KOBAYASHI HYPERBOLICITY, NEGATIVITY OF THE CURVATURE AND POSITIVITY OF THE CANONICAL BUNDLE

SIMONE DIVERIO

Abstract.

1. Introduction

Let X be a compact complex manifold. An *entire curve* traced in X is by definition a non constant holomorphic map $f\colon \mathbb{C} \to X$. By Brody's criterion X is Kobayashi-hyperbolic if and only if X does not admit any entire curve.

At the very beginning of the theory, in the early 70's, very few examples of (higher dimensional) compact complex manifolds where known: mainly compact quotients of bounded domains in $\mathbb{C}^n$. Suppose to be in the smooth case, and consider a compact complex manifold X whose universal cover is a bounded domain Ω in $\mathbb{C}^n$. Then, since Ω admits the Bergman metric ω_B, and this metric is invariant BLABLABLA [Kobayashi]. In particular, K_X is ample and X is projective.

2. Complex differential geometric background and hyperbolicity

The material in this section is somehow standard, but we take the opportunity here to fix notations and explain some remarkable facts which are not necessarily in everybody's background. We refer to [Dem12, Huy05, Zhe00] for an excellent and more systematic treatise of the subject.

Let X be a complex manifold of complex dimension n, and let h be a hermitian metric on its tangent space T_X, which is considered as a complex vector bundle endowed with the standard complex structure J inherited from the holomorphic coordinates on X. Then, the real part g of $h = g - i\omega$ defines a riemannian metric on the underlying real manifold, while its imaginary part ω defines a 2-form on X.

Now, one can consider both the riemannian or the hermitian theory on X. On the one hand we have the existence of a unique connection ∇ on $T_X^{\mathbb{R}}$ —the Levi–Civita connection— which is both compatible with the metric g and without torsion. Here the superscript $\mathbb{R}$ is put on T_X to emphasize that we are looking at the real underlying manifold. We call the square of

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this connection $R = \nabla^2$, the riemannian curvature of $(T_X^{\mathbb{R}}, g)$. It is a 2-form with values in the endomorphisms of $T_X^{\mathbb{R}}$.

On the other hand, we can complexify T_X and decompose it as a direct sum of the eigenbundles for the complexified complex structure $J \otimes \mathrm{Id}_{\mathbb{C}}$ relatives to the eigenvalues $\pm i$:

$$T_X^{\mathbb{C}} = T_X \otimes \mathbb{C} \simeq T_X^{1,0} \oplus T_X^{0,1}.$$

We have a natural vector bundle isomorphism

$$\begin{aligned} \xi \colon T_X^{\mathbb{R}} &\to T_X^{1,0} \\ v &\mapsto \frac{1}{2}(v - iJv) \end{aligned}$$

which is moreover $\mathbb{C}$-linear: $\xi \circ J = i\xi$. There is a natural way to define a hermitian metric on $T_X^{\mathbb{C}}$, as follows. We first consider the $\mathbb{C}$-bilinear extension $g^{\mathbb{C}}$ of g, and then its sesquilinear form $\tilde{h}$ made up using complex conjugation in $T_X^{\mathbb{C}}$:

$$\tilde{h}(\bullet, \bullet) := g^{\mathbb{C}}(\bullet, \bar{\bullet}).$$

Such a hermitian metric realize the direct sum decomposition above as an orthogonal decomposition. The complexification of ω, which we still call ω by an abuse of notation, is then a real positive $(1,1)$-form. These three notions, namely a hermitian metric on T_X, a hermitian metric on $T_X^{1,0}$, and a real positive $(1,1)$-form are essentially the same, since there is a canonical way to pass from one to the other.

Now, we know that there exists a unique connection D on $T_X^{1,0}$ which is both compatible with $\tilde{h}$ and the complex structure: the Chern connection. We call the square of this connection $\Theta = D^2$, the Chern curvature of $(T_X^{1,0}, \tilde{h})$. It is a $(1,1)$-form with values in the anti-hermitian endomorphisms of $(T_X^{1,0}, \tilde{h})$.

A basic question is then: can we compare these two theories *via* ξ? The answer is classical and surprisingly simple. The riemannian theory and the hermitian one are the same if and only if the metric h is Kähler, *i.e.* if and only if $d\omega = 0$. In other words, the metric is Kähler if and only if

$$D = \xi \circ \nabla \circ \xi^{-1},$$

and of course, in this case, $\Theta = \xi \circ R \circ \xi^{-1}$.

2.1. **Notions of curvature in riemannian and hermitian geometry and their correlation in the Kähler case.** We know give a brief overview of the different notions of curvature in the setting respectively of riemannian geometry and hermitian geometry. Then, we shall compare them in the case of Kähler metrics, with particular attention to the "propagation" of signs.

2.1.1. *The riemannian case.* Let (M, g) be a riemannian manifold, ∇ its Levi–Civita connection and R its riemannian curvature. To this data it is attached the classical notion of *sectional curvature* $\mathcal{K}_g$ of g. It is a function which assigns to each 2-plane $\pi = \operatorname{Span}(v, w)$ in T_M the real number

$$\mathcal{K}_g(\pi) = -\frac{\langle R(v, w) \cdot v, w\rangle_g}{||v||_g^2 ||w||_g^2 - |\langle v, w\rangle_g|^2}.$$

One can verify that this function completely determines the riemannian curvature tensor. One usually also considers other "easier" tensors obtained by performing some type of contractions on R, for instance the *Ricci curvature* r_g and the *scalar curvature* s_g. The former is a symmetric 2-tensor defined by

$$r_g(u, v) = \operatorname{tr}_{T_M}\big(w \mapsto R(w, u) \cdot v\big).$$

The latter is the real function on M obtained by taking the trace of the Ricci curvature with respect to g:

$$s_g = \operatorname{tr}_g r_g.$$

One can straightforwardly show that, up to a positive factor which depends only on the dimension of M, the Ricci curvature can be obtained as an average of sectional curvatures and the scalar curvature as an average of Ricci curvatures. In particular the sign of the sectional curvature "dominates" the sign of the Ricci curvature which, in turn, "dominates" the sign of the scalar curvature.

2.1.2. *The hermitian case.* We now look at the complex case. We start more generally with the notion of *Griffiths curvature* for a holomorphic hermitian vector bundle (E, h) over a complex manifold X. In this situation, we also have a unique connection both compatible with the metric and the holomorphic structure on E, whose curvature we still call Θ.

It is a $(1, 1)$-form with values in the anti-hermitian endomorphisms of (E, h). The Griffiths curvature assigns to each pair $(v, \zeta) \in T^{1,0}_{X,x} \times E_x$, $x \in X$, the real number given by

$$\theta_{E,h}(v, \zeta) = \langle \Theta(v, \bar{v}) \cdot \zeta, \zeta\rangle_h.$$

It has the remarkable property (which is a special case of what is called more generally Griffiths formulae) that it decreases when passing to holomorphic subbundles. Suppose that $S \subseteq E$ is a holomorphic subbundle of E and endow it with the restriction metric $h|_S$. Then, given $x \in X$, $\zeta \in S_x \subseteq E_x$, and $v \in T^{1,0}_{X,x}$, we always have

$$\theta_{S,h|_S}(v, \zeta) \leq \theta_{E,h}(v, \zeta).$$

Now, we look more closely at the special case where $E = T^{1,0}_X$, and the hermitian metric is $\tilde{h}$ as above. In this case, the Griffiths curvature is nothing but (up to normalization) what is classically called *holomorphic bisectional curvature.* To be more precise, given a point $x \in X$ and two non-zero holomorphic tangent vectors $v, w \in T^{1,0}_{X,x} \setminus \{0\}$ we define the holomorphic bisectional curvature in the directions given by v, w as

$$\mathrm{HBC}_h(x,[v],[w]) = \frac{\theta_{T^{1,0}_X,\tilde h}(v,w)}{||v||^2_{\tilde h}||w||^2_{\tilde h}}.$$

By a slight abuse of notation, we may possibly confuse and interchange h, $\tilde h$ and ω in what follows.

The *holomorphic sectional curvature* is defined to be the restriction of the holomorphic bisectional curvature to the diagonal:

$$\mathrm{HSC}_h(x,[v]) = \mathrm{HBC}_h(x,[v],[v]) = \frac{\theta_{T^{1,0}_X,\tilde h}(v,v)}{||v||^4_{\tilde h}}.$$

In the spirit of the riemannian case, we can construct a closed real $(1,1)$-form, the *Chern–Ricci form*, by taking the trace with respect to the endomorphism part of the Chern curvature. We also normalize it in such a way that its cohomology class coincides with the first Chern class of the manifold, namely:

$$\mathrm{Ric}_\omega = \frac{i}{2\pi}\,\mathrm{tr}_{T^{1,0}_X}\Theta.$$

The new feature here is that the Chern–Ricci tensor is a 2-form always belonging to a fixed cohomology class, the first Chern class of X, independently of the choice of the metric. This is because, in general, the trace of the curvature of a vector bundle is the curvature of the induced connection on the determinant bundle, which in this case is the dual of the canonical bundle of X. By taking again the trace, but this time with respect to ω, we get what is called the *Chern scalar curvature*. Thus, it is by definition the unique real function $\mathrm{scal}_\omega\colon X\to\mathbb{R}$ such that

$$\mathrm{Ric}_\omega \wedge \frac{\omega^{n-1}}{(n-1)!} = \mathrm{scal}_\omega\,\frac{\omega^n}{n!}.$$

2.1.3. *Negativity of the holomorphic sectional curvature and hyperbolicity.* Before going further and explore —when the metric is Kähler— the links among the different notions of curvature we introduced in the riemannian and hermitian setting, we would like to explain here how the negativity of the holomorphic sectional curvature implies the Kobayashi hyperbolicity of the manifold. We want to do it here before entering the Kähler world, since this is a purely hermitian fact. For our purposes, it is sufficient to deal with the smooth compact case even if more general statements can be established.

Theorem 2.1. *Let (X,ω) be a compact hermitian manifold such that we have $\mathrm{HSC}_\omega<0$. Then, X is Kobayashi hyperbolic.*

In the next section we will see that the negativity of the holomorphic sectional curvature is a sufficient but not necessary condition for Kobayashi hyperbolicity.

Proof. By Brody's theorem [Bro78] (see [Kob98, (3.6.3) Theorem] for a more modern account), it is sufficient to show that every holomorphic map $f\colon \mathbb{C} \to X$ whose derivative is ω-bounded is constant. So let f be such a map a consider the function

$$F\colon \mathbb{C} \to \mathbb{R} \cup \{-\infty\}$$
$$t \mapsto \log ||f'(t)||^2_\omega,$$

which is clearly upper semi-continuous and bounded from above. Suppose by contradiction that F is not identically $-\infty$, which corresponds to the fact that f is not constant. Then, of course, the locus where $\log ||f'(t)||^2_\omega$ is $-\infty$ is a discrete set. We now check that $\log ||f'(t)||^2_\omega$ is a subharmonic function on the whole $\mathbb{C}$, which is moreover strictly subharmonic over $\{f' \neq 0\}$. Since any bounded subharmonic function on $\mathbb{C}$ is constant [Kli91, Proposition 2.7.3], this gives a contradiction, because a constant function cannot be strictly subharmonic somewhere.

First of all we show the subharmonicity of $\log ||f'||^2_\omega$, by showing that for all positive integer k the smooth functions defined on the whole complex plane $\psi_\varepsilon = \log(||f'||^2_\omega + \varepsilon)$ are subharmonic, *i.e.* $i\partial\bar\partial\psi_\varepsilon \geq 0$. For, since

$$\log ||f'||^2_\omega = \lim_{\varepsilon \to 0} \psi_\varepsilon$$

pointwise, and the sequence $\{\psi_\varepsilon\}$ is decreasing, then the subharmonicity of $\log ||f'||^2_\omega$ follows from [Kli91, Theorem 2.6.1, (ii)].

