VALUE DISTRIBUTION THEORY FOR PARABOLIC RIEMANN SURFACES

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ABSTRACT. Let $\mathcal{Y}$ be a parabolic Riemann surface, i.e. subharmonic functions defined on $\mathcal{Y}$ are constant. We discuss Nevanlinna's theory for holomorphic maps $f : \mathcal{Y} \to \mathbb{P}^1$. The results we obtain parallels the classical case $\mathcal{Y} = \mathbb{C}$, as we describe now.

Let X be a manifold of general type, and let A be an ample line bundle on X. It is known that there exists a holomorphic jet differential P (of order $k \gg 0$) with values in the dual of A. If the map f has infinite area and if $\mathcal{Y}$ has finite Euler characteristic, then we show that f satisfies the differential relation induced by P.

As a consequence, we obtain a generalization of Bloch Theorem concerning the Zariski closure of maps f with values in a complex torus.

We then study the degree of Nevanlinna's current $T[f]$ associated to a parabolic leaf f of a foliation $\mathcal{F}$ by Riemann surfaces on a compact complex manifold. We show that the degree of $T[f]$ on the tangent bundle of the foliation is bounded from below in terms of the counting function of f with respect to the singularities of $\mathcal{F}$, and the Euler characteristic of $\mathcal{Y}$. In the case of complex surfaces of general type, we obtain a complete analogue of McQuillan's result: a parabolic curve of infinite area and finite Euler characteristic tangent to $\mathcal{F}$ is not Zariski dense.

1. INTRODUCTION

Let X be a compact complex manifold. S. Kobayashi introduced a pseudo-distance, determined by the complex structure of X. We recall here its infinitesimal version, cf. [13].

Given a point $x \in X$ and a tangent vector $v \in T_{X,x}$ at X in x, the length of v with respect to the *Kobayashi-Royden pseudo-metric* is the following quantity

$$\mathbf{k}_{X,x}(v) := \inf\{\lambda > 0; \exists f : \mathbb{D} \to X, f(0) = x, \lambda f'(0) = v\},$$

where $\mathbb{D} \subset \mathbb{C}$ is the unit disk, and f is a holomorphic map.

We remark that it may very well happen that $\mathbf{k}_{X,x}(v) = 0$; however, thanks to Brody re-parametrization lemma, this situation has a geometric counterpart, as follows. If there exists a couple (x, v) as above

such that $\mathbf{k}_{X,x}(v) = 0$, then one can construct a holomorphic non-constant map $f : \mathbb{C} \to X$. The point x is not necessarily in the image of f.

In conclusion, if any entire curve drawn on X is constant, then the pseudo-distance defined above is a distance, and we say that X is *hyperbolic in the sense of Brody*, or simply *hyperbolic* (since most of the time we will be concerned with compact manifolds).

As a starting point for the questions with which we will be concerned with in this article, we have the following result.

Proposition 1.1. *Let X be a hyperbolic manifold, and let $\mathcal{C}$ be a Riemann surface. Let $E \subset \mathcal{C}$ be a closed, countable set. Then any holomorphic map $f : \mathcal{C} \setminus E \to X$ admits a (holomorphic) extension to the surface $\mathcal{C}$.*

In particular, in the case of the complex plane we infer that any holomorphic map $f : \mathbb{C} \setminus E \to X$ must be constant (under the hypothesis of Proposition 1.1). We will give a proof and discuss some related statements and questions in the first paragraph of this paper. Observe however that if the cardinal of E is at least 2, then $\mathbb{C} \setminus E$ is Kobayashi hyperbolic.

Our next remark is that the surface $\mathbb{C} \setminus E$ is a particular case of a *parabolic Riemann surface*; we recall here the definition. A Riemann surface $\mathcal{Y}$ is parabolic if any bounded subharmonic function defined on $\mathcal{Y}$ is constant. This is a large class of surfaces, including e.g. $Y \setminus \Lambda$, where Y is a compact Riemann surface of arbitrary genus and $\Lambda \subset Y$ is any closed polar set. It is well-known (cf. [1]) that a parabolic Riemann surface $\mathcal{Y}$ admits a smooth exhaustion function

$$\sigma : \mathcal{Y} \to [1, \infty[$$

such that:

• σ is strictly subharmonic in the complement of a compact set;

• $\tau := \log \sigma$ is harmonic in the complement of a compact set of $\mathcal{Y}$. Moreover, we impose the normalization

$$(1) \qquad\qquad \int_{\mathcal{Y}} dd^c \log \sigma = 1,$$

where the operator d^c is defined as follows

$$d^c := \frac{\sqrt{-1}}{4\pi}(\overline{\partial} - \partial).$$

On the boundary $S(r) := (\sigma = r)$ of the parabolic ball of radius r we have the induced measure

$$d\mu_r := d^c \log \sigma|_{S(r)}.$$

The measure $d\mu_r$ has total mass equal to 1, by the relation (1) combined with the Stokes formula.

Since we are dealing with general parabolic surfaces, the growth of the Euler characteristic of the balls $\mathbb{B}(r) = (\sigma < r)$ will appear very often in our estimates. We introduce the following notion.

Definition 1.2. *Let $(\mathcal{Y}, \sigma)$ be a parabolic Riemann surface, together with an exhaustion function as above. For each $t \geq 1$ such that $S(t)$ is non-singular we denote by $\chi_\sigma(t)$ the Euler characteristic of the domain $\mathbb{B}(t)$, and let*

$$\mathfrak{X}_\sigma(r) := \int_1^r \left|\chi_\sigma(t)\right| \frac{dt}{t}$$

be the (weighted) mean Euler characteristic of the ball of radius r.

If $\mathcal{Y} = \mathbb{C}$, then $\mathfrak{X}_\sigma(r)$ is bounded by $\log r$. The same type of bound is verified if $\mathcal{Y}$ is the complement of a finite number of points in $\mathbb{C}$. If $\mathcal{Y} = \mathbb{C} \setminus E$ where E is a closed polar set of infinite cardinality, then things are more subtle, depending on the density of the distribution of the points of E in the complex plane. However, an immediate observation is that the surface $\mathcal{Y}$ has finite Euler characteristic if and only if

$$(2) \qquad \mathfrak{X}_\sigma(r) = \mathcal{O}(\log r).$$

In the first part of this article we will extend a few classical results in hyperbolicity theory to the context of parabolic Riemann surfaces, as follows.

We will review the so-called "first main theorem" and the logarithmic derivative lemma for maps $f : \mathcal{Y} \to X$, where X is a compact complex manifold. We also give a version of the first main theorem with respect to an ideal $\mathcal{J} \subset \mathcal{O}_X$. This will be a convenient language when studying foliations with singularities.

As a consequence, we derive a vanishing result for jet differentials, similar to the one obtained in case $\mathcal{Y} = \mathbb{C}$, as follows.

Let $\mathcal{P}$ be a jet differential of order k and degree m on X, with values in the dual of an ample bundle (see [8]; we recall a few basic facts about this notion in the next section). Then we prove the following result.

Theorem 1.3. *Let $\mathcal{Y}$ be a parabolic Riemann surface. We consider a holomorphic map $f : \mathcal{Y} \to X$, such that the area of $f(\mathcal{Y}) \subset X$ (counted with multiplicities) is infinite. Let $\mathcal{P}$ be an invariant jet differential of order k and degree m, vanishing on the dual of an ample line bundle. Then we have*

$$\mathcal{P}\left(j_k(f)\right) = 0$$

identically on $\mathcal{Y}$, provided that

$$\limsup_{r \to \infty} \frac{\mathfrak{X}_\sigma(r)}{T_{f,\omega}(r)} = 0$$

For example, the requirement above is satisfied if $\mathcal{Y}$ has finite Euler characteristic. In the previous statement we denote by $j_k(f)$ the kth jet associated to the map f. If $\mathcal{Y} = \mathbb{C}$, then this result is well-known, starting with the seminal work of A. Bloch (cf. [2]; see also [7], [24] and the references therein), and it is extremely useful in the investigation of the hyperbolicity properties of projective manifolds. In this context, the above result says that the vanishing result still holds in the context of Riemann surfaces of (eventually) infinite Euler characteristic, provided that the growth of this topological invariant is slow when compared to $T_{f,\omega}(r)$.

Concerning the existence of jet differentials, we recall Theorem 0.1 in [9], see also [15].

Theorem 1.4 ([7]). *Let X be a manifold of general type. Then there exist a couple of integers $m \gg k \gg 0$ and a (non-zero) holomorphic jet differential $\mathcal{P}$ of order k and degree m with values in the dual of an ample line bundle A.*

Thus, our result 1.3 can be used in the context of the general type manifolds.

As a consequence of Theorem 1.3, we obtain the following analogue of Bloch's theorem.

Theorem 1.5. *Let $\mathbb{C}^N/\Lambda$ be a complex torus, and let $\mathcal{Y}$ be a parabolic Riemann surface of finite Euler characteristic. Then the smallest analytic subset containing the closure of a holomorphic map*

$$f : \mathcal{Y} \to \mathbb{C}^N/\Lambda$$

of infinite area is a translate of a sub-torus in $\mathbb{C}^N/\Lambda$.

In the second part of this paper our aim is to recast part of the work of M. McQuillan and M. Brunella concerning the Green-Griffiths conjecture in the parabolic setting. We first recall the statement of this problem.

Conjecture 1.6. ([10]) *Let X be a projective manifold of general type. Then there exists an algebraic subvariety $W \subsetneq X$ which contains the image of all holomorphic curves $f : \mathbb{C} \to X$.*

It is hard to believe that this conjecture is correct for manifolds X of dimension ≥ 3. On the other hand, given the results available for surfaces (i.e. $\dim X = 2$), it is rather safe to predict that this holds true.

Given a map $f : \mathcal{Y} \to X$ defined on a parabolic Riemann surface $\mathcal{Y}$, we can associate a a Nevanlinna-type closed positive current $T[f]$. If X is surface of general type and if $\mathcal{Y}$ has finite Euler characteristic, then there exists an integer k such that the k-jet of f satisfies an algebraic

relation. As a consequence, there exists a foliation $\mathcal{F}$ by Riemann surfaces on the space of k-jets X_k of X_0, such that the lift of f is tangent to $\mathcal{F}$. In conclusion we are naturally lead to consider the pairs $(X, \mathcal{F})$, where X is a compact manifold, and $\mathcal{F}$ is a foliation by curves on X. We denote by $T_\mathcal{F}$ the so-called tangent bundle of $\mathcal{F}$.

We derive a lower bound of the intersection number $\int_X T[f] \wedge c_1(T_\mathcal{F})$ in terms of a Nevanlinna-type counting function of the intersection of f with the singular points of $\mathcal{F}$. As a consequence, if X is a complex surface and $\mathcal{F}$ has reduced singularities, we show that $\int_X T[f] \wedge c_1(T_\mathcal{F}) \geq 0$. For this part we follow closely the original argument of [16].

When combined with a result by Y. Miyaoka, the preceding inequality shows that the classes $\{T[f]\}$ and $c_1(T_\mathcal{F})$ are orthogonal. Since the class of the current $T[f]$ is nef, we show by a direct argument that we have $\int_X \{T[f]\}^2 = 0$, and from this we infer that the Lelong numbers of the diffuse part of $\{T[f]\}$, say R are equal to zero at each point of X.

This regularity property of R is crucial, since it allows to show –via the Baum-Bott formula and an elementary fact from dynamics– that we have $\int_X T[f] \wedge c_1(N_\mathcal{F}) \geq 0$, where $N_\mathcal{F}$ is the normal bundle of the foliation, and $c_1(N_\mathcal{F})$ is the first Chern class of $N_\mathcal{F}$.

We then obtain the next result, in the spirit of [16].

Theorem 1.7. *Let X be a surface of general type, and consider a holomorphic map $f : \mathcal{Y} \to X$, where $\mathcal{Y}$ is a parabolic Riemann surface of finite Euler characteristic. We assume that f is tangent to a holomorphic foliation $\mathcal{F}$; then the dimension of the Zariski closure of $f(\mathcal{Y})$ is at most 1.*

Furthermore, we show that the index theorem of M. Brunella in [3] admits an immediate generalization. Let L be a line bundle on a complex surface X, such that $S^m T_X^\star \otimes L$ has a non-identically zero section. Then we show that

$$\int_X c_1(L) \wedge T[f] \geq 0.$$

Brunella's theorem corresponds to the case $m = 1$ and $L = N_\mathcal{F}$: indeed, a foliation on X can be seen as a section of $T_X^\star \otimes N_\mathcal{F}$ (or in a dual manner, as a section of $T_X \otimes T_\mathcal{F}^\star$).

If X is a minimal surface of general type, such that $c_1^2 > c_2$, we see that this implies Theorem 1.7 directly, i.e. without considering the $T[f]$-degree of the tangent of $\mathcal{F}$. It is a very interesting problem to generalize the inequality above in the framework of higher order jet differentials, see section 6 for a precise statement.

2. Preliminaries

2.1. **Motivation: an extension result.** To start with, we first give the proof of Proposition 1.1; we refer to [18] for further results in this direction.

Proof (of Proposition 1.1) A first observation is that it is enough to treat the case where E is a single point. Indeed, assume that this case is settled. We consider the set $E_0 \subset E$ such that the map f does not extends across E_0; our goal is to prove that we have $E_0 = \emptyset$. If this is not the case, then we remark that E_0 contains at least an isolated point –since it is countable and non-empty–, and thus we obtain a contradiction.

Thus we can assume that we have a holomorphic map

$$f : \mathbb{D}^\star \to X$$

where $\mathbb{D}^\star$ is the pointed unit disc. Let $g_\mathcal{P}$ be the Kobayashi metric on X; we remark that by hypothesis, $g_\mathcal{P}$ is non-degenerate. By the *distance decreasing property* of this metric we infer that

$$(3) \qquad |f'(t)|^2_{g_\mathcal{P}} \leq \frac{1}{|t|^2 \log^2 |t|^2}$$

for any $t \in \mathbb{D}^\star$. This is a crucial information, since now we can argue as follows. The inequality (3) implies that the area of the graph associated to our map $\Gamma^0_f \subset \mathbb{D}^\star \times X$ defined by

$$\Gamma^0_f := \{(t, x) \in \mathbb{D}^\star \times X : f(t) = x\}$$

is finite. By the theorem of Bishop-Skoda (cf. [23] and the references therein) this implies that there exists an analytic subset $\Gamma \subset \mathbb{D} \times X$ whose restriction to $\mathbb{D}^\star \times X$ is precisely Γ^0_f. Hence we infer that the fiber of the projection $\Gamma \to \mathbb{D}$ on the second factor is a point. Indeed, if this is not the case, then the area of the image (via f) of the disk of radius ε is bounded from below by a constant independent of $\varepsilon > 0$. This of course cannot happen, as one can see by integrating the inequality (3) over the disk of radius ε.

$$\square$$

In connection with this result, we recall the following conjecture proposed in [18].

Conjecture 2.1. *Let X be a Kobayashi hyperbolic compact manifold of dimension n. We denote by $\mathbb{B}$ the unit ball in $\mathbb{C}^p$, and let E be a closed polar subset of $\mathbb{B}$. Then any holomorphic map*

$$f : \mathbb{B} \setminus E \to X$$

extends across E.

In case where X is the quotient of a bounded domain in $\mathbb{C}^n$, a proof of this conjecture was proposed by M. Suzuki in [27]. In general, even if we assume that the holomorphic bisectional curvature of X is bounded from above by -1, the conjecture above seems to be open.

2.2. **Jet spaces.** We will recall here a few basic facts concerning the jet spaces associated to complex manifolds; we refer to [8], [13] for a more complete overview.

Let X be an l-dimensional complex space; we denote by $J^k(X)$ the space of k-jets of holomorphic discs, described as follows. Let f and g be two germs of analytic discs $(\mathbb{C}, 0) \to (X, x)$, we say that they define the same k jet at x if their derivatives at zero coincide up to order k, i. e.

$$f^{(j)}(0) = g^{(j)}(0)$$

for $j = 0, ... k$. The equivalence classes defined by this equivalence relation is denoted by $J^k(X, x)$; as a set, $J^k(X)$ is the union of $J^k(X, x)$ for all $x \in X$. We remark that if $x \in X_{\text{reg}}$ is a non-singular point of X, then $J^k(X, x)$ is isomorphic to $\mathbb{C}^{kl}$, via the identification

$$f \to \big(f'(0), ..., f^{(k)}(0)\big).$$

This map is not intrinsic, it depends on the choice of some local coordinate system needed to express the derivatives above; at a global level the projection map

$$J^k(X_{\text{reg}}) \to X_{\text{reg}}$$

is a holomorphic fiber bundle (which is not a vector bundle in general, since the transition functions are polynomial instead of linear).

