f^{(k)})\in\mathcal{O}(\overline{F}^{\boldsymbol{a}}E_{k,m}V^\star)$, we see by induction on k that Q defines a section of

$$\mathcal{O}_{P_1V}(a_1)\otimes(\pi_{1,k})_\star\big(\pi^\star_{2,k}\mathcal{O}_{P_2V}(a_2)\otimes\cdots\otimes\mathcal{O}_{P_kV}(a_k)\big)$$

on P_1V, and we conclude that we get a section of $(\pi_{0,k})_\star\mathcal{O}_{P_kV}(\boldsymbol{a})$ by taking the direct image by $(\pi_1)_\star$.

ii-iii) By induction on k, we construct a relatively ample line bundle L_{k-1} on $P_{k-1}V$ such that $\mathcal{O}_{P_kV}(1)\otimes\pi_k^\star L_{k-1}$ is relatively nef; by definition, this is equivalent to saying that the vector bundle $V^\star_{k-1}\otimes L_{k-1}$ is relatively nef (for the notion of a nef vector bundle, see e.g. [DPS94]). Since $\mathcal{O}_{P_1V}(1)$ is relatively ample, we can start with $L_0=\mathcal{O}_X$. Suppose that L_{k-1} has been constructed. The dual of (5.4) yields an exact sequence

$$0\longrightarrow\mathcal{O}_{P_kV}(1)\longrightarrow V_k^\star\longrightarrow T^\star_{P_kV/P_{k-1}V}\longrightarrow 0.$$

As an extension of nef vector bundles is nef, it is enough to select L_k in such a way that

$$\text{(6.17)}\qquad \mathcal{O}_{P_kV}(1)\otimes L_k\quad\text{and}\quad T^\star_{P_kV/P_{k-1}V}\otimes L_k\quad\text{are relatively nef.}$$

By taking the second wedge power of the central term in (5.4′), we get an injection

$$0\longrightarrow T_{P_kV/P_{k-1}V}\longrightarrow\Lambda^2\big(\pi_k^\star V_{k-1}\otimes\mathcal{O}_{P_kV}(1)\big).$$

By dualizing and twisting with $\mathcal{O}_{P_{k-1}V}(2)\otimes\pi_k^\star L_{k-1}^{\otimes 2}$, we find a surjection

$$\pi_k^\star\Lambda^2(V^\star_{k-1}\otimes L_{k-1})\longrightarrow T^\star_{P_kV/P_{k-1}V}\otimes\mathcal{O}_{P_kV}(2)\otimes\pi_k^\star L_{k-1}^{\otimes 2}\longrightarrow 0.$$

As $V_{k-1}^\star \otimes L_{k-1}$ is relatively nef by the induction hypothesis, we obtain that its quotient $T^\star_{P_kV/P_{k-1}V} \otimes \mathcal{O}_{P_kV}(2) \otimes \pi_k^\star L_{k-1}^{\otimes 2}$ is also relatively nef. Hence Condition (6.17) is achieved if we take $L_k \geqslant \pi_k^\star L_{k-1}$ and $L_k \geqslant \mathcal{O}_{P_kV}(2) \otimes \pi_k^\star L_{k-1}^{\otimes 2}$ (the ordering relation $\geqslant$ is the one given by the cone of relatively nef line bundles). We need only define L_k inductively by

$$L_k = \mathcal{O}_{P_kV}(2) \otimes \pi_k^\star L_{k-1}^{\otimes 3}.$$

The relative ampleness of L_k is then clear by induction, since $\mathcal{O}_{P_kV}(1) \otimes \pi_k^\star L_{k-1}$ is relatively nef over X and relatively ample over $P_{k-1}V$. The resulting formula for L_k is

$$L_k = \mathcal{O}_{P_kV}\big((2\cdot 3^{k-1}, 2\cdot 3^{k-2}, \ldots, 6, 2)\big).$$

By definition, we then find

$$\mathcal{O}_{P_kV}(1) \otimes \pi_k^\star L_{k-1} = \mathcal{O}_{P_kV}\big((2\cdot 3^{k-2}, 2\cdot 3^{k-3}, \ldots, 6, 2, 1)\big) \qquad \text{relatively nef.}$$

These properties imply ii) and iii) by taking suitable convex combinations. □

6.18. Remark. As in Green-Griffiths [GrGr80], Riemann's extension theorem shows that for every meromorphic map $\Phi : X \dashrightarrow Y$ there are well-defined pullback morphisms

$$\Phi^\star : H^0(Y, E_{k,m}^{\mathrm{GG}}) \to H^0(X, E_{k,m}^{\mathrm{GG}}), \qquad \Phi^\star : H^0(Y, E_{k,m}) \to H^0(X, E_{k,m}).$$

In particular the dimensions $h^0(X, E_{k,m}^{\mathrm{GG}})$ and $h^0(X, E_{k,m}^{\mathrm{GG}})$ are bimeromorphic invariants of X. The same is true for spaces of sections of any subbundle of $E_{k,m}^{\mathrm{GG}}$ or $E_{k,m}$ constructed by means of the canonical filtrations $F_s^\bullet$.

§7. *k*-jet metrics with negative curvature

The goal of this section is to show that hyperbolicity is closely related to the existence of k-jet metrics with suitable negativity properties of the curvature. The connection between these properties is in fact a simple consequence of the Ahlfors-Schwarz lemma. Such ideas have been already developed long ago by Grauert-Reckziegel [GRec65], Kobayashi [Kob75] for 1-jet metrics (i.e., Finsler metrics on T_X) and by Cowen-Griffiths [CoGr76], Green-Griffiths [GrGr80] and Grauert [Gra89] for higher order jet metrics. However, even in the standard case $V = T_X$, the definition given below differs from that of [GrGr80], in which the k-jet metrics are not supposed to be $\mathbb{G}'_k$-invariant. We prefer to deal here with $\mathbb{G}'_k$-invariant objects, because they reflect better the intrinsic geometry. Grauert [Gra89] actually deals with $\mathbb{G}'_k$-invariant metrics, but he apparently does not take care of the way the quotient space $J_k^{\mathrm{reg}}V/\mathbb{G}_k$ can be compactified; also, his metrics are always induced by the Poincaré metric, and it is not at all clear whether these metrics have the expected curvature properties (see 7.14 below). In the present situation, it is important to allow also hermitian metrics possessing some singularities ("singular hermitian metrics" in the sense of [Dem90]).

7.1. Definition. *Let $L \to X$ be a holomorphic line bundle over a complex manifold X. We say that h is a singular metric on L if for any trivialization $L_{\restriction U} \simeq U \times \mathbb{C}$ of L, the metric is given by $|\xi|_h^2 = |\xi|^2 e^{-\varphi}$ for some real valued weight function $\varphi \in L^1_{\mathrm{loc}}(U)$. The curvature current of L is then defined to be the closed $(1,1)$-current $\Theta_{L,h} = \frac{i}{2\pi}\partial\overline{\partial}\varphi$, computed in the sense of distributions. We say that h admits a closed subset $\Sigma \subset X$ as its degeneration set if φ is locally bounded on $X \smallsetminus \Sigma$ and is unbounded on a neighborhood of any point of Σ.*

An especially useful situation is the case when the curvature of h is positive definite. By this, we mean that there exists a smooth positive definite hermitian metric ω and a continuous positive function ε on X such that $\Theta_{L,h} \geqslant \varepsilon\omega$ in the sense of currents, and we write in this case $\Theta_{L,h} \gg 0$. We need the following basic fact (quite standard when X is projective algebraic; however we want to avoid any algebraicity assumption here, so as to be able the case of general complex tori in § 9).

7.2. Proposition. *Let L be a holomorphic line bundle on a compact complex manifold X.*

i) *L admits a singular hermitian metric h with positive definite curvature current $\Theta_{L,h} \gg 0$ if and only if L is big.*

Now, define B_m to be the base locus of the linear system $|H^0(X, L^{\otimes m})|$ and let

$$\Phi_m : X \smallsetminus B_m \to \mathbb{P}^N$$

be the corresponding meromorphic map. Let Σ_m be the closed analytic set equal to the union of B_m and of the set of points $x \in X \smallsetminus B_m$ such that the fiber $\Phi_m^{-1}(\Phi_m(x))$ is positive dimensional.

ii) *If $\Sigma_m \neq X$ and G is any line bundle, the base locus of $L^{\otimes k} \otimes G^{-1}$ is contained in Σ_m for k large. As a consequence, L admits a singular hermitian metric h with degeneration set Σ_m and with $\Theta_{L,h}$ positive definite on X.*

iii) *Conversely, if L admits a hermitian metric h with degeneration set Σ and positive definite curvature current $\Theta_{L,h}$, there exists an integer $m > 0$ such that the base locus B_m is contained in Σ and $\Phi_m : X \smallsetminus \Sigma \to \mathbb{P}_m$ is an embedding.*

iv) *Assume that L admits a singular hermitian metric h with positive definite curvature current, such that the degeneration set Σ is an analytic subset of X. Assume moreover that for each irreducible component Σ_j of Σ, $L_{|\Sigma_j}$ admits a singular hermitian metric h_j with positive definite curvature current on Σ_j and degeneration set $\Sigma_{j,k} \subset \Sigma_j$. Then L admits a singular hermitian metric $\widetilde{h}$ of positive curvature current on X, with degeneration set $\widetilde{\Sigma} = \bigcup_{j,k} \Sigma_{j,k}$.*

Proof. i) is proved e.g. in [Dem90, 92], so we will only briefly sketch the details. If L is big, then X is Moishezon and we can even assume that X is projective algebraic after taking a suitable modification $\widetilde{X}$ (apply Hironaka [Hir64]; observe moreover that the direct image of a strictly positive current is strictly positive). So, assume that X is projective algebraic. Then it is well-known that some large multiple of L can be written as $L^{\otimes m} = \mathcal{O}_X(D + A)$ with divisors D, A such that D is effective and A ample. The invertible sheaf $\mathcal{O}_X(D)$ can be viewed as a subsheaf of the sheaf of meromorphic functions. We get a singular metric $|s|^2$ on sections of $\mathcal{O}_X(D)$ by just taking the square of the modulus of s viewed as a complex valued (meromorphic) function. By the Lelong-Poincaré equation, the curvature current of that metric is equal to the current of integration $[D] \geqslant 0$ over the divisor D. We thus get $\Theta_L = \frac{1}{m}([D] + \Theta_A) \geqslant \frac{1}{m}\Theta_A \gg 0$ for a suitable choice of the metric on $\mathcal{O}_X(A)$. In the other direction, if $\Theta_{L,h}$ is positive, one can construct a "lot of" sections in $H^0(X, L^{\otimes m})$, $m \gg 0$, by using Hörmander's L^2 estimates; the Hörmander-Bombieri-Skoda technique implies that these sections can be taken to have arbitrary jets at all points in a given finite subset of $X \smallsetminus \Sigma$, if Σ is the degeneration set of h. This also proves property iii).

ii) The assumption $\Sigma_m \neq X$ shows that there is a generically finite meromorphic map from X to an algebraic variety, and this implies again that X is Moishezon. By blowing-up the ideal

$$\mathcal{I}_m = \mathrm{Im}\big(H^0(X, L^{\otimes m}) \otimes \mathcal{O}_X(L^{\otimes -m}) \to \mathcal{O}_X\big) \subset \mathcal{O}_X$$

and resolving the singularities, we obtain a smooth modification $\mu : \widetilde{X} \to X$ and a line bundle $\widetilde{L} = \mu^\star(L^{\otimes m}) \otimes \mathcal{O}_{\widetilde{X}}(-E)$ (where E is a μ-exceptional divisor with support in $\mu^{-1}(\Sigma_m)$, such that $\widetilde{L}$ is base point free; after possibly blowing-up again, we may assume furthermore that $\widetilde{X}$ is projective algebraic. Clearly, it is enough to prove the result for $\widetilde{L}$, and we are thus reduced to the case when L is base point free and X is projective algebraic. We may finally assume that G is very ample (other we add a large ample divisor to G to make it very ample). In this situation, we have a holomorphic map $\Phi_m : X \to \mathbb{P}^N$ such that $L^{\otimes m} = \Phi_m^{-1}(\mathcal{O}(1))$, and Φ_m is finite-to-one outside Σ_m. Hence, if $x \in X \smallsetminus \Sigma_m$, the set $\Phi_m^{-1}(\Phi_m(x))$ is finite, and we can take a smooth divisor $D \in |G|$ such that $D \cap \Phi_m^{-1}(\Phi_m(x)) = \emptyset$. Thus $\Phi_m(D) \not\ni \varphi_m(x)$ in $\mathbb{P}^N$. It follows that there exists a hypersurface $H = \sigma^{-1}(0) \in |\mathcal{O}_{\mathbb{P}^N}(k)|$ of sufficiently large degree k, such that H contains $\Phi_m(D)$ but does not pass through $\Phi_m(x)$. Then $\Phi_m^\star\sigma$ can be viewed as a section of $\Phi_m^\star\mathcal{O}_{\mathbb{P}^N}(k) \otimes \mathcal{O}_X(-D) = L^{\otimes km} \otimes G^{-1}$, and $\Phi_m^\star\sigma$ does not vanish at x. By the Noetherian property, there exists k_0 such that the base locus of $L^{\otimes km} \otimes G^{-1}$ is contained in Σ_m for $k \geqslant k_0$ large. Claim ii) follows.

iv) is obtained by extending the metric h_j to a metric $\widetilde{h}_j$ on a neighborhood of Σ_j (it is maybe necessary to modify $\widetilde{h}_j$ slightly by adding some "transversally convex terms" in the weight, so as to obtain positive curvature in all directions of T_X, on a suitable neighborhood of Σ_j), and then taking $\widetilde{h} = \min(h, \varepsilon\widetilde{h}_j)$ with $\varepsilon > 0$ small enough. □

We now come to the main definitions. By (5.6), every regular k-jet $f \in J_kV$ gives rise to an element $f'_{[k-1]}(0) \in \mathcal{O}_{P_kV}(-1)$. Thus, measuring the "norm of k-jets" is the same as taking a hermitian metric on $\mathcal{O}_{P_kV}(-1)$.

7.3. Definition. *A smooth, (resp. continuous, resp. singular) k-jet metric on a complex directed manifold (X, V) is a hermitian metric h_k on the line bundle $\mathcal{O}_{P_kV}(-1)$ over P_kV (i.e. a Finsler metric on the vector bundle V_{k-1} over $P_{k-1}V$), such that the weight functions φ representing the metric are smooth (resp. continuous, L^1_{loc}). We let $\Sigma_{h_k} \subset P_kV$ be the singularity set of the metric, i.e., the closed subset of points in a neighborhood of which the weight φ is not locally bounded.*

We will always assume here that the weight function φ is quasi psh. Recall that a function φ is said to be quasi psh if φ is locally the sum of a plurisubharmonic function and of a smooth function (so that in particular $\varphi \in L^1_{\mathrm{loc}}$). Then the curvature current

$$\Theta_{h_k^{-1}}(\mathcal{O}_{P_kV}(1)) = \frac{i}{2\pi}\partial\overline{\partial}\varphi.$$

is well defined as a current and is locally bounded from below by a negative $(1,1)$-form with constant coefficients.

7.4. Definition. *Let h_k be a k-jet metric on (X, V). We say that h_k has negative jet curvature (resp. negative total jet curvature) if $\Theta_{h_k}(\mathcal{O}_{P_kV}(-1))$ is negative definite along the subbundle $V_k \subset T_{P_kV}$ (resp. on all of T_{P_kV}), i.e., if there is $\varepsilon > 0$ and a smooth hermitian metric ω_k on T_{P_kV} such that*

$$\langle\Theta_{h_k^{-1}}(\mathcal{O}_{P_kV}(1))\rangle(\xi) \geqslant \varepsilon|\xi|^2_{\omega_k}, \qquad \forall\xi \in V_k \subset T_{P_kV} \quad (\textit{resp.} \quad \forall\xi \in T_{P_kV}).$$

(*If the metric h_k is not smooth, we suppose that its weights φ are quasi psh, and the curvature inequality is taken in the sense of distributions.*)

It is important to observe that for $k \geqslant 2$ there cannot exist any smooth hermitian metric h_k on $\mathcal{O}_{P_kV}(1)$ with positive definite curvature along $T_{X_k/X}$, since $\mathcal{O}_{P_kV}(1)$ is not relatively ample over X. However, it is relatively big, and Prop. 7.2 i) shows that $\mathcal{O}_{P_kV}(-1)$ admits a singular hermitian metric with negative total jet curvature (whatever the singularities of the metric are) if and only if $\mathcal{O}_{P_kV}(1)$ is big over P_kV. It is therefore crucial to allow singularities in the metrics in Def. 7.4.

7.5. Special case of 1-jet metrics. A 1-jet metric h_1 on $\mathcal{O}_{P_1V}(-1)$ is the same as a Finsler metric $N = \sqrt{h_1}$ on $V \subset T_X$. Assume until the end of this paragraph that h_1 is smooth. By the well known Kodaira embedding theorem, the existence of a smooth metric h_1 such that $\Theta_{h_1^{-1}}(\mathcal{O}_{P_1V}(1))$ is positive on all of T_{P_1V} is equivalent to $\mathcal{O}_{P_1V}(1)$ being ample, that is, $V^\star$ ample. In the absolute case $V = T_X$, there are only few examples of varieties X such that $T_X^\star$ is ample, mainly quotients of the ball $\mathbb{B}_n \subset \mathbb{C}^n$ by a discrete cocompact group of automorphisms. The 1-jet negativity condition considered in Definition 7.4 is much weaker. For example, if the hermitian metric h_1 comes from a (smooth) hermitian metric h on V, then formula (4.16) implies that h_1 has negative total jet curvature (i.e. $\Theta_{h_1^{-1}}(\mathcal{O}_{P_1V}(1))$ is positive) if and only if $\langle\Theta_{V,h}\rangle(\zeta\otimes v) < 0$ for all $\zeta \in T_X \smallsetminus \{0\}$, $v \in V \smallsetminus \{0\}$, that is, if (V,h) is *negative in the sense of Griffiths*. On the other hand, $V_1 \subset T_{P_1V}$ consists by definition of tangent vectors $\tau \in T_{P_1V,(x,[v])}$ whose horizontal projection ${}^H\tau$ is proportional to v, thus $\Theta_{h_1}(\mathcal{O}_{P_1V}(-1))$ is negative definite on V_1 if and only if $\Theta_{V,h}$ satisfies the much weaker condition that the *holomorphic sectional curvature* $\langle\Theta_{V,h}\rangle(v\otimes v)$ is negative on every complex line. □

We now come back to the general situation of jets of arbitrary order k. Our first observation is the fact that the k-jet negativity property of the curvature becomes actually weaker and weaker as k increases.

7.6. Lemma. *Let (X,V) be a compact complex directed manifold. If (X,V) has a $(k-1)$-jet metric h_{k-1} with negative jet curvature, then there is a k-jet metric h_k with negative jet curvature such that $\Sigma_{h_k} \subset \pi_k^{-1}(\Sigma_{h_{k-1}}) \cup D_k$. (The same holds true for negative total jet curvature).*

Proof. Let ω_{k-1}, ω_k be given smooth hermitian metrics on $T_{P_{k-1}V}$ and T_{P_kV}. The hypothesis implies

$$\langle\Theta_{h_{k-1}^{-1}}(\mathcal{O}_{P_{k-1}V}(1))\rangle(\xi) \geqslant \varepsilon|\xi|^2_{\omega_{k-1}}, \qquad \forall\xi \in V_{k-1}$$

for some constant $\varepsilon > 0$. On the other hand, as $\mathcal{O}_{P_kV}(D_k)$ is relatively ample over $P_{k-1}V$ (D_k is a hyperplane section bundle), there exists a smooth metric $\widetilde{h}$ on $\mathcal{O}_{P_kV}(D_k)$ such that

$$\langle\Theta_{\widetilde{h}}(\mathcal{O}_{P_kV}(D_k))\rangle(\xi) \geqslant \delta|\xi|^2_{\omega_k} - C|(\pi_k)_\star\xi|^2_{\omega_{k-1}}, \qquad \forall\xi \in T_{P_kV}$$

for some constants $\delta, C > 0$. Combining both inequalities (the second one being applied to $\xi \in V_k$ and the first one to $(\pi_k)_\star\xi \in V_{k-1}$), we get

$$\langle\Theta_{(\pi_k^\star h_{k-1})^{-p}\widetilde{h}}(\pi_k^\star\mathcal{O}_{P_{k-1}V}(p)\otimes\mathcal{O}_{P_kV}(D_k))\rangle(\xi) \geqslant$$
$$\geqslant \delta|\xi|^2_{\omega_k} + (p\varepsilon - C)|(\pi_k)_\star\xi|^2_{\omega_{k-1}}, \qquad \forall\xi \in V_k.$$

Hence, for p large enough, $(\pi_k^\star h_{k-1})^{-p}\widetilde{h}$ has positive definite curvature along V_k. Now, by (5.9), there is a sheaf injection

$$\mathcal{O}_{P_kV}(-p)=\pi_k^\star\mathcal{O}_{P_{k-1}V}(-p)\otimes\mathcal{O}_{P_kV}(-pD_k)\hookrightarrow\big(\pi_k^\star\mathcal{O}_{P_{k-1}V}(p)\otimes\mathcal{O}_{P_kV}(D_k)\big)^{-1}$$

obtained by twisting with $\mathcal{O}_{P_kV}((p-1)D_k)$. Therefore $h_k:=((\pi_k^\star h_{k-1})^{-p}\widetilde{h})^{-1/p}=(\pi_k^\star h_{k-1})\widetilde{h}^{-1/p}$ induces a singular metric on $\mathcal{O}_{P_kV}(1)$ in which an additional degeneration divisor $p^{-1}(p-1)D_k$ appears. Hence we get $\Sigma_{h_k}=\pi_k^{-1}\Sigma_{h_{k-1}}\cup D_k$ and

$$\Theta_{h_k^{-1}}(\mathcal{O}_{P_kV}(1))=\frac{1}{p}\Theta_{(\pi_k^\star h_{k-1})^{-p}\widetilde{h}}+\frac{p-1}{p}[D_k]$$

is positive definite along V_k. The same proof works in the case of negative total jet curvature. □

One of the main motivations for the introduction of k-jets metrics is the following list of algebraic sufficient conditions.

7.7. Algebraic sufficient conditions. We suppose here that X is projective algebraic, and we make one of the additional assumptions i), ii) or iii) below.

i) Assume that there exist integers $k,m>0$ and $\boldsymbol{b}\in\mathbb{N}^k$ such that the line bundle $\mathcal{O}_{P_kV}(m)\otimes\mathcal{O}_{P_kV}(-\boldsymbol{b}\cdot D^\star)$ is ample over P_kV. Set $A=\mathcal{O}_{P_kV}(m)\otimes\mathcal{O}_{P_kV}(-\boldsymbol{b}\cdot D^\star)$. Then there is a smooth hermitian metric h_A on A with positive definite curvature on P_kV. By means of the morphism $\mu:\mathcal{O}_{P_kV}(-m)\to A^{-1}$, we get an induced metric $h_k=(\mu^\star h_A^{-1})^{1/m}$ on $\mathcal{O}_{P_kV}(-1)$ which is degenerate on the support of the zero divisor $\mathrm{div}(\mu)=\boldsymbol{b}\cdot D^\star$. Hence $\Sigma_{h_k}=\mathrm{Supp}(\boldsymbol{b}\cdot D^\star)\subset P_kV^{\mathrm{sing}}$ and

$$\Theta_{h_k^{-1}}(\mathcal{O}_{P_kV}(1))=\frac{1}{m}\Theta_{h_A}(A)+\frac{1}{m}[\boldsymbol{b}\cdot D^\star]\geqslant\frac{1}{m}\Theta_{h_A}(A)>0.$$

In particular h_k has negative total jet curvature.

ii) Assume more generally that there exist integers $k,m>0$ and an ample line bundle L on X such that $H^0(P_kV,\mathcal{O}_{P_kV}(m)\otimes\pi_{0,k}^\star L^{-1})$ has non zero sections $\sigma_1,\ldots,\sigma_N$. Let $Z\subset P_kV$ be the base locus of these sections; necessarily $Z\supset P_kV^{\mathrm{sing}}$ by 6.8 iii). By taking a smooth metric h_L with positive curvature on L, we get a singular metric h'_k on $\mathcal{O}_{P_kV}(-1)$ such that

$$h'_k(\xi)=\Big(\sum_{1\leqslant j\leqslant N}|\sigma_j(w)\cdot\xi^m|^2_{h_L^{-1}}\Big)^{1/m},\qquad w\in P_kV,\quad\xi\in\mathcal{O}_{P_kV}(-1)_w.$$

Then $\Sigma_{h'_k}=Z$, and by computing $\frac{i}{2\pi}\partial\overline{\partial}\log h'_k(\xi)$ we obtain

$$\Theta_{h'^{-1}_k}(\mathcal{O}_{P_kV}(1))\geqslant\frac{1}{m}\pi_{0,k}^\star\Theta_L.$$

By (6.15) and 6.16 iii), there exists $\boldsymbol{b}\in\mathbb{Q}_+^k$ such that $\mathcal{O}_{P_kV}(1)\otimes\mathcal{O}_{P_kV}(-\boldsymbol{b}\cdot D^\star)$ is relatively ample over X. Hence $A=\mathcal{O}_{P_kV}(1)\otimes\mathcal{O}_{P_kV}(-\boldsymbol{b}\cdot D^\star)\otimes\pi_{0,k}^\star L^{\otimes p}$ is ample on X for $p\gg 0$. The arguments used in i) show that there is a k-jet metric h''_k on $\mathcal{O}_{P_kV}(-1)$ with $\Sigma_{h''_k}=\mathrm{Supp}(\boldsymbol{b}\cdot D^\star)=P_kV^{\mathrm{sing}}$ and

$$\Theta_{h''^{-1}_k}(\mathcal{O}_{P_kV}(1))=\Theta_A+[\boldsymbol{b}\cdot D^\star]-p\,\pi_{0,k}^\star\Theta_L,$$

where Θ_A is positive definite on P_kV. The metric $h_k = ({h'_k}^{mp} h''_k)^{1/(mp+1)}$ then satisfies $\Sigma_{h_k} = \Sigma_{h'_k} = Z$ and

$$\Theta_{h_k^{-1}}(\mathcal{O}_{P_kV}(1)) \geqslant \frac{1}{mp+1}\Theta_A > 0.$$

iii) If $E_{k,m}V^\star$ is ample, there is an ample line bundle L and a sufficiently high symmetric power such that $S^p(E_{k,m}V^\star) \otimes L^{-1}$ is generated by sections. These sections can be viewed as sections of $\mathcal{O}_{P_kV}(mp) \otimes \pi_{0,k}^\star L^{-1}$ over P_kV, and their base locus is exactly $Z = P_kV^{\rm sing}$ by 6.8 iii). Hence the k-jet metric h_k constructed in ii) has negative total jet curvature and satisfies $\Sigma_{h_k} = P_kV^{\rm sing}$. □

An important fact, first observed by [GRe65] for 1-jet metrics and by [GrGr80] in the higher order case, is that k-jet negativity implies hyperbolicity. In particular, the existence of enough global jet differentials implies hyperbolicity.

7.8. Theorem. *Let (X,V) be a compact complex directed manifold. If (X,V) has a k-jet metric h_k with negative jet curvature, then every entire curve $f : \mathbb{C} \to X$ tangent to V is such that $f_{[k]}(\mathbb{C}) \subset \Sigma_{h_k}$. In particular, if $\Sigma_{h_k} \subset P_kV^{\rm sing}$, then (X,V) is hyperbolic.*

Proof. The main idea is to use the Ahlfors-Schwarz lemma, following the approach of [GrGr80]. However we will give here all necessary details because our setting is slightly different. Assume that there is a k-jet metric h_k as in the hypotheses of Theorem 7.8. Let ω_k be a smooth hermitian metric on T_{P_kV}. By hypothesis, there exists $\varepsilon > 0$ such that

$$\langle\Theta_{h_k^{-1}}(\mathcal{O}_{P_kV}(1))\rangle(\xi) \geqslant \varepsilon|\xi|^2_{\omega_k} \qquad \forall\xi \in V_k.$$

Moreover, by (5.4), $(\pi_k)_\star$ maps V_k continuously to $\mathcal{O}_{P_kV}(-1)$ and the weight e^φ of h_k is locally bounded from above. Hence there is a constant $C > 0$ such that

$$|(\pi_k)_\star\xi|^2_{h_k} \leqslant C|\xi|^2_{\omega_k}, \qquad \forall\xi \in V_k.$$

Combining these inequalities, we find

$$\langle\Theta_{h_k^{-1}}(\mathcal{O}_{P_kV}(1))\rangle(\xi) \geqslant \frac{\varepsilon}{C}|(\pi_k)_\star\xi|^2_{h_k}, \qquad \forall\xi \in V_k.$$

Now, let $f : \Delta_R \to X$ be a non constant holomorphic map tangent to V on the disk Δ_R. We use the line bundle morphism (5.6)

$$F = f'_{[k-1]} : T_{\Delta_R} \to f_{[k]}^\star\mathcal{O}_{P_kV}(-1)$$

to obtain a pullback metric

$$\gamma = \gamma_0(t)\,dt \otimes d\bar t = F^\star h_k \qquad \text{on } T_{\Delta_R}.$$

If $f_{[k]}(\Delta_R) \subset \Sigma_{h_k}$ then $\gamma \equiv 0$. Otherwise, $F(t)$ has isolated zeroes at all singular points of $f_{[k-1]}$ and so $\gamma(t)$ vanishes only at these points and at points of the degeneration set $(f_{[k]})^{-1}(\Sigma_{h_k})$ which is a polar set in Δ_R. At other points, the Gaussian curvature of γ satisfies

$$\frac{i\,\partial\overline\partial\log\gamma_0(t)}{\gamma(t)} = \frac{-2\pi\,(f_{[k]})^\star\Theta_{h_k}(\mathcal{O}_{P_kV}(-1))}{F^\star h_k} = \frac{\langle\Theta_{h_k^{-1}}(\mathcal{O}_{P_kV}(1))\rangle(f'_{[k]}(t))}{|f'_{[k-1]}(t)|^2_{h_k}} \geqslant \frac{\varepsilon}{C},$$

since $f'_{[k-1]}(t) = (\pi_k)_\star f'_{[k]}(t)$. The Ahlfors-Schwarz lemma 3.2 implies that γ can be compared with the Poincaré metric as follows:

$$\gamma(t) \leqslant \frac{2C}{\varepsilon} \frac{R^{-2}|dt|^2}{(1-|t|^2/R^2)^2} \quad \Longrightarrow \quad |f'_{[k-1]}(t)|^2_{h_k} \leqslant \frac{2C}{\varepsilon} \frac{R^{-2}}{(1-|t|^2/R^2)^2}.$$

If $f : \mathbb{C} \to X$ is an entire curve tangent to V such that $f_{[k]}(\mathbb{C}) \not\subset \Sigma_{h_k}$, the above estimate implies as $R \to +\infty$ that $f_{[k-1]}$ must be a constant, hence also f. Now, if $\Sigma_{h_k} \subset P_k V^{\mathrm{sing}}$, the inclusion $f_{[k]}(\mathbb{C}) \subset \Sigma_{h_k}$ implies $f'(t) = 0$ at every point, hence f is a constant and (X, V) is hyperbolic. □

Combining Theorem 7.8 with 7.7 ii) and iii), we get the following consequences.

