Fibrations in algebraic geometry and applications

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Abstract

0 Introduction

This paper is devoted to a crucial tool for the study of algebraic varieties, namely fibrations. We will in this survey paper use the following definition.

Definition 0.1. *A (rational) fibration on a projective variety or compact complex manifold X is a dominant (rational or meromorphic) map $X \dashrightarrow Y$ with connected general fiber.*

The obvious interest of a fibration is that it allows to deduce properties of the total space from properties of the base and of the fibers. For example, the following facts hold:

1. If the base is Brody (resp. algebraically) hyperbolic, and all the fibers are Brody (resp. algebraically) hyperbolic then the total space is Brody hyperbolic.
2. If the base is of general type and the fiber is of general type, then the total space is of general type (see [36], [55]).

3. If the base is rationally connected and the fiber is rationally connected, so is the total space (see Section 3.2, [32]).

Note however that a general fibration, especially rational fibration, may not be very useful : Indeed, starting from any smooth projective variety X, and choosing a Lefschetz pencil of high degree hypersurfaces in X, we get a rational map $\widetilde{X} \dashrightarrow \mathbb{P}^1$ which does not say much about X since the base has Kodaira dimension $-\infty$ and the fiber is of general type. What happens in this case is the fact that the blown-up locus is of codimension 2 in X but of codimension 1 and ample in the fibers. The case of a morphism is much better but an interesting morphism to a smaller dimensional basis does not always exist and many interesting fibrations are only defined as rational maps. This paper will rather describe carefully constructed fibrations for which the fibers are instead simpler than the total space, reducing in principle the study to phenomena on the base. There are two ways of constructing fibrations: The first way, which is more classical in birational geometry, consists in exploiting the presence of sections of the adequate tensors, or divisors. This method was initiated by Iitaka for sections of line bundles, and the exploitation of holomorphic contravariant tensors has been developed over the time by Castenuovo-de Franchis, Bogomolov, Catanese, Campana. We will describe this method in Section 1.

The second way appears in [2], [19], [41] and consists in constructing geometrically the fiber through a general point x by imposing that it contains all points that can be reached from x by composing a certain number of allowed geometric processes. The two striking instances of this approach are the MRC fibration, where the fiber through a very general point contains all rational curves passing through this point, and the Shafarevich map or Γ-reduction, for which the fiber through a very general point contains all the varieties passing through this point and having a fundamental group with small image in the ambient space. This is described in Sections 3.2, 3.3.

A third method presented in [20] beautifully combines both approaches to produce the so-called core fibration which will be described in Section 4. This fibration into special varieties is conjecturally the one which allows to understand the degeneracy of the Kobayashi pseudodistance at the general point of a variety. For a long period, standard conjectures about the Kobayashi pseudodistance ([43], [37]) were the following:

Conjecture 0.2. *A variety of general type has its Kobayashi pseudometric nondegenerate at a general point.*

Very nice recent evidences for this conjecture have been obtained by Diverio-Merker-Rousseau [27] and Brotbek [13], who work in the case of hypersurfaces in projective space, and by Demailly [26].

Conjecture 0.3. *A variety with trivial canonical bundle has a vanishing Kobayashi pseudodistance.*

We refer to [56] for examples and discussions concerning Conjecture 0.3. A beautiful progress on this conjecture has been obtained recently by Verbitsky [54] in the case of hyper-Kähler manifolds (see also Diverio's contribution to this volume). The two conjectures together suggest a parallel between the Kodaira dimension of an algebraic variety and the degeneracy level of its Kobayashi pseudometric or pseudodistance. The great novelty of Campana's ideas developed in [20], [21] is to suggest that apart from the two extreme cases presented in the two conjectures above, the relationships between these two measures of positivity of the cotangent bundle is much more subtle. Indeed, Campana makes the following conjecture (cf. [20]);

Conjecture 0.4. *Special varieties have vanishing Kobayashi pseudodistance.*

As we will discuss in Section 4, special varieties may have all possible Kodaira dimensions except for the maximal one, i.e. they cannot be of general type. For example, most elliptic fibrations over $\mathbb{P}^{n-1}$ will have Kodaira dimension $n-1$ and will be special.

An essential feature of projective or compact Kähler geometry which makes the second strategy work is the fact that parameter spaces for compact closed algebraic (or analytic) subsets of a projective (or compact Kähler) manifold is compact. We refer for Section 2.1 for details. If we drop the Kähler assumption, then the local deformation theory of closed analytic subschemes is unchanged, but we completely loose the compactness, as shows the example of the twistor family of a $K3$ surface S (see Section 2.1, Example 2.2).

1 Fibrations and holomorphic forms

This section is devoted to the construction of rational fibrations used sections of line bundles or differential forms. The main applications concern the geometry of the canonical line bundle, with the beginning of a birational classification of algebraic varieties by the Kodaira dimension.

1.1 General facts on fibrations and holomorphic forms

Let $f : X \to Y$ be a surjective morphism, where X and Y are smooth complex projective varieties or compact Kähler manifolds.

Remark 1.1. Starting from a morphism $f : X \to Y$, up to replacing f by its Stein factorization $f_{St} : X \to Y_{St}$, one may assume the fibers are connected, but then Y_{St} is only normal and some extra work is needed if one also wants smoothness of Y.

The morphism f is proper and we have:

Lemma 1.2. *The following properties hold:*

1. *The morphism $f_* : \pi_1(X) \to \pi_1(Y)$ has image of finite index (and is surjective if the fibers are connected).*
2. *The morphisms $f^* : H^i(Y, \mathbb{Q}) \to H^i(X, \mathbb{Q})$ are injective morphisms of Hodge structures.*
3. *The morphisms $f^* : H^0(Y, \Omega_Y^i) \to H^0(X, \Omega_X^i)$ are isomorphisms for $i \geq 0$ if the smooth fibers X_s of f are connected and satisfy $H^0(X_s, \Omega_{X_s}^i) = 0$ for $i > 0$.*

Proof. (i) follows from the fact that there is a dense Zariski (or analytic-Zariski) open set $U \subset Y$ such that the restriction $f_U : X_U \to U$ is smooth, hence a locally topologically trivial fibration. Then the statements hold for $f_{U*} : \pi_1(X_U) \to \pi_1(U)$ and we conclude using the fact the map $\pi_1(U) \to \pi_1(Y)$ is surjective because Y is smooth.

For items 2 and 3, we use the fact that there is a left inverse on cohomology with real coefficients given by $\alpha \mapsto f_*(\omega^d \smile \alpha)$, where $d = \dim X - \dim Y$ is the relative dimension and ω is the class of a Kähler form on X with volume 1 along the fibers of f. It follows that both maps f^* are injective. Finally the proof of 3 uses the cotangent bundle sequence along smooth fibers

$$0 \to f^*\Omega_{Y,s} \to \Omega_{X|X_s} \to \Omega_{X_s} \to 0, \tag{1}$$

which induces a filtration of $\Omega^i_{X|X_s}$, showing that under our assumptions

$$H^0(X_s, \Omega^i_{X|X_s}) = H^0(X_s, f^*\Omega^i_{Y_s}).$$

We then conclude that any holomorphic differential i-form on X is a section of $f^*\Omega^i_Y$, at least over the open set Y^0 of regular values of f and more precisely is of the form $f^*\beta$, where β is a holomorphic differential i-form on Y^0. But then β extends to Y as $\beta' = f_*(\omega^d \wedge \alpha)$, and we thus conclude that $\alpha = f^*\beta'$ on $X^0 = f^{-1}(Y^0)$ hence everywhere. $\square$

Remark 1.3. The final argument given above fails if we replace holomorphic forms by pluridifferential holomorphic forms. We will see in Section 4 that for pluricanonical forms, it is not true that a section of $\Omega_X^{\otimes i}$, which is a section of $f^*\Omega_Y^{\otimes i} \subset \Omega_X^{\otimes i}$ along a dense Zariski open set $X_U = f^{-1}(U) \subset X$ is the pull-back of a section of $\Omega_Y^{\otimes i}$. However Lemma 1.2, 3 is also true for pluridifferential forms.

Remark 1.4. The exact sequence (1) is a particular case of the conormal bundle exact sequence

$$0 \to N^*_{X_s/X} \to \Omega_{X|X_s} \to \Omega_{X_s} \to 0, \tag{2}$$

where in this case $N_{X_s/X} = f^*T_{Y,s}$.

Note also the following standard fact which will apply more generally to a covering family of X by d-dimensional varieties $\mathcal{X}_s \subset X$, that can be seen as a diagram

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\phi} & X \\ \downarrow{\scriptstyle f} & & \\ Y & & \end{array} \qquad , \tag{3}$$

where f is a fibration and ϕ is dominant generically finite:

Lemma 1.5. *In case of a fibration $f : X \to Y$, one has $K_{X|X_s} = K_{X_s}$. For a covering family $(\mathcal{X}_s)_{s\in Y}$, for a general member $\mathcal{X}_s$ of the covering family, one has $K_{\mathcal{X}_s} = K_{X|\mathcal{X}_s} + D$, where D is effective on $\mathcal{X}_s$.*

Proof. The first statement follows from (1) by taking determinants, using the fact that $f^*\Omega_Y$ is trivial along the fiber $\mathcal{X}_s$. Only the second point needs to be proved. However we have by the first property $K_{\mathcal{X}_s} = K_{\mathcal{X}|\mathcal{X}_s}$ and on the other hand $K_{\mathcal{X}} = \phi^*K_X + R$, where R is the ramification divisor of ϕ. Thus the statement holds once $\mathcal{X}_s$ is not contained in R. $\square$

We end this section with a standard lemma that will be used (sometimes implicitly) throughout the paper:

Lemma 1.6. *The holomorphic pluridifferential forms, that is sections of $\Omega_X^{\otimes k}$, $k \geq 0$, are bimeromorphic invariants of the compact complex manifold X.*

Proof. Let $\phi : X \dashrightarrow Y$ be a bimeromorphic map, with X and Y compact. There exists a Zariski-analytic open set $U \subset X$ such that $\operatorname{codim} X \setminus U \geq 2$ and ϕ is well defined on U. We can thus define (because $k \geq 0$)

$$\phi^* : H^0(Y, \Omega_Y^{\otimes k}) \to H^0(U, \Omega_U^{\otimes k}).$$

This morphism is injective because ϕ is generically of maximal rank. By Hartogs' theorem, we have $H^0(U, \Omega_U^{\otimes k}) = H^0(X, \Omega_X^{\otimes k})$. We thus constructed an injective morphism ϕ^* : $H^0(Y, \Omega_{Y/K}^{\otimes k}) \to H^0(X, \Omega_{X/K}^{\otimes k})$, which admits as inverse $(\phi^{-1})^*$. $\square$

1.2 Iitaka fibration

Let X be a smooth projective variety and let L be a line bundle on X. The subset $M(L) \subset \mathbb{N}$ defined as

$$M(L) = \{k \in \mathbb{N},\ H^0(X, kL) \neq 0\}$$

is a submonoid of $\mathbb{N}$ hence except for small values of k, it agrees with the set of positive multiples of an integer k_0. We assume from now on that $k_0 \neq 0$. The integer k_0 has the property that for any sufficiently large number m divisible by k_0, there are nonzero sections of mL. Typical examples where k_0 is actually needed, that is, powers of L of order nondivisible by k_0 have no nonzero sections, are given by $X = E \times Y$, where E is an elliptic curve, and $L = L_0 \boxtimes L_Y$, where L_0 is a torsion line bundle of order k_0 on E and L_Y is an ample line bundle on Y. The Iitaka dimension $\kappa(X, L)$ of (X, L) is defined as $-\infty$ if $M(L) = \{0\}$ and as the maximal dimension of the images of the maps

$$\phi_{kL} : X \dashrightarrow \mathbb{P}^N$$

induced by the linear systems $|kL|$ for $k \in M(L)$. For example $\kappa(X, L) = 0$ is equivalent to the fact that $M(L) \neq \{0\}$ and $h^0(X, kL) = 1$ for any $k \in M(L)$. When $L = K_X$, the Iitaka dimension of L is the Kodaira dimension of X.

Theorem 1.7. *(Iitaka) There exists a rational fibration*

$$\phi_L : X \dashrightarrow Y \tag{4}$$

which is well-defined up to birational modifications of Y, and is characterized by the following conditions:

1. $\dim Y = \kappa(X, L)$.
2. *Let $\tilde{\phi}_L : \widetilde{X} \to Y$, $\tau : \widetilde{X} \to X$ be a desingularization of ϕ_L. Then the restriction of $\tau^* L$ to the general fibers of $\tilde{\phi}_L$ has Iitaka dimension* 0.