If $k = 1$, and $x \in X_{\text{reg}}$ is a regular point, then $J^1(X, x)$ is the tangent space of X at x. We also mention here that the structure of the analytic space $J^k(X)$ at a singular point of X is far more complicated.

We assume that X is a subset of a complex manifold M; then for each positive integer k we have a natural inclusion

$$J^k(X) \subset J^k(M)$$

and one can see that the space $J^k(X)$ is the Zariski closure of the analytic space $J^k(X_{\text{reg}})$ in the complex manifold $J^k(M)$ (note that this coincides with the topological closure)

Next we recall the definition of the main geometric objects we will use in the analysis of the structure of the subvarieties of complex tori which are Zariski closure of some parabolic image.

As before, let $x \in X_{\text{reg}}$ be a regular point of X; we consider a coordinate system $(x^1, ..., x^l)$ of X centered at x. We consider the symbols

$$dx^1, ..., dx^l, d^2x^1, ..., d^2x^l, ..., d^kx^1, ..., d^kx^l$$

and we say that the weight of the symbol d^px^r is equal to p, for any $r = 1, ..., l$. A *jet differential* of order k and degree m at x is a homogeneous

polynomial of degree m in $(d^p x^r)_{p=1,\ldots,k,r=1,\ldots,l}$; we denote by $E_{k,m}^{\mathrm{GG}}(X,x)$ the vector space of all such polynomials, and then the set

$$E_{k,m}^{\mathrm{GG}}\big(X_{\mathrm{reg}}\big) := \cup_{x \in X_{\mathrm{reg}}} E_{k,m}^{\mathrm{GG}}(X,x)$$

has a structure of vector bundle, whose global sections are called jet differentials of weight m and order k. A global section $\mathcal{P}$ of the bundle $E_{k,m}^{\mathrm{GG}}(X)$ can be written locally as

$$\mathcal{P} = \sum_{|\alpha_1| + \cdots + k|\alpha_k| = m} a_\alpha dx^{\alpha_1} \ldots d^k x^{\alpha_k};$$

here we use standard the multi-index notation.

Let $f : (\mathbb{C}, 0) \to (X, x)$ be a k-jet at x; we say that the operator $\mathcal{P}$ is *invariant* if

$$\mathcal{P}\big((f \circ \varphi)', \ldots, (f \circ \varphi)^{(k)}\big) = \varphi'^m \mathcal{P}\big(f', \ldots, f^{(k)}\big).$$

The bundle of invariant jet differentials is denoted by $E_{k,m}(X)$; we will recall next an alternative description of this bundle, which will be very useful in what follows.

The group $\mathbb{G}_k$ of k–jets of biholomorphisms of $(\mathbb{C}, 0)$ acts on $J_k(X)$, and along the next few lines, we indicate a compactification of the quotient $J_k^{reg}(X)/\mathbb{G}_k$ following [8], where $J_k^{reg}(X)$ denote the space of non-constants jets.

We start with the pair (X, V), where $V \subset T_X$ is a subbundle of the tangent space of X. Then we define $X_1 := \mathbb{P}(T_X)$, and the bundle $V_1 \subset T_{X_1}$ is defined fiberwise by

$$V_{1,(x,[v])} := \big\{\xi \in T_{X_1(x,[v])} : d\pi(\xi) \in \mathbb{C}v\big\}$$

where $\pi : X_1 \to X$ is the canonical projection and $v \in V$. It is easy to see that we have the following alternative description of V_1: consider a non-constant disk $u : (\mathbb{C}, 0) \to (X, x)$. We can lift it to X_1 and denote the resulting germ by u_1. Then the derivative of u_1 belong to the V_1 directions.

Inductively by this procedure we get a tower of manifolds (X_k, V_k), starting from (X, T_X) and it turns out that we have an embedding $J_k^{reg}/\mathbb{G}_k \to X_k$. On each manifold X_k, we have a tautological bundle $\mathcal{O}_{X_k}(-1)$, and the positivity of its dual plays an important role here.

We denote by $\pi_k : X_k \to X$ the projection, and consider the direct image sheaf $\pi_{k*}\big(\mathcal{O}_{X_k}(m)\big)$. The result is a vector bundle $E_{k,m}(X)$ whose sections are precisely the invariant jet differentials considered above.

The fiber of π_k at a non-singular point of X is denoted by $\mathcal{R}_{n,k}$; it is a rational manifold, and it is a compactification of the quotient $\mathbb{C}^{nk} \setminus 0/\mathbb{G}_k$.

The articles [2], [8], [24] (to quote only a few) show that the existence of jet differentials are crucial in the analysis of the entire maps $f : \mathbb{C} \to$

X. As we will see in the next sections, they play a similar role in the study of the images of the parabolic Riemann surfaces.

3. Basics of Nevanlinna Theory for Parabolic Riemann Surfaces

Let $\mathcal{Y}$ be a parabolic Riemann surface; as we have recalled in the introduction, this means that there exists a non-singular exhaustion function

$$\sigma : \mathcal{Y} \to [1, \infty[$$

such that:
• σ is strictly psh in the complement of the subset $(\sigma < r_0) \subset \mathcal{Y}$;

• The function $\tau := \log \sigma$ is harmonic in the complement of a compact subset of $\mathcal{Y}$, and we have $\int_{\mathcal{Y}} dd^c \tau = 1$.

We denote by $\mathbb{B}(r) \subset \mathcal{Y}$ the parabolic ball of radius r, that is to say

$$\mathbb{B}(r) := \{y \in \mathcal{Y} : \sigma(y) \leq r\}.$$

For almost every value $r \in \mathbb{R}$, the sphere $S(r) := \partial \mathbb{B}(r)$ is a smooth curve drawn on $\mathcal{Y}$. The induced length measure on $S(r)$ is equal to

$$d\mu_r := d^c \log \sigma|_{S(r)}.$$

Let $v : \mathcal{Y} \to [-\infty, \infty[$ be a function defined on $\mathcal{Y}$, such that locally near every point of $\mathcal{Y}$ it can be written as a difference of two subharmonic functions, i.e. $dd^c v$ is of order zero.

Then we recall here the following formula, which will be very useful in what follows.

Proposition 3.1. (Jensen formula) *For every $r \geq 1$ large enough we have*

$$\int_1^r \frac{dt}{t} \int_{\mathbb{B}(t)} dd^c v = \int_{S(r)} v \, d\mu_r - \int_{\mathbb{B}(r_0)} v \, dd^c \tau =$$

$$= \int_{S(r)} v \, d\mu_r + \mathcal{O}(1)$$

where we have $\tau = \log \sigma$.

Proof. The arguments are standard. To start with, we remark that for each regular value r of σ the function v is integrable with respect to the measure $d\mu_r$ over the sphere $S(r)$. Next, we have

$$\int_1^r \frac{dt}{t} \int_{\mathbb{B}(t)} dd^c v = \int_{\mathbb{B}(r)} \big(\log r - \log \sigma \big) dd^c v =$$

$$= \int \log^+ \frac{r}{\sigma} dd^c v = \int v dd^c \big(\log^+ \frac{r}{\sigma} \big)$$

$$= \int_{S(r)} v d\mu_r - \int_{\mathbb{B}(r)} v dd^c \tau.$$

$\square$

Remark 3.2. As we can see, the Jensen formula above holds true even without the assumption that the function τ is harmonic outside a compact set. The only difference is eventually as $r \to \infty$, since the term $\int_{\mathbb{B}(r)} v dd^c \tau$ may tend to infinity.

We reformulate next the notion of mean Euler characteristic in analytic terms. To this end, we first recall that the tangent bundle $T_{\mathcal{Y}}$ of a non-compact parabolic surface admits a trivializing global holomorphic section $v \in H^0(\mathcal{Y}, T_{\mathcal{Y}})$, cf. [11] (actually, any such Riemann surface admits a submersion into $\mathbb{C}$). Via Poincaré-Hopf index theorem, we obtain the following result.

Proposition 3.3. *Let $(\mathcal{Y}, \sigma)$ be a parabolic Riemann surface, so that $\log \sigma$ is harmonic in the complement of a compact set. Then we have*

$$\mathfrak{X}_\sigma(r) = \frac{1}{2} \int_{S(r)} \log^+ |d\sigma(v)|^2 d\mu_r + \mathcal{O}(\log r)$$

for any $r \geq 1$.

Proof. By Jensen formula we have

$$\int_{S(r)} \log^+ |d\sigma(v)|^2 d\mu_r = 2\log r + \int_{S(r)} \log^+ |d\log \sigma(v)|^2 d\mu_r$$

$$= \int_1^r \frac{dt}{t} \int_{\mathbb{B}(t)} dd^c \log^+ |d\log \sigma(v)|^2 + 2\log r + \mathcal{O}(1)$$

$$= \int_1^r \frac{dt}{t} \int_{\mathbb{B}(t)} dd^c \log |\partial \log \sigma(v)|^2 + \mathcal{O}(\log r)$$

where the last identity is due to the fact that the function $\partial \log \sigma(v)$ is holomorphic outside a compact set.

The term $\mathcal{O}(\log r)$ above depends on the exhaustion function σ and on the fixed vector field v, but the quantity

$$\int_1^r \frac{dt}{t} \int_{\mathbb{B}(t)} dd^c \log |\partial \log \sigma(v)|^2$$

is equal to the weighted Euler characteristic $\mathfrak{X}_\sigma(r)$ of the domains $\mathbb{B}(r)$, in particular it is independent of v, up to a bounded term. This can be seen as a consequence of the Poincaré-Hopf index theorem, combined with the fact that the function $\partial \log \sigma(v)$ is holomorphic. $\qquad\square$

Along the next lines, we obtain the *first main theorem* and the *logarithmic derivative lemma* of Nevanlinna theory in the parabolic setting. The results are variation on well-known techniques (see [26] and the references therein). But for the convenience of the reader, we will reproduce here the arguments.

Let X be a compact complex manifold, and let $L \to X$ be a line bundle on X, endowed with a smooth metric h. We make no particular assumptions concerning the curvature form $\Theta_h(L)$. Let s be a non-trivial section of L normalized such that $\sup_X |s| = 1$, and let $f : \mathcal{Y} \to X$ be a holomorphic map, where $\mathcal{Y}$ is parabolic.

We define the usual characteristic function of f with respect to $\Theta_h(L)$ as follows

$$T_{f,\Theta_h(L)}(r) := \int_{r_0}^r \frac{dt}{t} \int_{\mathbb{B}(t)} f^\star \Theta_h(L).$$

If the form $\Theta_h(L)$ is positive definite, then precisely as in the classical case $\mathcal{Y} = \mathbb{C}$, the area of the image of f will be finite if and only if $T_{f,\Theta_h(L)}(r) = \mathcal{O}(\log r)$ as $r \to \infty$.

Let

$$N_{f,s}(r) := \int_{r_0}^r n_{f,s}(t)\frac{dt}{t}$$

be the counting function, where $n_{f,s}(t)$ is the number of zeroes of $s \circ f$ in the parabolic ball of radius t (counted with multiplicities). Hence we assume implicitly that the image of f is not contained in the set $(s = 0)$. Moreover, in our context the proximity function becomes

$$m_{f,s}(r) := \frac{1}{2\pi} \int_{S(r)} \log \frac{1}{|s \circ f|_h} d\mu_r.$$

In the important case of a (meromorphic) function $F : \mathcal{Y} \to \mathbb{P}^1$, one usually takes $\Theta_h(L)$ to be the Fubini-Study metric, and s the section vanishing at infinity; the proximity function becomes

(4) $$m_{f,\infty}(r) := \frac{1}{2\pi} \int_{S(r)} \log_+ |f| d\mu_r,$$

where $F := [f_0 : f_1]$, $f = f_1/f_0$ and $\log_+ := \max(\log, 0)$.

As a consequence of Jensen formula, we derive the next result.

Theorem 3.4. *With the above notations, we have*

(5) $$T_{f,\Theta_h(L)}(r) = N_{f,s}(r) + m_{f,s}(r) + \mathcal{O}(1)$$

as $r \to \infty$.

Proof. The argument is similar to the usual one: we apply the Jensen formula ($\star$) to the function $v := \log |s \circ f|_h$. Recall that that the Poincaré-Lelong equation gives $dd^c \log |s|_h^2 = [s = 0] - \Theta_h(L)$, which implies

$$dd^c v = \sum_j m_j \delta_{a_j} - f^\star(\Theta_h(L))$$

so by integration we obtain (5). □

Remark 3.5. If the measure $dd^c\tau$ does not have a compact support, then the term $\mathcal{O}(1)$ in the equality (5) is to be replaced by

$$(6) \qquad\qquad -\int_{\mathbb{B}(r)} \log |s \circ f|_h^2 dd^c\tau.$$

We observe that, thanks to the normalization condition we impose to s, the term (6) is negative. In particular we infer that

$$(7) \qquad\qquad T_{f,\Theta_h(L)}(r) \geq N_{f,s}(r)$$

for any $r \geq r_0$.

We will discuss now a version of Theorem 3.4 which will be very useful in dealing with singular foliations. Let $\mathcal{J} \subset \mathcal{O}_X$ be a coherent ideal of holomorphic functions. We consider a finite covering of X with coordinate open sets $(U_\alpha)_{\alpha \in \Lambda}$, such that on U_α the ideal $\mathcal{J}$ is generated by the holomorphic functions $(g_{\alpha i})_{i=1...N_\alpha}$.

Then we can construct a function $\psi_\mathcal{J}$, such that for each $\alpha \in \Lambda$ the difference

$$(8) \qquad\qquad \psi_\mathcal{J} - \log(\sum_i |g_{\alpha i}|^2)$$

is bounded on U_α. Indeed, let ρ_α be a partition of unity subordinated to $(U_\alpha)_{\alpha \in \Lambda}$. We define the function $\psi_\mathcal{J}$ as follows

$$(9) \qquad\qquad \psi_\mathcal{J} := \sum_\alpha \rho_\alpha \log(\sum_i |g_{\alpha i}|^2)$$

and the boundedness condition (8) is verified, since there exists a constant $C > 0$ such that

$$C^{-1} \leq \frac{\sum_i |g_{\alpha i}|^2}{\sum_i |g_{\beta i}|^2} \leq C$$

holds on $U_\alpha \cap U_\beta$, for each pair of indexes α, β. In the preceding context, the function $\psi_\mathcal{J}$ corresponds to $\log |s|_h^2$.

We can define a counting function and a proximity function for a holomorphic map $f : \mathcal{Y} \to X$ with respect to the analytic set defined by $\mathcal{J}$, as follows. Let $(t_j) \subset \mathcal{Y}$ be the set of solutions of the equation

$$\exp\big(\psi_\mathcal{J} \circ f(y)\big) = 0.$$

For each $r > 0$ we can write

$$\psi_{\mathcal{J}} \circ f(y)|_{\mathbb{B}(r)} = \sum_{\sigma(t_j)<r} \nu_j \log|y - t_j|^2 + \mathcal{O}(1)$$

for a set of multiplicities ν_j, and then the counting function is defined as follows

$$(10) \qquad N_{f,\mathcal{J}}(r) := \sum_j \nu_j \log \frac{r}{\sigma(t_j)}.$$

In a similar way, the proximity function is defined as follows

$$(11) \qquad m_{f,\mathcal{J}}(r) := -\int_{S(r)} \psi_{\mathcal{J}} \circ f d\mu_r.$$

By the principalization theorem (cf. e.g. [14]), there exists a non-singular manifold $\widehat{X}$ together with a birational map $p : \widehat{X} \to X$ such that the inverse image of the ideal $\mathcal{J}$ is equal to $\mathcal{O}_{\widehat{X}}(-D)$, where $D := \sum_j e_j W_j$ is a simple normal crossing divisor on $\widehat{X}$. Recall that $\mathcal{O}_{\widehat{X}}(-D) \subset \mathcal{O}_{\widehat{X}}$ is the sheaf of holomorphic functions vanishing on D. In terms of the function $\psi_{\mathcal{J}}$ associated to the ideal $\mathcal{J}$, this can be expressed as follows

$$(12) \qquad \psi_{\mathcal{J}} \circ p = \sum_j e_j \log|s_j|^2_{h_j} + \tau$$

where $W_j = (s_j = 0)$, the metric h_j on $\mathcal{O}(W_j)$ is arbitrary (and non-singular), and where τ is a bounded function on $\widehat{X}$.