7.9. Corollary. *Assume that there exist integers $k, m > 0$ and an ample line bundle L on X such that $H^0(P_k V, \mathcal{O}_{P_k V}(m) \otimes \pi_{0,k}^\star L^{-1}) \simeq H^0(X, E_{k,m}(V^\star) \otimes L^{-1})$ has non zero sections $\sigma_1, \ldots, \sigma_N$. Let $Z \subset P_k V$ be the base locus of these sections. Then every entire curve $f : \mathbb{C} \to X$ tangent to V is such that $f_{[k]}(\mathbb{C}) \subset Z$. In other words, for every global $\mathbb{G}_k$-invariant polynomial differential operator P with values in L^{-1}, every entire curve f must satisfy the algebraic differential equation $P(f) = 0$.* □

7.10. Corollary. *Let (X, V) be a compact complex directed manifold. If $E_{k,m} V^\star$ is ample for some positive integers k, m, then (X, V) is hyperbolic.* □

7.11. Remark. Green and Griffiths [GrGr80] stated that Corollary 7.9 is even true with sections $\sigma_j \in H^0(X, E^{\mathrm{GG}}_{k,m}(V^\star) \otimes L^{-1})$, in the special case $V = T_X$ they consider. We refer to the recent preprint [SiYe96c] by Siu and Yeung for a detailed proof of this fact, based on a use of the well-known logarithmic derivative lemma in Nevanlinna theory (the original proof given in [GrGr80] does not seem to be complete, as it relies on an unsettled pointwise version of the Ahlfors-Schwarz lemma for general jet differentials); other proofs seem to have been circulating in the literature in the last years. We give here a very short proof for the case when f is supposed to have a bounded derivative (thanks to Brody's theorem, this is enough if one is merely interested in proving hyperbolicity, thus Corollary 7.10 will be valid with $E^{\mathrm{GG}}_{k,m} V^\star$ in place of $E_{k,m} V^\star$). In fact, if f' is bounded, one can apply the Cauchy inequalities to all components f_j of f with respect to a finite collection of coordinate patches covering X. As f' is bounded, we can do this on sufficiently small discs $D(t, \delta) \subset \mathbb{C}$ of constant radius $\delta > 0$. Therefore all derivatives f', f'', … $f^{(k)}$ are bounded. From this we conclude that $\sigma_j(f)$ is a bounded section of $f^\star L^{-1}$. Its norm $|\sigma_j(f)|_{L^{-1}}$ (with respect to any positively curved metric $|\ \ |_L$ on L) is a bounded subharmonic function, which is moreover strictly subharmonic at all points where $f' \neq 0$ and $\sigma_j(f) \neq 0$. This is a contradiction unless f is constant or $\sigma_j(f) \equiv 0$. □

The above results justify the following definition and problems.

7.12. Definition. *We say that X, resp. (X, V), has non degenerate negative k-jet curvature if there exists a k-jet metric h_k on $\mathcal{O}_{P_k V}(-1)$ with negative jet curvature such that $\Sigma_{h_k} \subset P_k V^{\mathrm{sing}}$.*

7.13. Conjecture. *Let (X, V) be a compact directed manifold. Then (X, V) is hyperbolic if and only if (X, V) has nondegenerate negative k-jet curvature for k large enough.*

This is probably a hard problem. In fact, we will see in the next section that the smallest admissible integer k must depend on the geometry of X and need not be uniformly bounded as soon as $\dim X \geqslant 2$ (even in the absolute case $V = T_X$). On the other hand, if (X, V) is hyperbolic, we get for each integer $k \geqslant 1$ a generalized Kobayashi-Royden metric $\mathbf{k}_{(P_{k-1}V, V_{k-1})}$ on V_{k-1} (see Definition 1.3), which can be also viewed as a k-jet metric h_k on $\mathcal{O}_{P_kV}(-1)$; we will call it the *Grauert k-jet metric* of (X, V), although it formally differs from the jet metric considered in [Gra89] (see also [DGr91]). By looking at the projection $\pi_k : (P_kV, V_k) \to (P_{k-1}V, V_{k-1})$, we see that the sequence h_k is monotonic, namely $\pi_k^\star h_k \leqslant h_{k+1}$ for every k. If (X, V) is hyperbolic, then h_1 is nondegenerate and therefore by monotonicity $\Sigma_{h_k} \subset P_kV^{\mathrm{sing}}$ for $k \geqslant 1$. Conversely, if the Grauert metric satisfies $\Sigma_{h_k} \subset P_kV^{\mathrm{sing}}$, it is easy to see that (X, V) is hyperbolic. The following problem is thus especially meaningful.

7.14. Problem. *Estimate the k-jet curvature $\Theta_{h_k^{-1}}(\mathcal{O}_{P_kV}(1))$ of the Grauert metric h_k on (P_kV, V_k) as k tends to $+\infty$.*

§8. Algebraic criterion for the negativity of jet curvature

Our goal is to show that the negativity of k-jet curvature implies strong restrictions of an algebraic nature, similar to property 2.1 ii). Using this we give examples, for any prescribed integer k, of hyperbolic projective surfaces which do not admit any k-jet metric of negative jet curvature.

8.1. Theorem. *Let (X, V) be a compact complex directed manifold and let ω be a hermitian metric on X. If (X, V) has negative k-jet curvature, there exists a constant $\varepsilon > 0$ such that every closed irreducible curve $C \subset X$ tangent to V satisfies*

$$-\chi(\overline{C}) = 2g(\overline{C}) - 2 \geqslant \varepsilon \deg_\omega(C) + \sum_{t \in \overline{C}} (m_{k-1}(t) - 1) > 0$$

where $g(\overline{C})$ is the genus of the normalization $\nu : \overline{C} \to C \subset X$, and $m_k(t)$ is the multiplicity at point t of the k-th lifting $\nu_{[k]} : \overline{C} \to P_kV$ of ν.

Proof. By (5.6), we get a lifting $\nu_{[k]} : \overline{C} \to P_kV$ of the normalization map ν, and there is a canonical map

$$\nu'_{[k-1]} : T_{\overline{C}} \to \nu_{[k]}^\star \mathcal{O}_{P_kV}(-1).$$

Let $t_j \in \overline{C}$ be the singular points of $\nu_{[k-1]}$, and let $m_j = m_{k-1}(t_j)$ be the corresponding multiplicity. Then $\nu'_{[k-1]}$ vanishes at order $m_j - 1$ at t_j and thus we find

$$T_{\overline{C}} \simeq \nu_{[k]}^\star \mathcal{O}_{P_kV}(-1) \otimes \mathcal{O}_{\overline{C}}\Big(-\sum (m_j - 1) p_j\Big).$$

Taking any k-jet metric h_k with negative jet curvature on $\mathcal{O}_{P_kV}(-1)$, the Gauss-Bonnet formula yields

$$2g(\overline{C}) - 2 = \int_{\overline{C}} \Theta_{T^\star_{\overline{C}}} = \sum (m_j - 1) + \int_{\overline{C}} \nu_{[k]}^\star \Theta_{h_k^{-1}}(\mathcal{O}_{P_kV}(1)).$$

Now, the curvature hypothesis implies

$$\langle \Theta_{h_k^{-1}}(\mathcal{O}_{P_kV}(1)) \rangle(\xi) \geqslant \varepsilon' |\xi|^2_{\omega_k} \geqslant \varepsilon'' |(\pi_{0,k})_\star \xi|^2_\omega \qquad \forall \xi \in V_k,$$

for some ε', $\varepsilon'' > 0$ and some smooth hermitian metric ω_k on P_kV. As $\pi_{0,k} \circ \nu_{[k]} = \nu$, we infer from this $\nu_{[k]}^\star \Theta_{h_k^{-1}}(\mathcal{O}_{P_kV}(1)) \geqslant \nu^\star\omega$, hence

$$\int_{\overline{C}} \nu_{[k]}^\star \Theta_{h_k^{-1}}(\mathcal{O}_{P_kV}(1)) \geqslant \frac{\varepsilon''}{2\pi} \int_{\overline{C}} \nu^\star\omega = \varepsilon \deg_\omega(C)$$

with $\varepsilon = \varepsilon''/2\pi$. Theorem 8.1 follows. □

8.2. Theorem. *Let $k \geqslant 1$ be any positive integer. Then there is a nonsingular algebraic surface X (depending on k) which is hyperbolic, but does not carry any nondegenerate k-jet metric with negative jet curvature. In fact, given any two curves Γ, Γ' of genus at least 2, the surface X may be constructed as a fibration $X \to \Gamma$ in which one of the fibers $C_{\mathbf{0}}$ is singular and has Γ' as its normalization.*

Proof. The idea is to construct X in such a way that the singular fiber C which is normalized by Γ' violates the inequality obtained in Theorem 8.1. For this we need only having a singular point t_0 such that $m_{k-1}(t_0) - 1 > 2g(\overline{C}) - 2$, i.e., $m_{k-1}(t_0) \geqslant 2g(\Gamma')$. Moreover, as Γ is hyperbolic, X will be hyperbolic if and only if all fibers of $X \to \Gamma$ have geometric genus at least 2.

We first construct from Γ' a singular curve Γ'' with normalization $\overline{\Gamma}'' = \Gamma'$, simply by modifying the structure sheaf $\mathcal{O}_{\Gamma'}$ at one given point $w_0 \in \Gamma'$. Let t be a holomorphic coordinate on Γ' at w_0. We replace $\mathcal{O}_{\Gamma',w_0} = \mathbb{C}\{t\}$ by $\mathcal{O}_{\Gamma'',w_0} = \mathbb{C}\{t^a, t^b\}$, where $a < b$ are relatively prime integers. The corresponding singularity is described by the germ of embedding $t \mapsto f(t) = (t^a, t^b)$ in $(\mathbb{C}^2, 0)$. Now, $f'(t) = (at^{a-1}, bt^{b-1})$, thus $[f'(t)] \in \mathbb{P}^1 \simeq \mathbb{C} \cup \{\infty\}$ is given by $[f'(t)] = \frac{b}{a} t^{b-a}$. By induction, we see that the singularity of the j-th lifting $f_{[j]}$ is described by the embedding

$$t \mapsto (t^a, t^b, c_1 t^{b-a}, \ldots, c_j t^{b-ja}) \in \mathbb{C}^{j+2}, \qquad c_j = a^{-j} b(b-a)\cdots(b-(j-1)a)$$

if $b > ja$. Then we have $m(f_{[j]}, 0) = \min(a, b-ja)$. If we take for instance $a = 2g(\Gamma')$ and $b = ka+1$, then $m(f_{[k-1]}, 0) = a$. We embed Γ'' in some projective space $\mathbb{P}^n$ and let $C = p(\Gamma'')$ to be a generic projection to a plane $\mathbb{P}^2 \subset \mathbb{P}^n$ in such a way that C has only $x_0 = p(w_0)$ and some nodes (ordinary double points) as its singular points. By construction, the Zariski tangent space to Γ'' at w_0 is 2-dimensional, so we may assume that p projects that plane injectively into $T_{\mathbb{P}^2}$. Then we get a curve $C \subset \mathbb{P}^2$ with $\overline{C} = \Gamma'$, such that $m(\nu_{[k-1]}, w_0) = a = 2g(\overline{C})$, if $\nu : \overline{C} \to \mathbb{P}^2$ is the normalization.

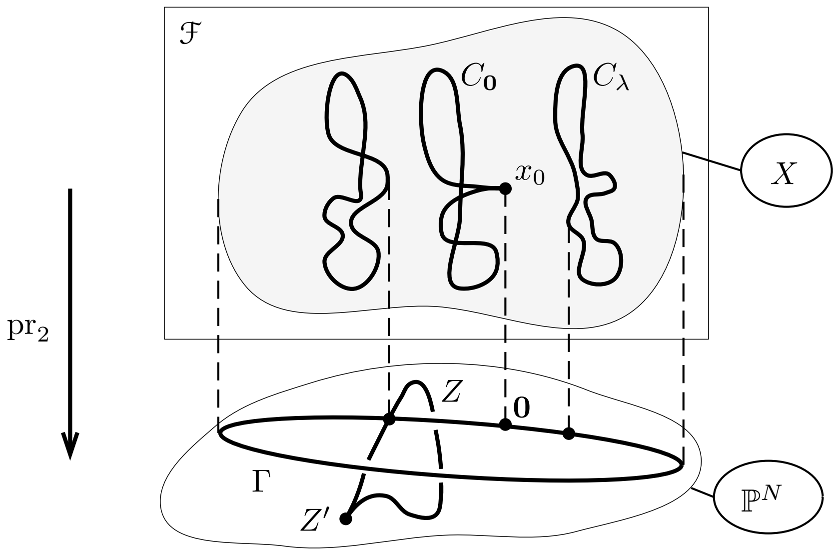

Figure 1. Construction of the surface X

Let $P_0(z_0, z_1, z_2) = 0$ be an equation of C in $\mathbb{P}^2$. Since C has geometric genus at least 2, we have $d = \deg P_0 \geqslant 4$. We complete P_0 into a basis $(P_0, \ldots, P_N)$ of the space of homogeneous polynomials of degree d, and consider the universal family

$$\mathcal{F} = \big\{([z_0 : z_1 : z_2], [\lambda_0, \lambda_1, \ldots, \lambda_N]) \in \mathbb{P}^2 \times \mathbb{P}^N \,;\, \sum \lambda_j P_j(z) = 0\big\} \subset \mathbb{P}^2 \times \mathbb{P}^N$$

of curves $C_\lambda = \{\sum \lambda_j P_j(z) = 0\}$ of degree d in $\mathbb{P}^2$. As is well known, the set Z of points $\lambda \in \mathbb{P}^N$ such that C_λ is a singular curve is an algebraic hypersurface, and the set $Z' \subset Z$ of points λ such that C_λ has not just a node in its singularity set satisfies $\operatorname{codim} Z' \geqslant 2$. The curve $C = C_{\mathbf{0}}$ itself corresponds to the point $\mathbf{0} = [1 : 0 : \cdots : 0] \in Z'$. Since $\operatorname{codim} Z' \geqslant 2$, we can embed Γ in $\mathbb{P}^N$ in such a way that $\Gamma \cap Z' = \{\mathbf{0}\}$. We then take $X \to \Gamma$ to be the family of curves $(C_\lambda)_{\lambda \in \Gamma}$. If X is singular, we move Γ by a generic automorphism of $\mathbb{P}^N$ leaving $\mathbf{0}$ fixed. Then, since $\mathcal{F}$ is smooth (it is a smooth $\mathbb{P}^{N-1}$ subbundle of $\mathbb{P}^2 \times \mathbb{P}^N$ over $\mathbb{P}^2$), Bertini's theorem implies that $X \smallsetminus C_{\mathbf{0}}$ will become nonsingular. That X will be also nonsingular near $C_{\mathbf{0}}$ depends only on the following first order condition: if $[1 : \alpha\lambda_1^0 : \cdots : \alpha\lambda_N^0]$, $\alpha \in \mathbb{C}$, is the tangent line to Γ at $\mathbf{0}$, then $\sum_{j \geqslant 1} \lambda_j^0 P_j(z)$ does not vanish at any of the singular points of $C_{\mathbf{0}}$. Now, all nonsingular fibers C_λ of the fibration $X \to \Gamma$ have genus $(d-1)(d-2)/2 \geqslant 3$, and the singular ones other than $C_{\mathbf{0}}$ only have one node, so their genus is $(d-1)(d-2)/2 - 1 \geqslant 2$. $\square$

If we make an assumption on the total jet curvature (as is the case with the algebraic sufficient conditions 7.7), Theorem 8.1 can be strengthened to curves which are not necessarily tangent to V, again by introducing the concept of deviation. We start with a general purpose statement.

8.3. Proposition. *Let (X, V) be a compact complex directed manifold and let L be a holomorphic line bundle over X. Assume that L is equipped with a singular hermitian metric h of degeneration set Σ_h, such that the curvature (computed in the sense of distributions) satisfies*

$$\Theta_{L,h} \geqslant \alpha, \qquad \alpha_{\restriction V} \geqslant \delta\omega_{\restriction V}$$

where δ is a positive constant, ω a smooth hermitian metric and α is a continuous real $(1,1)$-form on X. Then for every compact irreducible curve $C \subset X$ not contained in Σ_h, there exists a constant $\varepsilon > 0$ such that the following a priori inequality holds

$$\max\big(L\cdot C, \mathrm{dev}^2_\omega(C/V)\big) \geqslant \varepsilon\ \deg_\omega(C).$$

Proof. By the continuity of α and the compactness of X, our assumption $\alpha_{\restriction V} \geqslant \delta\omega$ implies that there is a constant $M > 0$ such that

$$\alpha + M\,\omega^{V^\perp} \geqslant \frac{\delta}{2}\omega$$

(to get this, one merely needs to apply the Cauchy-Schwarz inequality to mixed terms $V^\star \otimes (V^\perp)^\star$ in a hermitian form on V). In particular, we find

$$\Theta_{L,h} + M\,\omega^{V^\perp} \geqslant \frac{\delta}{2}\omega$$

This inequality gives rise to a corresponding numerical inequality on every irreducible curve $C \not\subset \Sigma_h$, for the difference has a well defined and nonnegative restriction to C (we use here the fact that the weight of h is quasi-psh and locally bounded at some point of C, hence locally integrable along C). From this we infer

$$L\cdot C + M\,\mathrm{dev}^2_\omega(C/V) \geqslant \frac{\delta}{2}\deg_\omega(C),$$

and the left hand side is at most equal to $(M+1)\max\big(L\cdot C, \mathrm{dev}^2_\omega(C/V)\big)$. $\square$

8.4. Proposition. *Let (X,V) be a compact complex directed manifold. Assume that there are integers $k, m > 0$ and $\boldsymbol{b} \in \mathbb{N}^k$ such that $\mathcal{O}_{P_kV}(m) \otimes \mathcal{O}_{P_kV}(-\boldsymbol{b}\cdot D^\star)$ is an ample line bundle over P_kV. Then (X,V) is hyperbolic and there exists $\varepsilon > 0$ such that every closed curve $C \subset X$ satisfies*

$$\max\Big(-\chi(\overline{C}) - \sum_{t\in\overline{C}}(m_{k-1}(t)-1), \mathrm{dev}^\infty_\omega(C/V)\Big) \geqslant \varepsilon\ \deg_\omega(C).$$

Proposition 8.4 is likely to be true also if we assume more generally that (X,V) has non degenerate total k-jet curvature but, in this case, some technical difficulties appear in the construction of the required singular hermitian metric h_k on $\mathcal{O}_{P_kT_X}(1)$ (see the proof below).

Proof. The hyperbolicity of (X,V) follows from 7.7 i) and Theorem 7.8. Now, the identity map defines a natural monomorphism $(X,V)\to(X,T_X)$ of directed manifolds and therefore induces an embedding $P_kV \hookrightarrow P_kT_X$ for each k. With respect to this embedding, we have

$$\begin{aligned}
&\mathcal{O}_{P_kT_X}(1)_{\restriction P_kV} = \mathcal{O}_{P_kV}(1),\\
&\mathcal{O}_{P_kT_X}(m)\otimes\mathcal{O}_{P_kT_X}(-\boldsymbol{b}\cdot D^\star)_{\restriction P_kV} = \mathcal{O}_{P_kV}(m)\otimes\mathcal{O}_{P_kV}(-\boldsymbol{b}\cdot D^\star)
\end{aligned}$$

By our assumptions, $\mathcal{O}_{P_kT_X}(m)\otimes\mathcal{O}_{P_kT_X}(-\boldsymbol{b}\cdot D^\star)$ is ample over P_kV and over the fibers of the projection $P_kT_X \to X$. Hence, we can find a smooth hermitian metric $h_{k,m,\boldsymbol{b}}$

on $\mathcal{O}_{P_kT_X}(m)\otimes\mathcal{O}_{P_kT_X}(-\boldsymbol{b}\cdot D^\star)$ such that the curvature form is positive definite on a neighborhood U of P_kV and satisfies

$$\Theta\big(\mathcal{O}_{P_kT_X}(m)\otimes\mathcal{O}_{P_kT_X}(-\boldsymbol{b}\cdot D^\star)\big)\geqslant -C\pi_{k,0}^\star\omega$$

for some Kähler metric ω over X. This metric $h_{k,m,\boldsymbol{b}}$ gives rise to a hermitian metric h_k on $\mathcal{O}_{P_kT_X}(1)$ with singularity set $\Sigma_{h_k}\subset P_k^{\rm sing}T_X$ and similar curvature properties, that is

$$\begin{cases}\Theta_{h_k}(\mathcal{O}_{P_kT_X}(1))\geqslant -C\pi_{k,0}^\star\omega & \text{on } P_kT_X,\\ \Theta_{h_k}(\mathcal{O}_{P_kT_X}(1))\geqslant \delta\omega_k\geqslant\delta'\pi_{k,0}^\star\omega & \text{on } U\supset P_kV,\end{cases}\tag{8.5}$$

where ω_k is a hermitian metric on P_kT_X and δ, $\delta'>0$. Now, assume that the conclusion of Prop. 8.4 is wrong. Then there would exist a sequence of curves (C_ℓ) and a sequence of positive numbers ε_ℓ converging to 0, such that

$$\mathcal{O}_{P_kT_X}(1)\cdot C_{\ell,[k]}\leqslant\varepsilon_\ell\deg_\omega(C_\ell),\qquad \mathrm{dev}_\omega^\infty(C_\ell/V)\big)\leqslant\varepsilon_\ell\deg_\omega(C_\ell)$$

where $C_{\ell,[k]}$ is the lifting of C_ℓ to P_kT_X [indeed, we have $\mathcal{O}_{P_kT_X}(1)\cdot C_{\ell,[k]}=-\chi(\overline{C}_\ell)-\sum(m_{k-1}(t)-1)$]. Let $\nu_\ell:\overline{C}_\ell\to X$ be the normalization map. As $\mathrm{dev}_\omega^\infty(C_\ell/V)\big)=\sup\nu_\ell^\star(\omega_{V^\perp})/d\widetilde\sigma$ where $d\sigma$ is the Poincaré metric and $d\widetilde\sigma$ the associated normalized metric, the second condition means

$$\sup\|\,\mathrm{pr}_{V^\perp}\nu_\ell'\|_{\sigma,\omega}^2=\sup\frac{\nu_\ell^\star(\omega_{V^\perp})}{d\sigma}\leqslant\frac{\varepsilon_\ell\deg_\omega(C_\ell)}{\int_{\overline{C}\ell}d\sigma}=\varepsilon_\ell\frac{\int_{\overline{C}\ell}\nu_\ell^\star\omega}{\int_{\overline{C}\ell}d\sigma}.$$

In addition to this, we have

$$\frac{\int_{\overline{C}\ell}\nu_\ell^\star\omega}{\int_{\overline{C}\ell}d\sigma}\leqslant R_\ell^2:=\sup\|\nu_\ell'\|_{\sigma,\omega}^2$$

and $R=\sup R_\ell<+\infty$, otherwise the proof of Prop. 2.9 would produce a non constant entire curve $g:\mathbb{C}\to X$ tangent to V, contradicting the hyperbolicity of (X,V). An application of the Cauchy inequalities to the components of $\mathrm{pr}_{V^\perp}$ on sufficiently small disks in the universal covering of $\overline{C}_\ell$ and in suitable trivializations of T_X/V shows that there is a constant $M_k\geqslant 0$ such that

$$\sup_{1\leqslant j\leqslant k}\|\,\mathrm{pr}_{V^\perp}\nu_\ell^{(j)}\|_{\sigma,\omega}^2\leqslant M_k\sup\|\,\mathrm{pr}_{V^\perp}\nu_\ell'\|_{\sigma,\omega}^2\leqslant M_k\varepsilon_\ell\frac{\int_{\overline{C}\ell}\nu_\ell^\star\omega}{\int_{\overline{C}\ell}d\sigma}.$$

As $\int_{\overline{C}_\ell}\|\nu_\ell'\|_{\sigma,\omega}^{-2}\nu_\ell^\star\omega=\int_{\overline{C}_\ell}d\sigma$, we infer

$$\int_{\overline{C}_\ell}\frac{\sup_{1\leqslant j\leqslant k}\|\,\mathrm{pr}_{V^\perp}\nu_\ell^{(j)}\|_{\sigma,\omega}^2}{\|\nu_\ell'\|_{\sigma,\omega}^2}\nu_\ell^\star\omega\leqslant M_k\varepsilon_\ell\int_{\overline{C}\ell}\nu_\ell^\star\omega.\tag{8.6}$$

Since U is a neighborhood of P_kV, there exists a constant $\eta>0$ such that

$$\frac{\sup_{1\leqslant j\leqslant k}\|\,\mathrm{pr}_{V^\perp}\nu_\ell^{(j)}(t)\|_{\sigma,\omega}^2}{\|\nu_\ell'(t)\|_{\sigma,\omega}^2}<\eta\quad\Longrightarrow\nu_{\ell,[k]}(t)\in U$$

for any $t \in \overline{C}_\ell$. By the integral estimate (8.6), the set S_η of "bad points" $t \in \overline{C}_\ell$ at which the left hand inequality does not hold has area $< M_k \varepsilon_\ell \deg_\omega(C_\ell)/\eta$ with respect to $\nu_\ell^\star\omega$. By (8.5), we then get

$$\begin{aligned}\mathcal{O}_{P_kT_X}(1)\cdot C_{\ell,[k]} &= \int_{\overline{C}_\ell\smallsetminus S_\eta} \nu_{\ell,[k]}^\star \Theta_{\mathcal{O}_{P_kT_X}(1)} + \int_{S_\eta} \nu_{\ell,[k]}^\star \Theta_{\mathcal{O}_{P_kT_X}(1)}\\ &\geqslant \delta' \int_{\overline{C}_\ell\smallsetminus S_\eta} \nu_\ell^\star\omega - C\int_{S_\eta}\nu_\ell^\star\omega\\ &= \big(\delta'(1-M_k\varepsilon_\ell/\eta) - CM_k\varepsilon_\ell/\eta\big)\deg_\omega(C_\ell).\end{aligned}$$

This contradicts our initial hypothesis that $\mathcal{O}_{P_kT_X}(1)\cdot C_{\ell,[k]} \leqslant \varepsilon_\ell \deg_\omega(C_\ell)$ when ε_ℓ is small enough. □

The above results lead in a natural way to the following questions, dealing with the "directed manifold case" of Kleiman's criterion (Kleiman's criterion states that a line bundle L on X is ample if and only if there exists $\varepsilon > 0$ such that $L\cdot C \geqslant \varepsilon \deg_\omega C$ for every curve $C \subset X$).

8.7. Questions. *Let (X,V) be a compact directed manifold and let L be a line bundle over X. Fix $p \in [2,+\infty]$.*

i) *Assume that*
$$\max\big(L\cdot C, \mathrm{dev}_\omega^p(C/V)\big) \geqslant \varepsilon \deg_\omega(C)$$
for every algebraic curve $C \subset X$ (and some $\varepsilon > 0$). Does L admit a smooth hermitian metric h with $(\Theta_{L,h})_{\restriction V}$ positive definite?

ii) *Assume more generally that there is an analytic subset $Y \supsetneq X$ such that* i) *holds for all curves $C \not\subset Y$. Does L admit a singular hermitian metric h with $(\Theta_{L,h})_{\restriction V}$ positive definite, and with degeneration set $\Sigma_h \subset Y$?*

iii) *Assume that there exists $\varepsilon > 0$ such that every closed curve $C \subset X$ satisfies*
$$\max\Big(-\chi(\overline{C}) - \sum_{t\in\overline{C}}(m_{k-1}(t)-1), \mathrm{dev}_\omega^p(C/V)\Big) \geqslant \varepsilon\ \deg_\omega(C).$$
Does it follow that (X,V) admits non degenerate negative k-jet (total) curvature?

The answer to 8.7 i) is positive if V is the vertical tangent sheaf of a smooth map $X \to S$, and in that case one can even restrict oneself to curves that are tangent to V (i.e. vertical curves): this is just the relative version of Kleiman's criterion. However, in general, it is not sufficient to deal only with curves tangent to V (if X is an abelian variety and V is a constant line subbundle of T_X with non closed leaves, the condition required for algebraic curves C is void, hence L can be taken negative on X ; then, of course, the curvature cannot be made positive along V.)

§9. Proof of the Bloch theorem

The core of the result can be expressed as a characterization of the Zariski closure of an entire curve drawn on a complex torus. The proof will be obtained as a simple consequence of the Ahlfors-Schwarz lemma (more specifically Theorem 7.8), combined with a jet bundle argument. Our argument works in fact without any algebraicity assumption on the complex tori under consideration (only the case of abelian or semi-abelian varieties seems to have been treated earlier).

9.1. Theorem. *Let Z be a complex torus and let $f : \mathbb{C} \to Z$ be a holomorphic map. Then the (analytic) Zariski closure $\overline{f(\mathbb{C})}^{\mathrm{Zar}}$ is a translate of a subtorus, i.e. of the form $a + Z'$, $a \in Z$, where $Z' \subset Z$ is a subtorus.*

The converse is of course also true: for any subtorus $Z' \subset Z$, we can choose a dense line $L \subset Z'$, and the corresponding map $f : \mathbb{C} \simeq a+L \hookrightarrow Z$ has Zariski closure $\overline{f(\mathbb{C})}^{\mathrm{Zar}} = a+Z'$.

Proof (based on the ideas of [GrGr80]). Let $f : \mathbb{C} \to Z$ be an entire curve and let X be the Zariski closure of $f(\mathbb{C})$. We denote by $Z_k = P_k(T_Z)$ the k-jet bundle of Z and by X_k the closure of $X_k^{\mathrm{reg}} = P_k(T_{X^{\mathrm{reg}}})$ in Z_k. As T_Z is trivial, we have $Z_k = Z \times \mathbb{R}_{n,k}$ where $\mathbb{R}_{n,k}$ is the rational variety introduced in § 5. By Proposition 6.16 iii), there is a weight $\boldsymbol{a} \in \mathbb{N}^k$ such that $\mathcal{O}_{Z_k}(\boldsymbol{a})$ is relatively very ample. This means that there is a very ample line bundle $\mathcal{O}_{\mathbb{R}_{n,k}}(\boldsymbol{a})$ over $\mathbb{R}_{n,k}$ such that $\mathcal{O}_{Z_k}(\boldsymbol{a}) = \mathrm{pr}_2^\star\,\mathcal{O}_{\mathbb{R}_{n,k}}(\boldsymbol{a})$. Consider the map $\Phi_k : X_k \to \mathbb{R}_{n,k}$ which is the restriction to X_k of the second projection $Z_k \to \mathbb{R}_{n,k}$. By fonctoriality, we have $\mathcal{O}_{X_k}(\boldsymbol{a}) = \Phi_k^\star \mathcal{O}_{\mathbb{R}_{n,k}}(\boldsymbol{a})$.

Define $B_k \subset X_k$ to be the set of points $x \in X_k$ such that the fiber of Φ_k through x is positive dimensional. Assume that $B_k \neq X_k$. By Proposition 7.2 ii), $\mathcal{O}_{X_k}(\boldsymbol{a})$ carries a hermitian metric with degeneration set B_k and with strictly positive definite curvature on X_k (if necessary, blow-up X_k along the singularities and push the metric forward). Theorem 7.8 shows that $f_{[k]}(\mathbb{C}) \subset B_k$, and this is of course also true if $B_k = X_k$. The inclusion $f_{[k]}(\mathbb{C}) \subset B_k$ means that through every point $f_{[k]}(t_0)$ there is a germ of positive dimensional variety in the fiber $\Phi_k^{-1}(\Phi_k(f_{[k]}(t_0)))$, say a germ of curve $t' \mapsto u(t') = (z(t'), j_k) \in X_k \subset Z \times \mathbb{R}_{n,k}$ with $u(0) = f_{[k]}(t_0) = (z_0, j_k)$ and $z_0 = f(t_0)$. Then $(z(t'), j_k)$ is the image of $f_{[k]}(t_0)$ by the k-th lifting of the translation $\tau_s : z \mapsto z + s$ defined by $s = z(t') - z_0$. Now, we have $f(\mathbb{C}) \not\subset X^{\mathrm{sing}}$ since X is the Zariski closure of $f(\mathbb{C})$, and we may therefore choose t_0 so that $f(t_0) \in X^{\mathrm{reg}}$ and $f(t_0)$ is a regular point. Let us define

$$A_k(f) = \big\{s \in Z \ :\ f_{[k]}(t_0) \in P_k(X) \cap P_k(\tau_{-s}(X))\big\}.$$

Clearly $A_k(f)$ is an analytic subset of Z containing the curve $t' \mapsto s(t') = z(t') - z_0$ through 0. Since

$$A_1(f) \supset A_2(f) \supset \cdots \supset A_k(f) \supset \cdots,$$

the Noetherian property shows that the sequence stabilizes at some $A_k(f)$. Therefore, there is a curve $D(0,r) \to Z$, $t' \mapsto s(t')$ such that the infinite jet j_∞ defined by f at t_0 is $s(t')$-translation invariant for all t'. By uniqueness of analytic continuation, we conclude that $s(t') + f(t) \in X$ for all $t \in \mathbb{C}$ and $t' \in D(0,r)$. As X is the Zariski closure of $f(\mathbb{C})$, we must have $s(t') + X \subset X$ for all $t' \in D(0,r)$; also, X is irreducible, thus we have in fact $s(t') + X = X$. Define

$$W = \big\{s \in Z\,;\ s + X = X\big\}.$$

Then W is a closed positive dimensional subgroup of Z. Let $p : Z \to Z/W$ be the quotient map. As Z/W is a complex torus with $\dim Z/W < \dim Z$, we conclude by induction on dimension that the curve $\widehat{f} = p \circ f : \mathbb{C} \to Z/W$ has its Zariski closure $\widehat{X} := \overline{\widehat{f}(\mathbb{C})}^{\mathrm{Zar}} = p(X)$ equal to a translate $\widehat{s} + \widehat{T}$ of some subtorus $\widehat{T} \subset Z/W$. Since X is W-invariant, we get $X = s + p^{-1}(\widehat{T})$, where $p^{-1}(\widehat{T})$ is a closed subgroup of Z. This implies that X is a translate of a subtorus, as expected. □

We now state two simple corollaries, and then the "Bloch theorem" itself (see also [Och77], [Nog77, 81, 84], [Kaw80] for other approaches in the algebraic case).