So, fix $\varepsilon > 0$ and consider a point $t_0 \in \mathbb{C}$. Call $x_0 = f(t_0) \in X$ and choose holomorphic coordinates $(z_1, \dots, z_n)$ for X centered at x_0 so that x_0 corresponds to $z = 0$, and write $f = (f_1, \dots, f_n)$ for f in these coordinates. Moreover, chose a normal coordinate frame $\{e_1, \dots, e_n\}$ for (T_X, ω) at x_0 [Dem12, (12.10) Proposition]. With this choice we have that

$$\langle e_l(z), e_m(z)\rangle_\omega = \delta_{lm} - \sum_{j,k=1}^{n} c_{jklm}\, z_j \bar z_k + O(|z|^3),$$

where the c_{jklm}'s are the coefficients of the Chern curvature of ω. Observe that, since the metric is not supposed to be Kähler, we can not chose holomorphic coordinates around x_0 such that the e_l's can be taken simply to be $\partial/\partial z_l$; nevertheless, by a constant change of coordinates, we can suppose that $e_l(x_0)$ equals $\partial/\partial z_l(x_0)$, at least at x_0. Now, of course there exists holomorphic functions φ_j, $j = 1, \dots, n$, defined on a neighborhood of t_0 such that

$$f'(t) = \sum_{j=1}^{n} \varphi_j(t)\, e_j(t),$$

so that around t_0 we have

$$||f'(t)||^2_\omega = |\varphi(t)|^2 - \sum_{j,k,l,m=1}^{n} c_{jklm}\, f_j(t)\overline{f_k(t)}\varphi_l(t)\overline{\varphi_m(t)} + O(|f|^3).$$

Moreover, we have $f'_j(t_0)=\varphi_j(t_0)$ for all j, since the e_l's and the $\partial/\partial z_l$'s agree at t_0.

Remark that, since X is compact, there exists a positive constant κ such that $\mathrm{HSC}_\omega<-\kappa$. This condition reads in our coordinates

$$\sum_{j,k,l,m=1}^n c_{jklm}\, v_j\bar v_k v_l \bar v_m < -\kappa|v|^4,\quad \forall v=(v_1,\dots,v_n)\in\mathbb{C}^n.$$

Now, we have to compute $\partial||f'||^2_\omega$, $\bar\partial||f'||^2_\omega$ and $\partial\bar\partial||f'||^2_\omega$ at t_0, since

$$\partial\bar\partial\psi_\varepsilon=\frac{-1}{(||f'||^2_\omega+\varepsilon)^2}\partial||f'||^2_\omega\wedge\bar\partial||f'||^2_\omega+\frac{1}{||f'||^2_\omega+\varepsilon}\partial\bar\partial||f'||^2_\omega.$$

We find

$$\begin{aligned}
\partial||f'||^2_\omega|_{t_0}&=\langle\varphi'(t_0),f'(t_0)\rangle\,dt,\\
\bar\partial||f'||^2_\omega|_{t_0}&=\langle f'(t_0),\varphi'(t_0)\rangle\,d\bar t,\\
i\partial\bar\partial||f'||^2_\omega|_{t_0}&=i\Bigg(|\varphi'(t_0)|^2-\sum_{j,k,l,m=1}^n c_{jklm}\, f'_j(t_0)\overline{f'_k(t_0)}f'_l(t_0)\overline{f'_m(t_0)}\Bigg)\,dt\wedge d\bar t\\
&> i\big(|\varphi'(t_0)|^2+\kappa|f'(t_0)|^4\big)\,dt\wedge d\bar t,
\end{aligned}$$

where the brackets just mean the standard hermitian product in $\mathbb{C}^n$. Putting all this together we obtain

$$\begin{aligned}
i\partial\bar\partial\psi_\varepsilon|_{t_0}&> i\Bigg(\frac{-|\langle f'(t_0),\varphi'(t_0)\rangle|^2}{(|f'(t_0)|^2+\varepsilon)^2}+\frac{|\varphi'(t_0)|^2+\kappa|f'(t_0)|^4}{|f'(t_0)|^2+\varepsilon}\Bigg)\,dt\wedge d\bar t\\
&\geq i\Bigg(\frac{-|f'(t_0)|^2|\varphi'(t_0)|^2}{(|f'(t_0)|^2+\varepsilon)^2}+\frac{|\varphi'(t_0)|^2+\kappa|f'(t_0)|^4}{|f'(t_0)|^2+\varepsilon}\Bigg)\,dt\wedge d\bar t\\
&= i\,\frac{\kappa|f'(t_0)|^6+\varepsilon\big(|\varphi'(t_0)|^2+\kappa|f'(t_0)|^4\big)}{(|f'(t_0)|^2+\varepsilon)^2}\,dt\wedge d\bar t\geq 0,
\end{aligned}$$

where we used Cauchy–Schwarz for the second inequality, and so ψ_ε is subharmonic at each point.

To conclude, observe that the very same computation with $\varepsilon=0$ makes sense away from points where $f'=0$, and give moreover strict positivity for $i\partial\bar\partial\log||f'||^2_\omega$ at these points. Which means that, away from $\{f'=0\}$, $\log||f'||^2_\omega$ is strictly subharmonic, as desired. □

2.1.4. *The Kähler case.* Suppose now that (X,ω) is a Kähler manifold, with $\omega=-\Im h$. Let $(X,g=\Re h)$ be the underlying riemannian manifold. We saw that this is precisely the case when Θ and R correspond to each other via ξ. In this setting, using ξ, one can show easily the following useful relation between the holomorphic bisectional curvature and the riemannian sectional curvature. Let $v, w \in T_{X,x}$ be two independent (real) tangent vectors. Then,

$$\begin{aligned}\mathrm{HBC}_\omega(x,[\xi(v)],[\xi(w)]) = \frac{||Jv||_g^2||w||_g^2 - |\langle Jv,w\rangle_g|^2}{||v||_g^2||w||^2}\,\mathcal{K}_g\big(\mathrm{Span}\{Jv,w\}\big) \\ + \frac{||v||_g^2||w||_g^2 - |\langle v,w\rangle_g|^2}{||v||_g^2||w||^2}\,\mathcal{K}_g\big(\mathrm{Span}\{v,w\}\big).\end{aligned}$$

In particular, if $\mathcal{K}_g$ has a sign at a certain point, so does HBC_ω, and the sign is the same. Moreover, by specializing at the diagonal, we get

$$\mathrm{HSC}_\omega(x,[\xi(v)]) = \mathcal{K}_g\big(\mathrm{Span}\{Jv,v\}\big),$$

that is, the holomorphic sectional curvature is nothing but the riemannian sectional curvature computed on complex 2-planes.

In the Kähler setting, not surprisingly, also the riemannian Ricci curvature and the Chern–Ricci forms as well as the corresponding scalar curvatures are related to each other. The precise relation is as follows, for $v, w \in T_{X,x}$:

$$\mathrm{Ric}_\omega\big(\xi(v),\overline{\xi(w)}\big) = \frac{i}{4\pi}\big(r_g(v,w) - i r_g(Jv,w)\big).$$

In particular, since for g the real part of a Kähler metric the Ricci tensor is J-invariant, we get that

$$-i\,\mathrm{Ric}_\omega\big(\xi(v),\overline{\xi(v)}\big) = \frac{1}{4\pi} r_g(v,v),$$

that is Ric_ω is a positive (resp. semi-positive, negative, semi-negative) $(1,1)$-form if and only if r_g is a positive (resp. semi-positive, negative, semi-negative) symmetric bilinear form (observe that $-i\,\mathrm{Ric}_\omega\big(\bullet,\bar\bullet\big)$ is nothing but the real quadratic form associated to the real $(1,1)$-form Ric_ω). From this we also infer that

$$\mathrm{scal}_\omega = \frac{1}{4\pi}\,s_g.$$

totscalcurv *Remark* 2.2. In the compact Kähler case, since both ω and Ric_ω are closed forms, the total scalar curvature becomes a cohomological invariant, namely

$$\begin{aligned}\int_X s_g\,dV_g &= 4\pi\int_X \mathrm{scal}_\omega\,\frac{\omega^n}{n!} \\ &= 4\pi\int_X \mathrm{Ric}_\omega\wedge\frac{\omega^{n-1}}{(n-1)!} \\ &= \frac{4\pi}{(n-1)!}\,c_1(X)\cdot[\omega]^{n-1}.\end{aligned}$$

In particular, the total scalar curvature of a compact Kähler manifold with vanishing first Chern class must always be zero. This observation will be useful later.

Now we explain, still in the Kähler case, the link between the signs respectively of the holomorphic bisectional curvature and Ricci curvature, and of the holomorphic sectional curvature with the scalar curvature.

average **Proposition 2.3.** *Let (X,ω) be compact Kähler manifold of complex dimension n, $x_0\in X$, and $v\in T^{1,0}_{X,x_0}\setminus\{0\}$. Then, we have*

$$-\frac{2\pi i}{n||v||^2_{\tilde h}}\operatorname{Ric}_\omega(v,\bar v)=\fint_{S^{2n-1}}\operatorname{HBC}_\omega([v],[w])\,d\sigma(w),$$

and

$$\frac{4\pi}{n(n+1)}\operatorname{scal}_\omega(x_0)=\fint_{S^{2n-1}}\operatorname{HSC}_\omega([v])\,d\sigma(v),$$

where $d\sigma$ is the Lebesgue measure on the $\tilde h$-unit sphere S^{2n-1} in $T^{1,0}_{X,x_0}$, and by $\fint$ we mean taking the average, i.e.

$$\fint_{S^{2n-1}}=\frac{(n-1)!}{2\pi^n}\int_{S^{2n-1}}.$$

In particular, we find that the sign of the holomorphic bisectional curvature of a Kähler metric dominates the sign of the Ricci curvature, while the sign of the holomorphic sectional curvature dominates the sign of the scalar curvature.

On the other hand, recall that the holomorphic sectional curvature of a Kähler metric *completely* determines the curvature tensor Zhe00 [Zhe00, Lemma 7.19], so that the point is really whether and how it does spread the sign.

Proof. Fix holomorphic coordinates around x_0 such that

$$\omega(x_0)=i\sum_{j=1}^n dz_j\wedge d\bar z_j,$$

i.e. such that $\{\partial/\partial z_j\}_{j=1,\dots,n}$ is a $\tilde h$-unitary basis at x_0. Now write the Chern curvature tensor at x_0 in these coordinates, to get

$$\Theta\big(T^{1,0}_X,\tilde h\big)_{x_0}=\sum_{j,k,l,m=1}^n c_{jklm}\,dz_j\wedge d\bar z_k\otimes\left(\frac{\partial}{\partial z_l}\right)^*\otimes\frac{\partial}{\partial z_m}.$$

Since ω is Kähler, and we have chosen a unitary basis at x_0, we have the well known Kähler symmetries for the coefficients c_{jklm}'s of the curvature:

$$c_{jklm}=c_{lmjk}=c_{lkjm}=c_{jmlk}=\overline{c_{kjml}}.$$

Now we express the holomorphic bisectional curvature in coordinates, to get

$$\operatorname{HBC}_\omega([v],[w])=\frac{1}{|v|^2|w|^2}\sum_{j,k,l,m=1}^n c_{jklm}\,v_j\bar v_k w_l\bar w_m,$$

where $v=\sum_j v_j\,\partial/\partial z_j$ and $w=\sum_j w_j\,\partial/\partial z_j$. In particular,

$$\operatorname{HSC}_\omega([v])=\frac{1}{|v|^4}\sum_{j,k,l,m=1}^n c_{jklm}\,v_j\bar v_k v_l\bar v_m.$$

Moreover, we get for the Chern–Ricci curvature and for the Chern–scalar curvature the following expressions:

$$\mathrm{Ric}_\omega = \frac{i}{2\pi}\sum_{j,k,l=1}^{n} c_{jkll}\, dz_j \wedge d\bar{z}_k,$$

and

$$\mathrm{scal}_\omega = \frac{1}{2\pi}\sum_{j,l} c_{jjll}.$$

Now, we compute

$$\fint_{S^{2n-1}} \mathrm{HBC}_\omega([v],[w])\, d\sigma(w) = \frac{1}{|v|^2}\sum_{j,k,l,m=1}^{n} c_{jklm}\, v_j \bar{v}_k \fint_{S^{2n-1}} w_l \bar{w}_m\, d\sigma(w).$$

The integral on the right hand side is easily seen to be zero unless $l = m$, in which case gives $1/n$. Thus, we obtain

$$\begin{aligned}\fint_{S^{2n-1}} \mathrm{HBC}_\omega([v],[w])\, d\sigma(w) &= \frac{1}{n|v|^2}\sum_{j,k,l=1}^{n} c_{jkll}\, v_j \bar{v}_k \\ &= -\frac{2\pi i}{n||v||^2_{\tilde{h}}}\,\mathrm{Ric}_\omega(v,\bar{v}).\end{aligned}$$

For the holomorphic sectional curvature we have instead

$$\fint_{S^{2n-1}} \mathrm{HSC}_\omega([v])\, d\sigma(v) = \sum_{j,k,l,m=1}^{n} c_{jklm} \fint_{S^{2n-1}} v_j \bar{v}_k v_l \bar{v}_m\, d\sigma(v).$$

Once again, it is easy to see that the integrals on the right hand side must be zero unless $j = k$ and $l = m$, or $j = m$ and $k = l$. We have (see for instance [Ber66] or [Div16, Lemma 2.2] for a detailed and more general computation):

