GENERIC POSITIVITY AND APPLICATIONS TO HYPERBOLICITY OF MODULI SPACES

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ABSTRACT. This chapter is an exposition of Campana and Păun's proof of Viehweg's celebrated hyperbolicity conjecture. The proof is a consequence of a vast generalisation of Miyaoka's generic semipositivity result for non-uniruled varieties to the context of pairs.

CONTENTS

1. INTRODUCTION

In 1962 Shafarevich conjectured that a smooth family $f^\circ : X^\circ \to Y^\circ$ of complex projective curves of genus at least equal to 2, parameterized by $Y^\circ = \mathbb{P}^1$, $\mathbb{C}$, $\mathbb{C}^*$, or an elliptic curve E is isotrivial, so that is there is no variation in the algebraic structure of the members of the family. Equivalently this conjecture can be expressed as the prediction that the base Y° of any smooth, non-isotrivial family of projective curves with $g \geq 2$ is of log-general type. In other words, we have $\kappa(Y, K_Y + D) = 1$ for any smooth compactification (Y, D) of Y°, with snc boundary divisor D. Shafarevich conjecture was shown by Parshin and Arakelov.

To generalise the Shafarevich conjecture to higher dimensional fibres and parametrizing spaces, Viehweg considered the hyperbolicity properties of the moduli

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stack of canonically-polarised manifolds. Recall that the moduli functor $\mathcal{M}$ of canonically-polarised manifolds with fixed Hilbert polynomial, is equipped with a natural transformation

$$\Psi : \mathcal{M}(\cdot) \to \operatorname{Hom}(\cdot, \mathfrak{M}),$$

where $\mathfrak{M}$ denotes the coarse moduli scheme associated with $\mathcal{M}$. The scheme $\mathfrak{M}$ was proved by Viehweg to be quasi-projective, cf. [Vie95]. Also recall that a complex analytic space U is said to be Brody hyperbolic if there are no non-constant holomorphic maps $f : \mathbb{C} \to U$. In the spirit of this definition, Shafarevich's conjecture is equivalent to the assertion that the base Y° of non-isotrivial, smooth, projective families of high genus curves is algebraically Brody hyperbolic in the sense that there are no non-constant morphisms from $\mathbb{C}^*$ to Y°.

Generalising Shafarevich's conjecture, Viehweg predicted that the moduli stack of canonically-polarised manifolds is not only algebraically Brody hyperbolic but that it is Brody hyperbolic. More precisely, a smooth quasi-projective variety Y° admitting a generically finite morphism $\mu : Y^\circ \to \mathfrak{M}$, must be Brody hyperbolic. This conjecture was settled by Viehweg and Zuo in [VZ03]. On the other hand, a long-standing conjecture of Lang predicts that for a quasi-projective Y°, Kobayashi hyperbolicity (which is equivalent to Brody hyperbolicity for projective varieties) implies that all subvarieties of Y°, including Y°, are of log-general type. In the light of Lang's problem, Viehweg extended his question on the hyperbolic nature of the moduli stack of canonically-polarised manifolds to the following conjecture.

Conjecture 1.1 (Viehweg's hyperbolicity conjecture)**.** *Let Y° be a smooth quasi-projective variety admitting a generically finite morphism $\mu : Y^\circ \to \mathfrak{M}$. Then, the smooth compactification (Y, D) of Y° is of log-general type.*

Viehweg's conjecture has attracted the interest of many algebraic geometers for a long time. We refer the reader to the survey [Keb13] for more details, including references to earlier results that are not mentioned here for lack of space.

1.1. **Viehweg's hyperbolicity conjecture according to Viehweg-Zuo and Campana-Păun.** A general strategy to prove Conjecture 1.1 consists of two main steps. Combining deep results of analytic [Zuo00], algebraic [Vie83] and Hodge theoretic [Gri84] nature, Viehweg and Zuo construct in a first step a subsheaf of the sheaf of pluri-log differential forms of the base whose birational positivity captures the variation[1] in the family.