Since we assume that the image of the map f is not contained in the zero set of the ideal $\mathcal{J}$, we can define the lift $\widehat{f} : \mathcal{Y} \to \widehat{X}$ of f to $\widehat{X}$ such that $p \circ \widehat{f} = f$. We have the next result.

Theorem 3.6. *Let $\mathcal{O}(D)$ be the line bundle associated to the divisor D; we endow it with the metric induced by (h_j), and let Θ_D be the associated curvature form. Then we have*

$$T_{\widehat{f},\Theta_D}(r) = N_{f,\mathcal{J}}(r) + m_{f,\mathcal{J}}(r) + \mathcal{O}(1)$$

as $r \to \infty$.

The argument is completely similar to the one given for Theorem 3.4 (by using the relation (12)). We basically apply the first main theorem for each s_j and add up the contributions. $\square$

We will treat now another important result in Nevanlinna theory, namely the *logarithmic derivative lemma* in the parabolic context. To this end, we will suppose that as part of the data we are given a vector field

$$\xi \in H^0(\mathcal{Y}, T_{\mathcal{Y}})$$

which is nowhere vanishing –hence it trivializes the tangent bundle of our surface $\mathcal{Y}$. We denote by f' the section $df(\xi)$ of the bundle $f^\star T_X$. For example, if $\mathcal{Y} = \mathbb{C}$, then we can take $\xi = \dfrac{\partial}{\partial z}$.

In the proof of the next result, we will need the following form of the co-area formula. Let ψ be a 1-form defined on the surface $\mathcal{Y}$; then we have

$$(13) \qquad \int_{\mathbb{B}(r)} d\sigma \wedge \psi = \int_0^r dt \int_{S(t)} \psi$$

for any $r > 1$. In particular, we get

$$(14) \qquad \frac{d}{dr} \int_{\mathbb{B}(r)} d\sigma \wedge \psi = \int_{S(r)} \psi$$

We have the following version of the classical logarithmic derivative lemma.

Theorem 3.7. *Let $f : \mathcal{Y} \to \mathbb{P}^1$ be a meromorphic map defined on a parabolic Riemann surface $\mathcal{Y}$. The inequality*

$$m_{f'/f,\infty}(r) \leq C\big(\log T_f(r) + \log r\big) + \mathfrak{X}_\sigma(r)$$

holds true for all r outside a set of small measure. We also get a similar estimate for higher order derivatives.

Proof. Within the framework of Nevanlinna theory, this kind of results can be derived in many ways if $\mathcal{Y}$ is the complex plane; the proof presented here follows an argument due to Selberg in [21].

On the complex plane $\mathbb{C} \subset \mathbb{P}^1$ we consider the coordinate w corresponding to $[1 : w]$ in homogeneous coordinates on $\mathbb{P}^1$. The form

$$(15) \qquad \Omega := \frac{1}{|w|^2(1 + \log^2 |w|)} \frac{\sqrt{-1}}{2\pi} dw \wedge d\overline{w}$$

on $\mathbb{C}$ has finite volume, as one can easily check by a direct computation. In what follows, we will use the same letter to denote the expression of the meromorphic function $\mathcal{Y} \to \mathbb{C}$ induced by f. For each $t \geq 0$, we denote by $n(t, f, w)$ the number of zeroes of the function $z \to f(z) - w$ in the parabolic ball $\mathbb{B}(t)$ and we have

$$(16) \qquad \int_{\mathbb{B}(t)} f^\star \Omega = \int_{\mathbb{C}} n(t, f, w)\Omega$$

by the change of variables formula.

Next, by integrating the relation (16) above and using Theorem 3.4, we infer the following

$$(17) \qquad \int_1^r \frac{dt}{t} \int_{\mathbb{B}(t)} f^\star \Omega = \int_{\mathbb{C}} N_{f-w,\infty}(r)\Omega \leq \int_{\mathbb{C}} T_f(r)\Omega;$$

by Remark 3.4. This last quantity is smaller than $C_0 T_f(r)$, where C_0 is a positive constant. Thus, we have

$$(18) \qquad \int_1^r \frac{dt}{t} \int_{\mathbb{B}(t)} f^\star \Omega \le C_0 T_f(r).$$

A simple algebraic computation shows the next inequality

$$\log \left(1 + \frac{|df(\xi)|^2}{|f(z)|^2} \right) \le \log \left(1 + \frac{|df(\xi)|^2}{|f(z)|^2 (1 + \log^2 |f(z)|^2) |d\sigma(\xi)|^2} \right) +$$
$$+ \log(1 + \log^2 |f(z)|^2) + \log(1 + |d\sigma(\xi)|^2)$$

and therefore we get

$$m_{f'/f,\infty}(r) \le \frac{1}{4\pi} \int_{S(r)} \log \left(1 + \frac{|df(\xi)|^2}{|f(z)|^2} \right) d\mu_r \le$$

$$\le \log^+ \int_{S(r)} \frac{|df(\xi)|^2}{|f(z)|^2 (1 + \log^2 |f(z)|)} \frac{1}{|d\sigma_z(\xi)|^2} d\mu_r +$$

$$+ \int_{S(r)} \log(1 + \log^2 |f(z)|^2) d\mu_r +$$

$$+ \frac{1}{4\pi} \int_{S(r)} \log(1 + |d\sigma_z(\xi)|^2) d\mu_r + C_1,$$

where C_1 is a positive constant; here we use the concavity of the log function.

By the formula (14) we obtain

$$\int_{S(r)} \frac{|df(\xi)|^2}{|f(z)|^2 (1 + \log^2 |f(z)|)} \frac{1}{|d\sigma_z(\xi)|^2} d\mu_r =$$

$$= \frac{1}{r} \frac{d}{dr} \int_{\mathbb{B}(r)} \frac{|df(\xi)|^2}{|f(z)|^2 (1 + \log^2 |f(z)|)} \frac{1}{|d\sigma_z(\xi)|^2} d\sigma \wedge d^c \sigma.$$

Next we show that we have

$$\frac{|df(\xi)|^2}{|f(z)|^2 (1 + \log^2 |f(z)|)} \frac{1}{|d\sigma_z(\xi)|^2} d\sigma \wedge d^c \sigma = f^\star \Omega.$$

Indeed this is clear, since we can choose a local coordinate z such that $\xi = \frac{\partial}{\partial z}$ and moreover we have $d\sigma \wedge d^c \sigma = \frac{\sqrt{-1}}{2\pi} \left| \frac{\partial \sigma}{\partial z} \right|^2 dz \wedge d\bar{z}$ (we are using here the fact that f is holomorphic).

Let H be a positive, strictly increasing function defined on $(0, \infty)$. It is immediate to check that the set of numbers $s \in \mathbb{R}_+$ such that the inequality

$$H'(s) \le H^{1+\delta}(s)$$

is not verified, is of finite Lebesgue measure. By applying this calculus lemma to the function $H(r) := \int_1^r \frac{dt}{t} \int_{\mathbb{B}(t)} f^\star \Omega$ we obtain

$$
\begin{aligned}
\log^+ \frac{1}{r} \frac{d}{dr} \int_{\mathbb{B}(r)} f^\star \Omega \leq & \log^+ \frac{1}{r} \Big(\int_{\mathbb{B}(r)} f^\star \Omega \Big)^{1+\delta} + \mathcal{O}(1) \\
\leq & \log^+ \Big(r^\delta \Big[\frac{d}{dr} \int_1^r \frac{dt}{t} \int_{\mathbb{B}(t)} f^\star \Omega \Big]^{1+\delta} \Big) + \mathcal{O}(1) \\
\leq & \delta \log r + \frac{1}{2} \log^+ \Big(\int_1^r \frac{dt}{t} \int_{\mathbb{B}(t)} f^\star \Omega \Big)^{(1+\delta)^2} + \mathcal{O}(1)
\end{aligned}
$$

for all r outside a set of finite measure.

The term

$$
\int_{S(r)} \log(1 + \log^2 |f(z)|^2) d\mu_r
$$

is bounded -up to a constant- by $\log T_f(r)$; combined with Proposition 3.3, this implies the desired inequality. $\qquad\square$

It is a simple matter to deduce the so-called *second main theorem* of Nevanlinna theory starting from the logarithmic derivative lemma (cf. e.g. [7]). The parabolic version of this result can be stated as follows.

Theorem 3.8. *Let $f : (\mathcal{Y}, \sigma) \to \mathbb{P}^1$ be a meromorphic function. We denote by $N_{R_f}(r)$ the Nevanlinna counting function for the ramification divisor associated to f. Then for any set of distinct points $(a)_{1 \leq j \leq p}$ in $\mathbb{P}^1$ there exists a set $\Lambda \subset \mathbb{R}_+$ of finite Lebesgue measure such that*

$$
N_{R_f}(r) + \sum_{j=1}^p m_{f,a_j}(r) \leq T_{f,\omega}(r) + \mathfrak{X}_\sigma(r) + \mathcal{O}\big(\log r + \log^+ T_{f,\omega}(r) \big) +
$$

for all $r \in \mathbb{R}_+ \setminus \Lambda$.

For the proof we refer e.g. to [7]; as we have already mentioned, it is a direct consequence of the logarithmic derivative lemma. As a consequence, we have

$$
\sum_{j=1}^p \delta_f(a_j) \leq 2 + \overline{\lim}_{r \to \infty} \frac{\mathfrak{X}_\sigma(r)}{T_{f,\omega}(r)}
$$

provided that $T_{f,\omega}(r) \gg \log r$. In the inequality above we use the classical notation $\delta_f(a) := \underline{\lim}_r \frac{m_{f,a}(r)}{T_{f,\omega}(r)} = 1 - \overline{\lim}_r \frac{N_a(r)}{T_{f,\omega}(r)}$.

4. The Vanishing Theorem

Let $\mathcal{P}$ be an invariant jet differential of order k and degree m. We assume that it has values in the dual of an ample line bundle, that is

to say

$$\mathcal{P} \in H^0(X, E_{k,m}T_X^\star \otimes A^{-1})$$

where A is an ample line bundle on X.

Let $f : \mathcal{Y} \to X$ be a parabolic curve on X; assume that we are given the exhaustion function σ, a vector field ξ and that

(19) $$\overline{\lim}_{r\to\infty} \frac{\mathfrak{X}_\sigma(r)}{T_{f,\omega}(r)} = 0$$

As we have already recalled in the preliminaries, the operator $\mathcal{P}$ can be seen as section of $\mathcal{O}_{X_k}(m)$ on X_k.

On the other hand, the curve f admits a canonical lift to X_k as follows. One first observes that the derivative $df : T_\mathcal{Y} \to f^\star T_X$ induces a map

$$f_1 : \mathcal{Y} \to \mathbb{P}(T_X);$$

we remark that to do so we do not need any supplementary data, since $df(v_1)$ and $df(v_2)$ are proportional, provided that $v_j \in T_{\mathcal{Y},t}$ are tangent vectors at the same point. It turns out that the curve f_1 is tangent to $V_1 \subset T_{X_1}$, so that we can continue this procedure and define inductively $f_k : \mathcal{Y} \to X_k$.

We prove next the following result.

Theorem 4.1. *We assume that the curve f has infinite area, and that the condition (19) holds. Then the image of f_k is contained in the zero set of the section of $\mathcal{O}_{X_k}(m) \otimes A^{-1}$ defined by the jet differential $\mathcal{P}$.*

Proof. We observe that we have

$$df_{k-1}(\xi) : \mathcal{Y} \to f_k^\star\big(\mathcal{O}_{X_k}(-1)\big)$$

that is to say, the image of the derivative of f_{k-1} belongs to the tautological bundle. Thus the quantity

$$\mathcal{P}_{f_k}\big(df_{k-1}(\xi)^{\otimes m}\big)$$

is a section of $f_k^\star(A^{-1})$; as a consequence, if ω_A is the curvature form of A, we have

(20) $$\sqrt{-1}\partial\overline{\partial} \log |\mathcal{P}_{f_k}\big(df_{k-1}(\xi)^{\otimes m}\big)|^2 \geq f_k^\star(\omega_A).$$

The missing term involves the Dirac masses at the critical points of f. We observe that the positivity of the bundle A is fully used at this point: we obtain an upper bound for the characteristic function of f. By integrating and using Jensen formula, we infer that we have

$$\int_{S(r)} \log |\mathcal{P}_{f_k}\big(df_{k-1}(\xi)^{\otimes m}\big)|^2 d\mu_r \geq T_{f,\omega_A}(r) + \mathcal{O}(1)$$

as $r \to \infty$.

Now we follow the arguments in [7]: there exists a finite set of rational functions $u_j : X_k \to \mathbb{P}^1$ and a positive constant C such that we have

$$\log^+ |\mathcal{P}_{f_k}\big(df_{k-1}(\xi)^{\otimes m}\big)|^2 \leq C \sum_j \log^+ \frac{|d(u_j \circ f_{k-1})(\xi)|^2}{|u_j \circ f_{k-1}|^2}$$

pointwise on $\mathcal{Y}$. We invoke next the logarithmic derivative lemma (Theorem 3.7) established in the previous section, and so we infer that we have

$$(21) \quad \int_{S(r)} \log |\mathcal{P}_{f_k}\big(df_{k-1}(\xi)^{\otimes m}\big)|^2 d\mu_r \leq C(\log T_{f_{k-1}(r)} + \log r + \mathfrak{X}_\sigma(r)).$$

It is not difficult to see that the characteristic function corresponding to f_{k-1} is the smaller than $CT_f(r)$, for some constant C; by combining the relations (20) and (21) we have

$$T_f(r) \leq C\big(\log T_f(r) + \log r + \mathfrak{X}_\sigma(r)\big).$$

Since the area of f in infinite, and since $\mathfrak{X}_\sigma(r) = o\big(T_f(r)\big)$, we obtain

$$\lim_r \frac{\log r}{T_f(r)} = 0$$

that is to say, a contradiction. Therefore, if the image of f_k is not contained in the zero set of $\mathcal{P}$, then the area of f is finite. $\square$

Let $E \subset \mathbb{C}$ be a polar subset of the complex plane. In the case

$$\mathcal{Y} = \mathbb{C} \setminus E,$$

we show that we have the following version of the previous result in the context of arbitrary jet differentials.

Theorem 4.2. *Let $f : \mathbb{C} \setminus E \to X$ be a holomorphic curve; we assume that the area of f is infinite, and that condition (19) is satisfied. Then $\mathcal{P}(f', \ldots, f^{(k)}) \equiv 0$ for any holomorphic jet differential $\mathcal{P}$ of degree m and order k with values in the dual of an ample line bundle.*

Proof. The argument is similar to the proof of the preceding Theorem 4.1, except that we use the pointwise inequality

$$\log^+ |\mathcal{P}(f', \ldots, f^{(k)})| \leq C \sum_j \sum_{l=1}^{k} \log |d^l \log(u_j \circ f)|$$

combined with Theorem 3.7 in order to derive a contradiction. $\square$

To state our next result, we consider the following data. Let X be a non-singular, projective manifold and let $D = Y_1 + \cdots + Y_l$ be an effective divisor, such that the pair (X, D) is log-smooth (this last condition means that the hypersurfaces Y_j are non-singular, and that they have transverse intersections). In some cases we have

$$(22) \qquad\qquad H^0\big(X, E_{k,m}T_X^\star\langle D\rangle \otimes A^{-1}\big) \neq 0$$

where $E_{k,m}T_X^\star\langle D\rangle$ is the log version of the space of invariant jet differentials of order k and degree m (for example, when $K_X + D$ is big)

We have the following result, which is a more general version of Theorem 4.2.

Theorem 4.3. *Let $f : \mathbb{C}\setminus E \to X\setminus D$ be a non-algebraic, holomorphic map. If the parabolic Riemann surface $\mathcal{Y} := \mathbb{C}\setminus E$ verifies*

$$\mathfrak{X}_\sigma(r) = o\big(T_f(r)\big)$$

then for any invariant log-jet differential

$$\mathcal{P} \in H^0\big(X, E_{k,m}T_X^\star\langle D\rangle \otimes A^{-1}\big)$$

we have $\mathcal{P}(f',\ldots,f^{(k)}) \equiv 0$.

