ON ALGEBRAIC HYPERBOLICITY

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ABSTRACT. Conjectures of Lang and Vojta state that varieties of general type should be *weakly algebraically hyperbolic*. In particular, curves of bounded geometric genus lying on such a variety form a bounded family. We describe some recent results related to algebraic hyperbolicity. We focus on some connections between recent results of Miyaoka on the canonical degree of curves on surfaces of general type and the theory of orbifold pairs developed by Campana.

CONTENTS

1. INTRODUCTION

Following ideas of Lang, it is generally expected that Kobayashi hyperbolicity, which is of analytic nature, could be characterized by purely algebraic properties. In this direction, Demailly [Dem97] made the following observation.

Theorem 1.1. *Let X be a Kobayashi hyperbolic complex projective variety. Then there exists $\epsilon > 0$ such that every irreducible algebraic curve $\mathcal{C} \subset X$ satisfies*

$$-\chi(\tilde{\mathcal{C}}) = 2g(\tilde{\mathcal{C}}) - 2 \geq \epsilon \deg \tilde{\mathcal{C}}, \qquad (1.1.1)$$

where $\tilde{\mathcal{C}}$ is the normalization of $\mathcal{C}$.

This motivates the following purely algebraic definition.

Definition 1.2. Let X be a complex projective variety. X is said to be *algebraically hyperbolic* if 1.1.1 holds for all irreducible algebraic curve $\mathcal{C} \subset X$.

One of the interest of this definition is that it is related to Lang-Vojta's conjectures on function fields. Let us formulate one of these geometric Lang-Vojta's conjectures in the setting of logarithmic pairs of general type, making clear the connection with algebraic hyperbolicity.

Conjecture 1.3. *Let X be a complex projective manifold, $D \subset X$ a normal crossing divisor. If (X, D) is of log-general type then there exists a proper subvariety $Z \subset X$ and real numbers A and B such that*

$$\deg f(C) \leq A(2g(C) - 2 + |S|) + B, \qquad (1.3.1)$$

for all smooth projective curves C, finite morphisms $f : C \to X$ and finite subsets $S \subset C$ such that $f^{-1}(D) \subset S$ and $f(C) \not\subset Z$.

Complex manifols satisfying the weaker condition 1.3.1 are said to be *weakly algebraically hyperbolic*.

The surface case in Conjecture 1.3 is still open, even when $X = \mathbb{P}^2$. Nevertheless, important results have been obtained towards this geometric Lang-Vojta's conjecture.

In the case of $X = \mathbb{P}^n$, the conjecture is solved independently by [Che04] and [PR07] for *very general* normal crossing divisors $D \subset X$ of degree $\deg D \geq 2n + 1$. For $n = 2$, some results have been obtained when $\deg D = 4$ using arithmetic methods on function fields. The four line case follows from an extension of Mason's ABC theorem [BM86] and the three components case can be reduced to a S-unit *gcd* problem [CZ08].

Several interesting results have also been obtained on quotients of bounded symmetric domains (see the interesting paper [ACLG12] for a discussion of conjecture 1.3 in this context). In [Fal83], Faltings establishes the following boundedness results for families $p : X \to C \setminus S$ of principally polarized abelian varieties of relative dimension g with level structures $n \geq 3$: for all such induced morphisms to the moduli space $\phi : C \to \overline{\mathcal{A}}_{g,n}$ one has the inequality $\deg \phi^*(K + D) \leq g(3g(C) + |S| + 1)$, where K is the canonical divisor of $\overline{\mathcal{A}}_{g,n}$ and D is the compactification divisor which can be assumed to be normal crossing. This was improved later by Kim [Kim98] obtaining the inequality $\deg \phi^*(K + D) \leq \frac{g(g+1)}{2}(2g(C) - 2 + |S|)$ which is exactly conjecture 1.3 in this setting. Recently, a similar result has been obtained in [RT15] for families of abelian varieties with real multiplication, thus establishing conjecture 1.3 for Hilbert modular varieties.

In dimension 2, for the compact case (i.e. $D = 0$), the first striking result is a theorem of Bogomolov [Bog77] proving conjecture 1.3 for surfaces of general type with positive second Segre number $s_2 := c_1^2 - c_2$. Recently, in the same setting, Miyaoka [Miy08] gives an alternative proof of this statement obtaining *effective* constants as functions of c_1^2 and c_2 in the inequality 1.3.1. Moreover, when the curve $C \subset X$ is supposed to be *smooth*, Miyaoka [Miy08] shows that $K_X.C \leq \frac{3}{2}(2g - 2) + o(g)$. These results are not only striking illustrations of Lang-Vojta's conjectures but we will try to explain that the method of proof is also interesting since it can be translated into an application of the theory advertised by Campana [Cam04] of the *orbifold* category.