9.2. Corollary. *Let X be a complex analytic subvariety in a complex torus Z. Then X is hyperbolic if and only if X does not contain any translate of a subtorus.*

9.3. Corollary. *Let X be a complex analytic subvariety of a complex torus Z. Assume that X is not a translate of a subtorus. Then every entire curve drawn in X is analytically degenerate.*

9.4. Bloch theorem. *Let X be a compact complex Kähler variety such that the irregularity $q = h^0(X, \Omega^1_X)$ is larger than the dimension $n = \dim X$. Then every entire curve drawn in X is analytically degenerate.*

Here X may be singular and Ω^1_X can be defined in any reasonable way (direct image of the $\Omega^1_{\widehat{X}}$ of a desingularization $\widehat{X}$ or direct image of Ω^1_U where U is the set of regular points in the normalization of X).

Proof. By blowing-up, we may assume that X is smooth. Then the Albanese map $\alpha : X \to \mathrm{Alb}(X)$ sends X onto a proper subvariety $Y \subset \mathrm{Alb}(X)$ (as $\dim Y \leqslant \dim X < \dim \mathrm{Alb}(X)$), and Y is not a translate of a subtorus by the universal property of the Albanese map. Hence, for every entire curve $f : \mathbb{C} \to X$ we infer that $\alpha \circ f : \mathbb{C} \to Y$ is analytically degenerate; it follows that f itself is analytically degenerate. □

§10. Projective meromorphic connections and Wronskians

We describe here an important method introduced by Siu [Siu87] and later developped by Nadel [Nad89], which is powerful enough to provide explicit examples of algebraic hyperbolic surfaces. It yields likewise interesting results about the algebraic degeneration of entire curves in higher dimensions. The main idea is to use meromorphic connections with low pole orders, and the associated Wronskian operators. In this way, Nadel produced examples of hyperbolic surfaces in $\mathbb{P}^3$ for any degree of the form $p = 6k + 3 \geqslant 21$. We present here a variation of Nadel's method, based on the more general concept of *partial projective connection*, which allows us to extend his result to all degrees $p \geqslant 11$. This approach is inspired from a recent work of J. El Goul [EG96], and is in some sense a formalization of his strategy.

Let X be a complex n-dimensional manifold. A *meromorphic connection* ∇ on T_X is a $\mathbb{C}$-linear sheaf morphism

$$\mathcal{M}(U, T_X) \longrightarrow \mathcal{M}(U, \Omega^1_X \otimes T_X)$$

(where $\mathcal{M}(U, \bullet)$ stands for meromorphic sections over U), satisfying the Leibnitz rule

$$\nabla(fs) = df \otimes s + f\nabla s$$

whenever $f \in \mathcal{M}(U)$ (resp. $s \in \mathcal{M}(U, T_X)$) is a meromorphic function (resp. section of T_X). Let $(z_1, \ldots, z_n)$ be holomorphic local coordinates on an open set $U \subset X$. The *Christoffel symbols* of ∇ with respect to these coordinates are the coefficients $\Gamma^\lambda_{j\mu}$ such that

$$\Gamma^\lambda_\mu = \sum_{1 \leqslant j \leqslant n} \Gamma^\lambda_{j\mu} dz_j = \lambda\text{-th component of } \nabla\Big(\frac{\partial}{\partial z_\mu}\Big).$$

The associated *connection form* on U is the tensor

$$\Gamma = \sum_{1\leqslant j,\lambda,\mu\leqslant n} \Gamma^\lambda_{j\mu}\, dz_j \otimes dz_\mu \otimes \frac{\partial}{\partial z_\lambda} \in \mathcal{M}(U, T^\star_X \otimes T^\star_X \otimes T_X).$$

Then, for all local sections $v = \sum_{1\leqslant\lambda\leqslant n} v_\lambda \frac{\partial}{\partial z_\lambda}$, $w = \sum_{1\leqslant\lambda\leqslant n} w_\lambda \frac{\partial}{\partial z_\lambda}$ of $\mathcal{M}(U, T_X)$, we get

$$\nabla v = \sum_{1\leqslant\lambda\leqslant n} \Big(dv_\lambda + \sum_{1\leqslant\mu\leqslant n} \Gamma^\lambda_\mu v_\mu\Big)\frac{\partial}{\partial z_\lambda} = dv + \Gamma\cdot v,$$
$$\nabla_w v = \sum_{1\leqslant j,\lambda\leqslant n} \Big(w_j \frac{\partial v_\lambda}{\partial z_j} + \sum_{1\leqslant\mu\leqslant n} \Gamma^\lambda_{j\mu} w_j v_\mu\Big)\frac{\partial}{\partial z_\lambda} = d_w v + \Gamma\cdot(w,v).$$

The connection ∇ is said to be *symmetric* if it satisfies $\nabla_v w - \nabla_w v = [v,w]$, or equivalently, if the Christoffel symbols $\Gamma^\lambda_{j\mu} = \Gamma^\lambda_{\mu j}$ are symmetric in j, μ.

We now turn ourselves to the important concept of *Wronskian operator*. Let B be the divisor of poles of ∇, that is, the divisor of the least common multiple of all denominators occuring in the meromorphic functions $\Gamma^\lambda_{j\mu}$. If $\beta \in H^0(X, \mathcal{O}(B))$ is the canonical section of divisor B, then the operator $\beta\nabla$ has holomorphic coefficients. Given a holomorphic curve $f : D(0,r) \to X$ whose image does not lie in the support $|B|$ of B, one can define inductively a sequence of covariant derivatives

$$f', \quad f''_\nabla = \nabla_{f'}(f'),\ \ldots\ ,\ f^{(k+1)}_\nabla := \nabla_{f'}(f^{(k)}_\nabla).$$

These derivatives are given in local coordinates by the explicit inductive formula

$$f^{(k+1)}_\nabla(t)_\lambda = \frac{d}{dt}\big(f^{(k)}_\nabla(t)_\lambda\big) + \sum_{1\leqslant\mu\leqslant n} (\Gamma^\lambda_{j\mu}\circ f)\, f'_j\, f^{(k)}_\nabla(t)_\mu. \tag{10.1}$$

Therefore, if $\operatorname{Im} f \not\subset |B|$, one can define the *Wronskian* of f relative to ∇ as

$$W_\nabla(f) = f' \wedge f''_\nabla \wedge \cdots \wedge f^{(n)}_\nabla. \tag{10.2}$$

Clearly, $W_\nabla(f)$ is a meromorphic section of $f^\star(\Lambda^n T_X)$. By induction $\beta(f)^{k-1} f^{(k)}_\nabla$ is holomorphic for all $k \geqslant 1$. We infer that $\beta(f)^{n(n-1)/2} W_\nabla(f)$ is holomorphic and can be seen as a holomorphic section of the line bundle $f^\star(\Lambda^n T_X \otimes \mathcal{O}_X(\frac12 n(n-1)B)$. From (10.1) and (10.2) we see that $P = \beta^{n(n-1)/2} W_\nabla$ is a global holomorphic polynomial operator $f \mapsto P(f', f'', \ldots, f^{(n)})$ of order n and total degree $n(n+1)/2$, with values in $\Lambda^n T_X \otimes \mathcal{O}_X(\frac12 n(n-1)B)$. Moreover, if we take a biholomorphic reparametrization φ, we get inductively

$$(f\circ\varphi)^{(k)}_\nabla = (\varphi')^k f^{(k)}_\nabla \circ \varphi + \text{linear combination of } f^{(j)}_\nabla \circ \varphi,\ \ j < k.$$

Therefore

$$W_\nabla(f\circ\varphi) = (\varphi')^{n(n+1)} W_\nabla(f)$$

and $\beta^{n(n-1)/2} W_\nabla$ can be viewed as a section

$$\beta^{n(n-1)/2} W_\nabla \in H^0(X, E_{n,n(n+1)/2} T^\star_X \otimes L^{-1}), \tag{10.3}$$

where L is the line bundle

$$L = K_X \otimes \mathcal{O}_X\Big(-\frac{1}{2}n(n-1)B\Big).$$

From this, we get the following theorem, which is essentially due to [Siu87] (with a more involved proof based on suitable generalizations of Nevanlinna's second main theorem).

10.4. Theorem (Y.T. Siu). *Let X be a compact complex manifold equipped with a meromorphic connection ∇ of pole divisor B. If $K_X \otimes \mathcal{O}_X(-\frac{1}{2}n(n-1)B)$ is ample, then for every non constant entire curve $f : \mathbb{C} \to X$, one has either $f(\mathbb{C}) \subset |B|$ or $W_\nabla(f) \equiv 0$.*

Proof. By Corollary 7.9 applied with $P = \beta^{n(n-1)/2} W_\nabla$, we conclude that

$$\beta^{n(n-1)/2}(f) W_\nabla(f) \equiv 0,$$

whence the result. □

10.5. Basic observation. It is not necessary to know all Christoffel coefficients of the meromorphic connection ∇ in order to be able to compute its Wronskian W_∇. In fact, assume that $\widetilde{\nabla}$ is another connection such that there are meromorphic 1-forms α, β with

$$\begin{aligned} \widetilde{\nabla} &= \nabla + \alpha \otimes \mathrm{Id}_{T_X} + (\beta \otimes \mathrm{Id}_{T_X})_{\tau_{12}}, \qquad \text{i.e.,} \\ \widetilde{\nabla}_w v &= \nabla_w v + \alpha(w) v + \beta(v) w, \end{aligned}$$

where τ_{12} means transposition of first and second arguments in the tensors of $T_X^\star \otimes T_X^\star \otimes T_X$. Then $W_\nabla = W_{\widetilde{\nabla}}$. Indeed, the defining formula $f_{\widetilde{\nabla}}^{(k+1)} = \widetilde{\nabla}_{f'}(f_{\widetilde{\nabla}}^{(k)})$ implies that $f_{\widetilde{\nabla}}^{(k+1)} = \nabla_{f'}(f_{\widetilde{\nabla}}^{(k)}) + \alpha(f') f_{\widetilde{\nabla}}^{(k)} + \beta(f_{\widetilde{\nabla}}^{(k)}) f'$, and an easy induction then shows that the $\widetilde{\nabla}$ derivatives can be expressed as linear combinations with meromorphic coefficients

$$f_{\widetilde{\nabla}}^{(k)}(t) = f_\nabla^{(k)}(t) + \sum_{1 \leqslant j < k} \gamma_j(t)\, f_\nabla^{(j)}(t). \qquad \square$$

The essential consequence of Remark 10.5 is that we need only have a "partial projective connection" ∇ on X, in the following sense.

10.6. Definition. *A (meromorphic) partial projective connection ∇ on X is a section of the quotient sheaf of meromorphic connections modulo addition of meromorphic tensors in $(\Omega_X^1 \otimes \mathrm{Id}_{T_X}) \oplus (\Omega_X^1 \otimes \mathrm{Id}_{T_X})_{\tau_{12}}$. In other words, it can be defined as a collection of meromorphic connections ∇_j relative to an open covering (U_j) of X, satisfying the compatibility conditions*

$$\nabla_k - \nabla_j = \alpha_{jk} \otimes \mathrm{Id}_{T_X} + (\beta_{jk} \otimes \mathrm{Id}_{T_X})_{\tau_{12}}$$

for suitable meromorphic 1-forms α_{jk}, β_{jk} on $U_j \cap U_k$.

If we have similar more restrictive compatibility relations with $\beta_{jk} = 0$, the connection form Γ is just defined modulo $\Omega_X^1 \otimes \mathrm{Id}_{T_X}$ and can thus be seen as a 1-form with values in the Lie algebra $\mathfrak{pgl}(n, \mathbb{C}) = \mathfrak{sl}(n, \mathbb{C})$ rather than in $\mathfrak{gl}(n, \mathbb{C})$. Such objects are sometimes referred to as "projective connections", although this terminology has been also employed in a completely different meaning. In any event, Proposition 10.4 extends (with a completely identical proof) to the more general case where ∇ is just a partial projective connection. Accordingly, the pole divisor B can be taken to be the pole divisor of the trace free part

$$\Gamma^0 = \Gamma \mod (\Omega^1_X \otimes \mathrm{Id}_{T_X}) \oplus (\Omega^1_X \otimes \mathrm{Id}_{T_X})_{\tau_{12}}.$$

Such partial projective connections occur in a natural way when one considers quotient varieties under the action of a Lie group. Indeed, let W be a complex manifold in which a connected complex Lie group G acts freely and properly (on the left, say), and let $X = W/G$ be the quotient complex manifold. We denote by $\pi : W \to X$ the projection. Given a connection $\widetilde\nabla$ on W and a local section $\sigma : U \to W$ of π, one gets an induced connection on $T_{X|U}$ by putting

$$\nabla = \pi_\star \circ (\sigma^\star \widetilde\nabla), \tag{10.7}$$

where $\sigma^\star\widetilde\nabla$ is the induced connection on $\sigma^\star T_W$ and $\pi_\star : T_W \to \pi^\star T_X$ is the projection. Of course, the connection ∇ may depend on the choice of σ, but we nevertheless have the following simple criterion ensuring that it yields an intrinsic partial projective connection.

10.8. Lemma. *Let $\widetilde\nabla = d+\widetilde\Gamma$ be a meromorphic connection on W. Assume that $\widetilde\nabla$ satisfies the following conditions:*

i) *$\widetilde\nabla$ is G-invariant;*

ii) *there are meromorphic 1-forms α, $\beta \in \mathcal{M}(W, T_{W/X})$ along the relative tangent bundle of $X \to W$, such that for all G-invariant holomorphic vector fields v, τ on W (possibly only defined locally over X) such that τ is tangent to the G-orbits, the vector fields*

$$\widetilde\nabla_\tau v - \alpha(\tau)v, \qquad \widetilde\nabla_v \tau - \beta(\tau)v$$

are again tangent to the G-orbits (α and β are thus necessarily G-invariant, and $\alpha = \beta$ if $\widetilde\nabla$ is symmetric).

Then Formula (10.7) yields a partial projective connection ∇ which is globally defined on X and independent of the choice of the local sections σ.

Proof. Since the expected conclusions are local with respect to X, it is enough to treat the case when $W = X \times G$ and G acts on the left on the second factor. Then $W/G \simeq X$ and $\pi : W \to X$ is the first projection. If d_G is the canonical left-invariant connection on G, we can write $\widetilde\nabla$ as

$$\widetilde\nabla = d_X + d_G + \widetilde\Gamma, \qquad \widetilde\Gamma = \widetilde\Gamma(x,g), \quad x \in X,\ g \in G,$$

where d_X is some connection on X, e.g. the "coordinate derivative" taken with respect to given local coordinates $(z_1, \ldots, z_n)$ on X. Then $\widetilde\nabla$ is left invariant on $W = X \times G$ if and only if $\widetilde\Gamma(x,g) = \Gamma(x)$ is independent of $g \in G$ (this is meaningful since the tangent bundle to G is trivial), and condition ii) means that

$$\Gamma(x)\cdot(\tau, v) - \alpha(\tau)v \quad \text{and} \quad \Gamma(x)\cdot(v,\tau) - \beta(\tau)v$$

are tangent to the G-orbits. A local section $\sigma : U \to W$ of π can be written $\sigma(x) = (x, h(x))$ for some holomorphic function $h : U \to G$. Formula (10.7) says more explicitly that

$$\nabla_w v = \pi_\star\big((\sigma^\star\widetilde\nabla)_w v\big) = \pi_\star\big(d_{\sigma_\star w}\sigma_\star v + (\widetilde\Gamma\circ\sigma)\cdot(\sigma_\star w, \sigma_\star v)\big).$$

Let $v = \sum v_j(z)\partial/\partial z_j$, $w = \sum w_j(z)\partial/\partial z_j$ be local vector fields on $U \subset X$. Since $\sigma_\star v = v + dh(v)$, we get

$$\begin{aligned}(\sigma^\star\widetilde{\nabla})_w v &= d_{w+dh(w)}(v + dh(v)) + \widetilde{\Gamma}(x, h(x)) \cdot \big(w + dh(w), v + dh(v)\big)\\ &= d_w v + d^2h(w, v) + \Gamma(x) \cdot (w + dh(w), v + dh(v)).\end{aligned}$$

As v, w, $dh(v)$, $dh(w)$ depend only on X, they can be seen as G-invariant vector fields over W, and $dh(v)$, $dh(w)$ are tangent to the G-orbits. Hence

$$\Gamma(x) \cdot (dh(w), v) - \alpha(dh(w))v, \quad \Gamma(x) \cdot (w, dh(v)) - \beta(dh(v))w, \quad \Gamma(x) \cdot (dh(w), dh(v))$$

are tangent to the G-orbits, i.e., in the kernel of $\pi_\star$. We thus obtain

$$\nabla_w v = \pi_\star\big((\sigma^\star\widetilde{\nabla})_w v\big) = d_w v + \Gamma(x) \cdot (w, v) + \alpha(dh(w))v + \beta(dh(v))w.$$

From this it follows by definition that the local connections $\nabla_{\restriction U_j}$ defined by various sections $\sigma_j : U_j \to W$ can be glued together to define a global partial projective connection ∇ on X. □

10.9. Remark. Lemma 10.8 is also valid when $\widetilde{\nabla}$ is a partial projective connection. Hypothesis 10.8 ii) must then hold with local meromorphic 1-forms α_j, $\beta_j \in \mathcal{M}(\widetilde{U}_j, T_{W/X})$ relatively to some open covering $\widetilde{U}_j$ of W. □

In the special case $\mathbb{P}^n = (\mathbb{C}^{n+1} \smallsetminus \{0\})/\mathbb{C}^\star$, we get

10.10. Corollary. *Let $\widetilde{\nabla} = d + \widetilde{\Gamma}$ be a meromorphic connection on $\mathbb{C}^{n+1}$. Let $\varepsilon = \sum z_j\partial/\partial z_j$ be the Euler vector field on $\mathbb{C}^{n+1}$ and $\pi : \mathbb{C}^{n+1} \smallsetminus \{0\} \to \mathbb{P}^n$ be the canonical projection. Then $\widetilde{\nabla}$ induces a meromorphic partial projective connection on $\mathbb{P}^n$ provided that*

i) *the Christoffel symbols $\Gamma^\lambda_{j\mu}$ are homogeneous rational functions of degree -1 (homothety invariance of the connection $\widetilde{\nabla}$);*

ii) *there are meromorphic functions α, β and meromorphic 1-forms γ, η such that*

$$\widetilde{\Gamma} \cdot (\varepsilon, v) = \alpha v + \gamma(v)\varepsilon, \qquad \widetilde{\Gamma} \cdot (w, \varepsilon) = \beta w + \eta(w)\varepsilon$$

for all vector fields v, w. □

Now, our goal is to study certain hypersurfaces Y of sufficiently high degree in $\mathbb{P}^n$. Assume for the moment that Y is an hypersurface in some n-dimensional manifold X, and that Y is defined locally by a holomorphic equation $s = 0$. We say that Y is *totally geodesic* with respect to a meromorphic connection ∇ on X if Y is not contained in the pole divisor $|B|$ of ∇, and for all pairs (v, w) of (local) vector fields tangent to Y the covariant derivative $\nabla_w v$ is again tangent to Y. (Notice that this concept also makes sense when ∇ is a partial projective connection.) If Y is totally geodesic, the ambient connection ∇ on T_X induces by restriction a connection $\nabla_{\restriction Y}$ on T_Y.

We now want to derive explicitly a condition for the hypersurface $Y = \{s = 0\}$ to be totally geodesic in (X, ∇). A vector field v is tangent to Y if and only if $ds \cdot v = 0$ along $s = 0$. By taking the differential of this identity along another vector field w tangent to Y, we find

$$d^2s \cdot (w, v) + ds \cdot (d_w v) = 0 \tag{10.11}$$

along $s=0$ (this is meaningful only with respect to some local coordinates). On the other hand, the condition that $\nabla_w v = d_w v + \Gamma\cdot(w,v)$ is tangent to Y is

$$ds\cdot\nabla_w v = ds\cdot(d_w v) + ds\circ\Gamma\cdot(w,v) = 0.$$

By subtracting the above from (10.11), we get the following equivalent condition: $(d^2s - ds\circ\Gamma)\cdot(w,v)=0$ for all vector fields v,w in the kernel of ds along $s=0$. Therefore we obtain the

10.12. Characterization of totally geodesic hypersurfaces. *The hypersurface $Y=\{s=0\}$ is totally geodesic with respect to ∇ if and only if there are holomorphic 1-forms $a=\sum a_j dz_j$, $b=\sum b_j dz_j$ and a 2-form $c=\sum c_{j\mu}dz_j\otimes dz_\mu$ such that*

$$\nabla^\star(ds) = d^2s - ds\circ\Gamma = a\otimes ds + ds\otimes b + s\,c$$

in a neighborhood of every point of Y (here $\nabla^\star$ is the induced connection on $T^\star_Y$).

From this, we derive the following useful lemma.

10.13. Lemma. *Let $Y\subset X$ be an analytic hypersurface which is totally geodesic with respect to a meromorphic connection ∇, and let $n=\dim X=\dim Y+1$. Let $f:D(0,R)\to X$ be a holomorphic curve such that $W_\nabla(f)\equiv 0$. Assume that there is a point $t_0\in D(0,R)$ such that*

i) *$f(t_0)$ is not contained in the poles of ∇;*

ii) *the system of vectors $(f'(t), f''_\nabla(t),\ldots,f^{(n-1)}_\nabla(t))$ achieves its generic rank (i.e. its maximal rank) at $t=t_0$;*

iii) *$f(t_0)\in Y$ and $f'(t_0)$, $f''_\nabla(t_0),\ldots,f^{(n-1)}_\nabla(t_0)\in T_{Y,f(t_0)}$.*

Then $f(D(0,R))\subset Y$.

Proof. Since $W_\nabla(f)\equiv 0$, the vector fields f', $f''_\nabla,\ldots,f^{(n)}_\nabla$ are linearly dependent and satisfy a non trivial relation

$$u_1(t)f'(t)+u_2(t)f''_\nabla(t)+\cdots+u_n(t)f^{(n)}_\nabla(t)=0$$

with suitable meromorphic coefficients $u_j(t)$ on $D(0,R)$. If u_n happens to be $\equiv 0$, we take ∇-derivatives in the above relation so as to reach another relation with $u_n\not\equiv 0$. Hence we can always write

$$f^{(n)}_\nabla = v_1 f' + v_2 f''_\nabla + \cdots + v_{n-1}f^{(n-1)}_\nabla$$

for some meromorphic functions $v_1,\ldots,v_{n-1}$. We can even prescribe the v_j to be 0 eXcept for indices $j=j_k\in\{1,\ldots,n-1\}$ such that $(f^{(j_k)}_\nabla(t))$ is a minimal set of generators at $t=t_0$. Then the coefficients v_j are uniquely defined and are holomorphic near t_0. By taking further derivatives, we conclude that $f^{(k)}_\nabla(t_0)\in T_{X,f(t_0)}$ for all k. We now use the assumption that X is totally geodesic to prove the following claim: if $s=0$ is a local equation of Y, the k-th derivative $\frac{d^k}{dt^k}(s\circ f(t))$ can be expressed as a holomorphic linear combination

$$\frac{d^k}{dt^k}\big(s\circ f(t)\big) = \gamma_{0k}(t)\,s\circ f(t) + \sum_{1\leqslant j\leqslant k}\gamma_{jk}(t)\,ds_{f(t)}\cdot f^{(j)}_\nabla(t)$$

on a neighborhood of t_0. This will imply $\frac{d^k}{dt^k}(s\circ f)(t_0) = 0$ for all $k \geqslant 0$, hence $s\circ f \equiv 0$. Now, the above claim is clearly true for $k = 0, 1$. By taking the derivative and arguing inductively, we need only show that

$$\frac{d}{dt}\big(ds_{f(t)} \cdot f_\nabla^{(j)}(t)\big)$$

is again a linear combination of the same type. However, Leibnitz's rule for covariant differentiations together with 10.12 yield

$$\begin{aligned}\frac{d}{dt}\big(ds_{f(t)} \cdot f_\nabla^{(j)}(t)\big) &= ds_{f(t)} \cdot \Big(\frac{\nabla}{dt} f_\nabla^{(j)}(t)\Big) + \nabla^\star(ds)_{f(t)} \cdot \big(f'(t), f_\nabla^{(j)}(t)\big)\\ &= ds \cdot f_\nabla^{(j+1)}(t) + (a\cdot f'(t))\big(ds \cdot f_\nabla^{(j)}(t)\big)\\ &\qquad + (ds\cdot f'(t))\big(b\cdot f_\nabla^{(j)}(t)\big) + (s\circ f(t))\big(c\cdot (f'(t), f_\nabla^{(j)}(t))\big),\end{aligned}$$

as desired. □

If $Y = \{s = 0\} \subset X$ is given and a connection ∇ on X is to be found so that Y is totally geodesic, condition 10.12 amounts to solving a highly underdetermined linear system of equations

$$\frac{\partial^2 s}{\partial z_j \partial z_\mu} - \sum_{1\leqslant\lambda\leqslant n} \Gamma_{j\mu}^\lambda \frac{\partial s}{\partial z_\lambda} = a_j \frac{\partial s}{\partial z_\mu} + b_\mu \frac{\partial s}{\partial z_j} + s\, c_{j\mu}, \qquad 1 \leqslant j, \mu \leqslant n,$$

in terms of the unknowns $\Gamma_{j\mu}^\lambda$, a_j, b_μ and $c_{j\mu}$. Nadel's idea is to take advantage of this indeterminacy to achieve that all members in a large linear system (Y_α) of hypersurfaces are totally geodesic with respect to ∇. The following definition is convenient.

10.14. Definition. *For any $(n+2)$-tuple of integers $(p, k_0, k_1 \ldots, k_n)$ with $0 < k_j < p/2$, let $\mathcal{S}_{p;\, k_0,\ldots,k_n}$ be the space of homogeneous polynomials $s \in \mathbb{C}[z_0, z_1, \ldots, z_n]$ of degree p such that every monomial of s is a product of a power $z_j^{p-k_j}$ of one of the variables with a lower degree monomial of degree k_j. Any polynomial $s \in \mathcal{S}_{p;\, k_0,\ldots,k_n}$ admits a unique decomposition*

$$s = s_0 + s_1 + \cdots + s_n, \qquad s_j \in \mathcal{S}_{p;\, k_0,\ldots,k_n}$$

where s_j is divisible by $z_j^{p-k_j}$.

Given a homogeneous polynomial $s = s_0 + s_1 + \cdots + s_n \in \mathcal{S}_{p;\, k_0,\ldots,k_n}$, we consider the linear system

$$Y_\alpha = \big\{\alpha_0 s_0 + \alpha_1 s_1 + \cdots + \alpha_n s_n = 0\big\}, \quad \alpha = (\alpha_0, \ldots, \alpha_n) \in \mathbb{C}^n. \tag{10.15}$$

Our goal is to study smooth varieties Z which arise as complete intersections $Z = Y_{\alpha^1} \cap \cdots \cap Y_{\alpha^q}$ of members in the linear system (the α^j being linearly independent elements in $\mathbb{C}^{n+1}$). For this, we want to construct a (partial projective) meromorphic connection ∇ on $\mathbb{P}^n$ such that all Y_α are totally geodesic. Corollary 10.10 shows that it is enough to construct a meromorphic connection $\widetilde\nabla = d + \widetilde\Gamma$ on $\mathbb{C}^{n+1}$ satisfying 10.10 i) and ii), such that the conic affine varieties $\widetilde Y_\alpha \subset \mathbb{C}^{n+1}$ lying over the Y_α are totally geodesic with respect to $\widetilde\nabla$. Now, Characterization 10.12 yields a sufficient condition in terms of the linear system of equations

$$\sum_{0\leqslant\lambda\leqslant n} \widetilde\Gamma_{j\mu}^\lambda \frac{\partial s_\kappa}{\partial z_\lambda} = \frac{\partial^2 s_\kappa}{\partial z_j \partial z_\mu}, \qquad 0 \leqslant j, \kappa, \mu \leqslant n. \tag{10.16}$$

(We just fix the choice of a_j, b_μ and $c_{j\mu}$ to be 0). This linear system can be considered as a collection of decoupled linear systems in the unknowns $(\widetilde{\Gamma}^\lambda_{j\mu})_\lambda$, when j and μ are fixed. Each of these has format $(n+1)\times(n+1)$ and can be solved by Cramer's rule if the principal determinant

$$\delta := \det\Big(\frac{\partial s_\kappa}{\partial z_\lambda}\Big)_{0\leqslant\kappa,\lambda\leqslant n} \not\equiv 0 \tag{10.17}$$

is not identically zero. We always assume in the sequel that this non degeneracy assumption is satisfied. As $\partial s_\kappa/\partial z_\lambda$ is homogeneous of degree $p-1$ and $\partial^2 s_\kappa/\partial z_j\partial z_\mu$ is homogeneous of degree $p-2$, the solutions $\widetilde{\Gamma}^\lambda_{j\mu}(z)$ are homogeneous rational functions of degree -1 (condition 10.10 i)). Moreover, $\widetilde{\nabla}$ is symmetric, for $\partial^2 s/\partial z_j\partial z_\mu$ is symmetric in j,μ. Finally, if we multiply (10.16) by z_j and take the sum, Euler's identity yields

$$\sum_{0\leqslant j,\lambda\leqslant n} z_j\widetilde{\Gamma}^\lambda_{j\mu}\frac{\partial s_\kappa}{\partial z_\lambda} = \sum_{0\leqslant j\leqslant n} z_j\frac{\partial^2 s_\kappa}{\partial z_j\partial z_\mu} = (p-1)\frac{\partial s_\kappa}{\partial z_\mu}, \qquad 0\leqslant\kappa,\mu\leqslant n.$$

The non degeneracy assumption implies $(\sum_j z_j\widetilde{\Gamma}^\lambda_{j\mu})_{\lambda\mu} = (p-1)\,\mathrm{Id}$, hence

$$\widetilde{\Gamma}(\varepsilon,v) = \widetilde{\Gamma}(v,\varepsilon) = (p-1)v$$

and condition 10.10 ii) is satisfied. From this we infer

10.18. Proposition. *Let $s = s_0+\cdots+s_n \in \mathcal{S}_{p;\,k_0,\ldots,k_n}$ be satisfying the non degeneracy condition $\delta := \det(\partial s_\kappa/\partial z_\lambda)_{0\leqslant\kappa,\lambda\leqslant n}\not\equiv 0$. Then the solution $\widetilde{\Gamma}$ of the linear system* (10.16) *provides a partial projective meromorphic connection on $\mathbb{P}^n$ such that all hypersurfaces*

$$Y_\alpha = \{\alpha_0 s_0+\cdots+\alpha_n s_n = 0\}$$

are totally geodesic. Moreover, the divisor of poles B of ∇ has degree at most equal to $n+1+\sum k_j$.

Proof. Only the final degree estimate on poles has to be checked. By Cramer's rule, the solutions are expressed in terms of ratios

$$\widetilde{\Gamma}^\lambda_{j\mu} = \frac{\delta^\lambda_{j\mu}}{\delta},$$

where $\delta^\lambda_{j\mu}$ is the determinant obtained by replacing the column of $\det(\partial s_\kappa/\partial z_\lambda)_{0\leqslant\kappa,\lambda\leqslant n}$of index λ by the column $(\partial^2 s_\kappa/\partial z_j\partial z_\mu)_{0\leqslant\kappa\leqslant n}$. Now, $\partial s_\kappa/\partial z_\lambda$ is a homogeneous polynomial of degree $p-1$ which is divisible by $z_\kappa^{p-k_\kappa-1}$, hence δ is a homogeneous polynomial of degree $(n+2)(p-1)$ which is divisible by $\prod z_j^{p-k_j-1}$. Similarly, $\partial^2 s_\kappa/\partial z_j\partial z_\mu$ has degree $p-2$ and is divisible by $z_\kappa^{p-k_\kappa-2}$. This implies that $\delta^\lambda_{j\mu}$ is divisible by $\prod z_j^{p-k_j-2}$. After removing this common factor in the numerator and denominator, we are left with a denominator of degree

$$\sum_{0\leqslant j\leqslant n}\big((p-1)-(p-k_j-2)\big) = \sum(k_j+1) = n+1+\sum k_j,$$

as stated. □

An application of Theorem 10.4 then yields the following theorem on certain complete intersections in projective spaces.