Proof. We only have to notice that the "logarithmic derivative lemma" type argument used in the proof of the vanishing theorem is still valid in our context, despite of the fact that the jet differential has poles along D (see [19] for a complete treatment). Thus, the result follows as above. $\square$

As a corollary of this result, we obtain the following statement.

Corollary 4.4. *Let (X, D) be a pair as above, and let $f : \mathbb{C} \to X$ be a non-algebraic, holomorphic map; we define $E := f^{-1}(D)$. Moreover, we assume that the property (19) holds true for $\mathcal{Y} := \mathbb{C}\setminus E$, and that there exists a polynomial $\mathcal{P}$ such that $\mathcal{P}(f',\ldots,f^{(k)})$ is not identically zero. We denote by $N_E(r)$ the Nevanlinna counting function associated to E, i. e. $N_E(r) = \displaystyle\int_0^r \frac{dt}{t}\mathrm{card}\big(E\cap(\sigma < r)\big)$. Then we have*

$$(23) \qquad\qquad \liminf_r \frac{N_E(r)}{T_f(r)} > 0.$$

Proof. By the non-vanishing of the $\mathcal{P}(f',\ldots,f^{(k)})$ and the proof of the vanishing theorem we infer that the inequality

$$T_f(r) \le C\big(\log T_f(r) + \mathcal{O}(\log r) + \mathfrak{X}_\sigma(r)\big)$$

holds as $r \to \infty$. On the other hand, we have $\mathfrak{X}_\sigma(r) = N_E(r)$, so if the relation (23) is not verified, then we obtain a contradiction. $\square$

4.1. **A few examples.** At the end of this paragraph we will discuss some examples of parabolic surfaces; we will try to emphasize in particular the properties of the function $\mathfrak{X}_\sigma(r)$.

(1) Let $E \subset \mathbb{C}$ be a finite subset of the complex plane. Then clearly we have $\mathfrak{X}_\sigma(r) = \mathcal{O}(\log r)$ for $\mathcal{Y} := \mathbb{C}\setminus E$.

(2) We treat next the case of $\mathcal{Y} := \mathbb{C}\setminus E$, where $E = (a_j)_{j\ge 1}$ is a closed, countable set of points in $\mathbb{C}$. As we will see, in this case it is natural

to use the Jensen formula without assuming that the support of the measure $dd^c \log \sigma$ is compact, see Remark 3.2.

Let $(r_j)_{j \geq 1}$ be a sequence of positive real numbers, such that the Euclidean disks $\mathbb{D}(a_j, r_j)$ are disjoints. As in the preceding example, we define the exhaustion function σ such that

$$\log \sigma = \log^+ |z| + \sum_{j \geq 1} \log^+ \frac{r_j}{|z - a_j|};$$

the difference here is that $d\mu_r$ is no longer a probability measure. However, we have the following inequality

$$\frac{1}{\sigma} \left| \frac{\partial \sigma}{\partial z} \right| \leq \left(\sum_{j \geq 1} \frac{1}{|z - a_j|} \chi_{\mathbb{D}(a_i, 1)} + \frac{1}{|z|} \chi_{(|z| > 1)} \right)$$

which holds true on $\mathcal{Y}$. On the parabolic sphere $S(r)$ we have $|z| < r$ and $|z - a_j| > \dfrac{r_j}{r}$ so that we obtain

$$\mathfrak{X}_\sigma(r) \leq \log r + \sum_{\sigma(a_j + r_j e^{i\theta_j}) < r} \log \frac{r}{r_j} \leq$$

$$\leq \log r + \sum_{|a_j| < r} \log \frac{r}{r_j} := \log r + N_{(a_j)_{j \geq 1}}(r).$$

Therefore, we can bound the Euler characteristic by the counting function for $(a_j)_{j \geq 1}$.

Remark 4.5. Let $\mathcal{Y}$ be a parabolic Riemann surface, with an exhaustion σ normalized as in (1). In more abstract terms, the quantity $\mathfrak{X}_\sigma(r)$ can be estimated along the following lines, by using the same kind of techniques as in the proof of the logarithmic derivative lemma.

$$\mathfrak{X}_\sigma(r) = \int_{S(r)} \log^+ \left| \frac{\partial \sigma}{\partial z} \right|^2 d\mu_r \leq$$

$$\leq \frac{2}{\varepsilon} \log^+ \int_{S(r)} \left| \frac{\partial \sigma}{\partial z} \right|^\varepsilon d^c \log \sigma = \frac{2}{\varepsilon} \log^+ \frac{1}{r} \int_{S(r)} \left| \frac{\partial \sigma}{\partial z} \right|^\varepsilon d^c \sigma \leq$$

$$\leq \frac{2}{\varepsilon} \log^+ \frac{1}{r} \frac{d}{dr} \int_{B(r)} \left| \frac{\partial \sigma}{\partial z} \right|^\varepsilon d\sigma \wedge d^c \sigma \leq$$

$$\leq \frac{2}{\varepsilon} \log^+ \frac{1}{r} \left(\int_{B(r)} \left| \frac{\partial \sigma}{\partial z} \right|^{2+\varepsilon} dz \wedge d\bar{z} \right)^{1+\delta} \leq$$

$$\leq \frac{2}{\varepsilon} \log^+ r^\delta \left(\int_{B(r)} \frac{1}{r} \left| \frac{\partial \sigma}{\partial z} \right|^{2+\varepsilon} dz \wedge d\bar{z} \right)^{1+\delta} \leq$$

$$\leq \frac{2\delta}{\varepsilon} \log^+ r + \frac{(1+\delta)^2}{\varepsilon} \log^+ \left(\int_1^r \frac{dt}{t} \int_{B(t)} \left| \frac{\partial \sigma}{\partial z} \right|^{2+\varepsilon} dz \wedge d\bar{z} \right).$$

On the other hand, we remark that we have

$$(24) \qquad \int_{\mathbb{B}(t)} d\sigma \wedge d^c\sigma = t^2;$$

this can be verified e.g. by considering the derivative of the left hand side of the expression (24) above. However, this does not means that the surface $\mathcal{Y}$ is of finite mean Euler characteristic, since we cannot take $\varepsilon = 0$ in our previous computations–and as the example (2) above shows it, there is a good reason to that.

5. Bloch Theorem

Let T be a complex torus, and let $X \subset T$ be an irreducible analytic set. In some sense, the birational geometry of X was completely understood since the work of Ueno [29]. He has established the following result.

Theorem 5.1. [29] *Let X be a subvariety of a complex torus T. Then there exist a complex torus $T_1 \subset T$, a projective variety W and an abelian variety A such that*

(1) *We have $W \subset A$ and W is a variety of general type;*
(2) *There exists a dominant (reduction) map $\mathcal{R} : X \to W$ whose general fiber is isomorphic to T_1.*

Thus, via this theorem the study of an arbitrary submanifold X of a torus is reduced to the case where X is of general type.

We analyze next the case where X is the Zariski closure of a *parabolic Riemann surface of finite Euler characteristic*. The next result can be seen as the complete analogue of the classical theorem of Bloch [2]

Theorem 5.2. *Let $X \subset T$ be a submanifold of a complex torus T. We assume that X is the Zariski closure of a parabolic Riemann surface of finite mean Euler characteristic, whose area is infinite. Then up to a translation, X is a torus.*

Proof. We will follow the approach in [2], [24]; starting with $V := T_X$, we consider the tower of directed manifolds $(T_k, V_k)_{k\geq 0}$, whose construction was recalled in the 1st part of this article. Since T is flat, we have

$$T_k = T \times \mathcal{R}_{n,k}$$

where we recall that here $\mathcal{R}_{n,k}$ is the "universal" rational homogeneous variety, cf. [8]. The curve $f : \mathcal{Y} \to T$ lifts to T_k, as already explained; we denote by

$$f_k : \mathcal{Y} \to T_k$$

the lift of f.

Let X_k be the Zariski closure of the image of f_k, and let

$$\tau_k : X_k \to \mathcal{R}_{n,k}$$

be the composition of the injection $X_k \mapsto T_k$ with the projection on the second factor $T_k \to \mathcal{R}_{n,k}$.

Then either the generic fibers of τ_k are finite for all k, or there exists a value of k for which they are positive dimensional. The geometric counterpart of each of these eventualities in analyzed along the two following statements, due to A. Bloch.

Proposition 5.3. *We assume that for each $k \geq 1$ the fibers of τ_k are positive dimensional. Then the dimension of the subgroup A_X of T defined by*

$$A_X := \{a \in T : a + X = X\}$$

is strictly positive.

Proof. We fix a point $x_0 \in \mathcal{Y}$ such that $f_k(x_0)$ is a regular point of X_k, for each $k \geq 1$. It is clear that such a point exists, since the image of the curve f_k is dense in X_k, hence the inverse image of the singular set of X_k by f_k is at most countable. By the same argument, we can assume that the fiber of the map τ_k through $f_k(x_0)$ is "generic", meaning that it is positive dimensional.

Hence by this choice of the point x_0 the fiber

$$F_{k,0} := \tau_k^{-1}\tau_k\big(f_k(x_0)\big)$$

is positive dimensional, for each $k \geq 1$. Thus, there exists a curve $\gamma_k : (\mathbb{C}, 0) \to X_k$ such that

$$\gamma_k(t) = \big(z(t), \lambda\big)$$

such that $f_k(x_0)$ corresponds to the couple $\big(z(0), \lambda\big)$ according to the decomposition of T_k. In particular, this shows that the dimension of the analytic set

$$\Xi_k := \{a \in T : f_k(x_0) \in J^k(X) \cap J^k(a + X)\}$$

is strictly positive. This is so because for each t close enough to zero we can define an element a_t by the equality

$$f_k(x_0) = a_t + z(t)$$

and thus $a_t \in \Xi_k$ since the curve $t \to z(t)$ lies on X.

Next we see that we have the sequence of inclusions

$$\Xi_1 \supset \Xi_2 \supset \ldots \supset \Xi_k \supset \Xi_{k+1} \supset \ldots$$

therefore by Noetherian induction there exists a large enough positive integer k_0 such that

$$\Xi_k = \Xi_{k+1}$$

for each $k \geq k_0$, and such that the dimension of Ξ_k is strictly positive. But this means that for every $a \in \Xi_{k_0}$, the image of the curve f belongs to the translation $a + X$ of the set X (because this curve is tangent to an infinite order to the a-translation of X). Thus we will have $X = X + a$ given that the curve f is Zariski dense. The proposition is proved. $\square$

In the following statement we analyze the other possibility.

Proposition 5.4. *Let k be a positive integer such that the map*

$$\tau_k : X_k \to \mathcal{R}_{n,k}$$

has finite generic fibers. Then there exists a jet differential $\mathcal{P}$ of order k with values in the dual of an ample line bundle, and whose restriction to X_k is non-identically zero.

Proof. The proof relies on the following claim: *the restriction to X_k of the tautological bundle $\mathcal{O}_k(1)$ associated to T_k is big* This condition is equivalent to the fact that

$$H^0\big(X_k, \mathcal{O}_k(m) \otimes A^{-1}\big) \neq 0.$$

Indeed, in the first place we know that $\mathcal{O}_k(1) = \tau_k^{\star}\mathcal{O}(1)$ where $\mathcal{O}(1)$ is the tautological bundle on $\mathcal{R}_{n,k}$. It also turns out that $\mathcal{O}(1)$ is big (cf. [24]), and these two facts combined with the fact that the generic fibers of τ_k are of dimension zero proves our claim. $\qquad\square$

Theorem 5.2 follows from the two statements above: let

$$\varphi : \mathcal{Y} \to T$$

be a non-constant holomorphic map from a parabolic surface $\mathcal{Y}$ of finite mean Euler characteristic into a complex torus T. We denote by X the Zariski closure of its image. Thanks to the result of Ueno we can consider the reduction map $R : X \to W$ associated to X. We claim that under the hypothesis of Theorem 5.2 the base W is reduced to a point.

If this is not the case, then we can assume that X is of general type. By the vanishing theorem 4.1 we see that the hypothesis of Proposition 5.3 will never be verified, for any $k \geq 1$. Hence the hypothesis of the Proposition 5.4 are verified, and so X will be invariant by a positive dimensional sub-torus of T. Since X is assumed to be a manifold of general type, this cannot happen, so our result is proved. $\qquad\square$

Remark 5.5. The same arguments show that Theorem 5.2 still holds in a more general context: it is enough to assume that we have $\mathfrak{X}_\sigma(r) = o(T_f(r))$. We leave the details to the interested reader.

6. Parabolic Curves Tangent to Holomorphic Foliations

6.1. **Nevanlinna's Currents Associated to a Parabolic Riemann Surface.** Let $(\mathcal{Y}, \sigma)$ be a parabolic Riemann surface. We fix a Kähler metric ω on X, and let $\varphi : \mathcal{Y} \to X$ be holomorphic map. For an open set $S \subset \mathcal{Y}$ with smooth boundary we denote by

$$(25) \qquad \qquad \|\varphi(S)\| := \int_S \varphi^{\star}\omega$$

We define the (normalized) integration current

$$T_S := \frac{[\varphi_\star(S)]}{\|\varphi(S)\|}$$

which has bidimension (1,1) and total mass equal to 1. In general, the current T_S is positive but not closed. However, it may happen that for some accumulation point

$$T_\infty = \lim_k T_{S_k}$$

is a closed current, when $S_k \to \mathcal{Y}$.

Theorem 6.1. [6] *Let $(\mathcal{Y}, \sigma)$ be a parabolic Riemann surface; we denote by $S_j := \mathbb{B}(r_j)$ the parabolic balls of radius r_j, where $(r_j)_{j \geq 1}$ is a sequence of real numbers such that $r_j \to \infty$. We consider a holomorphic map $\varphi : \mathcal{Y} \to X$ of infinite area. Then there exists at least one accumulation point of the sequence of currents*

$$T_j := \frac{[\varphi_\star(S_j)]}{\|\varphi(S_j)\|}$$

which is a closed (positive) current, denoted by T_∞.

Proof. The arguments presented here are a quantitative version of the ones in [6].

We denote by $u := \log \sigma$ the log of the exhaustion function; by hypothesis, the measure $dd^c u$ has compact support. We define a function H on $\mathcal{Y}$ by the equality

$$\varphi^\star \omega := H \, du \wedge d^c u.$$

Let $A(t) := \int_{(u<t)} \varphi^\star \omega$ be the area of the parabolic ball of radius e^t with respect to the inverse image of the metric ω, and let

$$L(t) := \int_{(u=t)} \sqrt{H} d^c u;$$

geometrically, it represents the length of the parabolic sphere of radius e^t measured with respect to the metric induced by $H \, du \wedge d^c u$.

By Cauchy-Schwarz inequality we have

$$(26) \qquad L(t)^2 \leq \int_{(u=t)} H d^c u \int_{\mathbb{B}(t)} dd^c u = \int_{(u=t)} H d^c u$$

because $\int_{\mathbb{B}(t)} dd^c u = 1$. On the other hand we have

$$(27) \qquad \frac{d}{dt} A(t) = \int_{(u=t)} H d^c u$$

thus combining the inequalities (26) and (27) we obtain

$$L(t)^2 \leq \frac{d}{dt} A(t).$$

For every positive ε we have $\dfrac{d}{dt}A(t) \leq A^{1+2\varepsilon}(t)$ for any t belonging to the complement of a set Λ_ε of finite measure; as a result we infer that the inequality

(28) $$L(t) \leq A(t)^{1/2+\varepsilon}$$

holds true for any $t \in \mathbb{R}_+ \setminus \Lambda_\varepsilon$. In particular, this implies the existence of a current as an accumulation point of T_j, and the theorem in proved.

$\square$

By using similar arguments, combined with a few results by B. Kleiner [12] and B. Saleur [20] we obtain a result in the direction of the conjecture in paragraph 1. Let $E \subset \mathbb{D}$ be a polar subset of the unit disk, and let $f : \mathbb{D} \setminus E \to M$ be a holomorphic map with values in a compact, Kobayashi hyperbolic manifold M. As it is well-known (cf [28]) the set $\mathbb{D} \setminus E$ carries a local exhaustion function σ such that $u := \log \sigma$ is harmonic. Let $\chi(t)$ be the Euler characteristic of the domain $(\sigma < t)$; we define $A(t) := \int_{\sigma < t} f^\star \omega$.