2. The approach via foliations

Following ideas of Bogomolov [Bog77], one obtains a postive answer for some surfaces.

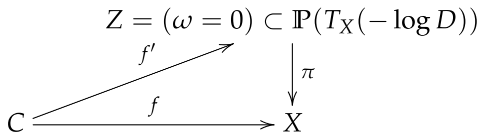
Theorem 2.1. *Let (X, D) be a log-smooth surface of log-general-type such that its log-Chern classes satisfy $c_1^2 > c_2$. Then (X, D) satisfies conjecture 1.3.*

Proof. Under the hypothesis $c_1^2 > c_2$, one obtains that $T_X^*(\log D)$ is big. Indeed, by Riemann-Roch

$$\chi(X, S^m T_X^*(\log D)) = \frac{m^3}{6}(c_1^2 - c_2) + O(m^2).$$

Therefore $h^0(X, S^m T_X^*(\log D)) + h^2(X, S^m T_X^*(\log D)) > cm^3$. Now, by Serre duality and the isomorphism $(K_X \otimes D) \otimes T_X(-\log D) = T_X^*(\log D)$, we have $h^2(X, S^m T_X^*(\log D)) = h^0(X, (K_X \otimes D)^{(-m)} \otimes K_X \otimes S^m T_X^*(\log D)) \leq h^0(X, S^m T_X^*(\log D))$. The last inequality comes from the fact that (X, D) is of general type and in particular, $(K_X \otimes D)^m \otimes K_X^{-1}$ is effectif for large m. Finally, we obtain $h^0(X, S^m T_X^*(\log D)) > \frac{c}{2} m^3$ and $T_X^*(\log D)$ is big.

So we have a section $\omega \in H^0(X, S^m T_X^*(\log D) \otimes A^{-1})$, where A is any line bundle. The morphism $f : C \to X$ induces a morphism $f' : C \to \mathbb{P}(T_X(-\log D))$.



By definition we have an inclusion $f'^*(\mathcal{O}(1)) \hookrightarrow K_C(f^*(D)_{red})$. So we easily obtain the algebraic tautological inequality

$$\deg_C(f'^*(\mathcal{O}(1)) \leq 2g(C) - 2 + N_1(f^*D).$$

If $f'(C) \not\subset Z$ then the previous inequality gives

$$\frac{1}{m} \deg f^*A \leq 2g(C) - 2 + N_1(f^*D).$$

Now, let us suppose that $f'(C) \subset Z$ and that Z is an irreducible horizontal surface. Then Z is equipped with a tautological holomorphic foliation by curves: if $z \in Z$ is a generic point, a neighbourhood U of z induces a foliation on a neighbourhood V of $x = \pi(z)$. Indeed, a point in $U \subset \mathbb{P}(T_X(-\log D))$ is of the form $(w, [t])$ where w is a point in X and t a tangent vector at this point. This foliation lifts through the isomorphism $U \to V$ induced by π. Leaves are just the derivatives of leaves on V. Tautologically, $f' : C \to Z$ is a leaf. By a theorem of Jouanolou [Jou78]: either Z has finitely many algebraic leaves or it is a fibration. In both cases, one obtains immediately that $\deg f^*(A)$ has to be bounded. □

Corollary 2.2. *Let $X = \mathbb{P}^2$ and $D = \sum_{i=1}^r C_i$ a normal crossing curve where C_i is a curve of degree d_i, $d_1 \leq d_2 \cdots \leq d_r$. Then $(\mathbb{P}^2, D)$ satisfies conjecture 1.3 if $r \geq 5$ or, $r = 4$ and $d_4 \geq 2$; $r = 3$ and $d_1 \geq 2$, $d_3 \geq 3$ or $d_1 = 1$, $d_2 \geq 3$, $d_3 \geq 4$; $r = 2$ and $d_1 \geq 5$ or $d_1 \geq 4$, $d_2 \geq 7$.*

Proof. Let $d := \sum d_i$. One has $c_1^2 - c_2 = 6 - 3d + \sum_{i<j} d_i d_j$. From theorem 2.1, one immediately obtains the result. □

3. Orbifold Bogomolov-Miyaoka-Yau inequalities

3.1. **The classical Bogomolov-Miyaoka-Yau inequality.** Let us start with the following classical statement.

Theorem 3.1. [Miy84] *Let (X, D) be a log-smooth surface with reduced boundary. Let $\mathcal{E}$ be a rank 2 reflexive subsheaf of $\Omega_X(\log D)$. If $c_1(\mathcal{E})$ is pseudoeffective, then*

$$c_1(\mathcal{E})^2 \leq 3c_2(\mathcal{E}). \tag{3.1.1}$$

We will give a simple proof of this theorem following [Lan01]. First, we need some lemmas.

