A SIMPLE PROOF OF THE KOBAYASHI CONJECTURE ON THE HYPERBOLICITY OF GENERAL ALGEBRAIC HYPERSURFACES

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Contents

0. Introduction

For a compact complex space X, a well known result of Brody [Bro78] asserts that hyperbolicity property introduced by Kobayashi [Kob67a, Kob67b] is equivalent to the nonexistence of nonconstant entire holomorphic curves $f : \mathbb{C} \to X$. The aim of this chapter is to describe some geometric techniques that are useful to investigate the existence or nonexistence of such curves. A central conjecture due to Green-Griffiths [GrGr79] and Lang [Lang86] stipulates that for every projective variety X of general type over $\mathbb{C}$, there exists a proper algebraic subvariety Y of X containing all nonconstant entire curves.

According to Green-Griffiths [GrGr79], jet bundles can be used to give sufficient conditions for Kobayashi hyperbolicity. As in [Dem95], we introduce the formalism of directed varieties and Semple towers [Sem54] to express these conditions in terms of intrinsic algebraic differential equations that entire curves must satisfy; see the "fundamental vanishing theorem" 3.23 below. An important application is a confirmation of an old-standing conjecture of Kobayashi (cf. [Kob70]): a general hypersurface X of complex projective space $\mathbb{P}^{n+1}$ of degree $d \geqslant d_n$ large enough is Kobayashi hyperbolic. The main arguments are based on techniques introduced in 2016 by Damian Brotbek [Brot17]; they make use of Wronskian differential operators and their associated multiplier ideals. Shortly afterwards, Ya Deng [Deng16] found how to make the method effective, and produced in this way an explicit value of d_n. We describe here a proof based on a simplification of their ideas, along with a further improvement of Deng's bound, namely $d_n = \lfloor \frac{1}{3}(en)^{2n+2} \rfloor$ (cf. [Dem18]). This extends in particular earlier results of Demailly-El Goul [DeEG97], McQuillan [McQ99], Păun [Pau08], Diverio-Merker-Rousseau [DMR10], Diverio-Trapani [DT10] and [Siu15]. According to work of Clemens[86], Zaidenberg [Zai87], Ein [Ein88, Ein91], Voisin [Voi96] and Pacienza [Pac04], every subvariety of a general algebraic hypersurface hypersurface X of $\mathbb{P}^{n+1}$ is of general type for degrees $d \geqslant \delta_n$, with an optimal lower bound given by $\delta_n = 2n + 1$ for $2 \leqslant n \leqslant 4$ and $\delta_n = 2n$ for $n \geqslant 5$ – that the same bound $d_n = \delta_n$ holds for Kobayashi hyperbolicity would then be a consequence of the Green-Griffiths-Lang conjecture.

In the same vein, we present a construction of hyperbolic hypersurfaces of $\mathbb{P}^{n+1}$ for all degrees $d \geqslant 4n^2$. The main idea is inspired from the method of Shiffman-Zaidenberg [ShZa01]; by using again Wronskians, it is possible to give a direct and self-contained argument.

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1. Hyperbolicity concepts

1.A. Kobayashi pseudodistance and pseudometric

We first recall a few basic facts concerning the concept of hyperbolicity, according to S. Kobayashi [Kob67a, Kob67b, Kob70, Kob76]. Let X be a complex space. Given two points $p, q \in X$, let us consider a *chain of analytic disks* from p to q, that is a sequence of holomorphic maps $f_0, f_1, \ldots, f_k : \mathbb{D} \to X$ from the unit disk $\mathbb{D} = D(0,1) \subset \mathbb{C}$ to X, together with pairs of points $a_0, b_0, \ldots, a_k, b_k$ of $\mathbb{D}$ such that

$$p = f_0(a_0), \quad q = f_k(b_k), \quad f_i(b_i) = f_{i+1}(a_{i+1}), \qquad i = 0, \ldots, k-1.$$

Denoting this chain by α, we define its length $\ell(\alpha)$ to be

$$\ell(\alpha) = d_P(a_1, b_1) + \cdots + d_P(a_k, b_k) \tag{1.1′}$$

where d_P is the Poincaré distance on $\mathbb{D}$, and the *Kobayashi pseudodistance* d_X^K on X to be

$$d_X^K(p,q) = \inf_\alpha \ell(\alpha). \tag{1.1″}$$

A *Finsler metric* (resp. *pseudometric*) on a vector bundle E is a homogeneous positive (resp. nonnegative) function N on the total space E, that is,

$$N(\lambda\xi) = |\lambda|\, N(\xi) \qquad \text{for all } \lambda \in \mathbb{C} \text{ and } \xi \in E,$$

but in general N is not assumed to be subbadditive (i.e. convex) on the fibers of E. A Finsler (pseudo-)metric on E is thus nothing but a hermitian (semi-)norm on the tautological line bundle $\mathcal{O}_{P(E)}(-1)$ of lines of E over the projectivized bundle $Y = P(E)$. The *Kobayashi-Royden infinitesimal pseudometric* on X is the Finsler pseudometric on the tangent bundle T_X defined by

$$\mathbf{k}_X(\xi) = \inf\big\{\lambda > 0\,;\, \exists f : \mathbb{D} \to X,\ f(0) = x,\ \lambda f'(0) = \xi\big\}, \qquad x \in X,\ \xi \in T_{X,x}. \tag{1.2}$$

If $\Phi : X \to Y$ is a morphism of complex spaces, by considering the compositions $\Phi \circ f : \mathbb{D} \to Y$, this definition immediately implies the monotonicity property $\Phi^*\mathbf{k}_Y \leqslant \mathbf{k}_X$, i.e.

$$\mathbf{k}_Y(\Phi_*\xi) \leqslant \mathbf{k}_X(\xi) \quad \text{for all } x \in X \text{ and } \xi \in T_{X,x}. \tag{1.3}$$

When X is a manifold, it follows from the work of H.L. Royden ([Roy71], [Roy74]) that d_X^K is the integrated pseudodistance associated with the pseudometric, i.e.

$$d_X^K(p,q) = \inf_\gamma \int_\gamma \mathbf{k}_X(\gamma'(t))\, dt, \tag{1.4}$$

where the infimum is taken over all piecewise smooth curves joining p to q ; in the case of complex spaces, a similar formula holds, involving jets of analytic curves of arbitrary order, cf. S. Venturini [Ven96].

1.5. Definition. *A complex space X is said to be* hyperbolic (*in the sense of Kobayashi*) *if d_X^K is actually a distance, namely if $d_X^K(p,q) > 0$ for all pairs of distinct points (p,q) in X.*

1.B. Brody criterion

In the above context, we have the following well-known result of Brody [Bro78]. Its main interest is to relate hyperbolicity to the non existence of entire curves.

1.6. Brody reparametrization lemma. *Let ω be a hermitian metric on X and let $f:\mathbb{D}\to X$ be a holomorphic map. For every $\varepsilon>0$, there exists a radius $R\geqslant(1-\varepsilon)\|f'(0)\|_\omega$ and a homographic transformation ψ of the disk $D(0,R)$ onto $(1-\varepsilon)\mathbb{D}$ such that*

$$\|(f\circ\psi)'(0)\|_\omega=1,\qquad \|(f\circ\psi)'(t)\|_\omega\leqslant\frac{1}{1-|t|^2/R^2}\quad \textit{for every } t\in D(0,R).$$

Proof. Select $t_0\in\mathbb{D}$ such that $(1-|t|^2)\|f'((1-\varepsilon)t)\|_\omega$ reaches its maximum for $t=t_0$. The reason for this choice is that $(1-|t|^2)\|f'((1-\varepsilon)t)\|_\omega$ is the norm of the differential $f'((1-\varepsilon)t):T_{\mathbb{D}}\to T_X$ with respect to the Poincaré metric $|dt|^2/(1-|t|^2)^2$ on $T_{\mathbb{D}}$, which is conformally invariant under $\mathrm{Aut}(\mathbb{D})$. One then adjusts R and ψ so that $\psi(0)=(1-\varepsilon)t_0$ and $|\psi'(0)|\,\|f'(\psi(0))\|_\omega=1$. As $|\psi'(0)|=\frac{1-\varepsilon}{R}(1-|t_0|^2)$, the only possible choice for R is

$$R=(1-\varepsilon)(1-|t_0|^2)\|f'(\psi(0))\|_\omega\geqslant(1-\varepsilon)\|f'(0)\|_\omega.$$

The inequality for $(f\circ\psi)'$ follows from the fact that the Poincaré norm is maximum at the origin, where it is equal to 1 by the choice of R. Using the Ascoli-Arzelà theorem we obtain immediately:

1.7. Corollary (Brody)**.** *Let (X,ω) be a compact complex hermitian manifold. Given a sequence of holomorphic mappings $f_\nu:\mathbb{D}\to X$ such that $\lim\|f_\nu'(0)\|_\omega=+\infty$, one can find a sequence of homographic transformations $\psi_\nu:D(0,R_\nu)\to(1-1/\nu)\mathbb{D}$ with $\lim R_\nu=+\infty$, such that, after passing possibly to a subsequence, $(f_\nu\circ\psi_\nu)$ converges uniformly on every compact subset of $\mathbb{C}$ towards a nonconstant holomorphic map $g:\mathbb{C}\to X$ with $\|g'(0)\|_\omega=1$ and $\sup_{t\in\mathbb{C}}\|g'(t)\|_\omega\leqslant1$.*

An entire curve $g:\mathbb{C}\to X$ such that $\sup_{\mathbb{C}}\|g'\|_\omega=M<+\infty$ is called a *Brody curve*; this concept does not depend on the choice of ω when X is compact, and one can always assume $M=1$ by rescaling the parameter t.

1.8. Brody criterion. *Let X be a* compact *complex manifold. The following properties are equivalent.*

(a) *X is hyperbolic.*

(b) *X does not possess any entire curve $f:\mathbb{C}\to X$.*

(c) *X does not possess any Brody curve $g:\mathbb{C}\to X$.*

(d) *The Kobayashi infinitesimal metric $\mathbf{k}_X$ is uniformly bounded below, namely*

$$\mathbf{k}_X(\xi)\geqslant c\|\xi\|_\omega,\qquad c>0,$$

for any hermitian metric ω on X.

When property (b) *holds, X is said to be* Brody hyperbolic*.*

Proof. (a) $\Rightarrow$ (b) If X possesses an entire curve $f:\mathbb{C}\to X$, then by looking at arbitrary large analytic disks $f:D(t_0,R)\subset\mathbb{C}$ and rescaling them on $\mathbb{D}$ as $t\mapsto f(t_0+Rt)$, it is easy to see that the Kobayashi distance of any two points in $f(\mathbb{C})$ is zero, so X is not hyperbolic.

(b) $\Rightarrow$ (c) is trivial.

(c) $\Rightarrow$ (d) If (d) does not hold, there exists a sequence of tangent vectors $\xi_\nu\in T_{X,x_\nu}$ with $\|\xi_\nu\|_\omega=1$ and $\mathbf{k}_X(\xi_\nu)\to0$. By definition, this means that there exists an analytic curve $f_\nu:\mathbb{D}\to X$ with $f(0)=x_\nu$ and $\|f_\nu'(0)\|_\omega\geqslant(1-\frac{1}{\nu})/\mathbf{k}_X(\xi_\nu)\to+\infty$. One can then produce a Brody curve $g=\mathbb{C}\to X$ by Corollary 1.7, contradicting (c).

(d) $\Rightarrow$ (a). In fact (d) implies after integrating that $d_X^K(p,q)\geqslant c\,d_\omega(p,q)$ where d_ω is the geodesic distance associated with ω, so d_X^K must be non degenerate. □

As a consequence, any projective variety containing a rational curve C (i.e. a curve normalized by $\overline{C}\simeq\mathbb{P}^1_{\mathbb{C}}\simeq\mathbb{C}\cup\{\infty\}$ or an elliptic curve (i.e. a curve normalized by a nonsingular elliptic curve $\mathbb{C}/(\mathbb{Z}\oplus\mathbb{Z}\tau)$) is non hyperbolic. An immediate consequence of the Brody criterion is the openness property of hyperbolicity for the metric topology:

1.9. Proposition. *Let $\pi : \mathcal{X} \to S$ be a holomorphic family of compact complex manifolds. Then the set of $s \in S$ such that the fiber $X_s = \pi^{-1}(s)$ is hyperbolic is open in the metric topology.*

Proof. Let ω be an arbitrary hermitian metric on $\mathcal{X}$, $(X_{s_\nu})_{s_\nu \in S}$ a sequence of non hyperbolic fibers, and $s = \lim s_\nu$. By the Brody criterion, one obtains a sequence of entire maps $f_\nu : \mathbb{C} \to X_{s_\nu}$ such that $\|f'_\nu(0)\|_\omega = 1$ and $\|f'_\nu\|_\omega \leqslant 1$. Ascoli's theorem shows that there is a subsequence of f_ν converging uniformly to a limit $f : \mathbb{C} \to X_s$, with $\|f'(0)\|_\omega = 1$. Hence X_s is not hyperbolic and the collection of non hyperbolic fibers is closed in S. □

1.C. Relationship of hyperbolicity with algebraic properties

In the case of projective algebraic varieties, Kobayashi hyperbolicity is expected to be an algebraic property. In fact, the following classical conjectures would give a necessary and sufficient algebraic characterization. Recall that a projective variety X of dimension $n = \dim_{\mathbb{C}} X$ is said to be of *general type* if the canonical bundle $K_{\widetilde{X}} = \Lambda^n T^*_{\widetilde{X}}$ of some desingularization $\widetilde{X}$ of X is big. When $n = \dim_{\mathbb{C}} X = 1$, this is equivalent to say that X is not rational or elliptic.

1.10. Some classical conjectures. *Let X be a projective variety.*

(i) (Green-Griffiths-Lang conjecture) *If X is of general type, there should exist a proper algebraic variety $Y \subsetneq X$ (possibly empty) containing all nonconstant entire curves $f : \mathbb{C} \to X$.*

(ii) *Conversely, if X is Kobayashi hyperbolic and nonsingular, it is expected that K_X should be ample. More generally, if X is singular, any desingularization $\widetilde{X}$ should be of general type.*

(iii) (Conjectural algebraic characterization of Kobayashi hyperbolicity). *A projective variety X is Kobayashi hyperbolic if and only if every positive dimensional algebraic subvariety $Y \subset X$ (including X itself) is of general type.*

In fact, since every analytic subspace of Kobayashi hyperbolic space is again hyperbolic by definition, it is not difficult to see by induction on dimension that 1.10 (iii) would follow formally from 1.10 (i) and (ii) [the "if" part is a consequence of 1.10 (i), and the "only if" part follows from 1.10 (ii)]. Thanks to fundamental work of Clemens [Cle86], Ein [Ein88, Ein91] and Voisin [Voi96], it is known that every subvariety Y of a generic algebraic hypersurface $X \subset \mathbb{P}^{n+1}$ of degree $d \geqslant 2n+1$ is of general type for $n \geqslant 2$; Pacienza [Pac04] has even shown that this holds for $d \geqslant 2n$ when $n \geqslant 5$. The Green-Griffiths-Lang conjecture would then imply that these hypersurfaces are Kobayashi hyperbolic.

1.11. Definition. *Let X be a projective algebraic manifold, and A a very ample line bundle on X. We say that X is* algebraically hyperbolic *if there exists $\varepsilon > 0$ such that every closed irreducible curve $C \subset X$ has a normalization $\overline{C}$ such that its Euler characteristic satisfies*

$$-\chi(\overline{C}) = 2g(\overline{C}) - 2 \geqslant \varepsilon \deg_A(C)$$

where $g(\overline{C})$ is the genus and $\deg_A(C) = C \cdot A = \int_C c_1(A)$.

1.12. Theorem. *Every Kobayashi hyperbolic projective variety is algebraically hyperbolic. More generally, if X is a hyperbolic compact complex manifold equipped with a hermitian metric ω, there exists $\varepsilon > 0$ such that every closed irreducible curve $C \subset X$ satisfies*

$$2g(\overline{C}) - 2 \geqslant \varepsilon \deg_\omega(C) \quad \textit{where } \deg_\omega(C) = \int_C \omega.$$

Proof ([Dem95]). When Γ is a nonsingular compact curve of genus at least 2, the uniformization theorem implies that the universal cover $\rho : \widehat{\Gamma} \to \Gamma$ is isomorphic to the unit disk $\mathbb{D}$, and one then sees that the Kobayashi metric $\mathbf{k}_\Gamma$ is induced by he Kobayashi metric of the disk, i.e.

$$\mathbf{k}_{\mathbb{D}}^2 = \frac{idz \wedge d\overline{z}}{(1 - |z|^2)^2}.$$

These metrics have constant negative curvature $-\frac{i}{2\pi}\partial\overline{\partial}\log\mathbf{k}_\Gamma^2 = -\frac{1}{\pi}\mathbf{k}_\Gamma^2$, hence

$$\frac{1}{\pi}\int_\Gamma \mathbf{k}_\Gamma^2 = -\chi(\Gamma) = 2g(\Gamma) - 2$$

by the Gauss-Bonnet formula. Now, if X is hyperbolic and $C \subset X$ is a closed analytic curve, the monotonicity formula (1.3) applied to the normalization map $\nu : \overline{C} \to X$ implies $\mathbf{k}_{\overline{C}} \geqslant \nu^*\mathbf{k}_X$, and we also have $\mathbf{k}_X^2 \geqslant c^2\omega$ for some $c > 0$ by 1.8 (d). Therefore

$$2g(\overline{C}) - 2 = \frac{1}{\pi}\int_{\overline{C}} \mathbf{k}_{\overline{C}}^2 \geqslant \frac{1}{\pi}\int_{\overline{C}} \nu^*\mathbf{k}_X^2 = \frac{1}{\pi}\int_C \mathbf{k}_X^2 \geqslant \frac{c^2}{\pi}\int_C \omega = \frac{c^2}{\pi}\deg_\omega(C).$$

It is not very difficult to check that the proof can be extended to the case of singular hyperbolic compact complex spaces (a smooth hermitian metric on X being a metric that has extensions with respect to local embeddings of X in open sets $U \subset \mathbb{C}^N$). □

1.13. Proposition. *Let $\mathcal{X} \to S$ be an algebraic family of projective algebraic manifolds, given by a projective morphism $\mathcal{X} \to S$. Then the set of $t \in S$ such that the fiber X_t is algebraically hyperbolic is open with respect to the countable Zariski topology*

Proof. After replacing S by a Zariski open subset, we may assume that the total space $\mathcal{X}$ itself is quasi-projective. Let ω be the Kähler metric on $\mathcal{X}$ obtained by pulling back the Fubini-Study metric via an embedding in a projective space. If integers $d > 0$, $g \geqslant 0$ are fixed, the set $A_{d,g}$ of $t \in S$ such that X_t contains an algebraic 1-cycle $C = \sum m_j C_j$ with $\deg_\omega(C) = d$ and $g(\overline{C}) = \sum m_j\, g(\overline{C}_j) \leqslant g$ is a closed algebraic subset of S (this follows from the existence of a relative cycle space of curves of given degree, and from the fact that the geometric genus is Zariski lower semicontinuous). Now, the set of non algebraically hyperbolic fibers is by definition

$$\bigcap_{k>0}\ \bigcup_{2g-2<d/k} A_{d,g}.$$

This concludes the proof. □

It is expected that the concepts of Kobayashi hyperbolicity and algebraic hyperbolicity coincide for projective varieties. This would of course imply that Kobayashi hyperbolicity is an open property with respect to the countable Zariski topology, a generalized form of the Kobayashi conjecture.

2. Semple tower associated to a directed manifold

2.A. Category of directed varieties

Let us consider a pair (X, V) consisting of a n-dimensional complex manifold X equipped with a *linear subspace* $V \subset T_X$: assuming X connected, this is by definition an irreducible closed analytic subspace of the total space of T_X such that each fiber $V_x = V \cap T_{X,x}$ is a vector subspace of $T_{X,x}$; the rank $x \mapsto \dim_{\mathbb{C}} V_x$ is Zariski lower semicontinuous, and it may a priori jump.