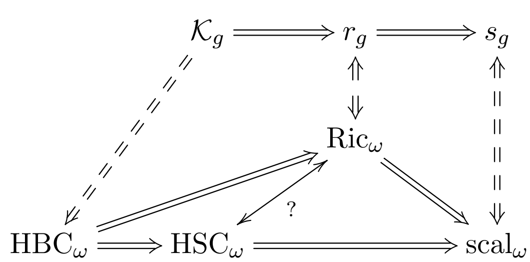
$$\fint_{S^{2n-1}} |v_j|^2 |v_k|^2\, d\sigma(v) = \begin{cases} \frac{2}{n(n+1)} & \text{if } j = k \\ \frac{1}{n(n+1)} & \text{otherwise,} \end{cases}$$

so that

$$\begin{aligned}\sum_{j,k,l,m=1}^{n} c_{jklm} \fint_{S^{2n-1}} v_j \bar v_k v_l \bar v_m \, d\sigma(v) &= \sum_{j,l=1}^{n} c_{jjll} \fint_{S^{2n-1}} |v_j|^2 |v_l|^2 \, d\sigma(v) \\ &\quad + \sum_{j,k=1}^{n} c_{jkkj} \fint_{S^{2n-1}} |v_j|^2 |v_k|^2 \, d\sigma(v) \\ &\quad - \sum_{j=1}^{n} c_{jjjj} \fint_{S^{2n-1}} |v_j|^4 \, d\sigma(v) \\ &= 2 \sum_{j,l=1}^{n} c_{jjll} \fint_{S^{2n-1}} |v_j|^2 |v_l|^2 \, d\sigma(v) \\ &\quad - \sum_{j=1}^{n} c_{jjjj} \fint_{S^{2n-1}} |v_j|^4 \, d\sigma(v) \\ &= 2 \sum_{j \neq l} c_{jjll} \fint_{S^{2n-1}} |v_j|^2 |v_l|^2 \, d\sigma(v) \\ &\quad + \sum_{j=1}^{n} c_{jjjj} \fint_{S^{2n-1}} |v_j|^4 \, d\sigma(v) \\ &= \frac{2}{n(n+1)} \sum_{j,l=1}^{n} c_{jjll} \\ &= \frac{4\pi}{n(n+1)} \operatorname{scal}_\omega(x_0),\end{aligned}$$

where, for the second equality, we have used the Kähler symmetry $c_{jkkj} = c_{jjkk}$. □

The situation can be therefore summarized as follows:



The arrows ⇒ in the diagram mean that the positivity (resp. semi-positivity, negativity, semi-negativity) of the source curvature implies the positivity (resp. semi-positivity, negativity, semi-negativity) of the target curvature. These arrows are always valid, even in the non Kähler setting. On the other hand, the dashed arrows are valid in the Kähler case only. It is however *a priori* unclear if and how the sign of the holomorphic sectional curvature propagates and determines the signs of the Ricci curvature.

So, Yau's conjecture deals exactly with this issue, at least in the case of negativity: if (X, ω) is a compact Kähler manifold such that $\mathrm{HSC}_\omega < 0$, then there exists a (possibly different) Kähler metric ω' on X such that $\mathrm{Ric}_{\omega'} < 0$. Note that, in particular, this implies that K_X is ample and therefore, by Kodaira's embedding theorem, that X is a projective algebraic manifold.

Remark 2.4. It is somehow embarrassing, but we don't dispose —at our best knowledge— any example of a (compact) Kähler manifold (X, ω) such that HSC_ω is negative but HBC_ω or Ric_ω do not have a sing. It would be of course highly desirable to have such an example, if any.

3. Motivations from birational geometry

We take the opportunity here to reproduce a quite standard argument in order to see how to reduce the Kobayashi conjecture on the ampleness of the canonical bundle of compact projective hyperbolic manifolds to showing that projective manifolds X with trivial first real Chern class are not hyperbolic. The same strategy can also be applied to have a proof, birational in spirit, of the Yau conjecture. All this, provided the abundance conjecture is true. Indeed, a lightly more general statement can be obtained and also the same kind of arguments can be applied to compact Kähler manifolds, as we shall see.

First of all, if the canonical bundle K_X is nef, then the abundance conjecture predicts that K_X should be semi-ample, *i.e.* some big tensor power of K_X should be generated by its global sections. This is known in dimension at most three.

So, let X be a smooth Kobayashi hyperbolic projective manifold. By the celebrated criterion of Mori, K_X is nef – otherwise X would contain a rational curve. Thus, K_X is already in the closure of the ample cone. Suppose now, and for the rest of the section, that the abundance conjecture holds true (in particular all we are saying hold in dimension at most three, unconditionally). Thus, we have that K_X is semi-ample. We are therefore able to use the following for the canonical bundle.

Theorem 3.1 (Semiample Iitaka fibrations [Laz04, Theorem 2.1.27])**.** *Let X be a normal projective variety and $L \to X$ a semi-ample line bundle on X. Then, there is an algebraic fiber space (i.e. a projective surjective mapping with connected fibers)*

$$\phi \colon X \to Y,$$

with $\dim Y = \kappa(L)$*, having the property that, for all sufficiently big and divisible integers m, it coincides with the map associated to the complete linear system $|L^{\otimes m}|$. Furthermore, there is an integer f and an ample line bundle A on Y such that $L^{\otimes f} \simeq \phi^* A$.*

Now, since K_X is semi-ample, we get an algebraic fiber space on X

$$\phi \colon X \to Y,$$

with $\dim Y = \kappa(X)$, and such that some power, say $K_X^{\otimes f}$, of the canonical bundle is the pull-back of an ample divisor A on Y. In particular, for every general (hence smooth) fiber F of ϕ, we have by taking the determinant of the short exact sequence

$$0 \to T_F \to T_X|_F \to \mathcal{O}_F^{\oplus \kappa(X)} \to 0,$$

that $K_F \simeq K_X|_F$. But then, $K_F^{\otimes f} \simeq K_X^{\otimes f}|_F \simeq \phi^* A|_F \simeq \mathcal{O}_F$. Thus, K_F is torsion, and the general fiber has Kodaira dimension zero and trivial first Chern class in real cohomology.

Suppose to be able to show that projective manifolds with trivial real first Chern class are not hyperbolic. We claim that this implies that the Kodaira dimension of a projective Kobayashi hyperbolic manifold X must be maximal, that is, X is of general type. Indeed, if $1 \le \kappa(X) < \dim X$, then ϕ has positive dimensional fibers and the general ones have zero Kodaira dimension. So we would, by our assumptions, find a non-Kobayashi hyperbolic positive dimensional subvariety of X, contradiction.

Now, if K_X is big and there are no rational curves on X it is not difficult to show that K_X is ample (cf. Lemma 5.1), and we are done.

Next, how to prove that a projective manifold X with trivial real first Chern class is not hyperbolic? By the Beauville–Bogomolov decomposition theorem [Bea83], a compact Kähler manifold with vanishing real first Chern class is, up to finite étale covers, a product of complex tori, Calabi–Yau manifolds and irreducible holomorphic symplectic manifolds. Since complex tori are obviously not Kobayashi hyperbolic, one is reduced to showing that Calabi–Yau manifolds and irreducible holomorphic symplectic manifolds are not Kobayashi hyperbolic (since Kobayashi hyperbolicity is preserved under étale covers). Very recently, in the spectacular paper [Ver15], Verbitsky has shown —among other things— that irreducible holomorphic symplectic manifolds with second Betti number greater than three (a condition that should indeed conjecturally hold for every irreducible holomorphic symplectic manifold) are not Kobayashi hyperbolic.

Thus, one of the main challenge is to show non hyperbolicity of Calabi–Yau manifolds. For such manifolds, much more is expected to be true: they should always contain rational curves! For several results in this direction, at least for Calabi–Yau manifolds with large Picard number, we refer the reader to [Wil89, Pet91, HBW92, Ogu93, DF14, DFM16], just to cite a few.

Coming back to curvature, of course possessing a Kähler metric whose holomorphic sectional curvature is negative implies Kobayashi hyperbolicity and thus having ample canonical bundle by the above discussion, provided the abundance conjecture is true. Therefore, this settles Yau's conjecture under the assumptions that abundance conjecture is true.

Now, what about compact Kähler manifolds with merely non positive holomorphic sectional curvature? Surely, they do not contain any rational curve (cf. Theorem 4.1). Thus, if X is projective, we conclude that K_X is nef as before by Mori. If X is merely Kähler, one needs to work more but the same conclusion of nefness for K_X holds true, thank to a very recent result by Tosatti and Yang [TY17] (which is a slight modification of the original Wu and Yau method [WY16]). Anyway, such a condition is not strong enough in order to obtain positivity of the canonical bundle, as flat complex tori immediately show. A less obvious but still easy counterexample is given by the product (with the product metric) of a flat torus and, say, a compact Riemann surface of genus greater than or equal to two endowed with its Poincaré metric. In this example, over each point there are some directions with strictly negative holomorphic sectional curvature but always some flat directions, too (we refer the reader to the recent paper [HLW14] for some nice results about this merely non positive case). So, if we look for the weakest condition, as long as the sign of holomorphic sectional curvature is concerned, for which one can hope to obtain the positivity of the canonical bundle, we are led to give the following (standard, indeed) definition.

Definition 3.2. The holomorphic sectional curvature is said to be *quasi-negative* if $\mathrm{HSC}_\omega \leq 0$ and moreover there exists at least one point $x \in X$ such that $\mathrm{HSC}_\omega(x,[v]) < 0$ for every $v \in T_{X,x} \setminus \{0\}$.

Remark 3.3. We shall see in the last section a slightly subtler condition on holomorphic sectional curvature, based on the notion of "truly flat" directions, very recently introduced by Heier, Lu, Wong, and Zheng in [HLWZ17], that should also work for this kind of purposes.

Now, why should we hope that such a condition would be sufficient? The reason comes again from the birational geometry of complex Kähler manifolds, and in particular again from the abundance conjecture. Let us illustrate why.

We begin with the following elementary observation.

prop:average **Proposition 3.4.** *Let (X,ω) be a compact Kähler manifold with $\mathrm{HSC}_\omega \leq 0$, and suppose there exists a direction $[v] \in P(T_{X,x_0})$ such that $\mathrm{HSC}_\omega(x_0,[v]) < 0$, for some $x_0 \in X$. Then, $c_1(X) \in H^2(X,\mathbb{R})$ cannot be zero.*

Proof. By Proposition 2.3, we know that scal_ω is everywhere non positive, and moreover, as an average, it is strictly negative at x_0. In particular, the total scalar curvature of ω is strictly negative. The conclusion follows from Remark 2.2. □

As a direct consequence, if X is moreover projective and $\mathrm{Pic}(X)$ is infinite cyclic, then K_X must be ample. This gives back (and slightly generalize) a result of [WWY12].

Now, let (X,ω) be a compact Kähler manifold with quasi-negative holomorphic sectional curvature. Then, Proposition 3.4 implies that X cannot have trivial first real Chern class. Moreover, since being quasi-negative is stronger than being non positive, we saw that, thanks to [TY17], K_X is nef.

Once again, suppose that the abundance conjecture holds true, but now also for compact Kähler manifolds. Then, K_X is semi-ample and we can consider exactly as before the semi-ample Iitaka fibration for K_X. Since X has non trivial first real Chern class, we must have that $\kappa(X) > 0$, otherwise some power of the canonical bundle would be a pull-back of a (ample) line bundle over point, and thus would be trivial!

If $\kappa(X) = \dim X$, then X would be birational to a projective variety, *i.e.* would be a Moishezon manifold. By Moishezon's theorem, a compact Kähler Moishezon manifold is projective. Moreover, X is without rational curves and of general type, and we conclude as before that K_X must be ample.

Next, suppose by contradiction that $1 \leq \kappa(X) \leq \dim X - 1$ so that if we call F the general fiber of ϕ, we have that F is a smooth compact Kähler manifold of positive dimension and different from X itself. Now, on the one hand, the short exact sequence of the fibration shows that $K_F \simeq K_X|_F$ and therefore it follows that $c_1(F)$ must be zero in real cohomology. On the other hand, the classical Griffiths' formulae for curvature of holomorphic vector bundles imply that the holomorphic sectional curvature decreases when passing to submanifolds, that is for every $x \in F \subset X$

$$\mathrm{HSC}_{\omega|_F}(x,[v]) \leq \mathrm{HSC}_{\omega}(x,[v]),$$

where $v \in T_{F,x}$ and, in the right hand side, v is seen as a tangent vector to X.

The quasi-negativity of the holomorphic sectional curvature implies, since F is a general fiber, that there exists a tangent vector to F along which the holomorphic sectional curvature of $\omega|_F$ is strictly negative. Thus, Proposition 3.4 implies that F cannot have trivial first real Chern class, which is absurd.