Lemma 3.2. *If $h^0(X, \mathcal{E}(-C)) \neq 0$ and $L(C - \frac{1}{2}N)) > 0$ for some nef divisor L then $C.P \leq c_2(\mathcal{E}) - \frac{1}{4}N^2$.*

Proof. Consider the exact sequence $0 \to \mathcal{O}(C) \to \mathcal{E} \to \mathcal{E}/\mathcal{O}(C) \to 0$. Then $c_2(\mathcal{E}) = c_1(\mathcal{O}(C)).c_1(\mathcal{E}/\mathcal{O}(C)) = C(c_1(\mathcal{E}) - C) = PC + \frac{1}{4}N^2 - (C - \frac{1}{2}N)^2$. By a classical lemma of Bogomolov, $\kappa(X, C - \frac{1}{2}N) \leq \kappa(X, C) \leq 1$. Now, by Riemann-Roch, it follows that as $L(C - \frac{1}{2}N)) > 0$ then $(C - \frac{1}{2}N)^2 \leq 0$, which concludes the proof. □

We need the following generalization

Lemma 3.3. *If* $h^0(X, S^n\mathcal{E}(-C)) \neq 0$ *and* $L(C - \frac{n}{2}N)) > 0$ *for some nef divisor* L *then* $C.P \leq n(c_2(\mathcal{E}) - \frac{1}{4}N^2)$.

Proof. Let $s \in h^0(X, S^n\mathcal{E}(-C))$. Let us take a finite morphism $f : Y \to X$ such that $f^*s = s_1 \dots s_n$ where $s_i \in h^0(X, f^*\mathcal{E}(-C_i))$. We note $f^*C = \sum C_i$. Therefore $(\sum C_i - \frac{n}{2}f^*N))f^*L = \deg f.(C - \frac{n}{2}N)).L > 0$. By the preceding lemma, if $(C_i - \frac{1}{2}f^*N)f^*L > 0$ (which has to be verified for at least one i) or $C_i f^*P = (C_i - \frac{1}{2}f^*N)f^*P > 0$ then

$$C_i.f^*P \leq c_2(f^*\mathcal{E}) - \frac{1}{4}(f^*N)^2 = \deg f(c_2(\mathcal{E}) - \frac{1}{4}N^2).$$

In the possible remaining cases where $C_i.f^*P = 0$ one has also $C_i.f^*P = 0 \leq c_2(f^*\mathcal{E}) - \frac{1}{4}(f^*N)^2$. To finish, we take the sum of all these inequalities. □

Now, we can prove theorem 3.1

Proof. Let us prove that $c_1(\mathcal{E})^2 \leq 3c_2(\mathcal{E}) + \frac{1}{4}N^2$ i.e.

$$\frac{1}{3}P^2 \leq c_2(\mathcal{E}) - \frac{1}{4}N^2.$$

If $h^0(X, S^n\mathcal{E}(-(\frac{n}{2}N - naP - H)) \neq 0$ for some ample divisor H and $a \geq \frac{1}{3}$ then $\frac{1}{3}nP^2 \leq naP^2 \leq P(naP + H) \leq n(c_2(\mathcal{E}) - \frac{1}{4}N^2)$ by the above lemma.

So we assume $h^0(X, S^n\mathcal{E}(-(\frac{n}{2}N - naP - H)) = 0$ for all $a \geq \frac{1}{3}$.

$$\begin{aligned} h^2(X, S^n\mathcal{E}(-(\frac{n}{2}N - naP - H)) &= h^0(X, S^n\mathcal{E}(-\frac{n}{2}N + (na - n)P + H + K_X)) \\ &\leq h^0(X, S^n\mathcal{E}(-\frac{n}{2}N - n(1-a)P - H)) + O(n^2). \end{aligned}$$

So for $a = \frac{1}{3}$, we have $\chi(X, S^n\mathcal{E}(-(\frac{n}{2}N - \frac{n}{3}P - H)) \leq O(n^2)$.

By Riemann-Roch,

$$\begin{aligned} &\chi(X, S^n\mathcal{E}(-(\frac{n}{2}N - \frac{n}{3}P - H)) \\ &= \frac{n^3}{6}(c_1^2(\mathcal{E}(-(\frac{1}{2}N - \frac{1}{3}P))) - c_2(\mathcal{E}(-(\frac{1}{2}N - \frac{1}{3}P))) + O(n^2) \\ &= \frac{n^3}{6}(\frac{1}{4}N^2 + \frac{1}{3}P^2 - c_2(\mathcal{E})) + O(n^2). \end{aligned}$$

□

3.2. **Orbifold Bogomolov-Miyaoka-Yau inequalities.** Let us start by introducing some notations. Following Campana [Cam04] and the recent survey [Cla15], we will call *orbifold* the data (X, Δ) of a log-smooth pair where X is a smooth complex manifold, the support of the divisor Δ is normal crossing and its coefficients are rational numbers belonging to $[0, 1]$ such that

$$\Delta = \sum_i (1 - \frac{b_i}{a_i})\Delta_i.$$

The general philosophy is that we will attach to this pair all the objects that are usually attached to manifolds, defining them on suitable coverings.