Theorem 1.2 (Existence of pluri-logarithmic forms in the base, cf. [VZ02, Thm. 1.4])**.** *If the smooth family of canonically-polarised manifolds f° has maximal variation, then there exist a positive integer N an invertible subsheaf $\mathscr{L} \subseteq \operatorname{Sym}^N\big(\Omega^1_Y \log(D)\big)$ such that $\kappa(Y, \mathscr{L}) = \dim Y$.*

Theorem 1.2 immediately resolves the original conjecture of Shafarevich. The goal in the second step is to trace a connection between the birational positivity (bigness) of $\mathscr{L}$ in Theorem 1.2 and that of $K_Y + D$, thus resolving Conjecture 1.1. Working along these lines, the second author and Kovács established Conjecture 1.1 for moduli stacks of dimension two and three, [KK08, KK10] and see [Keb13] for an overview. The work relied, among other things, on the log-abundance theorem for surfaces and threefolds. In the absence of these methods in higher dimensions, for instance a complete solution to the abundance problem, Campana and Păun devised an additional tool, namely a vast generalisation of the famous generic semipositivity result of Miyaoka to the context of pairs with

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[1] A family $f^\circ : X^\circ \to Y^\circ$ of canonically-polarised manifolds is said to have *maximal variation* if the moduli map $\Psi(f^\circ) : Y^\circ \to \mathfrak{M}$ is generically finite.

rational coefficients. Here, we state their result in its simplest form and we refer the reader to Section 5 for a general statement.

Theorem 1.3 (Logarithmic generic semipositivity, cf. [CP15b, Thm. 2.1])**.** *Let (X, D) be a reduced, projective, snc pair. If $K_X + D$ is pseudo-effective, then for every ample divisor H on X and every torsion free quotient $\mathscr{Q}$ of $\Omega^1_X(\log D)$ we have*

$$c_1(\mathscr{Q}) \cdot [H]^{n-1} \geq 0. \qquad \square$$

Despite its importance, we found the paper [CP15b] rather hard to read. This chapter is meant to serve as an exposition of Campana and Păun's proof of Theorem 1.3 and its application to resolving Conjecture 1.1.

1.2. **Structure of the current chapter.** In Section 2 we gather some preliminary definitions and notions that are used throughout this chapter. In Section 3 we review some of the basics of the theory of orbifolds. In Section 4 we delve deeper into some technical details that will be crucial to the proof of the generic semipositivity result in Section 8. In Section 5 we state Theorem 1.3 in its full generality. Section 6 sketches the proof of Conjecture 1.1 using this result. Part II is devoted to the proof of the semipositivity result of Campana and Păun.

1.3. **A note on further results.** Constructing degenerate Kähler-Einstein metrics, Campana and Păun have established a second proof of Theorem 5.2 that works for Kähler manifolds, [CP14].

More recently, they strengthened Theorems 1.3 and 5.2 also in another direction, by proving the pseudo-effectivity of torsion free quotients, [CP15a]. This latter result is specially significant for the proof of Viehweg's conjecture, as it makes the LMMP methods redundant. For a concise exposition of [CP15a] and its application to Viehweg's problem we refer the reader to the first author's notes written for the Bourbaki seminar, [Cla15].

In a slightly different, but closely related, direction a more general version of Viehweg's conjecture, that is perhaps closer to the spirit of the original conjecture of Shafarevich, was formulated by Campana. In this conjecture Campana proposed the so-called *special* varieties as higher dimensional analogues of $\mathbb{C}$, $\mathbb{C}^*$, $\mathbb{P}^1$ and E in Shafarevich conjecture. We refer the reader to the original paper of Campana, [Cam04], for the basic definitions and background in the theory of special varieties.

Conjecture 1.4 (The isotriviality conjecture)**.** *Any smooth family of canonically-polarised manifolds $f^\circ : X^\circ \to Y^\circ$ parametrised by a special quasi-projective variety Y° is isotrivial.*

Following the strategy of Campana and Păun and by using the result of [JK11b], Conjecture 1.4 has been settled in [Taj16]. More recently, in [PS15], Popa and Schnell have proved a vast generalisation of Conjecture 1.1 by extending Theorem 1.2 to smooth projective families of varieties of general type. Where their strategy follows the same two-steps approach discussed above, the main breakthrough in their result comes from an interesting use of the theory of Hodge modules to extend some crucial Hodge theoretic tools used in [VZ03].