In the case where $L = K_X$, the statement is particularly interesting because first of all, by Lemma 1.6, the sections of $K_X^{\otimes r}$ agree with the sections of $K_{\widetilde{X}}^{\otimes r}$ for all $r \geq 0$, which implies that the map $\phi_{K_X} : \widetilde{X} \to Y$ identifies with the Iitaka fibration of $K_{\widetilde{X}}$, and secondly, by Lemma 1.5, the restriction of $K_{\widetilde{X}}$ to the smooth fibers $\widetilde{X}_b$ of ϕ is nothing but the canonical bundle of $\widetilde{X}_b$. Thus we get in this case:

Corollary 1.8. *Any variety of nonnegative Kodaira dimension κ has a canonical fibration over a base of dimension κ with general fibers of Kodaira dimension* 0.

Proof of Theorem 1.7. We refer to [44] for a more detailed proof. First of all, constructing the rational fibration is immediate since for any $k \in M(L)$, choosing a nonzero section $\sigma \in H^0(X, k_0 L)$, there is an inclusion $\sigma : H^0(X, kL) \to H^0(X, (k + k_0)L)$ and thus a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\phi_{(k+k_0)L}} & \mathbb{P}^{N'} \\ \downarrow{=} & & \downarrow{\pi} \\ X & \xrightarrow{\phi_{kL}} & \mathbb{P}^N \end{array} \tag{5}$$

where the horizontal map are rational maps, and the second vertical map is a linear projection. It follows that if we take k large enough so that $\dim \operatorname{Im} \phi_{kL} = \kappa(X, L)$, the map π induces a generically finite rational map $\operatorname{Im} \phi_{(k+k_0)L} \dashrightarrow \operatorname{Im} \phi_{\Phi_{kL}}$, and as all these varieties are dominated by X, these rational maps must be birational when k is large enough. Let Y be a smooth projective model of the varieties $\operatorname{Im} \phi_{kL}$ for k large enough divisible by k_0. Let $\phi : \widetilde{X} \to Y$ be a desingularization of ϕ_{kL} (seen as rational map to Y). What remains to be proved is the fact that the map ϕ has irreducible (or equivalently connected) general fibers, and that the Iitaka dimension of the pull-back $\widetilde{L}$ of L to $\widetilde{X}$, restricted to the general fiber of ϕ is 0. Note that it is not true that the line bundle $\widetilde{L}$ is the pull-back of a line bundle on Y. However we observe that by definition, there is a nonzero section of $L^{\otimes k} \otimes \phi_{kL}^* \mathcal{O}_{Y_k}(-1)$ on X, where $Y_k = \operatorname{Im} \phi_{kL}$. It follows that denoting $\mathcal{O}_Y(1)$ the pull-back to Y of $\mathcal{O}_{Y_k}(1)$, there is a nonzero section α of $\widetilde{L}^{\otimes k} \otimes \phi^* \mathcal{O}_Y(-1)$ on $\widetilde{X}$. The line bundle $\mathcal{O}_Y(1)$ being big, for any $r \geq 0$, there is an $s > 0$ such that the sheaf $(R^0 \phi_* \widetilde{L}^{\otimes r}) \otimes \mathcal{O}_Y(s)$ is generically globally generated on Y. Hence, if there are two independent sections σ_1, σ_2 of $\widetilde{L}^{\otimes r}$ on the general fiber of ϕ, there are two sections τ_1, τ_2 of $\widetilde{L}^{\otimes r} \otimes \phi^* \mathcal{O}_Y(s)$ which restrict to σ_1, σ_2 on the general fiber. Multiplying by α^s, we produce two sections τ'_1, τ'_2 of $\widetilde{L}^{\otimes r+sk}$ with independent restrictions on the general fiber of ϕ, which contradicts the construction of the fibration ϕ. Hence we conclude that $\widetilde{L}^{\otimes r}$ has at most one section on the general fiber of ϕ, which implies in particular that it is connected. □

Remark 1.9. In the case where $L = K_X$, note that the line bundle $\mathcal{O}_Y(1)$ and its multiples have in general nothing to do with the canonical bundle of Y, which in some sense stops the analysis. We will come back to this in Section 4 describing results of Campana to analyze further the situation.

1.3 Castelnuovo-de Franchis and Bogomolov theorems

Castelnuovo-de Franchis Lemma is the following statement:

Lemma 1.10. *Let X be a smooth projective (or compact Kähler) manifold and let α, $\beta \in H^0(X, \Omega_X)$ be two independent 1-forms on X such that $\alpha \wedge \beta = 0$ in $H^0(X, \Omega_X^2)$. Then there exist a morphism $\phi : X \to C$, where C is a smooth curve of genus ≥ 2, and two holomorphic 1-forms α_0, β_0 on C such that $\alpha = \phi^*\alpha_0$, $\beta = \phi^*\beta_0$. More generally, if there are g independent $(1,0)$-forms α_i on X such that $\alpha_i \wedge \alpha_j = 0$ in $H^0(X, \Omega_X^2)$, X admits a morphism to a curve C of genus $\geq g$ and the α_i's are pulled-back from C.*

Proof. This immediately follows from the fact that holomorphic forms on X are closed. The two 1-forms α, β are pointwise proportional. Writing $\alpha = f\beta$, where f is a rational (or meromorphic) function on X, we thus get $df \wedge \beta = 0$ and thus df is also proportional to α and β. The two 1-forms α and β thus vanish along the fibers of $f : X \dashrightarrow \mathbb{P}^1$. Let $\tilde{f} : \widetilde{X} \to \mathbb{P}^1$ be a desingularization of f and let $F : \widetilde{X} \to C$ be the Stein factorization of f, where C is a smooth projective curve admitting a morphism $r : C \to \mathbb{P}^1$ such that $r \circ F = \tilde{f}$. The respective pull-backs $\tilde{\alpha}$, $\tilde{\beta}$ of α, β to X are closed holomorphic 1-forms which are sections of the bundle $F^*\Omega_C \subset \Omega_{\widetilde{X}}$, at least on the open set $U \subset \widetilde{X}$ where F has maximal rank. This implies that there exists holomorphic 1-forms α_0, β_0 on C such that $\tilde{\alpha} = F^*\alpha_0$, $\tilde{\beta} = F^*\beta_0$ (we can take for α_0 the form $F_*\omega^{d-1} \wedge \tilde{\alpha}$, where ω is a Kähler form on $\widetilde{X}$ which has integral 1 on the fibers of F, and similarly for β_0). It follows that $g(C)$ has genus at least 2, so that the morphism F is already defined on X (this follows from the fact that the irreducible components of the fibers of the birational map $\widetilde{X} \to X$ are rationally connected (see section 3.1 for the definition), hence contracted by F) since a rational curve cannot have a nonconstant morphism to a curve of positive genus. The general case is treated similarly. □

Catanese [24] gave a topological version of Castenuovo-de Franchis result, showing that the existence of a dominating map to a curve of genus ≥ 2 is a topological property of compact Kähler manifolds :

Theorem 1.11. *Let X be a compact Kähler manifold, and assume there are two independent degree 1 cohomology classes β, $\beta' \in H^1(X, \mathbb{C})$ such that $\beta \smile \beta' = 0$ in $H^2(X, \mathbb{C})$. Then there is a morphism $X \to C$ to a curve of genus ≥ 2 such that β, β' are pulled-back from degree 1 cohomology classes on C. More generally, the existence of a dominant morphism to a curve of genus $\geq g$ is equivalent to the existence of a vector subspace $V \subset H^1(X, \mathbb{C}$ of dimension g such that $\bigwedge^2 V \to H^2(X, \mathbb{C})$ is identically 0.*

Proof. We will just prove the case $g = 2$. Write the Hodge decomposition of β, β':

$$\beta = \beta^{1,0} + \beta^{0,1},\ \beta' = \beta'^{1,0} + \beta'^{0,1},$$

where $\beta^{1,0}$, $\overline{\beta^{0,1}}$, $\beta'^{1,0}$, $\overline{\beta'^{0,1}}$ are (classes of) holomorphic 1-forms on X. Then by type reasons,

$$\beta^{1,0} \smile \beta'^{1,0} = 0 \text{ in } H^2(X, \mathbb{C}),\ \beta^{0,1} \smile \beta'^{0,1} = 0 \text{ in } H^2(X, \mathbb{C}). \tag{6}$$

As exact holomorphic forms vanish identically, these equalities also say that

$$\beta^{1,0} \wedge \beta'^{1,0} = 0 \text{ in } H^0(X, \Omega_X^2),\ \overline{\beta^{0,1}} \wedge \overline{\beta'^{0,1}} = 0 \text{ in } H^0(X, \Omega_X^2). \tag{7}$$

Thus we can apply Lemma 1.10 if either $\beta^{1,0}$ and $\beta'^{1,0}$ are not proportional or $\overline{\beta^{0,1}}$ and $\overline{\beta'^{0,1}}$ are not proportional. Otherwise, we have $\beta'^{1,0} = \lambda\beta^{1,0}$, $\beta'^{0,1} = \mu\beta^{0,1}$ with $\lambda \neq \mu$ since β and β' are not proportional. Next $\beta \smile \beta' = 0$ also says that $\mu\beta^{1,0} \smile \beta^{0,1} + \lambda\beta^{0,1} \smile \beta^{1,0} = 0$ in $H^{1,1}(X)$ and as $\mu \neq \lambda$, this implies that

$$\beta^{1,0} \smile \beta^{0,1} = 0 \text{ in } H^{1,1}(X). \tag{8}$$

Wee claim that (8) implies the vanishing

$$\beta^{1,0} \wedge \overline{\beta^{0,1}} = 0 \text{ in } H^{2,0}(X). \tag{9}$$

Indeed, let $\eta = \beta^{1,0} \wedge \overline{\beta^{0,1}} \in H^{2,0}(X)$. Then from (8), we deduce $\eta \smile \overline{\eta} = 0$ in $H^{2,2}(X)$. The second Hodge-Riemann bilinear relations [57, 6.3.2] then imply that $\eta = 0$, which proves the claim. Note finally that $\beta^{0,1}$ cannot be proportional to $\overline{\beta^{1,0}}$ as otherwise (8) would contradict the the Hodge-Riemann relations. Using the vanishing (9), we can now apply Lemma 1.10 which concludes the proof. □

An important consequence of Catanese's theorem is the following result due independently to Beauville [6] and Siu [51] :

Corollary 1.12. *Let X be compact Kähler. The existence of a nonconstant map $F : X \to C$, where C is a smooth projective curve of genus at least 2, is equivalent to the existence of a group morphism $\alpha : \pi_1(X) \to \pi_1(C)$ whose image has finite index.*

Proof. Indeed, as C is a $K(\pi, 1)$, such a group morphism α is induced by a continuous map $f : X \to C$. As $\operatorname{Im}\alpha$ has finite index, the induced map $f_* : H_1(X, \mathbb{Z}) \to H_1(C, \mathbb{Z})$ has finite cokernel, hence the pull-back map $f^* : H^1(C, \mathbb{C}) \to H^1(X, \mathbb{C})$ is injective, and we have a commutative diagram

$$\begin{array}{ccc} H^1(C,\mathbb{C}) \otimes H^1(C,\mathbb{C}) & \longrightarrow & H^2(C,\mathbb{C}) \\ \downarrow{\scriptstyle f^*\otimes f^*} & & \downarrow{\scriptstyle f^*} \\ H^1(X,\mathbb{C}) \otimes H^1(X,\mathbb{C}) & \longrightarrow & H^2(X,\mathbb{C}) \end{array}, \tag{10}$$

where the horizontal maps are given by cup-product. Consider the subspace $V = f^*H^{1,0}(C) \subset H^1(X, \mathbb{C})$. It has dimension $g(C)$ because f^* is injective. Moreover, one has $\alpha \smile \alpha' = 0$ in $H^2(C, \mathbb{C})$ for any α, $\alpha' \in H^{1,0}(C)$, hence by (10), $f^*\alpha \smile f^*\alpha' = 0$ in $H^2(X, \mathbb{C})$. One can then apply Catanese's theorem 1.11. In the other direction, we already proved that a surjective map between compact Kähler manifolds induces a map between their fundamental groups which has finite index image (see Lemma 1.2, 1). □

Lemma 1.10 can be given many different generalizations: The following higher dimensional generalization is due to Bogomolov [8], subsequently generalized by Campana [17].

Theorem 1.13. *Let X be a compact Kähler manifold, and let $L \subset \Omega_X^k$ be a line bundle (not necessarily saturated). Assume $\kappa(L) \geq k$. Then there exists a rational map $\phi : X \dashrightarrow B$, where B is smooth projective of dimension k, such that L coincide with $\phi^* K_B \subset \Omega_X^k$ on a Zariski open set of X.*

Remark 1.14. It is not true that B has to be of general type but ϕ has to be of general type in the sense of Campana. The problem is that $\phi^* K_B$ can be nonsaturated and its saturation L can have a different Iitaka dimension than $\phi^* K_B$. A typical such example is described in Section 4 (see Lemma 4.1).