Definition 3.4. An adapted cover to the pair (X, Δ) is tha data of a Galois cover $\pi : Y \to X$ such that:

(3.4.1) Y is smooth projective.
(3.4.2) π ramifies exactly at order a_i over Δ_i: $\pi^*(\Delta_i) = \sum_j a_i D^i_j$.
(3.4.3) the support of the divisor $\pi^*\Delta + Ram(\pi)$ is normal crossing as well as the branch locus.

Remark 3.5. It is now classical that such coverings always exist (see [Laz04], prop. 4.1.12).

In local coordinates, the description of these coverings is quite simple. It is of the following form:

$$\pi(w_1,\dots,w_n) = (w_1^{a_1},\dots,w_k^{a_k},w_{k+1},\dots,w_{n-j},w_{n-j+1}^{m_j},\dots,w_n^{m_1}).$$

Let us now define the orbifold cotangent bundle. The idea is that this sheaf should be generated by $\frac{dz_i}{z_i^{1-b_i/a_i}}$.

The key observation is that we have a residue map $\pi^*\Omega_X(\log\lceil\Delta\rceil) \to \bigoplus_i \mathcal{O}_{b_iD^i}$, where $D^i := \frac{1}{a_i}\pi^*\Delta_i$.

Definition 3.6. We define the orbifold cotangent bundle $\Omega(\pi,\Delta)$ to be the kernel of this residue map. In other words, it is defined by the exact sequence

$$0 \to \Omega(\pi,\Delta) \to \pi^*\Omega_X(\log\lceil\Delta\rceil) \to \bigoplus_i \mathcal{O}_{b_iD^i} \to 0.$$

Let us define orbifold Chern classes as the Chern classes of the orbifold cotangent bundle.

Definition 3.7.

$$c_i(X,\Delta) := c_i(\Omega(\pi,\Delta)).$$

Now, we want to compute orbifold Chern classes in terms of the pair (X,Δ).

Proposition 3.8. *Let (X,Δ) be an orbifold surface. Then:*

$$c_1(X,\Delta) := \pi^*(K_X+\Delta),$$

$$c_2(X,\Delta) := \deg\pi.(c_2(X)-\chi(\Delta)),$$

where

$$\chi(\Delta) = \sum_i (1-\frac{b_i}{a_i})\chi(\Delta_i) - \sum_{i<j}(1-\frac{b_i}{a_i})(1-\frac{b_j}{a_j})\Delta_i\Delta_j.$$

Proof. Let $\pi : Y \to X$ be an adapted cover. We have the exact sequence

$$0 \to \Omega(\pi,\Delta) \to \pi^*\Omega_X(\log\lceil\Delta\rceil) \to \bigoplus_i \mathcal{O}_{b_iD^i} \to 0.$$

Therefore we have the following equality between total Chern classes

$$c(\pi^*\Omega_X(\log\lceil\Delta\rceil)) = c(\Omega(\pi,\Delta)).\prod_i c(\mathcal{O}_{b_iD^i}).$$

Using the exact sequence

$$0 \to \mathcal{O}(-b_iD^i) \to \mathcal{O}_Y \to \mathcal{O}_{b_iD^i} \to 0,$$

one obtains that $c_1(\mathcal{O}_{b_iD^i}) = b_iD^i$, and $c_2(\mathcal{O}_{b_iD^i}) = b_i^2(D^i)^2$.

For the first Chern class, we immediately obtain,

$$c_1(\Omega(\pi,\Delta)) = \pi^*(K_X+\lceil\Delta\rceil - \sum_i \frac{b_i}{a_i}\Delta_i).$$