Corollary 6.2. *If we have* $\dfrac{|\chi(t)|}{A(t)} \to 0$ *as* $t \to \infty$, *then the map* f *admits an extension through* E.

Proof. The argument relies heavily on the following result, which uses a technique due to B. Kleiner [12].

Theorem 6.3. [20] *Let* (M, ω) *be a compact Kobayashi hyperbolic manifold. There exist two constants* C_1, C_2 *such that for every holomorphic map* $f : \Sigma \to M$ *defined on a Riemann surface with smooth boundary* $\partial\Sigma$ *we have*

(29) $$\mathrm{Area}\big(f(\Sigma)\big) \leq C_1 |\chi(\Sigma)| + C_2 \mathrm{L}\big(f(\partial\Sigma)\big).$$

Coming back to the domains $(\sigma < t)$, inequality (28) shows that we have

$$L(t) \leq A(t)^{1/2+\varepsilon}.$$

When combined with the inequality (29) of the preceding theorem, we get

$$A(t) \leq C_1 |\chi(t)| + C_2 A(t)^{1/2+\varepsilon}.$$

Given the hypothesis concerning the growth of the Euler characteristic, we infer the existence of a constant C_3 such that

$$A(t) \leq C_3$$

and for the rest of the proof we will follow the argument given in Proposition 1.1. Indeed, the current associated to the graph $\Gamma_f \subset \mathbb{D} \setminus E \times M$ of the map f has finite mass near the polar set $E \times M \subset \mathbb{D} \times M$. By using Skoda-ElMir extension theorem (for a simple proof, see [23]), the current $[\Gamma]$ extends to $\mathbb{D} \times M$ with no mass on $E \times M$. But this

implies that the graph Γ extends as an analytic subset of $\mathbb{D} \times M$; in other words, f extends as a meromorphic map. It follows that in fact f is holomorphic, since f is defined on a 1-dimensional disk. $\qquad \square$

Remark 6.4. We obtain the same results if $(\mathcal{Y}, \sigma)$ is not necessarily parabolic, but we have to assume that we have

$$\int_{\mathbb{B}(t)} dd^c u = \mathcal{O}\big(A(t)^{1-\varepsilon}\big)$$

for some $\varepsilon > 0$.

Remark 6.5. We can also consider a version of the current T_∞ in the above statement. For each $r > 0$, the expression

$$T_r := \frac{1}{T_f(r)} \int_0^r \frac{dt}{t} [\varphi_\star(\mathbb{B}_t)]$$

defines a positive current on X. One can show that there exists a sequence r_k such that the limit points of (T_{r_k}) are positive and closed. Any such limit will be called a Nevanlinna current associated to φ, and will be denoted by $T[f]$. If we consider a lift of f to $\mathbb{P}(T_X)$, we get -with the same construction- a current denoted $T[f']$. Let $\pi : \mathbb{P}(T_X) \to X$ be the projection; then we can assume that $\pi_\star(T[f']) = T[f]$, as we will see later.

6.2. **Metrics on the tangent bundle of a holomorphic foliation by disks.** Let $\mathcal{F}$ be a 1-dimensional holomorphic foliation (possibly with singularities) on a manifold X. This means that we are given a finite open covering $(U_\alpha)_\alpha$ of X with coordinates charts, and a family of associated vector fields $v_\alpha \in H^0(U_\alpha, T_X|_{U_\alpha})$ such that there exists $g_{\alpha\beta} \in \mathcal{O}^\star(U_\alpha \cap U_\beta)$ with the property that

$$v_\alpha = g_{\alpha\beta} d\pi_{\alpha\beta}(v_\beta)$$

on the intersection of U_α and U_β; here we denote by $(\pi_{\alpha\beta})$ the transition functions of X, corresponding to the covering (U_α). The (analytic) set of zeros of (v_α) is supposed to have codimension at least two, and it is denoted by $\mathcal{F}_{\text{sing}}$. The functions $(g_{\alpha\beta})$ verify the cocycle property, and they define the cotangent bundle corresponding to the foliation $\mathcal{F}$, denoted by $T_\mathcal{F}^\star$.

From the global point of view, the family of vector fields $(v_\alpha)_\alpha$ corresponds to a section V of the vector bundle $T_X \otimes T_\mathcal{F}^\star$.

Let ω be a metric on X (which is allowed to be singular). We will show next that ω induces a metric h_s on the tangent bundle $T_\mathcal{F}$ (as we will see, the induced metric may be singular even if the reference metric ω is smooth).

Let $x \in X$, and let $\xi \in T_{\mathcal{F},x}$ be an element of the fiber at x of the tangent bundle corresponding to $\mathcal{F}$. Then we define its norm as follows

(30) $$|\xi|_{h_s}^2 := |V_x(\xi)|_\omega^2.$$

The local weights of the metric h_s on the set U_α are described as follows. Let $z^1, \ldots z^n$ be local coordinates on X centered at x. We write

$$v_\alpha = \sum_{i=1}^n a_\alpha^i \frac{\partial}{\partial z^i}$$

where a_α^i are holomorphic functions defined on U_α; we assume that their common zero set has codimension at least 2 in X.

The local weight ϕ_α of the metric h_s is given by the expression

$$\phi_\alpha = -\log \sum_{i,j} a_\alpha^i \overline{a_\alpha^j} \omega_{i\bar{j}}$$

where $\omega_{i\bar{j}}$ are the coefficients of the metric ω with respect to the local coordinates $(z^j)_{j=1,\ldots n}$. Indeed, let θ be a local trivialization of the bundle $T_{\mathcal{F}}$. Then according to the formula (30) we have

$$(31) \qquad |\xi|_{h_s}^2 = \Big(\sum_{i,j} a_\alpha^i \overline{a_\alpha^j} \omega_{i\bar{j}} \Big) |\theta(\xi)|^2,$$

which clarifies the formula for the local weight of h_s.

In some cases, the previous construction can be further refined, as follows.

Let $B = \sum_{j=1}^N W_j$ be a divisor on X. We assume that the following requirements are fulfilled.

(a) At each point of $x \in \mathrm{Supp}(B)$ the local equations of the analytic sets

$$(W_j, x)_{j=1,\ldots,k}$$

can be completed to a local coordinate system centered at x. Here we denote by k the number of hypersurfaces in the support of B containing the point x (and we make a slight abuse of notation). In the language of algebraic geometry, one calls such a pair (X, B) *log-smooth*.

(b) We assume that each component W_j of $\mathrm{Supp}(B)$ is invariant by the foliation $\mathcal{F}$.

If the condition (a) above is verified, then we recall that the *logarithmic tangent bundle* of (X, B) is the subsheaf of $\mathcal{O}(T_X)$ defined locally as follows.

Let $U \subset X$ be a coordinate open set. We assume that we have a coordinate system $z_1, \ldots, z_n$ on U, such that

$$\mathrm{Supp}(B) \cap U = \big(z_1 z_2 \ldots z_k = 0 \big).$$

Then the logarithmic tangent bundle $T_X\langle B \rangle$ corresponding to the pair (X, B) is the subsheaf of T_X whose local sections on U are given by

$$v = \sum_{j=1}^k v_j z_1 \frac{\partial}{\partial z_1} + \sum_{p=k+1}^n v_p \frac{\partial}{\partial z_p}.$$

In other words, the local sections of $T_X\langle B\rangle|_U$ are the vector fields of $T_X|_U$ which are tangent to B when restricted to B. We note that the $T_X\langle B\rangle$ is a vector bundle of rank n, and the local model of a hermitian metric on it is given by

$$\omega_U \equiv \sqrt{-1}\sum_{j=1}^{k}\frac{dz_j \wedge d\overline{z}_j}{|z_j|^2} + \sum_{j=k+1}^{n} dz_j \wedge d\overline{z}_j$$

i.e. a metric with logarithmic poles along B. So, we have

$$|v|^2_{\omega_U} = \sum_j |v_j|^2.$$

From a global point of view, a hermitian metric $\omega_{X,B}$ on $T_X\langle B\rangle$ can be written as

$$\omega_{X,B}|_U = \sqrt{-1}\sum_{j,i=1}^{k}\omega_{j\overline{i}}\frac{dz_j \wedge d\overline{z}_i}{z_j\overline{z}_i} + 2\mathrm{Re}\sqrt{-1}\sum_{j>k\geq i}\omega_{j\overline{i}}\frac{dz_j \wedge d\overline{z}_i}{\overline{z}_i}+$$
$$+\sqrt{-1}\sum_{j,i\geq k+1}\omega_{j\overline{i}}dz_j \wedge d\overline{z}_i$$

where the Hermitian matrix $(\omega_{j\overline{i}})$ is positive definite.

If moreover the condition (b) is fulfilled, then the family of vector fields v_α defining the foliation $\mathcal{F}$ can be seen as a global section V_B of the bundle

$$T_X\langle B\rangle \otimes T_{\mathcal{F}}^{\star}$$

and we have the following version of the metric constructed above. For each vector $\xi \in T_{\mathcal{F},x}$ we define

$$\|\xi\|^2_{h_{s,B}} := |V_{B,x}(\xi)|^2_{\omega_{X,B}}.$$

As in the case discussed before, we can give the local expression of the metric on $T_{\mathcal{F}}$, as follows. Let

$$v_\alpha = \sum_{i=1}^{k} a_\alpha^i z_i \frac{\partial}{\partial z^i} + \sum_{i=k+1}^{n} a_\alpha^i \frac{\partial}{\partial z^i}$$

be a logarithmic vector field trivializing the tangent bundle of the foliation on a coordinate set U_α. Then the local weight $\phi_{\alpha,B}$ of the metric $h_{s,B}$ induced by the metric $\omega_{X,B}$ is given by the expression

$$(32) \qquad\qquad \phi_{\alpha,B} = -\log \sum_{i,j} a_\alpha^i \overline{a_\alpha^j}\omega_{i\overline{j}}.$$

In particular we see that this weight is *less singular than* the one in the expression (31). This will be crucial in the applications.

As far as the curvature current is concerned, the metric h_s as well as its logarithmic variant $h_{s,B}$ seem useless: given the definition above,

its associated curvature is neither positive nor negative. Indeed, ϕ_α may tend to infinity along the singular set of the foliation $\mathcal{F}$, and it may tend to minus infinity along the singularities of the metric ω. However, we will present a few applications of this construction in the next paragraphs.

6.3. **Degree of currents associated to parabolic Riemann surfaces on the tangent bundle of foliations.** Let (X, ω) be a compact complex hermitian manifold, and let $\mathcal{F}$ be a holomorphic foliation on X of dimension 1. Let $f : \mathcal{Y} \to X$ be a holomorphic map, where $(\mathcal{Y}, \sigma)$ is a parabolic Riemann surface tangent to $\mathcal{F}$, and let

$$T[f] := \lim_r T_r[f]$$

be a Nevanlinna current associated to it.

In this section we will derive a lower bound in arbitrary dimension for the quantity

$$\int_X T[f] \wedge c_1(T_\mathcal{F}),$$

in the same spirit as [16], [3]. Prior to this, we introduce a few useful notations.

Let $\mathcal{J}_{\mathcal{F}_s}$ be the coherent ideal associated to the singularities of $\mathcal{F}$; this means that locally on U_α the generators of $\mathcal{J}_{\mathcal{F}_s}$ are precisely the coefficients (a_α) of the vector v_α defining $\mathcal{F}$, i.e.

$$v_\alpha = \sum_{i=1}^n a_\alpha^i \frac{\partial}{\partial z^i}.$$

As we have already mentioned in paragraph 3, there exists a function ψ_{sing} defined on X and having the property that locally on each open set U_α we have

$$\psi_{\text{sing}} \equiv \log |v_\alpha|_\omega^2$$

modulo a bounded function.

Let $B = \sum_j W_j$ be a divisor on X, such that the pair (X, B) satisfies the requirements (a) and (b) in the preceding paragraph. Then the local generator of $T_\mathcal{F}$ can be written in this case as

$$(33) \qquad v_{\alpha,B} = \sum_{i=1}^k a_\alpha^i z_i \frac{\partial}{\partial z^i} + \sum_{i=k+1}^n a_\alpha^i \frac{\partial}{\partial z^i}.$$

We denote by $\mathcal{J}_{\mathcal{F}_{s,B}}$ the coherent ideal defined by the functions (a_α^i) in (33). Then we have

$$\mathcal{J}_{\mathcal{F}_s} \subset \mathcal{J}_{\mathcal{F}_{s,B}},$$

and the inclusion may be strict. We denote by $\psi_{\text{sing},B}$ the associated function.

The counting function with respect to the ideal defined by $\mathcal{F}_{\text{sing}}$ will be denoted

$$N_{f,\mathcal{J}_{\mathcal{F}_{\text{s}}}}(r) = \sum_{0<\sigma(t_j)<r} \nu_j \log \frac{r}{\sigma(t_j)} = \int_0^r \frac{dt}{t} \int_{B(t)} \left(dd^c \psi_{\text{sing}} \circ f\right)_s,$$

with $f(t_j) \in \text{Supp}(\mathcal{F}_{\text{sing}})$, and the subscript s above denotes the singular part od the measure in question. Its normalized expression will be written as

$$(34) \qquad \nu^T(f, \mathcal{F}_{\text{sing}})(r) := \frac{1}{T_f(r)} N_{f,\mathcal{J}_{\mathcal{F}_{\text{s}}}}(r).$$

The upper limit of the expression above will be denoted by

$$\nu^T(f, \mathcal{F}_{\text{sing}}) := \overline{\lim}_r \nu^T(f, \mathcal{F}_{\text{sing}})(r).$$

If Ξ is an arbitrary analytic subset of X, we will denote by $\nu^T(f, \Xi)$ the quantity defined in a similar manner by using the function ψ_Ξ instead of ψ_{sing}.

We define the counting function with respect to $\mathcal{F}_{\text{s,B}}$ as

$$N_{f,\mathcal{J}_{\mathcal{F}_{\text{s,B}}}}(r) = \sum_{0<\sigma(t_j)<r} \nu_j \log \frac{r}{\sigma(t_j)},$$

with $f(t_j) \in \text{Supp}(\mathcal{F}_{\text{s,B}})$, together with it normalized expression

$$(35) \qquad \nu^T(f, \mathcal{F}_{\text{sing}}, B)(r) := \frac{1}{T_f(r)} N_{f,\mathcal{J}_{\mathcal{F}_{\text{s,B}}}}(r).$$

The following truncated counting function will appear in our next computations:

$$(36) \qquad N^{(1)}_{f,\mathcal{J}_{\mathcal{F}_{\text{s}}} \cap B}(r) = \sum_{0<\sigma(t_j)<r, f(t_j)\in B} \log \frac{r}{\sigma(t_j)}.$$

and let $\nu_1^T(f, \mathcal{F}_{\text{sing}} \cap B)$ be its normalized upper limit.

We also recall the definition of

$$m^T(f, \mathcal{F}_{\text{sing}}) := \overline{\lim}_r \frac{1}{T_f(r)} \int_{S(r)} -\psi_{\text{sing}} \circ f d\mu_r$$

which is the (normalized) asymptotic proximity function for f with respect to the ideal $\mathcal{J}_{\mathcal{F}_{\text{s}}}$, together with its logarithmic variant

$$m^T(f, \mathcal{F}_{\text{sing,B}}) := \overline{\lim}_r \frac{1}{T_f(r)} \int_{S(r)} -\psi_{\text{sing,B}} \circ f d\mu_r.$$

The ramification function corresponding to f is

$$(37) \qquad R_f(r) = \sum_{0<\sigma(t'_j)<r} \mu_j \log \frac{r}{\sigma(t'_j)};$$

so that μ_j is the vanishing order of f' at t'_j. The curve f is tangent to $\mathcal{F}$, therefore for each open set $\Omega \subset \mathcal{Y}$ such that $f(\Omega) \subset U_\alpha$ for some index α we can write

$$f'(t) = \lambda(t)v_{\alpha,f(t)}$$

for some function λ which is holomorphic on $\Omega \setminus f^{-1}(\mathcal{F}_{\text{sing}})$. We remark that if $f(t_j) \notin \mathcal{F}_{\text{sing}}$, then the multiplicities μ_j above coincide with the vanishing order of λ evaluated at the critical points of f. It will be useful in what follows to have the decomposition

$$R_f(r) := M_f(r) + N_f(\text{Ram}, r)$$

according to the possibility that the critical value $f(t'_j)$ of f belongs to the set $\mathcal{F}_{\text{sing}}$ or not. We are using the notations

$$N_f(\text{Ram}, r) := \sum_{0 < \sigma(t'_j) < r, f(t'_j) \notin \mathcal{F}_{\text{sing}}} \mu_j \log \frac{r}{\sigma(t'_j)},$$

and

$$M_f(r) := \sum_{0 < \sigma(t'_j) < r, f(t'_j) \in \mathcal{F}_{\text{sing}}} \mu_j \log \frac{r}{\sigma(t'_j)}.$$

Finally, the asymptotic normalized ramification of f is denoted by

$$\overline{\nu}(\text{Ram}, f) := \overline{\lim}_r \frac{1}{T_f(r)} R_f(r).$$

We establish next the following general result, which gives an estimate of the quantity $\int_X T[f] \wedge c_1(T_{\mathcal{F}})$ in terms of the intersection of f with the singularities of the foliation. Our statement is a quantitative expression of the fact that the derivative of f can be seen as a meromorphic section of $f^\star T_{\mathcal{F}}$.