Remark 1.15. Note that Theorem 1.13 implies in particular that L is generically decomposable, that is, generically equal to $\bigwedge^k F$ for some rank k subsheaf F of Ω_X.

Proof of Theorem 1.13. Assume first that the Iitaka fibration of L is given by sections of L. Let $0 \neq s_0, \ldots, s_N$ be sections generating L generically and giving a dominating rational map $\phi : X \dashrightarrow B$, with $\dim B = k$. Using the inclusion $L \subset \Omega_X^k$ the s_i's give holomorphic k-forms $\alpha_0, \ldots \alpha_N$ on X. We have $\alpha_i = \phi_i \alpha_0$ for some rational function ϕ_i. Using the fact that $d\alpha_i = 0$, $d\alpha_0 = 0$, we get that $d\phi_i \wedge \alpha_0 = 0$. At the generic point of X, the $d\phi_i$ generate a rank k subbundle of Ω_X, namely $\phi^*\Omega_B$. We then apply the following easy linear algebra fact:

Lemma 1.16. *Let W be a vector space and $0 \neq u \in \bigwedge^k W$. Then the vector space $V := \{v \in W, v \wedge u = 0\}$ has dimension $\leq k$, with equality if and only if u is a generator of $\bigwedge^k V$ (in particular u is decomposable).*

This lemma implies that at the generic point of X, L coincides with $\bigwedge^k(\phi^*\Omega_B) = \phi^*\Omega_B^k$, which proves the theorem in this case.

In general, (and this is the case considered by Campana), we use the well-known but important fact that given a line bundle L on X and a section s of $L^{\otimes N}$, there exist a generically finite dominating map $r : X' \to X$ and a section s' of r^*L such that $r_*(\operatorname{div} s') = \operatorname{div} s$. X' is obtained by desingularizing the cyclic cover of X of order N ramified along $\operatorname{div} s$. Iterating this process, it follows that we can find a generically finite cover $r : X' \to X$ with the property that $r^*L \subset \Omega_{X'}^k$ and the Iitaka fibration of r^*L is given by sections of r^*L. We then conclude by the previous argument that there exists a rational fibration $\phi' : X' \dashrightarrow Y'$, with Y' smooth of dimension k such that r^*L identifies with $\phi^*\Omega_Y^k$ generically on X'. The previous proof also shows that the map $\phi' = \phi_{r^*L}$ is in fact given by the Iitaka fibration associated with the line bundle r^*L. On the other hand, r^*L has Iitaka dimension 0 on the irreducible components of the varieties $r^{-1}(X_s)$ where X_s is the general fiber of the Iitaka fibration of X associated with L. It follows that we have a commutative diagram of Iitaka fibrations

$$\begin{array}{ccc} X' & \xrightarrow{\phi'} & Y' \\ \downarrow r & & \downarrow r' \\ X & \xrightarrow{\phi_L} & Y \end{array} \quad , \tag{11}$$

with r' generically finite, showing that L identifies generically on X with $\phi_L^*\Omega_Y^k$. $\square$

Definition 1.17. *Following Campana, we will call a rank 1 subsheaf $\mathcal{L} \subset \Omega_X^k$ a Bogomolov subsheaf if $\kappa(X, \mathcal{L}) = k$, with $k > 0$.*

Remark 1.18. By Lemma 1.6, the existence of a Bogomolov sheaf on X is a bimeromorphically or birationally invariant property of X (assumed to be compact or projective). More precisely, for any rank 1 saturated subsheaf $\mathcal{L} \subset \Omega_X^k$, and any birational map $\phi : X' \dashrightarrow X$, the saturation of $\phi^*\mathcal{L} \subset \Omega_{X'}^k$ has the same Iitaka dimension as $\mathcal{L}$.

2 Fibrations from families of cycles

The results developed in the previous section say nothing in the case of varieties of Kodaira dimension $-\infty$ or with no nonzero pluridifferential forms. We are going to present in this and the following sections various constructions of geometric fibrations explaining, at least conjecturally, the shape of pluridifferential forms. The main conjecture of the field says that the MRC fibration, a fibration with rationally connected fibers that we are going to construct in Section 3.2), is nontrivial (in the sense that its base has dimension $< \dim X$) for varieties X with Kodaira dimension $-\infty$. In a different direction, we will see in Section 4 that the classification of varieties by Kodaira dimension is not (at least conjecturally) the right one from the viewpoint of hyperbolicity. Campana introduced in [20] the notion of "special" variety, and constructed on any projective variety X a natural rational fibration with "special" fibers, that is conjectured to explain the degeneracy of the Kobayashi pseudometric of X. A posteriori, this fibration, called the "core" fibration, can be recovered from Bogomolov sheaves on X. We will also describe the Gamma-fibration, which is a crucial tool to study the fundamental group of a variety.

2.1 Generalities about Hilbert schemes and Chow varieties

In what follows, a very general point in a complex algebraic variety (or complex manifold) X is a point taken away from a specified countable union of closed algebraic (or analytic) subsets in an algebraic (or analytic) variety. Note that the bad set removed depends on the specific considered problem. A typical example is as follows: Let X be a complex projective variety. The theory of the Hilbert scheme tells us the following:

Theorem 2.1. *There are countably many projective schemes $\mathcal{Z}$ equiped with a flat morphism f to a projective scheme Y and a morphism ϕ to X*

$$\begin{array}{ccc} \mathcal{Z} & \xrightarrow{\phi} & X \\ \downarrow{\scriptstyle f} & & \\ Y & & \end{array} \quad , \tag{12}$$

which is an embedding on the fibers of f, and such that any subscheme $Z \subset X$ is $f(\mathcal{Z}_y)$ for such a family and for some point $y \in Y$.

A similar result holds if X is compact Kähler (with closed algebraic subsets or subschemes replaced with closed analytic subsets or ringed subspaces), except that the basis Y will be then only a compact analytic space with Kähler desingularization (see [4]).

The proof of Theorem 2.1 is much more difficult in the analytic context. In the algebraic situation, we observe that it suffices to construct the Hilbert schemes for $\mathbb{P}^N$ itself, since it is clear that the Hilbert schemes of $X \subset \mathbb{P}^N$ are closed algebraic subsets of $\mathrm{Hilb}\,\mathbb{P}^N$ defined by the conditions that the defining equations for X vanish on the considered subscheme. For $\mathbb{P}^N$, we first have to show that subschemes of bounded degree form a bounded family. This means that they are finitely many families parameterized by quasiprojective varieties exhausting all subschemes of a given degree in projective space. This can be proved by showing that closed algebraic subsets of degree d in $\mathbb{P}^N$ are schematically defined by equations of degree d', where d' depends only on d (we can take in fact $d' = d$). This boundedness allows to get uniform vanishing for the cohomology groups $H^i(\mathbb{P}^N, \mathcal{I}_{Z/\mathbb{P}^N}(k)) = 0$ for $i > 0$, $k \geq k_0$ for all subschemes Z of $\mathbb{P}^N$ of degree d, with k_0 depending only on d. It then follows that for $k \geq k_0$, $h^0(\mathbb{P}^N, \mathcal{I}_{Z/\mathbb{P}^N}(k)) = P_{\mathcal{I}_Z}(k)$, where $P_{\mathcal{I}_Z}$ is the Hilbert polynomial of $\mathcal{I}_{Z/\mathbb{P}^N}$. There are countably many Hilbert polynomials, and fixing one, we can then imbed the Hilbert scheme parameterizing subschemes Z of $\mathbb{P}^N$ with Hilbert polynomial $P_{\mathcal{I}_Z} = P$ into a Grassmannian $G(r_k, M)$ where $r_k = P_{\mathcal{I}_Z}(k)$, and $M = h^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(k))$, by the map

$$[Z] \mapsto I_Z(k) \subset H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(k)).$$

The last step in the construction of the Hilbert scheme is the proof that for k large, the image of this map is a closed algebraic subset of the Grassmannian. The algebraic equations for this algebraic subset of the Grassmannian describe the fact that the image of the multiplication map

$$I_Z(k) \otimes H^0(\mathcal{O}_{\mathbb{P}^N}(1)) \to H^0(\mathcal{O}_{\mathbb{P}^N}(k+1))$$

has image contained in $I_Z(k+1)$ hence rank $\leq P_{\mathcal{I}_Z}(k+1)$.

A key point in Theorem 2.1 is the projectivity or compactness of Y, which is where the projectivity or Kählerness of X is needed. The local existence of analytic spaces parameterizing deformations of closed subvarieties (resp. closed analytic subspaces) works without these assumptions (see [4]). The compactness due to Fujiki uses Bishop compactness [7]. Let us describe an example where closed analytic subspaces (in fact, rational curves) do not have a compact deformation space, even allowing of course degenerations, in a compact complex submanifold which is the twistor family of a $K3$ surface:

Example 2.2. Choose a positive closed $(1,1)$ form ω which is the Kähler form of a Kähler-Einstein metric h on a $K3$ surface S. Then if σ_S is the holomorphic 2-form on S, σ_S, ω and the operator I of almost complex structure on S are parallel with respect to the Levi-Civita connection associated to the metric $g = \operatorname{Re} h$. Then consider the conic in $\mathbb{P}^2$ with coordinates α, β, γ defined by

$$\int_S (\alpha\omega + \beta\sigma_S + \gamma\overline{\sigma_S})^2 = 0. \tag{13}$$

For a point $0 \neq t = (\alpha, \beta, \gamma)$ in this conic, the 2-form $\omega_t := \alpha\omega + \beta\sigma_S + \gamma\overline{\sigma_S}$ is parallel on S, hence so is its square. Equation (13) then implies that ω_t is everywhere degenerate on S. On the other hand, it is nowhere 0, hence it defines a rank 2 vector subbundle $F \subset T_{S,\mathbb{C}}$ which is in fact transverse to $T_{S,\mathbb{R}}$. To see this last point, observe that if $\omega_t \neq 0$ then $\omega_t \wedge \overline{\omega}_t > 0$ (this is true pointwise as forms on S as one easily checks). We then have an almost complex structure I_t such that F is the sheaf of $(1,0)$-tangent vectors for I_t and it is easy to show, using the flatness for the Levi-Civita connection, that I_t is integrable, giving rise to a deformation S_t of the complex structure on S. Working a little more, this construction provides a compact complex threefold $\mathcal{X}$ with a morphism $f : \mathcal{X} \to \mathbb{P}^1 = C$ whose fibers are the $K3$ surfaces S_t. $\mathcal{X}$ is diffeomorphic to $S \times C$ and the curves $x \times C$, for $x \in S$ are holomorphic $\mathbb{P}^1$'s contained in X. The family of these $\mathbb{P}^1$'s is not holomorphic however; it is the real part of a 4-dimensional holomorphic family $\mathcal{M}$ of deformations of these $\mathbb{P}^1$'s. The fact that there is no compact family $\overline{\mathcal{M}}$ of such curves and their degenerations follows from the observation (used by Campana in [18]) that the deformation space $\mathcal{M}_x$ of such $\mathbb{P}^1$ passing through one point $x \in \mathcal{X}$ is 2-dimensional. Assume $\overline{\mathcal{M}}$ is compact, then we have accordingly a compact 2-dimensional family $\overline{\mathcal{M}}_x$ of rational curves in X passing through x. This implies by Mori's argument [25] that there is a reducible $\mathbb{P}^1$ passing through this point. As the degree of these $\mathbb{P}^1$'s over C is 1, this is possible only if the limiting rational curve has an irreducible component which is a rational curve in S passing through x. Choosing for x a point which does not lie on a rational curve contained in S, we get a contradiction.

Remark 2.3. Campana used in [18] these rational curves and their *Brody* reparametrization to show that in the twistor family of any hyper-Kähler manifold, at least one fiber is not Kobayashi hyperbolic. This result is now essentially subsumed by Verbitsky's work [54]. The limits are thus entire curves and not closed analytic subsets.

An immediate consequence of Theorem 2.1 is the following result that will be crucial for the construction of fibrations:

Theorem 2.4. *Let X be a smooth projective variety (resp. a compact Kähler manifold). Let $x \in X$ be a very general point of X. Then if $Z \subset X$ is a closed subvariety of X passing through x, there exists a covering family*

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\phi} & X \\ \downarrow{\scriptstyle f} & & \\ Y & & \end{array} \quad , \tag{14}$$

where f is a fibration, $\mathcal{Z}$ and Y are smooth projective (resp. compact Kähler manifolds) and ϕ is surjective, such that for some point $s \in Y$, $\phi_{|\mathcal{X}_s}$ identifies to the natural map $\widetilde{Z} \to X$, where $\widetilde{Z}$ is a desingularization of Z in X.