2.1. Definition. *We will refer to such a pair (X, V) where $V \subset T_X$ is a linear subspace as being a* (*complex*) directed manifold*. A morphism $\Phi : (X, V) \to (Y, W)$ in the category of* (*complex*) *directed manifolds is a holomorphic map such that $\Phi_*(V) \subset W$.*

The rank $r \in \{0, 1, \ldots, n\}$ of V is by definition the dimension of V_x at a generic point. The dimension may be larger at non generic points; this happens e.g. on $X = \mathbb{C}^n$ for the rank 1 linear space V generated by the Euler vector field: $V_z = \mathbb{C}\sum_{1\leqslant j\leqslant n} z_j\frac{\partial}{\partial z_j}$ for $z \neq 0$, and $V_0 = \mathbb{C}^n$. The absolute situation is the case $V = T_X$ and the relative situation is the case when $V = T_{X/S}$ is the relative tangent space to a smooth holomorphic map $X \to S$. In general, we can associate to V a sheaf $\mathcal{V} = \mathcal{O}(V) \subset \mathcal{O}(T_X)$ of holomorphic sections. These sections need not generate the fibers of V at singular points, as one sees already in the case of the Euler vector field when $n \geqslant 2$. However, $\mathcal{V}$ is a *saturated* subsheaf of $\mathcal{O}(T_X)$, i.e. $\mathcal{O}(T_X)/\mathcal{V}$ has no torsion: in fact, if the components of a section have a common divisorial component, one can always simplify this divisor and produce a new section without any such common divisorial component. Instead of defining directed manifolds by picking a linear space V, one could equivalently define them by considering saturated coherent subsheaves $\mathcal{V} \subset \mathcal{O}(T_X)$. One could also take the dual viewpoint, looking at arbitrary quotient morphisms $\Omega^1_X \to \mathcal{W} = \mathcal{V}^*$ (and recovering $\mathcal{V} = \mathcal{W}^* = \mathrm{Hom}_{\mathcal{O}}(\mathcal{W}, \mathcal{O})$, as $\mathcal{V} = \mathcal{V}^{**}$ is reflexive). We want to stress here that no assumption need be made on the Lie bracket tensor $[\bullet, \bullet] : \mathcal{V} \times \mathcal{V} \to \mathcal{O}(T_X)/\mathcal{V}$, i.e. we do not assume any kind of integrability for $\mathcal{V}$ or $\mathcal{W}$. Even though we will not consider such situations here, one can even generalize the concept of directed structure to the case when X is a singular (say reduced) complex space X. In fact $V_{\restriction X'}$ should then be a holomorphic vector subbundle of $T_{X'}$ on some analytic Zariski open set $X' \subset X_{\mathrm{reg}}$, and if $U \hookrightarrow Z$ is an embedding of an open neighborhood $U \subset X$ of a point $x_0 \in X$ into an open set $Z \subset \mathbb{C}^N$, we demand that the directed structure $V_{\restriction U}$ be a (closed and analytic) subspace of T_Z, obtained as the closure of $V_{\restriction X' \cap U}$ in T_Z via the obvious "inclusion morphism" $(X' \cap U, V'_{\restriction X' \cap U}) \hookrightarrow (Z, T_Z)$. A morphism $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$ in the category of directed varieties is the same as a holomorphic curve $t \mapsto f(t)$ that is *tangent to* V, i.e. $f'(t) \in V_{f(t)}$ for all t. The concept of Koabayashi hyperbolicity can be extended to directed varieties as follows.

2.2. Definition. *Let (X, V) be a complex directed manifold. The Kobayashi-Royden infinitesimal metric of (X, V) is the Finsler metric on V defined for any $x \in X$ and $\xi \in V_x$ by*

$$\mathbf{k}_{(X,V)}(\xi) = \inf \big\{ \lambda > 0\,;\, \exists f : (\mathbb{D}, T_{\mathbb{D}}) \to (X, V),\ f(0) = x,\ \lambda f'(0) = \xi \big\}.$$

We say that (X, V) is *infinitesimally hyperbolic* if $\mathbf{k}_{(X,V)}$ is positive definite on every fiber V_x and satisfies a uniform lower bound $\mathbf{k}_{(X,V)}(\xi) \geqslant \varepsilon \|\xi\|_\omega$ in terms of any smooth hermitian metric ω on X, when x runs over a compact subset of X. When X is compact, the Brody criterion shows that this is equivalent to the nonexistence of nonconstant entire curves $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$, or even to the nonexistence of entire curves $g : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$ with $\sup \|g'(t)\|_\omega = \|g'(0)\|_\omega = 1$. In this context we have the

2.3. Generalized Green-Griffiths-Lang conjecture. *Let (X, V) be a projective directed manifold where $V \subset T_X$ is nonsingular (i.e. a subbundle of T_X). Assume that (X, V) is of "general type" in the sense that $K_V := \det V^*$ is a big line bundle. Then there should exist a proper algebraic subvariety $Y \subsetneq X$ containing the images $f(\mathbb{C})$ of all entire curves $f : \mathbb{C} \to X$ tangent to V.*

A similar statement can be made when V is singular, but then K_V has to be replaced by a certain (nonnecessarily invertible) rank 1 sheaf of "locally bounded" forms of $\mathcal{O}(\det V^*)$, with respect to a smooth hermitian form ω on T_X. The reader will find a more precise definition in [Dem18].

2.B. The 1-jet fonctor

The basic idea is to introduce a fonctorial process which produces a new complex directed manifold $(\widetilde{X}, \widetilde{V})$ from a given one (X, V). The new structure $(\widetilde{X}, \widetilde{V})$ plays the role of a space of 1-jets over X. First assume that V is *non singular*. We let

$$\widetilde{X} = P(V), \qquad \widetilde{V} \subset T_{\widetilde{X}} \tag{2.4}$$

be the projectivized bundle of lines of V, together with a subbundle $\widetilde{V}$ of $T_{\widetilde{X}}$ defined as follows: for every point $(x, [v]) \in \widetilde{X}$ associated with a vector $v \in V_x \smallsetminus \{0\}$,

$$\widetilde{V}_{(x,[v])} = \big\{ \xi \in T_{\widetilde{X},\,(x,[v])}\,;\, \pi_* \xi \in \mathbb{C}v \big\}, \qquad \mathbb{C}v \subset V_x \subset T_{X,x}, \tag{2.4$'$}$$

where $\pi : \widetilde{X} = P(V) \to X$ is the natural projection and $\pi_* : T_{\widetilde{X}} \to \pi^* T_X$ is its differential. On $\widetilde{X} = P(V)$ we have the tautological line bundle $\mathcal{O}_{\widetilde{X}}(-1) \subset \pi^* V$ such that $\mathcal{O}_{\widetilde{X}}(-1)_{(x,[v])} = \mathbb{C}v$. The bundle $\widetilde{V}$ is characterized by the two exact sequences

$$0 \longrightarrow T_{\widetilde{X}/X} \longrightarrow \widetilde{V} \xrightarrow{\ \pi_*\ } \mathcal{O}_{\widetilde{X}}(-1) \longrightarrow 0, \tag{2.5}$$

$$0 \longrightarrow \mathcal{O}_{\widetilde{X}} \longrightarrow \pi^* V \otimes \mathcal{O}_{\widetilde{X}}(1) \longrightarrow T_{\widetilde{X}/X} \longrightarrow 0, \tag{2.5$'$}$$

where $T_{\widetilde{X}/X}$ denotes the relative tangent bundle of the fibration $\pi : \widetilde{X} \to X$. The first sequence is a direct consequence of the definition of $\widetilde{V}$, whereas the second is a relative version of the Euler exact sequence describing the tangent bundle of the fibers $P(V_x)$. From these exact sequences we infer

$$\dim \widetilde{X} = n + r - 1, \qquad \operatorname{rank} \widetilde{V} = \operatorname{rank} V = r, \tag{2.6}$$

and by taking determinants we find $\det(T_{\widetilde{X}/X}) = \pi^* \det V \otimes \mathcal{O}_{\widetilde{X}}(r)$, thus

$$\det \widetilde{V} = \pi^* \det V \otimes \mathcal{O}_{\widetilde{X}}(r-1). \tag{2.7}$$

By definition, $\pi : (\widetilde{X}, \widetilde{V}) \to (X, V)$ is a morphism of complex directed manifolds. Clearly, our construction is fonctorial, i.e., for every morphism of directed manifolds $\Phi : (X, V) \to (Y, W)$, there is a commutative diagram

$$\begin{array}{ccc} (\widetilde{X}, \widetilde{V}) & \xrightarrow{\pi} & (X, V) \\ \widetilde{\Phi} \downarrow & & \downarrow \Phi \\ (\widetilde{Y}, \widetilde{W}) & \xrightarrow{\pi} & (Y, W) \end{array}$$

where the left vertical arrow is the meromorphic map $P(V) \dashrightarrow P(W)$ induced by the differential $\Phi_* : V \to \Phi^* W$ ($\widetilde{\Phi}$ is actually holomorphic if $\Phi_* : V \to \Phi^* W$ is injective).

2.C. Lifting of curves to the 1-jet bundle

Suppose that we are given a holomorphic curve $f : D_R \to X$ parametrized by the disk D_R of centre 0 and radius R in the complex plane, and that f is a tangent curve of the directed manifold, i.e., $f'(t) \in V_{f(t)}$ for every $t \in D_R$. If f is nonconstant, there is a well defined and unique tangent line $[f'(t)]$ for every t, even at stationary points, and the map

$$\widetilde{f} : D_R \to \widetilde{X}, \qquad t \mapsto \widetilde{f}(t) := (f(t), [f'(t)]) \tag{2.8}$$

is holomorphic (at a stationary point t_0, we just write $f'(t) = (t - t_0)^s u(t)$ with $s \in \mathbb{N}^*$ and $u(t_0) \neq 0$, and we define the tangent line at t_0 to be $[u(t_0)]$, hence $\widetilde{f}(t) = (f(t), [u(t)])$ near t_0; even for $t = t_0$, we still denote $[f'(t_0)] = [u(t_0)]$ for simplicity of notation). By definition $f'(t) \in \mathcal{O}_{\widetilde{X}}(-1)_{\widetilde{f}(t)} = \mathbb{C}\, u(t)$, hence the derivative f' defines a section

$$f' : T_{D_R} \to \widetilde{f}^* \mathcal{O}_{\widetilde{X}}(-1). \tag{2.9}$$

Moreover $\pi \circ \widetilde{f} = f$, therefore

$$\pi_* \widetilde{f}'(t) = f'(t) \in \mathbb{C} u(t) \Longrightarrow \widetilde{f}'(t) \in \widetilde{V}_{(f(t), u(t))} = \widetilde{V}_{\widetilde{f}(t)}$$

and we see that $\widetilde{f}$ is a tangent trajectory of $(\widetilde{X}, \widetilde{V})$. We say that $\widetilde{f}$ is the *canonical lifting* of f to $\widetilde{X}$. Conversely, if $g : D_R \to \widetilde{X}$ is a tangent trajectory of $(\widetilde{X}, \widetilde{V})$, then by definition of $\widetilde{V}$ we see that $f = \pi \circ g$ is a tangent trajectory of (X, V) and that $g = \widetilde{f}$ (unless g is contained in a vertical fiber $P(V_x)$, in which case f is constant).

For any point $x_0 \in X$, there are local coordinates $(z_1, \ldots, z_n)$ on a neighborhood Ω of x_0 such that the fibers $(V_z)_{z \in \Omega}$ can be defined by linear equations

$$V_z = \Big\{ \xi = \sum_{1 \leqslant j \leqslant n} \xi_j \frac{\partial}{\partial z_j}\, ;\, \xi_j = \sum_{1 \leqslant k \leqslant r} a_{jk}(z) \xi_k \text{ for } j = r+1, \ldots, n \Big\}, \tag{2.10}$$

where (a_{jk}) is a holomorphic $(n-r) \times r$ matrix. It follows that a vector $\xi \in V_z$ is completely determined by its first r components $(\xi_1, \ldots, \xi_r)$, and the affine chart $\xi_j \neq 0$ of $P(V)_{\restriction \Omega}$ can be described by the coordinate system

$$\Big(z_1, \ldots, z_n; \frac{\xi_1}{\xi_j}, \ldots, \frac{\xi_{j-1}}{\xi_j}, \frac{\xi_{j+1}}{\xi_j}, \ldots, \frac{\xi_r}{\xi_j} \Big). \tag{2.11}$$

Let $f \simeq (f_1, \ldots, f_n)$ be the components of f in the coordinates $(z_1, \ldots, z_n)$ (we suppose here R so small that $f(D_R) \subset \Omega$). It should be observed that f is uniquely determined by its initial value

x and by the first r components $(f_1,\ldots,f_r)$. Indeed, as $f'(t)\in V_{f(t)}$, we can recover the other components by integrating the system of ordinary differential equations

$$f'_j(t)=\sum_{1\leqslant k\leqslant r} a_{jk}(f(t))f'_k(t),\qquad j>r, \tag{2.12}$$

on a neighborhood of 0, with initial data $f(0)=x$. We denote by $m=m(f,t_0)$ the *multiplicity* of f at any point $t_0\in D_R$, that is, $m(f,t_0)$ is the smallest integer $m\in\mathbb{N}^*$ such that $f_j^{(m)}(t_0)\neq 0$ for some j. By (2.12), we can always suppose $j\in\{1,\ldots,r\}$, for example $f_r^{(m)}(t_0)\neq 0$. Then $f'(t)=(t-t_0)^{m-1}u(t)$ with $u_r(t_0)\neq 0$, and the lifting $\widetilde f$ is described in the coordinates of the affine chart $\xi_r\neq 0$ of $P(V)_{\restriction\Omega}$ by

$$\widetilde f\simeq\Big(f_1,\ldots,f_n;\frac{f'_1}{f'_r},\ldots,\frac{f'_{r-1}}{f'_r}\Big). \tag{2.13}$$

2.D. The Semple tower

Let X be a complex n-dimensional manifold. Following ideas of Green-Griffiths [GrGr79], we let $J_kX\to X$ be the bundle of k-jets of germs of parametrized curves in X, that is, the set of equivalence classes of holomorphic maps $f:(\mathbb{C},0)\to(X,x)$, with the equivalence relation $f\sim g$ if and only if all derivatives $f^{(j)}(0)=g^{(j)}(0)$ coincide for $0\leqslant j\leqslant k$, when computed in some local coordinate system of X near x. The projection map $J_kX\to X$ is simply $f\mapsto f(0)$. If $(z_1,\ldots,z_n)$ are local holomorphic coordinates on an open set $\Omega\subset X$, the elements f of any fiber J_kX_x, $x\in\Omega$, can be seen as $\mathbb{C}^n$-valued maps

$$f=(f_1,\ldots,f_n):(\mathbb{C},0)\to\Omega\subset\mathbb{C}^n,$$

and they are completetely determined by their Taylor expansion of order k at $t=0$

$$f(t)=x+t\,f'(0)+\frac{t^2}{2!}f''(0)+\cdots+\frac{t^k}{k!}f^{(k)}(0)+O(t^{k+1}).$$

In these coordinates, the fiber J_kX_x can thus be identified with the set of k-tuples of vectors $(\xi_1,\ldots,\xi_k)=(f'(0),\ldots,f^{(k)}(0))\in(\mathbb{C}^n)^k$. It follows that J_kX is a holomorphic fiber bundle with typical fiber $(\mathbb{C}^n)^k$ over X (however, J_kX is not a vector bundle for $k\geqslant 2$, because of the nonlinearity of coordinate changes. According to the philosophy of directed structures, one can also introduce the concept of jet bundle in the general situation of complex directed manifolds. If X is equipped with a holomorphic subbundle $V\subset T_X$, one associates to V a k-jet bundle J_kV as follows.

2.14. Definition. *Let (X,V) be a complex directed manifold. We define $J_kV\to X$ to be the bundle of k-jets of curves $f:(\mathbb{C},0)\to X$ which are tangent to V, i.e., such that $f'(t)\in V_{f(t)}$ for all t in a neighborhood of 0, together with the projection map $f\mapsto f(0)$ onto X.*

It is easy to check that J_kV is actually a subbundle of J_kX. In fact, by using (2.10) and (2.12), we see that the fibers J_kV_x are parametrized by

$$\big((f'_1(0),\ldots,f'_r(0));(f''_1(0),\ldots,f''_r(0));\ldots;(f_1^{(k)}(0),\ldots,f_r^{(k)}(0))\big)\in(\mathbb{C}^r)^k$$

for all $x\in\Omega$, hence J_kV is a locally trivial $(\mathbb{C}^r)^k$-subbundle of J_kX. Alternatively, we can pick a local holomorphic connection ∇ on V such that for any germs $w=\sum_{1\leqslant j\leqslant n}w_j\frac{\partial}{\partial z_j}\in\mathcal{O}(T_{X,x})$ and $v=\sum_{1\leqslant\lambda\leqslant r}v_\lambda e_\lambda\in\mathcal{O}(V)_x$ in a local trivializing frame $(e_1,\ldots,e_r)$ of $V_{\restriction\Omega}$ we have

$$\nabla_w v(x)=\sum_{1\leqslant j\leqslant n,\,1\leqslant\lambda\leqslant r} w_j\frac{\partial v_\lambda}{\partial z_j}e_\lambda(x)+\sum_{1\leqslant j\leqslant n,\,1\leqslant\lambda,\mu\leqslant r}\Gamma^\mu_{j\lambda}(x)w_jv_\lambda\,e_\mu(x). \tag{2.15}$$

We can of course take the frame obtained from (2.10) by lifting the vector fields $\partial/\partial z_1,\ldots,\partial/\partial z_r$, and the "trivial connection" given by the zero Christoffel symbolds $\Gamma=0$. One then obtains a trivialization $J^kV_{\restriction\Omega} \simeq V_{\restriction\Omega}^{\oplus k}$ by considering

$$J_kV_x \ni f \mapsto (\xi_1, \xi_2, \ldots, \xi_k) = (\nabla f(0), \nabla^2 f(0), \ldots, \nabla^k f(0)) \in V_x^{\oplus k}$$

and computing inductively the successive derivatives $\nabla f(t) = f'(t)$ and $\nabla^s f(t)$ via

$$\nabla^s f = (f^*\nabla)_{d/dt}(\nabla^{s-1} f) = \sum_{1\leqslant\lambda\leqslant r} \frac{d}{dt}\Big(\nabla^{s-1} f\Big)_\lambda e_\lambda(f) + \sum_{1\leqslant j\leqslant n,\, 1\leqslant\lambda,\mu\leqslant r} \Gamma^\mu_{j\lambda}(f) f'_j \Big(\nabla^{s-1} f\Big)_\lambda e_\mu(f).$$

This identification depends of course on the choice of ∇ and cannot be defined globally in general (unless we are in the rare situation where V has a global holomorphic connection. □

We now describe a convenient process for constructing "projectivized jet bundles", which will later appear as natural quotients of our jet bundles J_kV (or rather, as suitable desingularized compactifications of the quotients). Such spaces have already been considered since a long time, at least in the special case $X = \mathbb{P}^2$, $V = T_{\mathbb{P}^2}$ (see Gherardelli [Ghe41], Semple [Sem54]), and they have been mostly used as a tool for establishing enumerative formulas dealing with the order of contact of plane curves (see [Coll88], [CoKe94]); the article [ASS97] is also concerned with such generalizations of jet bundles, as well as [LaTh96] by Laksov and Thorup. One defines inductively the *projectivized k-jet bundle* X_k (or *Semple k-jet bundle*) and the associated subbundle $V_k \subset T_{X_k}$ by

$$(X_0, V_0) = (X, V), \qquad (X_k, V_k) = (\widetilde{X}_{k-1}, \widetilde{V}_{k-1}). \tag{2.16}$$

In other words, (X_k, V_k) is obtained from (X, V) by iterating k-times the lifting construction $(X, V) \mapsto (\widetilde{X}, \widetilde{V})$ described in § 2.B. By (2.4–2.9), we find

$$\dim X_k = n + k(r-1), \qquad \operatorname{rank} V_k = r, \tag{2.17}$$

together with exact sequences

$$0 \longrightarrow T_{X_k/X_{k-1}} \longrightarrow V_k \xrightarrow{(\pi_k)_*} \mathcal{O}_{X_k}(-1) \longrightarrow 0, \tag{2.18}$$

$$0 \longrightarrow \mathcal{O}_{X_k} \longrightarrow \pi_k^* V_{k-1} \otimes \mathcal{O}_{X_k}(1) \longrightarrow T_{X_k/X_{k-1}} \longrightarrow 0. \tag{2.18'}$$

where π_k is the natural projection $\pi_k : X_k \to X_{k-1}$ and $(\pi_k)_*$ its differential. Formula (5.4) yields

$$\det V_k = \pi_k^* \det V_{k-1} \otimes \mathcal{O}_{X_k}(r-1). \tag{2.19}$$

Every nonconstant tangent trajectory $f : D_R \to X$ of (X, V) lifts to a well defined and unique tangent trajectory $f_{[k]} : D_R \to X_k$ of (X_k, V_k). Moreover, the derivative $f'_{[k-1]}$ gives rise to a section

$$f'_{[k-1]} : T_{D_R} \to f^*_{[k]} \mathcal{O}_{X_k}(-1). \tag{2.20}$$

In coordinates, one can compute $f_{[k]}$ in terms of its components in the various affine charts (5.9) occurring at each step: we get inductively

$$f_{[k]} = (F_1, \ldots, F_N), \qquad f_{[k+1]} = \Big(F_1, \ldots, F_N, \frac{F'_{s_1}}{F'_{s_r}}, \ldots, \frac{F'_{s_{r-1}}}{F'_{s_r}}\Big) \tag{2.21}$$

where $N = n + k(r-1)$ and $\{s_1, \ldots, s_r\} \subset \{1, \ldots, N\}$. If $k \geqslant 1$, $\{s_1, \ldots, s_r\}$ contains the last $r-1$ indices of $\{1, \ldots, N\}$ corresponding to the "vertical" components of the projection $X_k \to X_{k-1}$, and in general, s_r is an index such that $m(F_{s_r}, 0) = m(f_{[k]}, 0)$, that is, F_{s_r} has the smallest vanishing order among all components F_s (s_r may be vertical or not, and the choice of $\{s_1, \ldots, s_r\}$ need not be unique).