For the second Chern class,

$$
\begin{aligned}
c_2(\Omega(\pi,\Delta)) &= c_2(\pi^*\Omega_X(\log\lceil\Delta\rceil) - \sum_{i<j} b_ib_jD_iD_j - \sum_i b_i^2(D^i)^2 - c_1.\sum_i b_iD_i \\
&= \deg\pi(c_2(X) - \sum_i \chi(\Delta_i) + \sum_{i<j}\Delta_i.\Delta_j - \sum_{i<j}\frac{b_i}{a_i}\frac{b_j}{a_j}\Delta_i.\Delta_j \\
&\quad - \sum_i \frac{b_i^2}{a_i^2}\Delta_i^2 - (K_X+\Delta).\sum_i\frac{b_i}{a_i}\Delta_i) \\
&= \deg\pi(c_2(X) - \sum_i \chi(\Delta_i) + \sum_{i<j}\Delta_i.\Delta_j - \sum_{i<j}\frac{b_i}{a_i}\frac{b_j}{a_j}\Delta_i.\Delta_j - \sum_i \frac{b_i^2}{a_i^2}\Delta_i^2 \\
&\quad - \sum_i \frac{b_i}{a_i}K_X.\Delta_i - \sum_{i,j}\frac{b_i}{a_i}\Delta_i.\Delta_j + \sum_{i,j}\frac{b_i}{a_i}\frac{b_j}{a_j}\Delta_i.\Delta_j). \\
&= \deg\pi(c_2(X) - \sum_i \chi(\Delta_i) + \sum_{i<j}\Delta_i.\Delta_j + \sum_{i<j}\frac{b_i}{a_i}\frac{b_j}{a_j}\Delta_i.\Delta_j \\
&\quad - \sum \frac{b_i}{a_i}(K_X+\Delta_i).\Delta_i - \sum_{i<j}(\frac{b_i}{a_i}+\frac{b_j}{a_j})\Delta_i.\Delta_j) \\
&= \deg\pi(c_2(X) - \chi(\Delta)).
\end{aligned}
$$

□

Now we can state the following orbifold Bogomolov-Miyaoka-Yau inequality.

Corollary 3.9. *Let (X,Δ) be an orbifold surface. If $K_X+\Delta$ is pseudoeffective then:*

(3.9.1) $$(K_X+\Delta)^2 \leq 3(c_2(X) - \chi(\Delta)).$$

Proof. One just applies theorem 3.1 to the orbifold cotangent bundle $\Omega(\pi,\Delta)$, use proposition 3.8 and divide by $\deg\pi$. □

3.3. **The higher dimensional case.** We will extend the orbifold Bogomolov-Miyaoka-Yau inequality to higher dimension.

Theorem 3.10. *Let (X,Δ) be an orbifold where X is a complex manifold of dimension n. If $K_X+\Delta$ is nef then for arbitrary ample divisors $H_1,\dots,H_{n-2}$ and each adapted cover $\pi: Y \to X$:*

(3.10.1) $$c_1(X,\Delta)^2.f^*(H_1\dots H_{n-2}) \leq 3c_2(X,\Delta).f^*(H_1\dots H_{n-2}).$$

Remark 3.11. In the logarithmic case (i.e. all coefficients of Δ are equal to 1), a much stronger inequality than 3.10.1 is claimed in [LM97] since K_X+D is only assumed to be pseudo-effective. Unfortunately, a gap in the proof has been recently found.

We will adapt the proof of Miyaoka [Miy87] using the recent result of [CP13].

Theorem 3.12. *Let (X,Δ) be an orbifold with $K_X+\Delta$ pseudoeffective and $\pi: Y\to X$ an adapted cover. Then $\Omega(\pi,\Delta)$ is generically semipositive in the following sense: let $H_1,\dots,H_{n-1}$ ba ample divisors, then any quotient $\Omega(\pi,\Delta)^{\otimes m} \twoheadrightarrow Q$ has nonnegative degree on $\pi^*(H_1\dots H_{n-1})$.*

Let us now give the proof of theorem 3.10.

Proof. Let $\mathscr{E} := \Omega(\pi,\Delta)$. Let $0 = \mathscr{E}_0 \subset \mathscr{E}_1 \subset \cdots \subset \mathscr{E}_s = \mathscr{E}$ be the $\pi^*(H_1,\dots,H_{n-2},K_X+\Delta)$-semistable filtration. Put $\mathcal{G}_i = \mathscr{E}_i/\mathscr{E}_{i-1}, r_i = \operatorname{rank}\mathcal{G}_i$, and α_i such that

$$r_i.\alpha_i = \frac{c_1(\mathcal{G}_i).\pi^*(H_1\dots H_{n-2}(K_X+\Delta))}{c_1(\mathscr{E}).\pi^*(H_1\dots H_{n-2}(K_X+\Delta))}.$$

Then we have $r_1.\alpha_1 + \cdots + r_s.\alpha_s = 1$ and $\alpha_1 > \cdots > \alpha_s \geq 0$, the last inequality coming from semipositivity of $\mathscr{E}$ (3.12).