Proof. We apply Theorem 2.1. We then have countably families of subvarieties or subschemes in X parameterized as in (12). For each of them, the morphism ϕ is projective, hence has Zariski closed image. Let us declare that the bad set is the countable union of the $\mathrm{Im}\,\phi$ which are not the whole of X. Let now $x \in X$ be taken away from this set, and let $Z \subset X$ be a subvariety (or subscheme!) of X passing through x. We know by Theorem 2.1 that there exists a family

$$\begin{array}{ccc} \mathcal{Z} & \xrightarrow{\phi} & X \\ \downarrow{\scriptstyle f} & & \\ Y & & \end{array} \quad , \tag{15}$$

such that Z identifies with one fiber $\mathcal{Z}_y \hookrightarrow X$. We can assume that Y is smooth by desingularization. By definition of the bad set, the morphism ϕ has to be surjective. The conclusion follows by taking for $\mathcal{X}$ a desingularization of $\mathcal{Z}$, using the fact that Z is reduced, hence generically smooth, so that $\mathcal{Z}$ is generically smooth along $\mathcal{Z}_y$. □

Remark 2.5. If the original variety Z is smooth, the variety $\mathcal{Z}$ above is smooth along $\mathcal{Z}_y$, hence we can assume that $\mathcal{X} \cong \mathcal{Z}$ along $\mathcal{Z}_y$, which is then isomorphic to Z.

Remark 2.6. As the general fiber Z is reduced, we can also if needed assume, by restricting and desingularizing Y, that the morphism $\mathcal{Z} \to X$ is generically finite. Indeed, let $z \in \mathcal{Z}$ be a smooth point such that Y is smooth at $y = f(z)$ and the fiber $\mathcal{Z}_y$ of f passing through z is smooth and ϕ is a submersion at z. Then as $\phi_{|\mathcal{Z}_y}$ is an immersion, for a general vector subspace $V \subset T_{Y,y}$ of dimension $k := \mathrm{codim}\,\mathcal{Z}_y \subset X$, the differential $\phi_* : f_{*,z}^{-1}(V) \to T_{X,\phi(z)}$ is an isomorphism, hence for any subvariety Y' of Y of dimension k with tangent space V at y, the map $\phi_{|\mathcal{Z}_{Y'}}$ is generically finite onto X.

2.2 First application: fibrations associated with a dominant selfmap

Let X be smooth projective and let $f : X \to X$ be a dominant rational selfmap. (Some nontrivial examples with X a projective hyper-Kähler or Calabi-Yau manifold with Picard number 1 have been constructed in [56].) The following appears in [2].

Theorem 2.7. *Either for the very general point $x \in X$, the orbit $\{x, f(x), \ldots, f^k(x) \ldots\}$ is Zariski dense in X, or there exists a nontrivial fibration $\phi : X \dashrightarrow B$ such that f acts on X over B: $\phi \circ f = \phi$.*

Proof. Let us assume for simplicity that f is a morphism. For any $x \in X$, we can consider the closed algebraic subset O_x which is defined as the Zariski closure of $\{x, f(x), \ldots, f^k(x) \ldots\}$. O_x is stable under f, hence its irreducible components Γ of a given dimension are preperiodic, that is $f^l(\Gamma)$ is stable under f^k for some $l \geq 0$, $k > 0$ which can be taken independent of x, by a countability argument and because x is very general. Let Γ_x be an irreducible component of O_x passing through x. If $\Gamma_x = X$, then we are done. Assume thus Γ_x is different from X. Then $f^l(\Gamma_x)$ is stable under f^k and passes through $f^l(x)$. Note that, as x is very general and f is dominant, the point $f^l(x)$ is also a very general point of X. We thus know that through the general point $f^l(x)$ of X, there is a variety $f^l(\Gamma_x)$ which is invariant under f^k for some $k > 0$. We can thus apply Theorem 2.4, and conclude that these varieties form a family, that is, X is swept-out by a family of varieties invariant under a power f^k. We now observe that the varieties of minimal dimension, irreducible and invariant under some power f^k passing through the general point of X form a meromorphic fibration of X. This means that there is only one such variety passing through the general point of X. This is obvious since each irreducible component of $\Gamma_1 \cap \Gamma_2$ passing through x satisfies these properties if Γ_1 and Γ_2 do (maybe up to changing k). We thus constructed a fibration of X which is invariant under some power f^k, for some $k > 0$. It is then immediate that f acts meromorphically on the base B of this fibration, providing $g : B \dashrightarrow B$ such that $g^k = Id_B$. Then the composite rational map $X \dashrightarrow B \dashrightarrow B/<g>$ satisfies the desired properties. □

3 MRC and Γ-fibrations

3.1 Rationally connected varieties

Rationally connected varieties were introduced by Kollár, Miyaoka and Mori in [40]. There are in fact several notions which are equivalent for a smooth projective variety over $\mathbb{C}$. The basic one is chain rational connectedness.

Definition 3.1. *A projective variety X is chain rationally connected if given any two points $x, y \in X$, there exists a chain of rational curves in X, that is rational maps $f_i : \mathbb{P}^1 \to X$, $i = 1, \dots, N$, such that $f_1(0) = x$, $f_N(\infty) = y$ and for $i = 1, \dots, N-1$, $f_i(\infty) = f_{i+1}(0)$. X is said to be rationally connected if one can join any two general points by a single rational curve.*

For example, the projective cone C_X over any projective variety X is chain rationally connected, but if X does not contain any rational curve, the only way to join x to y by a chain in C_X is to use the lines passing through x and y and meeting at the vertex O of the cone. This shows that one cannot in general assume that a single rational curve contains both x and y.

Example 3.2. The example of the cone also shows that a projective variety can be chain rationally connected without having any chain rationally connected smooth projective model. In particular, this is not a birationally invariant property. To the contrary, rational connectedness is clearly a birationally invariant property.

However, when X is smooth, one has the following:

Theorem 3.3. *(See [40]) Let X be smooth projective chain rationally connected over $\mathbb{C}$. Then for x, y two general points of X, there exists an irreducible rational curve $f : \mathbb{P}^1 \to X$ passing through x and y, that is $f(0) = x$, $f(\infty) = y$, so X is rationally connected. Equivalently, there exists a rational curve $f : \mathbb{P}^1 \to X$ such that f^*T_X is a positive vector bundle on $\mathbb{P}^1$. Thus a projective variety is rationally connected if and only if one (or any) of its desingularizations is chain rationally connected.*

Here we use Grothendieck's theorem saying that any vector bundle E on $\mathbb{P}^1$ is a direct sum of $\mathcal{O}_{\mathbb{P}^1}(a_i)$. E is said to be positive if all a_i are positive. The fact that the second statement is equivalent to the first follows from the observation that the first order deformations of $\mathbb{P}^1 \to X$ with the condition $f(0) = x$ for fixed x are parameterized by $H^0(\mathbb{P}^1, f^*T_X(-0))$ and that the differential of the evaluation map at (f, ∞) identifies then to the evaluation map

$$H^0(\mathbb{P}^1, f^*T_X(-0)) \to (f^*T_X(-0))_{|\infty}. \tag{16}$$

Thus if this differential is surjective, which by Sard's theorem will be the case for a general f if the general rational curve through x which is a deformation of f passes through the general point of X, the evaluation map (16) is surjective, which says that $f^*T_X(-0)$ is generically generated by sections, hence f^*T_X is positive. The first statement in Theorem 3.3 (passing from chains to reducible curves) is more tricky as one needs to smoothify the chain of $\mathbb{P}^1$'s. One ingredient is the following lemma:

Lemma 3.4. *Let X be smooth projective, and let $x \in X$ be a very general point. Let $f : \mathbb{P}^1 \to X$ be a rational curve passing through x. Then f^*T_X is semi-positive.*

Proof. Indeed, by Theorem 2.4, there is a family of rational curves $F : B \times \mathbb{P}^1 \to X$ which dominates X, and such that $F_b = f$ for some $b \in B$. As x is general and F is dominating, the differential of F can furthermore be assumed to be surjective at $(b, 0)$, where $F_b(0) = x$. This differential factors as before through an evaluation map

$$T_{B,b} \to H^0(\mathbb{P}^1, F_b^* T_X) \to (F_b^* T_X)_{|\infty}$$

and the surjectivity of the composed map thus implies that $f^* T_X$ is generically generated by sections, that is, semipositive. □

Remark 3.5. A rational curve $f : \mathbb{P}^1 \to X$ satisfying the conclusion of Lemma 3.4 is said to be free. Free rational curves have a very good deformation theory, since they are unobstructed.

The difficulty left in proving Theorem 3.3 is the following: We have for x, y general points of X a chain of rational curves from x to y. By Lemma 3.4, the first curve $f_1 : \mathbb{P}^1 \to X$ and the last curve $f_N : \mathbb{P}^1 \to X$ both satisfy the property that $f_i^* T_X$ is semipositive. Unfortunately, it could be very well that the intermediate curves do not sweep-out X and in fact have some negative summands in $f_i^* T_X$. What is done in [40] is to attach to these intermediate curves more "legs", coming for example from the deformations of C_1, C_N. This process of attaching free rational curves increases the positivity of the normal bundle. One can then smoothify progressively $C_1 \cup C_2$, $C_1 \cup C_2 \cup C_3$ etc...

3.2 The MRC fibration

The maximal rationally connected fibration of a smooth projective variety X is constructed by Campana [19]. The sketch of proof given here uses further ingredients coming from [40], [32].

Theorem 3.6. *Given any smooth projective variety X, there exists a rational map $\phi : X \dashrightarrow B$ which has the following properties:*

(i) For a very general point $x \in X$, any rational curve $C \subset X$ passing through a general point x' of the fiber X_x of ϕ through x is contained in X_x.

(ii) The fibers of ϕ are rationally connected.

(iii) The rational map ϕ is almost holomorphic, which means that it is well-defined along the general fiber.

Remark 3.7. Using property (iii), the general fiber of ϕ is smooth.

Remark 3.8. Point (iii) is essential and will be obtained as a consequence of a stronger version of (i), namely: *(i)' For a very general point $x \in X$, any rational curve $C \subset X$ intersecting X_x is contained in X_x.*

Note that we cannot impose a better condition than (i)', removing the generality assumption on x. For example, it is not possible in general to construct a morphism or rational map with rationally connected general fiber and contracting all rational curves. Indeed, consider the case of an elliptic surface S with Kodaira dimension ≥ 0. Then S is not ruled. On the other hand, many such surfaces contain countably many rational curves. It suffices for this to have one rational curve, whose intersection with the general fiber S_t is not proportional in $\operatorname{Pic} S_t$ to $h_{|S_t}$, where h is a given ample line bundle on S, of degree d on fibers S_t. Then we can use the rational maps $\phi_k : S \dashrightarrow S,\ S_t \ni x \mapsto kh_{|S_t} - (kd-1)x$ to construct infinitely many rational curves in S. No nonconstant rational map can contract all of them. Indeed, a generically finite rational map $S \dashrightarrow S'$ contracts only finitely many curves. For a rational map $S \dashrightarrow B$ to a curve, either there are finitely many rational curves contained in fibers, or all fibers are rational, since there are finitely many singular fibers. As our surface is not ruled, the second possibility is also excluded.