10.19. Theorem. *Let $s \in \mathcal{S}_{p;\,k_0,\ldots,k_{n+q}} \subset \mathbb{C}[z_0, z_1, \ldots, z_{n+q}]$ be a homogeneous polynomial satisfying the non degeneracy assumption* $\det(\partial s_\kappa/\partial z_\lambda) \not\equiv 0$ *in $\mathbb{C}^{n+q+1}$. Let*

$$Y_\alpha = \big\{\alpha_0 s_0 + \alpha_1 s_1 + \cdots + \alpha_{n+q} s_{n+q} = 0\big\} \subset \mathbb{P}^{n+q}$$

be the corresponding linear system, and let

$$Z = Y_{\alpha^1} \cap \cdots \cap Y_{\alpha^q} \subset \mathbb{P}^{n+q}$$

be a smooth n-dimensional complete intersection, for some linearly independent elements $\alpha^j \in \mathbb{C}^{n+q+1}$ such that $ds_{\alpha^1} \wedge \cdots \wedge ds_{\alpha^q}$ does not vanish along Z. Assume that Z is not contained in the set of poles $|B|$ of the meromorphic connection ∇ defined by (10.16)*, nor in any of the coordinate hyperplanes $z_j = 0$, and that*

$$pq > n + q + 1 + \frac{1}{2} n(n-1)\Big(n + q + 1 + \sum k_j\Big).$$

Then every nonconstant entire curve $f : \mathbb{C} \to Z$ is algebraically degenerate and satisfies either

i) $f(\mathbb{C}) \subset Z \cap |B|$ *or*

ii) $f(\mathbb{C}) \subset Z \cap Y_\alpha$ *for some member Y_α which does not contain Z.*

Proof. By Proposition 10.18, the pole divisor of ∇ has degree at most equal to $n+q+1+\sum k_j$, hence, if we let $B = \mathcal{O}(n + q + 1 + \sum k_j)$, we can find a section $\beta \in H^0(\mathbb{P}^{n+q}, B)$ such that the operator $f \mapsto \beta^{n(n+1)/2}(f)\, W_{Z,\nabla}(f)$ is holomorphic. Moreover, as Z is smooth, the adjunction formula yields

$$K_Z = \big(K_{\mathbb{P}^{n+q}} \otimes \mathcal{O}(pq)\big)_{\restriction Z} = \mathcal{O}_Z(pq - n - q - 1).$$

By (10.3), the differential operator $\beta^{n(n-1)/2}(f)\, W_{Z,\nabla}(f)$ defines a section in $H^0(Z, E_{n,n(n+1)/2} T^\star_Z \otimes L^{-1})$ with

$$\begin{aligned} L &= K_Z \otimes \mathcal{O}_Z\Big(-\frac{1}{2} n(n-1) B\Big) \\ &= \mathcal{O}_Z\Big(pq - n - q - 1 - \frac{1}{2} n(n-1)\Big(n + q + 1 + \sum k_j\Big)\Big). \end{aligned}$$

Hence, if $f(\mathbb{C}) \not\subset |B|$, we know by Theorem 10.4 that $W_{Z,\nabla}(f) \equiv 0$. Fix a point $t_0 \in \mathbb{C}$ such that $f(t_0) \notin |B|$ and $(f'(t_0), f''_\nabla(t_0), \ldots, f^{(n)}_\nabla(t_0))$ is of maximal rank $r < n$. There must exist an hypersurface $Y_\alpha \not\supset Z$ such that

$$f(t_0) \in Y_\alpha, \quad f'(t_0),\ f''_\nabla(t_0), \ldots,\ f^{(n)}_\nabla(t_0) \in T_{Y_\alpha, f(t_0)}.$$

In fact, these conditions amount to solve a linear system of equations

$$\sum_{0 \leqslant j \leqslant n+q} \alpha_j s_j(f(t_0)) = 0, \qquad \sum_{0 \leqslant j \leqslant n+q} \alpha_j ds_j(f^{(j)}_\nabla(t_0)) = 0$$

in the unknowns $(\alpha_0, \alpha_1, \ldots, \alpha_{n+q}) = \alpha$, which has rank $\leqslant r + 1 \leqslant n$. Hence the solutions form a vector space Sol of dimension at least $q + 1$, and we can find a solution α which is linearly independent from $\alpha^1, \ldots, \alpha^q$. We complete $(\alpha, \alpha^1, \ldots, \alpha^q)$ into a basis of $\mathbb{C}^{n+q+1}$ and use the fact that the determinant $\delta = \det(\partial s_\kappa/\partial s_\lambda)$ does not vanish identically on Z, since

$$Z \cap \{\delta = 0\} \subset Z \cap \big(|B| \cup \{\textstyle\prod z_j = 0\}\big) \subsetneq Z.$$

From this we see that $\sum \alpha_j ds_j$ does not vanish identically on Z, in particular $Z \not\subset Y_\alpha$. By taking a generic element $\alpha \in \mathrm{Sol}$, we get a smooth n-dimensional hypersurface $Z_\alpha = Y_\alpha \cap Y_{\alpha^2} \cap \cdots \cap Y_{\alpha^q}$ in $W = Y_{\alpha^2} \cap \cdots \cap Y_{\alpha^q}$. Lemma 10.13 applied to the pair (Z_α, W) shows that $f(\mathbb{C}) \subset Z_\alpha$, hence $f(\mathbb{C}) \subset Z \cap Z_\alpha = Z \cap Y_\alpha$, as desired. $\square$

If we want to decide whether Z is hyperbolic, we are thus reduced to decide whether the hypersurfaces $Z \cap |B|$ and $Z \cap Y_\alpha$ are hyperbolic. This may be a very hard problem, especially if $Z \cap |B|$ and $Z \cap Y_\alpha$ are singular. (In the case of a smooth intersection $Z \cap Y_\alpha$, we can of course apply the theorem again to $Z' = Z \cap Y_\alpha$ and try to argue by induction). However, when Z is a surface, $Z \cap |B|$ and $Z \cap Y_\alpha$ are curves and the problem can in principle be solved directly through explicit genus calculations.

10.20. Examples.

i) Consider the Fermat hypersurface of degree p

$$Z = \big\{z_0^p + z_1^p + \cdots + z_{n+1}^p = 0\big\}$$

in $\mathbb{P}^{n+1}$, which is defined by an element in $\mathcal{S}_{p;\,0,\ldots,0}$. A simple calculation shows that $\delta = p^{n+2} \prod z_j^{p-1} \not\equiv 0$ and that the Christoffel symbols are given by $\widetilde{\Gamma}^j_{jj} = (p-1)/z_j$ (with all other coefficients being equal to 0). Theorem 10.19 shows that all nonconstant entire curves $f : \mathbb{C} \to Y$ are algebraically degenerate when

$$p > n + 2 + \frac{1}{2} n(n-1)(n+2).$$

In fact the term $\frac{1}{2} n(n-1)(n+2)$ coming from the pole order estimate of the Wronskian is by far too pessimistic. A more precise calculation shows in that case that $(z_0 \cdots z_{n+1})^{n-1}$ can be taken as a denominator for the Wronskian. Hence the algebraic degeneracy occurs for $p > n+2+(n+2)(n-1)$, i.e., $p \geqslant (n+1)^2$. However, the Fermat hypersurfaces are not hyperbolic. For instance, when $n = 2$, they contain rational lines $z_1 = \omega z_0$, $z_3 = \omega' z_2$ associated with any pair (ω, ω') of p-th roots of -1.

ii) Following J. El Goul ([EG96, 97]), let us consider surfaces $Z \subset \mathbb{P}^3$ of the form

$$Z = \big\{z_0^p + z_1^p + z_2^p + z_3^{p-2}(\varepsilon_0 z_0^2 + \varepsilon_1 z_1^2 + \varepsilon_2 z_2^2 + z_3^2) = 0\big\},$$

defined by the element in $\mathcal{S}_{p;\,0,0,0,2}$ such that $s_3 = z_3^{p-2}(\varepsilon_0 z_0^2 + \varepsilon_1 z_1^2 + \varepsilon_2 z_2^2 + z_3^2)$ and $s_j = z_j^p$ for $0 \leqslant j \leqslant 2$. One can check that Z is smooth provided that

$$\sum_{j \in J} \varepsilon_j^{\frac{p}{p-2}} \neq \frac{2}{p-2}\Big(-\frac{p}{2}\Big)^{\frac{p}{p-2}}, \qquad \forall J \subset \{0,1,2\}, \tag{10.21}$$

for any choice of complex roots of order $p-2$. The connection $\widetilde{\nabla} = d + \widetilde{\Gamma}$ is computed by solving linear systems with principal determinant $\delta = \det(\partial s_\kappa/\partial z_\lambda)$ equal to

$$\begin{vmatrix} p z_0^{p-1} & 0 & 0 & 0 \\ 0 & p z_1^{p-1} & 0 & 0 \\ 0 & 0 & p z_2^{p-1} & 0 \\ 2\varepsilon_0 z_0 z_3^{p-2} & 2\varepsilon_1 z_1 z_3^{p-2} & 2\varepsilon_2 z_2 z_3^{p-2} & (p-2) z_3^{p-3}\big(\varepsilon_0 z_0^2 + \varepsilon_1 z_1^2 + \varepsilon_2 z_2^2 + \frac{p}{p-2} z_3^2\big) \end{vmatrix}$$

$$= p^3 (p-2) z_0^{p-1} z_1^{p-1} z_2^{p-1} z_3^{p-3} \Big(\varepsilon_0 z_0^2 + \varepsilon_1 z_1^2 + \varepsilon_2 z_2^2 + \frac{p}{p-2} z_3^2\Big) \not\equiv 0.$$

The numerator of $\widetilde\Gamma^\lambda_{j\mu}$ is obtained by replacing the column of index λ of δ by $(\partial^2 s_\kappa/\partial z_j\partial z_\mu)_{0\leqslant\kappa\leqslant 3}$, and $z_0^{p-2}z_1^{p-2}z_2^{p-2}z_3^{p-4}$ cancels in all terms. Hence the pole order of $\widetilde\nabla$ and of $W_{\widetilde\nabla}$ is 6 (as given by Proposition 10.18), with

$$z_0z_1z_2z_3\Big(\varepsilon_0z_0^2+\varepsilon_1z_1^2+\varepsilon_2z_2^2+\frac{p}{p-2}z_3^2\Big)$$

as the denominator, and its zero divisor as the divisor B. The condition on p we get is $p>n+2+6=10$. An explicit calculation shows that all curves $Z\cap|B|$ and $Z\cap Y_\alpha$ have geometric genus $\geqslant 2$ under the additional hypothesis

$$\text{(10.22)}\qquad \begin{cases}\text{none of the pairs } (\varepsilon_i,\varepsilon_j) \text{ is equal to } (0,0),\\ \varepsilon_i/\varepsilon_j\neq-\theta^2 \text{ whenever } \theta \text{ is a root of } \theta^p=-1.\end{cases}$$

[(10.22) excludes the existence of lines in the intersections $Z\cap Y_\alpha$.]

10.23. Corollary. *Under conditions* (10.21) *and* (10.22)*, the algebraic surface*

$$Z=\big\{z_0^p+z_1^p+z_2^p+z_3^{p-2}(\varepsilon_0z_0^2+\varepsilon_1z_1^2+\varepsilon_2z_2^2+z_3^2)=0\big\}\subset\mathbb{P}^3$$

is smooth and hyperbolic for all $p\geqslant 11$. □

Another question which has raised considerable interest is to decide when the complement $\mathbb{P}^2\smallsetminus C$ of a plane curve C is hyperbolic. If $C=\{\sigma=0\}$ is defined by a polynomial $\sigma(z_0,z_1,z_2)$ of degree p, we can consider the surface X in $\mathbb{P}^3$ defined by $z_3^p=\sigma(z_0,z_1,z_2)$. The projection

$$\rho:X\to\mathbb{P}^2,\qquad (z_0,z_1,z_2,z_3)\mapsto(z_0,z_1,z_2)$$

is a finite $p:1$ morphism, ramified along C. It follows that $\mathbb{P}^2\smallsetminus C$ is hyperbolic if and only if its unramified covering $X\smallsetminus\rho^{-1}(C)$ is hyperbolic; hence a sufficient condition is that X itself is hyperbolic. If we take $\varepsilon_2=0$ in Cor. 10.23 and exchange the roles of z_2, z_3, we get the following

10.24. Corollary. *Consider the plane curve*

$$C=\big\{z_0^p+z_1^p+z_2^{p-2}(\varepsilon_0z_0^2+\varepsilon_1z_1^2+z_2^2)=0\big\}\subset\mathbb{P}^2,\qquad \varepsilon_0,\,\varepsilon_1\in\mathbb{C}^\star.$$

Assume that neither of the numbers ε_0, ε_1, $\varepsilon_0+\varepsilon_1$ is equal to $\frac{2}{p-2}\big(-\frac{p}{2}\big)^{\frac{p}{p-2}}$ and that $\varepsilon_1/\varepsilon_0\neq-\theta^2$ whenever $\theta^p=-1$. Then $\mathbb{P}^2\smallsetminus C$ is hyperbolic. □

§11. Morse inequalities and the Green-Griffiths-Lang conjecture

The goal of this section is to study the existence and properties of entire curves $f:\mathbb{C}\to X$ drawn in a complex irreducible n-dimensional variety X, and more specifically to show that they must satisfy certain global algebraic or differential equations as soon as X is projective of general type. By means of holomorphic Morse inequalities and a probabilistic analysis of the cohomology of jet spaces, we are able to prove a significant step of a generalized version of the Green-Griffiths-Lang conjecture on the algebraic degeneracy of entire curves.

11.A. Introduction

Let X be a complex n-dimensional manifold; most of the time we will assume that X is compact and even projective algebraic. By an "entire curve" we always mean a non constant holomorphic map defined on the whole complex line $\mathbb{C}$, and we say that it is algebraically degenerate if its image is contained in a proper algebraic subvariety of the ambient variety. If $\mu : \widetilde{X} \to X$ is a modification and $f : \mathbb{C} \to X$ is an entire curve whose image $f(\mathbb{C})$ is not contained in the image $\mu(E)$ of the exceptional locus, then f admits a unique lifting $\widetilde{f} : \mathbb{C} \to \widetilde{X}$. For this reason, the study of the algebraic degeneration of f is a birationally invariant problem, and singularities do not play an essential role at this stage. We will therefore assume that X is non singular, possibly after performing a suitable composition of blow-ups. We are interested more generally in the situation where the tangent bundle T_X is equipped with a *linear subspace* $V \subset T_X$, that is, an irreducible complex analytic subset of the total space of T_X such that

(11.1) all fibers $V_x := V \cap T_{X,x}$ are vector subspaces of $T_{X,x}$.

Then the problem is to study entire curves $f : \mathbb{C} \to X$ which are tangent to V, i.e. such that $f_* T_{\mathbb{C}} \subset V$. We will refer to a pair (X, V) as being a *directed variety* (or *directed manifold*). A morphism of directed varieties $\Phi : (X, V) \to (Y, W)$ is a holomorphic map $\Phi : X \to Y$ such that $\Phi_* V \subset W$; by the irreducibility, it is enough to check this condition over the dense open subset $X \smallsetminus \mathrm{Sing}(V)$ where V is actually a subbundle. Here $\mathrm{Sing}(V)$ denotes the indeterminacy set of the associated meromorphic map $\alpha : X \dashrightarrow G_r(T_X)$ to the Grassmannian bbundle of r-planes in T_X, $r = \mathrm{rank}\, V$; we thus have $V_{|X \smallsetminus \mathrm{Sing}(V)} = \alpha^* S$ where $S \to G_r(T_X)$ is the tautological subbundle of $G_r(T_X)$. In that way, we get a category, and we will be mostly interested in the subcategory whose objects (X, V) are projective algebraic manifolds equipped with algebraic linear subspaces. Notice that an entire curve $f : \mathbb{C} \to X$ tangent to V is just a morphism $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$.

The case where $V = T_{X/S}$ is the relative tangent space of some fibration $X \to S$ is of special interest, and so is the case of a foliated variety (this is the situation where the sheaf of sections $\mathcal{O}(V)$ satisfies the Frobenius integrability condition $[\mathcal{O}(V), \mathcal{O}(V)] \subset \mathcal{O}(V)$); however, it is very useful to allow as well non integrable linear subspaces V. We refer to $V = T_X$ as being the *absolute case*. Our main target is the following deep conjecture concerning the algebraic degeneracy of entire curves, which generalizes similar statements made in [GrGr79] (see also [Lang86, Lang87]).

11.2. Generalized Green-Griffiths-Lang conjecture. *Let (X, V) be a projective directed manifold such that the canonical sheaf K_V is big (in the absolute case $V = T_X$, this means that X is a variety of general type, and in the relative case we will say that (X, V) is of general type). Then there should exist an algebraic subvariety $Y \subsetneq X$ such that every non constant entire curve $f : \mathbb{C} \to X$ tangent to V is contained in Y.*

The precise meaning of K_V and of its bigness will be explained below – our definition *does not coincide* with other frequently used definitions and is in our view better suited to the study of entire curves of (X, V). One says that (X, V) is Brody-hyperbolic when there are no entire curves tangent to V. According to (generalized versions of) conjectures of Kobayashi [Kob70, Kob76] the hyperbolicity of (X, V) should imply that K_V is big, and even possibly ample, in a suitable sense. It would then follow from conjecture (11.2) that (X, V) is hyperbolic if and only if for every irreducible variety $Y \subset X$, the linear subspace $V_{\widetilde{Y}} = \overline{T_{\widetilde{Y} \smallsetminus E} \cap \mu_*^{-1} V} \subset T_{\widetilde{Y}}$ has a big canonical sheaf whenever $\mu : \widetilde{Y} \to Y$ is a desingularization and E is the exceptional locus.

The most striking fact known at this date on the Green-Griffiths-Lang conjecture is a recent result of Diverio, Merker and Rousseau [DMR10] in the absolute case, confirming the statement when $X \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ is a generic non singular hypersurface of large degree d, with a (non optimal) sufficient lower bound $d \geqslant 2^{n^5}$. Their proof is based in an essential way on a strategy developed by Siu [Siu02, Siu04], combined with techniques of [Dem95]. Notice that if the Green-Griffiths-Lang conjecture holds true, a much stronger and probably optimal result would be true, namely all smooth hypersurfaces of degree $d \geqslant n+3$ would satisfy the expected algebraic degeneracy statement. Moreover, by results of Clemens [Cle86] and Voisin [Voi96], a (very) generic hypersurface of degree $d \geqslant 2n+1$ would in fact be hyperbolic for every $n \geqslant 2$. Such a generic hyperbolicity statement has been obtained unconditionally by McQuillan [McQu98, McQu99] when $n = 2$ and $d \geqslant 35$, and by Demailly-El Goul [DeEG00] when $n = 2$ and $d \geqslant 21$. Recently Diverio-Trapani [DT10] proved the same result when $n = 3$ and $d \geqslant 593$. By definition, proving the algebraic degeneracy means finding a non zero polynomial P on X such that all entire curves $f : \mathbb{C} \to X$ satisfy $P(f) = 0$. All known methods of proof are based on establishing first the existence of certain algebraic differential equations $P(f\,;\, f', f'', \ldots, f^{(k)}) = 0$ of some order k, and then trying to find enough such equations so that they cut out a proper algebraic locus $Y \subsetneq X$.

Let $J_k V$ be the space of k-jets of curves $f : (\mathbb{C}, 0) \to X$ tangent to V. One defines the sheaf $\mathcal{O}(E^{\mathrm{GG}}_{k,m} V^*)$ of jet differentials of order k and degree m to be the sheaf of holomorphic functions $P(z; \xi_1, \ldots \xi_k)$ on $J_k V$ which are homogeneous polynomials of degree m on the fibers of $J^k V \to X$ with respect to local coordinate derivatives $\xi_j = f^{(j)}(0)$ (see below in case V has singularities). The degree m considered here is the weighted degree with respect to the natural $\mathbb{C}^*$ action on $J^k V$ defined by $\lambda \cdot f(t) := f(\lambda t)$, i.e. by reparametrizing the curve with a homothetic change of variable. Since $(\lambda \cdot f)^{(j)}(t) = \lambda^j f^{(j)}(\lambda t)$, the weighted action is given in coordinates by

$$\lambda \cdot (\xi_1, \xi_2, \ldots, \xi_k) = (\lambda \xi_1, \lambda^2 \xi_2, \ldots, \lambda^k \xi_k). \tag{11.3}$$

One of the major tool of the theory is the following result due to Green-Griffiths [GrGr79] (see also [Blo26], [Dem95, Dem97], [SiYe96a, SiYe96c], [Siu97]), which is a strenghtening of Theorem 7.8 and Corollary 7.9.

11.4. Fundamental vanishing theorem. *Let (X, V) be a directed projective variety and $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$ an entire curve tangent to V. Then for every global section $P \in H^0(X, E^{\mathrm{GG}}_{k,m} V^* \otimes \mathcal{O}(-A))$ where A is an ample divisor of X, one has $P(f\,;\, f', f'', \ldots, f^{(k)}) = 0$.*

Let us give the proof of (11.4) in a special case. We interpret here $E^{\mathrm{GG}}_{k,m} V^* \otimes \mathcal{O}(-A)$ as the bundle of differential operators whose coefficients vanish along A. By Brody's lemma [Bro78], for every entire curve $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$, one can extract a convergent "renormalized" sequence $g = \lim f \circ h_\nu$ where h_ν are suitable homographic functions, in such a way that g is an entire curve with bounded derivative $\sup_{t \in \mathbb{C}} \|g'(t)\|_\omega < +\infty$ (with respect to any given Hermitian metric ω on X); the image $g(\mathbb{C})$ is then contained in the cluster set $\overline{f(\mathbb{C})}$, but it is possible that $\overline{g(\mathbb{C})} \subsetneq \overline{f(\mathbb{C})}$. Then Cauchy inequalities imply that all derivatives $g^{(j)}$ are bounded, and therefore, by compactness of X, $u = P(g\,;\, g', g'', \ldots, g^{(k)})$ is a bounded holomorphic function on $\mathbb{C}$. However, after raising P to a power, we may assume that A is very ample, and after moving $A \in |A|$, that $\operatorname{Supp} A$ intersects $g(\mathbb{C})$. Then u vanishes somewhere, hence $u \equiv 0$ by Liouville's theorem. The proof for the general case is more subtle and makes use of Nevanlinna's second main theorem (see the above references).

It is expected that the global sections of $H^0(X, E_{k,m}^{\mathrm{GG}}V^* \otimes \mathcal{O}(-A))$ are precisely those which ultimately define the algebraic locus $Y \subsetneq X$ where the curve f should lie. The problem is then reduced to the question of showing that there are many non zero sections of $H^0(X, E_{k,m}^{\mathrm{GG}}V^* \otimes \mathcal{O}(-A))$, and further, understanding what is their joint base locus. The first part of this program is the main result of this chapter.

11.5. Theorem. *Let (X,V) be a directed projective variety such that K_V is big and let A be an ample divisor. Then for $k \gg 1$ and $\delta \in \mathbb{Q}_+$ small enough, $\delta \leqslant c(\log k)/k$, the number of sections $h^0(X, E_{k,m}^{\mathrm{GG}}V^* \otimes \mathcal{O}(-m\delta A))$ has maximal growth, i.e. is larger that $c_k m^{n+kr-1}$ for some $m \geqslant m_k$, where c, $c_k > 0$, $n = \dim X$ and $r = \operatorname{rank} V$. In particular, entire curves $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X,V)$ satisfy (many) algebraic differential equations.*

The statement is very elementary to check when $r = \operatorname{rank} V = 1$, and therefore when $n = \dim X = 1$. In higher dimensions $n \geqslant 2$, only very partial results were known at this point, concerning merely the absolute case $V = T_X$. In dimension 2, Theorem 11.5 is a consequence of the Riemann-Roch calculation of Green-Griffiths [GrGr79], combined with a vanishing theorem due to Bogomolov [Bog79] – the latter actually only applies to the top cohomology group H^n, and things become much more delicate when extimates of intermediate cohomology groups are needed. In higher dimensions, Diverio [Div08, Div09] proved the existence of sections of $H^0(X, E_{k,m}^{\mathrm{GG}}V^* \otimes \mathcal{O}(-1))$ whenever X is a hypersurface of $\mathbb{P}^{n+1}_{\mathbb{C}}$ of high degree $d \geqslant d_n$, assuming $k \geqslant n$ and $m \geqslant m_n$. More recently, Merker [Mer10] was able to treat the case of arbitrary hypersurfaces of general type, i.e. $d \geqslant n+3$, assuming this time k to be very large. The latter result is obtained through explicit algebraic calculations of the spaces of sections, and the proof is computationally very intensive. Bérczi [Ber10] also obtained related results with a different approach based on residue formulas, assuming $d \geqslant 2^{7n\log n}$.

All these approaches are algebraic in nature. Here, however, our techniques are based on more elaborate curvature estimates in the spirit of Cowen-Griffiths [CoGr76]. They require holomorphic Morse inequalities (see 11.10 below) – and we do not know how to translate our method in an algebraic setting. Notice that holomorphic Morse inequalities are essentially insensitive to singularities, as we can pass to non singular models and blow-up X as much as we want: if $\mu : \widetilde{X} \to X$ is a modification then $\mu_*\mathcal{O}_{\widetilde{X}} = \mathcal{O}_X$ and $R^q\mu_*\mathcal{O}_{\widetilde{X}}$ is supported on a codimension 1 analytic subset (even codimension 2 if X is smooth). As already observed in Section III 3, it follows from the Leray spectral sequence that the cohomology estimates for L on X or for $\widetilde{L} = \mu^*L$ on $\widetilde{X}$ differ by negligible terms, i.e.

$$h^q(\widetilde{X}, \widetilde{L}^{\otimes m}) - h^q(X, L^{\otimes m}) = O(m^{n-1}).$$

Finally, singular holomorphic Morse inequalities (see Setion III 6) allow us to work with singular Hermitian metrics h; this is the reason why we will only require to have big line bundles rather than ample line bundles. In the case of linear subspaces $V \subset T_X$, we introduce singular Hermitian metrics as follows.

11.6. Definition. *A singular Hermitian metric on a linear subspace $V \subset T_X$ is a metric h on the fibers of V such that the function $\log h : \xi \mapsto \log|\xi|_h^2$ is locally integrable on the total space of V.*

Such a metric can also be viewed as a singular Hermitian metric on the tautological line bundle $\mathcal{O}_{P(V)}(-1)$ on the projectivized bundle $P(V) = V \smallsetminus \{0\}/\mathbb{C}^*$, and therefore its dual metric h^* defines a curvature current $\Theta_{\mathcal{O}_{P(V)}(1),h^*}$ of type $(1,1)$ on $P(V)\subset P(T_X)$, such that

$$p^*\Theta_{\mathcal{O}_{P(V)}(1),h^*}=\frac{i}{2\pi}\partial\overline\partial\log h,\qquad \text{where } p:V\smallsetminus\{0\}\to P(V).$$

If $\log h$ is quasi-plurisubharmonic (or quasi-psh, which means psh modulo addition of a smooth function) on V, then $\log h$ is indeed locally integrable, and we have moreover

$$\Theta_{\mathcal{O}_{P(V)}(1),h^*}\geqslant -C\omega \tag{11.7}$$

for some smooth positive $(1,1)$-form on $P(V)$ and some constant $C>0$; conversely, if (11.7) holds, then $\log h$ is quasi-psh.

11.8. Definition. *We will say that a singular Hermitian metric h on V is* admissible *if h can be written as $h=e^{\varphi}h_{0|V}$ where h_0 is a smooth positive definite Hermitian on T_X and φ is a quasi-psh weight with analytic singularities on X, as in* (11.6)*. Then h can be seen as a singular Hermitian metric on $\mathcal{O}_{P(V)}(1)$, with the property that it induces a smooth positive definite metric on a Zariski open set $X'\subset X\smallsetminus\mathrm{Sing}(V)$; we will denote by* $\mathrm{Sing}(h)\supset\mathrm{Sing}(V)$ *the complement of the largest such Zariski open set X'.*

If h is an admissible metric, we define $\mathcal{O}_h(V^*)$ to be the sheaf of germs of holomorphic sections sections of $V^*_{|X\smallsetminus\mathrm{Sing}(h)}$ which are h^*-bounded near $\mathrm{Sing}(h)$; by the assumption on the analytic singularities, this is a coherent sheaf (as the direct image of some coherent sheaf on $P(V)$), and actually, since $h^*=e^{-\varphi}h_0^*$, it is a subsheaf of the sheaf $\mathcal{O}(V^*):=\mathcal{O}_{h_0}(V^*)$ associated with a smooth positive definite metric h_0 on T_X. If r is the generic rank of V and m a positive integer, we define similarly $K^m_{V,h}$ to be sheaf of germs of holomorphic sections of $(\det V^*_{|X'})^{\otimes m}=(\Lambda^rV^*_{|X'})^{\otimes m}$ which are $\det h^*$-bounded, and $K^m_V:=K^m_{V,h_0}$.

If V is defined by $\alpha:X\dashrightarrow G_r(T_X)$, there always exists a modification $\mu:\widetilde X\to X$ such that the composition $\alpha\circ\mu:\widetilde X\to G_r(\mu^*T_X)$ becomes holomorphic, and then $\mu^*V_{|\mu^{-1}(X\smallsetminus\mathrm{Sing}(V))}$ extends as a locally trivial subbundle of μ^*T_X which we will simply denote by μ^*V. If h is an admissible metric on V, then μ^*V can be equipped with the metric $\mu^*h=e^{\varphi\circ\mu}\mu^*h_0$ where μ^*h_0 is smooth and positive definite. We may assume that $\varphi\circ\mu$ has divisorial singularities (otherwise just perform further blow-ups of $\widetilde X$ to achieve this). We then see that there is an integer m_0 such that for all multiples $m=pm_0$ the pull-back $\mu^*K^m_{V,h}$ is an invertible sheaf on $\widetilde X$, and $\det h^*$ induces a smooth non singular metric on it (when $h=h_0$, we can even take $m_0=1$). By definition we always have $K^m_{V,h}=\mu_*(\mu^*K^m_{V,h})$ for any $m\geqslant 0$. In the sequel, however, we think of $K_{V,h}$ not really as a coherent sheaf, but rather as the "virtual" $\mathbb{Q}$-line bundle $\mu_*(\mu^*K^{m_0}_{V,h})^{1/m_0}$, and we say that $K_{V,h}$ is big if $h^0(X,K^m_{V,h})\geqslant cm^n$ for $m\geqslant m_1$, with $c>0$, i.e. if the invertible sheaf $\mu^*K^{m_0}_{V,h}$ is big in the usual sense.

At this point, it is important to observe that "our" canonical sheaf K_V differs from the sheaf $\mathcal{K}_V:=i_*\mathcal{O}(K_V)$ associated with the injection $i:X\smallsetminus\mathrm{Sing}(V)\hookrightarrow X$, which is usually referred to as being the "canonical sheaf", at least when V is the space of tangents to a foliation. In fact, $\mathcal{K}_V$ is always an invertible sheaf and there is an obvious inclusion $K_V\subset\mathcal{K}_V$. More precisely, the image of $\mathcal{O}(\Lambda^rT^*_X)\to\mathcal{K}_V$ is equal to $\mathcal{K}_V\otimes_{\mathcal{O}_X}\mathcal{J}$ for a certain coherent ideal $\mathcal{J}\subset\mathcal{O}_X$, and the condition to have h_0-bounded sections on $X\smallsetminus\mathrm{Sing}(V)$ precisely means that our sections are bounded by $\mathrm{Const}\sum|g_j|$ in terms of the generators (g_j) of $\mathcal{K}_V\otimes_{\mathcal{O}_X}\mathcal{J}$, i.e. $K_V=\mathcal{K}_V\otimes_{\mathcal{O}_X}\overline{\mathcal{J}}$ where $\overline{\mathcal{J}}$ is the integral closure of $\mathcal{J}$. More generally,

$$K^m_{V,h}=\mathcal{K}^m_V\otimes_{\mathcal{O}_X}\overline{\mathcal{J}}^{\,m/m_0}_{h,m_0}$$

where $\overline{\mathcal{J}}_{h,m_0}^{m/m_0} \subset \mathcal{O}_X$ is the (m/m_0)-integral closure of a certain ideal sheaf $\mathcal{J}_{h,m_0} \subset \mathcal{O}_X$, which can itself be assumed to be integrally closed; in our previous discussion, μ is chosen so that $\mu^*\mathcal{J}_{h,m_0}$ is invertible on $\widetilde{X}$.