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2. DEFINITIONS AND NOTATION

In the current section we gather some very basic definitions and concepts needed for the arguments in the later parts of this chapter. For the more standard definitions, we refer to [Har77]. The reader who is familiar with these preliminaries may wish to skip Subsections 2.1 to 2.5 and move on Subsection 2.6. In this chapter, all varieties are defined over $\mathbb{C}$.

2.1. **Varieties, subsets, sheaves and pairs.** Let us begin by introducing the most basic objects, recurrent throughout this chapter.

Notation 2.1 (Small and big sets). Let X be a variety. A subset $S \subseteq X$ is called *small* if its Zariski closure satisfies $\operatorname{codim}_X \overline{S} \geq 2$. A subset $U \subseteq X$ is called *big* if its complement is small.

Notation 2.2 (Families of curves on projective varieties). Let X be a projective variety. A family of curves is a smooth subvariety $T \subseteq \operatorname{Hilb}(X)$ whose associated subschemes $(C_t)_{t \in T}$ are reduced, irreducible and of dimension one. We say that the family *dominates* X if $\cup_{t \in T} C_t$ is dense in X. We say that the family *avoids small sets* if, given any small set $S \subset X$, there exists a dense open $T^\circ \subset T$ such that $C_t \cap S = \emptyset$, for all $t \in T^\circ$.

Definition 2.3 (Pair)**.** *A* pair (X, Δ) *consists of a normal variety X and a $\mathbb{Q}$-Weil divisor Δ on X with coefficients in $[0, 1] \cap \mathbb{Q}$. A pair (X, Δ) is called* snc *if X is smooth and if the support of Δ has simple normal crossings only. We denote the maximal open subset of X where (X, Δ) is smooth by $(X, \Delta)_{\text{snc}}$. Note that this is a big subset of X. The fractional part of Δ is written as $\{\Delta\}$.*

Notation 2.4 (Reflexive hull). Given a normal, quasi-projective variety X and a coherent sheaf $\mathscr{E}$ on X, write

$$\Omega_X^{[p]} := \left(\Omega_X^p\right)^{**}, \quad \mathscr{E}^{[m]} := \left(\mathscr{E}^{\otimes m}\right)^{**} \quad \text{and} \quad \det \mathscr{E} := \left(\wedge^{\operatorname{rank} \mathscr{E}} \mathscr{E}\right)^{**}.$$

Given any morphism $f : Y \to X$, write $f^{[*]}\mathscr{E} := (f^* \mathscr{E})^{**}$, etc.

2.2. **Morphisms.** Let us now briefly review some basic notions and properties of morphisms between normal varieties.

Construction 2.5 (Push-forward of Weil divisor). Let $f : X \to Z$ be a morphism of normal varieties. Recall from "Zariski's Main Theorem in the form of Grothendieck"[2] that there exists a unique, normal variety $\overline{X}$ and a unique factorisation $f = \beta \circ \alpha$ as follows,

$$\begin{tikzcd} X \arrow[r, "\alpha", "\text{open immersion}"'] & \overline{X} \arrow[r, "\beta", "\text{proper morphism}"'] & Z. \end{tikzcd}$$

Taking Zariski-closures yields a push-forward morphism $\alpha_* : \operatorname{WDiv}(X) \to \operatorname{WDiv}(\overline{X})$. Composing with the standard push-forward morphism of the proper morphism β, cf. [Ful98, I Sect. 1.4], we obtain a map $f_* : \operatorname{WDiv}(X) \to \operatorname{WDiv}(Z)$.

Remark 2.6 (Push-forward vs. linear equivalence). The map f_* of Construction 2.5 will in general not respect linear equivalence, unless one of the following folds.

(2.6.1) The morphism f is proper.

(2.6.2) The variety X is a big open subset of Z and f is the inclusion. In this case, f_* is an isomorphism.

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[2] See [Gro66] Zariski's main theorem in the form of Grothendieck and [GKP16, Thm. 3.8] for the precise statement used here. A full proof is found in the extended version of [GKP16], available on the arXiv.

2.2.1. *Galois covers.* As we will be working with G-sheaves, Galois morphisms are of particular interest.