By definition, there is a canonical injection $\mathcal{O}_{X_k}(-1) \hookrightarrow \pi_k^* V_{k-1}$, and a composition with the projection $(\pi_{k-1})_*$ (analogue for order $k-1$ of the arrow $(\pi_k)_*$ in the sequence (2.18)) yields for all $k \geqslant 2$ a canonical line bundle morphism

$$\mathcal{O}_{X_k}(-1) \longrightarrow \pi_k^* V_{k-1} \xrightarrow{(\pi_k)^*(\pi_{k-1})_*} \pi_k^* \mathcal{O}_{X_{k-1}}(-1), \tag{2.22}$$

which admits precisely $D_k = P(T_{X_{k-1}/X_{k-2}}) \subset P(V_{k-1}) = X_k$ as its zero divisor (clearly, D_k is a hyperplane subbundle of X_k). Hence we find

$$\mathcal{O}_{X_k}(1) = \pi_k^*\mathcal{O}_{X_{k-1}}(1) \otimes \mathcal{O}(D_k). \tag{2.23}$$

Now, we consider the composition of projections

$$\pi_{j,k} = \pi_{j+1} \circ \cdots \circ \pi_{k-1} \circ \pi_k : X_k \longrightarrow X_j. \tag{2.24}$$

Then $\pi_{0,k} : X_k \to X_0 = X$ is a locally trivial holomorphic fiber bundle over X, and the fibers $X_{k,x} = \pi_{0,k}^{-1}(x)$ are k-stage towers of $\mathbb{P}^{r-1}$-bundles. Since we have (in both directions) morphisms $(\mathbb{C}^r, T_{\mathbb{C}^r}) \leftrightarrow (X, V)$ of directed manifolds which are bijective on the level of bundle morphisms, the fibers are all isomorphic to a "universal" non singular projective algebraic variety of dimension $k(r-1)$ which we will denote by $\mathbb{R}_{r,k}$; it is not hard to see that $\mathbb{R}_{r,k}$ is rational (as will indeed follow from the proof of Theorem 3.11 below).

2.25. Remark. When (X, V) is singular, one can easily extend the construction of the Semple tower by fonctoriality. In fact, assume that X is a closed analytic subset of some open set $Z \subset \mathbb{C}^N$, and that $X' \subset X$ is a Zariski open subset on which $V_{\restriction X'}$ is a subbundle of $T_{X'}$. Then we consider the injection of the nonsingular directed manifold (X', V') into the absolute structure (Z, W), $W = T_Z$. This yields an injection $(X'_k, V'_k) \hookrightarrow (Z_k, W_k)$, and we simply define (X_k, V_k) to be the closure of (X'_k, V'_k) into (Z_k, W_k). It is not hard to see that this is indeed a closed analytic subset of the same dimension $n + k(r-1)$, where $r = \operatorname{rank} V'$.

3. Jet differentials and Green-Griffiths bundles

3.A. Green-Griffiths jet differentials

We first introduce the concept of jet differentials in the sense of Green-Griffiths [GrGr79]. The goal is to provide an intrinsic geometric description of holomorphic differential equations that a germ of curve $f : (\mathbb{C}, 0) \to X$ may satisfy. In the sequel, we fix a directed manifold (X, V) and suppose implicitly that all germs of curves f are tangent to V.

Let $\mathbb{G}_k$ be the group of germs of k-jets of biholomorphisms of $(\mathbb{C}, 0)$, that is, the group of germs of biholomorphic maps

$$t \mapsto \varphi(t) = a_1 t + a_2 t^2 + \cdots + a_k t^k, \qquad a_1 \in \mathbb{C}^*,\ a_j \in \mathbb{C},\ j \geqslant 2,$$

in which the composition law is taken modulo terms t^j of degree $j > k$. Then $\mathbb{G}_k$ is a k-dimensional nilpotent complex Lie group, which admits a natural fiberwise right action on $J_k V$. The action consists of reparametrizing k-jets of maps $f : (\mathbb{C}, 0) \to X$ by a biholomorphic change of parameter $\varphi : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$, that is, $(f, \varphi) \mapsto f \circ \varphi$. There is an exact sequence of groups

$$1 \to \mathbb{G}'_k \to \mathbb{G}_k \to \mathbb{C}^* \to 1$$

where $\mathbb{G}_k \to \mathbb{C}^*$ is the obvious morphism $\varphi \mapsto \varphi'(0)$, and $\mathbb{G}'_k = [\mathbb{G}_k, \mathbb{G}_k]$ is the group of k-jets of biholomorphisms tangent to the identity. Moreover, the subgroup $\mathbb{H} \simeq \mathbb{C}^*$ of homotheties $\varphi(t) = \lambda t$ is a (non normal) subgroup of $\mathbb{G}_k$, and we have a semidirect decomposition $\mathbb{G}_k = \mathbb{G}'_k \ltimes \mathbb{H}$. The corresponding action on k-jets is described in coordinates by

$$\lambda \cdot (f', f'', \ldots, f^{(k)}) = (\lambda f', \lambda^2 f'', \ldots, \lambda^k f^{(k)}).$$

Following [GrGr79], we introduce the vector bundle $E^{\mathrm{GG}}_{k,m} V^* \to X$ whose fibers are complex valued polynomials $Q(f', f'', \ldots, f^{(k)})$ on the fibers of $J_k V$, of weighted degree m with respect to the $\mathbb{C}^*$ action defined by H, that is, such that

$$Q(\lambda f', \lambda^2 f'', \ldots, \lambda^k f^{(k)}) = \lambda^m Q(f', f'', \ldots, f^{(k)}) \tag{3.1}$$

for all $\lambda \in \mathbb{C}^*$ and $(f', f'', \ldots, f^{(k)}) \in J_k V$. Here we view $(f', f'', \ldots, f^{(k)})$ as indeterminates with components

$$\big((f'_1, \ldots, f'_r); (f''_1, \ldots, f''_r); \ldots; (f^{(k)}_1, \ldots, f^{(k)}_r)\big) \in (\mathbb{C}^r)^k.$$

Notice that the concept of polynomial on the fibers of J_kV makes sense, for all coordinate changes $z \mapsto w = \Psi(z)$ on X induce polynomial transition automorphisms on the fibers of J_kV, given by a formula

$$(3.2)\qquad (\Psi\circ f)^{(j)} = \Psi'(f)\cdot f^{(j)} + \sum_{s=2}^{s=j}\ \sum_{j_1+j_2+\cdots+j_s=j} c_{j_1\ldots j_s}\Psi^{(s)}(f)\cdot(f^{(j_1)},\ldots,f^{(j_s)})$$

with suitable integer constants $c_{j_1\ldots j_s}$ (this is easily checked by induction on s). In the case $V = T_X$, we get the bundle of "absolute" jet differentials $E^{\rm GG}_{k,m}T^*_X$. If $Q \in E^{\rm GG}_{k,m}V^*$ is decomposed into multihomogeneous components of multidegree $(\ell_1,\ell_2,\ldots,\ell_k)$ in $f',f'',\ldots,f^{(k)}$ (the decomposition is of course coordinate dependent), these multidegrees must satisfy the relation

$$\ell_1 + 2\ell_2 + \cdots + k\ell_k = m.$$

The bundle $E^{\rm GG}_{k,m}V^*$ will be called the *bundle of jet differentials of order k and weighted degree m*. It is clear from (3.2) that a coordinate change $f \mapsto \Psi\circ f$ transforms every monomial $(f^{(\bullet)})^\ell = (f')^{\ell_1}(f'')^{\ell_2}\cdots(f^{(k)})^{\ell_k}$ of partial weighted degree $|\ell|_s := \ell_1 + 2\ell_2 + \cdots + s\ell_s$, $1 \leqslant s \leqslant k$, into a polynomial $((\Psi\circ f)^{(\bullet)})^\ell$ in $(f',f'',\ldots,f^{(k)})$ which has the same partial weighted degree of order s if $\ell_{s+1} = \cdots = \ell_k = 0$, and a larger or equal partial degree of order s otherwise. Hence, for each $s = 1,\ldots,k$, we get a well defined (i.e., coordinate invariant) decreasing filtration $F^\bullet_s$ on $E^{\rm GG}_{k,m}V^*$ as follows:

$$(3.3)\qquad F^p_s(E^{\rm GG}_{k,m}V^*) = \left\{\begin{matrix} Q(f',f'',\ldots,f^{(k)}) \in E^{\rm GG}_{k,m}V^* \text{ involving} \\ \text{only monomials } (f^{(\bullet)})^\ell \text{ with } |\ell|_s \geqslant p \end{matrix}\right\},\qquad \forall p\in\mathbb{N}.$$

The graded terms $\mathrm{Gr}^p_{k-1}(E^{\rm GG}_{k,m}V^*)$ associated with the filtration $F^p_{k-1}(E^{\rm GG}_{k,m}V^*)$ are precisely the homogeneous polynomials $Q(f',\ldots,f^{(k)})$ whose monomials $(f^\bullet)^\ell$ all have partial weighted degree $|\ell|_{k-1} = p$ (hence their degree ℓ_k in $f^{(k)}$ is such that $m - p = k\ell_k$, and $\mathrm{Gr}^p_{k-1}(E^{\rm GG}_{k,m}V^*) = 0$ unless k divides $m-p$). The transition automorphisms of the graded bundle are induced by coordinate changes $f\mapsto\Psi\circ f$, and they are described by substituting the arguments of $Q(f',\ldots,f^{(k)})$ according to formula (3.2), namely $f^{(j)} \mapsto (\Psi\circ f)^{(j)}$ for $j<k$, and $f^{(k)}\mapsto\Psi'(f)\circ f^{(k)}$ for $j=k$ (when $j=k$, the other terms fall in the next stage F^{p+1}_{k-1} of the filtration). Therefore $f^{(k)}$ behaves as an element of $V\subset T_X$ under coordinate changes. We thus find

$$(3.4)\qquad G^{m-k\ell_k}_{k-1}(E^{\rm GG}_{k,m}V^*) = E^{\rm GG}_{k-1,m-k\ell_k}V^*\otimes S^{\ell_k}V^*.$$

Combining all filtrations $F^\bullet_s$ together, we find inductively a filtration $F^\bullet$ on $E^{\rm GG}_{k,m}V^*$ such that the graded terms are

$$(3.5)\qquad \mathrm{Gr}^\ell(E^{\rm GG}_{k,m}V^*) = S^{\ell_1}V^*\otimes S^{\ell_2}V^*\otimes\cdots\otimes S^{\ell_k}V^*,\qquad \ell\in\mathbb{N}^k,\quad |\ell|_k = m.$$

The bundles $E^{\rm GG}_{k,m}V^*$ have other interesting properties. In fact,

$$E^{\rm GG}_{k,\bullet}V^* := \bigoplus_{m\geqslant 0} E^{\rm GG}_{k,m}V^*$$

is in a natural way a bundle of graded algebras (the product is obtained simply by taking the product of polynomials). There are natural inclusions $E^{\rm GG}_{k,\bullet}V^*\subset E^{\rm GG}_{k+1,\bullet}V^*$ of algebras, hence $E^{\rm GG}_{\infty,\bullet}V^* = \bigcup_{k\geqslant 0}E^{\rm GG}_{k,\bullet}V^*$ is also an algebra. Moreover, the sheaf of holomorphic sections $\mathcal{O}(E^{\rm GG}_{\infty,\bullet}V^*)$ admits a canonical derivation $D^{\rm GG}$ given by a collection of $\mathbb{C}$-linear maps

$$D^{\rm GG}:\mathcal{O}(E^{\rm GG}_{k,m}V^*)\to\mathcal{O}(E^{\rm GG}_{k+1,m+1}V^*),$$

constructed in the following way. A holomorphic section of $E^{\mathrm{GG}}_{k,m}V^*$ on a coordinate open set $\Omega\subset X$ can be seen as a differential operator on the space of germs $f:(\mathbb{C},0)\to\Omega$ of the form

$$Q(f)=\sum_{|\alpha_1|+2|\alpha_2|+\cdots+k|\alpha_k|=m} a_{\alpha_1\ldots\alpha_k}(f)\,(f')^{\alpha_1}(f'')^{\alpha_2}\cdots(f^{(k)})^{\alpha_k} \tag{3.6}$$

in which the coefficients $a_{\alpha_1\ldots\alpha_k}$ are holomorphic functions on Ω. Then $D^{\mathrm{GG}}Q$ is given by the formal derivative $(D^{\mathrm{GG}}Q)(f)(t)=d(Q(f))/dt$ with respect to the 1-dimensional parameter t in $f(t)$. For example, in dimension 2, if $Q\in H^0(\Omega,\mathcal{O}(E^{\mathrm{GG}}_{2,4}))$ is the section of weighted degree 4

$$Q(f)=a(f_1,f_2)\,f_1'^3f_2'+b(f_1,f_2)\,f_1''^2,$$

we find that $D^{\mathrm{GG}}Q\in H^0(\Omega,\mathcal{O}(E^{\mathrm{GG}}_{3,5}))$ is given by

$$\begin{aligned}(D^{\mathrm{GG}}Q)(f)&=\frac{\partial a}{\partial z_1}(f_1,f_2)\,f_1'^4f_2'+\frac{\partial a}{\partial z_2}(f_1,f_2)\,f_1'^3f_2'^2+\frac{\partial b}{\partial z_1}(f_1,f_2)\,f_1'f_1''^2\\&\quad+\frac{\partial b}{\partial z_2}(f_1,f_2)\,f_2'f_1''^2+a(f_1,f_2)\,\big(3f_1'^2f_1''f_2'+f_1'^3f_2''\big)+b(f_1,f_2)\,2f_1''f_1'''.\end{aligned}$$

Associated with the graded algebra bundle $E^{\mathrm{GG}}_{k,\bullet}V^*$, we define an analytic fiber bundle

$$X_k^{\mathrm{GG}}:=\mathrm{Proj}(E^{\mathrm{GG}}_{k,\bullet}V^*)=(J_kV\smallsetminus\{0\})/\mathbb{C}^* \tag{3.7}$$

over X, which has weighted projective spaces $\mathbb{P}(1^{[r]},2^{[r]},\ldots,k^{[r]})$ as fibers (these weighted projective spaces are singular for $k>1$, but they only have quotient singularities, see [Dol81] ; here $J_kV\smallsetminus\{0\}$ is the set of nonconstant jets of order k ; we refer e.g. to Hartshorne's book [Har77] for a definition of the Proj fonctor). As such, it possesses a canonical sheaf $\mathcal{O}_{X_k^{\mathrm{GG}}}(1)$ such that $\mathcal{O}_{X_k^{\mathrm{GG}}}(m)$ is invertible when m is a multiple of $\mathrm{lcm}(1,2,\ldots,k)$. Under the natural projection $\pi_k:X_k^{\mathrm{GG}}\to X$, the direct image $(\pi_k)_*\mathcal{O}_{X_k^{\mathrm{GG}}}(m)$ coincides with polynomials

$$P(z\,;\xi_1,\ldots,\xi_k)=\sum_{\alpha_\ell\in\mathbb{N}^r,\,1\leqslant\ell\leqslant k} a_{\alpha_1\ldots\alpha_k}(z)\,\xi_1^{\alpha_1}\ldots\xi_k^{\alpha_k} \tag{3.8}$$

of weighted degree $|\alpha_1|+2|\alpha_2|+\ldots+k|\alpha_k|=m$ on J^kV with holomorphic coefficients; in other words, we obtain precisely the sheaf of sections of the bundle $E^{\mathrm{GG}}_{k,m}V^*$ of jet differentials of order k and degree m.

3.9. Proposition. *By construction, if $\pi_k:X_k^{\mathrm{GG}}\to X$ is the natural projection, we have the direct image formula*

$$(\pi_k)_*\mathcal{O}_{X_k^{\mathrm{GG}}}(m)=\mathcal{O}(E^{\mathrm{GG}}_{k,m}V^*)$$

for all k and m.

3.B. Invariant jet differentials

In the geometric context, we are not really interested in the bundles $(J_kV\smallsetminus\{0\})/\mathbb{C}^*$ themselves, but rather on their quotients $(J_kV\smallsetminus\{0\})/\mathbb{G}_k$ (would such nice complex space quotients exist!). We will see that the Semple bundle X_k constructed in § 2.D plays the role of such a quotient. First we introduce a canonical bundle subalgebra of $E^{\mathrm{GG}}_{k,\bullet}V^*$.

3.10. Definition. *We introduce a subbundle $E_{k,m}V^*\subset E^{\mathrm{GG}}_{k,m}V^*$, called the bundle of invariant jet differentials of order k and degree m, defined as follows: $E_{k,m}V^*$ is the set of polynomial differential operators $Q(f',f'',\ldots,f^{(k)})$ which are invariant under arbitrary changes of parametrization, i.e., for every $\varphi\in\mathbb{G}_k$*

$$Q\big((f\circ\varphi)',(f\circ\varphi)'',\ldots,(f\circ\varphi)^{(k)}\big)=\varphi'(0)^mQ(f',f'',\ldots,f^{(k)}).$$

Alternatively, $E_{k,m}V^* = (E^{\rm GG}_{k,m}V^*)^{\mathbb{G}'_k}$ is the set of invariants of $E^{\rm GG}_{k,m}V^*$ under the action of $\mathbb{G}'_k$. Clearly, $E_{\infty,\bullet}V^* = \bigcup_{k\geqslant 0}\bigoplus_{m\geqslant 0} E_{k,m}V^*$ is a subalgebra of $E^{\rm GG}_{k,m}V^*$ (observe however that this algebra is not invariant under the derivation $D^{\rm GG}$, since e.g. $f''_j = D^{\rm GG}f_j$ is not an invariant polynomial).