Sketch of the proof of Theorem 3.6. We first use Lemma 3.4 which says that if $x \in X$ is a very general point, then any rational curve $f : \mathbb{P}^1 \to X$ passing through x is free. We also observe that similarly, a very general point x' of the curve $C_x = f(\mathbb{P}^1)$ is a very general point of X, and thus Lemma 3.4 applies to x'. Assume there is now a rational curve $C'_{x'}$, $C' = \operatorname{Im} f'$, $f' : \mathbb{P}^1 \to X$, passing through x', which is different from C. Then we will get at least a surface $S = \bigcup_{x' \in \mathbb{P}^1} C'_{x'}$ passing through x. This surface is clearly chain rationally connected since any general point in it is joined to x by a chain of two rational curves, a curve C'_x and the curve C), but it is not clear that it is rationally connected: as it has been constructed, it could be singular along the curve C and after normalization becomes a ruled surface over a nonrational covering of C. To circumvent this problem, one has to use the fact that both C and the curves $C'_{x'}$ are free. Note that one can also assume that the two curves C and $C'_{x'}$ are smooth and transverse at x'. The following deformation lemma is needed (see [40]):

Lemma 3.9. *Let $f : \mathbb{P}^1 \to X$, $f' : \mathbb{P}^1 \to X$ be two free rational curves such that $f(0) = f'(0')$. Then there is a smoothification of $(f, f') : C'' \to X$, where C'' is the union of C and C' glued via $0 = 0'$, that is a morphism $g : \mathcal{C} \to X$, where $\mathcal{C} \to B$ is a smooth surface over the smooth curve B, with smooth fiber isomorphic to $\mathbb{P}^1$ over $b \neq b_0$, and with fiber $\mathcal{C}_{b_0} \cong C''$, $g_{|\mathcal{C}_{b_0}} = (f, f')$. Furthermore, the morphism g_b is free for general b.*

This deformation lemma is called "weak gluing lemma" in [48] where it is even mentioned that one can assume that the deformations g_b all pass through the given point $f(0) = f'(0')$. Applying this lemma to the general chain $C \cup C'$ of two free rational curves, we conclude that there is for very general $x \in X$ at least a surface S_x in X consisting of points joined to x by a free rational curve. We can then continue arguing with C_x replaced by S_x, until the process stops. When the process stops, one gets a variety X_x passing through the very general point $x \in X$, with the property (i). One then proves that these varieties give a fibration of X. The very general point x' of a variety X_x is a very general point of X and $X_x = X_{x'}$ has the property that a general point y of X_x is joined to x' by a rational curve, hence X_x is rationally connected. Hence (i) and (ii) are proved. Point (iii) can be obtained as an application of Theorem 3.10 below. Indeed, the fibration whose construction is sketched above gives a morphism $\tilde{\phi} : \widetilde{X} \to B$, where $\tau : \widetilde{X} \to X$ is constructed by a sequence of blow-ups starting from X. The irreducible components of the positive dimensional fibers of τ are rationally connected. Hence if E is an exceptional divisor of τ, either E does not dominate B or the morphism $\tilde{\phi} : E \to B$ factors through τ. One thus concludes that generically over B, the morphism ϕ is well-defined. □

An essential property satisfied by the MRC fibration map, and already used above to prove (iii) in Theorem 3.6 and (i)' in Remark 3.8, has been established by Graber-Harris-Starr [32].

Theorem 3.10. *Let X be smooth and projective and let $\phi : X \dashrightarrow B$ be the MRC fibration of X. Then the variety B (which is defined up to birational equivalence) is not uniruled.*

Remark 3.11. Item (iii) had been given a direct proof (see [19]) before Theorem 3.10 was proved.

Theorem 3.10 is obtained as a consequence of the general properties (i), (ii) of the MRC fibrations and of the following fundamental result:

Theorem 3.12. *Let $f : Y \to C$ be a projective morphism with rationally connected general fiber, where C is a smooth curve. Then there is a section $C \to Y$ of f. Furthermore, there is a section of f passing through the general point of the general fiber of f.*

Proof of the implication Theorem 3.12 ⇒ Theorem 3.10. Let $\phi : X \dashrightarrow B$ be the MRC fibration of X and let b be a general point of B. Up to a birational transformation, we may assume f is a morphism. Assume there is a rational curve, that is a nonconstant map $\alpha : \mathbb{P}^1 \to B$ passing through b, hence a point $0 \in \mathbb{P}^1$ such that $\alpha(0) = b$. Then as b is general,

$X_\alpha := X \times_B \mathbb{P}^1$ has only one irreducible component $X_{\alpha,d}$ dominating $\mathbb{P}^1$. Furthermore $X_{\alpha,d}$ is smooth along the fiber over 0 of the induced morphism $\phi_\alpha : X_{\alpha,d} \to \mathbb{P}^1$ which identifies with the fiber X_b, hence is rationally connected. Thus, even after desingularization of $X_{\alpha,d}$, the morphism $\tilde{\phi}_\alpha : \widetilde{X_{\alpha,d}} \to \mathbb{P}^1$ has rationally connected fibers and we can apply to it Theorem 3.12. The variety $\widetilde{X_{\alpha,d}}$ thus contains rational curves not contained in the fiber over 0 and passing through the general point of the fiber over 0. The morphism $\widetilde{X_{\alpha,d}} \to X$ is finite onto its image near the fiber over 0, hence we conclude that X contains rational curves not contained in the fiber X_b and passing through the general point of X_b. This contradicts property (i) in Theorem 3.6. □

3.2.1 Mumford's conjecture

Let us start with the following observation:

Lemma 3.13. *Let X be a rationally connected variety. Then we have $H^0(X, \Omega_X^{\otimes k}) = 0$ for $k > 0$.*

Proof. Indeed, by Theorem 3.3, there exists a morphism $f : \mathbb{P}^1 \to X$ such that $f^*T_X = \oplus_i \mathcal{O}_{\mathbb{P}^1}(a_i)$, where all a_i's are > 0. The deformations of this morphism on the other hand sweep-out X since they are unobstructed and the first order evaluation map

$$f_\epsilon : \mathbb{P}^1 \times B_\epsilon \to X$$

has surjective differential at any point of $\mathbb{P}^1$. Note that the generic deformation $f_t : \mathbb{P}^1 \to X$ also satisfies the property that $f_t^*T_X = \oplus_i \mathcal{O}_{\mathbb{P}^1}(a'_i)$, where all a_i's are > 0, since this is a Zariski open property on families of vector bundles on $\mathbb{P}^1$. Next, as $f^*T_X = \oplus_i \mathcal{O}_{\mathbb{P}^1}(a_i)$, with $a_i > 0$ for any i, we get that $H^0(\mathbb{P}^1, f^*\Omega_X^{\otimes k}) = 0$ for $k > 0$ and similarly $H^0(\mathbb{P}^1, f_t^*\Omega_X^{\otimes k}) = 0$ for $k > 0$ for a general deformation f_t of f. Thus $H^0(X, \Omega_X^{\otimes k}) = 0$ for $k > 0$ since the curves $f_t(\mathbb{P}^1)$ sweep-out a dense Zariski open set of X. □

The converse statement is the following conjecture attributed to Mumford.

Conjecture 3.14. *Let X be smooth projective such that $H^0(X, \Omega_X^{\otimes k}) = 0$ for $k > 0$. Then X is rationally connected.*

Using the MRC fibration, this conjecture has been reduced in [32] to the following conjecture, which is known to hold true in dimension ≤ 3 (see [48]):

Conjecture 3.15. *Let X be a variety with Kodaira dimension $-\infty$. Then X is uniruled.*

The reduction to Conjecture 3.15 goes as follows: Assume Conjecture 3.15 and let X be such that $H^0(X, \Omega_X^{\otimes k}) = 0$ for $k > 0$. Let $X \dashrightarrow B$ be the MRC fibration of X. Then $H^0(B, K_B^{\otimes l})$ injects into $H^0(X, \Omega_X^{\otimes l \dim B})$ for $l > 0$, hence $H^0(B, K_B^{\otimes l}) = 0$ for $k > 0$. As Conjecture 3.15 is assumed to hold, B is uniruled unless it is a point. But by Theorem 3.10, the basis of the MRC fibration of any variety is never uniruled. So B is a point and X is rationally connected.

A line bundle L on a projective variety X is pseudoeffective if it is a "limit" of effective line bundles, which means concretely that there are sequences of positive integers (k_n, k'_n) with $\lim_{m\to\infty} k'_m/k_m = 0$ such that $k_m L + k'_m H$ is effective for all m, where H is a given ample line bundle on X. Pseudoeffective line bundles or divisors D have a nonnegative intersection with moving curves, i.e. curves whose deformations sweep-out the ambient variety. Indeed, we have

$$[D] = \lim_{m\to\infty} [D] + \frac{k'_m}{k_m}[H] \text{ in } H^2(X, \mathbb{R}$$

where $[D] + \frac{k'_m}{k_m}[H] = \frac{k'_m}{k_m}[E_m]$, with E_m effective. Thus $[D] \cdot C = \lim_{m\to\infty} [E_m] \cdot C \geq 0$ if C is moving. In particular, a pseudoeffective divisor on a smooth projective variety X has nonnegative intersection with the class $c_1(L)^{n-1}$, where L is any ample line bundle on X and $\dim X = n$. Working a little more, this intersection number is in fact positive if $[D] \neq 0$.

An important recent progress on Conjecture 3.15 has been obtained in [12] which proves:

Theorem 3.16. *Let X be a smooth projective variety such that K_X is not pseudoeffective. Then X is uniruled.*

The question left is thus whether Kodaira dimension $-\infty$ implies that the canonical bundle is not pseudoeffective, which is one instance of the abundance conjecture.

3.2.2 An application to Calabi-Yau manifolds

Recall first from [5] that a smooth projective variety with trivial canonical bundle has a finite étale cover which is a product of irreducible hyper-Kähler manifolds, simply connected Calabi-Yau manifolds, and abelian varieties. The Ricci flat Kähler metric existing on such variety is the product metric, and on the Calabi-Yau factors, say of dimension k, the holonomy is $SU(k)$, while on a hyper-Kähler factor of dimension $2k$, the holonomy is $Sp(k)$. A K-trivial manifold will be said irreducible Calabi-Yau if, for the decomposition above, it has only one factor. The following theorem is proved in [45] (for Lagrangian fibrations on hyper-Kähler manifolds, it follows from [60] combined with [47]).

Theorem 3.17. *Let X be a projective irreducible simply connected Calabi-Yau manifold. Then for any dominant rational map $\phi : X \dashrightarrow Y$, where Y is smooth projective and $\dim Y < \dim X$, Y is rationally connected (or a point).*

Proof. We first observe that it suffices to prove that under the assumptions above, Y is uniruled. Indeed, starting from $\phi : X \dashrightarrow Y$, we have the dominating composite rational map $\phi' : X \dashrightarrow Y \dashrightarrow B$, where the second map is the MRC fibration of Y. We know by Theorem 3.10 that if B is not a point, then B is not uniruled, hence this produces a dominating rational map from X to a not uniruled variety. Once this is excluded, we find that B has to be a point hance Y is rationally connected. Next by Theorem 3.16, Y is uniruled if and only if its canonical bundle is not pseudoeffective. So we only have to prove that given a dominant rational map $\phi : X \dashrightarrow Y$, K_Y is not pseudoeffective. Assume the contrary: The map ϕ provides a rank 1 subsheaf $\phi^*\Omega_Y^k$, $k = \dim Y$, of Ω_X^k on the Zariski open set $U \subset Y$ where ϕ is defined. There is thus a line bundle $D \subset \Omega_X^k$ which is the saturation of $\phi^*\Omega_Y^k$. If we choose an ample line bundle H on X, we know by the existence of Kähler-Einstein metric on X with Kähler class $c_1(H)$ that Ω_X is H-stable and that Ω^k is H-polystable (see [38], [46]). More precisely, in the case of $SU(n)$ holonomy, the vector bundle Ω_X^k is stable, while in the case of $Sp(n)$ holonomy, there is a decomposition

$$\Omega_X^k = \bigoplus_{2r \leq k} \sigma_X^r \wedge \Omega_{X,0}^{k-2r}, \tag{17}$$

where $2n = \dim X$ and $\Omega_{X,0}^{k-2r} := \mathrm{Ker}\,(\Omega_{X,0}^{k-2r} \xrightarrow{\wedge \sigma_X^{n-k+2r+1}} \Omega_X^{2n-k+2r+2})$. Here σ_X denotes the holomorphic 2-form of X, and the summands $\Omega_{X,0}^{k-2r}$ are stable. The line bundle $D \subset \Omega_X^k$ being pseudoeffective has nonnegative H-degree. This is possible only if D is one of the summands in the decomposition (17) of Ω_X^k as a direct sum of H-stable bundles. However the only rank 1 such summand is $\sigma_X^{k/2}$ when k is even and it does not correspond to a locally decomposable subsheaf of Ω_X^k when $k < 2n$. This is a contradiction which concludes the proof. □

3.3 Shafarevich maps (or Γ-reductions) and Shafarevich conjecture

Recall that a complex manifold M is said to be holomorphically convex if the holomorphic functions on M give a proper holomorphic map $M \to U$ where U is Stein. Let now X be a smooth projective variety (or compact Kähler manifold). The Shafarevich conjecture states the following:

Conjecture 3.18. *The universal cover* $\widetilde{X}$ *is holomorphically convex.*

Note that the conjecture is wrong for intermediate covers, as shows example 4.5 in [49], attributed to Narasimhan: For a general lattice $\mathbb{Z}^4 \cong \Lambda \subset \mathbb{C}^2$, let X be the complex torus $\mathbb{C}^2/\Lambda$. Then for a general rank 3 submodule $\Lambda' \subset \Lambda$, the quotient variety $\mathbb{C}^2/\Lambda'$ is a cover of X which is not holomorphically convex: it is not compact, but has no nonconstant holomorphic function.