As a consequence, me may indeed hope to extend Wu–Yau–Tosatti–Yang theorem to the optimal, quasi-negative case. This is precisely the main contribution of the paper [DT16].

thm:main **Theorem 3.5** ([DT16, Theorem 1.2])**.** *Let (X,ω) be a connected compact Kähler manifold. Suppose that the holomorphic sectional curvature of ω is quasi-negative. Then, K_X is ample. In particular, X is projective.*

We shall spend some words on this result in the last section.

Remark 3.6. To finish this section with, unfortunately, we must confess that we are not aware of any example of a compact Kähler manifold with a Kähler metric whose holomorphic sectional curvature is quasi-negative but which does not posses any Kähler metric with strictly negative holomorphic sectional curvature. In other word, is Theorem 3.5 a true generalization of Wu–Yau–Tosatti–Yang result? We believe so. Then, such an example, if any, would be urgently needed!

4. An example by J.-P. Demailly

In this section we would like to explain, following [Dem97, §8], how one can construct examples of compact hyperbolic projective manifolds which nevertheless do not admit any hermitian metric of negative holomorphic sectional curvature. Such examples can be generalized to higher order analogues — namely "k-jet curvature"— of holomorphic sectional curvature: this will be mentioned at the end of the section, and related conjectures that come out from this picture will be discussed at the end of the chapter.

The first observation is the following algebraic criterium for the nonexistence of a metric with negative holomorphic sectional curvature. Let X be a complex manifold, C be a compact Riemann surface, and $F\colon C \to X$ be a non constant holomorphic map. Let $m_p \in \mathbb{N}$ be the multiplicity at $p \in C$ of F. Clearly, $m_p = 1$ except possibly at finitely many points of C, and $m_p \geq 2$ if and only if F is not an immersion at p.

algcrit **Theorem 4.1** (Demailly [Dem97, Special case of Theorem 8.1])**.** *Consider (X,ω) a compact hermitian manifold and let $F\colon C \to X$ be a non constant holomorphic map from a compact Riemann surface C of genus $g = g(C)$ to X. Suppose that* $\mathrm{HSC}_\omega \leq -\kappa$ *for some $\kappa \geq 0$. Then,*

$$2g-2 \geq \frac{\kappa}{2\pi}\deg_\omega C + \sum_{p\in C}(m_p-1),$$

where $\deg_\omega C = \int_C F^*\omega > 0$ is the degree of C with respect to ω.

We shall use this theorem especially in the case where C is the normalization of a singular curve in X and F the normalization map. Observe that, in particular, we recover the well-known fact that on a compact hermitian manifold with negative holomorphic sectional curvature there are no rational nor elliptic curves (even singular), and that there are no rational (possibly singular) curves on a compact hermitian manifold with non positive holomorphic sectional curvature.

Proof. The differential F' of F gives us a map $F'\colon T_C \to F^*T_X$. This map is injective at the level of sheaves, but not necessarily at the level of vector bundle, since F' may vanish at some point. Taking into account these vanishing points counted with multiplicities, we obtain the following injection of vector bundles

$$F'\colon T_C \otimes \mathcal{O}_C(D) \to F^*T_X,$$

where we defined the effective divisor D to be $\sum_{p\in C}(m_p-1)\,p$. Thus, via F', we realized $T_C \otimes \mathcal{O}_C(D)$ as a subbundle of the hermitian vector bundle $(F^*T_X, F^*\omega)$, and —as such— we can endow it with the induced metric $h = F^*\omega|_{T_C\otimes\mathcal{O}_C(D)}$ (observe the we consider $F^*\omega$ not as a pull-back of differential forms, but as a pull-back of hermitian metrics).

Now, a local holomorphic frame for $T_C \otimes \mathcal{O}_C(D)$ around a point $p \in C$ is given by $\eta(t) = 1/t^{m_p-1}\,\frac{\partial}{\partial t}$, where t is a holomorphic coordinate centered at

p. Call $\xi(t) = F'(\eta(t)) \in (F^*T_X)_t = T_{X,F(t)}$, so that ξ is a local holomorphic frame for $T_C \otimes \mathcal{O}_C(D)$ when seen as a subbundle of F^*T_X. We have, for the Griffiths curvature of $(T_C \otimes \mathcal{O}_C(D), h)$,

$$\begin{aligned}&\langle \Theta(T_C \otimes \mathcal{O}_C(D), h)(\partial/\partial t, \partial/\partial \bar t\,) \cdot \xi, \xi\rangle_h \\ &\qquad\qquad = \Theta(T_C \otimes \mathcal{O}_C(D), h)(\partial/\partial t, \partial/\partial \bar t\,)\, \underbrace{||\xi||_h^2}_{=||\xi||_\omega^2}.\end{aligned}$$

By the classical Griffiths' formulae, we have the following decreasing property for the Griffiths curvatures:

$$\begin{aligned}&\langle \Theta(T_C \otimes \mathcal{O}_C(D), h)(\partial/\partial t, \partial/\partial \bar t\,) \cdot \xi, \xi\rangle_h \\ &\qquad \le \langle \Theta(F^*T_X, F^*\omega)(\partial/\partial t, \partial/\partial \bar t\,) \cdot \xi, \xi\rangle_{F^*\omega} \\ &\qquad = \langle F^*\Theta(T_X, \omega)(\partial/\partial t, \partial/\partial \bar t\,) \cdot \xi, \xi\rangle_{F^*\omega} \\ &\qquad = \langle \Theta(T_X, \omega)(F'(\partial/\partial t), \overline{F'(\partial/\partial t)}\,) \cdot \xi, \xi\rangle_\omega \\ &\qquad = |t^{m_p-1}|^2\, \langle \Theta(T_X, \omega)(\xi, \bar\xi\,) \cdot \xi, \xi\rangle_\omega \le -\kappa |t^{m_p-1}|^2 ||\xi||_\omega^4,\end{aligned}$$

where the last inequality holds since $\langle \Theta(T_X, \omega)(\xi, \bar\xi\,) \cdot \xi, \xi\rangle_\omega = ||\xi||_\omega^4\ \mathrm{HSC}_\omega(\xi)$. Therefore, we obtain

$$\Theta(T_C \otimes \mathcal{O}_C(D), h)(\partial/\partial t, \partial/\partial \bar t\,) \le -\kappa |t^{m_p-1}|^2 ||\xi||_\omega^2 = i\kappa\,(F^*\omega)(\partial/\partial t, \partial/\partial \bar t\,),$$

where by $F^*\omega$ here we mean the pull-back at the level of differential forms. Summing up, we have obtained that

$$i\,\Theta(T_C \otimes \mathcal{O}_C(D), h) \le -\kappa\, F^*\omega,$$

as real $(1,1)$-forms. But then,

$$\int_C \frac{i}{2\pi}\,\Theta(T_C \otimes \mathcal{O}_C(D), h) \le -\frac{\kappa}{2\pi}\int_C F^*\omega = -\frac{\kappa}{2\pi}\deg_\omega C,$$

and

$$\int_C \frac{i}{2\pi}\,\Theta(T_C \otimes \mathcal{O}_C(D), h) = \deg(T_C \otimes \mathcal{O}_C(D)) = 2 - 2g + \sum_{p\in C}(m_p - 1),$$

since $\deg(T_C) = 2 - 2g$ by Hurwitz's formula. The statement follows. $\square$

Following Demailly, we shall now exhibit a smooth projective surface which is Kobayashi hyperbolic, with ample canonical bundle, but which cannot admit any hermitian metric with negative holomorphic sectional curvature. It will be constructed as a fibration of Kobayashi hyperbolic curves onto a Kobayashi hyperbolic curve, with at least one "very" singular fiber, which will violate the above criterium.

Proposition 4.2 (Cf. [Dem97, 8.2. Theorem]). *There is a smooth projective surface S which is hyperbolic (and hence with ample canonical bundle K_S) but does not carry any hermitian metric with negative holomorphic sectional curvature. Moreover, given any two smooth compact hyperbolic Riemann surfaces Γ, Γ', such a surface can be obtained as a fibration $S \to \Gamma$,* *with hyperbolic fibers, in which (at least) one of the fibers is singular and has Γ' as its normalization.*

Proof. Take any compact hyperbolic Riemann surface Γ', and let $g = g(\Gamma') \geq 2$ be its genus. Now, we modify it into a singular compact Riemann surface Γ'' of the same genus, whose normalization is Γ'.

In order to do so, consider a pair of positive relatively prime integers (a, b), with $a < b$, an the associated affine plane curve C in $\mathbb{C}^2$ given by the equation $y^a - x^b = 0$, which has a monomial singularity of type (a, b) at $0 \in \mathbb{C}^2$. Its normalization is given by $\mathbb{C} \ni t \mapsto (t^a, t^b) \in \mathbb{C}^2$. Choose integers n, m such that $na+mb = 1$. Then, the restriction of the rational function on $\mathbb{C}^2$ defined by $(x, y) \mapsto x^n y^m$ to C gives a holomorphic coordinate on it minus the singular point (this is actually the inverse map of the normalization map outside the singularity). In particular, the set of points $(x, y) \in C$ such that $0 < |x| < 1$ is biholomorphic to the punctured unit disc.

Now, take a point $x_0 \in \Gamma'$ and choose a holomorphic coordinate centered at x_0 such that we can select a neighborhood of x_0 whose image is the unit disc *via* this coordinate. Finally, remove the point x_0 in order to obtain a holomorphic coordinate chart whose image is the punctured unit disc. By identifying with the punctured unit disc constructed above, we replace this neighborhood of x_0 with the set of point $(x, y) \in C$ such that $|x| < 1$, thus creating the desired singularity at x_0. Call the resulting curve Γ''. By construction, the normalization of Γ'' is exactly Γ', and Γ'' has one single singular point, whose singularity type is plane and monomial of type (a, b) (for an excellent and very elementary discussion around this subject we refer the reader to [Mir95, Chapter III, Section 2]).

Next, we embed Γ'' in some large projective space, and then we project it to $\mathbb{P}^2$, in such a way that the singular point is left untouched and outside it we create at most a finite number of nodes (*i.e.* plane monomial singularity of type $(2, 2)$). Call the resulting projective plane curve C_0, whose normalization is of course again Γ'. Observe that the normalization map $\nu\colon \Gamma' \to C_0$ is an immersion outside the (single) preimage of the first singular point we created. On the other hand, at this point it has multiplicity a.

In order to obtain the desired surface S, we select then a so that $a - 1 > 2g-2$, *i.e.* $a \geq 2g$. Such a surface S then does contain a curve which violates the criterium given in Theorem 4.1, and we are done.

Take a (reduced) homogeneous polynomial equation $P_0(z_0, z_1, z_2) = 0$ for C_0 in $\mathbb{P}^2$. Then, we necessarily have $d = \deg P_0 \geq 4$, since otherwise C_0 would be normalized by a rational or an elliptic curve. Next, complete P_0 into a basis $\{P_0, P_1, \dots, P_N\}$ of the space $H^0(\mathbb{P}^2, \mathcal{O}(d))$ of homogeneous polynomials of degree d in three variables, and consider the corresponding universal family

$$\mathcal{U}=\Big\{\big([z_0:z_1:z_2],[\alpha_0:\dots:\alpha_N]\big)\in\mathbb{P}^2\times\mathbb{P}^N\mid\sum_{j=0}^N\alpha_j\,P_j(z)=0\Big\}\subset\mathbb{P}^2\times\mathbb{P}^N,$$

of curves of degree d in $\mathbb{P}^2$, together with the projection $\pi\colon\mathcal{U}\to\mathbb{P}^N$. Our starting curve C_0 is then the fiber $U_{[1:0:\cdots:0]}$ over the point $[1:0:\cdots:0]\in\mathbb{P}^N$. Now, we embed the first curve Γ into $\mathbb{P}^N$ (this is of course possible since $N\geq 3$) in such a way that $[1:0:\cdots:0]\in\Gamma$. The desired fibration $S\to\Gamma$ will be obtained as the pull-back family

$$\begin{array}{ccc} S=\mathcal{U}\times_{\mathbb{P}^N}\Gamma & \longrightarrow & \mathcal{U} \\ \downarrow & & \downarrow \\ \Gamma & \hookrightarrow & \mathbb{P}^N. \end{array}$$

Of course, we have to select carefully the embedding of Γ into $\mathbb{P}^N$, so that S will be non singular, and in such a way that we have a good control of the singular fibers out of $U_{[1:0:\cdots:0]}$.