3.11. Theorem. *Suppose that V has rank $r\geqslant 2$. Let $\pi_{0,k}: X_k \longrightarrow X$ be the Semple jet bundles constructed in section 2.B, and let $J_kV^{\rm reg}$ be the bundle of regular k-jets of maps $f:(\mathbb{C},0)\to X$, that is, jets f such that $f'(0)\neq 0$.*

(i) *The quotient $J_kV^{\rm reg}/\mathbb{G}_k$ has the structure of a locally trivial bundle over X, and there is a holomorphic embedding $J_kV^{\rm reg}/\mathbb{G}_k \hookrightarrow X_k$ over X, which identifies $J_kV^{\rm reg}/\mathbb{G}_k$ with $X_k^{\rm reg}$ (thus X_k is a relative compactification of $J_kV^{\rm reg}/\mathbb{G}_k$ over X).*

(ii) *The direct image sheaf*

$$(\pi_{0,k})_*\mathcal{O}_{X_k}(m) \simeq \mathcal{O}(E_{k,m}V^*)$$

can be identified with the sheaf of holomorphic sections of $E_{k,m}V^*$.

(iii) *For every $m>0$, the relative base locus of the linear system $|\mathcal{O}_{X_k}(m)|$ is equal to the set $X_k^{\rm sing}$ of singular k-jets. Moreover, $\mathcal{O}_{X_k}(1)$ is relatively big over X.*

Proof. (i) For $f\in J_kV^{\rm reg}$, the lifting $\widetilde{f}$ is obtained by taking the derivative $(f,[f'])$ without any cancellation of zeroes in f', hence we get a uniquely defined $(k-1)$-jet $\widetilde{f}:(\mathbb{C},0)\to\widetilde{X}$. Inductively, we get a well defined $(k-j)$-jet $f_{[j]}$ in X_j, and the value $f_{[k]}(0)$ is independent of the choice of the representative f for the k-jet. As the lifting process commutes with reparametrization, i.e., $(f\circ\varphi)^\sim = \widetilde{f}\circ\varphi$ and more generally $(f\circ\varphi)_{[k]} = f_{[k]}\circ\varphi$, we conclude that there is a well defined set-theoretic map

$$J_kV^{\rm reg}/\mathbb{G}_k \to X_k^{\rm reg}, \qquad f \ \mathrm{mod}\ \mathbb{G}_k \mapsto f_{[k]}(0).$$

This map is better understood in coordinates as follows. Fix coordinates $(z_1,\ldots,z_n)$ near a point $x_0\in X$, such that $V_{x_0} = \mathrm{Vect}(\partial/\partial z_1,\ldots,\partial/\partial z_r)$. Let $f=(f_1,\ldots,f_n)$ be a regular k-jet tangent to V. Then there exists $i\in\{1,2,\ldots,r\}$ such that $f'_i(0)\neq 0$, and there is a unique reparametrization $t=\varphi(\tau)$ such that $f\circ\varphi = g = (g_1,g_2,\ldots,g_n)$ with $g_i(\tau)=\tau$ (we just express the curve as a graph over the z_i-axis, by means of a change of parameter $\tau = f_i(t)$, i.e. $t=\varphi(\tau) = f_i^{-1}(\tau)$). Suppose $i=r$ for the simplicity of notation. The space X_k is a k-stage tower of $\mathbb{P}^{r-1}$-bundles. In the corresponding inhomogeneous coordinates on these $\mathbb{P}^{r-1}$'s, the point $f_{[k]}(0)$ is given by the collection of derivatives

$$\big((g'_1(0),\ldots,g'_{r-1}(0));(g''_1(0),\ldots,g''_{r-1}(0));\ldots;(g^{(k)}_1(0),\ldots,g^{(k)}_{r-1}(0))\big).$$

[Recall that the other components $(g_{r+1},\ldots,g_n)$ can be recovered from $(g_1,\ldots,g_r)$ by integrating the differential system (5.10)]. Thus the map $J_kV^{\rm reg}/\mathbb{G}_k\to X_k$ is a bijection onto $X_k^{\rm reg}$, and the fibers of these isomorphic bundles can be seen as unions of r affine charts $\simeq(\mathbb{C}^{r-1})^k$, associated with each choice of the axis z_i used to describe the curve as a graph. The change of parameter formula $\frac{d}{d\tau} = \frac{1}{f'_r(t)}\frac{d}{dt}$ expresses all derivatives $g_i^{(j)}(\tau) = d^jg_i/d\tau^j$ in terms of the derivatives $f_i^{(j)}(t) = d^jf_i/dt^j$

$$(3.12)\qquad \begin{aligned}(g'_1,\ldots,g'_{r-1}) &= \Big(\frac{f'_1}{f'_r},\ldots,\frac{f'_{r-1}}{f'_r}\Big);\\ (g''_1,\ldots,g''_{r-1}) &= \Big(\frac{f''_1f'_r - f''_rf'_1}{f'^3_r},\ldots,\frac{f''_{r-1}f'_r - f''_rf'_{r-1}}{f'^3_r}\Big);\ \ldots\ ;\\ (g^{(k)}_1,\ldots,g^{(k)}_{r-1}) &= \Big(\frac{f^{(k)}_1f'_r - f^{(k)}_rf'_1}{f'^{k+1}_r},\ldots,\frac{f^{(k)}_{r-1}f'_r - f^{(k)}_rf'_{r-1}}{f'^{k+1}_r}\Big) + (\text{order} < k).\end{aligned}$$

Also, it is easy to check that $f'^{2k-1}_r g^{(k)}_i$ is an invariant polynomial in $f',f'',\ldots,f^{(k)}$ of total degree $2k-1$, i.e., a section of $E_{k,2k-1}$.

(ii) Since the bundles X_k and $E_{k,m}V^*$ are both locally trivial over X, it is sufficient to identify sections σ of $\mathcal{O}_{X_k}(m)$ over a fiber $X_{k,x} = \pi_{0,k}^{-1}(x)$ with the fiber $E_{k,m}V^*_x$, at any point $x \in X$. Let $f \in J_kV_x^{\rm reg}$ be a regular k-jet at x. By (6.6), the derivative $f'_{[k-1]}(0)$ defines an element of the fiber of $\mathcal{O}_{X_k}(-1)$ at $f_{[k]}(0) \in X_k$. Hence we get a well defined complex valued operator

$$Q(f', f'', \ldots, f^{(k)}) = \sigma(f_{[k]}(0)) \cdot (f'_{[k-1]}(0))^m. \tag{3.13}$$

Clearly, Q is holomorphic on $J_kV_x^{\rm reg}$ (by the holomorphicity of σ), and the $\mathbb{G}_k$-invariance condition of Definition 3.10 is satisfied since $f_{[k]}(0)$ does not depend on reparametrization and

$$(f\circ\varphi)'_{[k-1]}(0) = f'_{[k-1]}(0)\varphi'(0).$$

Now, $J_kV_x^{\rm reg}$ is the complement of a linear subspace of codimension n in J_kV_x, hence Q extends holomorphically to all of $J_kV_x \simeq (\mathbb{C}^r)^k$ by Riemann's extension theorem (here we use the hypothesis $r \geqslant 2$; if $r = 1$, the situation is anyway not interesting since $X_k = X$ for all k). Thus Q admits an everywhere convergent power series

$$Q(f', f'', \ldots, f^{(k)}) = \sum_{\alpha_1,\alpha_2,\ldots,\alpha_k\in\mathbb{N}^r} a_{\alpha_1\ldots\alpha_k}\,(f')^{\alpha_1}(f'')^{\alpha_2}\cdots(f^{(k)})^{\alpha_k}.$$

The $\mathbb{G}_k$-invariance (3.10) implies in particular that Q must be multihomogeneous in the sense of (3.1), and thus Q must be a polynomial. We conclude that $Q \in E_{k,m}V^*_x$, as desired.

Conversely, for all w in a neighborhood of any given point $w_0 \in X_{k,x}$, we can find a holomorphic family of germs $f_w : (\mathbb{C}, 0) \to X$ such that $(f_w)_{[k]}(0) = w$ and $(f_w)'_{[k-1]}(0) \neq 0$ (just take the projections to X of integral curves of (X_k, V_k) integrating a nonvanishing local holomorphic section of V_k near w_0). Then every $Q \in E_{k,m}V^*_x$ yields a holomorphic section σ of $\mathcal{O}_{X_k}(m)$ over the fiber $X_{k,x}$ by putting

$$\sigma(w) = Q(f'_w, f''_w, \ldots, f_w^{(k)})(0) \cdot \big((f_w)'_{[k-1]}(0)\big)^{-m}. \tag{3.14}$$

(iii) By what we saw in (i)–(ii), every section σ of $\mathcal{O}_{X_k}(m)$ over the fiber $X_{k,x}$ is given by a polynomial $Q \in E_{k,m}V^*_x$, and this polynomial can be expressed on the Zariski open chart $f'_r \neq 0$ of $X_{k,x}^{\rm reg}$ as

$$Q(f', f'', \ldots, f^{(k)}) = f_r'^m \widehat{Q}(g', g'', \ldots, g^{(k)}), \tag{3.15}$$

where $\widehat{Q}$ is a polynomial and g is the reparametrization of f such that $g_r(\tau) = \tau$. In fact $\widehat{Q}$ is obtained from Q by substituting $f'_r = 1$ and $f_r^{(j)} = 0$ for $j \geqslant 2$, and conversely Q can be recovered easily from $\widehat{Q}$ by using the substitutions (3.12).

In this context, the jet differentials $f \mapsto f'_1, \ldots, f \mapsto f'_r$ can be viewed as sections of $\mathcal{O}_{X_k}(1)$ on a neighborhood of the fiber $X_{k,x}$. Since these sections vanish exactly on $X_k^{\rm sing}$, the relative base locus of $\mathcal{O}_{X_k}(m)$ is contained in $X_k^{\rm sing}$ for every $m > 0$. We see that $\mathcal{O}_{X_k}(1)$ is big by considering the sections of $\mathcal{O}_{X_k}(2k-1)$ associated with the polynomials $Q(f', \ldots, f^{(k)}) = f_r'^{2k-1} g_i^{(j)}$, $1 \leqslant i \leqslant r-1$, $1 \leqslant j \leqslant k$; indeed, these sections separate all points in the open chart $f'_r \neq 0$ of $X_{k,x}^{\rm reg}$.

Now, we check that every section σ of $\mathcal{O}_{X_k}(m)$ over $X_{k,x}$ must vanish on $X_{k,x}^{\rm sing}$. Pick an arbitrary element $w \in X_k^{\rm sing}$ and a germ of curve $f : (\mathbb{C}, 0) \to X$ such that $f_{[k]}(0) = w$, $f'_{[k-1]}(0) \neq 0$ and $s = m(f, 0) \gg 0$ (such an f exists by Corollary 6.14). There are local coordinates $(z_1, \ldots, z_n)$ on X such that $f(t) = (f_1(t), \ldots, f_n(t))$ where $f_r(t) = t^s$. Let Q, $\widehat{Q}$ be the polynomials associated with σ in these coordinates and let $(f')^{\alpha_1}(f'')^{\alpha_2}\cdots(f^{(k)})^{\alpha_k}$ be a monomial occurring in Q, with $\alpha_j \in \mathbb{N}^r$, $|\alpha_j| = \ell_j$, $\ell_1 + 2\ell_2 + \cdots + k\ell_k = m$. Putting $\tau = t^s$, the curve $t \mapsto f(t)$ becomes a Puiseux expansion $\tau \mapsto g(\tau) = (g_1(\tau), \ldots, g_{r-1}(\tau), \tau)$ in which g_i is a power series in $\tau^{1/s}$, starting with exponents of τ at least equal to 1. The derivative $g^{(j)}(\tau)$ may involve negative powers of τ, but the exponent is always $\geqslant 1 + \frac{1}{s} - j$. Hence the Puiseux expansion of $\widehat{Q}(g', g'', \ldots, g^{(k)})$ can only involve powers of τ of exponent $\geqslant \ -\max_\ell((1-\frac{1}{s})\ell_2+\cdots+(k-1-\frac{1}{s})\ell_k)$. Finally $f_r'(t)=st^{s-1}=s\tau^{1-1/s}$, thus the lowest exponent of τ in $Q(f',\ldots,f^{(k)})$ is at least equal to

$$\Big(1-\frac{1}{s}\Big)m-\max_\ell\Big(\Big(1-\frac{1}{s}\Big)\ell_2+\cdots+\Big(k-1-\frac{1}{s}\Big)\ell_k\Big)$$
$$\geqslant \min_\ell \Big(1-\frac{1}{s}\Big)\ell_1+\Big(1-\frac{1}{s}\Big)\ell_2+\cdots+\Big(1-\frac{k-1}{s}\Big)\ell_k$$

where the minimum is taken over all monomials $(f')^{\alpha_1}(f'')^{\alpha_2}\cdots(f^{(k)})^{\alpha_k}$, $|\alpha_j|=\ell_j$, occurring in Q. Choosing $s\geqslant k$, we already find that the minimal exponent is positive, hence $Q(f',\ldots,f^{(k)})(0)=0$ and $\sigma(w)=0$ by (3.14). □

Theorem 3.11 (iii) shows that $\mathcal{O}_{X_k}(1)$ is never relatively ample over X for $k\geqslant 2$. In order to overcome this difficulty, we define for every $a_\bullet=(a_1,\ldots,a_k)\in\mathbb{Z}^k$ a line bundle $\mathcal{O}_{X_k}(a_\bullet)$ on X_k such that

$$\mathcal{O}_{X_k}(a_\bullet)=\pi_{1,k}^*\mathcal{O}_{X_1}(a_1)\otimes\pi_{2,k}^*\mathcal{O}_{X_2}(a_2)\otimes\cdots\otimes\mathcal{O}_{X_k}(a_k). \tag{3.16}$$

By (6.9), we have $\pi_{j,k}^*\mathcal{O}_{X_j}(1)=\mathcal{O}_{X_k}(1)\otimes\mathcal{O}_{X_k}(-\pi_{j+1,k}^*D_{j+1}-\cdots-D_k)$, thus by putting $D_j^*=\pi_{j+1,k}^*D_{j+1}$ for $1\leqslant j\leqslant k-1$ and $D_k^*=0$, we find an identity

$$\begin{aligned}(3.17)\qquad &\mathcal{O}_{X_k}(a_\bullet)=\mathcal{O}_{X_k}(b_k)\otimes\mathcal{O}_{X_k}(-b_\bullet\cdot D^*),\qquad\text{where}\\ &b_\bullet=(b_1,\ldots,b_k)\in\mathbb{Z}^k,\quad b_j=a_1+\cdots+a_j,\\ &b_\bullet\cdot D^*=\sum_{1\leqslant j\leqslant k-1}b_j\,\pi_{j+1,k}^*D_{j+1}.\end{aligned}$$

In particular, if $b_\bullet\in\mathbb{N}^k$, i.e., $a_1+\cdots+a_j\geqslant 0$, we get a morphism

$$\mathcal{O}_{X_k}(a_\bullet)=\mathcal{O}_{X_k}(b_k)\otimes\mathcal{O}_{X_k}(-b_\bullet\cdot D^*)\to\mathcal{O}_{X_k}(b_k). \tag{3.18}$$

The following result gives a sufficient condition for the relative nefness or ampleness of weighted jet bundles.

3.19. Proposition. *Take a very ample line bundle A on X, and consider on X_k the line bundle*

$$L_k=\mathcal{O}_{X_k}(3^{k-1},3^{k-2},\ldots,3,1)\otimes\pi_{k,0}^*A^{\otimes 3^k}$$

defined inductively by $L_0=A$ and $L_k=\mathcal{O}_{X_k}(1)\otimes\pi_{k,k-1}^*L_{k-1}^{\otimes 3}$. Then $V_k^*\otimes L_k^{\otimes 2}$ is a nef vector bundle on X_k, which is in fact generated by its global sections, for all $k\geqslant 0$. Equivalently

$$L_k'=\mathcal{O}_{X_k}(1)\otimes\pi_{k,k-1}^*L_{k-1}^{\otimes 2}=\mathcal{O}_{X_k}(2\cdot 3^{k-2},2\cdot 3^{k-3},\ldots,6,2,1)\otimes\pi_{k,0}^*A^{\otimes 2\cdot 3^{k-1}}$$

is nef over X_k (and generated by sections) for all $k\geqslant 1$.

Let us recall that a line bundle $L\to X$ on a projective variety X is said to nef if $L\cdot C\geqslant 0$ for all irreducible algebraic curves $C\subset X$, and that a vector bundle $E\to X$ is said to be nef if $\mathcal{O}_{\mathbb{P}(E)}(1)$ is nef on $\mathbb{P}(E):=P(E^*)$; any vector bundle generated by global sections is nef (cf. [DePS94] for more details). The statement concerning L_k' is obtained by projectivizing the vector bundle $E=V_{k-1}^*\otimes L_{k-1}^{\otimes 2}$ on X_{k-1}, whose associated tautological line bundle is $\mathcal{O}_{\mathbb{P}(E)}(1)=L_k'$ on $\mathbb{P}(E)=P(V_{k-1})=X_k$. Also one gets inductively that

$$L_k=\mathcal{O}_{\mathbb{P}(V_{k-1}\otimes L_{k-1}^{\otimes 2})}(1)\otimes\pi_{k,k-1}^*L_{k-1}\quad\textit{is very ample on } X_k. \tag{3.20}$$

Proof. Let $X\subset\mathbb{P}^N$ be the embedding provided by A, so that $A=\mathcal{O}_{\mathbb{P}^N}(1)_{\restriction X}$. As is well known, if Q is the tautological quotient vector bundle on $\mathbb{P}^N$, the twisted cotangent bundle

$$T_{\mathbb{P}^N}^*\otimes\mathcal{O}_{\mathbb{P}^N}(2)=\Lambda^{N-1}Q$$

is nef; hence its quotients $T^*_X \otimes A^{\otimes 2}$ and $V^*_0 \otimes L_0^{\otimes 2} = V^* \otimes A^{\otimes 2}$ are nef (any tensor power of nef vector bundles is nef, and so is any quotient). We now proceed by induction, assuming $V^*_{k-1} \otimes L_{k-1}^{\otimes 2}$ to be nef, $k \geqslant 1$. By taking the second wedge power of the central term in (6.4′), we get an injection

$$0 \longrightarrow T_{X_k/X_{k-1}} \longrightarrow \Lambda^2\big(\pi_k^\star V_{k-1} \otimes \mathcal{O}_{X_k}(1)\big).$$

By dualizing and twisting with $\mathcal{O}_{X_{k-1}}(2) \otimes \pi_k^\star L_{k-1}^{\otimes 2}$, we find a surjection

$$\pi_k^\star \Lambda^2(V^\star_{k-1} \otimes L_{k-1}) \longrightarrow T^\star_{X_k/X_{k-1}} \otimes \mathcal{O}_{X_k}(2) \otimes \pi_k^\star L_{k-1}^{\otimes 2} \longrightarrow 0.$$

By the induction hypothesis, we see that $T^\star_{X_k/X_{k-1}} \otimes \mathcal{O}_{X_k}(2) \otimes \pi_k^\star L_{k-1}^{\otimes 2}$ is nef. Next, the dual of (6.4) yields an exact sequence

$$0 \longrightarrow \mathcal{O}_{X_k}(1) \longrightarrow V_k^\star \longrightarrow T^\star_{X_k/X_{k-1}} \longrightarrow 0.$$

As an extension of nef vector bundles is nef, the nefness of $V_k^* \otimes L_k^{\otimes 2}$ will follow if we check that $\mathcal{O}_{X_k}(1) \otimes L_k^{\otimes 2}$ and $T^\star_{X_k/X_{k-1}} \otimes L_k^{\otimes 2}$ are both nef. However, this follows again from the induction hypothesis if we observe that the latter implies

$$L_k \geqslant \pi^*_{k,k-1} L_{k-1} \quad \text{and} \quad L_k \geqslant \mathcal{O}_{X_k}(1) \otimes \pi^*_{k,k-1} L_{k-1}$$

in the sense that $L'' \geqslant L'$ if the "difference" $L'' \otimes (L')^{-1}$ is nef. All statements remain valid if we replace "nef" with "generated by sections" in the above arguments. □

3.21. Corollary. *A $\mathbb{Q}$-line bundle $\mathcal{O}_{X_k}(a_\bullet) \otimes \pi^*_{k,0} A^{\otimes p}$, $a_\bullet \in \mathbb{Q}^k$, $p \in \mathbb{Q}$, is nef (resp. ample) on X_k as soon as*

$$a_j \geqslant 3a_{j+1} \text{ for } j = 1, 2, \ldots, k-2 \text{ and } a_{k-1} \geqslant 2a_k \geqslant 0,\ p \geqslant 2 \textstyle\sum a_j,$$

resp.

$$a_j \geqslant 3a_{j+1} \text{ for } j = 1, 2, \ldots, k-2 \text{ and } a_{k-1} > 2a_k > 0,\ p > 2 \textstyle\sum a_j.$$

Proof. This follows easily by taking convex combinations of the L_j and L'_j and applying Proposition 3.19 and our observation (3.20). □

3.22. Remark. As $\mathbb{G}_k$ is a non reductive group, it is a priori unclear whether the graded ring $\mathcal{A}_{n,k,r} = \bigoplus_{m\in\mathbb{Z}} E_{k,m} V^\star$ (taken pointwise over X) is finitely generated. This can be checked manually ([Dem07a], [Dem07b]) for $n = 2$ and $k \leqslant 4$. Rousseau [Rou06] also checked the case $n = 3$, $k = 3$, and then Merker [Mer08, Mer10] proved the finiteness for $n = 2, 3, 4$, $k \leqslant 4$ and $n = 2$, $k = 5$. Recently, Bérczi and Kirwan [BeKi12] made an attempt to prove the finiteness in full generality, but it appears that the general case is still unsettled.