Theorem 6.6. *Let $(X, \mathcal{F})$ be a compact complex manifold endowed with a holomorphic 1-dimensional foliation $\mathcal{F}$. Let $(\mathcal{Y}, \sigma)$ be a parabolic Riemann surface of finite Euler characteristic, and let $f : \mathcal{Y} \to X$ be a holomorphic map whose image is tangent to $\mathcal{F}$. We assume that the image of f is Zariski dense. Then we have*

$$(\star) \quad \int_X T[f] \wedge c_1(T_{\mathcal{F}}) \geq -\nu^T(f, \mathcal{F}_{\text{sing}}) - m^T(f, \mathcal{F}_{\text{sing}}) + \overline{\nu}(\text{Ram}, f)$$

Proof. Let ω be a smooth metric on X, and let h_s be the metric induced on $T_{\mathcal{F}}$ by the procedure described in the preceding sub-section.

Let $r > t > 0$; we begin by evaluating the quantity

$$\int_X T_r[f] \wedge \Theta_{h_s}(T_{\mathcal{F}})$$

and to this end we introduce the notations

$$\mathbb{B}_{\alpha,\varepsilon}(t) := \{z \in \mathbb{B}(t) : f(z) \in U_\alpha, \text{ and } d_\omega(f(z), \mathcal{F}_{\text{sing}}) \geq \varepsilon\}$$

as well as its complement set inside the parabolic ball of radius t

$$\mathbb{B}^c_{\alpha,\varepsilon}(t) := \mathbb{B}_\alpha(t) \setminus \mathbb{B}_{\alpha,\varepsilon}(t)$$

where $\mathbb{B}_\alpha(t) := \mathbb{B}(t) \cap f^{-1}(U_\alpha)$. Let (ρ_α) be a partition of unit corresponding to the cover (U_α).

In the definition of the metric h_s we use the smooth Kähler metric ω we have fixed on X, and we have.

$$\int_X T_r[f] \wedge \Theta_{h_s}(T_\mathcal{F}) = -\sum_\alpha \frac{1}{T_f(r)} \int_1^r \frac{dt}{t} \int_{B^c_{\alpha,\varepsilon}(t)} \rho_\alpha(f) f^\star dd^c \log |v_\alpha|^2_\omega$$

$$-\sum_\alpha \frac{1}{T_f(r)} \int_1^r \frac{dt}{t} \int_{B_{\alpha,\varepsilon}(t)} \rho_\alpha(f) f^\star dd^c \log |v_\alpha|^2_\omega.$$

We remark that for each $t < r$ and for each index α we have

$$\int_{B_{\alpha,\varepsilon}(t)} \rho_\alpha(f) f^\star dd^c \log |v_\alpha|^2_\omega = \int_{B_{\alpha,\varepsilon}(t)} \rho_\alpha(f) dd^c \log |f'|^2_\omega$$

$$- \sum_{0<\sigma(t'_j)<t, f(t'_j)\notin\mathcal{F}_{\mathrm{sing}}} \rho_\alpha\big(f(t'_j)\big) \mu_j \delta_{t'_j}.$$

The equality in the formula above is due to the fact that locally at each point in the complement of the set $\mathcal{F}_{\mathrm{sing}}$ we have

$$f'(t) = \lambda v_{f(t)}$$

for some holomorphic function λ. We also remark that the relation above is valid for any $\varepsilon > 0$, and if we let $\varepsilon \to 0$, we have

$$\lim_{\varepsilon\to 0} \int_1^r \frac{dt}{t} \int_{\mathbb{B}_{\alpha,\varepsilon}(t)} \rho_\alpha(f) dd^c \log |f'|^2_\omega = \int_1^r \frac{dt}{t} \int_{\mathbb{B}_\alpha(t)} \rho_\alpha(f) dd^c \log |f'|^2_\omega$$

$$- \sum_{f(t'_j)\in\mathcal{F}_{\mathrm{sing}}} \rho_\alpha\big(f(t_j)\big) \mu_j \log \frac{r}{\sigma(t_j)}$$

as well as

$$\lim_{\varepsilon\to 0} \int_1^r \frac{dt}{t} \int_{\mathbb{B}^c_{\alpha,\varepsilon}(t)} \rho_\alpha(f) f^\star dd^c \log |v_\alpha|^2_\omega = \sum_{t_j\in\mathbb{B}_\alpha(t)} \rho_\alpha\big(f(t_j)\big) \nu_j \log \frac{r}{\sigma(t_j)}.$$

Therefore we obtain

$$\langle T_r[f], \Theta_{h_s}(T_\mathcal{F})\rangle \geq -\nu^T(f, \mathcal{F}_{\mathrm{sing}})(r) + \frac{1}{T_f(r)} M_f(r)$$

$$-\frac{1}{T_f(r)} \int_1^r \frac{dt}{t} \int_{\mathbb{B}(t)} dd^c \log |f'|^2_\omega + \frac{1}{T_f(r)} N_f(\mathrm{Ram}, r) =$$

$$= -\nu^T(f, \mathcal{F}_{\mathrm{sing}})(r) + \frac{1}{T_f(r)} R_f(r) -$$

$$\frac{1}{T_f(r)} \int_{S(r)} \log |f'|^2_\omega d\mu_r.$$

Let $h := h_s \exp(-\psi_{\text{sing}})$; it is a metric with *bounded* weights of $T_{\mathcal{F}}$, hence we can use it in order to compute the quantity we are interested in, namely

$$\int_X T[f] \wedge c_1(T_{\mathcal{F}}) = \lim_r \langle T_r[f], \Theta_h(T_{\mathcal{F}}) \rangle.$$

We recall that we have the formula

$$\Theta_h(T_{\mathcal{F}}) = \Theta_{h_s}(T_{\mathcal{F}}) + dd^c \psi_{\text{sing}},$$

so as a consequence we infer that we have

$$\int_X T_r[f] \wedge \Theta_h(T_{\mathcal{F}}) = \langle T_r[f] \wedge \Theta_{h_s}(T_{\mathcal{F}}) \rangle + \frac{1}{T_f(r)} \int_{S(r)} \psi_{\text{sing}} \circ f d\mu_r + o(1).$$

By combining the relations above we infer that we have

$$\int_X T_r[f] \wedge \Theta_h(T_{\mathcal{F}}) \geq - \nu^T(f, \mathcal{F}_{\text{sing}})(r) + \frac{1}{T_f(r)} \int_{S(r)} \psi_{\text{sing}} \circ f d\mu_r$$

$$+ \frac{1}{T_f(r)} R_f(r) - \frac{1}{T_f(r)} \int_{S(r)} \log |f'|_\omega^2 d\mu_r + o(1).$$

By the logarithmic derivative lemma, the last term of the preceding relation tends to a positive value, as $r \to \infty$, hence we obtain

$$\int_X T_r[f] \wedge \Theta_h(T_{\mathcal{F}}) \geq -\nu^T(f, \mathcal{F}_{\text{sing}}) - m^T(f, \mathcal{F}_{\text{sing}}) + \overline{\nu}(\text{Ram}, f)$$

and Theorem 6.6 is proved. □

Before stating a version of Theorem 6.6, we note the following observations. The lower bound obtained in Theorem 6.6 admits an easy interpretation, as follows.

Let $\mathcal{J}$ be the ideal sheaf defined by the scheme $\mathcal{F}_{\text{sing}}$; locally, this ideal is generated by the coefficients of the vectors (v_α) defining the foliation $\mathcal{F}$. Let $p : \widehat{X} \to X$ be a principalization of $\mathcal{J}$, so that $p^\star(\mathcal{J}) = \mathcal{O}(-D)$ for some (normal crossing) effective divisor D on $\widehat{X}$. According to Theorem 3.5, we have

$$T_{\widehat{f}, \Theta_D}(r) = N_{\widehat{f}, \mathcal{J}}(r) + m_{\widehat{f}, \mathcal{J}}(r) + \mathcal{O}(1)$$

where $\widehat{f}$ is the lifting of the map f to $\widehat{X}$. As a consequence, we infer the relation

$$\int_{\widehat{X}} T[\widehat{f}] \wedge c_1(D) \geq \nu^T(f, \mathcal{F}_{\text{sing}}) + m^T(f, \mathcal{F}_{\text{sing}})$$

and therefore Theorem 6.6 can be restated as follows.

Corollary 6.7. *We have the inequality*

(38) $$\int_{\widehat{X}} T[\widehat{f}] \wedge \big(c_1(T_{\widehat{\mathcal{F}}}) + c_1(D) \big) \geq 0.$$

We turn next to the logarithmic version of Theorem 6.6.

Theorem 6.8. *Let (X, B) be a log-smooth pair, such that every component of the support of B is invariant by the foliation $\mathcal{F}$. Let $(\mathcal{Y}, \sigma)$ be a parabolic Riemann surface of finite characteristic, and let $f : \mathcal{Y} \to X$ be a holomorphic map whose image is tangent to $\mathcal{F}$, and it is not contained in the set $\mathrm{Supp}(B)$. Then we have the inequality*
$(\star_{\mathrm{B}})$
$$\int_X T[f] \wedge c_1(T_{\mathcal{F}}) \geq -\nu^T(f, \mathcal{F}_{\mathrm{sing},\mathrm{B}}) - \nu_1^T(f, \mathcal{F}_{\mathrm{sing}} \cap B) - m^T(f, \mathcal{F}_{\mathrm{sing},\mathrm{B}}).$$

The proof of Theorem 6.8 follows from the arguments we have used for 6.6. The additional negative term in the statement is due to the singularities of the metric $\omega_{X,B}$:

$$\lim_{\varepsilon \to 0} \int_1^r \frac{dt}{t} \int_{\mathbb{B}_{\alpha,\varepsilon}(t)} \rho_\alpha(f) dd^c \log |f'|^2_{\omega_{X,B}} = \int_1^r \frac{dt}{t} \int_{\mathbb{B}_\alpha(t)} \rho_\alpha(f) dd^c \log |f'|^2_{\omega_{X,B}}$$
$$- \sum_{f(t'_j) \in \mathcal{F}_{\mathrm{sing}} \setminus B} \rho_\alpha\big(f(t'_j)\big) \nu_j \log \frac{r}{\sigma(t'_j)}$$
$$+ \sum_{f(t''_j) \in \mathcal{F}_{\mathrm{sing}} \cap B} \rho_\alpha\big(f(t''_j)\big) \log \frac{r}{\sigma(t''_j)}.$$

where we denote by t'_j the critical points of f. The points t''_j appearing in the last expression above are not necessarily critical. We remark that the limit
$$\overline{\lim}_r \frac{1}{T_f(r)} \int_{S(r)} \log |f'|^2_{\omega_{X,B}} d\mu_r$$
is non-positive, as it follows from the logarithmic derivative lemma, i.e. this term is not affected by the poles of $\omega_{X,B}$.

In conclusion, the presence of a log-smooth divisor on X invariant by $\mathcal{F}$ improves substantially the lower bound we have obtained in Theorem 6.6, since the main negative terms are defined by $\mathcal{J}_{\mathcal{F}_s,B}$.   □

6.4. **Foliations with reduced singularities on surfaces.** In this subsection we assume that the dimension of X is equal to $n = 2$. As an application of the results in the preceding paragraph, we obtain here a complete analogue of some results originally due to Michael McQuillan [16].

The twisted vector field V defining the foliation $\mathcal{F}$ is locally given by the expression
$$v_\alpha = a_{\alpha 1} \frac{\partial}{\partial z} + a_{\alpha 2} \frac{\partial}{\partial w}.$$
In this paragraph we will assume that *the singularities of $\mathcal{F}$ are reduced,* i.e. the linearization of the vector field at a singular point has at least a non-zero eigenvalue. A result by Seidenberg implies that this situation can be achieved after finitely many monoidal transformations. We extract next the following important consequences from the classification theory of foliations with reduced singularities in dimension two [22].

(s_1) *A singular point x_0 of $\mathcal{F}$ is called non-degenerate if we have*

$$C^{-1} \leq \frac{|a_{\alpha 1}(z,w)|^2 + |a_{\alpha 2}(z,w)|^2}{|z|^2 + |w|^2} \leq C$$

for some coordinate system (z,w) centered at x_0.

(s_2) *A singular point x_1 of $\mathcal{F}$ is called degenerate if we have*

$$C^{-1} \leq \frac{|a_{\alpha 1}(z,w)|^2 + |a_{\alpha 2}(z,w)|^2}{|z|^2 + |w|^{2k}} \leq C$$

for some coordinate system (z,w) centered at x_1, where $k \geq 2$ is an integer.

(s_3) *Any singular point of $\mathcal{F}$ is either non-degenerate or degenerate.*

(s_4) *For any blow-up $p : \widehat{X} \to X$ of a point x_0 on the surface X we denote by $\widehat{\mathcal{F}}$ the induced foliation on $\widehat{X}$. Then we have*

$$p^\star T_{\mathcal{F}} = T_{\widehat{\mathcal{F}}}$$

and moreover, the foliations $\mathcal{F}$ and $\widehat{\mathcal{F}}$ have the same number of degenerate singular points. In addition, the number "k" appearing in the inequality (s_2) is invariant.

For a proof of the preceding claims (s_1)–(s_3) we refer to the paper [22]. As for the property (s_4), we can verify it by an explicit computation, as follows.

Locally near the point x_0 the equations of the blow-up map p are given by

$$(x,y) \to (x, xy) \quad \text{or } (x,y) \to (xy, x)$$

corresponding to the two charts covering $\mathbb{P}^1$. Then the expression of the vector field defining the foliation $\widehat{\mathcal{F}}$ on the first chart is as follows

$$a_{\alpha 1}(x, xy)\frac{\partial}{\partial x} + \left(\frac{a_{\alpha 2}(x, xy)}{x} - y\frac{a_{\alpha 1}(x, xy)}{x}\right)\frac{\partial}{\partial y}.$$

We denote by $A_{\alpha 1}$ and $A_{\alpha 2}$ the coefficient of $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ in the expression above, respectively. They are holomorphic functions, and the set of their common zeroes is discrete. Indeed, this is clear if the singularity x_0 is non-degenerate. If x_0 is a degenerate singularity of $\mathcal{F}$, then this can be verified by an explicit computation, given the normal form [22]

$$v_\alpha = \left(z(1 + \tau w^k) + wF(z,w)\right)\frac{\partial}{\partial z} + w^{k+1}\frac{\partial}{\partial w},$$

where F is a function vanishing to order k, and τ is a complex number. In conclusion, the transition functions for the tangent bundle of $\widehat{\mathcal{F}}$ are the same as the ones corresponding to $\mathcal{F}$, modulo composition with the blow-up map p, so our statement is proved.

6.4.1. *Intersection with the tangent bundle.* In the context of foliations with reduced singularities, the lower bound obtained in Theorem 6.6 can be improved substantially, as follows.