Definition 2.7 (Cover and covering morphism)**.** *Finite, surjective morphisms $\gamma : Y \to X$ between normal varieties X, Y are called* covers *or* covering morphisms*.*

Definition 2.8 (Galois morphism)**.** *A covering map $\gamma : X \to Y$ of normal varieties is called* Galois *if there exists a finite group $G \subset \mathrm{Aut}(X)$ such that γ is isomorphic to the quotient map.*

Warning 2.9 (Galois $\not\Rightarrow$ étale). Definition 2.8 does not require γ to be étale. This will be of crucial importance for nearly everything that follows.

2.3. **Equidimensional morphisms.** In a normal variety, Weil divisors need not be Cartier. Still, it is possible to define a pull-back map, at least for equidimensional morphisms.

Definition 2.10 (Equidimensional morphism)**.** *Let $f : X \to Z$ be a dominant morphism of varieties. We say that f is* equidimensional *if there exists a number d such for any $x \in X$, the associated fibre $f^{-1}f(x)$ is of pure dimension d. The number d is called* relative dimension*.*

Remark 2.11 (Preimages of big and small sets). In the setting of Definition 2.10, if $Z' \subsetneq Z$ is any algebraic set, then $\mathrm{codim}_X f^{-1}(Z') \geq \mathrm{codim}_Z Z'$. In particular, preimages of small sets are small, and preimages of big sets are big.

Lemma 2.12 (Equidimensional morphism and normalisation)**.** *Let $f : X \to Z$ be a dominant, equidimensional morphism of varieties. If X is normal and Z' the normalisation of Z, then the natural morphism $f' : X \to Z'$ is likewise equidimensional.*

Proof. Recalling that the normalisation morphism $\eta : Z' \to Z$ is finite, it follows that for any $x \in X$, the fibre $F' := (f')^{-1}f'(x)$ is a union of connected components of $F = f^{-1}f(x)$. If F is of pure dimension d, then so is F'. □

2.3.1. *Pull-back.* We now explain the construction of pull-back maps for Weil divisors in a normal variety.

Construction 2.13 (Pull-back of Weil divisor). Let $f : X \to Z$ be an equidimensional morphism between normal varieties. We define a pull-back morphism $f^* : \mathrm{WDiv}(Z) \to \mathrm{WDiv}(X)$ as the composition of the following morphisms,

$$\begin{tikzcd}
\mathrm{WDiv}(Z) \arrow[r, "\cong", "\text{by (2.6.2) since } Z_{\mathrm{reg}} \text{ is big}"'] & \mathrm{WDiv}(Z_{\mathrm{reg}}) \arrow[r, "\cong"] & \mathrm{CDiv}(Z_{\mathrm{reg}}) \arrow[r, "\left(f|_{f^{-1}(Z_{\mathrm{reg}})}\right)^*"] & {} \\
\mathrm{CDiv}\big(f^{-1}(Z_{\mathrm{reg}})\big) \arrow[r] & \mathrm{WDiv}\big(f^{-1}(Z_{\mathrm{reg}})\big) \arrow[r, "\cong", "\text{by (2.6.2) and Rem. 2.11}"'] & \mathrm{WDiv}(X).
\end{tikzcd}$$

Since all morphisms respect linear equivalence, so does f^*.

2.3.2. *Multiplicities, ramification and branch divisors.* We briefly review various notions of multiplicities that appear in the following sections.

Definition 2.14 (Multiplicities)**.** *Let $f : X \to Z$ be an equidimensional morphism between normal varieties. If $\Delta \in \mathrm{WDiv}(X)$ is prime, define the* multiplicity of f along D *as*

$$\mathrm{mult}_\Delta f := \mathrm{mult}_\Delta f^*D, \quad \textit{where} \quad D := (f_*\Delta)_{\mathrm{red}}.$$

Remark 2.15. In Definition 2.14, either $\mathrm{supp}\,\Delta$ dominates Z, or $f_*\Delta \neq 0$ and $\mathrm{mult}_\Delta f \geq 1$.