3.C. Fundamental vanishing theorem

We prove here a fundamental vanishing theorem due to Siu and Yeung ([SiYe96, SiYe97], [Siu97]). Their original proof makes use of Nevanlinna theory, especially of the logarithmic derivative lemma, see also [Dem97] for a more detailed account (in French). An alternative simpler proof based on the Ahlfors lemma and on algebraic properties of jet differentials can be found in [Dem18] (cf. also [Dem95]).

3.23. Fundamental vanishing theorem. *Let (X, V) be a projective directed manifold and A an ample divisor on X. Then $P(f\,;\, f', f'', \ldots, f^{(k)}) = 0$ for every entire curve $f : (\mathbb{C}, T_\mathbb{C}) \to (X, V)$ and every global section $P \in H^0(X, E^{\mathrm{GG}}_{k,m} V^* \otimes \mathcal{O}(-A))$.*

Proof. We first give a proof of 3.23 in the special case where f is a Brody curve, i.e. $\sup_{t\in\mathbb{C}} \|f'(t)\|_\omega < +\infty$ with respect to a given Hermitian metric ω on X. In fact, the proof is much simpler in that case, and thanks to the Brody criterion 1.8, this is sufficient to establish the hyperbolicity of (X, V). After raising P to a power P^s and replacing $\mathcal{O}(-A)$ with $\mathcal{O}(-sA)$, one can always assume that A

is a very ample divisor. We interpret $E^{\mathrm{GG}}_{k,m}V^*\otimes\mathcal{O}(-A)$ as the bundle of complex valued differential operators whose coefficients $a_\alpha(z)$ vanish along A.

Fix a finite open covering of X by coordinate balls $B(p_j,R_j)$ such that the balls $B_j(p_j,R_j/4)$ still cover X. As f' is bounded, there exists $\delta>0$ such that for $f(t_0)\in B(p_j,R_j/4)$ we have $f(t)\in B(p_j,R_j/2)$ whenever $|t-t_0|<\delta$, uniformly for every $t_0\in\mathbb{C}$. The Cauchy inequalities applied to the components of f in each of the balls imply that the derivatives $f^{(j)}(t)$ are bounded on $\mathbb{C}$, and therefore, since the coefficients $a_\alpha(z)$ of P are also uniformly bounded on each of the balls $B(p_j,R_j/2)$ we conclude that $g:=P(f\,;\,f',f'',\ldots,f^{(k)})$ is a bounded holomorphic function on $\mathbb{C}$. After moving A in the linear system $|A|$, we may further assume that $\operatorname{Supp}A$ intersects $f(\mathbb{C})$. Then g vanishes somewhere, hence $g\equiv 0$ by Liouville's theorem, as expected.

Next we consider the case where $P\in H^0(X,E_{k,m}V^*\otimes\mathcal{O}(-A))$ is an invariant differential operator. We may of course assume $P\neq 0$. Then we get an associated non zero section $\sigma\in H^0(X_k,\mathcal{O}_{X_k}(m)\otimes\pi^*_{k,0}\mathcal{O}(-A))$. Thanks to Corollary 3.21, the line bundle

$$L=\mathcal{O}_{X_k}(a_\bullet)\otimes\pi^*_{k,0}\mathcal{O}(pA)=\mathcal{O}_{X_k}(m')\otimes\mathcal{O}_{X_k}(-b_\bullet\cdot D^*)\otimes\pi^*_{k,0}\mathcal{O}(pA)$$

is ample on X_k for suitable $b_\bullet\geqslant 0$ and $m',p>0$. Let h_L be a smooth metric on L such that $\omega_k=\Theta_{L,h_L}$ is a Kähler metric on X_k. Then we can produce a singular hermitian metric h on $\mathcal{O}_{X_k}(-1)$ by putting

$$\|\xi\|_h=(\|\sigma^p\cdot\xi^{pm+m'}\|_{h_L^{-1}})^{1/(pm+m')},\xi\in\mathcal{O}_{X_k}(-1),$$

and viewing $\sigma^p\cdot\xi^{pm+m'}$ as an element in $\mathcal{O}_{X_k}(-m')\otimes\pi^*_{k,0}\mathcal{O}(-pA)\subset\mathcal{O}(L^{-1})$. The metric h has a weight e^φ that is continuous, with zeroes contained in the union of $\{\sigma=0\}$ and of the vertical divisor D^*. Moreover the curvature tensor $\Theta_{\mathcal{O}_{X_k}(1),h^{-1}}=\frac{i}{2\pi}\partial\overline{\partial}\log h$ satisfies by construction $\Theta_{\mathcal{O}_{X_k}(1),h^{-1}}\geqslant(pm+m')^{-1}\omega_k$. On the other hand, the continuity of the weight of h and the compactness of X_k imply that there exists a constant $C>0$ such that $\|d\pi_{k,k-1}(\eta)\|_h\leqslant C\|\eta\|_{\omega_k}$ for all vectors $\eta\in V_k$ (notice that $\xi=d\pi_{k,k-1}(\eta)\in\mathcal{O}_{X_k}(-1)$). Now, the derivative $f'_{[k-1]}$ can be seen as a section of $f^*_{[k]}\mathcal{O}_{X_k}(-1)$, and we use this to define a singular hermitian metric $\gamma(t)\,i\,dt\wedge d\overline{t}$ on $\mathbb{C}$ by taking

$$\gamma(t)=\|f'_{[k-1]}(t)\|^2_{h(f_{[k]}(t))}.$$

If $f_{[k]}(\mathbb{C})$ is not contained in the divisor $\{\sigma=0\}$, then γ is not identically zero and, in the sense of distributions, we find

$$\frac{i}{2\pi}\partial\overline{\partial}\log\gamma\geqslant f^*_{[k]}\Theta_{\mathcal{O}_{X_k}(1),h^{-1}}\geqslant(pm+m')^{-1}f^*_{[k]}\omega_k\geqslant C^{-1}(pm+m')^{-1}\gamma.$$

The final inequality comes from the inequality relating h and ω_k when we take $\eta=f'_{[k]}(t)$ and $\xi=f'_{[k-1]}(t)$. However, the Ahlfors lemma shows that a hermitian metric on $\mathbb{C}$ with negative curvature bounded away from 0 cannot exist, thus we must have $f_{[k]}(\mathbb{C})\subset\{\sigma=0\}$. This proves our vanishing theorem in the case where P is invariant. The general case of a nonnecessarily invariant operator P will not be used here; a proof can be obtained by decomposing P into invariant parts and using an induction on m (cf. [Dem18] for details), or alternatively by means of Nevanlinna theory arguments ([SiYe97], [Siu97], see also [Dem97]). □

Especially, we can apply the above vanishing theorem for any global invariant jet differential $P\in H^0(X,E_{k,m}V^*\otimes\mathcal{O}(-A))$. In that case, P corresponds bijectively to a section

$$\sigma\in H^0(X_k,\mathcal{O}_{X_k}(m)\otimes\pi^*_{k,0}\mathcal{O}(-A)),\tag{3.24}$$

and assuming $P\neq 0$, the vanishing theorem can be reinterpreted by stating that $f_{[k]}(\mathbb{C})$ is contained in the zero divisor $Z_\sigma\subset X_k$. Let $\Delta_k=\bigcup_{2\leqslant\ell\leqslant k}\pi^{-1}_{k,\ell}(D_\ell)$ be the union of the vertical divisors (see

(2.22) and (2.23)). Then $f_{[k]}(\mathbb{C})$ cannot be contained in Δ_k (as otherwise we would have $f'(t) = 0$ identically). We define the k-stage Green-Griffiths locus of (X, V) to be the Zariski closure

$$\mathrm{GG}_k(X,V) = \overline{(X_k \smallsetminus \Delta_k) \cap \bigcap_{m\in\mathbb{N}} \Big(\text{base locus of } \mathcal{O}_{X_k}(m) \otimes \pi_{k,0}^*\mathcal{O}(-A)\Big)} \tag{3.25}$$

(trivially independent of the choice of A), and

$$\mathrm{GG}(X,V) = \bigcap_{k\in\mathbb{N}^*} \pi_{k,0}\big(\mathrm{GG}_k(X,V)\big). \tag{3.26}$$

Then Theorem 3.23 implies that $f_{[k]}(\mathbb{C})$ must be contained in $\mathrm{GG}_k(X, V)$ for every entire curve $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$, and also that $f(\mathbb{C}) \subset \mathrm{GG}(X, V)$.

3.27. Corollary. *If $\mathrm{GG}(X, V) = \emptyset$, then (X, V) is hyperbolic. In particular, if there exists $k \geqslant 1$ and a weight $a_\bullet \in \mathbb{N}^k$ such that $\mathcal{O}_{X_k}(a_\bullet)$ is ample on X_k, then (X, V) is hyperbolic.*

It should be observed that Corollary 3.27 yields a sufficient condition for hyperbolicity, but this is not a necessary condition. In fact, if we take $X = C_1 \times C_2$ to be a product of curves of genus $\geqslant 2$ and $V = T_X$, it is easily checked that $\mathrm{GG}(X) = \mathrm{GG}(X, T_X) = X$. More general examples have been found by Diverio and Rousseau [DR15]. In a similar way, the Green-Griffiths-Lang conjecture holds for (X, V) if $Y := \mathrm{GG}(X, V) \subsetneq X$, but this is only a sufficient condition. The following fundamental existence theorem, however, has been proved in [Dem11], using holomorphic Morse inequalities of [Dem85] as an essential tool. We only state the main result, as it will not be used here.

3.28. Theorem. *Let (X, V) be a projective directed manifold of general type, in the sense that the sheaf K_V of locally bounded sections of $\mathcal{O}(\det V^*)$ is big. Let A be an ample $\mathbb{Q}$-divisor on X such that $\mathcal{O}(\det V^*) \otimes \mathcal{O}(-A)$ is still ample. Then*

$$H^0\bigg(X_k, \mathcal{O}_{X_k}(m) \otimes \pi_{k,0}^*\mathcal{O}\Big(-\frac{m}{kr}\Big(1 + \frac{1}{2} + \ldots + \frac{1}{k}\Big)A\Big)\bigg) \neq 0$$

for $m \gg k \gg 1$ and m sufficiently divisible (so that the multiple of A is an integral divisor). In particular $\mathrm{GG}_k(X, V) \subsetneq X_k$ for $k \gg 1$.

4. Proof of the Kobayashi conjecture on the hyperbolicity of general hypersurfaces

We give here a simple proof of the Kobayashi conjecture, combining ideas of Green-Griffiths [GrGr79], Nadel [Nad89], Masuda-Noguchi [MaNo96], Demailly [Dem95], Siu-Yeung [SiYe96], Shiffman-Zaidenberg [ShZa01], Brotbek [Brot17], Ya Deng [Deng16], in chronological order. Related ideas had been used earlier in [Xie15] and [BrDa17] to establish Debarre's conjecture on the ampleness of the cotangent bundle of generic complete intersections, when their codimension is at least equal to the dimension.

4.A. General Wronskian operators

This section follows closely the work of D. Brotbek [Brot17]. Let U be an open set of a complex manifold X, $\dim X = n$, and $s_0, \ldots, s_k \in \mathcal{O}_X(U)$ be holomorphic functions. To these functions, we can associate a Wronskian operator of order k defined by

$$W_k(s_0, \ldots, s_k)(f) = \begin{vmatrix} s_0(f) & s_1(f) & \ldots & s_k(f) \\ D(s_0(f)) & D(s_1(f)) & \ldots & D(s_k(f)) \\ \vdots & & & \vdots \\ D^k(s_0(f)) & D^k(s_1(f)) & \ldots & D^k(s_k(f)) \end{vmatrix} \tag{4.1}$$

where $f : t \mapsto f(t) \in U \subset X$ is a germ of holomorphic curve (or a k-jet of curve), and $D = \frac{d}{dt}$. For a biholomorphic change of variable φ of $(\mathbb{C}, 0)$, we find by induction on ℓ a polynomial differential operator $Q_{\ell,s}$ of order $\leqslant \ell$ acting on φ satisfying

$$D^\ell(s_j(f\circ\varphi)) = \varphi'^\ell D^\ell(s_j(f))\circ\varphi + \sum_{s<\ell} p_{\ell,s}(\varphi) D^s(s_j(f))\circ\varphi.$$

It follows easily from there that

$$W_k(s_0,\ldots,s_k)(f\circ\varphi) = (\varphi')^{1+2+\cdots+k} W_k(s_0,\ldots,s_k)(f)\circ\varphi,$$

hence $W_k(s_0,\ldots,s_k)(f)$ is an invariant differential operator of degree $k' = \frac{1}{2}k(k+1)$. Especially, we get in this way a section that we denote

$$W_k(s_0,\ldots,s_k) = \begin{vmatrix} s_0 & s_1 & \ldots & s_k \\ D(s_0) & D(s_1) & \ldots & D(s_k) \\ \vdots & & & \vdots \\ D^k(s_0) & D^k(s_1) & \ldots & D^k(s_k) \end{vmatrix} \in H^0(U, E_{k,k'}T^*_X). \tag{4.2}$$

4.3. Proposition. *These Wronskian operators satisfy the following properties.*

(a) *$W_k(s_0,\ldots,s_k)$ is $\mathbb{C}$-multilinear and alternate in $(s_0,\ldots,s_k)$.*

(b) *For any $g\in\mathcal{O}_X(U)$, we have*

$$W_k(gs_0,\ldots,gs_k) = g^{k+1}W_k(s_0,\ldots,s_k).$$

Property 4.3 (b) is an easy consequence of the Leibniz formula

$$D^\ell(g(f)s_j(f)) = \sum_{k=0}^{\ell}\binom{\ell}{k} D^k(g(f))D^{\ell-k}(s_j(f)),$$

by performing linear combinations of rows in the determinants. This property implies in its turn that one can define more generally an operator

$$W_k(s_0,\ldots,s_k)\in H^0(U, E_{k,k'}T^*_X\otimes L^{k+1}) \tag{4.5}$$

for any $(k+1)$-tuple of sections $s_0,\ldots,s_k\in H^0(U,L)$ of a holomorphic line bundle $L\to X$. In fact, when we compute the Wronskian in a local trivialization of $L_{\restriction U}$, Property 4.3 (b) shows that the determinant is independent of the trivialization. Moreover, if $g\in H^0(U,G)$ for some line bundle $G\to X$, we have

$$W_k(gs_0,\ldots,gs_k) = g^{k+1}W_k(s_0,\ldots,s_k)\in H^0(U, E_{k,k'}T^*_X\otimes L^{k+1}\otimes G^{k+1}). \tag{4.6}$$

Now, let $\Sigma\subset H^0(X,L)$ be a vector subspace such that $W_k(s_0,\ldots,s_k)\not\equiv 0$ for generic elements $s_0,\ldots,s_k\in\Sigma$. We view here $W_k(s_0,\ldots,s_k)$ as a section of $H^0(X_k,\mathcal{O}_{X_k}(k')\otimes\pi^*_{k,0}L^{k+1})$ on the k-stage of the Semple tower. As the Wronskian is alternate and multilinear, we get a meromorphic map $X_k \dashrightarrow P(\Lambda^{k+1}\Sigma^*)$ by sending a k-jet $\gamma = f_{[k]}(0)\in X_k$ to the point $[W_k(u_{i_0},\ldots,u_{i_k})(f)(0)]_{I\subset J}$ where $(u_j)_{j\in J}$ is a basis of Σ. This assignment factorizes through the Plücker embedding as a meromorphic map $\Phi : X_k \dashrightarrow \mathrm{Gr}_{k+1}(\Sigma)$ into the Grassmannian of dimension $k+1$ subspaces of Σ^* (or codimension $k+1$ subspaces of Σ, alternatively). In fact, if $L_{\restriction U}\simeq U\times\mathbb{C}$ is a trivialization of L in a neighborhood of a point $x_0 = f(0)\in X$, we can consider the map $\Psi_U : X_k\to\mathrm{Hom}(\Sigma,\mathbb{C}^{k+1})$ given by

$$\pi^{-1}_{k,0}(U)\ni f_{[k]}\mapsto \big(s\mapsto (D^\ell(s(f))_{0\leqslant\ell\leqslant k})\big),$$

and associate either the kernel $\Xi\subset\Sigma$ of $\Psi_U(f_{[k]})$, seen as a point $\Xi\in\mathrm{Gr}_{k+1}(\Sigma)$, or $\Lambda^{k+1}\Xi^\perp\subset\Lambda^{k+1}\Sigma^*$, seen as a point of $P(\Lambda^{k+1}\Sigma^*)$ (assuming that we are at a point where the rank is equal to $k+1$). Let $\mathcal{O}_{\mathrm{Gr}}(1)$ be the tautological very ample line bundle on $\mathrm{Gr}_{k+1}(\Sigma)$ (equal to the restriction of $\mathcal{O}_{P(\Lambda^{k+1}\Sigma^*)}(1)$). By construction, Φ is induced by the linear system of sections

$$W_k(u_{i_0},\ldots,u_{i_k})\in H^0(X_k,\mathcal{O}_{X_k}(k')\otimes L^{k+1}),$$

and we thus get a natural isomorphism

$$\mathcal{O}_{X_k}(k') \otimes L^{k+1} \simeq \Phi^*\mathcal{O}_{\mathrm{Gr}}(1) \quad \text{on } X_k \smallsetminus B_k \tag{4.7}$$

where $B_k \subset X_k$ is the base locus of our linear system of Wronskians. In order to avoid the indeterminacy set, we have to introduce the ideal sheaf $\mathcal{J}_{k,\Sigma} \subset \mathcal{O}_{X_k}$ generated by the linear system, and take a modification $\mu_{k,\Sigma} : \widehat{X}_{k,\Sigma} \to X_k$ in such a way that $\mu_{k,\Sigma}^*\mathcal{J}_{k,\Sigma} = \mathcal{O}_{\widehat{X}_{k,\Sigma}}(-F_{k,\Sigma})$ for some divisor $F_{k,\Sigma}$ in $\widehat{X}_{k,\Sigma}$. Then Φ is resolved into a morphism $\Phi \circ \mu_{k,\Sigma} : \widehat{X}_{k,\Sigma} \to \mathrm{Gr}_{k+1}(\Sigma)$, and on $\widehat{X}_{k,\Sigma}$, (4.7) becomes an everywhere defined isomorphism

$$\mu_{k,\Sigma}^*\big(\mathcal{O}_{X_k}(k') \otimes L^{k+1}\big) \otimes \mathcal{O}_{\widehat{X}_{k,\Sigma}}(-F_{k,\Sigma}) \simeq (\Phi \circ \mu_{k,\Sigma})^*\mathcal{O}_{\mathrm{Gr}}(1). \tag{4.8}$$

In fact, we can simply take $\widehat{X}_k$ to be the normalized blow-up of $\mathcal{J}_{k,\Sigma}$, i.e. the normalization of the closure $\Gamma \subset X_k \times \mathrm{Gr}_{k+1}(\Sigma)$ of the graph of Φ and $\mu_{k,\Sigma} : \widehat{X}_k \to X_k$ to be the composition of the normalization map $\widehat{X}_k \to \Gamma$ with the first projection $\Gamma \to X_k$; notice that this construction is completely universal, so it can be applied fonctorially, and by definition $\widehat{X}_k$ is normal (one could also apply an equivariant Hironaka desingularization procedure to replace $\widehat{X}_k$ by a non singular modification, and then turn $F_{k,\Sigma}$ into a simple normal crossing divisor, but this will not be needed here). In this context, there is a maximal universal ideal sheaf $\mathcal{J}_k \supset \mathcal{J}_{k,\Sigma}$ achieved by linear systems Σ that generate k-jets of sections at every point. In fact, according to an idea of Ya Deng [Deng16], the bundle $X_k \to X$ is turned into a locally trivial product $U \times \mathbb{R}_{n,k}$ when we fix local coordinates $(z_1, \ldots, z_n)$ on U (cf. the end of § 2.D). In this setting, $\mathcal{J}_k$ is the pull-back by the second projection $U \times \mathbb{R}_{n,k} \to \mathbb{R}_{n,k}$ of an ideal sheaf defined on the fibers $\mathbb{R}_{n,k}$ of $X_k \to X$; therefore, in order to get the largest possible ideal $\mathcal{J}_k$, we need only consider all possible Wronskian sections

$$\mathbb{R}_{n,k} \simeq \pi_{k,0}^{-1}(x_0) \ni f_{[k]} \mapsto W_k(s_0, \ldots, s_k)(f) \in \mathcal{O}_{X_k}(k')_{\restriction \mathbb{R}_{n,k}},$$

associated with germs of sections s_j, but they clearly depend only on the k-jets of the s_j's at x_0 (even though we might have to pick for f a Taylor expansion of higher order to reach all points of the fiber). We can therefore summarize this discussion by the following statement.