Assume $\widetilde{X}$ is holomorphically convex, and let $F : \widetilde{X} \to U$ be the proper map given by holomorphic functions on $\widetilde{X}$. The fibers of F should thus be exactly the compact closed analytic subsets of $\widetilde{X}$. But via the natural holomorphic map $\widetilde{X} \to X$, the later correspond to closed analytic subsets Z of X having the property that $\pi_1(Z') \to \pi_1(X)$ has finite image where Z' is a desingularization of Z. Note here that there is a delicate and crucial issue on whether we consider Z or one of its desingularizations Z'. If the map $\pi_1(Z) \to \pi_1(X)$ has finite image, one concludes that the inverse image of Z in $\widetilde{X}$ is a countable *disjoint* union of compact closed analytic subsets. If we only know that the map $\pi_1(Z') \to \pi_1(X)$ has finite image, where Z' is a desingularization of Z, then the inverse image of Z in $\widetilde{X}$ is a countable but a priori not disjoint union of compact closed analytic subsets. This problem has been suggested as a possible reason for a negative answer to Shafarevich conjecture (cf. [10]). More generally, given a group morphism $\rho : \pi_1(X) \to \Gamma$, one considers closed analytic subsets Z of X having the property that $\pi_1(Z) \to \pi_1(X) \stackrel{\rho}{\to} \Gamma$ has finite image. The Γ-reduction was invented independently by Campana in [15] and Kollár in [41] (under the alternative name of Shafarevich map) in order to understand closed analytic subsets Z satisfying this property.

Theorem 3.19. *Let X be a smooth projective variety or compact Kähler manifold and let $\rho : \pi_1(X) \to \Gamma$ be a group morphism. There exists a rational fibration $\phi_\rho : X \dashrightarrow Y$ which is well defined up to birational transformations of Y, and satisfies the following properties:*

(i) For a very general point $x \in X$, the fiber X_x of ϕ_ρ passing through x has the property that the natural morphism $\pi_1(\widetilde{X}_x) \to \Gamma$ has finite image, where $\widetilde{X}_x$ is a (any) desingularization of X_ρ.

(ii) For a very general point $x \in X$ and any subvariety $Z \subset X$ passing through x with desingularization $\widetilde{Z}$, such the map $\pi_1(\widetilde{Z}) \to \Gamma$ has finite image, Z is contained in X_x.

(iii) The rational map ϕ_ρ is almost holomorphic.

Remark 3.20. It follows from (iii) that the general fiber X_s of ϕ_ρ is smooth, so that in (i), we can replace $\widetilde{X}_x$ by X_x.

Sketch of proof of Theorem 3.19. Point (iii) is an immediate consequence of (ii). Let $\tau : \widetilde{X} \to X$ be a smooth model of X on which ϕ_ρ is a morphism. If ϕ_ρ is not almost holomorphic, there is an exceptional divisor $E \subset \widetilde{X}$ which dominates Y and such that the fibers of $\tau_{|E}$ are not contracted by ϕ_ρ. The fibers of $\tau_{|E}$ are rationally connected and it follows that there is a rational curve $\mathbb{P}^1 \cong R \to \widetilde{X}$ such that its image $\mathbb{P}^1 \cong R \to Y$ passes through the very general point of Y. Then any desingularization $\widetilde{X}_R$ of the variety $R \times_Y \widetilde{X}$ has a dominant map p_1 to R by the first projection (note that, ϕ_ρ having irreducible general fiber, $R \times_Y \widetilde{X}$ has only one irreducible component dominating R, and $\widetilde{X}_R$ will be in fact a desingularization of this component). Let $R^0 \subset R$ be the regular locus of the composite map $p_1 : \widetilde{X}_R \to R \times_Y \widetilde{X} \to R$. We have the exact sequence of π_1's

$$\pi_1(\widetilde{X}_t) \to \pi_1(\widetilde{X}_{R^0}) \to \pi_1(R^0) \to 1,\, t \in \widetilde{R}^0.$$

On the other hand, there is a natural copy of R which is contained in $R \times_Y \widetilde{X}$, so by desingularization, we get a curve $R' \subset \widetilde{X}_R$ which maps to R (or rather its image) in $\widetilde{X}$.

In particular it is contracted by τ and thus the map $\pi_1(R'^0) \to \pi_1(X)$ is trivial. On the other hand, the natural map $\pi_1(R'^0) \to \pi_1(R^0)$ has image of finite index. As the natural map $\pi_1(\widetilde{X}_t) \to \Gamma$ has finite image, so does the map $\pi_1(\widetilde{X}_{R^0}) \to \Gamma$, which contradicts the maximality of the fibers (property (ii)), using the fact that $\pi_1(\widetilde{X}_{R^0}) \to \pi_1(\widetilde{X}_R)$ is surjective since $\widetilde{X}_R$ is smooth.

In order to construct the desired fibration, we will need the following lemma 3.21 (which has been in fact implicitly used in the previous proof). Let X, $\rho : \pi_1(X) \to \Gamma$ be as in the theorem. In what follows, $\widetilde{Y}$ will denote a smooth model of the variety Y.

Lemma 3.21. *Let $Z \subset X$ be a subvariety and let $\phi_W : \mathcal{W} \to X$, $\pi : \mathcal{W} \to B$ be a family of closed subvarieties $\mathcal{W}_b$ of X parameterized by a quasiprojective basis B such that:*

(i) For a general $b \in B$, the map $\rho \circ i_{b*} : \pi_1(\widetilde{\mathcal{W}_b}) \to \Gamma$ has finite image, where $i_b : \widetilde{\mathcal{W}_b} \to X$ is the natural map.

(ii) For any $b \in B$, $\mathcal{W}_b$ intersects Z.

(iii) The map $\rho \circ j_* : \pi_1(\widetilde{Z}) \to \Gamma$, where $j : \widetilde{Z} \to X$ is the natural morphism, has finite image.

Then for any smooth projective model $\widetilde{W}$ of the Zariski closure W in X of the union $\cup_{b \in B} \mathcal{W}_b$, the natural map $\pi_1(\widetilde{W}) \to \Gamma$ has finite image.

Proof. Let $\phi_W : \widetilde{\widetilde{\mathcal{W}}} \to X$ be a smooth and projective model of $\mathcal{W}$ mapping to X. Then by definition $\widetilde{W}$ is a desingularization of the image $\phi_W(\widetilde{\widetilde{\mathcal{W}}}) \subset X$ and we may assume by further blowing-up $\widetilde{\widetilde{\mathcal{W}}}$ that the morphism ϕ_W factors through a dominant morphism $\psi_W : \widetilde{\widetilde{\mathcal{W}}} \to \widetilde{W}$. By Lemma 1.2, 1, the map $\psi_{W*} : \pi_1(\widetilde{\widetilde{\mathcal{W}}}) \to \pi_1(\widetilde{W})$ has finite index, so it suffices to show that $\rho \circ \phi_{W*} : \pi_1(\widetilde{\widetilde{\mathcal{W}}}) \to \Gamma$ has finite image. The variety $\widetilde{\widetilde{\mathcal{W}}}$ contains a Zariski open set $\widetilde{\widetilde{\mathcal{W}}}^0$ which is a desingularization of $\mathcal{W}$, and in particular maps to B. Shrinking B, we can even assume that $\widetilde{\widetilde{\mathcal{W}}}^0 \to B$ is a smooth proper fibration. The map $\pi_1(\widetilde{\widetilde{\mathcal{W}}}^0) \to \pi_1(\widetilde{\widetilde{\mathcal{W}}})$ is surjective so we can restrict to $\widetilde{\widetilde{\mathcal{W}}}^0 =: \widetilde{\mathcal{W}}$. We now observe that by assumption (ii), the variety $\widetilde{\mathcal{W}}$ contains a subvariety Σ which dominates B via the morphism π, can be assumed to be smooth and maps to Z via ϕ_W. Up to further blow-ups and shrinking, we can also assume that the morphism $\phi_\Sigma : \Sigma \to Z$ factors through a morphism $\psi_\Sigma : \Sigma \to \widetilde{Z}$. The map $\pi_* : \pi_1(\Sigma) \to \pi_1(B)$ has finite index image. On the other hand, by (iii), the map $\rho \circ \phi_{W*}$ maps $\pi_1(\Sigma)$ to the subgroup $\rho \circ j_*(\pi_1(\widetilde{Z}))$ of Γ which by assumption is finite. Writing then the homotopy exact sequence

$$\pi_1(\widetilde{\mathcal{W}_b}) \to \pi_1(\widetilde{\mathcal{W}}) \to \pi_1(B^0) \to 1$$

and using the fact that $\rho \circ i_{b*}(\pi_1(\widetilde{\mathcal{W}_b}))$ has finite image, we conclude that $\mathrm{Im}\,(\rho \circ \phi_{W*} : \pi_1(\widetilde{\mathcal{W}_b}) \to \Gamma)$ has finite index in $\mathrm{Im}\,(\rho \circ \phi_{W*} : \pi_1(\widetilde{\mathcal{W}}) \to \Gamma)$. By property (i), $\mathrm{Im}\,(\rho \circ i_{b*} = \mathrm{Im}\,(\rho \circ \phi_{W*})_{|\pi_1(\widetilde{\mathcal{W}_b})} : \pi_1(\widetilde{\mathcal{W}_b}) \to \Gamma)$ is finite and we are done. □

The proof of Theorem 3.19 is now quite easy. For a very general point $x \in X$, let k be the maximal integer such that there exists an irreducible closed subvariety $Z_x \subset X$ passing through x and such that the natural map $\pi_1(\widetilde{Z}_x) \to \Gamma$ has finite image. We claim that Z_x is unique and that it is the fiber of a fibration $X \dashrightarrow B$ satisfying the desired assumptions. First of all, we apply Theorem 2.4 which provides

$$\begin{array}{ccc} \mathcal{Z} & \xrightarrow{\phi} & X \\ \downarrow{\scriptstyle f} & & \\ Y & & \end{array} \quad , \qquad (18)$$

where f is proper, ϕ is dominant and may be assumed to be finite, and Z_x corresponds to the inclusion in X of one fiber $\mathcal{Z}_y$ of f. Note that we can assume that $f(\mathcal{Z}_y) \neq f(\mathcal{Z}_{y'})$ for

y general and $y' \neq y$, as otherwise there is a map $Y \to Y'$ such that the family of varieties $f(\mathcal{Z}_y)$ factors through a family parameterized by Y'. We now desingularize $\mathcal{Z}$ to $\widetilde{\mathcal{Z}}$ and restrict to the Zariski open set Y^0 of Y over which f is smooth. As x is very general, we can assume that $y \in Y^0$, and thus the locally constant map $\pi_1(\widetilde{\mathcal{Z}}_y) \to \Gamma$ has finite image for any $y \in Y^0$. We claim that the map $f : \mathcal{Z} \to X$ is birational. To see this, consider $f^{-1}(f(\mathcal{Z}_y)) \subset \mathcal{Z}$. It clearly contains $\mathcal{Z}_y$. If f is not birational, $f^{-1}(f(\mathcal{Z}_y))$ has another irreducible component T dominating $\mathcal{Z}_y$, whose image B in Y has positive dimension. We now apply Lemma 3.21 to $\mathcal{Z}_B := \phi^{-1}(B)$. It satisfies properties (i), (ii) (with $Z = \mathcal{Z}_y$) and (iii), and thus we conclude that its image $\phi(\mathcal{Z}_B)$ is a closed algebraic subset Z' of X, passing through x and such that $\pi_1(\widetilde{Z}') \to \Gamma$ has finite image. This contradicts the fact that Z had maximal dimension. □

Remark 3.22. Exactly as in the case of the Iitaka-Kodaira fibration where it was observed that the basis needs not be of general type and even can have negative Kodaira fibration, it is quite possible that the basis of the Shafarevich map has trivial fundamental group. Here is an example: Let S be a $K3$ surface with an involution ι_S acting freely (so the quotient is an Enriques surface), and let C be a hyperelliptic curve with its hyperelliptic involution ι_C. Then let $X := S \times C/(\iota_S, \iota_C)$. The fundamental group of X is infinite because X has $S \times C$ as étale cover. On the other hand, X admits a natural map $f : X \to C/\iota_C = \mathbb{P}^1$ with general fiber isomorphic to S which is simply connected. Thus the very general fibers S_t must by definition of the Shafarevich map be the fibers of the Shafarevich map which is thus equal to f. In this case the basis of the Shafarevich map is thus $\mathbb{P}^1$.