The discrepancy already occurs e.g. with the rank 1 linear space $V \subset T_{\mathbb{P}^n_{\mathbb{C}}}$ consisting at each point $z \neq 0$ of the tangent to the line $(0z)$ (so that necessarily $V_0 = T_{\mathbb{P}^n_{\mathbb{C}},0}$). As a sheaf (and not as a linear space), $i_*\mathcal{O}(V)$ is the invertible sheaf generated by the vector field $\xi = \sum z_j\partial/\partial z_j$ on the affine open set $\mathbb{C}^n \subset \mathbb{P}^n_{\mathbb{C}}$, and therefore $\mathcal{K}_V := i_*\mathcal{O}(V^*)$ is generated over $\mathbb{C}^n$ by the unique 1-form u such that $u(\xi) = 1$. Since ξ vanishes at 0, the generator u is *unbounded* with respect to a smooth metric h_0 on $T_{\mathbb{P}^n_{\mathbb{C}}}$, and it is easily seen that K_V is the non invertible sheaf $K_V = \mathcal{K}_V \otimes \mathfrak{m}_{\mathbb{P}^n_{\mathbb{C}},0}$. We can make it invertible by considering the blow-up $\mu : \widetilde{X} \to X$ of $X = \mathbb{P}^n_{\mathbb{C}}$ at 0, so that μ^*K_V is isomorphic to $\mu^*\mathcal{K}_V \otimes \mathcal{O}_{\widetilde{X}}(-E)$ where E is the exceptional divisor. The integral curves C of V are of course lines through 0, and when a standard parametrization is used, their derivatives do not vanish at 0, while the sections of $i_*\mathcal{O}(V)$ do – another sign that $i_*\mathcal{O}(V)$ and $i_*\mathcal{O}(V^*)$ are the *wrong objects* to consider. Another standard example is obtained by taking a generic pencil of elliptic curves $\lambda P(z) + \mu Q(z) = 0$ of degree 3 in $\mathbb{P}^2_{\mathbb{C}}$, and the linear space V consisting of the tangents to the fibers of the rational map $\mathbb{P}^2_{\mathbb{C}} \dashrightarrow \mathbb{P}^1_{\mathbb{C}}$ defined by $z \mapsto Q(z)/P(z)$. Then V is given by

$$0 \longrightarrow i_*\mathcal{O}(V) \longrightarrow \mathcal{O}(T_{\mathbb{P}^2_{\mathbb{C}}}) \xrightarrow{PdQ-QdP} \mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(6) \otimes \mathcal{J}_S \longrightarrow 0$$

where $S = \mathrm{Sing}(V)$ consists of the 9 points $\{P(z) = 0\} \cap \{Q(z) = 0\}$, and $\mathcal{J}_S$ is the corresponding ideal sheaf of S. Since $\det \mathcal{O}(T_{\mathbb{P}^2}) = \mathcal{O}(3)$, we see that $\mathcal{K}_V = \mathcal{O}(3)$ is ample, which seems to contradict (11.2) since all leaves are elliptic curves. There is however no such contradiction, because $K_V = \mathcal{K}_V \otimes \mathcal{J}_S$ is not big in our sense (it has degree 0 on all members of the elliptic pencil). A similar example is obtained with a generic pencil of conics, in which case $\mathcal{K}_V = \mathcal{O}(1)$ and $\operatorname{card} S = 4$.

For a given admissible Hermitian structure (V, h), we define similarly the sheaf $E^{\mathrm{GG}}_{k,m}V^*_h$ to be the sheaf of polynomials defined over $X \smallsetminus \mathrm{Sing}(h)$ which are "h-bounded". This means that when they are viewed as polynomials $P(z\,;\,\xi_1, \ldots, \xi_k)$ in terms of $\xi_j = (\nabla^{1,0}_{h_0})^j f(0)$ where $\nabla^{1,0}_{h_0}$ is the $(1,0)$-component of the induced Chern connection on (V, h_0), there is a uniform bound

$$\big|P(z\,;\,\xi_1, \ldots, \xi_k)\big| \leqslant C\Big(\sum \|\xi_j\|_h^{1/j}\Big)^m \tag{11.9}$$

near points of $X \smallsetminus X'$ (see section 2 for more details on this). Again, by a direct image argument, one sees that $E^{\mathrm{GG}}_{k,m}V^*_h$ is always a coherent sheaf. The sheaf $E^{\mathrm{GG}}_{k,m}V^*$ is defined to be $E^{\mathrm{GG}}_{k,m}V^*_h$ when $h = h_0$ (it is actually independent of the choice of h_0, as follows from arguments similar to those given in section 2). Notice that this is exactly what is needed to extend the proof of the vanishing theorem 11.4 to the case of a singular linear space V; the value distribution theory argument can only work when the functions $P(f\,;\,f', \ldots, f^{(k)})(t)$ do not exhibit poles, and this is guaranteed here by the boundedness assumption.

Our strategy can be described as follows. We consider the Green-Griffiths bundle of k-jets $X^{\mathrm{GG}}_k = J^kV \smallsetminus \{0\}/\mathbb{C}^*$, which by (11.3) consists of a fibration in *weighted projective spaces*, and its associated tautological sheaf

$$L = \mathcal{O}_{X^{\mathrm{GG}}_k}(1),$$

viewed rather as a virtual $\mathbb{Q}$-line bundle $\mathcal{O}_{X^{\mathrm{GG}}_k}(m_0)^{1/m_0}$ with $m_0 = \mathrm{lcm}(1, 2, \ldots, k)$. Then, if $\pi_k : X^{\mathrm{GG}}_k \to X$ is the natural projection, we have

$$E^{\mathrm{GG}}_{k,m} = (\pi_k)_*\mathcal{O}_{X^{\mathrm{GG}}_k}(m) \quad \text{and} \quad R^q(\pi_k)_*\mathcal{O}_{X^{\mathrm{GG}}_k}(m) = 0 \text{ for } q \geqslant 1.$$

Hence, by the Leray spectral sequence we get for every invertible sheaf F on X the isomorphism

$$H^q(X, E_{k,m}^{\mathrm{GG}} V^* \otimes F) \simeq H^q(X_k^{\mathrm{GG}}, \mathcal{O}_{X_k^{\mathrm{GG}}}(m) \otimes \pi_k^* F).$$

The latter group can be evaluated thanks to holomorphic Morse inequalities. Let us recall the main statement.

11.10. Holomorphic Morse inequalities ([Dem85]). *Let X be a compact complex manifolds, $E \to X$ a holomorphic vector bundle of rank r, and (L, h) a hermitian line bundle. The dimensions $h^q(X, E \otimes L^k)$ of cohomology groups of the tensor powers $E \otimes L^k$ satisfy the following asymptotic estimates as $k \to +\infty$:*

(WM) *Weak Morse inequalities*:

$$h^q(X, E \otimes L^k) \leqslant r \frac{k^n}{n!} \int_{X(L,h,q)} (-1)^q \Theta_{L,h}^n + o(k^n) \ .$$

(SM) *Strong Morse inequalities*:

$$\sum_{0 \leqslant j \leqslant q} (-1)^{q-j} h^j(X, E \otimes L^k) \leqslant r \frac{k^n}{n!} \int_{X(L,h,\leqslant q)} (-1)^q \Theta_{L,h}^n + o(k^n) \ .$$

(RR) *Asymptotic Riemann-Roch formula*:

$$\chi(X, E \otimes L^k) := \sum_{0 \leqslant j \leqslant n} (-1)^j h^j(X, E \otimes L^k) = r \frac{k^n}{n!} \int_X \Theta_{L,h}^n + o(k^n) \ .$$

Moreover (cf. Bonavero's PhD thesis [Bon93]), if $h = e^{-\varphi}$ is a singular hermitian metric with analytic singularities, the estimates are still true provided all cohomology groups are replaced by cohomology groups $H^q(X, E \otimes L^k \otimes \mathcal{I}(h^k))$ twisted with the multiplier ideal sheaves

$$\mathcal{I}(h^k) = \mathcal{I}(k\varphi) = \big\{ f \in \mathcal{O}_{X,x},\ \ \exists V \ni x,\ \int_V |f(z)|^2 e^{-k\varphi(z)} d\lambda(z) < +\infty \big\}.$$

The special case of 11.10 (SM) when $q = 1$ yields a very useful criterion for the existence of sections of large multiples of L.

11.11. Corollary. *Under the above hypotheses, we have*

$$h^0(X, E \otimes L^k) \geqslant h^0(X, E \otimes L^k) - h^1(X, E \otimes L^k) \geqslant r \frac{k^n}{n!} \int_{X(L,h,\leqslant 1)} \Theta_{L,h}^n - o(k^n) \ .$$

Especially L is big as soon as $\int_{X(L,h,\leqslant 1)} \Theta_{L,h}^n > 0$ for some hermitian metric h on L.

Now, given a directed manifold (X, V), we can associate with any admissible metric h on V a metric (or rather a natural family) of metrics on $L = \mathcal{O}_{X_k^{\mathrm{GG}}}(1)$. The space X_k^{GG} always possesses quotient singularities if $k \geqslant 2$ (and even some more if V is singular), but we do not really care since Morse inequalities still work in this setting thanks to Bonavero's generalization. As we will see, it is then possible to get nice asymptotic formulas as $k \to +\infty$. They appear to be of a *probabilistic nature* if we take the components of the k-jet (i.e. the successive derivatives $\xi_j = f^{(j)}(0)$, $1 \leqslant j \leqslant k$) as random variables. This probabilistic behaviour was somehow already visible in the Riemann-Roch calculation of [GrGr79]. In this way, assuming K_V big, we produce a lot of sections $\sigma_j = H^0(X_k^{\mathrm{GG}}, \mathcal{O}_{X_k^{\mathrm{GG}}}(m) \otimes \pi_k^* F)$, corresponding to certain divisors $Z_j \subset X_k^{\mathrm{GG}}$. The hard problem which is left in order to complete a proof of the generalized Green-Griffiths-Lang conjecture is to compute the base locus $Z = \bigcap Z_j$ and to show that $Y = \pi_k(Z) \subset X$ must be a proper algebraic variety.

11.B. Hermitian geometry of weighted projective spaces

The goal of this section is to introduce natural Kähler metrics on weighted projective spaces, and to evaluate the corresponding volume forms. Here we put $d^c = \frac{i}{4\pi}(\overline{\partial} - \partial)$ so that $dd^c = \frac{i}{2\pi}\partial\overline{\partial}$. The normalization of the d^c operator is chosen such that we have precisely $(dd^c \log |z|^2)^n = \delta_0$ for the Monge-Ampère operator in $\mathbb{C}^n$; also, for every holomorphic or meromorphic section σ of a Hermitian line bundle (L, h) the Lelong-Poincaré can be formulated

$$dd^c \log |\sigma|_h^2 = [Z_\sigma] - \Theta_{L,h},$$

where $\Theta_{L,h} = \frac{i}{2\pi} D_{L,h}^2$ is the $(1,1)$-curvature form of L and Z_σ the zero divisor of σ. The closed $(1,1)$-form $\Theta_{L,h}$ is a representative of the first Chern class $c_1(L)$. Given a k-tuple of "weights" $a = (a_1, \ldots, a_k)$, i.e. of integers $a_s > 0$ with $\gcd(a_1, \ldots, a_k) = 1$, we introduce the weighted projective space $P(a_1, \ldots, a_k)$ to be the quotient of $\mathbb{C}^k \smallsetminus \{0\}$ by the corresponding weighted $\mathbb{C}^*$ action:

$$P(a_1, \ldots, a_k) = \mathbb{C}^k \smallsetminus \{0\}/\mathbb{C}^*, \qquad \lambda \cdot z = (\lambda^{a_1} z_1, \ldots, \lambda^{a_k} z_k). \tag{11.12}$$

As is well known, this defines a toric $(k-1)$-dimensional algebraic variety with quotient singularities. On this variety, we introduce the possibly singular (but almost everywhere smooth and non degenerate) Kähler form $\omega_{a,p}$ defined by

$$\pi_a^* \omega_{a,p} = dd^c \varphi_{a,p}, \qquad \varphi_{a,p}(z) = \frac{1}{p} \log \sum_{1 \leqslant s \leqslant k} |z_s|^{2p/a_s}, \tag{11.13}$$

where $\pi_a : \mathbb{C}^k \smallsetminus \{0\} \to P(a_1, \ldots, a_k)$ is the canonical projection and $p > 0$ is a positive constant. It is clear that $\varphi_{p,a}$ is real analytic on $\mathbb{C}^k \smallsetminus \{0\}$ if p is an integer and a common multiple of all weights a_s. It is at least C^2 if p is real and $p \geqslant \max(a_s)$, which will be more than sufficient for our purposes (but everything would still work for any $p > 0$). The resulting metric is in any case smooth and positive definite outside of the coordinate hyperplanes $z_s = 0$, and these hyperplanes will not matter here since they are of capacity zero with respect to all currents $(dd^c \varphi_{a,p})^\ell$. In order to evaluate the volume $\int_{P(a_1,\ldots,a_k)} \omega_{a,p}^{k-1}$, one can observe that

$$\begin{aligned}
\int_{P(a_1,\ldots,a_k)} \omega_{a,p}^{k-1} &= \int_{z \in \mathbb{C}^k,\, \varphi_{a,p}(z) = 0} \pi_a^* \omega_{a,p}^{k-1} \wedge d^c \varphi_{a,p} \\
&= \int_{z \in \mathbb{C}^k,\, \varphi_{a,p}(z) = 0} (dd^c \varphi_{a,p})^{k-1} \wedge d^c \varphi_{a,p} \\
&= \frac{1}{p^k} \int_{z \in \mathbb{C}^k,\, \varphi_{a,p}(z) < 0} (dd^c e^{p\varphi_{a,p}})^k. \qquad (11.14)
\end{aligned}$$

The first equality comes from the fact that $\{\varphi_{a,p}(z) = 0\}$ is a circle bundle over $P(a_1, \ldots, a_k)$, together with the identities $\varphi_{a,p}(\lambda \cdot z) = \varphi_{a,p}(z) + \log |\lambda|^2$ and

$\int_{|\lambda|=1} d^c \log|\lambda|^2 = 1$. The third equality can be seen by Stokes formula applied to the $(2k-1)$-form

$$(dd^c e^{p\varphi_{a,p}})^{k-1} \wedge d^c e^{p\varphi_{a,p}} = e^{p\varphi_{a,p}} (dd^c \varphi_{a,p})^{k-1} \wedge d^c \varphi_{a,p}$$

on the pseudoconvex open set $\{z \in \mathbb{C}^k\,;\ \varphi_{a,p}(z) < 0\}$. Now, we find

$$(dd^c e^{p\varphi_{a,p}})^k = \Big(dd^c \sum_{1\leqslant s\leqslant k} |z_s|^{2p/a_s}\Big)^k = \prod_{1\leqslant s\leqslant k} \Big(\frac{p}{a_s}|z_s|^{\frac{p}{a_s}-1}\Big)(dd^c|z|^2)^k, \tag{11.15}$$

$$\int_{z\in\mathbb{C}^k,\,\varphi_{a,p}(z)<0} (dd^c e^{p\varphi_{a,p}})^k = \prod_{1\leqslant s\leqslant k} \frac{p}{a_s} = \frac{p^k}{a_1\ldots a_k}. \tag{11.16}$$

In fact, (11.15) and (11.16) are clear when $p = a_1 = \ldots = a_k = 1$ (this is just the standard calculation of the volume of the unit ball in $\mathbb{C}^k$); the general case follows by substituting formally $z_s \mapsto z_s^{p/a_s}$, and using rotational invariance together with the observation that the arguments of the complex numbers z_s^{p/a_s} now run in the interval $[0, 2\pi p/a_s[$ instead of $[0, 2\pi[$ (say). As a consequence of (11.14) and (11.16), we obtain the well known value

$$\int_{P(a_1,\ldots,a_k)} \omega_{a,p}^{k-1} = \frac{1}{a_1\ldots a_k}, \tag{11.17}$$

for the volume. Notice that this is independent of p (as it is obvious by Stokes theorem, since the cohomology class of $\omega_{a,p}$ does not depend on p). When p tends to $+\infty$, we have $\varphi_{a,p}(z) \mapsto \varphi_{a,\infty}(z) = \log\max_{1\leqslant s\leqslant k} |z_s|^{2/a_s}$ and the volume form $\omega_{a,p}^{k-1}$ converges to a rotationally invariant measure supported by the image of the polycircle $\prod\{|z_s| = 1\}$ in $P(a_1,\ldots,a_k)$. This is so because not all $|z_s|^{2/a_s}$ are equal outside of the image of the polycircle, thus $\varphi_{a,\infty}(z)$ locally depends only on $k-1$ complex variables, and so $\omega_{a,\infty}^{k-1} = 0$ there by log homogeneity.

Our later calculations will require a slightly more general setting. Instead of looking at $\mathbb{C}^k$, we consider the weighted $\mathbb{C}^*$ action defined by

$$\mathbb{C}^{|r|} = \mathbb{C}^{r_1} \times \ldots \times \mathbb{C}^{r_k}, \qquad \lambda\cdot z = (\lambda^{a_1} z_1, \ldots, \lambda^{a_k} z_k). \tag{11.18}$$

Here $z_s \in \mathbb{C}^{r_s}$ for some k-tuple $r = (r_1,\ldots,r_k)$ and $|r| = r_1 + \ldots + r_k$. This gives rise to a weighted projective space

$$\begin{aligned} &P(a_1^{[r_1]},\ldots,a_k^{[r_k]}) = P(a_1,\ldots,a_1,\ldots,a_k,\ldots,a_k),\\ &\pi_{a,r} : \mathbb{C}^{r_1}\times\ldots\times\mathbb{C}^{r_k} \smallsetminus \{0\} \longrightarrow P(a_1^{[r_1]},\ldots,a_k^{[r_k]}) \end{aligned} \tag{11.19}$$

obtained by repeating r_s times each weight a_s. On this space, we introduce the degenerate Kähler metric $\omega_{a,r,p}$ such that

$$\pi_{a,r}^*\omega_{a,r,p} = dd^c\varphi_{a,r,p}, \qquad \varphi_{a,r,p}(z) = \frac{1}{p}\log\sum_{1\leqslant s\leqslant k} |z_s|^{2p/a_s} \tag{11.20}$$

where $|z_s|$ stands now for the standard Hermitian norm $(\sum_{1\leqslant j\leqslant r_s} |z_{s,j}|^2)^{1/2}$ on $\mathbb{C}^{r_s}$. This metric is cohomologous to the corresponding "polydisc-like" metric $\omega_{a,p}$ already defined, and therefore Stokes theorem implies

$$\int_{P(a_1^{[r_1]},\ldots,a_k^{[r_k]})} \omega_{a,r,p}^{|r|-1} = \frac{1}{a_1^{r_1}\ldots a_k^{r_k}}. \tag{11.21}$$

Since $(dd^c \log|z_s|^2)^{r_s} = 0$ on $\mathbb{C}^{r_s} \smallsetminus \{0\}$ by homogeneity, we conclude as before that the weak limit $\lim_{p\to+\infty} \omega_{a,r,p}^{|r|-1} = \omega_{a,r,\infty}^{|r|-1}$ associated with

$$\varphi_{a,r,\infty}(z) = \log \max_{1\leqslant s\leqslant k} |z_s|^{2/a_s} \tag{11.22}$$

is a measure supported by the image of the product of unit spheres $\prod S^{2r_s-1}$ in $P(a_1^{[r_1]}, \ldots, a_k^{[r_k]})$, which is invariant under the action of $U(r_1) \times \ldots \times U(r_k)$ on $\mathbb{C}^{r_1} \times \ldots \times \mathbb{C}^{r_k}$, and thus coincides with the Hermitian area measure up to a constant determined by condition (11.21). In fact, outside of the product of spheres, $\varphi_{a,r,\infty}$ locally depends only on at most $k-1$ factors and thus, for dimension reasons, the top power $(dd^c\varphi_{a,r,\infty})^{|r|-1}$ must be zero there. In the next section, the following change of variable formula will be needed. For simplicity of exposition we restrict ourselves to continuous functions, but a standard density argument would easily extend the formula to all functions that are Lebesgue integrable with respect to the volume form $\omega_{a,r,p}^{|r|-1}$.

11.23. Proposition. *Let $f(z)$ be a bounded function on $P(a_1^{[r_1]}, \ldots, a_k^{[r_k]})$ which is continuous outside of the hyperplane sections $z_s = 0$. We also view f as a $\mathbb{C}^*$-invariant continuous function on $\prod(\mathbb{C}^{r_s} \smallsetminus \{0\})$. Then*

$$\begin{aligned}&\int_{P(a_1^{[r_1]},\ldots,a_k^{[r_k]})} f(z)\,\omega_{a,r,p}^{|r|-1}\\ &= \frac{(|r|-1)!}{\prod_s a_s^{r_s}} \int_{(x,u)\in\Delta_{k-1}\times\prod S^{2r_s-1}} f(x_1^{a_1/2p}u_1, \ldots, x_k^{a_k/2p}u_k) \prod_{1\leqslant s\leqslant k} \frac{x_s^{r_s-1}}{(r_s-1)!}\,dx\,d\mu(u)\end{aligned}$$

where Δ_{k-1} is the $(k-1)$-simplex $\{x_s \geqslant 0, \sum x_s = 1\}$, $dx = dx_1 \wedge \ldots \wedge dx_{k-1}$ its standard measure, and where $d\mu(u) = d\mu_1(u_1)\ldots d\mu_k(u_k)$ is the rotation invariant probability measure on the product $\prod_s S^{2r_s-1}$ of unit spheres in $\mathbb{C}^{r_1} \times \ldots \times \mathbb{C}^{r_k}$. As a consequence

$$\lim_{p\to+\infty} \int_{P(a_1^{[r_1]},\ldots,a_k^{[r_k]})} f(z)\,\omega_{a,r,p}^{|r|-1} = \frac{1}{\prod_s a_s^{r_s}} \int_{\prod S^{2r_s-1}} f(u)\,d\mu(u).$$

Proof. The area formula of the disc $\int_{|\lambda|<1} dd^c|\lambda|^2 = 1$ and a consideration of the unit disc bundle over $P(a_1^{[r_1]}, \ldots, a_k^{[r_k]})$ imply that

$$I_p := \int_{P(a_1^{[r_1]},\ldots,a_k^{[r_k]})} f(z)\,\omega_{a,r,p}^{|r|-1} = \int_{z\in\mathbb{C}^{|r|},\varphi_{a,r,p}(z)<0} f(z)\,(dd^c\varphi_{a,r,p})^{|r|-1} \wedge dd^c e^{\varphi_{a,r,p}}.$$

Now, a straightforward calculation on $\mathbb{C}^{|r|}$ gives

$$\begin{aligned}(dd^c e^{p\varphi_{a,r,p}})^{|r|} &= \Big(dd^c \sum_{1\leqslant s\leqslant k} |z_s|^{2p/a_s}\Big)^{|r|}\\ &= \prod_{1\leqslant s\leqslant k} \Big(\frac{p}{a_s}\Big)^{r_s+1} |z_s|^{2r_s(p/a_s-1)} (dd^c|z|^2)^{|r|}.\end{aligned}$$

On the other hand, we have $(dd^c|z|^2)^{|r|} = \frac{|r|!}{r_1!\ldots r_k!} \prod_{1\leqslant s\leqslant k} (dd^c|z_s|^2)^{r_s}$ and

$$\begin{aligned}(dd^c e^{p\varphi_{a,r,p}})^{|r|} &= \big(p\,e^{p\varphi_{a,r,p}}(dd^c\varphi_{a,r,p} + p\,d\varphi_{a,r,p} \wedge d^c\varphi_{a,r,p})\big)^{|r|}\\ &= |r|p^{|r|+1} e^{|r|p\varphi_{a,r,p}} (dd^c\varphi_{a,r,p})^{|r|-1} \wedge d\varphi_{a,r,p} \wedge d^c\varphi_{a,r,p}\\ &= |r|p^{|r|+1} e^{(|r|p-1)\varphi_{a,r,p}} (dd^c\varphi_{a,r,p})^{|r|-1} \wedge dd^c e^{\varphi_{a,r,p}},\end{aligned}$$

thanks to the homogeneity relation $(dd^c\varphi_{a,r,p})^{|r|}=0$. Putting everything together, we find

$$I_p=\int_{z\in\mathbb{C}^{|r|},\,\varphi_{a,r,p}(z)<0}\frac{(|r|-1)!\,p^{k-1}f(z)}{(\sum_s|z_s|^{2p/a_s})^{|r|-1/p}}\prod_s\frac{(dd^c|z_s|^2)^{r_s}}{r_s!\,a_s^{r_s+1}|z_s|^{2r_s(1-p/a_s)}}.$$

A standard calculation in polar coordinates with $z_s=\rho_s u_s$, $u_s\in S^{2r_s-1}$, yields

$$\frac{(dd^c|z_s|^2)^{r_s}}{|z_s|^{2r_s}}=2r_s\frac{d\rho_s}{\rho_s}\,d\mu_s(u_s)$$

where μ_s is the $U(r_s)$-invariant probability measure on S^{2r_s-1}. Therefore

$$\begin{aligned}I_p&=\int_{\varphi_{a,r,p}(z)<0}\frac{(|r|-1)!\,p^{k-1}f(\rho_1u_1,\ldots,\rho_ku_k)}{(\sum_{1\leqslant s\leqslant k}\rho_s^{2p/a_s})^{|r|-1/p}}\prod_s\frac{2\rho_s^{2pr_s/a_s}\frac{d\rho_s}{\rho_s}d\mu_s(u_s)}{(r_s-1)!\,a_s^{r_s+1}}\\&=\int_{u_s\in S^{2r_s-1},\,\sum t_s<1}\frac{(|r|-1)!\,p^{-1}f(t_1^{a_1/2p}u_1,\ldots,t_k^{a_k/2p}u_k)}{(\sum_{1\leqslant s\leqslant k}t_s)^{|r|-1/p}}\prod_s\frac{t_s^{r_s-1}dt_s\,d\mu_s(u_s)}{(r_s-1)!\,a_s^{r_s}}\end{aligned}$$

by putting $t_s=|z_s|^{2p/a_s}=\rho_s^{2p/a_s}$, i.e. $\rho_s=t_s^{a_s/2p}$, $t_s\in\,]0,1]$. We use still another change of variable $t_s=tx_s$ with $t=\sum_{1\leqslant s\leqslant k}t_s$ and $x_s\in\,]0,1]$, $\sum_{1\leqslant s\leqslant k}x_s=1$. Then

$$dt_1\wedge\ldots\wedge dt_k=t^{k-1}\,dx\,dt\qquad\text{where } dx=dx_1\wedge\ldots\wedge dx_{k-1}.$$

The $\mathbb{C}^*$ invariance of f shows that

$$\begin{aligned}I_p&=\int_{\substack{u_s\in S^{2r_s-1}\\ \sum x_s=1,\,t\in]0,1]}}(|r|-1)!f(x_1^{a_s/2p}u_1,\ldots,x_k^{a_k/2p}u_k)\prod_{1\leqslant s\leqslant k}\frac{x_s^{r_s-1}d\mu_s(u_s)}{(r_s-1)!\,a_s^{r_s}}\frac{dx\,dt}{p\,t^{1-1/p}}\\&=\int_{\substack{u_s\in S^{2r_s-1}\\ \sum x_s=1}}(|r|-1)!f(x_1^{a_s/2p}u_1,\ldots,x_k^{a_k/2p}u_k)\prod_{1\leqslant s\leqslant k}\frac{x_s^{r_s-1}d\mu_s(u_s)}{(r_s-1)!\,a_s^{r_s}}\,dx.\end{aligned}$$

This is equivalent to the formula given in Proposition 11.23. We have $x_s^{2a_s/p}\to1$ as $p\to+\infty$, and by Lebesgue's bounded convergence theorem and Fubini's formula, we get

$$\lim_{p\to+\infty}I_p=\frac{(|r|-1)!}{\prod_s a_s^{r_s}}\int_{(x,u)\in\Delta_{k-1}\times\prod S^{2r_s-1}}f(u)\prod_{1\leqslant s\leqslant k}\frac{x_s^{r_s-1}}{(r_s-1)!}\,dx\,d\mu(u).$$

It can be checked by elementary integrations by parts and induction on k, $r_1,\ldots,r_k$ that

(11.24) $$\int_{x\in\Delta_{k-1}}\prod_{1\leqslant s\leqslant k}x_s^{r_s-1}dx_1\ldots dx_{k-1}=\frac{1}{(|r|-1)!}\prod_{1\leqslant s\leqslant k}(r_s-1)!\ .$$

This implies that $(|r|-1)!\prod_{1\leqslant s\leqslant k}\frac{x_s^{r_s-1}}{(r_s-1)!}\,dx$ is a probability measure on Δ_{k-1} and that

$$\lim_{p\to+\infty}I_p=\frac{1}{\prod_s a_s^{r_s}}\int_{u\in\prod S^{2r_s-1}}f(u)\,d\mu(u).$$

Even without an explicit check, Formula (11.24) also follows from the fact that we must have equality for $f(z)\equiv1$ in the latter equality, if we take into account the volume formula (11.21). □

11.C. Probabilistic estimate of the curvature of k-jet bundles

Let (X,V) be a compact complex directed non singular variety. To avoid any technical difficulty at this point, we first assume that V is a holomorphic vector subbundle of T_X, equipped with a smooth Hermitian metric h.