4.9. Proposition. *Assume that L generates all k-jets of sections (e.g. take $L = A^p$ with A very ample and $p \geqslant k$), and let $\Sigma \subset H^0(X, L)$ be a linear system that also generates k-jets of sections at any point of X. Then we have a universal isomorphism*

$$\mu_k^*\big(\mathcal{O}_{X_k}(k') \otimes L^{k+1}\big) \otimes \mathcal{O}_{\widehat{X}_{k,\Sigma}}(-F_k) \simeq (\Phi \circ \mu_k)^*\mathcal{O}_{\mathrm{Gr}}(1)$$

where $\mu_k : \widehat{X}_k \to X_k$ is the normalized blow-up of the (maximal) ideal sheaf $\mathcal{J}_k \subset \mathcal{O}_{X_k}$ associated with order k Wronskians, and F_k the universal divisor of $\widehat{X}_k$ resolving $\mathcal{J}_k$.

4.B. Existence of hyperbolic hypersurfaces of low degree

Let Z be a non singular $(n+1)$-dimensional projective variety, and let A be a very ample divisor on Z ; the fundamental example is of course $Z = \mathbb{P}^{n+1}$ and $A = \mathcal{O}_{\mathbb{P}^{n+1}}(1)$. Our goal is to show that a well chosen (n-dimensional) hypersurface $X = \{x \in Z\,;\ \sigma(x) = 0\}$ defined by a section $\sigma \in H^0(Z, A^d)$, $d \gg 1$, is Kobayashi hyperbolic. The construction explained below follows closely the ideas of Shiffman-Zaidenberg [ShZa01] and is based similarly on a use of Fermat-Waring type hypersurfaces. Our proof is however completely self-contained. The reader can consult Brody-Green [BrGr77], Nadel [Nad89] and Masuda-Noguchi [MaNo96] for constructions based on other techniques.

4.10. Theorem. *Let Z be a non singular $(n+1)$-dimensional projective variety, A a very ample divisor on Z, and $\tau_j \in H^0(Z, A)$, $0 \leqslant j \leqslant N$, sufficiently general sections. Then for $N \geqslant 2n$ and $d \geqslant N^2$, the hypersurface $X = \sigma^{-1}(0)$ associated with $\sigma = \sum_{0 \leqslant j \leqslant N} \tau_j^d \in H^0(Z, A^d)$ is Kobayashi hyperbolic.*

In particular, Theorem 4.10 provides examples of hyperbolic hypersurfaces of $\mathbb{P}^{n+1}$ for all $n \geqslant 1$ and all degrees $d \geqslant 4n^2$. A substantially improved bound $d \geqslant \lceil (n+3)^2/4 \rceil$ has been obtained recently by [DTH16] via a deformation argument for certain unions of hyperplanes, but the methods are quite different from the techniques used here. As in [ShZa01], the main step of our proof is the following proposition due to Toda [Toda71], Fujimoto [Fuj74] and Green [Gre75].

4.11. Proposition. *Let $g_j : \mathbb{C} \to \mathbb{C}$, $0 \leqslant j \leqslant N$, be non zero entire functions such that the curve $g = [g_0 : \ldots : g_N] : \mathbb{C} \to \mathbb{P}^N$ satisfies $\sum_{0 \leqslant j \leqslant N} g_j^d = 0$. If $d \geqslant N^2$, there exists a partition $J_1, \ldots, J_q$ of $\{0, 1, \ldots, N\}$ such that $|J_s| \geqslant 2$, g_j/g_i is constant for all $i, j \in J_s$, and $\sum_{j \in J_s} g_j^d = 0$ for all $s = 1, 2, \ldots, q$. If g is nonconstant, we must have $q \geqslant 2$.*

Proof. The result is true for $N = 1$ (with a single $J_1 = \{0, 1\}$), and for higher values $N \geqslant 2$ we apply induction and use vanishing arguments for Wronskians. The map $g = [g_0 : \ldots : g_N] : \mathbb{C} \to \mathbb{P}^N$ can be seen as an entire curve drawn in the (smooth, irreducible) Fermat hypersurface $Y = \sum_{0 \leqslant j \leqslant N} z_j^d$ of $\mathbb{P}^N$. We set $k = N - 1$ and consider on Y the Wronskian operator

$$W_k(s_0, \ldots, s_k) \quad \text{where } s_j(z) = z_j^d, \quad s_j \in H^0(Y, \mathcal{O}(d)).$$

Then

$$W_{N-1}(s_0, \ldots, s_{N-1}) \in H^0(Y, E_{k,k'} T_Y^* \otimes \mathcal{O}(Nd)).$$

Since $D^\ell(s_j)$ is divisible by z_j^{d-k} for $\ell \leqslant k$, we conclude that $W_{N-1}(s_0, \ldots, s_{N-1})$ is divisible by $\prod_{j<N} z_j^{d-k}$. However, as $s_0 = -(s_1 + \ldots + s_N)$ on Y, we get

$$W_{N-1}(s_0, \ldots, s_{N-1}) = (-1)^N W_{N-1}(s_1, \ldots, s_N)$$

and conclude that $W_{N-1}(s_0, \ldots, s_{N-1})$ must be also divisible by z_N^{d-k}. Since the $\{z_j = 0\}$, $0 \leqslant j \leqslant N$, form a normal crossing divisor on Y, we infer that

$$\widetilde{W} := \prod_{0 \leqslant j \leqslant N} z_j^{-(d-k)} W_{N-1}(s_0, \ldots, s_{N-1}) \in H^0(Y, E_{k,k'} T_Y^* \otimes \mathcal{O}(Nd - (N+1)(d-k)))$$

i.e. $\widetilde{W} \in H^0(Y, E_{k,k'} T_Y^* \otimes \mathcal{O}(N^2 - 1 - d))$. By the fundamental vanishing theorem, we must have $\widetilde{W}(g) = 0$. Since this is equivalent to the vanishing of the determinant $\det(D^\ell(g_j^d))$, we conclude that the functions $g_0^d, \ldots, g_{N-1}^d$ must be linearly dependent. After eliminating zero coefficients, we find a linear relation $\sum_{0 \leqslant k \leqslant p} c_k g_{j(k)}^d = 0$ with $c_k \in \mathbb{C}^*$, $j(k) \leqslant N - 1$ and $1 \leqslant p \leqslant N - 1$. The induction hypothesis applied to the functions $c_k^{1/d} g_{j(k)}$, implies that at least two of them are proportional. By grouping together the g_j's that are proportional in the identity $\sum_{0 \leqslant j \leqslant N} g_j^d = 0$, we find a partition $(J_s)_{1 \leqslant s \leqslant q}$ of $\{0, 1, \ldots, N\}$ and relations of the form $\sum_{j \in J_s} g_j^d = \lambda_s g_{j_s}^d$, $j_s \in J_s$. Moreover we get $\sum_{1 \leqslant s \leqslant q} \lambda_s g_{j_s}^d = 0$ with strictly less than $N + 1$ functions g_{j_s} involved, all of them being pairwise nonproportional. This contradicts the induction hypothesis unless all coefficients λ_s are zero, and we must then have $|J_s| \geqslant 2$. The case $q = 1$ corrresponds to g being constant. Proposition 4.11 follows. □

Proof of Theorem 4.10. We argue by induction on $n \geqslant 1$. For $n = 1$, an easy adjunction argument shows that it is enough to take $d \geqslant 4$: sections of A can be used to embed the polarized surface (Z, A) in $\mathbb{P}^N$ (e.g. with $N = 5$), and whenever $X = \sigma^{-1}(0)$ is a smooth curve, we have $K_X = K_{Z \restriction X} \otimes A^d$ and a surjective restriction morphism $\Omega^2_{\mathbb{P}^N} \to K_Z = \Lambda^2 T_Z^*$. As $\Omega^2_{\mathbb{P}^N} \otimes \mathcal{O}(3) = \Lambda^{N-2}(T_{\mathbb{P}^N} \otimes \mathcal{O}(-1))$ is generated by sections, one sees that $K_Z \otimes A^3$ is also generated by sections, hence K_X is ample for $d \geqslant 4$.

Now, assume that the result is already proved for $n - 1$ and consider a (non constant) entire curve $f : \mathbb{C} \to X$ where $X = \{\sum_{0 \leqslant j \leqslant N} \tau_j^d = 0\} \subset Z$. For suitably chosen sections $\tau_j \in H^0(Z, A)$, $0 \leqslant j \leqslant N$ and $N \geqslant \dim Z = n + 1$, the map $\tau := [\tau_0 : \ldots : \tau_N] : Z \to \mathbb{P}^N$ can be taken to be a generically finite morphism. If $\tau_j \circ f$ vanishes for some j, say $j = N$, then f is drawn in the hypersurface X' of $Z' = \tau_N^{-1}(0)$ associated with $\sigma' = \sum_{0\leqslant j\leqslant N-1}\tau_j^d$. We can suppose that Z' is smooth and, by the induction hypothesis for $(n-1, N-1)$, that X' is hyperbolic (notice that $N-1 \geqslant 2(n-1)$ and $d \geqslant (N-1)^2$); this is a contradiction.

Without loss of generality, we can thus assume that all sections $g_j := \tau_j \circ f$ are non zero. Also suppose that $g = \tau\circ f$ is nonconstant. By definition of X, we have $\sum_{0\leqslant j\leqslant N} g_j^d = 0$, and Proposition 4.11 shows that there exists a partition $J = \{J_1, \ldots, J_q\}$ of $\{0, 1, \ldots, N\}$ such that $q \geqslant 2$, $|J_s| \geqslant 2$, and the ratios $g_{j'}/g_j$ are constant for $j, j' \in J_s$, and $\sum_{j\in J_s} g_j^d = 0$ for all $s = 1, 2, \ldots, q$. Set $j_s = \min J_s$ and $w_j = g_j/g_{j_s} \in \mathbb{C}^*$ for $j \in J_s \smallsetminus \{j_s\}$. Then $g = [g_0 : \ldots : g_N] = \tau\circ f$ is drawn in a projective linear subspace $Y_{J,w} \subset \mathbb{P}^{N-1}$ of dimension $q-1$ defined by the equations

$$Y_{J,w} : z_j = w_j z_{j_s} \quad \text{for } j \in J_s \smallsetminus \{j_s\}, \quad 1 + \sum_{j\in J_s\smallsetminus\{j_s\}} w_j^d = 0, \quad 1 \leqslant s \leqslant q. \tag{4.12}$$

Theorem 4.10 is now a consequence of the following lemma, which forces $g = \tau\circ f$, and hence f, to be constant. □

4.13. Lemma. *For $N \geqslant 2n$ and $\tau_j \in H^0(Z, A)$ sufficiently general, $0 \leqslant j \leqslant N$, the hypersurface $X = \{\sum_{0\leqslant j\leqslant N}\tau_j^d = 0\}$ is smooth and the map $\tau = [\tau_0, \ldots : \tau_N] : Z \to \mathbb{P}^N$ has a restriction $\tau : X \to \mathbb{P}^N$ that is a finite morphism. Moreover, for all partitions $J = \{J_s\}$ and all choices of $w = (w_j) \in (\mathbb{C}^*)^{N+1-q}$ as in (4.12), the set $\tau^{-1}(Y_{J,w})$ is finite.*

Proof. Let $(\sigma_1, \ldots, \sigma_m)$ be a basis of $H^0(Z, A)$. We write $\tau_j = \sum_{1\leqslant\ell\leqslant m} a_{j\ell}\sigma_\ell$ and consider the matrix $a = (a_{j\ell}) \in \mathbb{C}^{m(N+1)}$. The singular locus of $X = \{\sum_{0\leqslant j\leqslant N}\tau_j^d = 0\}$ is described by the equations

$$\sum_{0\leqslant j\leqslant N}\Big(\sum_{1\leqslant\ell\leqslant p} a_{j\ell}\sigma_\ell(x)\Big)^d = 0, \quad \frac{\partial}{\partial x_s}\bigg(\sum_{0\leqslant j\leqslant N}\Big(\sum_{1\leqslant\ell\leqslant m} a_{j\ell}\sigma_\ell(x)\Big)^d\bigg) = 0, \quad 1 \leqslant s \leqslant n+1$$

in coordinates. As the σ_ℓ's generate all 1-jets at every point $x \in X$, we have $(n+2)$ independent equations in terms of a, hence the bad locus L of points $(x, a) \in Z \times \mathbb{C}^{m(N+1)}$ admits a fibration $\mathrm{pr}_1 : L \to Z$ whose fibers are of dimension $m(N+1) - (n+2)$ in $\mathbb{C}^{m(N+1)}$. Therefore we get $\dim L \leqslant m(N+1) - 1$ and $\mathrm{pr}_2(L)$ does not cover $\mathbb{C}^{m(N+1)}$. Any matrix a taken in the complement $\mathbb{C}^{m(N+1)} \smallsetminus \mathrm{pr}_2(L)$ will produce a smooth hypersurface X.

Similary, as the σ_ℓ's separate points of Z, the set S of triples $(x_1, x_2, a) \in Z \times Z \times \mathbb{C}^{m(N+1)}$ with $x_1, x_2 \in X$, $x_1 \neq x_2$ and $\tau(x_1) = \tau(x_2)$ is such that the fibers of $S \to Z \times Z$ in $\mathbb{C}^{m(N+1)}$ are described by $N+1$ independent equations

$$\sum_{0\leqslant j\leqslant N}\Big(\sum_{1\leqslant\ell\leqslant p} a_{j\ell}\sigma_\ell(x_1)\Big)^d = 0, \quad \Big[\sum_{1\leqslant\ell\leqslant m} a_{j\ell}\,\sigma_\ell(x_1)\Big]_{0\leqslant j\leqslant N} = \Big[\sum_{1\leqslant\ell\leqslant m} a_{j\ell}\,\sigma_\ell(x_2)\Big]_{0\leqslant j\leqslant N} \in \mathbb{P}^N.$$

Therefore $\dim S = \dim(Z\times Z) + m(N+1) - (N+1) \leqslant m(N+1) + 1$ and the projection $S \to \mathbb{C}^{m(N+1)}$ has a fiber of dimension at most 1 over a generic point $a \in \mathbb{C}^{m(N+1)}$. For such a choice of a, if $F = \tau^{-1}(y)$ is a fiber of $\tau : X \to \mathbb{P}^N$, then S contains $F \times F \smallsetminus \Delta_F$, hence we must have $\dim F = 0$, and all fibers F are finite.

In order to study the finiteness of $\tau^{-1}(Y_{J,w})$, we look at the incidence variety V_J of 4-tuples $(x_1, x_2, a, w) \in Z^2 \times \mathbb{C}^{m(N+1)} \times W_J$ such that $x_1 \neq x_2$ and $\tau(x_1) = \tau(x_2) \in Y_{J,w}$, where W_J is the set of points $w = (w_j)$ such that $1 + \sum_{j\in J_s\smallsetminus\{j_s\}} w_j^d = 0$, $1 \leqslant s \leqslant q$. Notice that we have only finitely many subvarieties W_J involved, and that $\dim W_J = \sum(|J_s| - 2) = N + 1 - 2q$. The variety

V_J is defined by $2(N+1-q)+q-1$ linear equations in the $a_{j\ell}$:

$$\sum_{1\leqslant\ell\leqslant m}(a_{j\ell}-w_ja_{j_s\ell})\sigma_\ell(x_i)=0,\quad j\in J_s\smallsetminus\{j_s\},\quad 1\leqslant s\leqslant q,\quad i=1,2,$$
$$\Big[\sum_{1\leqslant\ell\leqslant m}a_{j_s\ell}\,\sigma_\ell(x_1)\Big]_{1\leqslant s\leqslant q}=\Big[\sum_{1\leqslant\ell\leqslant m}a_{j_s\ell}\,\sigma_\ell(x_2)\Big]_{1\leqslant s\leqslant q}\in\mathbb{P}^{q-1}.$$

These equations are independent: this is again a consequence of the fact that the σ_ℓ's separate points of Z. The dimension of V_J is thus

$$\begin{aligned}\dim V_J&=m(N+1)+2(n+1)+(N+1-2q)-\big(2(N+1-q)+(q-1)\big)\\&=m(N+1)+2n+2-N-q.\end{aligned}$$

For $q\geqslant 2$ and $N\geqslant 2n$, we have $\dim V_J\leqslant n(N+1)$, therefore the projection $V_J\to\mathbb{C}^{m(N+1)}$ has finite fibers over a Zariski open set $\mathbb{C}^{m(N+1)}\smallsetminus S_J$. Hence, for $a\in\mathbb{C}^{m(N+1)}\smallsetminus\bigcup S_J$, we infer that all sets $\tau^{-1}(Y_{J,w})$ are finite. (For $N\geqslant 2n+1$, we could even take a outside of the projections of the incidence varieties V_J, and in that case, for a generic, the sets $\tau^{-1}(Y_{J,w})$ have at most one point). □

4.C. Hyperbolicity of general hypersurfaces of large degree

Let again Z be a non singular $(n+1)$-dimensional projective variety polarized with a very ample divisor A. In this section, our more ambitious goal is to show that a sufficiently general hypersurface $X=\{x\in Z\,;\,\sigma(x)=0\}$ defined by $\sigma\in H^0(Z,A^d)$ is Kobayashi hyperbolic when d is large – we no longer want to restrict ourselves to the special case of Fermat-Waring hypersurfaces $\sum_{0\leqslant j\leqslant N}\tau_j^d=0$, $\tau_j\in H^0(Z,A)$. Brotbek's main idea developed in [Brot17] is that a carefully selected hypersurface may have enough Wronskian sections to directly imply the ampleness of some tautological jet line bundle – a Zariski open property. Here, we take σ be a sum of terms

$$\sigma=\sum_{0\leqslant j\leqslant N}a_jm_j^\delta,\quad a_j\in H^0(Z,A^\rho),\ m_j\in H^0(Z,A^b),\ n<N\leqslant k\leqslant\rho,\ d=\delta b+\rho,\tag{4.14}$$

where each m_j is a product of b "linear" sections $\tau_I\in H^0(Z,A)$, $\delta\gg 1$, and the factors a_j are general enough. The products m_j will be chosen in such a way that for suitable $c\in\mathbb{N}$, $1\leqslant c\leqslant N$, any subfamily of c terms m_j shares a common factor τ_I. To this end, we consider all subsets $I\subset\{0,1,\ldots,N\}$ with $\mathrm{card}\,I=c$; there are $B=\binom{N+1}{c}$ subsets of this type. For all such I, we select sections $\tau_I\in H^0(Z,A)$ such that $\prod_I\tau_I=0$ is a simple normal crossing divisor in Z (with all of its components of multiplicity 1). For $j=0,1,\ldots,N$ given, the number of subsets I containing j is $b=\binom{N}{c-1}$. We put

$$m_j=\prod_{I\ni j}\tau_I\in H^0(Z,A^b).\tag{4.15}$$

The first step consists in checking that we can achieve X to be smooth with these constraints.