In order to do so, the first observation is that —as it is well-known— the locus Z in $\mathbb{P}^N$ which corresponds to singular curve is an algebraic hypersurface and, moreover, the locus $Z'\subset Z$ which corresponds to curves which have not only one node in their singularity set is of codimension 2 in $\mathbb{P}^N$. In particular, by possibly moving Γ with a generic projective automorphism of $\mathbb{P}^N$ leaving fixed $[1:0:\cdots:0]$, we can suppose that $\Gamma\cap Z'=\{[1:0:\cdots:0]\}$, so that all the fibers of S, except from C_0, are either smooth, or with a single node. If such an S were non singular, we would be done. Indeed, by Plücker's formula, the smooth fibers have genus $(d-1)(d-2)/2\geq 3$, $U_{[1:0:\cdots:0]}$ has genus $g\geq 2$ by construction, and the other singular fibers have genus $(d-1)(d-2)/2-1\geq 2$, since they have only one node. Therefore, S is a fibration onto a hyperbolic Riemann surface with all hyperbolic fibers and is then hyperbolic (and hence with ample canonical bundle), with a fiber which contradicts Theorem 4.1.

So we are left to check the smoothness of S, knowing that we can possibly use again generic automorphisms of $\mathbb{P}^N$ leaving fixed $[1:0:\cdots:0]$ to move Γ. Thus, since Γ is embedded in $\mathbb{P}^N$, we can think at S as included in $\mathcal{U}$, and since $\mathcal{U}$ is smooth, Bertini's theorem immediately implies that S can be chosen non singular outside $U_{[1:0:\cdots:0]}$. Now, what about points along $U_{[1:0:\cdots:0]}$? Fix such a point $([z_0:z_1:z_2],[1:0:\cdots:0])\in U_{[1:0:\cdots:0]}$, and suppose, just to fix ideas, that $z_0\neq 0$. Take the corresponding affine coordinates, say $((z,w),(a_1,\dots,a_N))$ around this point, set $p_j(z,w)=P_j(1,z,w)$ to be the dehomogenization of the P_j's, and let $f_1(a),\dots,f_r(a)$ be affine equations of the curve Γ. Then, we have to check the rank of the following Jacobian matrix at the point $\big((z,w),(0,\dots,0)\big)$, the affine equation for $\mathcal{U}$

being $p_0(z,w) + \sum_{j=1}^N a_j\, p_j(z,w) = 0$:

$$)\,($$

□

5. The Wu–Yau theorem and its generalizations

In this section we go into the details of the proof of Wu–Yau's theorem on the positivity of the canonical class for projective manifolds endowed with a Kähler metric of negative holomorphic sectional curvature. We will present a proof which follows, for the first part, almost *verbatim* the original proof of Wu and Yau. On the other hand, the conclusion will be achieved with an approach which is more pluripotential in flavor, taken from [DT16]. Finally, we shall discuss at the end of this section several generalizations of this result (including the Kähler case, and weaker notions of negativity).

The proof is achieved in essentially three steps, after a reduction as follows. As we have seen, the negativity of the curvature (or even its non-positivty) implies the non existence of rational curves on X. Then, by Mori's Cone Theorem, we deduce that the canonical bundle of X is nef. But then, it is sufficient to prove that $c_1(K_X)^n > 0$, which in this case means that the canonical bundle is big. Indeed, if K_X is big and there are no rational curves on X one can conclude the ampleness of the canonical bundle via the following standard lemma.

lem:ratcurv **Lemma 5.1** (Exercise 8, page 219 of [Deb01])**.** *Let X be a smooth projective variety of general type which contains no rational curves. Then, K_X is ample.*

Here is a proof, for the sake of completeness.

Proof. Since there are no rational curves on X, Mori's theorem implies as above that K_X is nef. Since K_X is big and nef, the Base Point Free theorem tells us that K_X is semi-ample. If K_X were not ample, then the morphism defined by (some multiple of) K_X would be birational but not an isomorphism. In particular, there would exist an irreducible curve $C \subset X$ contracted by this morphism. Therefore, $K_X \cdot C = 0$. Now, take any very ample divisor H. For any $\varepsilon > 0$ rational and small enough, $K_X - \varepsilon H$ remains big and thus some large positive multiple, say $m(K_X - \varepsilon H)$, of $K_X - \varepsilon H$ is linearly equivalent to an effective divisor D. Set $\Delta = \varepsilon' D$, where $\varepsilon' > 0$ is a rational number. We have:

$$\begin{aligned}(K_X + \Delta) \cdot C &= \varepsilon'\, D \cdot C \\ &= \varepsilon' m (K_X - \varepsilon H) \cdot C \\ &= -\varepsilon\varepsilon' m\, H \cdot C < 0.\end{aligned}$$

Finally, if ε' is small enough, then (X, Δ) is a klt pair. Thus, the (logarithmic version of the) Cone Theorem would give the existence of an extremal ray generated by the class of a rational curve in X, contradiction. □

Keeping in mind that what we have to show is then that $c_1(K_X)^n > 0$, we illustrate now the steps of the proof.

Step 1: Solving an approximate Kähler–Einstein equation. Let ω be our fixed Kähler metric (with negative holomorphic sectional curvature, but we shall not use this hypothesis for the moment).

step1 *Claim* 5.2. For each $\varepsilon > 0$ there exists a unique smooth function $u_\varepsilon\colon X \to \mathbb{R}$ such that

$$\omega_\varepsilon := \varepsilon\omega - \operatorname{Ric}_\omega + i\partial\bar\partial u_\varepsilon$$

is a positive $(1,1)$-form (hence Kähler, belonging to the cohomology class $c_1(K_X) + \varepsilon[\omega]$) form satisfying the Monge–Ampère equation

$$\omega_\varepsilon^n = e^{u_\varepsilon}\,\omega^n.$$

In particular,

$$\operatorname{Ric}(\omega_\varepsilon) = -\omega_\varepsilon + \varepsilon\omega,$$

whence the terminology "approximate Kähler–Einstein", and we have the following uniform upper bound:

$$\sup_X u_\varepsilon \le C,$$

where the constant C depends only on ω and $n = \dim X$. Observe finally, that in particular, $\operatorname{Ric}(\omega_\varepsilon) \ge -\omega_\varepsilon$.

Step 2: A laplacian estimate involving the holomorphic sectional curvature. This step is somehow a refinement of the laplacian estimate needed in order to achieve the classical C^2-estimates to solve the complex Monge–Ampère equation on compact Kähler manifolds. In the classical setting an upper bound for the holomorphic bisectional curvature is used. Here we shall employ a lemma due to Royden in order to use only the weaker information given by the bound on the holomorphic sectional curvature, as in the hypotheses. The crucial part of this step is the following.

step2 *Claim* 5.3. Suppose $-\kappa < 0$ is an upper bound for the holomorphic sectional curvature of ω. Suppose moreover that ω' is another Kähler metric on X whose Ricci curvature is comparable with ω and ω' as follows:

$$\operatorname{Ric}(\omega') \ge -\lambda\omega' + \mu\omega,$$

where λ, μ are non negative constants. Define a smooth function $S\colon X \to \mathbb{R}_{>0}$ to be the trace of ω with respect to ω', *i.e.*

$$S := \operatorname{tr}_{\omega'}\omega = n\,\frac{\left(\omega'\right)^{n-1}\wedge\omega}{\left(\omega'\right)^n}.$$

Then, the following differential inequality holds:

diffineq (1) $$-\Delta_{\omega'}\log S \ge \left(\frac{\kappa(n+1)}{2n} + 2\pi\frac{\mu}{n}\right)S(x_0) - 2\pi\lambda.$$

Observing that

$$\int_X \Delta_{\omega'} \log S \left(\omega'\right)^n = 0,$$

we shall use the inequality above with $\omega' = \omega_\varepsilon$, $\lambda = 1$, and $\mu = 0$, in the following integral form:

intineq

$$\int_X \frac{(n+1)\kappa}{2n} S_\varepsilon \, \omega_\varepsilon^n \leq 2\pi \int_X \omega_\varepsilon^n, \tag{2}$$

where we added the subscript ε to S in order to emphasize the dependence of S from ε.

Step 3: Proof of the key inequality. One wants to show that $c_1(K_X)^n > 0$. Since $\omega_\varepsilon = -\operatorname{Ric}(\omega_\varepsilon) + \varepsilon\omega$, we have that

$$\omega_\varepsilon^n = \big(-\operatorname{Ric}(\omega_\varepsilon)\big)^n + O(\varepsilon).$$

But then,

$$\int_X \omega_\varepsilon^n = \int_X \big(-\operatorname{Ric}(\omega_\varepsilon)\big)^n + O(\varepsilon).$$

On the other hand,

$$\int_X \big(-\operatorname{Ric}(\omega_\varepsilon)\big)^n = c_1(K_X)^n$$

is independent from ε, and thus

$$c_1(K_X)^n = \lim_{\varepsilon \to 0^+} \int_X \omega_\varepsilon^n.$$

What we want is therefore to show the positivity of such a limit.

step3

Claim 5.4. The limit

$$\lim_{\varepsilon \to 0^+} \int_X \omega_\varepsilon^n$$

is strictly positive.

This is what we call the key inequality. During the proof of the main result, this will be the only step where what we present here differs from Wu–Yau's original approach.

We now proceed with the proof of the various claims stated above.

Proof of Claim step1 *5.2.* The first observation is that, since K_X is nef, for each $\varepsilon > 0$ the cohomology class $c_1(K_X) + \varepsilon\,[\omega] = -c_1(X) + \varepsilon\,[\omega]$ is a Kähler class. This implies, thanks to the $\partial\bar\partial$-lemma, that there exists a smooth real function f_ε on X, unique up to an additive constant, such that

$$\omega_{f_\varepsilon} := \varepsilon\,\omega - \operatorname{Ric}_\omega + \frac{i}{2\pi}\,\partial\bar\partial f_\varepsilon$$

is a Kähler form on X.

Now we use the following theorem, in order to obtain an approximate Kähler–Einstein metric on X. We give here Yau's original general statement, which is the key ingredient to get his celebrated solution of the Calabi conjecture.

solCalabi **Theorem 5.5** (Yau [Yau78]). *Let (X,ω_0) be a compact Kähler manifold, and $F\colon X\times\mathbb{R}_t\to\mathbb{R}$ smooth function such that $\partial F/\partial t\geq 0$. Suppose that there exists smooth function $\psi\colon X\to\mathbb{R}$ such that*

$$\int_X e^{F(x,\psi(x))}\,\omega_0^n=\int_X\omega_0^n.$$

Then, there exists a unique smooth function $\varphi\colon X\to\mathbb{R}$, such that

$$\begin{cases}\omega_0+\frac{i}{2\pi}\,\partial\bar\partial\varphi>0,\\ \left(\omega_0+\frac{i}{2\pi}\,\partial\bar\partial\varphi\right)^n=e^{F(x,\varphi(x))}\,\omega_0^n.\end{cases}$$

From this statement one can derive easily both the existence of a Kähler metric in a fixed Kähler class with prescribed volume form (or, equivalently, Ricci tensor), and the existence of Kähler–Einstein metrics on compact Kähler manifold with negative (resp. zero) real first Chern class.

Now, we fix $\varepsilon>0$, and define a smooth real function α_ε on X implicitly by

$$\omega_{f_\varepsilon}^n=e^{-\alpha_\varepsilon}\,\omega^n.$$

We then apply the theorem above with the following data:

$$\omega=\omega_{f_\varepsilon},\quad F(x,t)=t+\alpha_\varepsilon(x)+f_\varepsilon(x).$$

Then, there exists a unique smooth real function v_ε such that

$$\begin{aligned}\left(\omega_{f_\varepsilon}+\frac{i}{2\pi}\,\partial\bar\partial v_\varepsilon\right)^n&=e^{v_\varepsilon+\alpha_\varepsilon+f_\varepsilon}\,\omega_{f_\varepsilon}^n\\ &=e^{v_\varepsilon+f_\varepsilon}\,\omega^n,\end{aligned}$$

and

$$\omega_\varepsilon:=\omega_{f_\varepsilon}+\frac{i}{2\pi}\,\partial\bar\partial v_\varepsilon>0$$

on X. Now, define u_ε to be the sum $f_\varepsilon+v_\varepsilon$, so that it holds

$$\omega_\varepsilon^n=e^{u_\varepsilon}\,\omega^n.$$

Thus, we get for the Ricci curvature of ω_ε

$$\begin{aligned}\mathrm{Ric}_{\omega_\varepsilon}&=-\frac{i}{2\pi}\partial\bar\partial\log\omega_\varepsilon^n\\ &=-\frac{i}{2\pi}\partial\bar\partial v_\varepsilon\underbrace{-\frac{i}{2\pi}\partial\bar\partial f_\varepsilon\overbrace{-\frac{i}{2\pi}\partial\bar\partial\log\omega^n}^{=\mathrm{Ric}_\omega}}_{=\varepsilon\,\omega-\omega_{f_\varepsilon}}\\ &=\varepsilon\,\omega-\omega_\varepsilon.\end{aligned}$$

In particular, $\mathrm{Ric}_{\omega_\varepsilon}\geq-\omega_\varepsilon$.