The Shafarevich fibration is well-understood for certain types of groups:

Theorem 3.23. *(Campana [16], see also [35]) If X has nilpotent fundamental group, the Shafarevich fibration is given by the Stein factorization $X \to Y_{St}$ of the Albanese map $X \to Y \subset \operatorname{Alb} X$. Equivalently, let $\phi : X' \to Y'$ be a desingularization of $X \to Y_{St}$ (so that X' and Y' are smooth and ϕ is a morphism), then $\pi_1(X) = \pi_1(X')$ and $\phi_* : \pi_1(X') \to \pi_1(Y')$ is surjective with finite kernel.*

Proof. We follow Campana's proof. The fact that ϕ_* is surjective is Lemma 1.2, 1 since the fibers of $_phi$ are connected. Observe next that the map $\phi_* : H_1(X', \mathbb{Z}) \to H_1(Y', \mathbb{Z})$ is surjective with finite kernel (indeed, the map $\mathrm{alb}_* : H_1(X', \mathbb{Z}) \to H_1(\operatorname{Alb} X', \mathbb{Z})$ has finite kernel). Furthermore, the map $\phi^* : H^2(Y', \mathbb{Q}) \to H^2(X', \mathbb{Q}) \to$ is injective by Lemma 1.2, 2. We claim now that if $H = \pi_1(Y')$, the natural map $n_{Y'} : H^2(H, \mathbb{Q}) \to H^2(Y', \mathbb{Q})$ is injective. This can be proved as follows: Let $E_H \to B_H = E_H/H$ be the classifying space of H, with E_H contractible. Then Y' is homotopically equivalent to $(\widetilde{Y'} \times E_H)/H$ where $\widetilde{Y'}$ is the universal cover of Y', and denoting by $u_{Y'}$ the continuous map $(\widetilde{Y} \times E_H)/H \to B_H$, the map $n_{Y'}$ is equal to

$$u_{Y'}^* : H^2(B_H, \mathbb{Q}) \to H^2(Y', \mathbb{Q}) = H^2((\widetilde{Y'} \times E_H)/H, \mathbb{Q}).$$

It is clear that the fibers of $u_{Y'}$ are homeomorphic to $\widetilde{Y}$, hence are simply connected. The Leray spectral sequence of $u_{Y'}$ then shows that $u_{Y'}^* : H^2(B_H, \mathbb{Q}) \to H^2((\widetilde{Y'} \times E_H)/H, \mathbb{Q})$ is injective.

We can thus conclude by dualizing and composing the two surjective maps that the map $\phi_* : H_2(X, \mathbb{Q}) \to H_2(H, \mathbb{Q})$ is surjective. Let $G = \pi_1(X)$. It follows that the map $\phi_* : H_2(G, \mathbb{Q}) \to H_2(H, \mathbb{Q})$ is also surjective, using the following commutative diagram.

$$\begin{array}{ccc} H_2(X, \mathbb{Q}) & \longrightarrow & H_2(Y, \mathbb{Q}) \\ \downarrow & & \downarrow \\ H_2(G, \mathbb{Q}) & \longrightarrow & H_2(H, \mathbb{Q}) \end{array} , \qquad (19)$$

The proof of Theorem 3.23 is then concluded with the following lemma due to Stallings [52]. For any group Γ, let us denote by $\Gamma_n \subset \Gamma$ the n-th term of the lower central series: $\Gamma_n = [\Gamma, \Gamma_{n-1}]$.

Lemma 3.24. *Let $\alpha : G \to H$ be a morphism between finitely generated groups. Assume that α induces an isomorphism $H_1(G, \mathbb{Q}) \cong H_1(H, \mathbb{Q})$ and a surjective map $H_2(G, \mathbb{Q}) \to H_2(H, \mathbb{Q})$. Then for any n, the natural map $G/G_n \to H/H_n$ has finite kernel and cokernel.*

□

Corollary 3.25. *[35] If $\pi_1(X)$ has a finite index subgroup which is nilpotent, then the Shafarevich conjecture is true for X.*

Proof. Replacing X by a finite étale cover, we immediately reduce to the case where X has nilpotent fundamental group. We use the same notation as before: Y' is a smooth model of the Stein factorization of $\mathrm{alb}_X : X \to \mathrm{Alb}\, X$ and $a : X' \to Y'$ is a desingularization of the rational map $X \dashrightarrow Y'$. Let $\widetilde{A}$ be the universal cover of $\mathrm{Alb}\, X = \mathrm{Alb}\, Y'$. Then the cover $Y'_{ab} := Y' \times_{\mathrm{Alb(X)}} \widetilde{A}$ maps in a proper way to the Stein analytic space $\widetilde{A}$. Theorem 3.23 shows that the morphism $a_* : \pi_1(X') \to \pi_1(Y')$ of fundamental groups is surjective with finite kernel. Denoting by $\widetilde{X'}_{univ}$, $\widetilde{Y'}_{univ}$ the universal covers of X', Y' respectively, it follows that via the natural map $a_{univ} : \widetilde{X'}_{univ} \to \widetilde{Y'}_{univ}$, $\widetilde{X'}_{univ}$ identifies with a finite étale cover of $X' \times_{Y'} \widetilde{Y'}_{univ}$. In particular, a_{univ} is proper. On the other hand, $\widetilde{Y'}_{univ}$ maps in a proper way to the universal cover of the Stein manifold $Y' \times_{\mathrm{Alb(X)}} \widetilde{A}$. We then use the fact that the universal cover of a Stein manifold is Stein [53]. Thus $\widetilde{X'}_{univ}$ has a proper holomorphic map to a Stein manifold. □

Although Theorem 3.19 is not precise enough for applications to the Shafarevich conjecture, the idea of the Shafarevich map has played a crucial role in the proof of the Shafarevich conjecture under extra assumptions on the fundamental group. The case of linear fundamental groups has been solved in [29].

Theorem 3.26. *Let X be a smooth projective variety such that $\pi_1(X)$ admits a faithful linear representation. Then $\widetilde{X}$ is holomorphically convex.*

The case where the fundamental group admits a reductive representation had been treated by Eyssidieux in [28]. Let us say that a projective or compact Kähler variety has a large fundamental group if for any positive dimensional closed algebraic subset $Z \subset X$, with desingularization $\widetilde{Z} \to X$, the natural map $\pi_1(\widetilde{Z}) \to \pi_1(X)$ has infinite image. We will say that the fundamental of X is generically large if the same is true for varieties passing through the very general point of X. Theorem 3.19 can be rephrased saying that a variety has generically large fundamental group if and only if the Shafarevich fibration is the trivial fibration $X \to X$. As we mentioned in Remark 3.22, it is unfortunately not the case that the base of the Shafarevich fibration has generically large fundamental group. Furthermore, only its very general fiber is controled, and it could even be that special fibers have configurations of their components leading to a disproof of the Shafarevich conjecture. The paper [29] constructs in a more controled way, using nonabelian Hodge theory, the Shafarevich fibration associated with a given linear representation of the fundamental group, and introduces an orbifold structure Y_{orb} on the base, such that the given representation of the fundamental group of the original variety X factors through the orbifold fundamental group. This is the starting point of the works [28], [29], the second crucial ingredient being again anabelian Hodge theory related to the given faithful representation of $\pi_1(X)$. This allows the authors to construct a plurisubharmonic exhaustion function on the universal cover of the orbifold or stack Y_{orb}, in order to show that it is Stein.

4 Special varieties and the core

4.1 Morphisms of general type

The starting point of Campana's work [20] is illustrated by the following example: Consider a hyperelliptic curve C of genus $g \geq 2$ with hyperelliptic involution ι and an elliptic curve

E with nonzero 2-torsion point η. Then the involution $\sigma := (\iota, \eta)$ acts freely on $C \times E$. The quotient surface $\Sigma : C \times E/\sigma$ admits a morphism to the elliptic curve E/η, and a morphism $f : \Sigma \to \mathbb{P}^1 = C/\iota$ whose fibers are elliptic curves.

Lemma 4.1. *(i) There is no dominant morphism $\phi : \Sigma \to B$, where B is a variety of general type.*

(ii) The morphism f has multiple fibers over the branch divisor $D \subset \mathbb{P}^1$ of degree $2g+2$ of $C/\mathbb{P}^1$.

(iii) The twisted canonical bundle $f^*K_{\mathbb{P}^1}(\frac{1}{2}D)$ pulls-back to $pr_1^*K_C$ on the surface $C \times E$.

Proof. (i) As the fibers of f are elliptic, Σ has Kodaira dimension 1 and does not dominate a surface of general type. If $\phi : \Sigma \to B$ is dominant with B a curve of genus > 1, the fibers of f cannot dominate B via ϕ because they are elliptic, hence are contracted by ϕ. Thus ϕ factors through f, and B is dominated by $\mathbb{P}^1$, which is absurd.

(ii) As σ has no fixed points, the local form of the map f is the same as the local form of the map $f \circ q : C \times E \to \mathbb{P}^1$, where q is the quotient map $C \times E \to \Sigma$. But the fibers of $f \circ q$ are clearly multiples fibers $2E \times x$ over each fixed point of the hyperelliptic involution ι.

(iii) This follows from the Hurwitz formula which says that $K_C = r^*K_{\mathbb{P}^1}(\frac{1}{2}D)$, where $r : C \to \mathbb{P}^1$ is the natural map. $\square$

Note also that the surface Σ behaves as the surface $C \times E$ from the viewpoint of hyperbolicity. Indeed, any holomorphic map from a disk or $\mathbb{C}$ to Σ lifts, by simple connectedness, to $C \times E$. Hence we conclude that the Kobayashi pseudodistance of Σ, although degenerate, is not 0: it is the pull-back of a certain pseudodistance on $\mathbb{P}^1$ which takes into account the divisor D.

Campana introduced special varieties in [20]. The idea is that these varieties should not map in any way to a variety of general type and the conjecture is that these varieties are those which have a trivial Kobayashi pseudodistance. However Lemma 4.1 above shows that the surface Σ constructed above should not be considered as special. The crucial corrected definition 4.8 below was introduced by Campana. We first define a *morphism of general type.*

Definition 4.2. *Let $\phi : X \to Y$ be a dominant proper morphism with X and Y smooth of dimension $k > 0$. ϕ is said to be of general type if the saturated rank 1 subsheaf $\mathcal{L}_\phi := (\phi^*K_Y)_{sat} \subset \Omega_X^k$ has Iitaka dimension k.*

Remark 4.3. In the situation above, Bogomolov's theorem 1.13 says that the Iitaka dimension of $(\phi^*K_Y)_{sat}$ is at most k.

For a rational map $\phi : X \dashrightarrow Y$, we will say that ϕ is of general type if a (in fact any) desingularization $\tilde{\phi} : \widetilde{X} \to Y$ of ϕ is of general type. As noticed above, the notion of morphism or rational map of general type does not say much on the base. However, Campana realized that there is an orbifold structure on the base which under extra assumptions allows to interpret the morphism ϕ being of general type as a certain orbifold or log structure on Y being of general type. Let $\phi : X \to Y$ be a morphism. The orbifold structure on Y will be determined by the $\mathbb{Q}$-divisor Δ_ϕ on Y of multiple fibers of ϕ defined as follows: For any irreducible reduced divisor $D \subset Y$, we can write the effective divisor $\phi^*(D)$ as $\sum_E m_E E + D'$, where the divisor D' does not dominate D. Let $m_D := \mathrm{Inf}_E m_E$. We then define

$$\Delta_\phi = \sum_D \frac{m_D - 1}{m_D} D. \tag{20}$$

The multiplicities appearing in the divisor D are inspired from the case of a morphism between curves. The local structure of the morphism is $v_l : U \to V, z \mapsto z^l$, where U, V are open neighborhood of 0 in $\mathbb{C}$, and the local multiplicity ν is then l. On the other hand, we have locally

$$v_l^*(K_V) = K_U(-(l-1)0),\ v_l^*(0) = l0,$$

that is $v_l^*(K_V(\frac{l-1}{l}0) = K_U$.