Now, we have

$$\begin{aligned}(2c_2(\mathscr{E}) - c_1^2(\mathscr{E})).\pi^*(H_1 \dots H_{n-2}) = & \quad (\sum_i 2c_2(\mathcal{G}_i) - c_1^2(\mathcal{G}_i)).\pi^*(H_1 \dots H_{n-2}) \\ \geq & \quad (\sum \frac{-1}{r_i} c_1^2(\mathcal{G}_i)).\pi^*(H_1 \dots H_{n-2}).\end{aligned}$$

So, we deduce

$$\begin{aligned}(6c_2(\mathscr{E}) - 2c_1^2(\mathscr{E}))\pi^*(H_1 \dots H_{n-2}) \geq \\ \left(3(\sum_{i>1} \frac{-1}{r_i} c_1^2(\mathcal{G}_i)) + 6c_2(\mathscr{E}_1) - 3c_1^2(\mathscr{E}_1) + c_1^2(\mathscr{E})\right).\pi^*(H_1 \dots H_{n-2})\end{aligned}$$

And finally,

(3.12.1)

$$(6c_2(\mathscr{E}) - 2c_1^2(\mathscr{E}))\pi^*(H_1 \dots H_{n-2}) \geq ((1 - 3\sum_{i>1} r_i\alpha_i^2).c_1^2(\mathscr{E}) + 6c_2(\mathscr{E}_1) - 3c_1^2(\mathscr{E}_1)).\pi^*(H_1 \dots H_{n-2}).$$

There are three possibilities: $r_1 \geq 3$, $r_1 = 2$ and $r_1 = 1$.

If $r_1 \geq 3$, using Bogomolov-Gieseker inequality and the Hodge index theorem, we obtain

$$\begin{aligned}(6c_2(\mathscr{E}) - 2c_1^2(\mathscr{E}))\pi^*(H_1 \dots H_{n-2}) \geq \\ ((1 - 3\sum_{i>1} r_i\alpha_i^2).c_1^2(\mathscr{E}) - 3\frac{1}{r_1}c_1^2(\mathscr{E}_1)).\pi^*(H_1 \dots H_{n-2}) \geq \\ (1 - 3\sum_i r_i\alpha_i^2).c_1^2(\mathscr{E})\pi^*(H_1 \dots H_{n-2}) \geq (1 - 3\alpha_1)c_1^2(\mathscr{E})\pi^*(H_1 \dots H_{n-2}) \geq 0.\end{aligned}$$

since $3\alpha_1 \leq r_1\alpha_1 \leq \sum_i r_i\alpha_i = 1$.

If $r_1 = 2$, let S be a general complete intersection surface of the linear systems $\pi^*|m_iH_i|$. We choose S general enough so that $\mathscr{E}_{1|S}$ injects into $\Omega_S(\log \pi^{-1}\lceil\Delta\rceil_{|S})$.

Using theorem 3.1.1, we have either $\kappa(S, c_1(\mathscr{E}_{1|S})) \leq 0$ or $c_1^2(\mathscr{E}_{1|S}) \leq 3c_2(\mathscr{E}_{1|S})$.

In the case $\kappa(S, c_1(\mathscr{E}_{1|S})) \leq 0$, since $c_1(\mathscr{E}_{1|S}).\pi^*(K_X + \Delta) > 0$, we have $c_1^2(\mathscr{E}_{1|S}) \leq 0$.

Applying Bogomolov-Gieseker inequality to 3.12.1:

$$\begin{aligned}(6c_2(\mathscr{E}) - 2c_1^2(\mathscr{E}))\pi^*(H_1 \dots H_{n-2}) \geq \\ ((1 - 3\sum_{i>1} r_i\alpha_i^2).c_1^2(\mathscr{E}) - \frac{3}{2}c_1^2(\mathscr{E}_1)).\pi^*(H_1 \dots H_{n-2}) \geq \\ (1 - 3\sum_{i>1} r_i\alpha_i^2).c_1^2(\mathscr{E}).\pi^*(H_1 \dots H_{n-2}) \geq \\ (1 - 3\alpha_2\sum_{i>1} r_i\alpha_i).c_1^2(\mathscr{E}).\pi^*(H_1 \dots H_{n-2}) = \\ (1 - 3\alpha_2(1 - 2\alpha_1)).c_1^2(\mathscr{E}).\pi^*(H_1 \dots H_{n-2}) \geq \\ (1 - 3\alpha_1(1 - 2\alpha_1)).c_1^2(\mathscr{E}).\pi^*(H_1 \dots H_{n-2}) = \\ \left(6(\alpha_1 - \frac{1}{4})^2 + \frac{5}{8}\right).c_1^2(\mathscr{E}).\pi^*(H_1 \dots H_{n-2}) \geq 0.\end{aligned}$$