Theorem 6.9. *Let $(X, \mathcal{F})$ be a non-singular compact complex surface, endowed with a foliation. Let $f : \mathcal{Y} \to X$ be a holomorphic map tangent to $\mathcal{F}$; here $(\mathcal{Y}, \sigma)$ is a parabolic Riemann surface not necessarily of finite Euler characteristic. If the singularities of $\mathcal{F}$ are reduced, then we have*

$$(39) \qquad \int_X T[f] \wedge c_1(T_\mathcal{F}) \geq 0.$$

Proof. If $\mathcal{Y} = \mathbb{C}$, then Theorem 6.9 is one of the key results established in [16]. The original arguments in this article can be adapted to the parabolic setting we are interested in, as we will sketch next.

• Let $x_0 \in \mathcal{F}_{\mathrm{sing}}$ be a singular point of the foliation. We will assume for simplicity that p is non-degenerate; the general case is a little bit more complicated technically, but the main ideas are the same. We denote by $\pi : X_1 \to X$ the blow-up of X at x_0, and let E_1 be the corresponding exceptional divisor. Let $\mathcal{F}_1$ be the foliation $\pi^\star \mathcal{F}$ on X_1; then E_1 is an invariant curve. The foliation $\mathcal{F}_1$ has two singularities on E, say $\widehat{x}_1$ and $\widehat{y}_1$, both non-degenerate.

We repeat this procedure, and blow-up $\widehat{x}_1$ and $\widehat{y}_1$. On the surface X_2 obtained in this way, the inverse image of x_0 is equal to

$$B := \widehat{E}_1 + E_2 + E_3,$$

where $\widehat{E}_1$ is the proper transform of E_1 and E_2, E_3 are the exceptional divisors. In the new configuration, we have 4 singular points of the induced foliation $\mathcal{F}_2$. Two of them belong to $\widehat{E}_1 \cap E_2$ and $\widehat{E}_1 \cap E_3$, respectively, and we denote $\widehat{x}_2$ and $\widehat{y}_2$ the other ones.

We have the injection of sheaves

$$(40) \qquad 0 \to T_{\widehat{\mathcal{F}}} \to T_{\widehat{X}}\langle B \rangle$$

that is to say, the tangent bundle of $\mathcal{F}_2$ is a subsheaf of the logarithmic tangent bundle of (X, B). The metric on $T_{\widehat{\mathcal{F}}}$ induced by the morphism above is *non-singular* at each of the four singular points above, and moreover, the image of the lift of the curve to X_2 do not intersect $\widehat{E}_1 \cap E_2$ or $\widehat{E}_1 \cap E_3$. Indeed, these singularities of the foliation $\widehat{\mathcal{F}}$ have the property that both separatrices containing them are algebraic sets. So if one of these points belong to the image of the curve, then the curve is automatically contained in $\widehat{E}_1, E_2$ or E_3 (since they are leaves of the foliation).

Therefore, by Theorem 6.8 we do not have any negative contribution in $(\star_B)$ due to these two points; the only term we have to understand is

$$(41) \qquad \nu_1^T(f, \widehat{x}_2) + \nu_1^T(f, \widehat{y}_2)$$

i.e. the truncated counting function corresponding to $\widehat{x}_2$ and $\widehat{y}_2$.

• Let $T[f]$ be a Nevanlinna current associated to a Zariski-dense parabolic curve on the surface X. Let $\pi : \widehat{X} \to X$ be the blow-up of X at x; we denote by $\widehat{f}$ the lifting of f. Then there exists $T[\widehat{f}]$ a Nevanlinna current associated to $\widehat{f}$ such that we have

$$(42) \qquad\qquad T[\widehat{f}] = \pi^\star T[f] - \rho[E]$$

where $\rho = \displaystyle\int_{\widehat{X}} T[\widehat{f}] \wedge c_1(E)$ is a positive number, smaller than the Lelong number of T at p. We remark that we have

$$\rho^2 + \int_{\widehat{X}} \{T[\widehat{f}]\}^2 = \int_X \{T[f]\}^2$$

where we denote by $\{T[\widehat{f}]\}$ the cohomology class of $T[\widehat{f}]$. Indeed, we have $\int_{\widehat{X}} \pi^\star T[f] \wedge c_1(E) = 0$.

• If we iterate the blow-up procedure, the quantities ρ_j we obtain as in (42) verify

$$(43) \qquad\qquad \sum_j \rho_j^2 \leq \int_X \{T[f]\}^2,$$

that is to say, the preceding sum is convergent.

• Let $p \in \mathcal{F}_{\text{sing}}$ be a singular point of the foliation $\mathcal{F}$. We blow-up the points x_2 and y_2 and we obtain x_3 and y_3, plus two singular points at the intersection of rational curves. After iterating k times the blow-up procedure described above, the only negative factor we have to deal with is

$$(44) \qquad\qquad -\nu_1^T(f, \widehat{x}_k) - \nu_1^T(f, \widehat{y}_k)$$

where X_k is the surface obtained after iterating k times the procedure described above, f_k is the induced parabolic curve and $\mathcal{F}_k$ is the induced folia tion. We emphasize that even if the number of singular points of the induced foliation has increased, the corresponding negative terms we have to take into account in $(\star_B)$ remains the same, i.e. the algebraic intersection of the lifting of f with the two "extremal" singular points. By using the notations in (44) above, the convergence of the sum (43) implies that we have

$$\sum_{k \geq 1} \nu_1^T(f_k, \widehat{x}_k)^2 + \nu_1^T(f_k, \widehat{y}_k)^2 < \infty$$

since $\nu_1^T(f_k, \widehat{x}_k) = \displaystyle\int_{X_k} T[f_k] \wedge c_1(E_k)$. Hence the algebraic multiplicity term (44) tends to zero as $k \to \infty$.

• We therefore have

$$\int_{X_k} T[f_k] \wedge c_1(T_{\mathcal{F}_k}) \geq -\varepsilon_k$$

Since the singularities of $\mathcal{F}$ are reduced, we use the property s_4 and we have

$$\int_{X_k} T[f_k] \wedge c_1(T_{\mathcal{F}_k}) = \int_{X_k} T[f_k] \wedge \pi_k^{\star} c_1(T_{\mathcal{F}})$$

and this last term is simply

$$\int_X T[f] \wedge c_1(T_{\mathcal{F}})$$

by the projection formula. $\square$

The theorem above is particularly interesting when coupled with the following very special case of a result due to Y. Miyaoka, cf. []

Theorem 6.10. *Let X be a projective surface, whose canonical bundle K_X is big. Let $L \to X$ be a line bundle such that $H^0\big(X, S^m T_X \otimes L\big) \neq 0$. Then L is pseudo-effective.*

We apply this result for $L = T_{\mathcal{F}}^{\star}$, so we infer that $T_{\mathcal{F}}^{\star}$ is pseudo-effective (indeed, the bundle $T_X \otimes T_{\mathcal{F}}^{\star}$ has a non-trivial section).

Combined with Theorem 6.9, we obtain

(45) $$\int_X T[f] \wedge c_1(T_{\mathcal{F}}) = 0.$$

We derive the following consequence.

Theorem 6.11. *We consider the data $(X, \mathcal{F})$ and $f : \mathcal{Y} \to X$ as in Theorem 6.9; in addition, we assume that K_X is big. Then we have*

(46) $$\int_X \{T[f]\}^2 = 0.$$

If R denotes the diffuse part of $T[f]$, then we have $\int_X \{R\}^2 = 0$. In particular, since $T[f]$ as already nef, we infer that $\nu(R, x) = 0$ at each point $x \in X$.

Proof. If the equality (46) does not hold, we show next that $\int_X T[f] \wedge c_1(T_{\mathcal{F}}) < 0$, contradicting the relation (45).

We will only discuss here the case K_X ample; the general case (i.e. K_X big) is obtained in a similar manner.

Let ω be a metric on X, such that $\mathrm{Ricci}_\omega \leq -\varepsilon_0 \omega$. We have a sequence of Kähler classes $(\alpha_k)_{k \geq 1}$ whose limit is $\{T[f]\}$. It follows from Yau's theorem [**?**] that there exists a sequence of Kähler metrics $(\omega_k)_{k \geq 1}$, such that $\omega_k \in \alpha_k$ for each $k \geq 1$, and such that we have

$$\omega_k^2 = \lambda_k \omega^2.$$

The sequence of normalizing constants λ_k is bounded from above, and also from below *away from zero*, by our assumption.

We denote by V the tautological section of $T_X \otimes T_{\mathcal{F}}^\star$. The Poincaré-Lelong formula gives

$$\sqrt{-1}\partial\overline{\partial}\log|V|_k^2 \geq \Theta_h(T_{\mathcal{F}}) - \frac{\langle\Theta_{\omega_k}(T_X)V, V\rangle}{|V|_k^2}$$

where we denote by $|V|_k$ the norm of V measured with respect to ω_k and an arbitrary metric h on $T_{\mathcal{F}}$.

By considering the wedge product with ω_k and integrating over X we obtain

$$\int_X \omega_k \wedge c_1(T_{\mathcal{F}}) \leq \int_X \frac{\mathrm{Ricci}_{\omega_k}(V, \overline{V})}{|V|_{\omega_k}^2}\omega_k^2.$$

We remark that the expression $\dfrac{\mathrm{Ricci}_{\omega_k}(V, \overline{V})}{|V|_{\omega_k}^2}$ under the integral sign is well-defined, even if V is only a vector field with values in $T_{\mathcal{F}}^\star$. Given the Monge-Ampère equation satisfied by ω_k we have

$$\int_X \frac{\mathrm{Ricci}_{\omega_k}(V, \overline{V})}{|V|_{\omega_k}^2}\omega_k^2 = \lambda_k \int_X \frac{\mathrm{Ricci}_\omega(V, \overline{V})}{|V|_{\omega_k}^2}\omega^2$$

Since Ricci_ω is negative definite, we have

$$\int_X \frac{\mathrm{Ricci}_\omega(V, \overline{V})}{|V|_{\omega_k}^2}\omega^2 \leq \int_{U_k} \frac{\mathrm{Ricci}_\omega(V, \overline{V})}{|V|_{\omega_k}^2}\omega^2$$

for any open set $U_k \subset X$.

We have $\displaystyle\int_X \omega_k \wedge \omega \leq C$ for some constant C independent of k, there exists an open set U_k of large volume (with respect to ω), on which the trace of ω_k with respect to ω is bounded uniformly with respect to k, i.e. there exists a constant C_1 such that

$$\omega_k|_{U_k} \leq C_1\omega|_{U_k}$$

for any $k \geq 1$. With this choice of U_k, we will have

$$\alpha_k \wedge c_1(T_{\mathcal{F}}) \leq \int_{U_k} \frac{\mathrm{Ricci}_\omega(V, \overline{V})}{|V|_{\omega_k}^2}\omega^2 \leq C_2 \int_{U_k} \frac{\mathrm{Ricci}_\omega(V, \overline{V})}{|V|_\omega^2}\omega^2 < -\delta$$

for some strictly positive quantity δ, and the first part of Theorem 6.11 is established by taking $k \to \infty$.

We write next the Siu decomposition of $T[f]$

$$(46) \qquad\qquad T[f] = \sum_j \tau^j[C_j] + R$$

and we remark that the class of the current R is nef. Indeed, this is a consequence of the fact that $R = \lim R_\varepsilon$, where for each $\varepsilon > 0$ the current R_ε is closed, positive, and non-singular in the complement of a

finite set of points. Then we have $\int_X \{T[f]\}^2 \geq \int_X \{R\}^2$, and it follows that $\int_X \{R\}^2 = 0$. Thus we infer that

$$(47) \qquad\qquad \nu(R, x) = 0$$

for any $x \in X$, i.e. the Lelong number of R at each point of X is equal to zero. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

6.4.2. *Intersection with the normal bundle,* I. Our aim in this part is to evaluate the degree of the curve f on the normal bundle of the foliation $\mathcal{F}$. To this end we follow [3], [16] up to a certain point; their approach relies on the Baum-Bott formula [], which is what we will survey next.

Actually, we will first give the precise expression of a representative of the Chern class

$$(48) \qquad\qquad c_1\big(N_{\mathcal{F},B}\big)$$

in arbitrary dimension, where $B = B_1 + \cdots + B_N$ be a divisor on X, such that (X, B) verifies the conditions (a) and (b) in subsection 6.2. The vector bundle in (48) is defined by the exact sequence

$$(49) \qquad\qquad 0 \to T_{\mathcal{F}} \to T_X\langle B\rangle \to N_{\mathcal{F},B} \to 0.$$

We remark that in the case of surfaces, this bundle equals $N_{\mathcal{F}} \otimes \mathcal{O}(-B)$.

Let $U_\alpha \subset X$ be an open coordinate set, and let $(z_1, \ldots, z_n)$ be a coordinate system on U_α. We assume that

$$\mathrm{Supp}(B) \cap U_\alpha = (z_1 \ldots z_p = 0).$$

Since B is invariant by $\mathcal{F}$, the vector field giving the local trivialization of $T_{\mathcal{F}}$ is written as

$$(50) \qquad\qquad v_\alpha = \sum_{j=1}^{p} z_j F_{j\alpha}(z)\frac{\partial}{\partial z_j} + \sum_{i=p+1}^{n} F_{i\alpha}(z)\frac{\partial}{\partial z_i}$$

where $F_{j\alpha}$ are holomorphic functions defined on U_α.

With respect to the coordinates (z) on U_α, the canonical bundle associated to (X, B) is locally trivialized by

$$(51) \qquad\qquad \Omega_\alpha := \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_p}{z_p} \wedge dz_{p+1} \wedge \cdots \wedge dz_n,$$

and therefore we obtain a local trivialization of the bundle $\det\big(N^\star_{\mathcal{F},B}\big)$ by contracting the form (51) with the vector field (50), i.e.

$$(52) \qquad\qquad \omega_\alpha := \Lambda_{v_\alpha}(\Omega_\alpha),$$

that is to say

$$\omega_\alpha = \sum_{j=1}^{p}(-1)^{j+1}F_{j\alpha}(z)\frac{dz_1}{z_1} \wedge \cdots \frac{\widehat{dz_j}}{z_j} \wedge \cdots \wedge \frac{dz_p}{z_p} \wedge dz_{p+1} \wedge \cdots \wedge dz_n$$

$$+ \sum_{i=p+1}^{n}(-1)^{i+1}F_{i\alpha}(z)\frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_p}{z_p} \wedge dz_{p+1} \wedge \cdots \wedge \widehat{dz_i} \wedge \cdots dz_n$$

We define the differential 1-form ξ_α on U_α by the formula

$$(53) \qquad\qquad \xi_{\alpha,\varepsilon} := \frac{F_\alpha^{(1)}}{\|G_\alpha\|^2}\sum_j \overline{G}_{j\alpha}dz_j$$

where we have used the following notations. If the index j is smaller than p, then $G_{j\alpha} := z_j F_{j\alpha}$, and if $j > p$, then we define $G_{j\alpha} := F_{j\alpha}$. Also, we write

$$F_\alpha^{(1)}(z) := \sum_{j=1}^{p}z_j F_{j\alpha,z_j}(z) + \sum_{i=p+1}^{n}F_{i\alpha,z_i}(z)$$

where $F_{k\alpha,z_r}$ is the partial derivative of $F_{k\alpha}$ with respect to z_r, and

$$\|G_\alpha\|^2 := \sum_i |G_{i\alpha}|^2.$$

Then we have the equality

$$(54) \qquad\qquad d\omega_\alpha = \xi_\alpha \wedge \omega_\alpha$$

as one can check by a direct computation.

Let α, β be two indexes such that $U_\alpha \cap U_\beta \neq \emptyset$. Then we have

$$(55) \qquad\qquad \omega_\alpha = g_{\alpha\beta}\omega_\beta$$

on $U_\alpha \cap U_\beta$; here $g_{\alpha\beta}$ are the transition functions for the determinant bundle of $N_{\mathcal{F}}^\star(B)$. By differentiating the relation (55), we obtain

$$(56) \qquad\qquad d\omega_\alpha = d\log g_{\alpha\beta} \wedge \omega_\alpha + g_{\alpha\beta}d\omega_\beta$$

hence from the relation (54) we get

$$(57) \qquad\qquad \left(\xi_\alpha - \xi_\beta - d\log g_{\alpha\beta}\right) \wedge \omega_\alpha = 0$$

on $U_\alpha \cap U_\beta$.

For the rest of this section, we assume that the singularities of $\mathcal{F}$ are *isolated and non-degenerate*.