According to the notation already specified in the introduction, we denote by J^kV the bundle of k-jets of holomorphic curves $f:(\mathbb{C},0)\to X$ tangent to V at each point. Let us set $n=\dim_{\mathbb{C}} X$ and $r=\mathrm{rank}_{\mathbb{C}} V$. Then $J^kV\to X$ is an algebraic fiber bundle with typical fiber $\mathbb{C}^{rk}$ (see below). It has a canonical $\mathbb{C}^*$-action defined by $\lambda\cdot f:(\mathbb{C},0)\to X$, $(\lambda\cdot f)(t)=f(\lambda t)$. Fix a point x_0 in X and a local holomorphic coordinate system $(z_1,\ldots,z_n)$ centered at x_0 such that V_{x_0} is the vector subspace $\langle\partial/\partial z_1,\ldots,\partial/\partial z_r\rangle$ at x_0. Then, in a neighborhood U of x_0, V admits a holomorphic frame of the form

$$\frac{\partial}{\partial z_\beta}+\sum_{r+1\leqslant\alpha\leqslant n}a_{\alpha\beta}(z)\frac{\partial}{\partial z_\alpha},\qquad 1\leqslant\beta\leqslant r,\quad a_{\alpha\beta}(0)=0. \tag{11.25}$$

Let $f(t)=(f_1(t),\ldots,f_n(t))$ be a k-jet of curve tangent to V starting from a point $f(0)=x\in U$. Such a curve is entirely determined by its initial point and by the projection $\widetilde{f}(t):=(f_1(t),\ldots,f_r(t))$ to the first r-components, since the condition $f'(t)\in V_{f(t)}$ implies that the other components must satisfy the ordinary differential equation

$$f'_\alpha(t)=\sum_{1\leqslant\beta\leqslant r}a_{\alpha\beta}(f(t))f'_\beta(t).$$

This implies that the k-jet of f is entirely determined by the initial point x and the Taylor expansion

$$\widetilde{f}(t)-\widetilde{x}=\xi_1t+\xi_2t^2+\ldots+\xi_kt^k+O(t^{k+1}) \tag{11.26}$$

where $\xi_s=(\xi_{s\alpha})_{1\leqslant\alpha\leqslant r}\in\mathbb{C}^r$. The $\mathbb{C}^*$ action $(\lambda,f)\mapsto\lambda\cdot f$ is then expressed in coordinates by the weighted action

$$\lambda\cdot(\xi_1,\xi_2,\ldots,\xi_k)=(\lambda\xi_1,\lambda^2\xi_2,\ldots,\lambda^k\xi_k) \tag{11.27}$$

associated with the weight $a=(1^{[r]},2^{[r]},\ldots,k^{[r]})$. The quotient projectivized k-jet bundle

$$X_k^{\mathrm{GG}}:=(J^kV\smallsetminus\{0\})/\mathbb{C}^* \tag{11.28}$$

considered by Green and Griffiths [GrGr79] is therefore in a natural way a $P(1^{[r]},2^{[r]},\ldots,k^{[r]})$ weighted projective bundle over X. As such, it possesses a canonical sheaf $\mathcal{O}_{X_k^{\mathrm{GG}}}(1)$ such that $\mathcal{O}_{X_k^{\mathrm{GG}}}(m)$ is invertible when m is a multiple of $\mathrm{lcm}(1,2,\ldots,k)$. Under the natural projection $\pi_k:X_k^{\mathrm{GG}}\to X$, the direct image $(\pi_k)_*\mathcal{O}_{X_k^{\mathrm{GG}}}(m)$ coincides with the sheaf of sections of the bundle $E_{k,m}^{\mathrm{GG}}V^*$ of jet differentials of order k and degree m, namely polynomials

$$P(z\,;\xi_1,\ldots,\xi_k)=\sum_{\alpha_\ell\in\mathbb{N}^r,\,1\leqslant\ell\leqslant k}a_{\alpha_1\ldots\alpha_k}(z)\,\xi_1^{\alpha_1}\ldots\xi_k^{\alpha_k} \tag{11.29}$$

of weighted degree $|\alpha_1|+2|\alpha_2|+\ldots+k|\alpha_k|=m$ on J^kV with holomorphic coefficients. The jet differentials operate on germs of curves as differential operators

$$P(f)(t)=\sum a_{\alpha_1\ldots\alpha_k}(f(t))\;f'(t)^{\alpha_1}\ldots f^{(k)}(t)^{\alpha_k}. \tag{11.30}$$

In the sequel, we do not make any further use of coordinate frames as (11.25), because they need not be related in any way to the Hermitian metric h of V. Instead, we choose a local holomorphic coordinate frame $(e_\alpha(z))_{1\leqslant\alpha\leqslant r}$ of V on a neighborhood U of x_0, such that

$$\langle e_\alpha(z), e_\beta(z)\rangle = \delta_{\alpha\beta} + \sum_{1\leqslant i,j\leqslant n,\,1\leqslant\alpha,\beta\leqslant r} c_{ij\alpha\beta} z_i \overline{z}_j + O(|z|^3) \tag{11.31}$$

for suitable complex coefficients $(c_{ij\alpha\beta})$. It is a standard fact that such a normalized coordinate system always exists, and that the Chern curvature tensor $\frac{i}{2\pi}D^2_{V,h}$ of (V,h) at x_0 is then given by

$$\Theta_{V,h}(x_0) = -\frac{i}{2\pi}\sum_{i,j,\alpha,\beta} c_{ij\alpha\beta}\, dz_i\wedge d\overline{z}_j \otimes e^*_\alpha\otimes e_\beta. \tag{11.32}$$

Also, instead of defining the vectors $\xi_s\in\mathbb{C}^r$ as in (11.26), we consider a local holomorphic connection ∇ on $V_{|U}$ (e.g. the one which turns (e_α) into a parallel frame), and take $\xi_k = \nabla^k f(0)\in V_x$ defined inductively by $\nabla^1 f = f'$ and $\nabla^s f = \nabla_{f'}(\nabla^{s-1}f)$. This is just another way of parametrizing the fibers of J^kV over U by the vector bundle $V^k_{|U}$. Notice that this is highly dependent on ∇ (the bundle J^kV actually does not carry a vector bundle or even affine bundle structure); however, the expression of the weighted action (11.27) is unchanged in this new setting. Now, we fix a finite open covering $(U_\alpha)_{\alpha\in I}$ of X by open coordinate charts such that $V_{|U_\alpha}$ is trivial, along with holomorphic connections ∇_α on $V_{|U_\alpha}$. Let θ_α be a partition of unity of X subordinate to the covering (U_α). Let us fix $p>0$ and small parameters $1=\varepsilon_1\gg\varepsilon_2\gg\ldots\gg\varepsilon_k>0$. Then we define a global weighted exhaustion on J^kV by putting for any k-jet $f\in J^k_xV$

$$\Psi_{h,p,\varepsilon}(f) := \Big(\sum_{\alpha\in I}\theta_\alpha(x)\sum_{1\leqslant s\leqslant k}\varepsilon_s^{2p}\|\nabla^s_\alpha f(0)\|^{2p/s}_{h(x)}\Big)^{1/p} \tag{11.33}$$

where $\|\ \ \|_{h(x)}$ is the Hermitian metric h of V evaluated on the fiber V_x, $x=f(0)$. The function $\Psi_{h,p,\varepsilon}$ satisfies the fundamental homogeneity property

$$\Psi_{h,p,\varepsilon}(\lambda\cdot f) = \Psi_{h,p,\varepsilon}(f)\,|\lambda|^2 \tag{11.34}$$

with respect to the $\mathbb{C}^*$ action on J^kV, in other words, it induces a Hermitian metric on the dual L^* of the tautological $\mathbb{Q}$-line bundle $L_k=\mathcal{O}_{X_k^{\rm GG}}(1)$ over $X_k^{\rm GG}$. The curvature of L_k is given by

$$\pi_k^*\Theta_{L_k,\Psi^*_{h,p,\varepsilon}} = dd^c\log\Psi_{h,p,\varepsilon} \tag{11.35}$$

where $\pi_k : J^kV\smallsetminus\{0\}\to X_k^{\rm GG}$ is the canonical projection. Our next goal is to compute precisely the curvature and to apply holomorphic Morse inequalities to $L\to X_k^{\rm GG}$ with the above metric. It might look a priori like an untractable problem, since the definition of $\Psi_{h,p,\varepsilon}$ is a rather unnatural one. However, the "miracle" is that the asymptotic behavior of $\Psi_{h,p,\varepsilon}$ as $\varepsilon_s/\varepsilon_{s-1}\to 0$ is in some sense uniquely defined and very natural. It will lead to a computable asymptotic formula, which is moreover simple enough to produce useful results.

11.36. Lemma. *On each coordinate chart U equipped with a holomorphic connection ∇ of $V_{|U}$, let us define the components of a k-jet $f\in J^kV$ by $\xi_s=\nabla^s f(0)$, and consider the rescaling transformation*

$$\rho_{\nabla,\varepsilon}(\xi_1,\xi_2,\ldots,\xi_k) = (\varepsilon_1^1\xi_1,\varepsilon_2^2\xi_2,\ldots,\varepsilon_k^k\xi_k)\quad \textit{on } J^k_xV,\ x\in U$$

(*it commutes with the* $\mathbb{C}^*$*-action but is otherwise unrelated and not canonically defined over* X *as it depends on the choice of* ∇). *Then, if* p *is a multiple of* $\mathrm{lcm}(1,2,\ldots,k)$ *and* $\varepsilon_s/\varepsilon_{s-1} \to 0$ *for all* $s = 2,\ldots,k$*, the rescaled function* $\Psi_{h,p,\varepsilon} \circ \rho_{\nabla,\varepsilon}^{-1}(\xi_1,\ldots,\xi_k)$ *converges towards*

$$\Big(\sum_{1\leqslant s\leqslant k} \|\xi_s\|_h^{2p/s}\Big)^{1/p}$$

on every compact subset of $J^kV_{|U} \smallsetminus \{0\}$*, uniformly in* C^∞ *topology.*

Proof. Let $U \subset X$ be an open set on which $V_{|U}$ is trivial and equipped with some holomorphic connection ∇. Let us pick another holomorphic connection $\widetilde\nabla = \nabla + \Gamma$ where $\Gamma \in H^0(U, \Omega^1_X \otimes \mathrm{Hom}(V,V)$. Then $\widetilde\nabla^2 f = \nabla^2 f + \Gamma(f)(f')\cdot f'$, and inductively we get

$$\widetilde\nabla^s f = \nabla^s f + P_s(f\,;\,\nabla^1 f,\ldots,\nabla^{s-1}f)$$

where $P(x\,;\,\xi_1,\ldots,\xi_{s-1})$ is a polynomial with holomorphic coefficients in $x \in U$ which is of weighted homogeneous degree s in $(\xi_1,\ldots,\xi_{s-1})$. In other words, the corresponding change in the parametrization of $J^kV_{|U}$ is given by a $\mathbb{C}^*$-homogeneous transformation

$$\widetilde\xi_s = \xi_s + P_s(x\,;\,\xi_1,\ldots,\xi_{s-1}).$$

Let us introduce the corresponding rescaled components

$$(\xi_{1,\varepsilon},\ldots,\xi_{k,\varepsilon}) = (\varepsilon_1^1\xi_1,\ldots,\varepsilon_k^k\xi_k), \qquad (\widetilde\xi_{1,\varepsilon},\ldots,\widetilde\xi_{k,\varepsilon}) = (\varepsilon_1^1\widetilde\xi_1,\ldots,\varepsilon_k^k\widetilde\xi_k).$$

Then

$$\begin{aligned}\widetilde\xi_{s,\varepsilon} &= \xi_{s,\varepsilon} + \varepsilon_s^s\,P_s(x\,;\,\varepsilon_1^{-1}\xi_{1,\varepsilon},\ldots,\varepsilon_{s-1}^{-(s-1)}\xi_{s-1,\varepsilon})\\ &= \xi_{s,\varepsilon} + O(\varepsilon_s/\varepsilon_{s-1})^s\,O(\|\xi_{1,\varepsilon}\| + \ldots + \|\xi_{s-1,\varepsilon}\|^{1/(s-1)})^s\end{aligned}$$

and the error terms are thus polynomials of fixed degree with arbitrarily small coefficients as $\varepsilon_s/\varepsilon_{s-1} \to 0$. Now, the definition of $\Psi_{h,p,\varepsilon}$ consists of glueing the sums

$$\sum_{1\leqslant s\leqslant k} \varepsilon_s^{2p}\|\xi_k\|_h^{2p/s} = \sum_{1\leqslant s\leqslant k} \|\xi_{k,\varepsilon}\|_h^{2p/s}$$

corresponding to $\xi_k = \nabla_\alpha^s f(0)$ by means of the partition of unity $\sum\theta_\alpha(x) = 1$. We see that by using the rescaled variables $\xi_{s,\varepsilon}$ the changes occurring when replacing a connection ∇_α by an alternative one ∇_β are arbitrary small in C^∞ topology, with error terms uniformly controlled in terms of the ratios $\varepsilon_s/\varepsilon_{s-1}$ on all compact subsets of $V^k \smallsetminus \{0\}$. This shows that in C^∞ topology, $\Psi_{h,p,\varepsilon}\circ\rho_{\nabla,\varepsilon}^{-1}(\xi_1,\ldots,\xi_k)$ converges uniformly towards $(\sum_{1\leqslant s\leqslant k}\|\xi_k\|_h^{2p/s})^{1/p}$, whatever the trivializing open set U and the holomorphic connection ∇ used to evaluate the components and perform the rescaling are. □

Now, we fix a point $x_0 \in X$ and a local holomorphic frame $(e_\alpha(z))_{1\leqslant\alpha\leqslant r}$ satisfying (11.31) on a neighborhood U of x_0. We introduce the rescaled components $\xi_s = \varepsilon_s^s\nabla^s f(0)$ on $J^kV_{|U}$ and compute the curvature of

$$\Psi_{h,p,\varepsilon}\circ\rho_{\nabla,\varepsilon}^{-1}(z\,;\,\xi_1,\ldots,\xi_k) \simeq \Big(\sum_{1\leqslant s\leqslant k}\|\xi_s\|_h^{2p/s}\Big)^{1/p}$$

(by Lemma 11.36, the errors can be taken arbitrary small in C^∞ topology). We write $\xi_s = \sum_{1\leqslant\alpha\leqslant r} \xi_{s\alpha} e_\alpha$. By (11.31) we have

$$\|\xi_s\|_h^2 = \sum_\alpha |\xi_{s\alpha}|^2 + \sum_{i,j,\alpha,\beta} c_{ij\alpha\beta} z_i \overline{z}_j \xi_{s\alpha}\overline{\xi}_{s\beta} + O(|z|^3|\xi|^2).$$

The question is to evaluate the curvature of the weighted metric defined by

$$\begin{aligned}\Psi(z\,;\,\xi_1,\ldots,\xi_k) &= \Big(\sum_{1\leqslant s\leqslant k} \|\xi_s\|_h^{2p/s}\Big)^{1/p}\\ &= \Big(\sum_{1\leqslant s\leqslant k}\Big(\sum_\alpha |\xi_{s\alpha}|^2 + \sum_{i,j,\alpha,\beta} c_{ij\alpha\beta} z_i\overline{z}_j\xi_{s\alpha}\overline{\xi}_{s\beta}\Big)^{p/s}\Big)^{1/p} + O(|z|^3).\end{aligned}$$

We set $|\xi_s|^2 = \sum_\alpha |\xi_{s\alpha}|^2$. A straightforward calculation yields

$$\begin{aligned}&\log\Psi(z\,;\,\xi_1,\ldots,\xi_k) =\\ &= \frac{1}{p}\log\sum_{1\leqslant s\leqslant k}|\xi_s|^{2p/s} + \sum_{1\leqslant s\leqslant k}\frac{1}{s}\frac{|\xi_s|^{2p/s}}{\sum_t |\xi_t|^{2p/t}}\sum_{i,j,\alpha,\beta} c_{ij\alpha\beta} z_i\overline{z}_j \frac{\xi_{s\alpha}\overline{\xi}_{s\beta}}{|\xi_s|^2} + O(|z|^3).\end{aligned}$$

By (11.35), the curvature form of $L_k = \mathcal{O}_{X_k^{\rm GG}}(1)$ is given at the central point x_0 by the following formula.

11.37. Proposition. *With the above choice of coordinates and with respect to the rescaled components $\xi_s = \varepsilon_s^s \nabla^s f(0)$ at $x_0 \in X$, we have the approximate expression*

$$\Theta_{L_k,\Psi^*_{h,p,\varepsilon}}(x_0,[\xi]) \simeq \omega_{a,r,p}(\xi) + \frac{i}{2\pi}\sum_{1\leqslant s\leqslant k}\frac{1}{s}\frac{|\xi_s|^{2p/s}}{\sum_t |\xi_t|^{2p/t}}\sum_{i,j,\alpha,\beta} c_{ij\alpha\beta}\frac{\xi_{s\alpha}\overline{\xi}_{s\beta}}{|\xi_s|^2}\,dz_i\wedge d\overline{z}_j$$

where the error terms are $O(\max_{2\leqslant s\leqslant k}(\varepsilon_s/\varepsilon_{s-1})^s)$ uniformly on the compact variety $X_k^{\rm GG}$. Here $\omega_{a,r,p}$ is the (degenerate) Kähler metric associated with the weight $a = (1^{[r]}, 2^{[r]}, \ldots, k^{[r]})$ of the canonical $\mathbb{C}^*$ action on J^kV.

Thanks to the uniform approximation, we can (and will) neglect the error terms in the calculations below. Since $\omega_{a,r,p}$ is positive definite on the fibers of $X_k^{\rm GG} \to X$ (at least outside of the axes $\xi_s = 0$), the index of the $(1,1)$ curvature form $\Theta_{L_k,\Psi^*_{h,p,\varepsilon}}(z,[\xi])$ is equal to the index of the $(1,1)$-form

$$\gamma_k(z,\xi) := \frac{i}{2\pi}\sum_{1\leqslant s\leqslant k}\frac{1}{s}\frac{|\xi_s|^{2p/s}}{\sum_t |\xi_t|^{2p/t}}\sum_{i,j,\alpha,\beta} c_{ij\alpha\beta}(z)\frac{\xi_{s\alpha}\overline{\xi}_{s\beta}}{|\xi_s|^2}\,dz_i\wedge d\overline{z}_j \tag{11.38}$$

depending only on the differentials $(dz_j)_{1\leqslant j\leqslant n}$ on X. The q-index integral of $(L_k, \Psi^*_{h,p,\varepsilon})$ on $X_k^{\rm GG}$ is therefore equal to

$$\begin{aligned}&\int_{X_k^{\rm GG}(L_k,q)}\Theta^{n+kr-1}_{L_k,\Psi^*_{h,p,\varepsilon}} =\\ &\qquad= \frac{(n+kr-1)!}{n!(kr-1)!}\int_{z\in X}\int_{\xi\in P(1^{[r]},\ldots,k^{[r]})}\omega^{kr-1}_{a,r,p}(\xi)\mathbb{1}_{\gamma_k,q}(z,\xi)\gamma_k(z,\xi)^n\end{aligned}$$

where $\mathbb{1}_{\gamma_k,q}(z,\xi)$ is the characteristic function of the open set of points where $\gamma_k(z,\xi)$ has signature $(n-q,q)$ in terms of the dz_j's. Notice that since $\gamma_k(z,\xi)^n$ is a determinant, the product $\mathbb{1}_{\gamma_k,q}(z,\xi)\gamma_k(z,\xi)^n$ gives rise to a continuous function on $X_k^{\rm GG}$. Formula 11.24 with $r_1=\ldots=r_k=r$ and $a_s=s$ yields the slightly more explicit integral

$$\int_{X_k^{\rm GG}(L_k,q)}\Theta^{n+kr-1}_{L_k,\Psi^*_{h,p,\varepsilon}}=\frac{(n+kr-1)!}{n!(k!)^r}\times$$
$$\int_{z\in X}\int_{(x,u)\in\Delta_{k-1}\times(S^{2r-1})^k}\mathbb{1}_{g_k,q}(z,x,u)g_k(z,x,u)^n\,\frac{(x_1\ldots x_k)^{r-1}}{(r-1)!^k}\,dx\,d\mu(u),$$

where $g_k(z,x,u)=\gamma_k(z,x_1^{1/2p}u_1,\ldots,x_k^{k/2p}u_k)$ is given by

$$g_k(z,x,u)=\frac{i}{2\pi}\sum_{1\leqslant s\leqslant k}\frac{1}{s}x_s\sum_{i,j,\alpha,\beta}c_{ij\alpha\beta}(z)\,u_{s\alpha}\overline{u}_{s\beta}\,dz_i\wedge d\overline{z}_j \tag{11.39}$$

and $\mathbb{1}_{g_k,q}(z,x,u)$ is the characteristic function of its q-index set. Here

$$d\nu_{k,r}(x)=(kr-1)!\,\frac{(x_1\ldots x_k)^{r-1}}{(r-1)!^k}\,dx \tag{11.40}$$

is a probability measure on Δ_{k-1}, and we can rewrite

$$\begin{aligned}&\int_{X_k^{\rm GG}(L_k,q)}\Theta^{n+kr-1}_{L_k,\Psi^*_{h,p,\varepsilon}}=\frac{(n+kr-1)!}{n!(k!)^r(kr-1)!}\times\\&\int_{z\in X}\int_{(x,u)\in\Delta_{k-1}\times(S^{2r-1})^k}\mathbb{1}_{g_k,q}(z,x,u)g_k(z,x,u)^n\,d\nu_{k,r}(x)\,d\mu(u).\end{aligned}\tag{11.41}$$

Now, formula (11.39) shows that $g_k(z,x,u)$ is a "Monte Carlo" evaluation of the curvature tensor, obtained by averaging the curvature at random points $u_s\in S^{2r-1}$ with certain positive weights x_s/s; we should then think of the k-jet f as some sort of random parameter such that the derivatives $\nabla^kf(0)$ are uniformly distributed in all directions. Let us compute the expected value of $(x,u)\mapsto g_k(z,x,u)$ with respect to the probability measure $d\nu_{k,r}(x)\,d\mu(u)$. Since $\int_{S^{2r-1}}u_{s\alpha}\overline{u}_{s\beta}d\mu(u_s)=\frac{1}{r}\delta_{\alpha\beta}$ and $\int_{\Delta_{k-1}}x_s\,d\nu_{k,r}(x)=\frac{1}{k}$, we find

$$\mathbf{E}(g_k(z,\bullet,\bullet))=\frac{1}{kr}\sum_{1\leqslant s\leqslant k}\frac{1}{s}\cdot\frac{i}{2\pi}\sum_{i,j,\alpha}c_{ij\alpha\alpha}(z)\,dz_i\wedge d\overline{z}_j.$$

In other words, we get the normalized trace of the curvature, i.e.

$$\mathbf{E}(g_k(z,\bullet,\bullet))=\frac{1}{kr}\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)\Theta_{\det(V^*),\det h^*}, \tag{11.42}$$

where $\Theta_{\det(V^*),\det h^*}$ is the $(1,1)$-curvature form of $\det(V^*)$ with the metric induced by h. It is natural to guess that $g_k(z,x,u)$ behaves asymptotically as its expected value $\mathbf{E}(g_k(z,\bullet,\bullet))$ when k tends to infinity. If we replace brutally g_k by its expected value in (11.41), we get the integral

$$\frac{(n+kr-1)!}{n!(k!)^r(kr-1)!}\frac{1}{(kr)^n}\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)^n\int_X\mathbb{1}_{\eta,q}\eta^n,$$

where $\eta := \Theta_{\det(V^*),\det h^*}$ and $\mathbb{1}_{\eta,q}$ is the characteristic function of its q-index set in X. The leading constant is equivalent to $(\log k)^n/n!(k!)^r$ modulo a multiplicative factor $1+O(1/\log k)$. By working out a more precise analysis of the deviation, we will prove the following result.

11.43. Probabilistic estimate. *Fix smooth Hermitian metrics h on V and $\omega = \frac{i}{2\pi}\sum \omega_{ij} dz_i \wedge d\overline{z}_j$ on X. Denote by $\Theta_{V,h} = -\frac{i}{2\pi}\sum c_{ij\alpha\beta} dz_i \wedge d\overline{z}_j \otimes e_\alpha^* \otimes e_\beta$ the curvature tensor of V with respect to an h-orthonormal frame (e_α), and put*

$$\eta(z) = \Theta_{\det(V^*),\det h^*} = \frac{i}{2\pi}\sum_{1\leqslant i,j\leqslant n} \eta_{ij} dz_i \wedge d\overline{z}_j, \qquad \eta_{ij} = \sum_{1\leqslant \alpha\leqslant r} c_{ij\alpha\alpha}.$$

Finally consider the k-jet line bundle $L_k = \mathcal{O}_{X_k^{\mathrm{GG}}}(1) \to X_k^{\mathrm{GG}}$ equipped with the induced metric $\Psi^*_{h,p,\varepsilon}$ (as defined above, with $1 = \varepsilon_1 \gg \varepsilon_2 \gg \ldots \gg \varepsilon_k > 0$). When k tends to infinity, the integral of the top power of the curvature of L_k on its q-index set $X_k^{\mathrm{GG}}(L_k, q)$ is given by

$$\int_{X_k^{\mathrm{GG}}(L_k,q)} \Theta^{n+kr-1}_{L_k,\Psi^*_{h,p,\varepsilon}} = \frac{(\log k)^n}{n!\,(k!)^r}\bigg(\int_X \mathbb{1}_{\eta,q}\eta^n + O((\log k)^{-1})\bigg)$$

for all $q = 0, 1, \ldots, n$, and the error term $O((\log k)^{-1})$ can be bounded explicitly in terms of Θ_V, η and ω. Moreover, the left hand side is identically zero for $q > n$.

The final statement follows from the observation that the curvature of L_k is positive along the fibers of $X_k^{\mathrm{GG}} \to X$, by the plurisubharmonicity of the weight (this is true even when the partition of unity terms are taken into account, since they depend only on the base); therefore the q-index sets are empty for $q > n$. We start with three elementary lemmas.

11.44. Lemma. *The integral*

$$I_{k,r,n} = \int_{\Delta_{k-1}} \Big(\sum_{1\leqslant s\leqslant k} \frac{x_s}{s}\Big)^n d\nu_{k,r}(x)$$

is given by the expansion

$$\text{(a)} \qquad I_{k,r,n} = \sum_{1\leqslant s_1,s_2,\ldots,s_n\leqslant k} \frac{1}{s_1 s_2 \ldots s_n}\,\frac{(kr-1)!}{(r-1)!^k}\,\frac{\prod_{1\leqslant i\leqslant k}(r-1+\beta_i)!}{(kr+n-1)!}.$$

where $\beta_i = \beta_i(s) = \mathrm{card}\{j\,;\ s_j = i\}$, $\sum \beta_i = n$, $1 \leqslant i \leqslant k$. The quotient

$$I_{k,r,n}\Big/ \frac{r^n}{kr(kr+1)\ldots(kr+n-1)}\Big(1+\frac12+\ldots+\frac1k\Big)^n$$

is bounded below by 1 and bounded above by

$$\text{(b)} \qquad 1 + \frac13 \sum_{m=2}^{n} \frac{2^m n!}{(n-m)!}\Big(1+\frac12+\ldots+\frac1k\Big)^{-m} = 1 + O((\log k)^{-2})$$

As a consequence

$$\text{(c)}\qquad I_{k,r,n} = \frac{1}{k^n}\Big(\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)^n + O((\log k)^{n-2})\Big) = \frac{(\log k+\gamma)^n + O((\log k)^{n-2})}{k^n}$$

where γ is the Euler-Mascheroni constant.

Proof. Let us expand the n-th power $\big(\sum_{1\leqslant s\leqslant k}\frac{x_s}{s}\big)^n$. This gives

$$I_{k,r,n} = \sum_{1\leqslant s_1,s_2,\ldots,s_n\leqslant k}\frac{1}{s_1s_2\ldots s_n}\int_{\Delta_{k-1}} x_1^{\beta_1}\ldots x_k^{\beta_k}\,d\nu_{k,r}(x)$$

and by definition of the measure $\nu_{k,r}$ we have

$$\int_{\Delta_{k-1}} x_1^{\beta_1}\ldots x_k^{\beta_k}\,d\nu_{k,r}(x) = \frac{(kr-1)!}{(r-1)!^k}\int_{\Delta_{k-1}} x_1^{r+\beta_1-1}\ldots x_k^{r+\beta_k-1}\,dx_1\ldots dx_k.$$

By Formula (11.24), we find

$$\begin{aligned}\int_{\Delta_{k-1}} x_1^{\beta_1}\ldots x_k^{\beta_k}\,d\nu_{k,r}(x) &= \frac{(kr-1)!}{(r-1)!^k}\,\frac{\prod_{1\leqslant i\leqslant k}(r+\beta_i-1)!}{(kr+n-1)!}\\ &= \frac{r^n\prod_{i,\,\beta_i\geqslant 1}(1+\frac{1}{r})(1+\frac{2}{r})\ldots(1+\frac{\beta_i-1}{r})}{kr(kr+1)\ldots(kr+n-1)},\end{aligned}$$

and (11.44 a) follows from the first equality. The final product is minimal when $r=1$, thus

$$\frac{r^n}{kr(kr+1)\ldots(kr+n-1)} \leqslant \int_{\Delta_{k-1}} x_1^{\beta_1}\ldots x_k^{\beta_k}\,d\nu_{k,r}(x) \leqslant \frac{r^n\prod_{1\leqslant i\leqslant k}\beta_i!}{kr(kr+1)\ldots(kr+n-1)}. \tag{11.45}$$

Also, the integral is maximal when all β_i vanish except one, in which case one gets

$$\int_{\Delta_{k-1}} x_j^n\,d\nu_{k,r}(x) = \frac{r(r+1)\ldots(r+n-1)}{kr(kr+1)\ldots(kr+n-1)}. \tag{11.46}$$

By (11.45), we find the lower and upper bounds

$$I_{k,r,n} \geqslant \frac{r^n}{kr(kr+1)\ldots(kr+n-1)}\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)^n, \tag{11.47}$$

$$I_{k,r,n} \leqslant \frac{r^n}{kr(kr+1)\ldots(kr+n-1)}\sum_{1\leqslant s_1,\ldots,s_n\leqslant k}\frac{\beta_1!\ldots\beta_k!}{s_1\ldots s_n}. \tag{11.48}$$

In order to make the upper bound more explicit, we reorganize the n-tuple $(s_1,\ldots,s_n)$ into those indices $t_1<\ldots<t_\ell$ which appear a certain number of times $\alpha_i=\beta_{t_i}\geqslant 2$, and those, say $t_{\ell+1}<\ldots<t_{\ell+m}$, which appear only once. We have of course $\sum\beta_i = n-m$, and

each choice of the t_i's corresponds to $n!/\alpha_1!\ldots\alpha_\ell!$ possibilities for the n-tuple $(s_1,\ldots,s_n)$. Therefore we get

$$\sum_{1\leqslant s_1,\ldots,s_n\leqslant k}\frac{\beta_1!\ldots\beta_k!}{s_1\ldots s_n}\leqslant n!\sum_{m=0}^{n}\ \sum_{\ell,\,\Sigma\alpha_i=n-m}\ \sum_{(t_i)}\frac{1}{t_1^{\alpha_1}\ldots t_\ell^{\alpha_\ell}}\frac{1}{t_{\ell+1}\ldots t_{\ell+m}}.$$

A trivial comparison series vs. integral yields

$$\sum_{s<t<+\infty}\frac{1}{t^\alpha}\leqslant\frac{1}{\alpha-1}\frac{1}{s^{\alpha-1}}$$

and in this way, using successive integrations in t_ℓ, $t_{\ell-1}$, ... , we get inductively

$$\sum_{1\leqslant t_1<\ldots<t_\ell<+\infty}\frac{1}{t_1^{\alpha_1}\ldots t_\ell^{\alpha_\ell}}\leqslant\frac{1}{\prod_{1\leqslant i\leqslant\ell}(\alpha_{\ell-i+1}+\ldots+\alpha_\ell-i)}\leqslant\frac{1}{\ell!},$$

since $\alpha_i\geqslant 2$ implies $\alpha_{\ell-i+1}+\ldots+\alpha_\ell-i\geqslant i$. On the other hand

$$\sum_{1\leqslant t_{\ell+1}<\ldots<t_{\ell+m}\leqslant k}\frac{1}{t_{\ell+1}\ldots t_{\ell+m}}\leqslant\frac{1}{m!}\sum_{1\leqslant s_1,\ldots,s_m\leqslant k}\frac{1}{s_1\ldots s_m}=\frac{1}{m!}\left(1+\frac{1}{2}+\ldots+\frac{1}{k}\right)^m.$$

Since partitions $\alpha_1+\ldots+\alpha_\ell=n-m$ satisfying the additional restriction $\alpha_i\geqslant 2$ correspond to $\alpha_i'=\alpha_i-2$ satisfying $\sum\alpha_i'=n-m-2\ell$, their number is equal to

$$\binom{n-m-2\ell+\ell-1}{\ell-1}=\binom{n-m-\ell-1}{\ell-1}\leqslant 2^{n-m-\ell-1}$$

and we infer from this

$$\sum_{1\leqslant s_1,\ldots,s_n\leqslant k}\frac{\beta_1!\ldots\beta_k!}{s_1\ldots s_n}\leqslant\sum_{\substack{\ell\geqslant1\\2\ell+m\leqslant n}}\frac{2^{n-m-\ell-1}n!}{\ell!\,m!}\left(1+\frac{1}{2}+\ldots+\frac{1}{k}\right)^m+\left(1+\frac{1}{2}+\ldots+\frac{1}{k}\right)^n$$

where the last term corresponds to the special case $\ell=0$, $m=n$. Therefore

$$\begin{aligned}\sum_{1\leqslant s_i\leqslant k}\frac{\beta_1!\ldots\beta_k!}{s_1\ldots s_n}&\leqslant\frac{e^{1/2}-1}{2}\sum_{m=0}^{n-2}\frac{2^{n-m}n!}{m!}\left(1+\frac{1}{2}+\ldots+\frac{1}{k}\right)^m+\left(1+\frac{1}{2}+\ldots+\frac{1}{k}\right)^n\\&\leqslant\frac{1}{3}\sum_{m=2}^{n}\frac{2^m n!}{(n-m)!}\left(1+\frac{1}{2}+\ldots+\frac{1}{k}\right)^{n-m}+\left(1+\frac{1}{2}+\ldots+\frac{1}{k}\right)^n.\end{aligned}$$

This estimate combined with (11.47, 11.48) implies the upper bound (11.44 b) (the lower bound 1 being now obvious). The asymptotic estimate (11.44 c) follows immediately. □

11.49. Lemma. *If A is a Hermitian $n\times n$ matrix, set $\mathbb{1}_{A,q}$ to be equal to 1 if A has signature $(n-q,q)$ and 0 otherwise. Then for all $n\times n$ Hermitian matrices A, B we have the estimate*

$$\big|\mathbb{1}_{A,q}\det A-\mathbb{1}_{B,q}\det B\big|\leqslant\|A-B\|\sum_{0\leqslant i\leqslant n-1}\|A\|^i\|B\|^{n-1-i},$$

where $\|A\|$, $\|B\|$ are the Hermitian operator norms of the matrices.