With this construction of the orbifold base, the above computation shows that if $\phi : X \to Y$ is finite, the divisor $K_X - (\phi^* K_Y(\Delta_\phi))$ is effective. For morphisms of relative dimension > 0, a similar interpretation is also possible but only after birational transformations. In fact, the Kodaira dimension of the pair (Y, Δ_ϕ) is not a birational invariant of (X, ϕ) if no assumptions are made on the morphism ϕ. This problem is addressed by Campana (see [22]) as follows:

Definition 4.4. *A morphism $\phi : X \to Y$ is neat if there exists a birational morphism $u : X \to X'$ with X' smooth, such that divisors D in X contracted by ϕ (in the sense that their image in Y has codimension ≥ 2) are contracted by u.*

Lemma 4.5. *Any fibration $X \to Y$ has a neat birational model, with X and Y smooth.*

Proof. Let

$$\begin{array}{ccc} X' & \xrightarrow{u'} & X \\ \downarrow{\scriptstyle \phi'} & & \downarrow{\scriptstyle \phi} \\ Y' & \xrightarrow{v'} & Y \end{array}\quad , \tag{21}$$

be a flattening of ϕ, that is u' and v' are projective and birational and ϕ' is flat. We can also assume Y' is smooth by resolving the singularities of Y' and base-changing. Let now $\tau : X'' \to X'$ be a resolution of singularities and let $\phi'' = \phi' \circ \tau$, $u = u' \circ \tau$. We claim that ϕ'' is neat. Indeed, if D is a divisor contracted by ϕ'', the image of D under τ is mapped by ϕ' to a codimension ≥ 2 closed algebraic subset of Y', and as ϕ' is is flat, it follows that $\tau(D)$ cannot be of codimension 1 in X', that is D is contracted by τ, and a fortiori by u. □

Proposition 4.6. *If the morphism ϕ is neat, the orbifold Kodaira dimension $\kappa(Y, \Delta_\phi)$ is equal to the Iitaka dimension $\kappa(X, \mathcal{L}_\phi)$, where $\mathcal{L}_\phi$ is the saturation of the rank 1 subsheaf $\phi^* K_Y \subset \Omega_X^p$, $p = \dim Y$.*

Proof. We prove that for m divisible enough, the sections of $\mathcal{L}_\phi^{\otimes m}$ on X identify via the pull-back with the sections of $m(K_Y + \Delta_\phi)$. Here we ask that m is divisible by all multiplicities appearing in Δ_ϕ so that $m(K_Y + \Delta_\phi)$ is a honest divisor. As $\mathcal{L}_\phi$ is saturated, we only have to worry about what happens in codimension 1 on X. The sheaves $\mathcal{L}_\phi$ and $\phi^* K_Y$ differ along divisors contracted by ϕ and along divisors of multiple fibers. For a contracted divisor $E = \sum_i n_i E_i$, as the E_i are contracted by $u : X \to X'$, with X' smooth, one can use the fact that pluridifferential forms on X are pulled-back from X' (see Lemma 1.6) and that $u(E)$ has codimension ≥ 2 in X' to conclude that it suffices to compare sections of $\mathcal{L}_\phi^{\otimes N}$ and $\phi^*(K_Y(\Delta_\phi)^{\otimes N})$ on $X \setminus E$, and more generally, over $Y \setminus W$ for any codimension ≥ 2 closed analytic subset W of Y. For the divisors of multiple fibers, it suffices to look by the above argument at what happens at the generic point of an irreducible component D of Δ_ϕ: If we compare $\mathcal{L}_\phi$ and $\phi^* K_Y(\Delta_\phi)$ we see that they coincide along at least one component D_1 of $\phi^{-1}(D)$, namely the one where the multiplicity ν of fibers is minimal. It then follows that the sections $m(K_Y + \Delta_\phi)$ and $\mathcal{L}_\phi^{\otimes m}$ coincide because otherwise the divisor $\phi^* m' D - m' \nu D_1$ is effective along the fibers of $\phi^{-1}(D) \to D$. □

A simple but interesting property of rational fibrations of general type is the following:

Lemma 4.7. *Let $\phi : X \dashrightarrow Y$ be of general type. Then ϕ is almost holomorphic.*

Proof. Let us assume that ϕ admits a desingularization $\phi : \widetilde{X} \to Y$ by a single blow-up along a smooth center $Z \subset X$ along which ϕ has indeterminacies. Let $E \to Z \subset X$ be the exceptional divisor of $\widetilde{X} \to X$. We have to show that $E \to Y$ cannot be dominant. Assume the contrary: The saturated sheaf $\mathcal{L}_\phi$ restricted to E then injects into Ω_E^p and its Iitaka dimension is at least $k = \dim Y$. We use then the fact that E is a projective bundle over Z, so that any Bogomolov sheaf on E comes from Z, using the fact that there are no nonzero pluridifferential forms on the fibers $\mathbb{P}^k$ and the relative cotangent exact sequence

$$0 \to \tau_E^*\Omega_Z \to \Omega_E \to \Omega_{E/Z} \to 0.$$

Thus the map $\phi_{|E}$ factors rationally through Z, which contradicts the fact that the morphism has indeterminacies generically along Z. □

Definition 4.8. *(Campana [20], see also [23]) A smooth projective variety X is special if no birational model $\widetilde{X}$ of X admits a morphism of general type $\widetilde{X} \to Y$. Equivalently, X admits no Bogomolov subsheaf (see Definition 1.17).*

By Proposition 4.6, X is special if and only if for any birational model $\widetilde{X}$ of X and morphism $\phi : \widetilde{X} \to Y$, with Y smooth of positive dimension, one has $\kappa(Y, \Delta_\phi) < \dim Y$.

The surface Σ considered above is not special as the morphism $f : \Sigma \to \mathbb{P}^1$ is of general type by Lemma 4.1.

Lemma 4.9. *(i) A rationally connected variety is special.*
(ii) A variety with trivial canonical bundle is special.

Proof. Point (i) follows from Lemma 3.13 while, if a variety X has a birational model $\widetilde{X}$ admitting a morphism of general type $\phi : \widetilde{X} \to Y$, then there exist for N large nonzero sections of $\mathcal{L}_\phi^{\otimes N} \subset (\Omega_{\widetilde{X}}^k)^{\otimes N}$, hence also nonzero sections of $(\Omega_X^k)^{\otimes N}$ by Lemma 1.6.

For point (ii), we use Yau's theorem on the existence of Kähler-Einstein metrics [59] which has as a consequence ([38], [46]) the polystability (with respect to any Kähler class) of the tangent bundle of a K-trivial projective manifold. It follows that Ω_X^k is also polystable and that its effective rank 1 subsheaves have Iitaka dimension ≤ 0. □

4.2 The core fibration

The following result has been obtained by Campana [20].

Theorem 4.10. *For any smooth projective variety X, there exists a canonically defined rational fibration $\phi : X \dashrightarrow Y$ which satisfies the following properties:*
(i) ϕ is of general type. In particular, ϕ is almost holomorphic by Lemma 4.7.
(ii) The general fiber of ϕ is special.
(iii) For a very general point $x \in X$, any subvariety $X' \subset X$ passing through x and with special desingularization is contained in the fiber of ϕ through x.
(iv) ϕ is universal in the following sense: Any rational map of general type $X \dashrightarrow Y'$ factors through Y.

Sketch of the proof. The construction of ϕ is given by the choice of a Bogomolov subsheaf $\mathcal{L} \subset \Omega_X^p$ with p maximal. If $p = 0$, X is special by definition and there is nothing to prove. Applying Theorem 1.13, one gets a rational map of general type $X \dashrightarrow Y$. One then has to prove that it satisfies all the properties (ii) to (iv). The key point is the proof of (ii), which will follow from Theorem 4.11 below by the following argument. Assume the general fiber is not special. Then by induction on dimension, it has a well-defined core fibration, so that one can construct a relative (over Y) core fibration

$$\psi : X \overset{\psi}{\dashrightarrow} Y' \overset{\chi}{\dashrightarrow} Y.$$

Here, over the very general point $t \in Y$, the morphism $\psi_t : X_t \dashrightarrow Y'_t$ is the core fibration of X_t. For an adequate birational model of ψ, the orbifold canonical bundle of (Y'_t, Δ_{ψ_t}) is then of general type by Proposition 4.6. We then have a fibration $\chi : Y' \to Y$ which we can assume to be neat. Furthermore Y' is equipped with an orbifold structure Δ_ψ whose general fibers are of general type. Finally, the morphism $Y' \to Y$ is easily seen to be of general type since $X \to Y$ is. It follows from the additivity Theorem 4.11 in the case of a base of general type that (Y', Δ_ψ) is of general type and since $\dim Y' > \dim Y$, we get a contradiction. □

Theorem 4.11. *Let (Y', D) be an orbifold, and let $Y' \to Y$ be a neat morphism of general type. Then $\kappa(Y', D) \geq \kappa(Y'_t, D_t) + \dim Y$.*

Here $\kappa(Y'_t, D_t)$ is the logarithmic Kodaira dimension of the general fiber Y'_t equipped with the restricted orbifold structure.

4.3 Conjectures

In the inspiring paper [43], the subject of Kobayashi hyperbolicity, which had been exposed in [37] from the viewpoint of complex analysis and geometry, is treated from the viewpoint of arithmetics. The case of curves is seen as a model, with Faltings finiteness theorem for points on curves of genus ≥ 2 defined over a number field [30]. There is a beautiful parallel developed by Lang between Brody curves on one hand and points defined over a number field on the other hand. The following conjectures are an example:

Conjecture 4.12. *(Green-Griffiths [33]) Let X be a complex projective manifold which is of general type. Then entire curves are not Zariski dense in X.*

This conjecture is optimal since even when the canonical bundle of X is ample, there might be rational curves in X. Note that it is not even known that surfaces with ample canonical bundle (eg quintic surfaces in $\mathbb{P}^3$) contain only finitely many rational curves (despite positive results by Bogomolov under extra assumptions on Chern numbers, see [9]). Conjecture 4.12 is also known to hold for subvarieties of abelian varieties by results of Kawamata [36] and Ochiai [50] (cf. [58]). In this case, one can use Brody curves (which are entire curves with bounded derivative) and observe that Brody curves in abelian varieties are projections of an affine line in the universal cover of A. It follows that the Zariski closure of a Brody curve in an abelian variety is a translate of an abelian subvariety. The arithmetic counterpart of the Green-Griffiths conjecture is the following:

Conjecture 4.13. *(Lang-Bombieri) Let X be a variety of general type which is defined over a number field K. Then K-points of X are not Zariski dense.*

This conjecture is Faltings' theorem in the case of subvarieties of abelian varieties.

What should be the generalization of these conjectures to varieties of any Kodaira dimension? Again, it is generally conjectured that for varieties of Kodaira dimension ≤ 0, the Kobayashi pseudodistance is trivial (see [37]). On the arithmetic side, the following analogue is completely open:

Conjecture 4.14. *(Campana) Let X be a variety of Kodaira dimension ≤ 0, defined on a field K which is either a number field or the function field of a complex curve. Then rational points are potentially dense on X, meaning that there is a finite extension L of K such that L-points of X are Zariski dense in X.*

Note that potential density is a property which is invariant under proper étale covers of projective varieties (Chevalley-Weil's theorem, see [34]). It follows that the Lang-Bombieri conjecture also implies :

Conjecture 4.15. *If X is smooth projective and there exists a dominant rational map from an étale cover X' of X to a variety of general type, then X is not potentially dense.*

In the situation of Conjecture 4.15, X' is not special and thus, as specialness is a property invariant under proper étale covers, X is not special. Campana goes further and proposes that the obstruction to potential density lies in the existence of a rational map of general type, or equivalently, in the fact that X is not special.

Conjecture 4.16. *(Campana) A smooth projective variety defined over a number field is potentially dense if and only if it is special.*

This is the arithmetic counterpart of the following conjecture on hyperbolicity:

Conjecture 4.17. *(Campana) A smooth projective variety defined over a number field has vanishing Kobayashi pseudo-distance if and only if it is special.*

Campana asked in [20] whether a variety which is not special has a finite étale cover admitting a dominant rational map to a variety of general type. The following example, due to Bogomolov and Tschinkel [11] provides a negative answer. In other words, the assumption of Conjecture 4.15 is stronger than being nonspecial.

Example 4.18. The examples constructed by Bogomolov and Tschinkel are elliptic threefolds $\phi : X \to B$ fibered over an elliptic surface B and satisfying the following conditions:

(i) $\pi_1(X) = \{1\}$.

(ii) The pair (B, Δ_ϕ) is of general type.

Such an X is constructed as follows: First of all one construct a simply connected elliptic surface $\mathcal{E} \to \mathbb{P}^1$ with one multiple fiber, say of multiplicity 2, over ∞. Then one chooses an elliptic surface S and a rational map $f : S \dashrightarrow \mathbb{P}^1$ given by an ample enough Lefschetz pencil of curves on S and smooth fiber S_∞ over ∞. Let B be a blow-up of S such that f becomes well-defined on B and let $X = B \times_{\mathbb{P}^1} \mathcal{E}$. The divisor Δ_ϕ for the morphism $X \to B$ contains the curve B_∞ with multiplicity $\frac{1}{2}$. Let F be the exceptional divisor of $\tau : B \to S$. One has $B_\infty = \tau^* S_\infty(-F)$ and $K_B = \tau^* K_S(F)$ hence $K_B(\frac{1}{2}B_\infty) = \tau^*(K_S(\frac{1}{2}S_\infty)) + \frac{1}{2}F$, hence the logarithmic Kodaira dimension of (B, Δ_ϕ) is as large as we want, by choosing the curve S_∞ positive enough. One then shows that X can be chosen simply connected.

Being fibered over B into elliptic curves, with B not a surface of general type and not admitting a morphism to a curve of genus ≥ 2, the variety X does not admit a dominant rational map to a variety of general type, and neither does any of its finite étale covers (since they are all isomorphic to X). On the other hand, X is not special since ϕ is of general type.

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