Then we can find a family of non-singular 1-forms γ_α so that we have

$$(58) \qquad\qquad \xi_\alpha - \xi_\beta - d\log g_{\alpha\beta} = \gamma_\alpha - \gamma_\beta;$$

for example, we can take

$$(59) \qquad\qquad \gamma_\alpha := \sum_\beta \rho_\beta\left(\xi_\alpha - \xi_\beta - d\log g_{\alpha\beta}\right),$$

where (ρ_α) is a partition of unit subordinate to the covering (U_α). Here we assume that each U_α contains at most one singular point of $\mathcal{F}$. Then the global 2-form whose restriction to U_α is

(60) $$\Theta|_{U_\alpha} := \frac{1}{2\pi\sqrt{-1}} d(\xi_\alpha - \gamma_\alpha)$$

represents the first Chern class of $N_{\mathcal{F},B}$.

Let T be a closed positive current of type $(n-1, n-1)$ on X; we assume that T is *diffuse and directed by the foliation* $\mathcal{F}$. This implies that for each index α there exists a positive measure τ_α on U_α such that

(61) $$T|_{U_\alpha} = \tau_\alpha \omega_\alpha \wedge \overline{\omega}_\alpha.$$

We intend to use the representative (60) in order to evaluate the degree of the current T on the normal bundle of the foliation. Prior to this, we have to regularize the forms ξ_α. This is done simply by replacing ξ_α with $h_\alpha \xi_\alpha$, where h_α is equal to 0 in a open set containing the singularity of $\mathcal{F}$, and it is equal to one out of a slightly bigger open set. A specific choice of such functions will be made shortly; we remark that the effect of the multiplication with h_α is that the equality (54) is only verified in the complement of an open set containing the singular point. The formula for the curvature becomes $\Theta|_{U_\alpha} := \dfrac{1}{2\pi\sqrt{-1}} d(h_\alpha \xi_\alpha - \gamma_\alpha)$.

As a consequence, we have the next statement, which is the main result of this subsection.

Lemma 6.12. *Let $\mathcal{F}$ be a foliation by curves with isolated singularities on X. We consider a closed positive current T of bidimension $(1,1)$, which is directed by the foliation $\mathcal{F}$. If $\mathcal{F}_{\mathrm{sing}} \cap \mathrm{Supp}(T)$ consists in non-degenerate points only, then we have*

(62) $$T \wedge \Theta \geq -C \sup_{x \in \mathcal{F}_{\mathrm{sing}}} \nu(T, x)$$

for some positive constant C.

Proof. We first observe that we have

(63) $$d\gamma_\alpha \wedge T = 0.$$

Indeed, ω_α is holomorphic and γ_α is a (1,0) form, so we have

(64) $$d\gamma_\alpha \wedge T = \tau_\alpha \overline{\partial}(\gamma_\alpha \wedge \omega_\alpha) \wedge \overline{\omega}_\alpha.$$

The equality (63) is therefore a consequence of (57)-(59).

In order to compute the other term of (62), we use the relation (54), and we infer that we have

(65) $$\overline{\partial}(h_\alpha \xi_\alpha) \wedge \omega_\alpha = \overline{\partial} h_\alpha \wedge \xi_\alpha \wedge \omega_\alpha$$

for any choice of h_α.

In what follows, we will drop the index α, and concentrate around a single singular point namely the origin of the coordinate system. Let

χ be a smooth function vanishing near zero, and which equals 1 for $x \geq 1/2$. We consider

(66) $$h_r(z) := \chi\big(\|z\|^r\big)$$

for each $r > 0$. We have then $0 \leq h_r \leq 1$, and as $r \to 0$, h_r converges to the the characteristic function of $U \setminus 0$.

If moreover the singular point $z = 0$ is non-degenerate, in the sense that we have

$$C^{-1} < \frac{\|G(z)\|}{\|z\|} < C$$

for some constant $C > 0$, then we have to bound from above the mass of the measure

$$r \frac{\chi'\big(\|z\|^r\big)}{\|z\|^{2-r}} \sigma_T$$

as $r \to 0$; here σ_T is the trace measure of T. In other words we have to obtain an upper bound for the integral

(67) $$r \int_{C_1 < \|z\|^r < C_2} \|z\|^{r-2} \sigma_T.$$

We observe that we have

$$\int_{C_1 < \|z\|^r < C_2} \|z\|^{r-2} \sigma_T = \int_0^\infty \sigma_T\Big(\big(\|z\|^{r-2} > s\big) \cap \big(C_1 < \|z\|^r < C_2\big)\Big) ds;$$

and up to a $\mathcal{O}(r)$ term, the quantity we have to evaluate is smaller than

(68) $$r \int_1^{C_1^{-\frac{2-r}{r}}} \sigma_T\big(\|z\| < s^{\frac{1}{r-2}}\big) ds.$$

If ν is Lelong number of T at 0, then we have $\sigma_T\big(\|z\| < \tau\big) \leq 2\nu\tau^2$, as soon as τ is small enough. Therefore, a quick computation shows that the integral (68) is bounded by $C\nu$, for some positive constant $C > 0$. $\qquad\square$

As one can see from the proof, this kind of arguments can only be used if the singular point is non-degenerate. However, this is sufficient for our purposes, given the next result.

Theorem 6.13. *Let X be a projective surface endowed with a holomorphic foliation $\mathcal{F}$. We assume that the singularities of $\mathcal{F}$ are reduced. Let R be a diffuse closed positive (1,1) current on X, directed by $\mathcal{F}$. Let $\{x_1, \ldots x_d\}$ be the set of degenerate singularities of $\mathcal{F}$. Then we have*

$$\mathrm{Supp}(R) \subset X \setminus \{x_1, \ldots x_d\}.$$

Proof. Suppose $(0,0)$ is a degenerate reduced singularity of $\mathcal{F}$. It is well-known that the holonomy map h at such a point is tangent to identity (eventually after a finit enumber of iterations). We assume that the separatrix is given by $w = 0$, and that the w-axis is transversal to $\mathcal{F}$.

The argument is based on the fact that the dynamics of such a map near the origin is well-understood. It is the Léau "flower theorem". There is a neighborhood of zero, say U, such that

$$U \setminus 0 = \bigcup_{1 \leq j \leq p} P_j^+ \cup P_j^-.$$

If the point z belongs to the petals P_j^+, then we have $h^n(z) \to 0$, where we denote by h^n the nth iteration of h. If z belongs to the petals P_j^-, then we have $h^{-n}(z) \to 0$.

Suppose that the current R has mass in a small open Ξ set near 0, which can be assumed to have the form $\Xi = P_1^+ \times \Delta$, where Δ is a disk. Let $x_0 := (0, w_0)$ be a point in $U \cap \mathrm{Supp}(R)$, such that $w_0 \neq 0$.

The current R can be written on Ξ as

$$R = \int_{P_1^+} [V_w] d\mu(w),$$

where V_w is the plaque through w.

We show now that μ has no mass out of the origin. Let $W \subset P_1^+$ be an open set containing w_0, such that $W \cap (w = 0) = \emptyset$. We construct a new open set $W_1 \subset P_1^+$ via the holonomy map, as follows. Without loss of generality we can assume that the petal P_1^+ is invariant by h. We define $W_1 := h^{n_1}(W)$ where n_1 is large enough in order to have $W_1 \cap W = \emptyset$.

We can restart this procedure with W_1, obtaining $W_2, \ldots, W_k$. In conclusion, we get a sequence of open sets $(W_k) \subset P_1^+$ such that $W_i \cap W_j = \emptyset$ if $i \neq j$. The mass of each set with respect to the transverse measure is preserved, since the measure is invariant by the holonomy map. This is equivalent to the fact that the current R is closed and directed by $\mathcal{F}$. Therefore we obtain a contradiction. $\qquad \square$

In conclusion, we obtain the next result.

Theorem 6.14. *Let X be a projective surface endowed with a holomorphic foliation $\mathcal{F}$. We assume that the singularities of $\mathcal{F}$ are reduced. Let T be a diffuse closed positive (1,1) current on X, directed by $\mathcal{F}$. Then there exists a constant $C > 0$ such that*

$$(69) \qquad \int_X T \wedge c_1\big(N_{\mathcal{F}}(-B)\big) \geq -C \sup_{x \in \mathcal{F}_{\mathrm{sing}}} \nu(T, x)$$

Proof. By Theorem 6.13, we infer that the local contribution of the representative (60) at a degenerate singular point to the evaluation of (62) is equal to zero.

In the case of a non-degenerate singularity, we use the relations (66) and (68) above. The theorem is thus proved.   □

As a conclusion of the considerations in the preceding subsections, we infer the following "parabolic version" of the result in [16].

Corollary 6.15. *Let $(X, \mathcal{F})$ be a surface of general type endowed with a holomorphic foliation $\mathcal{F}$, and let $f : \mathcal{Y} \to X$ be a holomorphic curve, where $\mathcal{Y}$ is a parabolic Riemann surface of finite Euler characteristic. If f is tangent to $\mathcal{F}$, then the dimension of the Zariski closure of the image of f is at most 1.*

Proof. The first thing to remark is that the hypothesis is stable by blow-up, hence we can assume that the singularities of $\mathcal{F}$ are reduced. We can also assume that the current associated to f can be written as

$$T[f] = \sum_j \nu_j [C_j] + R$$

where $C := C_1 + \ldots C_N$ verifies (a) and (b), and the Lelong numbers of R may be positive eventually at a finite subset of X. We argue by contradiction, so we add the hypothesis that the image of f is Zariski dense.

If so, by Theorem 6.9 we infer that $\int_X T[f] \wedge c_1(T_{\mathcal{F}}) = 0$, as well as $\int_X \{T[f]\}^2 = 0$. This implies that the Lelong numbers of the diffuse part R of $T[f]$ are equal to zero; by Theorem 6.14, we infer that

$$\int_X R \wedge c_1(N_{\mathcal{F}}) \geq 0.$$

The next step is to show the inequality

$$\int_X T[f] \wedge c_1(N_{\mathcal{F}}) \geq \int_X R \wedge c_1(N_{\mathcal{F}}(-C)).$$

We follow the presentation in [3]: we have

$$\int_{C_j} c_1(N_{\mathcal{F}}) = \int_X c_1(C_j)^2 + Z(C_j, \mathcal{F})$$

where $Z(C_j, \mathcal{F})$ is the multiplicity of the singularities of $\mathcal{F}$ along the curve C_j. We clearly have

$$Z(C_j, \mathcal{F}) \geq \sum_{j \neq k} c_1(C_j) \wedge c_1(C_k)$$

and thus we obtain

$$\sum_j \int_X \nu_j c_1(C_j) \wedge c_1(N_{\mathcal{F}}) \geq \sum_{j,k} \int_X \nu_j c_1(C_j) \wedge c_1(C_k).$$

The preceding inequality implies that

$$\int_X T[f] \wedge c_1(N_{\mathcal{F}}) \geq \int_X R \wedge c_1(N_{\mathcal{F}}) + \int_X (T[f] - R) \wedge c_1(C)$$

and given that $T[f]$ is nef, we get the desired inequality. As a consequence we obtain

$$\int_X T[f] \wedge c_1(N_{\mathcal{F}}) \geq 0.$$

Since $c_1(N_{\mathcal{F}}) + c_1(T_{\mathcal{F}}) = c_1(X)$, the inequalities above imply that $\int_X T[f] \wedge c_1(K_X) \leq 0$. This is absurd since K_X is big and $T[f]$ is nef. $\qquad\square$

6.4.3. *Intersection with the normal bundle,* II. We will offer here a few comments about the following beautiful result, due to M. Brunella [3]. The context is as follows: X is a compact complex surface, endowed with a holomorphic foliation $\mathcal{F}$ and f is a parabolic curve tangent to $\mathcal{F}$.

Theorem 6.16. [3] *If the curve $f : \mathcal{Y} \to X$ has a Zariski dense image, then we have*

(70) $$\int_X T[f] \wedge c_1(N_{\mathcal{F}}) \geq 0.$$

We remark that a similar statement appeared in the preceding subsection, but it was obtained in a very indirect way, under the assumption that the canonical bundle of X is big (and as part of an argument by contradiction...).

M. Brunella's arguments involve many considerations from dynamics, including a study at Siegel points of $\mathcal{F}_{\mathrm{sing}}$. They seem difficult to accommodate to higher dimensional case. Nevertheless, this is very strong statement; in order to illustrate it, we note that Theorem **??** has the following generalization.

Corollary 6.17. *Let L be a line bundle on X, such that*

$$H^0(X, S^m T_X^\star \otimes L) \neq 0.$$

Then we have

(71) $$\int_X T[f] \wedge c_1(L) \geq 0.$$

Proof. Indeed, if the inequality (71) above does not hold, then given a section u of $S^m T_X^\star \otimes L$ we have $u(f'^{\otimes m}) \equiv 0$, by the vanishing theorem. Let $\Gamma \subset \mathbb{P}(T_X)$ be the set of zeros of u, interpreted as a section of the bundle $\mathcal{O}(m) \otimes \pi^\star(L)$, where $\pi : \mathbb{P}(T_X) \to X$ is the natural projection. The natural lift of the curve f is contained in one of the irreducible components of Γ; let X_m be a desingularization of the component containing the image of f.

In conclusion, we have a projective surface X_m, endowed with a holomorphic foliation $\mathcal{F}_m$, a generically finite morphism $p : X_m \to X$, and a curve $f_m : \mathcal{Y} \to X_m$, such that $f = p \circ f_m$.

Let $u_m := p^\star(u)$ be the section of $S^m T^\star_{X_m} \otimes p^\star(L)$ obtained by taking the inverse image of u. We decompose the set $Z_m := (u_m = 0) \subset \mathbb{P}(T_{X_m})$ as follows

$$(72) \qquad Z_m = \sum_j m_j \Gamma_j$$

where $\Gamma_j \subset \mathbb{P}(T_{X_m})$ are irreducible hypersurfaces, and m_j are positive integers.

The image of the lift of f_m to $\mathbb{P}(T_{X_m})$ is contained in Z_m, and it is equally contained in the graph of $\mathcal{F}_m$. Since the curve f_m is supposed to be Zariski dense, its lift can be contained in at most one hypersurface of $\mathbb{P}(T_{X_m})$. We will henceforth assume that the graph of the foliation $\mathcal{F}_m$ coincides with Γ_1.

Numerically, we have

$$(73) \qquad \Gamma_j \equiv \nu_j \mathcal{O}(1) + \pi^\star_m(L_j)$$

where $\pi_m : \mathbb{P}(T_{X_m}) \to X_m$ is the projection, ν_j are positive integers, and L_j are line bundles on X_m. This is a consequence of the structure of the Picard group of $\mathbb{P}(T_{X_m})$.

The relations (72), (73) show that we have

$$p^\star(L) \equiv \sum m_j L_j.$$

Since the lift of the curve f_m is not contained in any Γ_j for $j \geq 2$, we have $\displaystyle\int_X T[f'_m] \wedge c_1(\Gamma_j) \geq 0$. The tautological inequality states that $\displaystyle\int_X T[f'_m] \wedge c_1\big(\mathcal{O}(1)\big) \leq 0$. so we obtain

$$(74) \qquad \int_{X_m} T[f_m] \wedge c_1(L_j) \geq 0$$

for each index $j \geq 2$.

But we have assumed that (71) does not hold, so we infer that

$$(75) \qquad \int_{X_m} T[f_m] \wedge c_1(L_1) < 0$$

which contradicts Theorem 6.16. $\qquad\qquad\square$

In particular, this gives a proof of the Green-Griffiths conjecture for minimal general type surfaces X such that $c_1^2(X) > c_2(X)$ without using any consideration about the tangent bundles of foliations! Indeed, under this hypothesis we have

$$H^0(X, S^m T^\star_X \otimes A^{-1}) \neq 0$$

where A is ample, and $m \gg 0$ is a positive integer (by a result due to F. Bogomolov, []), so if f is Zariski dense, we obtain a contradiction by the corollary above, given the strict positivity of A.

Given the aforementioned consequence of Brunella's theorem, it is very tempting to formulate the following problem.

Conjecture 6.18. *Let k, m be two positive integers, and let $L \to X$ be a line bundle, such that*

$$H^0\big(X, E_{k,m}T_X^\star \otimes L\big) \neq 0.$$

If $f : \mathcal{Y} \to X$ is a Zariski dense parabolic curve of finite Euler characteristic, then we have

$$(76) \qquad\qquad \int_X T[f] \wedge c_1(L) \geq 0.$$

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