Proof. We first check that the estimate holds for $|\det A - \det B|$. Let $\lambda_1 \leqslant \ldots \leqslant \lambda_n$ be the eigenvalues of A and $\lambda'_1 \leqslant \ldots \leqslant \lambda'_n$ be the eigenvalues of B. We have $|\lambda_i| \leqslant \|A\|$, $|\lambda'_i| \leqslant \|B\|$ and the minimax principle implies that $|\lambda_i - \lambda'_i| \leqslant \|A - B\|$. We then get the desired estimate by writing

$$\det A - \det B = \lambda_1 \ldots \lambda_n - \lambda'_1 \ldots \lambda'_n = \sum_{1\leqslant i\leqslant n} \lambda_1 \ldots \lambda_{i-1}(\lambda_i - \lambda'_i)\lambda'_{i+1} \ldots \lambda'_n.$$

This already implies (11.49) if A or B is degenerate. If A and B are non degenerate we only have to prove the result when one of them (say A) has signature $(n-q,q)$ and the other one (say B) has a different signature. If we put $M(t) = (1-t)A + tB$, the already established estimate for the determinant yields

$$\Big|\frac{d}{dt} \det M(t)\Big| \leqslant n\|A - B\|\ \|M(t)\| \leqslant n\|A - B\|\big((1-t)\|A\| + t\|B\|\big)^{n-1}.$$

However, since the signature of $M(t)$ is not the same for $t = 0$ and $t = 1$, there must exist $t_0 \in\ ]0,1[$ such that $(1-t_0)A + t_0B$ is degenerate. Our claim follows by integrating the differential estimate on the smallest such interval $[0, t_0]$, after observing that $M(0) = A$, $\det M(t_0) = 0$, and that the integral of the right hand side on $[0,1]$ is the announced bound. $\square$

11.50. Lemma. *Let Q_A be the Hermitian quadratic form associated with the Hermitian operator A on $\mathbb{C}^n$. If μ is the rotation invariant probability measure on the unit sphere S^{2n-1} of $\mathbb{C}^n$ and λ_i are the eigenvalues of A, we have*

$$\int_{|\zeta|=1} |Q_A(\zeta)|^2 d\mu(\zeta) = \frac{1}{n(n+1)}\Big(\sum \lambda_i^2 + \Big(\sum \lambda_i\Big)^2\Big).$$

The norm $\|A\| = \max |\lambda_i|$ satisfies the estimate

$$\frac{1}{n^2}\|A\|^2 \leqslant \int_{|\zeta|=1} |Q_A(\zeta)|^2 d\mu(\zeta) \leqslant \|A\|^2.$$

Proof. The first identity is an easy calculation, and the inequalities follow by computing the eigenvalues of the quadratic form $\sum \lambda_i^2 + \big(\sum \lambda_i\big)^2 - c\lambda_{i_0}^2$, $c > 0$. The lower bound is attained e.g. for $Q_A(\zeta) = |\zeta_1|^2 - \frac{1}{n}(|\zeta_2|^2 + \ldots + |\zeta_n|^2)$ when we take $i_0 = 1$ and $c = 1 + \frac{1}{n}$. $\square$

Proof of the Probabilistic estimate 11.43. Take a vector $\zeta \in T_{X,z}$, $\zeta = \sum \zeta_i \frac{\partial}{\partial z_i}$, with $\|\zeta\|_\omega = 1$, and introduce the trace free sesquilinear quadratic form

$$Q_{z,\zeta}(u) = \sum_{i,j,\alpha,\beta} \widetilde{c}_{ij\alpha\beta}(z)\, \zeta_i \overline{\zeta}_j\, u_\alpha \overline{u}_\beta, \qquad \widetilde{c}_{ij\alpha\beta} = c_{ij\alpha\beta} - \frac{1}{r}\eta_{ij}\delta_{\alpha\beta}, \qquad u \in \mathbb{C}^r$$

where $\eta_{ij} = \sum_{1\leqslant\alpha\leqslant r} c_{ij\alpha\alpha}$. We consider the corresponding trace free curvature tensor

$$\widetilde{\Theta}_V = \frac{i}{2\pi} \sum_{i,j,\alpha,\beta} \widetilde{c}_{ij\alpha\beta}\, dz_i \wedge d\overline{z}_j \otimes e^*_\alpha \otimes e_\beta. \tag{11.51}$$

As a general matter of notation, we adopt here the convention that the canonical correspondence between Hermitian forms and $(1,1)$-forms is normalized as $\sum a_{ij}dz_i \otimes d\overline{z}_j \leftrightarrow \frac{i}{2\pi}\sum a_{ij}dz_i \wedge d\overline{z}_j$, and we take the liberty of using the same symbols for both types of objects; we do so especially for $g_k(z,x,u)$ and $\eta(z) = \frac{i}{2\pi}\sum \eta_{ij}(z)dz_i \wedge d\overline{z}_j = \mathrm{Tr}\,\Theta_V(z)$. First observe that for all k-tuples of unit vectors $u = (u_1,\ldots,u_k) \in (S^{2r-1})^k$, $u_s = (u_{s\alpha})_{1\leqslant\alpha\leqslant r}$, we have

$$\int_{(S^{2r-1})^k}\Bigg|\sum_{1\leqslant s\leqslant k}\frac{1}{s}x_s\sum_{i,j,\alpha,\beta}\widetilde{c}_{ij\alpha\beta}(z)\,\zeta_i\overline{\zeta}_j u_{s\alpha}\overline{u}_{s\beta}\Bigg|^2 d\mu(u) = \sum_{1\leqslant s\leqslant k}\frac{x_s^2}{s^2}\mathbf{V}(Q_{z,\zeta})$$

where $\mathbf{V}(Q_{z,\zeta})$ is the variance of $Q_{z,\zeta}$ on S^{2r-1}. This is so because we have a sum over s of independent random variables on $(S^{2r-1})^k$, all of which have zero mean value (Lemma 11.50 shows that the variance $\mathbf{V}(Q)$ of a trace free Hermitian quadratic form $Q(u) = \sum_{1\leqslant\alpha\leqslant r}\lambda_\alpha|u_\alpha|^2$ on the unit sphere S^{2r-1} is equal to $\frac{1}{r(r+1)}\sum\lambda_\alpha^2$, but we only give the formula to fix the ideas). Formula (11.46) yields

$$\int_{\Delta_{k-1}} x_s^2 d\nu_{k,r}(x) = \frac{r+1}{k(kr+1)}.$$

Therefore, according to notation (11.39), we obtain the partial variance formula

$$\int_{\Delta_{k-1}\times(S^{2r-1})^k}\big|g_k(z,x,u)(\zeta) - \overline{g}_k(z,x)(\zeta)\big|^2 d\nu_{k,r}(x)d\mu(u)$$
$$= \frac{(r+1)}{k(kr+1)}\Bigg(\sum_{1\leqslant s\leqslant k}\frac{1}{s^2}\Bigg)\sigma_h(\widetilde{\Theta}_V(\zeta,\zeta))^2$$

in which

$$\overline{g}_k(z,x)(\zeta) = \sum_{1\leqslant s\leqslant k}\frac{1}{s}x_s\frac{1}{r}\sum_{ij\alpha}c_{ij\alpha\alpha}\zeta_i\overline{\zeta}_j = \Bigg(\sum_{1\leqslant s\leqslant k}\frac{1}{s}x_s\Bigg)\frac{1}{r}\eta(z)(\zeta),$$
$$\sigma_h(\widetilde{\Theta}_V(\zeta,\zeta))^2 = \mathbf{V}\big(u \mapsto \langle\widetilde{\Theta}_V(\zeta,\zeta)u,u\rangle_h\big) = \int_{u\in S^{2r-1}}\big|\langle\widetilde{\Theta}_V(\zeta,\zeta)u,u\rangle_h\big|^2 d\mu(u).$$

By integrating over $\zeta \in S^{2n-1} \subset \mathbb{C}^n$ and applying the left hand inequality in Lemma 11.50 we infer

$$\int_{\Delta_{k-1}\times(S^{2r-1})^k}\|g_k(z,x,u) - \overline{g}_k(z,x)\|_\omega^2 d\nu_{k,r}(x)d\mu(u)$$
$$\leqslant \frac{n^2(r+1)}{k(kr+1)}\Bigg(\sum_{1\leqslant s\leqslant k}\frac{1}{s^2}\Bigg)\sigma_{\omega,h}(\widetilde{\Theta}_V)^2 \tag{11.52}$$

where $\sigma_{\omega,h}(\widetilde{\Theta}_V)$ is the standard deviation of $\widetilde{\Theta}_V$ on $S^{2n-1}\times S^{2r-1}$:

$$\sigma_{\omega,h}(\widetilde{\Theta}_V)^2 = \int_{|\zeta|_\omega=1,\,|u|_h=1}\big|\langle\widetilde{\Theta}_V(\zeta,\zeta)u,u\rangle_h\big|^2 d\mu(\zeta)\,d\mu(u).$$

On the other hand, brutal estimates give the Hermitian operator norm estimates

$$\|\overline{g}_k(z,x)\|_\omega \leqslant \Bigg(\sum_{1\leqslant s\leqslant k}\frac{1}{s}x_s\Bigg)\frac{1}{r}\|\eta(z)\|_\omega, \tag{11.53}$$

$$\|g_k(z,x,u)\|_\omega \leqslant \Bigg(\sum_{1\leqslant s\leqslant k}\frac{1}{s}x_s\Bigg)\|\Theta_V\|_{\omega,h} \tag{11.54}$$

where

$$\|\Theta_V\|_{\omega,h} = \sup_{|\zeta|_\omega=1,\,|u|_h=1} \big|\langle\Theta_V(\zeta,\zeta)u,u\rangle_h\big|.$$

We use these estimates to evaluate the q-index integrals. The integral associated with $\overline{g}_k(z,x)$ is much easier to deal with than $g_k(z,x,u)$ since the characteristic function of the q-index set depends only on z. By Lemma 11.49 we find

$$\begin{aligned}
&\big|\mathbb{1}_{g_k,q}(z,x,u)\det g_k(z,x,u) - \mathbb{1}_{\eta,q}(z)\det\overline{g}_k(z,x)\big|\\
&\qquad\leqslant \big\|g_k(z,x,u)-\overline{g}_k(z,x)\big\|_\omega \sum_{0\leqslant i\leqslant n-1}\|g_k(z,x,u)\|_\omega^i\|\overline{g}_k(z,x)\|_\omega^{n-1-i}.
\end{aligned}$$

The Cauchy-Schwarz inequality combined with (11.52 – 11.54) implies

$$\begin{aligned}
&\int_{\Delta_{k-1}\times(S^{2r-1})^k}\big|\mathbb{1}_{g_k,q}(z,x,u)\det g_k(z,x,u) - \mathbb{1}_{\eta,q}(z)\det\overline{g}_k(z,x)\big|\,d\nu_{k,r}(x)d\mu(u)\\
&\leqslant\bigg(\int_{\Delta_{k-1}\times(S^{2r-1})^k}\big\|g_k(z,x,u)-\overline{g}_k(z,x)\big\|_\omega^2 d\nu_{k,r}(x)d\mu(u)\bigg)^{1/2}\times\\
&\quad\bigg(\int_{\Delta_{k-1}\times(S^{2r-1})^k}\Big(\sum_{0\leqslant i\leqslant n-1}\|g_k(z,x,u)\|_\omega^i\|\overline{g}_k(z,x)\|_\omega^{n-1-i}\Big)^2 d\nu_{k,r}(x)d\mu(u)\bigg)^{1/2}\\
&\leqslant\frac{n(1+1/r)^{1/2}}{(k(k+1/r))^{1/2}}\Big(\sum_{1\leqslant s\leqslant k}\frac{1}{s^2}\Big)^{1/2}\sigma_{\omega,h}(\widetilde{\Theta}_V)\sum_{1\leqslant i\leqslant n-1}\|\Theta_V\|_{\omega,h}^i\Big(\frac{1}{r}\|\eta(z)\|_\omega\Big)^{n-1-i}\\
&\qquad\times\bigg(\int_{\Delta_{k-1}}\Big(\sum_{1\leqslant s\leqslant k}\frac{x_s}{s}\Big)^{2n-2}d\nu_{k,r}(x)\bigg)^{1/2} = O\Big(\frac{(\log k)^{n-1}}{k^n}\Big)
\end{aligned}$$

by Lemma 11.44 with n replaced by $2n-2$. This is the essential error estimate. As one can see, the growth of the error mainly depends on the final integral factor, since the initial multiplicative factor is uniformly bounded over X. In order to get the principal term, we compute

$$\begin{aligned}
\int_{\Delta_{k-1}}\det\overline{g}_k(z,x)\,d\nu_{k,r}(x) &= \frac{1}{r^n}\det\eta(z)\int_{\Delta_{k-1}}\Big(\sum_{1\leqslant s\leqslant k}\frac{x_s}{s}\Big)^n d\nu_{k,r}(x)\\
&\sim\frac{(\log k)^n}{r^nk^n}\det\eta(z).
\end{aligned}$$

From there we conclude that

$$\begin{aligned}
&\int_{z\in X}\int_{(x,u)\in\Delta_{k-1}\times(S^{2r-1})^k}\mathbb{1}_{g_k,q}(z,x,u)g_k(z,x,u)^n\,d\nu_{k,r}(x)d\mu(u)\\
&\qquad\qquad=\frac{(\log k)^n}{r^nk^n}\int_X\mathbb{1}_{\eta,q}\eta^n + O\Big(\frac{(\log k)^{n-1}}{k^n}\Big)
\end{aligned}$$

The probabilistic estimate 11.43 follows by (11.41). □

11.55. Remark. If we take care of the precise bounds obtained above, the proof gives in fact the explicit estimate

$$\int_{X_k^{\mathrm{GG}}(L_k,q)}\Theta^{n+kr-1}_{L_k,\Psi^*_{h,p,\varepsilon}} = \frac{(n+kr-1)!\,I_{k,r,n}}{n!(k!)^r(kr-1)!}\Big(\int_X\mathbb{1}_{\eta,q}\eta^n+\varepsilon_{k,r,n}J\Big)$$

where

$$J = n\,(1+1/r)^{1/2}\Big(\sum_{s=1}^k \frac{1}{s^2}\Big)^{1/2}\int_X \sigma_{\omega,h}(\widetilde\Theta_V)\sum_{i=1}^{n-1} r^{i+1}\|\Theta_V\|^i_{\omega,h}\|\eta(z)\|^{n-1-i}_\omega\omega^n$$

and

$$\begin{aligned}|\varepsilon_{k,r,n}| &\leqslant \frac{\Big(\int_{\Delta_{k-1}}\Big(\sum_{s=1}^k\frac{x_s}{s}\Big)^{2n-2}d\nu_{k,r}(x)\Big)^{1/2}}{(k(k+1/r))^{1/2}\int_{\Delta_{k-1}}\Big(\sum_{s=1}^k\frac{x_s}{s}\Big)^{n}d\nu_{k,r}(x)}\\ &\leqslant \frac{\Big(1+\frac13\sum_{m=2}^{2n-2}\frac{2^m(2n-2)!}{(2n-2-m)!}\big(1+\frac12+\ldots+\frac1k\big)^{-m}\Big)^{1/2}}{1+\frac12+\ldots+\frac1k}\sim\frac{1}{\log k}\end{aligned}$$

by the lower and upper bounds of $I_{k,r,n}$, $I_{k,r,2n-2}$ obtained in Lemma 11.44. As $(2n-2)!/(2n-2-m)!\leqslant(2n-2)^m$, one easily shows that

$$|\varepsilon_{k,r,n}|\leqslant\frac{(31/15)^{1/2}}{\log k}\qquad\text{for } k\geqslant e^{5n-5}. \tag{11.56}$$

Also, we see that the error terms vanish if $\widetilde\Theta_V$ is identically zero, but this is of course a rather unexpected circumstance. In general, since the form $\widetilde\Theta_V$ is trace free, Lemma 11.50 applied to the quadratic form $u\mapsto\langle\Theta_V(\zeta,\zeta)u,u\rangle$ on $\mathbb{C}^r$ implies

$$\sigma_{\omega,h}(\widetilde\Theta_V)\leqslant(r+1)^{-1/2}\|\widetilde\Theta_V\|_{\omega,h}.$$

This yields the simpler bound

$$J\leqslant n\,r^{1/2}\Big(\sum_{s=1}^k\frac{1}{s^2}\Big)^{1/2}\int_X\|\widetilde\Theta_V\|_{\omega,h}\sum_{i=1}^{n-1}r^i\|\Theta_V\|^i_{\omega,h}\|\eta(z)\|^{n-1-i}_\omega\omega^n. \tag{11.57}$$

□

It will be useful to extend the above estimates to the case of sections of

$$L_k=\mathcal{O}_{X_k^{\rm GG}}(1)\otimes\pi_k^*\mathcal{O}\Big(\frac{1}{kr}\Big(1+\frac12+\ldots+\frac1k\Big)F\Big) \tag{11.58}$$

where $F\in\mathrm{Pic}_{\mathbb{Q}}(X)$ is an arbitrary $\mathbb{Q}$-line bundle on X and $\pi_k: X_k^{\rm GG}\to X$ is the natural projection. We assume here that F is also equipped with a smooth Hermitian metric h_F. In formulas (11.41–11.43), the renormalized curvature $\eta_k(z,x,u)$ of L_k takes the form

$$\eta_k(z,x,u)=\frac{1}{\frac{1}{kr}(1+\frac12+\ldots+\frac1k)}g_k(z,x,u)+\Theta_{F,h_F}(z), \tag{11.59}$$

and by the same calculations its expected value is

$$\eta(z):=\mathbf{E}(\eta_k(z,\bullet,\bullet))=\Theta_{\det V^*,\det h^*}(z)+\Theta_{F,h_F}(z). \tag{11.60}$$

Then the variance estimate for $\eta_k-\eta$ is unchanged, and the L^p bounds for η_k are still valid, since our forms are just shifted by adding the constant smooth term $\Theta_{F,h_F}(z)$. The probabilistic estimate 11.43 is therefore still true in exactly the same form, provided we use (11.58 – 11.60) instead of the previously defined L_k, η_k and η. An application of holomorphic Morse inequalities gives the desired cohomology estimates for

$$\begin{aligned} h^q\Big(X, E^{\rm GG}_{k,m}V^*\otimes\mathcal{O}\Big(\frac{m}{kr}\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)F\Big)\Big) \\ = h^q(X^{\rm GG}_k, \mathcal{O}_{X^{\rm GG}_k}(m)\otimes\pi_k^*\mathcal{O}\Big(\frac{m}{kr}\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)F\Big)\Big),\end{aligned}$$

provided m is sufficiently divisible to give a multiple of F which is a $\mathbb{Z}$-line bundle.

11.61. Theorem. *Let (X,V) be a directed manifold, $F\to X$ a $\mathbb{Q}$-line bundle, (V,h) and (F,h_F) smooth Hermitian structure on V and F respectively. We define*

$$\begin{aligned} L_k &= \mathcal{O}_{X^{\rm GG}_k}(1)\otimes\pi_k^*\mathcal{O}\Big(\frac{1}{kr}\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)F\Big),\\ \eta &= \Theta_{\det V^*,\det h^*}+\Theta_{F,h_F}.\end{aligned}$$

Then for all $q\geqslant 0$ and all $m\gg k\gg 1$ such that m is sufficiently divisible, we have

(a) $$h^q(X^{\rm GG}_k,\mathcal{O}(L_k^{\otimes m}))\leqslant\frac{m^{n+kr-1}}{(n+kr-1)!}\frac{(\log k)^n}{n!\,(k!)^r}\Big(\int_{X(\eta,q)}(-1)^q\eta^n+O((\log k)^{-1})\Big),$$

(b) $$h^0(X^{\rm GG}_k,\mathcal{O}(L_k^{\otimes m}))\geqslant\frac{m^{n+kr-1}}{(n+kr-1)!}\frac{(\log k)^n}{n!\,(k!)^r}\Big(\int_{X(\eta,\leqslant 1)}\eta^n-O((\log k)^{-1})\Big),$$

(c) $$\chi(X^{\rm GG}_k,\mathcal{O}(L_k^{\otimes m}))=\frac{m^{n+kr-1}}{(n+kr-1)!}\frac{(\log k)^n}{n!\,(k!)^r}\big(c_1(V^*\otimes F)^n+O((\log k)^{-1})\big).$$

Green and Griffiths [GrGr79] already checked the Riemann-Roch calculation (11.61 c) in the special case $V=T^*_X$ and $F=\mathcal{O}_X$. Their proof is much simpler since it relies only on Chern class calculations, but it cannot provide any information on the individual cohomology groups, except in very special cases where vanishing theorems can be applied; in fact in dimension 2, the Euler characteristic satisfies $\chi=h^0-h^1+h^2\leqslant h^0+h^2$, hence it is enough to get the vanishing of the top cohomology group H^2 to infer $h^0\geqslant\chi$; this works for surfaces by means of a well-known vanishing theorem of Bogomolov which implies in general

$$H^n\Big(X, E^{\rm GG}_{k,m}T^*_X\otimes\mathcal{O}\Big(\frac{m}{kr}\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)F\Big)\Big)\Big)=0$$

as soon as $K_X\otimes F$ is big and $m\gg 1$.

In fact, thanks to Bonavero's singular holomorphic Morse inequalities [Bon93], everything works almost unchanged in the case where $V\subset T_X$ has singularities and h is an admissible metric on V (see (11.8)). We only have to find a blow-up $\mu:\widetilde{X}_k\to X_k$ so that the resulting pull-backs μ^*L_k and μ^*V are locally free, and $\mu^*\det h^*$, $\mu^*\Psi_{h,p,\varepsilon}$ only have divisorial singularities. Then η is a $(1,1)$-current with logarithmic poles, and we have to deal with smooth metrics on $\mu^*L_k^{\otimes m}\otimes\mathcal{O}(-mE_k)$ where E_k is a certain effective divisor on X_k (which, by our assumption (11.8), does not project onto X). The cohomology groups involved are then the twisted cohomology groups

$$H^q(X^{\rm GG}_k,\mathcal{O}(L_k^{\otimes m})\otimes\mathcal{J}_{k,m})$$

where $\mathcal{J}_{k,m} = \mu_*(\mathcal{O}(-mE_k))$ is the corresponding multiplier ideal sheaf, and the Morse integrals need only be evaluated in the complement of the poles, that is on $X(\eta, q) \smallsetminus S$ where $S = \mathrm{Sing}(V) \cup \mathrm{Sing}(h)$. Since

$$(\pi_k)_*\big(\mathcal{O}(L_k^{\otimes m}) \otimes \mathcal{J}_{k,m}\big) \subset E_{k,m}^{\mathrm{GG}} V^* \otimes \mathcal{O}\Big(\frac{m}{kr}\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)F\Big)\Big)$$

we still get a lower bound for the H^0 of the latter sheaf (or for the H^0 of the un-twisted line bundle $\mathcal{O}(L_k^{\otimes m})$ on X_k^{GG}). If we assume that $K_V \otimes F$ is big, these considerations also allow us to obtain a strong estimate in terms of the volume, by using an approximate Zariski decomposition on a suitable blow-up of (X, V). The following corollary implies in particular Theorem 11.5.

11.62. Corollary. *If F is an arbitrary $\mathbb{Q}$-line bundle over X, one has*

$$\begin{aligned} &h^0\bigg(X_k^{\mathrm{GG}}, \mathcal{O}_{X_k^{\mathrm{GG}}}(m) \otimes \pi_k^*\mathcal{O}\Big(\frac{m}{kr}\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)F\Big)\bigg) \\ &\qquad \geqslant \frac{m^{n+kr-1}}{(n+kr-1)!}\,\frac{(\log k)^n}{n!\,(k!)^r}\Big(\mathrm{Vol}(K_V\otimes F) - O((\log k)^{-1})\Big) - o(m^{n+kr-1}),\end{aligned}$$

when $m \gg k \gg 1$, in particular there are many sections of the k-jet differentials of degree m twisted by the appropriate power of F if $K_V \otimes F$ is big.

Proof. The volume is computed here as usual, i.e. after performing a suitable modification $\mu : \widetilde{X} \to X$ which converts K_V into an invertible sheaf. There is of course nothing to prove if $K_V \otimes F$ is not big, so we can assume $\mathrm{Vol}(K_V \otimes F) > 0$. Let us fix smooth Hermitian metrics h_0 on T_X and h_F on F. They induce a metric $\mu^*(\det h_0^{-1} \otimes h_F)$ on $\mu^*(K_V \otimes F)$ which, by our definition of K_V, is a smooth metric. By the result of Fujita [Fuj94] on approximate Zariski decomposition, for every $\delta > 0$, one can find a modification $\mu_\delta : \widetilde{X}_\delta \to X$ dominating μ such that

$$\mu_\delta^*(K_V \otimes F) = \mathcal{O}_{\widetilde{X}_\delta}(A + E)$$

where A and E are $\mathbb{Q}$-divisors, A ample and E effective, with

$$\mathrm{Vol}(A) = A^n \geqslant \mathrm{Vol}(K_V \otimes F) - \delta.$$

If we take a smooth metric h_A with positive definite curvature form Θ_{A,h_A}, then we get a singular Hermitian metric $h_A h_E$ on $\mu_\delta^*(K_V \otimes F)$ with poles along E, i.e. the quotient $h_A h_E/\mu^*(\det h_0^{-1} \otimes h_F)$ is of the form $e^{-\varphi}$ where φ is quasi-psh with log poles $\log|\sigma_E|^2$ (mod $C^\infty(\widetilde{X}_\delta)$) precisely given by the divisor E. We then only need to take the singular metric h on T_X defined by

$$h = h_0 e^{\frac{1}{r}(\mu_\delta)^*\varphi}$$

(the choice of the factor $\frac{1}{r}$ is there to correct adequately the metric on $\det V$). By construction h induces an admissible metric on V and the resulting curvature current $\eta = \Theta_{K_V,\det h^*} + \Theta_{F,h_F}$ is such that

$$\mu_\delta^*\eta = \Theta_{A,h_A} + [E], \qquad [E] = \text{current of integration on } E.$$

Then the 0-index Morse integral in the complement of the poles is given by

$$\int_{X(\eta,0)\smallsetminus S} \eta^n = \int_{\widetilde{X}_\delta} \Theta_{A,h_A}^n = A^n \geqslant \mathrm{Vol}(K_V \otimes F) - \delta$$

and (11.62) follows from the fact that δ can be taken arbitrary small. $\square$

11.63. Example. In some simple cases, the above estimates can lead to very explicit results. Take for instance X to be a smooth complete intersection of multidegree $(d_1, d_2, \ldots, d_s)$ in $\mathbb{P}^{n+s}_{\mathbb{C}}$ and consider the absolute case $V = T_X$. Then

$$K_X = \mathcal{O}_X(d_1 + \ldots + d_s - n - s - 1).$$

Assume that X is of general type, i.e. $\sum d_j > n + s + 1$. Let us equip $V = T_X$ with the restriction of the Fubini-Study metric $h = \Theta_{\mathcal{O}(1)}$; a better choice might be the Kähler-Einstein metric but we want to keep the calculations as elementary as possible. The standard formula for the curvature tensor of a submanifold gives

$$\Theta_{T_X,h} = (\Theta_{T_{\mathbb{P}^{n+s}},h})_{|X} + \beta^* \wedge \beta$$

where $\beta \in C^\infty\big(\Lambda^{1,0}T^*_X \otimes \mathrm{Hom}(T_X, \bigoplus \mathcal{O}(d_j))\big)$ is the second fundamental form. In other words, by the well known formula for the curvature of projective space, we have

$$\langle \Theta_{T_X,h}(\zeta,\zeta)u, u\rangle = |\zeta|^2|u|^2 + |\langle \zeta, u\rangle|^2 - |\beta(\zeta)\cdot u|^2.$$

The curvature ρ of $(K_X, \det h^*)$ (i.e. the opposite of the Ricci form $\mathrm{Tr}\,\Theta_{T_X,h}$) is given by

$$\rho = -\,\mathrm{Tr}\,\Theta_{T_X,h} = \mathrm{Tr}(\beta\wedge\beta^*) - (n+1)h \geqslant -(n+1)h. \tag{11.64}$$

We take here $F = \mathcal{O}_X(-a)$, $a \in \mathbb{Q}_+$, and we want to determine conditions for the existence of sections

$$H^0\bigg(X, E^{\mathrm{GG}}_{k,m}T^*_X \otimes \mathcal{O}\Big(-a\frac{m}{kr}\Big(1 + \frac{1}{2} + \ldots + \frac{1}{k}\Big)\Big)\bigg), \qquad m \gg 1. \tag{11.65}$$

We have to choose $K_X \otimes \mathcal{O}_X(-a)$ ample, i.e. $\sum d_j > n+s+a+1$, and then (by an appropriate choice of the metric of $F = \mathcal{O}_X(-a)$), the form $\eta = \Theta_{K_X\otimes\mathcal{O}_X(-a)}$ can be taken to be any positive form cohomologous to $(\sum d_j - (n+s+a+1))h$. We use remark 11.55 and estimate the error terms by considering the Kähler metric

$$\omega = \rho + (n+s+2)h \equiv \Big(\sum d_j + 1\Big)h.$$

Inequality (11.64) shows that $\omega \geqslant 2h$ and also that $\omega \geqslant \mathrm{Tr}(\beta\wedge\beta^*)$. From this, one easily concludes that $\|\eta\|_\omega \leqslant 1$ by an appropriate choice of η, as well as $\|\Theta_{T_X,h}\|_{\omega,h} \leqslant 1$ and $\|\widetilde{\Theta}_{T_X,h}\|_{\omega,h} \leqslant 2$. By (11.57), we obtain for $n \geqslant 2$

$$J \leqslant n^{3/2}\frac{\pi}{\sqrt{6}} \times 2\,\frac{n^n - 1}{n-1}\int_X \omega^n < \frac{4\pi}{\sqrt{6}}\,n^{n+1/2}\int_X \omega^n$$

where $\int_X \omega^n = \big(\sum d_j + 1\big)^n \deg(X)$. On the other hand, the leading term $\int_X \eta^n$ equals $\big(\sum d_j - n - s - a - 1\big)^n \deg(X)$ with $\deg(X) = d_1 \ldots d_s$. By the bound (11.56) on the error term $\varepsilon_{k,r,n}$, we find that the leading coefficient of the growth of our spaces of sections is strictly controlled below by a multiple of

$$\Big(\sum d_j - n - s - a - 1\Big)^n - 4\pi\Big(\frac{31}{90}\Big)^{1/2}\frac{n^{n+1/2}}{\log k}\Big(\sum d_j + 1\Big)^n$$

if $k \geqslant e^{5n-5}$. A sufficient condition for the existence of sections in (11.65) is thus

$$k \geqslant \exp\Big(7.38\, n^{n+1/2}\Big(\frac{\sum d_j + 1}{\sum d_j - n - s - a - 1}\Big)^n\Big). \tag{11.66}$$

This is good in view of the fact that we can cover arbitrary smooth complete intersections of general type. On the other hand, even when the degrees d_j tend to $+\infty$, we still get a large lower bound $k \sim \exp(7.38\, n^{n+1/2})$ on the order of jets, and this is far from being optimal: Diverio [Div08, Div09] has shown e.g. that one can take $k = n$ for smooth hypersurfaces of high degree. It is however not unlikely that one could improve estimate (11.66) with more careful choices of ω, h. □

§11.D. Non probabilistic estimate of the Morse integrals

We assume here that the curvature tensor $(c_{ij\alpha\beta})$ satisfies a lower bound

$$\sum_{i,j,\alpha,\beta} c_{ij\alpha\beta}\xi_i\overline{\xi}_j u_\alpha \overline{u}_\beta \geqslant -\sum \gamma_{ij}\xi_i\overline{\xi}_j\, |u|^2, \qquad \forall \xi \in T_X,\ u \in X \tag{11.67}$$

for some semipositive $(1,1)$-form $\gamma = \frac{i}{2\pi}\sum \gamma_{ij}(z)\, dz_i \wedge d\overline{z}_j$ on X. This is the same as assuming that the curvature tensor of (V^*, h^*) satisfies the semipositivity condition

$$\Theta_{V^*,h^*} + \gamma \otimes \mathrm{Id}_{V^*} \geqslant 0 \tag{11.67$'$}$$

in the sense of Griffiths, or equivalently $\Theta_{V,h} - \gamma \otimes \mathrm{Id}_V \leqslant 0$. Thanks to the compactness of X, such a form γ always exists if h is an admissible metric on V. Now, instead of replacing Θ_V with its trace free part $\widetilde{\Theta}_V$ and exploiting a Monte Carlo convergence process, we replace Θ_V with $\Theta_V^\gamma = \Theta_V - \gamma \otimes \mathrm{Id}_V \leqslant 0$, i.e. $c_{ij\alpha\beta}$ by $c^\gamma_{ij\alpha\beta} = c_{ij\alpha\beta} + \gamma_{ij}\delta_{\alpha\beta}$. Also, we take a line bundle $F = A^{-1}$ with $\Theta_{A,h_A} \geqslant 0$, i.e. F seminegative. Then our earlier formulas (11.39), (11.58), (11.59) become instead

$$g_k^\gamma(z,x,u) = \frac{i}{2\pi}\sum_{1\leqslant s\leqslant k}\frac{1}{s}x_s \sum_{i,j,\alpha,\beta} c^\gamma_{ij\alpha\beta}(z)\, u_{s\alpha}\overline{u}_{s\beta}\, dz_i \wedge d\overline{z}_j \geqslant 0, \tag{11.68}$$

$$L_k = \mathcal{O}_{X_k^{\mathrm{GG}}}(1) \otimes \pi_k^*\mathcal{O}\Big(-\frac{1}{kr}\Big(1 + \frac{1}{2} + \ldots + \frac{1}{k}\Big)A\Big), \tag{11.69}$$

$$\Theta_{L_k} = \eta_k(z,x,u) = \frac{1}{\frac{1}{kr}(1 + \frac{1}{2} + \ldots + \frac{1}{k})} g_k^\gamma(z,x,u) - (\Theta_{A,h_A}(z) + r\gamma(z)). \tag{11.70}$$

In fact, replacing Θ_V by $\Theta_V - \gamma \otimes \mathrm{Id}_V$ has the effect of replacing $\Theta_{\det V^*} = \mathrm{Tr}\,\Theta_{V^*}$ by $\Theta_{\det V^*} + r\gamma$. The major gain that we have is that $\eta_k = \Theta_{L_k}$ is now expressed as a difference of semipositive $(1,1)$-forms, and we can exploit the following simple lemma, which is the key to derive algebraic Morse inequalities from their analytic form (cf. [Dem94], Theorem 12.3).