In the case $c_1^2(\mathscr{E}_{1|S}) \leq 3c_2(\mathscr{E}_{1|S})$, we have from 3.12.1:

$$\begin{aligned} (6c_2(\mathscr{E}) - 2c_1^2(\mathscr{E}))\pi^*(H_1 \dots H_{n-2}) &\geq \\ ((1 - 3\sum_{i>1} r_i\alpha_i^2).c_1^2(\mathscr{E}) - c_1^2(\mathscr{E}_1)).\pi^*(H_1 \dots H_{n-2}) &\geq \\ ((1 - 4\alpha_1^2 - 3\sum_{i>1} r_i\alpha_i^2).c_1^2(\mathscr{E})).\pi^*(H_1 \dots H_{n-2}) &\geq \\ ((1 - 4\alpha_1^2 - 3\alpha_2\sum_{i>1} r_i\alpha_i).c_1^2(\mathscr{E})).\pi^*(H_1 \dots H_{n-2}) &= \\ ((1 - 4\alpha_1^2 - 3\alpha_2(1-2\alpha_1)).c_1^2(\mathscr{E})).\pi^*(H_1 \dots H_{n-2}) &= \\ (1-2\alpha_1)(1+2\alpha_1 - 3\alpha_2).c_1^2(\mathscr{E})).\pi^*(H_1 \dots H_{n-2}). \end{aligned}$$

As $3\alpha_2 < r_1\alpha_1 + r_2\alpha_2 \leq 1$, we have

$$(6c_2(\mathscr{E}) - 2c_1^2(\mathscr{E}))\pi^*(H_1 \dots H_{n-2}) \geq 2\alpha_1(1-2\alpha_1)c_1^2(\mathscr{E}))\pi^*(H_1 \dots H_{n-2}) \geq 0.$$

Finally, if $r_1 = 1$, a classical lemma of Bogomolov implies that $\mathscr{E}_{1|S} \subset \Omega_S(\log \pi^{-1}\lceil\Delta\rceil_{|S})$ has Kodaira dimension at most one. Therefore $c_1^2(\mathscr{E}_{1|S}) \leq 0$. From 3.12.1, one obtains:

$$\begin{aligned} (6c_2(\mathscr{E}) - 2c_1^2(\mathscr{E}))\pi^*(H_1 \dots H_{n-2}) &\geq \\ ((1 - 3\sum_{i>1} r_i\alpha_i^2).c_1^2(\mathscr{E})).\pi^*(H_1 \dots H_{n-2}) &\geq \\ ((1 - 3\alpha_1\sum_{i>1} r_i\alpha_i).c_1^2(\mathscr{E})).\pi^*(H_1 \dots H_{n-2}) &= \\ ((1 - 3\alpha_1(1-\alpha_1)).c_1^2(\mathscr{E})).\pi^*(H_1 \dots H_{n-2}) &\geq \\ \left(1 - \frac{3}{2}(1-\frac{1}{2})\right).c_1^2(\mathscr{E}).\pi^*(H_1 \dots H_{n-2}) = \frac{1}{4}c_1^2(\mathscr{E})).\pi^*(H_1 \dots H_{n-2}) &\geq 0. \end{aligned}$$

□

4. APPLICATIONS

First, let us recall the immediate application of the classical inequality 3.1.1.

Corollary 4.1. *Let $C \subset X$ be a smooth curve in a surface X with pseudo-effective canonical bundle K_X. Then*

(4.1.1) $$K_X.C \leq 2(2g-2) + 3c_2 - c_1^2.$$

Proof. We apply the classical Bogomolov-Miyaoka-Yau inequality 3.1.1 to $\mathcal{E} = T_X^*(\log C)$. This gives $c_1^2(T_X^*(\log C)) \leq 3c_2(T_X^*(\log C))$, i.e. $(K_X + C)^2 \leq 3(\chi(X) - \chi(C))$. Finally, we obtain $K_X.C \leq 2(2g-2) + 3c_2 - c_1^2$. □

We will now explain how the orbifold inequality 3.9.1 inequality improves the estimate 4.1.1.

First, let us observe that the orbifold inequality 3.9.1 easily implies the following theorem of Miyaoka [Miy08].

Theorem 4.2. *Let X be a surface of non-negative Kodaira dimension and let C be a* smooth *curve of genus g on it. If $0 \leq \alpha \leq 1$ is a real number, then the inequality*

(4.2.1) $$\frac{\alpha^2}{2}(C^2 + 3CK_X - 6g + 6) - 2\alpha(CK_X - 3g + 3) + 3c_2 - c_1^2 \geq 0,$$

holds.

Proof. When $0 \leq \alpha \leq 1$ is a rational number, one applies the inequality 3.9.1 to the pair $(X, \alpha C)$ and easily gets 4.2.1. □

Remark 4.3. We see that if one takes $\alpha = 0$ then one recover the classical inequality $c_1^2 \leq 3c_2$, whereas if $\alpha = 1$ one finds the logarithmic inequality of corollary 4.1. Therefore the inequality 4.2.1 interpolates between the compact case and the logarithmic case.