Now, we use the maximum principle in order to obtain the desired uniform upper bound for u_ε. To do so, pick a point $x_0 \in X$ such that $\sup_X u_\varepsilon = u_\varepsilon(x_0)$. Then, at this point we have that the complex hessian of u_ε is negative semi-definite, *i.e.* $i\,\partial\bar\partial u_\varepsilon(x_0) \leq 0$. Thus,

$$\begin{aligned}\omega_\varepsilon(x_0) &= \big(\varepsilon\,\omega - \mathrm{Ric}_\omega + \frac{i}{2\pi}\,\partial\bar\partial u_\varepsilon\big)(x_0)\\ &\leq \big(\varepsilon\,\omega - \mathrm{Ric}_\omega\big)(x_0)\\ &\leq \big(\varepsilon_0\,\omega - \mathrm{Ric}_\omega\big)(x_0),\end{aligned}$$

if $\varepsilon_0 > \varepsilon$. Therefore,

$$\begin{aligned}e^{\sup_X u_\varepsilon} &= e^{u_\varepsilon(x_0)}\\ &= \frac{\big(\varepsilon\,\omega - \mathrm{Ric}_\omega + \frac{i}{2\pi}\,\partial\bar\partial u_\varepsilon\big)(x_0)}{\omega^n(x_0)}\\ &\leq \frac{\big(\varepsilon_0\,\omega - \mathrm{Ric}_\omega\big)(x_0)}{\omega^n(x_0)} =: e^C,\end{aligned}$$

so that

$$\sup_X u_\varepsilon \leq C, \quad \forall \varepsilon < \varepsilon_0.$$

This complete the proof of Claim step1 5.2. □

Proof of Claim step2 *5.3.* Let $x_0 \in X$ be a fixed point. Chose holomorphic normal coordinates $(z_1, \dots, z_n)$ with respect to ω, centered at x_0. Without loss of generality, by a constant ω-unitary change of variables, we may also suppose that ω' is diagonalized with respect to ω at x_0. Thus we write

$$\omega = i \sum_{l,m=1}^{n} \omega_{lm}\, dz_l \wedge d\bar z_m, \quad \omega_{lm}(z) = \delta_{lm} - \sum_{j,k=1}^{n} c_{jklm}\, z_j \bar z_k + O(|z^3|),$$

where the c_{jklm}'s are the coefficients of the Chern curvature tensor of (X, ω) at x_0, and

$$\omega' = i \sum_{l,m=1}^{n} \omega'_{lm}\, dz_l \wedge d\bar z_m, \quad \omega'_{lm}(z) = \lambda_l\, \delta_{lm} + O(|z|),$$

where the λ_j's are the eigenvalues at x_0 of ω' with respect to ω. In particular, $\lambda_j > 0$, $j = 1, \dots, n$. Next, call ρ'_{jk} the coefficients of the Ricci curvature of ω', so that

$$\mathrm{Ric}_{\omega'} = \frac{i}{2\pi} \sum_{j,k=1}^{n} \rho'_{jk}\, dz_j \wedge d\bar z_k,$$

where

$$\rho'_{jk} = \sum_{l=1}^{n} c'_{jkll}.$$

With these notations, the starting point is the following

Lemma 5.6 (See WYZ09 [WYZ09, pag. 371])**.** *The following differential equality holds:*

difeq
$$-\Delta_{\omega'} S(x_0) = \sum_{l=1}^{n} \frac{\rho'_{ll}}{(\lambda_l)^2} + \sum_{j,l,a=1}^{n} \frac{\left|\partial\omega'_{al}/\partial z_j\right|^2}{\lambda_j(\lambda_l)^2\lambda_a} - \sum_{j,l=1}^{n} \frac{c_{jjll}}{\lambda_j\lambda_l}, \tag{3}$$

where the right hand side is intended to be computed at x_0.

Proof. A straightforward computation, using the adjugate matrix method to obtain the inverse, shows that

$$S = \sum_{l,m=1}^{n} \Omega'_{ml}\,\omega_{lm},$$

where we define (Ω'_{lm}) to be the inverse matrix of (ω'_{lm}). We want to compute $\Delta_{\omega'} S$ at x_0. We have, by the basic commutation relations in Kähler geometry,

$$\Delta_{\omega'} S = \bar\partial^*\bar\partial S = -i\Lambda_{\omega'}\partial\bar\partial S,$$

and thus, since acting with $\Lambda_{\omega'}$ on real $(1,1)$-forms amounts to takeing the trace with respect to ω',

$$\Delta_{\omega'} S = -\operatorname{tr}_{\omega'} i\partial\bar\partial S = -\sum_{j,k=1}^{n} \Omega'_{kj}\,\frac{\partial^2 S}{\partial z_j\partial\bar z_k}.$$

Now,

$$\begin{aligned}\frac{\partial^2 S}{\partial z_j\partial\bar z_k} &= \frac{\partial^2}{\partial z_j\partial\bar z_k}\sum_{l,m=1}^{n} \Omega'_{ml}\,\omega_{lm}\\ &= \sum_{l,m=1}^{n} \omega_{lm}\frac{\partial^2\Omega'_{ml}}{\partial z_j\partial\bar z_k} + \Omega'_{ml}\frac{\partial^2\omega_{lm}}{\partial z_j\partial\bar z_k} + \underbrace{\frac{\partial\Omega'_{ml}}{\partial z_j}\frac{\partial\omega_{lm}}{\partial\bar z_k} + \frac{\partial\omega_{lm}}{\partial z_j}\frac{\partial\Omega'_{ml}}{\partial\bar z_k}}_{:=R_{jklm}}.\end{aligned}$$

At the end of the day, thanks to the choice of geodesic coordinates, the terms with only one derivative involved —which we called R_{jklm}— will disappear. Therefore, we only have to understand the summands with two derivatives of Ω'_{ml}, and express them in terms of the ω'_{lm}'s. In order to do this, call $H = (\omega'_{lm})$, so that $H^{-1} = (\Omega'_{lm})$ and observe that

partialH-1
$$0 \equiv \partial(HH^{-1}) = \partial H H^{-1} + H\partial H^{-1}, \tag{4}$$

barpartialH-1
$$0 \equiv \bar\partial(HH^{-1}) = \bar\partial H H^{-1} + H\bar\partial H^{-1}, \tag{5}$$

and

$$0 \equiv \partial\bar\partial(HH^{-1}) = \partial\bar\partial H H^{-1} - \bar\partial H \wedge \partial H^{-1} + \partial H \wedge \bar\partial H^{-1} + H\partial\bar\partial H^{-1}.$$

We obtain therefore the matrix identity

$$\partial\bar\partial H^{-1} = -H^{-1}\partial\bar\partial H H^{-1} - H^{-1}\bar\partial H \wedge H^{-1}\partial H H^{-1} + H^{-1}\partial H \wedge H^{-1}\bar\partial H H^{-1},$$

which gives us the following expression for the second derivatives of Ω'_{ml}:

$$\frac{\partial^2\Omega'_{ml}}{\partial z_j \partial\bar z_k} = -\sum_{a,b=1}^{n} \Omega'_{ma}\frac{\partial^2\omega'_{ab}}{\partial z_j\partial\bar z_k}\Omega'_{bl} + \sum_{a,b,p,q=1}^{n} \Omega'_{mp}\frac{\partial\omega'_{pq}}{\partial\bar z_k}\Omega'_{qa}\frac{\partial\omega'_{ab}}{\partial z_j}\Omega'_{bl} + \Omega'_{ma}\frac{\partial\omega'_{ab}}{\partial z_j}\Omega'_{bp}\frac{\partial\omega'_{pq}}{\partial\bar z_k}\Omega'_{ql}.$$

Thus, we get the following expression for the ω'-Laplacian:

exprlapl

$$\begin{aligned}
\Delta_{\omega'} S &= -\sum_{j,k,l,m=1}^{n} \Omega'_{kj}\left(\omega_{lm}\frac{\partial^2\Omega'_{ml}}{\partial z_j\partial\bar z_k} + \Omega'_{ml}\frac{\partial^2\omega_{lm}}{\partial z_j\partial\bar z_k} + R_{jklm}\right) \\
&= -\sum_{j,k,l,m=1}^{n} \Omega'_{kj}\Omega'_{ml}\frac{\partial^2\omega_{lm}}{\partial z_j\partial\bar z_k} + \Omega'_{kj}\omega_{lm}\frac{\partial^2\Omega'_{ml}}{\partial z_j\partial\bar z_k} + \Omega'_{kj}\, R_{jklm} \\
&= -\sum_{j,k,l,m=1}^{n} \Omega'_{kj}\Omega'_{ml}\frac{\partial^2\omega_{lm}}{\partial z_j\partial\bar z_k} + \Omega'_{kj}\, R_{jklm} \\
&\quad + \sum_{j,k,l,m,a,b=1}^{n} \omega_{lm}\Omega'_{kj}\Omega'_{ma}\Omega'_{bl}\frac{\partial^2\omega'_{ab}}{\partial z_j\partial\bar z_k} \\
&\quad - \sum_{j,k,l,m,a,b,p,q=1}^{n} \omega_{lm}\Omega'_{kj}\Omega'_{mp}\Omega'_{qa}\Omega'_{bl}\frac{\partial\omega'_{ab}}{\partial z_j}\frac{\partial\omega'_{pq}}{\partial\bar z_k} \\
&\quad - \sum_{j,k,l,m,a,b,p,q=1}^{n} \omega_{lm}\Omega'_{kj}\Omega'_{ma}\Omega'_{bp}\Omega'_{ql}\frac{\partial\omega'_{ab}}{\partial z_j}\frac{\partial\omega'_{pq}}{\partial\bar z_k}.
\end{aligned} \tag{6}$$

Now, still denoting by H the matrix (ω'_{lm}), we recall the well-known formula to determine in local coordinates the Chern curvature of ω', namely

$$\begin{aligned}
\Theta(T_X,\omega') &\simeq_{\mathrm{loc}} \bar\partial\big(\bar H^{-1}\partial\bar H\big) \\
&= \bar\partial\bar H^{-1}\wedge\partial\bar H + \bar H^{-1}\bar\partial\partial\bar H \\
&= -\bar H^{-1}\bar\partial\bar H\bar H^{-1}\wedge\partial\bar H^{-1} + \bar H^{-1}\bar\partial\partial\bar H,
\end{aligned}$$

where the last equality is obtain by using formula (barpartialH-1 5). So, if we write in these coordinates

$$\Theta(T_X,\omega') = \sum_{j,k,l,m} c'_{jklm}\, dz_j\wedge d\bar z_k \otimes \left(\frac{\partial}{\partial z_l}\right)^{*}\otimes\frac{\partial}{\partial\bar z_m},$$

we obtain the following expression for the coefficients of the Chern curvature tensor:

c'jklm

$$c'_{jkal} = -\sum_{b=1}^{n} \Omega'_{bl} \frac{\partial^2 \omega'_{ab}}{\partial z_j \partial \bar z_k} + \sum_{b,p,q=1}^{n} \Omega'_{pl} \Omega'_{bq} \frac{\partial \omega'_{ab}}{\partial z_j} \frac{\partial \omega'_{qp}}{\partial \bar z_k}. \tag{7}$$

We can now use the above identity (7) to replace in the right hand side of formula (6) the summand

$$\sum_{b=1}^{n} \Omega'_{bl} \frac{\partial^2 \omega'_{ab}}{\partial z_j \partial \bar z_k}$$

with

$$-c'_{jkal} + \sum_{b,p,q=1}^{n} \Omega'_{pl} \Omega'_{bq} \frac{\partial \omega'_{ab}}{\partial z_j} \frac{\partial \omega'_{qp}}{\partial \bar z_k}.$$