11.71. Lemma. *Let $\eta = \alpha - \beta$ be a difference of semipositive $(1,1)$-forms on an n-dimensional complex manifold X, and let $\mathbb{1}_{\eta,\leqslant q}$ be the characteristic function of the open set where η is non degenerate with a number of negative eigenvalues at most equal to q. Then*

$$(-1)^q \mathbb{1}_{\eta,\leqslant q}\, \eta^n \leqslant \sum_{0\leqslant j\leqslant q} (-1)^{q-j}\alpha^{n-j}\beta^j,$$

in particular

$$\mathbb{1}_{\eta,\leqslant 1}\ \eta^n \geqslant \alpha^n - n\alpha^{n-1}\wedge\beta \qquad \textit{for } q=1.$$

Proof. Without loss of generality, we can assume $\alpha>0$ positive definite, so that α can be taken as the base hermitian metric on X. Let us denote by

$$\lambda_1\geqslant\lambda_2\geqslant\ldots\geqslant\lambda_n\geqslant 0$$

the eigenvalues of β with respect to α. The eigenvalues of $\eta=\alpha-\beta$ are then given by

$$1-\lambda_1\leqslant\ldots\leqslant 1-\lambda_q\leqslant 1-\lambda_{q+1}\leqslant\ldots\leqslant 1-\lambda_n,$$

hence the open set $\{\lambda_{q+1}<1\}$ coincides with the support of $\mathbb{1}_{\eta,\leqslant q}$, except that it may also contain a part of the degeneration set $\eta^n=0$. On the other hand we have

$$\binom{n}{j}\alpha^{n-j}\wedge\beta^j=\sigma_n^j(\lambda)\,\alpha^n,$$

where $\sigma_n^j(\lambda)$ is the j-th elementary symmetric function in the λ_j's. Thus, to prove the lemma, we only have to check that

$$\sum_{0\leqslant j\leqslant q}(-1)^{q-j}\sigma_n^j(\lambda)-\mathbb{1}_{\{\lambda_{q+1}<1\}}(-1)^q\prod_{1\leqslant j\leqslant n}(1-\lambda_j)\geqslant 0.$$

This is easily done by induction on n (just split apart the parameter λ_n and write $\sigma_n^j(\lambda)=\sigma_{n-1}^j(\lambda)+\sigma_{n-1}^{j-1}(\lambda)\,\lambda_n$). □

We apply here Lemma 11.71 with

$$\alpha=g_k^\gamma(z,x,u),\qquad \beta=\beta_k=\frac{1}{kr}\Big(1+\frac12+\ldots+\frac1k\Big)(\Theta_{A,h_A}+r\gamma),$$

which are both semipositive by our assumption. The analogue of (11.41) leads to

$$\begin{aligned}&\int_{X_k^{\mathrm{GG}}(L_k,\leqslant 1)}\Theta^{n+kr-1}_{L_k,\Psi^*_{h,p,\varepsilon}}\\ &=\frac{(n+kr-1)!}{n!(k!)^r(kr-1)!}\int_{z\in X}\int_{(x,u)\in\Delta_{k-1}\times(S^{2r-1})^k}\mathbb{1}_{g_k^\gamma-\beta_k,\leqslant 1}\,(g_k^\gamma-\beta_k)^n\,d\nu_{k,r}(x)\,d\mu(u)\\ &\geqslant\frac{(n+kr-1)!}{n!(k!)^r(kr-1)!}\int_{z\in X}\int_{(x,u)\in\Delta_{k-1}\times(S^{2r-1})^k}((g_k^\gamma)^n-n(g_k^\gamma)^{n-1}\wedge\beta_k)\,d\nu_{k,r}(x)\,d\mu(u).\end{aligned}$$

The resulting integral now produces a "closed formula" which can be expressed solely in terms of Chern classes (at least if we assume that γ is the Chern form of some semipositive line bundle). It is just a matter of routine to find a sufficient condition for the positivity of the integral. One can first observe that g_k^γ is bounded from above by taking the trace of $(c_{ij\alpha\beta})$, in this way we get

$$0\leqslant g_k^\gamma\leqslant\Big(\sum_{1\leqslant s\leqslant k}\frac{x_s}{s}\Big)(\Theta_{\det V^*}+r\gamma)$$

where the right hand side no longer depends on $u \in (S^{2r-1})^k$. Also, g_k^γ can be written as a sum of semipositive $(1,1)$-forms

$$g_k^\gamma = \sum_{1\leqslant s\leqslant k} \frac{x_s}{s}\theta^\gamma(u_s), \qquad \theta^\gamma(u) = \sum_{i,j,\alpha,\beta} c^\gamma_{ij\alpha\beta} u_\alpha \overline{u}_\beta \, dz_i \wedge d\overline{z}_j,$$

hence for $k \geqslant n$ we have

$$(g_k^\gamma)^n \geqslant n! \sum_{1\leqslant s_1<\ldots<s_n\leqslant k} \frac{x_{s_1}\ldots x_{s_n}}{s_1\ldots s_n}\,\theta^\gamma(u_{s_1})\wedge\theta^\gamma(u_{s_2})\wedge\ldots\wedge\theta^\gamma(u_{s_n}).$$

Since $\int_{S^{2r-1}}\theta^\gamma(u)\,d\mu(u) = \frac{1}{r}\,\mathrm{Tr}(\Theta_{V^*}+\gamma) = \frac{1}{r}\Theta_{\det V^*}+\gamma$, we infer from this

$$\int_{(x,u)\in\Delta_{k-1}\times(S^{2r-1})^k} (g_k^\gamma)^n\, d\nu_{k,r}(x)\,d\mu(u)$$
$$\geqslant n! \sum_{1\leqslant s_1<\ldots<s_n\leqslant k} \frac{1}{s_1\ldots s_n}\Big(\int_{\Delta_{k-1}} x_1\ldots x_n\, d\nu_{k,r}(x)\Big)\Big(\frac{1}{r}\Theta_{\det V^*}+\gamma\Big)^n.$$

By putting everything together, we conclude:

11.72. Theorem. *Assume that* $\Theta_{V^*}\geqslant -\gamma\otimes \mathrm{Id}_{V^*}$ *with a semipositive* $(1,1)$*-form* γ *on* X. *Then the Morse integral of the line bundle*

$$L_k = \mathcal{O}_{X_k^{\mathrm{GG}}}(1)\otimes \pi_k^*\mathcal{O}\Big(-\frac{1}{kr}\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)A\Big), \qquad A\geqslant 0$$

satisfies for $k\geqslant n$ *the inequality*

$$\frac{1}{(n+kr-1)!}\int_{X_k^{\mathrm{GG}}(L_k,\leqslant 1)} \Theta^{n+kr-1}_{L_k,\Psi^*_{h,p,\varepsilon}}$$
$$(*)\quad \geqslant \frac{1}{n!(k!)^r(kr-1)!}\int_X c_{n,r,k}\big(\Theta_{\det V^*}+r\gamma\big)^n - c'_{n,r,k}\big(\Theta_{\det V^*}+r\gamma\big)^{n-1}\wedge\big(\Theta_{A,h_A}+r\gamma\big)$$

where

$$c_{n,r,k} = \frac{n!}{r^n}\Big(\sum_{1\leqslant s_1<\ldots<s_n\leqslant k}\frac{1}{s_1\ldots s_n}\Big)\int_{\Delta_{k-1}} x_1\ldots x_n\,d\nu_{k,r}(x),$$
$$c'_{n,r,k} = \frac{n}{kr}\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)\int_{\Delta_{k-1}}\Big(\sum_{1\leqslant s\leqslant k}\frac{x_s}{s}\Big)^{n-1} d\nu_{k,r}(x).$$

Especially we have a lot of sections in $H^0(X_k^{\mathrm{GG}}, mL_k)$, $m\gg 1$, *as soon as the difference occurring in* $(*)$ *is positive.*

The statement is also true for $k<n$, but then $c_{n,r,k}=0$ and the lower bound $(*)$ cannot be positive. By Corollary 11.11, it still provides a non trivial lower bound for $h^0(X_k^{\mathrm{GG}}, mL_k) - h^1(X_k^{\mathrm{GG}}, mL_k)$, though. For $k\geqslant n$ we have $c_{n,r,k}>0$ and $(*)$ will be positive if $\Theta_{\det V^*}$ is large enough. By Formula 11. 20 we have

$$(11.73)\qquad c_{n,r,k} = \frac{n!\,(kr-1)!}{(n+kr-1)!}\sum_{1\leqslant s_1<\ldots<s_n\leqslant k}\frac{1}{s_1\ldots s_n}\geqslant \frac{(kr-1)!}{(n+kr-1)!},$$

(with equality for $k=n$), and by ([Dem11], Lemma 2.20 (b)) we get the upper bound

$$\frac{c'_{n,k,r}}{c_{n,k,r}} \leqslant \frac{(kr+n-1)r^{n-2}}{k/n}\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)^n\Bigg[1+\frac{1}{3}\sum_{m=2}^{n-1}\frac{2^m(n-1)!}{(n-1-m)!}\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)^{-m}\Bigg].$$

The case $k=n$ is especially interesting. For $k=n\geqslant 2$ one can show (with $r\leqslant n$ and H_n denoting the harmonic sequence) that

$$\text{(11.74)}\qquad \frac{c'_{n,k,r}}{c_{n,k,r}} \leqslant \frac{n^2+n-1}{3}n^{n-2}\exp\Big(\frac{2(n-1)}{H_n}+n\log H_n\Big) \leqslant \frac{1}{3}\big(n\log(n\log 24n)\big)^n.$$

We will later need the particular values that can be obtained by direct calculations (cf. Formula (11.24) and [Dem11, Lemma 2.20]).

$$\text{(11.75}_2\text{)}\qquad c_{2,2,2}=\frac{1}{20},\qquad c'_{2,2,2}=\frac{9}{16},\qquad \frac{c'_{2,2,2}}{c_{2,2,2}}=\frac{45}{4},$$

$$\text{(11.75}_3\text{)}\qquad c_{3,3,3}=\frac{1}{990},\qquad c'_{3,3,3}=\frac{451}{4860},\qquad \frac{c'_{3,3,3}}{c_{3,3,3}}=\frac{4961}{54}.$$

§11.E. Global generation of the twisted tangent space of the universal family

In [Siu02, Siu04], Y.T. Siu developed a new stategy to produce jet differentials, involving meromorphic vector fields on the total space of jet bundles – these vector fields are used to differentiate the sections of $E^{\rm GG}_{k,m}$ so as to produce new ones with less zeroes. The approach works especially well on universal families of hypersurfaces in projective space, thanks to the good positivity properties of the relative tangent bundle, as shown by L. Ein [Ein88, Ein91] and C. Voisin [Voi96]. This allows at least to prove the hyperbolicity of generic surfaces and generic 3-dimensional hypersurfaces of sufficiently high degree. We reproduce here the improved approach given by [Pău08] for the twisted global generation of the tangent space of the space of vertical two jets. The situation of k-jets in arbitrary dimension n is substantially more involved, details can be found in [Mer09].

Consider the universal hypersurface $\mathcal{X}\subset\mathbb{P}^{n+1}\times\mathbb{P}^{N_d}$ of degree d given by the equation

$$\sum_{|\alpha|=d} A_\alpha\, Z^\alpha = 0,$$

where $[Z]\in\mathbb{P}^{n+1}$, $[A]\in\mathbb{P}^{N_d}$, $\alpha=(\alpha_0,\ldots,\alpha_{n+1})\in\mathbb{N}^{n+2}$ and

$$N_d=\binom{n+d+1}{d}-1.$$

Finally, we denote by $\mathcal{V}\subset\mathcal{X}$ the vertical tangent space, i.e. the kernel of the projection

$$\pi:\mathcal{X}\to U\subset\mathbb{P}^{N_d}$$

where U is the Zariski open set parametrizing smooth hypersurfaces, and by $J_k\mathcal{V}$ the bundle of k-jets of curves tangent to $\mathcal{V}$, i.e. curves contained in the fibers $X_s=\pi^{-1}(s)$. The goal is to describe certain meromorphic vector fields on the total space of $J_k\mathcal{V}$. In the special case $n=2$, $k=2$ considered by Păun [Pău08], one fixes the affine open set

$$\mathcal{U}_0=\{Z_0\neq 0\}\times\{A_{0d00}\neq 0\}\simeq\mathbb{C}^3\times\mathbb{C}^{N_d}$$

in $\mathbb{P}^3 \times \mathbb{P}^{N_d}$ with the corresponding inhomogeneous coordinates $(z_j = Z_j/Z_0)_{j=1,2,3}$ and $(a_\alpha = A_\alpha/A_{0d00})_{|\alpha|=d,\alpha_1<d}$. Since α_0 is determined by $\alpha_0 = d - (\alpha_1 + \alpha_2 + \alpha_3)$, with a slight abuse of notation in the sequel, α will be seen as a multiindex $(\alpha_1, \alpha_2, \alpha_3)$ in $\mathbb{N}^3$, with moreover the convention that $a_{d00} = 1$. On this affine open set we have

$$\mathcal{X}_0 := \mathcal{X} \cap \mathcal{U}_0 = \Big\{ z_1^d + \sum_{|\alpha|\leqslant d,\alpha_1<d} a_\alpha\, z^\alpha = 0 \Big\}.$$

We now write down equations for the open variety $J_2\mathcal{V}_0$, where we indicated with $\mathcal{V}_0$ the restriction of $\mathcal{V} \subset T_{\mathcal{X}}$, the kernel of the differential of the second projection, to $\mathcal{X}_0$: elements in $J_2\mathcal{V}_0$ are therefore 2-jets of germs of "vertical" holomorphic curves in $\mathcal{X}_0$, that is curves tangent to vertical fibers. The equations, which live in a natural way in $\mathbb{C}^3_{z_j} \times \mathbb{C}^{N_d}_{a_\alpha} \times \mathbb{C}^3_{z'_j} \times \mathbb{C}^3_{z''_j}$, stand as follows.

$$\begin{aligned}
&\sum_{|\alpha|\leqslant d} a_\alpha\, z^\alpha = 0,\\
&\sum_{1\leqslant j\leqslant 3}\ \sum_{|\alpha|\leqslant d} a_\alpha\, \frac{\partial z^\alpha}{\partial z_j}\, z'_j = 0,\\
&\sum_{1\leqslant j\leqslant 3}\ \sum_{|\alpha|\leqslant d} a_\alpha\, \frac{\partial z^\alpha}{\partial z_j}\, z''_j + \sum_{1\leqslant j,k\leqslant 3}\ \sum_{|\alpha|\leqslant d} a_\alpha\, \frac{\partial^2 z^\alpha}{\partial z_j \partial z_k}\, z'_j z'_k = 0.
\end{aligned}$$

Let $\mathcal{W}_0$ to be the closed algebraic subvariety of $J_2\mathcal{V}_0$ defined by

$$\mathcal{W}_0 = \{(z, a, z', z'') \in J_2\mathcal{V}_0 \mid z' \wedge z'' = 0\}$$

and let $\mathcal{W}$ be the Zariski closure of $\mathcal{W}_0$ in $J_2\mathcal{V}$: we call this set the *Wronskian locus* of $J_2\mathcal{V}$. Explicit calculations (cf. [Pău08]) then produce the following vector fields:

First family of tangent vector fields. For any multiindex α such that $\alpha_1 \geqslant 3$, consider the vector field

$$\theta^{300}_\alpha = \frac{\partial}{\partial a_\alpha} - 3z_1\, \frac{\partial}{\partial a_{\alpha-\delta_1}} + 3z_1^2\, \frac{\partial}{\partial a_{\alpha-2\delta_1}} - z_1^3\, \frac{\partial}{\partial a_{\alpha-3\delta_1}},$$

where $\delta_j \in \mathbb{N}^4$ is the multiindex whose j-th component is equal to 1 and the others are zero. For the multiindexes α which verify $\alpha_1 \geqslant 2$ and $\alpha_2 \geqslant 1$, define

$$\begin{aligned}
\theta^{210}_\alpha = {}& \frac{\partial}{\partial a_\alpha} - 2z_1\, \frac{\partial}{\partial a_{\alpha-\delta_1}} - z_2\, \frac{\partial}{\partial a_{\alpha-\delta_2}} + z_1^2\, \frac{\partial}{\partial a_{\alpha-2\delta_1}}\\
&+ 2z_1 z_2\, \frac{\partial}{\partial a_{\alpha-\delta_1-\delta_2}} - z_1^2 z_2\, \frac{\partial}{\partial a_{\alpha-2\delta_1-\delta_2}}.
\end{aligned}$$

Finally, for those α for which $\alpha_1, \alpha_2, \alpha_3 \geqslant 1$, set

$$\begin{aligned}
\theta^{111}_\alpha = {}& \frac{\partial}{\partial a_\alpha} - z_1\, \frac{\partial}{\partial a_{\alpha-\delta_1}} - z_2\, \frac{\partial}{\partial a_{\alpha-\delta_2}} - z_3\, \frac{\partial}{\partial a_{\alpha-\delta_3}}\\
&+ z_1 z_2\, \frac{\partial}{\partial a_{\alpha-\delta_1-\delta_2}} + z_1 z_3\, \frac{\partial}{\partial a_{\alpha-\delta_1-\delta_3}} + z_2 z_3\, \frac{\partial}{\partial a_{\alpha-\delta_2-\delta_3}}\\
&- z_1 z_2 z_3 \frac{\partial}{\partial a_{\alpha-\delta_1-\delta_2-\delta_3}}.
\end{aligned}$$

Second family of tangent vector fields. We construct here the holomorphic vector fields in order to span the $\partial/\partial z_j$-directions. For $j = 1, 2, 3$, consider the vector field

$$\frac{\partial}{\partial z_j} - \sum_{|\alpha+\delta_j|\leqslant d} (\alpha_j + 1) a_{\alpha+\delta_j} \frac{\partial}{\partial a_\alpha}.$$

Third family of tangent vector fields. In order to span the jet directions, consider a vector field of the following form:

$$\theta_B = \sum_{|\alpha|\leqslant d, \alpha_1 < d} p_\alpha(z, a, b) \frac{\partial}{\partial a_\alpha} + \sum_{1\leqslant j\leqslant 3} \sum_{k=1}^{2} \xi_j^{(k)} \frac{\partial}{\partial z_j^{(k)}},$$

where $\xi^{(k)} = B \cdot z^{(k)}$, $k = 1, 2$, and $B = (b_{jk})$ varies among 3×3 invertible matrices with complex entries. By studying more carefully these three families of vector fields, one obtains:

11.76. Theorem. *The twisted tangent space $T_{J_2\mathcal{V}} \otimes \mathcal{O}_{\mathbb{P}^3}(7) \otimes \mathcal{O}_{\mathbb{P}^{N_d}}(1)$ is generated over by its global sections over the complement $J_2\mathcal{V} \smallsetminus \mathcal{W}$ of the Wronskian locus $\mathcal{W}$. Moreover, one can choose generating global sections that are invariant with respect to the action of $\mathbb{G}_2$ on $J_2\mathcal{V}$.*

By similar, but more computationally intensive arguments [Mer09], one can investigate the higher dimensional case. The following result strengthens the initial announcement of [Siu04].

11.77. Theorem. *Let $J_k^{\mathrm{vert}}(\mathcal{X})$ be the space of vertical k-jets of the universal hypersurface*

$$\mathcal{X} \subset \mathbb{P}^{n+1} \times \mathbb{P}^{N_d}$$

parametrizing all projective hypersurfaces $X \subset \mathbb{P}^{n+1}$ of degree d. Then for $k = n$, there exist constants c_n and c'_n such that the twisted tangent bundle

$$T_{J_k^{\mathrm{vert}}(\mathcal{X})} \otimes \mathcal{O}_{\mathbb{P}^{n+1}}(c_n) \otimes \mathcal{O}_{\mathbb{P}^{N_d}}(c'_n)$$

is generated by its global $\mathbb{G}_k$-invariant sections outside a certain exceptional algebraic subset $\Sigma \subset J_k^{\mathrm{vert}}(\mathcal{X})$. One can take either $c_n = \frac{1}{2}(n^2 + 5n)$, $c'_n = 1$ and Σ defined by the vanishing of certain Wronskians, or $c_n = n^2 + 2n$ and a smaller set $\widetilde{\Sigma} \subset \Sigma$ defined by the vanishing of the 1-jet part.

11.F. General strategy of the proof of hyperbolicity properties

Let again $\mathcal{X} \subset \mathbb{P}^{n+1} \times \mathbb{P}^{N_d}$ be the universal hypersurface of degree d in $\mathbb{P}^{n+1}$.

(11.78) *Assume that we can prove the existence of a non zero polynomial differential operator*

$$P \in H^0(\mathcal{X}, E_{k,m}^{\mathrm{GG}} T^*_{\mathcal{X}} \otimes \mathcal{O}(-A)),$$

where A is an ample divisor on $\mathcal{X}$, at least over some Zariski open set U in the base of the projection $\pi : \mathcal{X} \to U \subset \mathbb{P}^{N_d}$.

Observe that we now have a lot of techniques to do this; the existence of P over the family follows from lower semicontinuity in the Zariski topology, once we know that such a section P exists on a generic fiber $X_s = \pi^{-1}(s)$. Let $\mathcal{Y} \subset \mathcal{X}$ be the set of points $x \in \mathcal{X}$ where $P(x) = 0$, as an element in the fiber of the vector bundle $E^{\mathrm{GG}}_{k,m} T^*_{\mathcal{X}} \otimes \mathcal{O}(-A))$ at x. Then $\mathcal{Y}$ is a proper algebraic subset of $\mathcal{X}$, and after shrinking U we may assume that $Y_s = \mathcal{Y} \cap X_s$ is a proper algebraic subset of X_s for every $s \in U$.

(11.79) *Assume also, according to Theorems* 11.76 *and* 11.77*, that we have enough global holomorphic $\mathbb{G}_k$-invariant vector fields θ_i on $J_k\mathcal{V}$ with values in the pull-back of some ample divisor B on $\mathcal{X}$, in such a way that they generate $T_{J_k\mathcal{V}} \otimes p_k^* B$ over the dense open set $(J_k\mathcal{V})^{\mathrm{reg}}$ of regular k-jets, i.e. k-jets with non zero first derivative* (*here $p_k : J_k\mathcal{V} \to \mathcal{X}$ is the natural projection*).

Considering jet differentials P as functions on $J_k\mathcal{V}$, the idea is to produce new ones by taking differentiations

$$Q_j := \theta_{j_1} \ldots \theta_{j_\ell} P, \qquad 0 \leqslant \ell \leqslant m,\ j = (j_1, \ldots, j_\ell).$$

Since the θ_j's are $\mathbb{G}_k$-invariant, they are in particular $\mathbb{C}^*$-invariant, thus

$$Q_j \in H^0(\mathcal{X}, E^{\mathrm{GG}}_{k,m} T^*_{\mathcal{X}} \otimes \mathcal{O}(-A + \ell B))$$

(and Q is in fact $\mathbb{G}'_k$ invariant as soon as P is). In order to be able to apply the vanishing theorems of § 8, we need $A - mB$ to be ample, so A has to be large compared to B. If $f : \mathbb{C} \to X_s$ is an entire curve contained in some fiber $X_s \subset \mathcal{X}$, its lifting $j_k(f) : \mathbb{C} \to J_k\mathcal{V}$ has to lie in the zero divisors of all sections Q_j. However, every non zero polynomial of degree m has at any point some non zero derivative of order $\ell \leqslant m$. Therefore, at any point where the θ_i generate the tangent space to $J_k\mathcal{V}$, we can find some non vanishing section Q_j. By the assumptions on the θ_i, the base locus of the Q_j's is contained in the union of $p_k^{-1}(\mathcal{Y}) \cup (J_k\mathcal{V})^{\mathrm{sing}}$; there is of course no way of getting a non zero polynomial at points of $\mathcal{Y}$ where P vanishes. Finally, we observe that $j_k(f)(\mathbb{C}) \not\subset (J_k\mathcal{V})^{\mathrm{sing}}$ (otherwise f is constant). Therefore $j_k(f)(\mathbb{C}) \subset p_k^{-1}(\mathcal{Y})$ and thus $f(\mathbb{C}) \subset \mathcal{Y}$, i.e. $f(\mathbb{C}) \subset Y_s = \mathcal{Y} \cap X_s$.

11.80. Corollary. *Let $\mathcal{X} \subset \mathbb{P}^{n+1} \times \mathbb{P}^{N_d}$ be the universal hypersurface of degree d in $\mathbb{P}^{n+1}$. If $d \geqslant d_n$ is taken so large that conditions* (11.78) *and* (11.79) *are met with $A - mB$ ample, then the generic fiber X_s of the universal family $\mathcal{X} \to U$ satisfies the Green-Griffiths conjecture, namely all entire curves $f : \mathbb{C} \to X_s$ are contained in a proper algebraic subvariety $Y_s \subset X_s$, and the Y_s can be taken to form an algebraic subset $\mathcal{Y} \subset \mathcal{X}$.*

This is unfortunately not enough to get the hyperbolicity of X_s, because we would have to know that Y_s itself is hyperbolic. However, one can use the following simple observation due to Diverio and Trapani [DT10]. The starting point is the following general, straightforward remark. Let $\mathcal{E} \to \mathcal{X}$ be a holomorphic vector bundle let $\sigma \in H^0(\mathcal{X}, \mathcal{E}) \neq 0$; then, up to factorizing by an effective divisor D contained in the common zeroes of the components of σ, one can view σ as a section

$$\sigma \in H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{O}_{\mathcal{X}}(-D)),$$

and this section now has a zero locus without divisorial components. Here, when $n \geqslant 2$, the very generic fiber X_s has Picard number one by the Noether-Lefschetz theorem, and so, after shrinking U if necessary, we can assume that $\mathcal{O}_{\mathcal{X}}(-D)$ is the restriction of $\mathcal{O}_{\mathbb{P}^{n+1}}(-p)$, $p \geqslant 0$ by the effectivity of D. Hence D can be assumed to be nef. After performing this simplification, $A - mB$ is replaced by $A - mB + D$, which is still ample if $A - mB$ is ample.

As a consequence, we may assume $\operatorname{codim} \mathcal{Y} \geqslant 2$, and after shrinking U again, that all Y_s have $\operatorname{codim} Y_s \geqslant 2$.

11.81. Additional statement. *In corollary* 11.80*, under the same hypotheses* (11.78) *and* (11.79)*, one can take all fibers Y_s to have* $\operatorname{codim} Y_s \geqslant 2$*.*

This is enough to conclude that X_s is hyperbolic if $n = \dim X_s \leqslant 3$. In fact, this is clear if $n = 2$ since the Y_s are then reduced to points. If $n = 3$, the Y_s are at most curves, but we know by Ein and Voisin that a generic hypersurface $X_s \subset \mathbb{P}^4$ of degree $d \geqslant 7$ does not possess any rational or elliptic curve. Hence Y_s is hyperbolic and so is X_s, for s generic. □

11.82. Corollary. *Assume that $n = 2$ or $n = 3$, and that $\mathcal{X} \subset \mathbb{P}^{n+1} \times \mathbb{P}^{N_d}$ is the universal hypersurface of degree $d \geqslant d_n \geqslant 2n+1$ so large that conditions* (11.78) *and* (11.79) *are met with $A - mB$ ample. Then the very generic hypersurface $X_s \subset \mathbb{P}^{n+1}$ of degree d is hyperbolic.*

§11.G. Proof of the Green-griffiths conjecture for generic hypersurfaces in $\mathbb{P}^{n+1}$

The most striking progress made at this date on the Green-Griffiths conjecture itself is a recent result of Diverio, Merker and Rousseau [DMR10], confirming the statement when $X \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ is a generic hypersurface of large degree d, with a (non optimal) sufficient lower bound $d \geqslant 2^{n^5}$. Their proof is based in an essential way on Siu's strategy as developed in § 11.E, combined with the earlier techniques of [Dem95]. Using our improved bounds from § 11.D, we obtain here a better estimate (actually of exponential order one $O(\exp(n^{1+\varepsilon})$ rather than order 5).

11.83. Theorem. *A generic hypersurface $X \subset \mathbb{P}^{n+1}$ of degree $d \geqslant d_n$ with*

$$d_2 = 286, \quad d_3 = 7316, \quad d_n = \left\lfloor \frac{n^4}{3}\big(n\log(n\log(24n))\big)^n \right\rfloor \quad \textit{for } n \geqslant 4,$$

satisfies the Green-Griffiths conjecture.

Proof. Let us apply Theorem 11.72 with $V = T_X$, $r = n$ and $k = n$. The main starting point is the well known fact that $T^*_{\mathbb{P}^{n+1}} \otimes \mathcal{O}_{\mathbb{P}^{n+1}}(2)$ is semipositive (in fact, generated by its sections). Hence the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^{n+1}}(-d) \to T^*_{\mathbb{P}^{n+1}|X} \to T^*_X \to 0$$

implies that $T^*_X \otimes \mathcal{O}_X(2) \geqslant 0$. We can therefore take $\gamma = \Theta_{\mathcal{O}(2)} = 2\omega$ where ω is the Fubini-Study metric. Moreover $\det V^* = K_X = \mathcal{O}_X(d-n-2)$ has curvature $(d-n-2)\omega$, hence $\Theta_{\det V^*} + r\gamma = (d+n-2)\omega$. The Morse integral to be computed when $A = \mathcal{O}_X(p)$ is

$$\int_X \Big(c_{n,n,n}(d+n-2)^n - c'_{n,n,n}(d+n-2)^{n-1}(p+2n)\Big)\omega^n,$$

so the critical condition we need is

$$d+n-2 > \frac{c'_{n,n,n}}{c_{n,n,n}}(p+2n).$$

On the other hand, Siu's differentiation technique requires $\frac{m}{n^2}(1+\frac{1}{2}+\ldots+\frac{1}{n})A - mB$ to be ample, where $B = \mathcal{O}_X(n^2+2n)$ by Merker's result 11.77. This ampleness condition yields

$$\frac{1}{n^2}\Big(1+\frac{1}{2}+\ldots+\frac{1}{n}\Big)p - (n^2+2n) > 0,$$

so one easily sees that it is enough to take $p = n^4 - 2n$ for $n \geqslant 3$. Our estimates (11.74) and (11.75) give the expected bound d_n. □

Thanks to 11.81, one also obtains the generic hyperbolicity of 2 and 3-dimensional hypersurfaces of large degree.

11.84. Theorem. *For $n = 2$ or $n = 3$, a generic hypersurface $X \subset \mathbb{P}^{n+1}$ of degree $d \geqslant d_n$ is Kobayashi hyperbolic.*

By using more explicit calculations of Chern classes (and invariant jets rather than Green-Griffiths jets) Diverio-Trapani [DT10] obtained the better lower bound $d \geqslant d_3 = 593$ in dimension 3. In the case of surfaces, Paun [Pău08] obtained $d \geqslant d_2 = 18$, using deep results of McQuillan [McQu98].

One may wonder whether it is possible to use jets of order $k < n$ in the proof of 11.83 and 11.84. Diverio [Div08] showed that the answer is negative (his proof is based on elementary facts of representation theory and a vanishing theorem of Brückmann-Rackwitz [BR90]):

11.85. Proposition ([Div08]). *Let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface. Then*

$$H^0(X, E_{k,m}^{\mathrm{GG}} T_X^*) = 0$$

for $m \geqslant 1$ and $1 \leqslant k < n$. More generally, if $X \subset \mathbb{P}^{n+s}$ is a smooth complete intersection of codimension s, there are no global jet differentials for $m \geqslant 1$ and $k < n/s$.

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