Remark 4.4. The remarkable fact established by Miyaoka [Miy08] is that the inequality 4.2.1 remains valid for *any* irreducible curve of genus g not necessarily smooth.

Let us see now the interesting consequences of this inequality.

Corollary 4.5. *Let X be a surface of non-negative Kodaira dimension and let C be an irreducible curve of genus g on it. If $C \not\simeq \mathbb{P}^1$ and $K_X.C > 3g-3$ then*

(4.5.1) $$2(K_X.C - 3g+3)^2 - (3c_2 - c_1^2)(C^2 + 3K_X.C - 6g+6) \leq 0$$

Proof. Since $C \not\simeq \mathbb{P}^1$, we have $C^2 + K_X.C \geq 0$. Let

$$Q(\alpha) = \frac{\alpha^2}{2}(C^2 + 3CK_X - 6g+6) - 2\alpha(CK_X - 3g+3) + 3c_2 - c_1^2.$$

It is a polynomial of degree 2 with a positive leading coefficient. The minimum is obtained at $\alpha_0 = \frac{2(C.K_X - 3g+3)}{C^2+3C.K_X-6g+6)}$ which belongs to $[0,1]$ since the denominator is equal to $2(C.K_X - 3g+3) + C^2 + K_X.C$. Now,

$$Q(\alpha_0) = -\frac{2(C.K_X - 3g+3)^2}{(C^2 + 3C.K_X - 6g+6)} + 3c_2 - c_1^2 \geq 0$$

by theorem 4.2. This is exactly the inequality we wanted to prove. □

Corollary 4.6. *Let X be a surface of non-negative Kodaira dimension and let C be a* smooth *curve of genus $g \geq 1$ on it. Then*

$$K_X.C \leq \frac{3}{2}(2g-2) + \frac{3c_2 - c_1^2}{2} + \frac{\sqrt{3c_2 - c_1^2}\sqrt{4g-4+3c_2-c_1^2}}{2}.$$

In particular, $K_X.C \leq \frac{3}{2}(2g-2) + o(g)$.

Proof. As C is smooth, we have $C^2 = 2g-2-C.K_X$. Therefore the inequality 4.5.1 gives $(K_X.C)^2 + (2(3-3g) - (3c_2 - c_1^2))K_X.C + (3-3g)^2 - (3c_2-c_1^2)(2-2g) \leq 0$. We look at the last quantity as a quadratic polynomial in K_XC. Therefore $K_X.C$ is smaller than the biggest root of this polynomial i.e.

$$K_X.C \leq \frac{3}{2}(2g-2) + \frac{3c_2 - c_1^2}{2} + \frac{\sqrt{3c_2 - c_1^2}\sqrt{4g-4+3c_2-c_1^2}}{2}.$$

□

We will now prove the following theorem

Theorem 4.7. *Let X be a complex projective minimal surface of general type with $c_1^2 > c_2$ and $C \subset X$ an irreducible curve of genus g. Then $K_X.C \leq a(2g-2) + b$ where a and b are functions of c_1^2 and c_2.*

Proof. We write the inequality 4.5.1 as

$$-2(K_X.C)^2 + 4K_X.C(3g-3) + 3(c_2 - c_1^2)K_X.C - 2(3-3g)^2 + (3c_2 - c_1^2)(6-6g) \geq -(3c_2 - c_1^2)C^2.$$

i.e.

$$(\frac{c_2}{c_1^2}-1)(K_X.C)^2+K_X.C(4(g-1)+3c_2-c_1^2)-2(g-1)(3(g-1)+3c_2-c_1^2)$$
$$\geq(\frac{c_2}{c_1^2}-\frac{1}{3})((K_X.C)^2-C^2.K_X^2).$$

As above, we look at the quantity on the left as a quadratic polynomial in $K_X.C$. Assuming $c_1^2 > c_2$, it has a negative leading coefficient. From the Hodge index theorem, we see that $K_X.C$ is smaller than the biggest root of this quadratic polynomial. □

We will now show an application of corollary 4.6 in the case of surfaces which are compact quotient of the bi-disc recovering a result of [BHK+13] on the finiteness of smooth Shimura curves on compact Hilbert modular surfaces.

Corollary 4.8. *Let X be a smooth compact quotient of the bi-disc. Then there exists finitely many smooth totally geodesic curves in X.*

Proof. Let $C \subset X$ be a smooth totally geodesic curve. Then $K_X.C = 2(2g-2) = -2C^2$. It follows from corollary 4.6 that the geometric genus g is bounded. Therefore smooth totally geodesic curves form a bounded family. In case of infinite number of them, they would deform contradicting $C^2 < 0$. □

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