4.16. Lemma. *Assume $N\geqslant c(n+1)$. Then, for a generic choice of the sections $a_j\in H^0(Z,A^\rho)$ and $\tau_I\in H^0(Z,A)$, the hypersurface $X=\sigma^{-1}(0)\subset Z$ defined by* (4.14-4.15) *is non singular. Moreover, under the same condition for N, the intersection of $\prod\tau_I=0$ with X can be taken to be a simple normal crossing divisor in X.*

Proof. As the properties considered in the Lemma are Zariski open properties in terms of the $(N+B+1)$-tuple (a_j,τ_I), it is sufficient to prove the result for a specific choice of the a_j's: we fix here $a_j=\tilde\tau_j\tau_{I(j)}^{\rho-1}$ where $\tilde\tau_j\in H^0(X,A)$, $0\leqslant j\leqslant N$ are new sections such that $\prod\tilde\tau_j\prod\tau_I=0$ is a simple normal crossing divisor, and $I(j)$ is any subset of cardinal c containing j. Let H be the hypersurface of degree d of $\mathbb{P}^{N+B}$ defined in homogeneous coordinates $(z_j, z_I) \in \mathbb{C}^{N+B+1}$ by $h(z) = 0$ where

$$h(z) = \sum_{0\leqslant j\leqslant N} z_j z_{I(j)}^{\rho-1} \prod_{I\ni j} z_I^\delta,$$

and consider the morphism $\Phi : Z \to \mathbb{P}^{N+B}$ such that $\Phi(x) = (\tilde\tau_j(x), \tau_I(x))$. With our choice of the a_j's, we have $\sigma = h \circ \Phi$. Now, when the $\tilde\tau_j$ and τ_I are general enough, the map Φ defines an embedding of Z into $\mathbb{P}^{N+B}$ (for this, one needs $N + B \geqslant 2 \dim Z + 1 = 2n + 3$, which is the case by our assumptions). Then, by definition, X is isomorphic to the intersection of H with $\Phi(Z)$. Changing generically the $\tilde\tau_j$ and τ_I's can be achieved by composing Φ with a generic automorphism $g \in \mathrm{Aut}(\mathbb{P}^{N+B}) = \mathrm{PGL}_{N+B+1}(\mathbb{C})$ (as $\mathrm{GL}_{N+B+1}(\mathbb{C})$ acts transitively on $(N + B + 1)$-tuples of linearly independent linear forms). As $\dim g \circ \Phi(Z) = \dim Z = n + 1$, Lemma 4.16 will follow from a standard Bertini argument if we can check that $\mathrm{Sing}(H)$ has codimension at least $n+2$ in $\mathbb{P}^{N+B}$. In fact, this condition implies $\mathrm{Sing}(H) \cap (g \circ \Phi(Z)) = \emptyset$ for g generic, while $g \circ \Phi(Z)$ can be chosen transverse to $\mathrm{Reg}(H)$. Now, a sufficient condition for smoothness is that one of the differentials dz_j appears with a non zero factor in $dh(z)$. We infer from this and the fact that $\delta \geqslant 2$ that $\mathrm{Sing}(H)$ consists of the locus defined by $\prod_{I\ni j} z_I = 0$ for all $j = 0, 1, \ldots, N$. It is the union of the linear subspaces $z_{I_0} = \ldots = z_{I_N} = 0$ for all possible choices of subsets I_j such that $I_j \ni j$. Since $\mathrm{card}\, I_j = c$, the equality $\bigcup I_j = \{0, 1, \ldots, N\}$ implies that there are at least $\lceil (N + 1)/c\rceil$ distinct subsets I_j involved in each of these linear subspaces, and the equality can be reached. Therefore $\mathrm{codim}\,\mathrm{Sing}(H) = \lceil (N + 1)/c\rceil \geqslant n + 2$ as soon as $N \geqslant c(n + 1)$. By the same argument, we can assume that the intersection of Z with at least $(n+2)$ distinct hyperplanes $z_I = 0$ is empty. In order that $\prod \tau_I = 0$ defines a normal crossing divisor at a point $x \in X$, it is sufficient to ensure that for any family $\mathcal{G}$ of coordinate hyperplanes $z_I = 0$, $I \in \mathcal{G}$, with $\mathrm{card}\,\mathcal{G} \leqslant n + 1$, we have a "free" index $j \notin \bigcup_{I\in\mathcal{G}} I$ such that $x_I \neq 0$ for all $I \ni j$, so that dh involves a non zero term $* \, dz_j$ independent of the dz_I, $I \in \mathcal{G}$. If this fails, there must be at least $(n + 2)$ hyperplanes $z_I = 0$ containing x, associated either with $I \in \mathcal{G}$, or with other I's covering $\complement\big(\bigcup_{I\in\mathcal{G}} I\big)$. The corresponding bad locus is of codimension at least $(n + 2)$ in $\mathbb{P}^{N+B}$ and can be avoided by $g(\Phi(Z))$ for a generic choice of $g \in \mathrm{Aut}(\mathbb{P}^{N+B})$. Then $X \cap \bigcap_{I\in\mathcal{G}} \tau_I^{-1}(0)$ is smooth of codimension equal to $\mathrm{card}\,\mathcal{G}$. □

Now, to any families s, $\hat\tau$ of sections $s_1, \ldots, s_r \in H^0(Z, A^k)$, $\hat\tau_1, \ldots, \hat\tau_r \in H^0(Z, A)$, and any subset $J \subset \{0, 1, \ldots, N\}$ with $\mathrm{card}\, J = c$, we associate a Wronskian operator of order k (i.e. a $(k + 1) \times (k + 1)$-determinant)

$$W_{k,s,\hat\tau,a,J} = W_k\big(s_1\hat\tau_1^{d-k}, \ldots, s_r\hat\tau_r^{d-k}, (a_j m_j^\delta)_{j\in\complement J}\big), \quad r = k + c - N. \tag{4.17}$$

We assume here again that the $\hat\tau_j$ are chosen so that $\prod \hat\tau_j \prod \tau_I = 0$ defines a simple normal crossing divisor in Z and X. Since $s_j\hat\tau_j^{d-k}$, $a_j m_j^\delta \in H^0(Z, A^d)$, formula (4.5) applied with $L = A^d$ implies that

$$W_{k,s,\hat\tau,a,J} \in H^0(Z, E_{k,k'}T_Z^* \otimes A^{(k+1)d}). \tag{4.18}$$

However, we are going to see that $W_{k,s,\hat\tau,a,J}$ and its restriction $W_{k,s,\hat\tau,a,J\restriction X}$ are divisible by monomials $\hat\tau^\alpha\tau^\beta$ of very large degree, where $\hat\tau$, resp. τ, denotes the collection of sections $\hat\tau_j$, resp. τ_I in $H^0(Z, A)$. In this way, we will see that we can even obtain a negative exponent of A after simplifying $\hat\tau^\alpha\tau^\beta$ in $W_{k,s,\hat\tau,a,J\restriction X}$. This simplification process is a generalization of techniques already considered by [Siu87] and [Nad89] (and later [DeEG97]), in relation with the use of meromorphic connections of low pole order.

4.19. Lemma. *Assume that $\delta \geqslant k$. Then the Wronskian operator $W_{k,s,\hat\tau,a,J}$, resp. $W_{k,s,\hat\tau,a,J\restriction X}$, is divisible by a monomial $\hat\tau^\alpha\tau^\beta$, resp. $\hat\tau^\alpha\tau^\beta\tau_J^{\delta-k}$ (with a multiindex notation $\hat\tau^\alpha\tau^\beta = \prod \hat\tau_j^{\alpha_j} \prod \tau_I^{\beta_I}$), and*

$$\alpha, \beta \geqslant 0, \quad |\alpha| = r(d - 2k), \quad |\beta| = (N + 1 - c)(\delta - k)b.$$

Proof. $W_{k,s,\hat\tau,a,J}$ is obtained as a determinant whose r first columns are the derivatives $D^\ell(s_j\hat\tau_j^{d-k})$ and the last $N+1-c$ columns are the $D^\ell(a_jm_j^\delta)$, divisible respectively by $\hat\tau_j^{d-2k}$ and $m_j^{\delta-k}$. As m_j is of the form τ^γ, $|\gamma|=b$, this implies the divisibility of $W_{k,s,\hat\tau,a,J}$ by a monomial of the form $\hat\tau^\alpha\tau^\beta$, as asserted. Now, we explain why one can gain the additional factor $\tau_J^{\delta-k}$ dividing the restriction $W_{k,s,\hat\tau,a,J\restriction X}$. First notice that τ_J *does not* appear as a factor in $\hat\tau^\alpha\tau^\beta$, precisely because the Wronskian involves only terms $a_jm_j^\delta$ with $j\notin J$, hence these m_j's do not contain τ_J. Let us pick $j_0=\min(\complement J)\in\{0,1,\ldots,N\}$. Since X is defined by $\sum_{0\leqslant j\leqslant N}a_jm_j^\delta=0$, we have identically

$$a_{j_0}m_{j_0}^\delta=-\sum_{i\in J}a_im_i^\delta-\sum_{i\in\complement J\smallsetminus\{j_0\}}a_im_i^\delta$$

in restriction to X, whence (by the alternate property of $W_k(\bullet)$)

$$W_{k,s,\hat\tau,a,J\restriction X}=-\sum_{i\in J}W_k\big(s_1\hat\tau_1^{d-k},\ldots,s_r\hat\tau_r^{d-k},a_im_i^\delta,(a_jm_j^\delta)_{j\in\complement J\smallsetminus\{j_0\}}\big)_{\restriction X}.$$

However, all terms m_i, $i\in J$, contain by definition the factor τ_J, and the derivatives $D^\ell(\bullet)$ leave us a factor $m_i^{\delta-k}$ at least. Therefore, the above restricted Wronskian is also divisible by $\tau_J^{\delta-k}$, thanks to the fact that $\prod\hat\tau_j\prod\tau_I=0$ forms a simple normal crossing divisor in X. □

4.20. Corollary. *For $\delta\geqslant k$, there exists a monomial $\hat\tau^{\alpha_J}\tau^{\beta_J}$ dividing $W_{k,s,\hat\tau,a,J\restriction X}$ such that*

$$|\alpha_J|+|\beta_J|=(k+c-N)(d-2k)+(N+1-c)(\delta-k)b+(\delta-k)$$

and we have

$$\widetilde W_{k,s,\hat\tau,a,J\restriction X}:=(\hat\tau^{\alpha_J}\tau^{\beta_J})^{-1}W_{k,s,\hat\tau,a,J\restriction X}\in H^0(X,E_{k,k'}T_X^*\otimes A^{-p})$$

where

$$p=|\alpha_J|+|\beta_J|-(k+1)d=(\delta-k)-(k+c-N)2k-(N+1+c)(kb+\rho). \tag{4.21}$$

In particular, we have $p>0$ for δ large enough (all other parameters being fixed or bounded), and under this assumption, the fundamental vanishing theorem implies that all entire curves $f:\mathbb C\to X$ are annihilated by these Wronskian operators.

Proof. In fact,

$$(k+1)d=(k+c-N)d+(N+1-c)d=(k+c-N)d+(N+1-c)(\delta b+\rho)$$

and we get (4.21) by subtraction. □

The next step is to control more precisely their base locus and to select suitable values of N, k, c, $d=b\delta+\rho$. Although we will not formally use it, the next lemma is useful to realize that the base locus is related to a natural rank condition.

4.22. Lemma. *Set $u_j:=a_jm_j^\delta$. The base locus in $X_k^{\rm reg}$ of the above Wronskians $W_{k,s,\hat\tau,a,J\restriction X}$, when s, $\hat\tau$ vary, consists of jets $f_{[k]}(0)\in X_k^{\rm reg}$ such that the matrix $(D^\ell(u_j\circ f)(0))_{0\leqslant\ell\leqslant k,\,j\in\complement J}$ is not of maximal rank (i.e., of rank $<\operatorname{card}\complement J=N+1-c$); if $\delta>k$, this includes all jets $f_{[k]}(0)$ such that $f(0)\in\bigcup_{I\neq J}\tau_I^{-1}(0)$. When J also varies, the base locus of all $W_{k,s,\hat\tau,a,J\restriction X}$ in the Zariski open set $X'_k:=X_k^{\rm reg}\smallsetminus\bigcup_{|I|=c}\tau_I^{-1}(0)$ consists of all k-jets such that* $\operatorname{rank}(D^\ell(u_j\circ f)(0))_{0\leqslant\ell\leqslant k,\,0\leqslant j\leqslant N}\leqslant N-c$.

Proof. If $\delta>k$ and $m_j\circ f(0)=0$ for some $j\in J$, we have in fact $D^\ell(u_j\circ f)(0)=0$ for all derivatives $\ell\leqslant k$, because the exponents involved in all factors of the differentiated monomial $a_jm_j^\delta$ are at least equal to $\delta-k>0$. Hence the rank of the matrix cannot be maximal. Now, assume that $m_j\circ f(0)\neq0$ for all $j\in\complement J$, i.e.

$$x_0:=f(0)\in X\smallsetminus\bigcup_{j\in\complement J}m_j^{-1}(0)=X\smallsetminus\bigcup_{I\neq J}\tau_I^{-1}(0). \tag{4.23}$$

We take sections $\hat\tau_j$ so that $\hat\tau_j(x_0)\neq 0$, and then adjust the k-jet of the sections $s_1,\ldots,s_r$ in order to generate any matrix of derivatives $(D^\ell(s_j(f)\hat\tau_j(f)^{d-k})(0))_{0\leqslant\ell\leqslant k,\,j\in\complement J}$ (the fact that $f'(0)\neq 0$ is used for this!). Therefore, by expanding the determinant according to the last $N+1-c$ columns, we see that the base locus is defined by the equations

$$\det(D^\ell(u_j(f))(0))_{\ell\in L,\,j\in\complement J}=0,\qquad \forall L\subset\{0,1,\ldots,k\},\ |L|=N+1-c, \tag{4.24}$$

equivalent to the non maximality of the rank. The last assertion follows by a simple linear algebra argument. □

For a finer control of the base locus, we adjust the family of coefficients

$$a=(a_j)_{0\leqslant j\leqslant N}\in S:=H^0(Z,A^\rho)^{\oplus(N+1)} \tag{4.25}$$

in our section $\sigma=\sum a_jm_j^\delta\in H^0(Z,A^d)$, and denote by $X_a=\sigma^{-1}(0)\subset Z$ the corresponding hypersurface. By Lemma 4.16, we know that there is a Zariski open set $U\subset S$ such that X_a is smooth and $\prod\tau_I=0$ is a simple normal crossing divisor in X_a for all $a\in U$. We consider the Semple tower $X_{a,k}:=(X_a)_k$ of X_a, the "universal blow-up" $\mu_{a,k}:\widehat X_{a,k}\to X_{a,k}$ of the Wronskian ideal sheaf $\mathcal J_{a,k}$ such that $\mu_{a,k}^*\mathcal J_{a,k}=\mathcal O_{\widehat X_{a,k}}(-F_{a,k})$ for some "Wronskian divisor" $F_{a,k}$ in $\widehat X_{a,k}$. By the universality of this construction, we can also embed $X_{a,k}$ in the Semple tower Z_k of Z, blow up the Wronskian ideal sheaf $\mathcal J_k$ of Z_k to get a Wronskian divisor F_k in $\widehat Z_k$ where $\mu_k:\widehat Z_k\to Z_k$ is the blow-up map. Then $F_{a,k}$ is the restriction of F_k to $\widehat X_{a,k}\subset\widehat Z_k$. Our section $\widetilde W_{k,a,\hat\tau,s,J\restriction X_a}$ is the restriction of a *meromorphic* section defined on Z, namely

$$(\hat\tau^{\alpha_J}\tau^{\beta_J})^{-1}W_{k,s,\hat\tau,a,J}=(\hat\tau^{\alpha_J}\tau^{\beta_J})^{-1}W_k\big(s_1\hat\tau_1^{d-k},\ldots,s_r\hat\tau_r^{d-k},(a_jm_j^\delta)_{j\in\complement J}\big). \tag{4.26}$$

It induces over the Zariski open set $Z'=Z\smallsetminus\bigcup_I\tau_I^{-1}(0)$ a holomorphic section

$$\sigma_{k,s,\hat\tau,a,J}\in H^0\big(\widehat Z'_k,\mu_k^*(\mathcal O_{Z_k}(k')\otimes\pi_{k,0}^*A^{-p})\otimes\mathcal O_{\widehat Z_k}(-F_k)\big) \tag{4.27}$$

(notice that the relevant factors $\hat\tau_j$ remain divisible on the whole variety Z). By construction, thanks to the divisibility property explained in Lemma 4.19, the restriction of this section to $\widehat X'_{a,k}=\widehat X_{a,k}\cap\widehat Z'_k$ extends holomorphically to $\widehat X_{a,k}$, i.e.

$$\sigma_{k,s,\hat\tau,a,J\restriction\widehat X_{a,k}}\in H^0\big(\widehat X_{a,k},\mu_{a,k}^*(\mathcal O_{X_{a,k}}(k')\otimes\pi_{k,0}^*A^{-p})\otimes\mathcal O_{\widehat X_{a,k}}(-F_{a,k})\big). \tag{4.28}$$

(Here the fact that we took $\widehat X_{k,a}$ to be normal avoids any potential issue in the division process, as $\widehat X_{k,a}\cap\mu_k^{-1}\big(\pi_{k,0}^{-1}\bigcap_{I\in\mathcal G}\tau_I^{-1}(0)\big)$ has the expected codimension $=\operatorname{card}\mathcal G$ for any family $\mathcal G$).

4.29. Lemma. *Let V be a finite dimensional vector space over $\mathbb C$, $\Psi:V^p\to\mathbb C$ a non zero alternating multilinear form, and let $m,c\in\mathbb N$, $c<m\leqslant p$, $r=p+c-m\geqslant 0$. Then the subset $T\subset V^m$ of vectors $(v_1,\ldots,v_m)\in V^m$ such that*

$(*)\qquad \Psi(h_1,\ldots,h_r,(v_j)_{j\in\complement J})=0$ *for all $J\subset\{1,\ldots,m\}$, $|J|=c$, and all $h_1,\ldots,h_r\in V$,*

is a closed algebraic subset of codimension $\geqslant(c+1)(r+1)$.