With this substitution, we obtain

exprlaplsemifinal

$$\begin{aligned} \Delta_{\omega'} S = & -\sum_{j,k,l,m=1}^{n} \Omega'_{kj} \Omega'_{ml} \frac{\partial^2 \omega_{lm}}{\partial z_j \partial \bar z_k} + \Omega'_{kj} R_{jklm} \\ & + \sum_{j,k,l,m,a=1}^{n} \omega_{lm} \Omega'_{kj} \Omega'_{ma} \left(-c'_{jkal} + \sum_{b,p,q=1}^{n} \Omega'_{pl} \Omega'_{bq} \frac{\partial \omega'_{ab}}{\partial z_j} \frac{\partial \omega'_{qp}}{\partial \bar z_k} \right) \\ & - \sum_{j,k,l,m,a,b,p,q=1}^{n} \omega_{lm} \Omega'_{kj} \Omega'_{mp} \Omega'_{qa} \Omega'_{bl} \frac{\partial \omega'_{ab}}{\partial z_j} \frac{\partial \omega'_{pq}}{\partial \bar z_k} \\ & - \sum_{j,k,l,m,a,b,p,q=1}^{n} \omega_{lm} \Omega'_{kj} \Omega'_{ma} \Omega'_{bp} \Omega'_{ql} \frac{\partial \omega'_{ab}}{\partial z_j} \frac{\partial \omega'_{pq}}{\partial \bar z_k} \\ = & -\sum_{j,k,l,m=1}^{n} \Omega'_{kj} \Omega'_{ml} \frac{\partial^2 \omega_{lm}}{\partial z_j \partial \bar z_k} + \Omega'_{kj} R_{jklm} \\ & - \sum_{j,k,l,m,a=1}^{n} \omega_{lm} \Omega'_{kj} \Omega'_{ma} c'_{jkal} \\ & - \sum_{j,k,l,m,a,b,p,q=1}^{n} \omega_{lm} \Omega'_{kj} \Omega'_{mp} \Omega'_{qa} \Omega'_{bl} \frac{\partial \omega'_{ab}}{\partial z_j} \frac{\partial \omega'_{pq}}{\partial \bar z_k}. \end{aligned} \tag{8}$$

Now, since $(\partial/\partial z_1, \dots, \partial/\partial z_n)$ is merely ω'-orthogonal but not necessarily ω'-unitary at x_0, the Kähler symmetries of the coefficients c'_{jklm}'s at x_0 read

$$c'_{jklm} \sqrt{\lambda_l} \sqrt{\lambda_m} = c'_{lmjk} \sqrt{\lambda_j} \sqrt{\lambda_k}.$$

In particular,

$$c'_{jjll} \lambda_l = c'_{lljj} \lambda_j.$$

We are now in a good position to conclude the proof of the lemma. Indeed, evaluating (8) at the point x_0 with our initial choice of coordinates gives

exprlaplsemifinal

exprlaplfinal

$$\Delta_{\omega'} S(x_0) = \sum_{j,l=1}^{n} \frac{c_{jjll}}{\lambda_j \lambda_l} - \sum_{j,l=1}^{n} \underbrace{\frac{c'_{jjll}}{\lambda_j \lambda_l}}_{=\frac{c'_{lljj}}{(\lambda_l)^2}} - \sum_{j,l,a=1}^{n} \frac{1}{\lambda_j (\lambda_l)^2 \lambda_a} \frac{\partial \omega'_{al}}{\partial z_j} \frac{\partial \omega'_{la}}{\partial \bar{z}_j} \tag{9}$$
$$= \sum_{j,l=1}^{n} \frac{c_{jjll}}{\lambda_j \lambda_l} - \sum_{l=1}^{n} \frac{\rho'_{ll}}{(\lambda_l)^2} - \sum_{j,l,a=1}^{n} \frac{|\partial \omega'_{al}/\partial z_j(x_0)|^2}{\lambda_j (\lambda_l)^2 \lambda_a}.$$

□

Our next task will be to estimate the three summands appearing on the right hand side of the differential equality of the above lemma. We begin with the term involving the Ricci curvature of ω'. Recall that the we are supposing that

$$\mathrm{Ric}(\omega') \geq -\lambda \omega' + \mu \omega.$$

term1

Lemma 5.7. *At the point $x_0 \in X$, we have*

$$\sum_{l=1}^{n} \frac{\rho'_{ll}}{(\lambda_l)^2} \geq 2\pi \left(-\lambda S + \frac{\mu}{n} S^2 \right).$$

Proof. The hypothesis on the Ricci curvature of ω', when red at the point x_0 with our choice of coordinates, gives

$$\rho'_{ll} \geq 2\pi(-\lambda \lambda_l + \mu).$$

Thus, we get

$$\sum_{l=1}^{n} \frac{\rho'_{ll}}{(\lambda_l)^2} \geq -2\pi\lambda \sum_{l=1}^{n} \frac{1}{\lambda_l} + 2\pi\mu \sum_{l=1}^{n} \frac{1}{(\lambda_l)^2}.$$

Now, since the λ_l's are the eigenvalues of ω' with respect to ω, the eigenvalues of omega with respect to ω' are $1/\lambda_l$, $l = 1, \dots, n$, and therefore $S = \sum_{l=1}^{n} 1/\lambda_l$. Moreover, by the standard inequality between 1-norm and 2-norm of vectors in $\mathbb{R}^n$, we have $\sum_{l=1}^{n} 1/(\lambda_l)^2 \geq 1/n \big(\sum_{l=1}^{n} 1/\lambda_l \big)^2$. We finally obtain

$$\sum_{l=1}^{n} \frac{\rho'_{ll}}{(\lambda_l)^2} \geq 2\pi \left(-\lambda S + \frac{\mu}{n} S^2 \right).$$

□

Now, we treat the term with the first order derivatives of the metric ω'. In doing this, we have to keep in mind that, at the end of the day, we want to estimate $\Delta_{\omega'}\log S$. This Laplacian is given in coordinates by

$$\Delta_{\omega'}\log S = -\sum_{j,k=1}^{n}\Omega'_{kj}\underbrace{\frac{\partial^2\log S}{\partial z_j\partial\bar z_k}}_{=\frac{\partial}{\partial z_j}\left(\frac{1}{S}\frac{\partial S}{\partial\bar z_k}\right)=-\frac{1}{S^2}\frac{\partial S}{\partial z_j}\frac{\partial S}{\partial\bar z_k}+\frac{1}{S}\frac{\partial^2 S}{\partial z_j\partial\bar z_k}}$$
$$=\frac{1}{S}\Delta_{\omega'}S+\frac{1}{S^2}\sum_{j,k=1}^{n}\Omega'_{kj}\frac{\partial S}{\partial z_j}\frac{\partial S}{\partial\bar z_k}.$$

Once computed at x_0, we have

laplacianlog
$$\Delta_{\omega'}\log S(x_0)=\frac{1}{S(x_0)}\Delta_{\omega'}S(x_0)+\frac{1}{S(x_0)^2}\sum_{j=1}^{n}\frac{1}{\lambda_j}\left|\frac{\partial S}{\partial z_j}(x_0)\right|^2. \tag{10}$$

What we want to do in the lemma below is then to try to express these first order derivatives in terms of first order derivatives of S.

term2
Lemma 5.8. *At the point $x_0\in X$, we have*

$$\sum_{j,l,a=1}^{n}\frac{\left|\partial\omega'_{al}/\partial z_j\right|^2}{\lambda_j(\lambda_l)^2\lambda_a}\geq\frac{1}{S(x_0)}\sum_{j=1}^{n}\frac{1}{\lambda_j}\left|\frac{\partial S}{\partial z_j}(x_0)\right|^2.$$

Proof. Since the sum we are dealing with is made up of non negative terms, we have by plain minoration

$$\sum_{j,l,a=1}^{n}\frac{\left|\partial\omega'_{al}/\partial z_j\right|^2}{\lambda_j(\lambda_l)^2\lambda_a}\geq\sum_{j,l=1}^{n}\frac{\left|\partial\omega'_{ll}/\partial z_j\right|^2}{\lambda_j(\lambda_l)^3}.$$

Now, let us compute $\partial S/\partial z_j$ at x_0. We have

$$\frac{\partial S}{\partial z_j}(x_0)=\frac{\partial}{\partial z_j}\sum_{l,m=1}^{n}\Omega'_{ml}\omega_{lm}\bigg|_{x_0}$$
$$=\sum_{l,m=1}^{n}\frac{\partial\Omega'_{ml}}{\partial z_j}\omega_{lm}+\Omega'_{ml}\frac{\partial\omega_{lm}}{\partial z_j}\bigg|_{x_0}=\sum_{l=1}^{n}\frac{\partial\Omega'_{ll}}{\partial z_j}(x_0).$$

Now, we use the identity (partialH-1 4) to replace $\partial\Omega'_{ll}/\partial z_j(x_0)$ with

$$-\sum_{a,b=1}^{n}\Omega'_{la}\frac{\partial\omega'_{ab}}{\partial z_j}\Omega'_{bl}\bigg|_{x_0}=-\frac{1}{(\lambda_l)^2}\frac{\partial\omega'_{ll}}{\partial z_j}(x_0).$$

Thus, we obtain

$$\frac{\partial S}{\partial z_j}(x_0)=-\sum_{l=1}^{n}\frac{1}{(\lambda_l)^2}\frac{\partial\omega'_{ll}}{\partial z_j}(x_0).$$

laplacianlog

Now, inspired by (10), we compute

$$\begin{aligned}
\sum_{j=1}^n \frac{1}{\lambda_j}\left|\frac{\partial S}{\partial z_j}(x_0)\right|^2 &= \sum_{j=1}^n \frac{1}{\lambda_j}\left|\sum_{l=1}^n \frac{1}{(\lambda_l)^2}\frac{\partial \omega'_{ll}}{\partial z_j}(x_0)\right|^2 \\
&= \sum_{j=1}^n \frac{1}{\lambda_j}\left|\sum_{l=1}^n \frac{1}{(\lambda_l)^{1/2}}\frac{\partial \omega'_{ll}/\partial z_j(x_0)}{(\lambda_l)^{3/2}}\right|^2 \\
&\leq \sum_{j=1}^n \frac{1}{\lambda_j}\sum_{k=1}^n \frac{1}{\lambda_k}\sum_{l=1}^n \frac{\left|\partial \omega'_{ll}/\partial z_j(x_0)\right|^2}{(\lambda_l)^3} \\
&= S(x_0)\sum_{l,j=1}^n \frac{\left|\partial \omega'_{ll}/\partial z_j(x_0)\right|^2}{\lambda_j(\lambda_l)^3},
\end{aligned}$$

where the inequality is given by Cauchy-Schwarz. The lemma follows. □

Finally, we estimate the term involving the curvature of ω, using the hypothesis on the negativity of the holomorphic sectional curvature.

term3

Lemma 5.9. *At the point $x_0 \in X$, we have*

$$\sum_{j,l=1}^n \frac{c_{jjll}}{\lambda_j\lambda_l} \leq -\frac{\kappa(n+1)}{2n}S^2.$$

Classically this term has been bounded in terms of a uniform bound on the holomorphic bisectional curvature of ω. In order to prove this lemma, we need to be able to transform an information on the sum of holomorphic bisectional curvature type terms into an estimate using holomorphic sectional curvature only. Next proposition in hermitian linear algebra is the key point to do that. It is due to Royden.

royden

Proposition 5.10 (Royden [Roy80]). *Let $\xi_1, \dots, \xi_\nu$ be mutually orthogonal (but not necessarily unitary) non-zero vectors of a hermitian vector space (V, h). Suppose that $\Theta(\xi, \eta, \zeta, \omega)$ is a symmetric "bi-hermitian" form, i.e. Θ is sesquilinear in the first two and last two variables having the same pointwise properties of the (contraction with the metric of the) Chern curvature of a Kähler metric. Suppose also that there exists a real constant K such that for all $\xi \in V$ one has*

$$\Theta(\xi,\xi,\xi,\xi) \leq K\,||\xi||_h^4.$$

Then,

$$\sum_{\alpha,\beta}\Theta(\xi_\alpha,\xi_\alpha,\xi_\beta,\xi_\beta) \leq \frac{1}{2}K\left(\left(\sum_\alpha ||\xi_\alpha||_h^2\right)^2 + \sum_\alpha ||\xi_\alpha||_h^4\right).$$

Moreover, if $K \leq 0$, then

$$\sum_{\alpha,\beta}\Theta(\xi_\alpha,\xi_\alpha,\xi_\beta,\xi_\beta) \leq \frac{\nu+1}{2\nu}K\left(\sum_\alpha ||\xi_\alpha||_h^2\right)^2.$$

We shall use this proposition with

$$\left\langle \Theta(T_X,\omega)(\bullet,\bar{\bullet})\cdot\bullet,\bullet\right\rangle_\omega$$

as the symmetric "bi-hermitian" form on T_{X,x_0} in the statement. In terms of holomorphic bisectional curvature it can be rephrased as follows, when $\nu = n = \dim X$.

Suppose that a Kähler metric ω has negative holomorphic sectional curvature at the point x_0, bounded above by a negative constant $-\kappa$. Then, if $\xi_1,\dots,\xi_n$ is a ω-orthogonal basis for T_{X,x_0} we have

$$\sum_{\alpha,\beta=1}^n ||\xi_\alpha||^2_\omega ||\xi_\beta||^2_\omega \ \mathrm{HBC}_\omega(\xi_\alpha,\xi_\beta) \le -\frac{\kappa(n+1)}{2n}\left(\sum_{\alpha=1}^n ||\xi_\alpha||^2_\omega\right)^2.$$

Here is the proof.