Proof. A typical example is $\Psi=\det$ on a p-dimensional vector space V, then T consists of m-tuples of vectors of rank $<p-r$, and the assertion concerning the codimension is well known (we will reprove it anyway). In general, the algebraicity of T is obvious. We argue by induction on p, the result being trivial for $p=1$ (the kernel of a non zero linear form is indeed of codimension $\geqslant 1$). If K is the kernel of Ψ, i.e. the subspace of vectors $v\in V$ such that $\Psi(h_1,\ldots,h_{p-1},v)=0$ for all $h_j\in V$, then Ψ induces an alternating multilinear form $\overline\Psi$ on V/K, whose kernel is equal to $\{0\}$. The proof is thus reduced to the case when $\operatorname{Ker}\Psi=\{0\}$. Notice that we must have $\dim V\geqslant p$, otherwise Ψ would vanish. If $\operatorname{card}\complement J=m-c=1$, condition $(*)$ implies that $v_j\in\operatorname{Ker}\Psi=\{0\}$ for all j, hence $\operatorname{codim}T=\dim V^m\geqslant mp=(c+1)(r+1)$, as desired. Now, assume $m-c\geqslant 2$, fix $v_m\in V\smallsetminus\{0\}$ and consider the non zero alternating multilinear form on V^{p-1} such that

$$\Psi'_{v_m}(w_1,\ldots,w_{p-1}):=\Psi(w_1,\ldots,w_{p-1},v_m).$$

If $(v_1, \ldots, v_m) \in T$, then $(v_1, \ldots, v_{m-1})$ belongs to the set T'_{v_m} associated with the new data $(\Psi'_{v_m}, p-1, m-1, c, r)$. The induction hypothesis implies that $\operatorname{codim} T'_{v_m} \geqslant (c+1)(r+1)$, and since the projection $T \to V$ to the first factor admits the T'_{v_m} as its fibers, we conclude that

$$\operatorname{codim} T \cap ((V \smallsetminus \{0\}) \times V^{m-1}) \geqslant (c+1)(r+1).$$

By permuting the arguments v_j, we also conclude that

$$\operatorname{codim} T \cap (V^{k-1} \times (V \smallsetminus \{0\}) \times V^{m-k}) \geqslant (c+1)(r+1)$$

for all $k = 1, \ldots, m$. The union $\bigcup_k (V^{k-1} \times (V \smallsetminus \{0\}) \times V^{m-k}) \subset V^m$ leaves out only $\{0\} \subset V^m$ whose codimension is at least $mp \geqslant (c+1)(r+1)$, so Lemma 4.29 follows. □

4.30. Proposition. *Consider in $U \times \widehat{Z}'_k$ the set Γ of pairs (a, ξ) such that $\sigma_{k,s,\hat\tau,a,J}(\xi) = 0$ for all choices of s, $\hat\tau$ and $J \subset \{0, 1, \ldots, N\}$ with* $\operatorname{card} J = c$. *Then Γ is an algebraic set of dimension*

$$\dim \Gamma \leqslant \dim S - (c+1)(k+c-N+1) + n + 1 + kn.$$

As a consequence, if $(c+1)(k+c-N+1) > n+1+kn$, there exists $a \in U \subset S$ such that the base locus of the family of sections $\sigma_{k,s,\hat\tau,a,J}$ in $\widehat{X}_{a,k}$ lies over $\bigcup_I X_a \cap \tau_I^{-1}(0)$.

Proof. The idea is similar to [Brot17, Lemma 3.8], but somewhat simpler in the present context. Let us consider a point $\xi \in \widehat{Z}'_k$ and the k-jet $f_{[k]} = \mu_k(\xi) \in Z'_k$, so that $x = f(0) \in Z' = Z \smallsetminus \bigcup_I \tau_I^{-1}(0)$. Let us take the $\hat\tau_j$ such that $\hat\tau_j(x) \neq 0$. Then, we do not have to pay attention to the non vanishing factors $\hat\tau^{\alpha_J} \tau^{\beta_J}$, and the k-jets of sections m_j and $\hat\tau_j^{d-k}$ are invertible near x. Let e_A be a local generator of A near x and $e_{\mathcal{L}}$ a local generator of the invertible sheaf

$$\mathcal{L} = \mu_k^* \mathcal{O}_{Z_k}(k') \otimes \mathcal{O}_{\widehat{Z}_k}(-F_k)$$

near $\xi \in \widehat{Z}'_k$. Let $J^k \mathcal{O}_{Z,x} = \mathcal{O}_{Z,x} / \mathfrak{m}_{Z,x}^{k+1}$ be the vector space of k-jets of functions on Z at x. By definition of the Wronskian ideal and of the associated divisor F_k, we have a *non zero* alternating multilinear form

$$\Psi : (J^k \mathcal{O}_{Z,x})^{k+1} \to \mathbb{C}, \qquad (g_0, \ldots, g_k) \mapsto \mu_k^* W_k(g_0, \ldots, g_k)(\xi)/e_{\mathcal{L}}(\xi).$$

The simultaneous vanishing of our sections at ξ is equivalent to the vanishing of

$$\Psi\big(s_1 \hat\tau_1^{d-k} e_A^{-d}, \ldots, s_r \hat\tau_r^{d-k} e_A^{-d}, (a_j m_j^\delta e_A^{-d})_{j \in \complement J}\big) \tag{4.31}$$

for all $(s_1, \ldots, s_r)$. Since A is very ample and $\rho \geqslant k$, the power A^ρ generates k-jets at every point $x \in Z$, hence the morphisms

$$H^0(Z, A^\rho) \to J^k \mathcal{O}_{Z,x}, \quad a \mapsto a m_j^\delta e_A^{-d} \quad \text{and} \quad H^0(Z, A^k) \to J^k \mathcal{O}_{Z,x}, \quad s \mapsto s \hat\tau_j^{d-k} e_A^{-d}$$

are surjective. Lemma 4.29 applied with $r = k + c - N$ and (p, m) replaced by $(k+1, N+1)$ implies that the codimension of families $a = (a_0, \ldots, a_N) \in S = H^0(Z, A^\rho)^{\oplus(N+1)}$ for which $\sigma_{k,s,\hat\tau,a,J}(\xi) = 0$ for all choices of s, $\hat\tau$ and J is at least $(c+1)(k+c-N+1)$, i.e. the dimension is at most $\dim S - (c+1)(k+c-N+1)$. When we let ξ vary over $\widehat{Z}'_k$ which has dimension $(n+1)+kn$ and take into account the fibration $(a, \xi) \mapsto \xi$, the dimension estimate of Proposition 4.30 follows. Under the assumption

$$(c+1)(k+c-N+1) > n+1+kn \tag{4.32}$$

we have $\dim \Gamma < \dim S$, hence the image of the projection $\Gamma \to S$, $(a, \xi) \mapsto a$ is a constructible algebraic subset distinct from S. This concludes the proof. □

Our final goal is to completely eliminate the base locus. For $x \in Z$, we let $\mathcal{G}$ be the family of hyperplane sections $\tau_I = 0$ that contain x. We introduce the set $P = \{0, 1, \ldots, N\} \smallsetminus \bigcup_{I \in \mathcal{G}} I$ and the smooth intersection

$$Z_{\mathcal{G}} = Z \cap \bigcap_{I \in \mathcal{G}} \tau_I^{-1}(0),$$

so that $N'+1 := \operatorname{card} P \geqslant N+1-c\operatorname{card}\mathcal{G}$ and $\dim Z_{\mathcal{G}} = n+1-\operatorname{card}\mathcal{G}$. If $a\in U$ is such that $x\in X_a$, we also look at the intersection

$$X_{\mathcal{G},a} = X_a \cap \bigcap_{I\in\mathcal{G}} \tau_I^{-1}(0),$$

which is a smooth hypersurface of $Z_{\mathcal{G}}$. In that situation, we consider Wronskians $W_{k,s,\hat\tau,a,J}$ as defined above, but we now take $J\subset P$, $\operatorname{card} J = c$, $\complement J = P\smallsetminus J$, $r' = k+c-N'$.

4.33. Lemma. *In the above setting, if we assume $\delta > k$, the restriction $W_{k,s,\hat\tau,a,J\restriction X_{\mathcal{G},a}}$ is still divisible by a monomial $\hat\tau^{\alpha_J}\tau^{\beta_J}$ such that*

$$|\alpha_J| + |\beta_J| = (k+c-N')(d-2k) + (N'+1-c)(\delta-k)b + (\delta-k).$$

Therefore, if

$$p' = |\alpha_J| + |\beta_J| - (k+1)d = (\delta-k) - (k+c-N')2k - (N'+1+c)(kb+\rho)$$

as in (4.21)*, we obtain again holomorphic sections*

$$\widetilde W_{k,s,\hat\tau,a,J\restriction X_{\mathcal{G},a}} := (\hat\tau^{\alpha_J}\tau^{\beta_J})^{-1} W_{k,s,\hat\tau,a,J\restriction X_{\mathcal{G},a}} \in H^0(X_{\mathcal{G},a}, E_{k,k'}T^*_X\otimes A^{-p'}),$$

$$\sigma_{k,s,\hat\tau,a,J\restriction \pi_{k,0}^{-1}(X_{\mathcal{G},a})} \in H^0\big(\pi_{k,0}^{-1}(X_{\mathcal{G},a}), \mu_{a,k}^*(\mathcal{O}_{X_{a,k}}(k')\otimes \pi_{k,0}^*A^{-p'})\otimes \mathcal{O}_{\widehat X_{a,k}}(-F_{a,k})\big).$$

Proof. The arguments are similar to those employed in the proof of Lemma 4.19. Let $f_{[k]}\in X_{a,k}$ be a k-jet such that $f(0)\in X_{\mathcal{G},a}$ (the k-jet need not be entirely contained in $X_{\mathcal{G},a}$). Putting $j_0 = \min(\complement J)$, we observe that we have on $X_{\mathcal{G},a}$ an identity

$$a_{j_0}m_{j_0}^\delta = -\sum_{i\in P\smallsetminus\{j_0\}} a_i m_i^\delta = -\sum_{i\in J} a_i m_i^\delta - \sum_{P\smallsetminus(J\cup\{j_0\})} a_i m_i^\delta$$

because $m_i = \prod_{I\ni i}\tau_I = 0$ on $X_{\mathcal{G},a}$ when $i\in\complement P = \bigcup_{I\in\mathcal{G}} I$ (one of the factors τ_I is such that $I\in\mathcal{G}$, hence $\tau_I = 0$). If we compose with a germ $t\mapsto f(t)$ such that $f(0)\in X_{\mathcal{G},a}$ (even though f does not necessarily lie entirely in $X_{\mathcal{G},a}$), we get

$$a_{j_0}m_{j_0}^\delta(f(t)) = -\sum_{i\in J} a_i m_i^\delta(f(t)) - \sum_{P\smallsetminus(J\cup\{j_0\})} a_i m_i^\delta(f(t)) + O(t^{k+1})$$

as soon as $\delta > k$. Hence we have an equality for all derivatives $D^\ell(\bullet)$, $\ell\leqslant k$ at $t=0$, and

$$W_{k,s,\hat\tau,a,J\restriction X\mathcal{G},a}(f_{[k]}) = -\sum_{i\in J} W_k\big(s_1\hat\tau_1^{d-k},\ldots,s_{r'}\hat\tau_{r'}^{d-k}, a_i m_i^\delta, (a_j m_j^\delta)_{j\in P\smallsetminus(J\cup\{j_0\})}\big)_{\restriction X_{\mathcal{G},a}}(f_{[k]}).$$

Then, again, $\tau_J^{\delta-k}$ is a new additional common factor of all terms in the sum, and we conclude as in Lemma 4.19 and Corollary 4.20. □

Now, we analyze the base locus of these new sections on

$$\bigcup_{a\in U}\mu_{a,k}^{-1}\pi_{k,0}^{-1}(X_{\mathcal{G},a}) \subset \mu_k^{-1}\pi_{k,0}^{-1}(Z_{\mathcal{G}}) \subset \widehat Z_k.$$

As x runs in $Z_{\mathcal{G}}$ and $N' < N$, Lemma 4.29 shows that (4.32) can be replaced by the less demanding condition

$$(c+1)(k+c-N'+1) > n+1-\operatorname{card}\mathcal{G} + kn = \dim\mu_k^{-1}\pi_{k,0}^{-1}(Z_{\mathcal{G}}). \tag{4.32'}$$

A proof entirely similar to that of Proposition 4.30 shows that for a generic choice of $a\in U$, the base locus of these sections on $\widehat X_{\mathcal{G},a,k}$ projects onto $\bigcup_{I\in\complement\mathcal{G}} X_{\mathcal{G},a}\cap\tau_I^{-1}(0)$. Arguing inductively on $\operatorname{card}\mathcal{G}$, the base locus can be shrinked step by step down to empty set (but it is in fact sufficient to stop when $X_{\mathcal{G},a}\cap\tau_I^{-1}(0)$ reaches dimension 0).

4.34. Corollary. *Under condition* (4.32) *and the hypothesis $p>0$ in* (4.21)*, the following properties hold.*

(a) *The line bundle*

$$\mathcal{L} := \mu_{a,k}^*(\mathcal{O}_{X_{a,k}}(k') \otimes \pi_{k,0}^* A^{-1}) \otimes \mathcal{O}_{\widehat{X}_{a,k}}(-F_{a,k})$$

is nef on $\widehat{X}_{a,k}$ *for general* $a \in U$.

(b) *Let* $\Delta_a = \sum_{2 \leqslant \ell \leqslant k} \lambda_\ell D_{a,\ell}$ *be a positive rational combination of vertical divisors of the Semple tower and* $q \in \mathbb{N}$, $q \gg 1$, *an integer such that*

$$\mathcal{L}' := \mathcal{O}_{X_{a,k}}(1) \otimes \mathcal{O}_{a,k}(-\Delta_a) \otimes \pi_{k,0}^* A^q$$

is ample on $X_{a,k}$. *Then the* $\mathbb{Q}$*-line bundle*

$$\mathcal{L}_{\varepsilon,\eta} := \mu_{a,k}^*(\mathcal{O}_{X_{a,k}}(k') \otimes \mathcal{O}_{X_{a,k}}(-\varepsilon\Delta_a) \otimes \pi_{k,0}^* A^{-1+q\varepsilon}) \otimes \mathcal{O}_{\widehat{X}_{a,k}}(-(1+\varepsilon\eta)F_{a,k})$$

is ample on $\widehat{X}_{a,k}$ *for general* $a \in U$, *some* $q \in \mathbb{N}$ *and* $\varepsilon, \eta \in \mathbb{Q}_{>0}$ *arbitrarily small.*

Proof. (a) This would be obvious if we had global sections generating $\mathcal{L}$ on the whole of $\widehat{X}_{a,k}$, but our sections are only defined on a stratification of $\widehat{X}_{a,k}$. In any case, if $C \subset \widehat{X}_{a,k}$ is an irreducible curve, we take a maximal family $\mathcal{G}$ such that $C \subset X_{\mathcal{G},a,k}$. Then, by what we have seen, for $a \in U$ general enough, we can find global sections of $\mathcal{L}$ on $\widehat{X}_{\mathcal{G},a,k}$ such that C is not contained in their base locus. Hence $\mathcal{L} \cdot C \geqslant 0$ and $\mathcal{L}$ is nef.

(b) The existence of Δ_a and q follows from Proposition 3.19 and Corollary 3.21, which even provide universal values for λ_ℓ and q. After taking the blow up $\mu_{a,k} : \widehat{X}_{a,k} \to X_{a,k}$ (cf. (4.8)), we infer that

$$\mathcal{L}'_\eta := \mu_{a,k}^* \mathcal{L}' \otimes \mathcal{O}_{\widehat{X}_{a,k}}(-\eta F_{a,k}) = \mu_{a,k}^*\big(\mathcal{O}_{X_{a,k}}(1) \otimes \mathcal{O}_{X_{a,k}}(-\Delta_a) \otimes \pi_{k,0}^* A^q\big) \otimes \mathcal{O}_{\widehat{X}_{a,k}}(-\eta F_{a,k})$$

is ample for $\eta > 0$ small. The result now follows by taking a combination $\mathcal{L}_{\varepsilon,\eta} = \mathcal{L}^{1-\varepsilon/k'} \otimes (\mathcal{L}'_\eta)^\varepsilon$. □

At this point, we fix our integer parameters to meet all conditions that have been found. We must have $N \geqslant c(n+1)$ by Lemma 4.16, and for such a large value of N, condition (4.32) can hold only when $c \geqslant n$, so we take $c = n$ and $N = n(n+1)$. Inequality (4.32) then requires k large enough, $k = n^3 + n^2 + 1$ being the smallest possible value. We find

$$b = \binom{N}{c-1} = \binom{n^2+n}{n-1} = n\frac{(n^2+n)\ldots(n^2+2)}{n!}.$$

We have $n^2 + k = n^2(1 + k/n^2) < n^2 \exp(k/n^2)$ and by Stirling's formula, $n! > \sqrt{2\pi n}\,(n/e)^n$, hence

$$b < \frac{n^{2n-1}\exp((2+\ldots+n)/n^2)}{\sqrt{2\pi n}\,(n/e)^n} < \frac{e^{n+1/2+1/2n}}{\sqrt{2\pi}}\, n^{n-3/2}.$$

Finally, we divide $d - k$ by b, get in this way $d - k = b\delta + \lambda$, $0 \leqslant \lambda < b$, and put $\rho = \lambda + k \geqslant k$. Then $\delta + 1 \geqslant (d-k+1)/b$ and formula (4.21) yields

$$\begin{aligned} p &= (\delta - k) - (n^3+1)2k - (n^2+2n+1)(kb+\rho) \\ &\geqslant (d-k+1)/b - 1 - (2n^3+3)k - (n^2+2n+1)(kb+k+b-1), \end{aligned}$$

therefore $p > 0$ is achieved as soon as

$$d \geqslant d_n = k + b\big(1 + (2n^3+3)k + (n^2+2n+1)(kb+k+b-1)\big)$$

where

$$k = n^3 + n^2 + 1, \quad b = \binom{n^2+n}{n-1}.$$

The dominant term in d_n is $k(n^2+2n+1)b^2 \sim e^{2n+1} n^{2n+2}/2\pi$. By means of more precise numerical calculations and of Stirling's asymptotic expansion for $n!$, one can show in fact that $d_n \leqslant \lfloor (n+4)\,(en)^{2n+1}/2\pi \rfloor$ for $n \geqslant 4$ (which is also an equivalent and a close approximation as $n \to +\infty$), while $d_1 = 61$, $d_2 = 6685$, $d_3 = 2825761$. We can now state the main result of this section.

4.35. Theorem. *Let Z be a projective $(n+1)$-dimensional manifold and A a very ample line bundle on Z. Then, for a general section $\sigma \in H^0(Z, A^d)$ and $d \geqslant d_n$, the hypersurface $X_\sigma = \sigma^{-1}(0)$ is Kobayashi hyperbolic. The bound d_n for the degree can be taken to be*

$$d_n = \lfloor (n+4)\,(en)^{2n+1}/2\pi \rfloor \quad \textit{for } n \geqslant 4,$$

and for $n \leqslant 3$, one can take $d_1 = 4$, $d_2 = 6685$, $d_3 = 2825761$.

A simpler (and less refined) choice is $\tilde{d}_n = \lfloor \frac{1}{3}(en)^{2n+2} \rfloor$, which is valid for all n. These bounds are only slightly better than the ones found by Ya Deng in his PhD thesis [Deng16], namely $\tilde{d}_n = (n+1)^{n+2}(n+2)^{2n+7} = O(n^{3n+9})$. A polynomial bound $d_n = O(n^C)$ would already be quite interesting – when $Z = \mathbb{P}^{n+1}$, the expected optimal bound is a linear one.

Proof. On the "Wronskian blow-up" $\widehat{X}_{\sigma,k}$ of $X_{\sigma,k}$, let us consider the line bundle

$$\mathcal{L}_{\sigma,\varepsilon,\eta} := \mu_{\sigma,k}^*(\mathcal{O}_{X_{\sigma,k}}(k') \otimes \mathcal{O}_{X_{\sigma,k}}(-\varepsilon\Delta_\sigma) \otimes \pi_{k,0}^* A^{-1+q\varepsilon}) \otimes \mathcal{O}_{\widehat{X}_{\sigma,k}}(-(1+\varepsilon\eta)F_{\sigma,k})$$

associated intrinsically with our k-jet constructions. By 4.34 (b), we can find $\sigma_0 \in H^0(Z, A^d)$ such that $X_{\sigma_0} = \sigma_0^{-1}(0)$ is smooth and $\mathcal{L}^m_{\sigma_0,\varepsilon,\eta}$ is an ample line bundle on $\widehat{X}_{\sigma_0,k}$ ($m \in \mathbb{N}^*$). As ampleness is a Zariski open condition, we conclude that $\mathcal{L}^m_{\sigma,\varepsilon,\eta}$ remains ample for a general section $\sigma \in H^0(Z, A^d)$, i.e. for $[\sigma]$ in some Zariski open set $\Omega \subset P(H^0(Z, A^d))$. Since $\mu_{\sigma,k}(F_{\sigma,k})$ is contained in the vertical divisor of $X_{\sigma,k}$, we conclude by Corollary 3.27 that X_σ is Kobayashi hyperbolic for $[\sigma] \in \Omega$. For $n = 1$, we have already seen in § 4.B that $d_1 = 4$ works (rather than the insane value $d_1 = 61$). □

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