Proof. Realize $\mathbb{Z}_4$ as the group of 4th roots of unity and set, for a vector $A = (\epsilon_1,\dots,\epsilon_\nu)\in\mathbb{Z}_4^\nu$,

$$\xi_A = \sum_\alpha \epsilon_\alpha\,\xi_\alpha.$$

Then, by orthogonality, $||\xi_A||^2_h = \sum_\alpha ||\xi_\alpha||^2_h$, and thus by hypothesis

$$\Theta(\xi_A,\xi_A,\xi_A,\xi_A) \le K\,||\xi_A||^4_h = K\left(\sum_\alpha ||\xi_\alpha||^2_h\right)^2.$$

Now, we take the sum over all possible $A\in\mathbb{Z}_4^\nu$ and get

$$\begin{aligned} K\left(\sum_\alpha ||\xi_\alpha||^2_h\right)^2 &\ge \frac{1}{4^\nu}\sum_A \Theta(\xi_A,\xi_A,\xi_A,\xi_A) \\ &= \frac{1}{4^\nu}\sum_A\sum_{\alpha,\beta,\gamma,\delta} \epsilon_\alpha\bar\epsilon_\beta\epsilon_\gamma\bar\epsilon_\delta\,\Theta(\xi_\alpha,\xi_\beta,\xi_\gamma,\xi_\delta) \\ &= \frac{1}{4^\nu}\sum_{\alpha,\beta,\gamma,\delta}\sum_A \frac{\epsilon_\alpha\epsilon_\gamma}{\epsilon_\beta\epsilon_\delta}\,\Theta(\xi_\alpha,\xi_\beta,\xi_\gamma,\xi_\delta). \end{aligned}$$

Now, fix a 4-tuple $(\alpha,\beta,\gamma,\delta)$. We claim that only the terms with $\alpha=\beta$ and $\gamma=\delta$ or $\alpha=\delta$ and $\beta=\gamma$ can survive after summing over all A. Thus, we are left only with the following terms

$$\begin{aligned} &\frac{1}{4^\nu}\sum_{\alpha,\beta,\gamma,\delta}\sum_A \frac{\epsilon_\alpha\epsilon_\gamma}{\epsilon_\beta\epsilon_\delta}\,\Theta(\xi_\alpha,\xi_\beta,\xi_\gamma,\xi_\delta) \\ &\qquad = \sum_\alpha \Theta(\xi_\alpha,\xi_\alpha,\xi_\alpha,\xi_\alpha) + \sum_{\alpha\neq\gamma}\Theta(\xi_\alpha,\xi_\alpha,\xi_\gamma,\xi_\gamma) + \Theta(\xi_\alpha,\xi_\gamma,\xi_\gamma,\xi_\alpha). \end{aligned}$$

The claim is straightforwardly verified, since for all the other terms, for each $A\in\mathbb{Z}_4^\nu$ one can find an $A'=(\epsilon'_1,\dots,\epsilon'_\nu)\in\mathbb{Z}_4^\nu$ such that $\frac{\epsilon_\alpha\epsilon_\gamma}{\epsilon_\beta\epsilon_\delta} = -\frac{\epsilon'_\alpha\epsilon'_\gamma}{\epsilon'_\beta\epsilon'_\delta}$.

Now, by symmetry of Θ, adding $\sum_\alpha \Theta(\xi_\alpha, \xi_\alpha, \xi_\alpha, \xi_\alpha)$ to both side and using the upper bound as in the hypotheses, we get

$$2 \sum_{\alpha,\gamma} \Theta(\xi_\alpha, \xi_\alpha, \xi_\gamma, \xi_\gamma) \leq K \left(\left(\sum_\alpha ||\xi_\alpha||_h^2 \right)^2 + \sum_\alpha ||\xi_\alpha||_h^4 \right).$$

To end the proof, observe that applying the Cauchy–Schwarz inequality in $\mathbb{R}^\nu$ to the vectors $(||\xi_1||_h^2, \dots, ||\xi_\nu||_h^2)$ and $(1, \dots, 1)$, we have

$$\left(\sum_\alpha ||\xi_\alpha||_h^2 \right)^2 \leq \nu \sum_\alpha ||\xi_\alpha||_h^4,$$

so that if $K \leq 0$, then

$$K \sum_\alpha ||\xi_\alpha||_h^4 \leq \frac{K}{\nu} \left(\sum_\alpha ||\xi_\alpha||_h^2 \right)^2,$$

and thus

$$\sum_{\alpha,\gamma} \Theta(\xi_\alpha, \xi_\alpha, \xi_\gamma, \xi_\gamma) \leq \frac{\nu+1}{2\nu} K \left(\sum_\alpha ||\xi_\alpha||_h^2 \right)^2,$$

as desired. □

We are now ready to give a

Proof of Lemma term3 *5.9.* Set

$$\xi_j := \frac{1}{\sqrt{\lambda_j}} \frac{\partial}{\partial z_j}, \quad j = 1, \dots n,$$

so that $\xi_1, \dots, \xi_n$ is a ω-orthogonal basis for T_{X,x_0}. Now, it suffices to observe that

$$\begin{aligned} \frac{c_{jjll}}{\lambda_j \lambda_l} &= \left\langle \Theta(T_X, \omega)(\xi_j, \bar{\xi}_j) \cdot \xi_l, \xi_l \right\rangle_\omega \\ &= ||\xi_j||_\omega^2 ||\xi_l||_\omega^2 \ \mathrm{HBC}_\omega(\xi_j, \xi_l). \end{aligned}$$

Now take the sum over all $j, l = 1, \dots, n$, to obtain

$$\sum_{j,l=1}^n \frac{c_{jjll}}{\lambda_j \lambda_l} \leq -\frac{\kappa(n+1)}{2n} \left(\sum_{\alpha=1}^n \frac{1}{\lambda_j} \right)^2 = -\frac{\kappa(n+1)}{2n} S^2.$$

□

Now, to conclude the proof of Claim step2 5.3, *i.e.* to show inequality (diffineq 1), we just put together the three estimates of the above lemmata, and plug them into formula (10). We get:

$$
\begin{aligned}
-\Delta_{\omega'}\log S(x_0) &= -\frac{1}{S(x_0)}\Delta_{\omega'}S(x_0) - \frac{1}{S(x_0)^2}\sum_{j=1}^{n}\frac{1}{\lambda_j}\left|\frac{\partial S}{\partial z_j}(x_0)\right|^2 \\
&= \frac{1}{S(x_0)}\left(\sum_{l=1}^{n}\frac{\rho'_{ll}}{(\lambda_l)^2} + \sum_{j,l,a=1}^{n}\frac{\left|\partial\omega'_{al}/\partial z_j(x_0)\right|^2}{\lambda_j(\lambda_l)^2\lambda_a} - \sum_{j,l=1}^{n}\frac{c_{jjll}}{\lambda_j\lambda_l}\right) \\
&\quad - \frac{1}{S(x_0)^2}\sum_{j=1}^{n}\frac{1}{\lambda_j}\left|\frac{\partial S}{\partial z_j}(x_0)\right|^2 \\
&\geq \frac{1}{S(x_0)}\left(2\pi\left(-\lambda S(x_0) + \frac{\mu}{n}S(x_0)^2\right)\right. \\
&\quad \left. + \frac{1}{S(x_0)}\sum_{j=1}^{n}\frac{1}{\lambda_j}\left|\frac{\partial S}{\partial z_j}(x_0)\right|^2 + \frac{\kappa(n+1)}{2n}S(x_0)^2\right) \\
&\quad - \frac{1}{S(x_0)^2}\sum_{j=1}^{n}\frac{1}{\lambda_j}\left|\frac{\partial S}{\partial z_j}(x_0)\right|^2 \\
&= \left(\frac{\kappa(n+1)}{2n} + 2\pi\frac{\mu}{n}\right)S(x_0) - 2\pi\lambda,
\end{aligned}
$$

as desired. □

Proof of Claim 5.4. We want to show that the limit

$$
\lim_{\varepsilon\to 0^+}\int_X \omega_\varepsilon^n = \lim_{\varepsilon\to 0^+}\int_X e^{u_\varepsilon}\,\omega^n
$$

is strictly positive. The first observation is that the functions u_ε are all ω'-plurisubharmonic for some fixed Kähler form ω' and $\varepsilon > 0$ small enough. For, let $\ell > 0$ be such that $\ell\omega - \mathrm{Ric}_\omega$ is positive and call $\omega' = \ell\omega - \mathrm{Ric}_\omega$. Thus, for all $0 < \varepsilon < \ell$, one has

$$
0 < \varepsilon\omega - \mathrm{Ric}_\omega + i\partial\bar\partial u_\varepsilon < \ell\omega - \mathrm{Ric}_\omega + i\partial\bar\partial u_\varepsilon = \omega' + i\partial\bar\partial u_\varepsilon.
$$

Therefore, by [GZ05, Proposition 2.6], either $\{u_\varepsilon\}$ converges uniformly to $-\infty$ on X or it is relatively compact in $L^1(X)$. Suppose for a moment that we are in the second case. Then, there exists a subsequence $\{u_{\varepsilon_k}\}$ converging in $L^1(X)$ and moreover the limit coincides *a.e.* with a uniquely determined ω'-plurisubharmonic function u. Up to pass to a further subsequence, we can also suppose that u_{ε_k} converges pointwise *a.e.* to u. But then, $e^{u_{\varepsilon_k}} \to e^u$ pointwise *a.e.* on X. On the other hand, by Claim 5.2, we have $e^{u_{\varepsilon_k}} \leq e^C$ so that, by dominated convergence, we also have $L^1(X)$-convergence and

$$
\lim_{k\to\infty}\int_X e^{u_{\varepsilon_k}}\,\omega^n = \int_X e^u\omega^n > 0.
$$

The upshot is that what we need to prove is that $\{u_\varepsilon\}$ does not converge uniformly to $-\infty$ on X. From now on, we shall suppose by contradiction that

$$\sup_X u_\varepsilon \to -\infty.$$

Now, recall that we defined the smooth positive function S_ε on X by

$$\omega \wedge \omega_\varepsilon^{n-1} = \frac{S_\varepsilon}{n}\,\omega_\varepsilon^n.$$

Now, set $T_\varepsilon = \log S_\varepsilon$. In other words, T_ε is the logarithm of the trace of ω with respect to ω_ε.

lem:sge-m **Lemma 5.11.** *The function T_ε satisfies the following inequality:*

$$T_\varepsilon > -\frac{u_\varepsilon}{n}.$$

In particular, if $\{u_\varepsilon\}$ converges uniformly to $-\infty$ on X, then T_ε converges uniformly to $+\infty$ on X.

Proof. Let $0 < \lambda_1 \leq \cdots \leq \lambda_n$ be the eigenvalues of ω_ε with respect to ω, so that $0 < 1/\lambda_n \leq \cdots \leq 1/\lambda_1$ are the eigenvalues of ω with respect to ω_ε. Then,

$$e^{T_\varepsilon} = \operatorname{tr}_{\omega_\varepsilon} \omega = \frac{1}{\lambda_1} + \cdots + \frac{1}{\lambda_n} > \frac{1}{\lambda_1}.$$

Thus, $e^{-T_\varepsilon} < \lambda_1$ so that $e^{-nT_\varepsilon} < (\lambda_1)^n \leq \lambda_1 \cdots \lambda_n$. But, $e^{u_\varepsilon}\omega^n = \omega_\varepsilon^n = \lambda_1 \cdots \lambda_n\, \omega^n$, and so we get $e^{-nT_\varepsilon} < e^{u_\varepsilon}$, or, in other words,

$$T_\varepsilon > -\frac{u_\varepsilon}{n}.$$

□

As announced, we now use the integral inequality (intineq 2). We write it as follows:

$$\frac{(n+1)\kappa}{2n}\int_X e^{T_\varepsilon + u_\varepsilon}\,\omega^n \leq 2\pi \int_X e^{u_\varepsilon}\,\omega^n.$$

Next, if we define $C_\varepsilon := \inf_X e^{-u_\varepsilon/n}$, we have that $e^{T_\varepsilon} > C_\varepsilon$, and thus

eq:final
$$C_\varepsilon\,\frac{(n+1)\kappa}{2n}\int_X e^{u_\varepsilon}\,\omega^n \leq 2\pi \int_X e^{u_\varepsilon}\,\omega^n. \tag{11}$$

But then, we obtain that

$$C_\varepsilon\,\frac{(n+1)\kappa}{2n} \leq 2\pi,$$

and this gives a contradiction, since we are assuming that $C_\varepsilon \to +\infty$ as $\varepsilon \to 0^+$. □

5.1. **The Kähler case and quasi-negative holomorphic sectional curvature.**

6. Where from here?

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Simone Diverio, Dipartimento di matematica "Guido Castelnuovo", SAPIENZA Università di Roma, Piazzale Aldo Moro 5, I-00185 Roma.

E-mail address: diverio@mat.uniroma1.it