Definition 2.16 (Ramification and branch divisors)**.** *Let $f : X \to Z$ be an equidimensional morphism between normal varieties. The* ramification *and* branch divisor *of f are defined as follows,*

$$\begin{aligned} \operatorname{Ramification} f &:= \sum_{\substack{\Delta \in \operatorname{WDiv}(X)\\ \textit{prime}}} \max\{0, \operatorname{mult}_\Delta f - 1\} \cdot \Delta \\ \operatorname{Branch} f &:= \sum_{\substack{D \in \operatorname{WDiv}(Z) \textit{ prime}\\ D \le f_* \operatorname{Ramification} f}} \operatorname{lcm}\{\operatorname{mult}_\Delta f^*D \mid \Delta \subseteq \operatorname{supp} f^*D \textit{ a prime div.}\} \cdot D \\ \operatorname{OrbiBranch} f &:= \sum_{\substack{D \in \operatorname{WDiv}(Z) \textit{ prime}\\ D \le \operatorname{Branch} f}} \frac{1 - \operatorname{mult}_D \operatorname{Branch} f}{\operatorname{mult}_D \operatorname{Branch} f} \cdot D \end{aligned}$$

Remark 2.17. In the setting of Definition 2.16, observe that

$$f\big(\operatorname{supp} \operatorname{Ramification} f\big) \subseteq \operatorname{supp} \operatorname{Branch} f.$$

If f is proper, then equality holds.

2.3.3. *Local normal form.* In this section we give explicit description of equidimensional morphisms in a suitably chosen local analytic coordinate system.

Construction 2.18 (Local normal form). Let $f : X \to Z$ be an equidimensional morphism of normal varieties, of relative dimension d. Let $Z^\circ \subseteq Z$ be the largest open set such that both Z° and $Z^\circ \cap (\operatorname{supp} \operatorname{Branch} f)$ are smooth. Let $X^\circ \subseteq f^{-1}(Z^\circ)$ be the largest open set such that both X° and $X^\circ \cap (\operatorname{supp} \operatorname{Ramification} f)$ are smooth, and such that the following restrictions of f are smooth morphisms,

$$\begin{aligned} X^\circ \setminus (\operatorname{supp} \operatorname{Ramification} f) &\to Z^\circ \\ X^\circ \cap (\operatorname{supp} \operatorname{Ramification} f) &\to Z^\circ \cap (\operatorname{supp} \operatorname{Branch} f). \end{aligned}$$

Remark 2.17 ensures that the second map is defined. Observe that both Z° and X° are big open subsets of Z and X, respectively. Let $\vec{x} \in X^\circ$ be any point and $\vec{z} := f(\vec{x}) \in Z^\circ$ be its image.

If $\vec{x}$ is *not* contained in the support of $\operatorname{Ramification}(f)$, then f is smooth at $\vec{x}$. If $z_0, \dots, z_n \in \mathscr{O}_{Z,\vec{z}}$ are local holomorphic coordinates on Z centred about $\vec{z}$, then $x_i := z_i \circ f \in \mathscr{O}_{X,\vec{x}}$ can be completed to a system of holomorphic coordinates on X centred about $\vec{x}$. In these coordinates, f takes the form

$$f : (x_0, \dots, x_n, x_{n+1}, \dots, x_{n+d}) \mapsto (x_0, \dots, x_n). \tag{2.18.1}$$

If $\vec{x}$ *is* contained in the support of $\operatorname{Ramification}(f)$, then there exists a holomorphic function $z_0 \in \mathscr{O}_{Z,\vec{z}}$ which locally generates the ideal of the smooth hypersurface $\big(\operatorname{supp} \operatorname{Branch}(f)\big)$. Near $\vec{x}$, there exists a holomorphic function $x_0 \in \mathscr{O}_{X^\circ,\vec{x}}$ such that $z_0 \circ f = x_0^m$, where m is the order of ramification of f along the unique component of $\operatorname{Ramification}(f)$ that contains $\vec{x}$. Completing, we obtain holomorphic coordinate functions of the following form,

$$z_0, \dots, z_n \in \mathscr{O}_{Z,\vec{z}} \quad \text{and} \quad x_0, \underbrace{x_1}_{:= z_1 \circ f}, \dots, \underbrace{x_n}_{:= z_n \circ f}, x_{n+1}, \dots, x_{n+d} \in \mathscr{O}_{X,\vec{x}}.$$

In these coordinates, f takes the form

$$f : (x_0, x_1, \dots, x_n, x_{n+1}, \dots, x_{n+d}) \mapsto (x_0^m, x_1, \dots, x_n). \tag{2.18.2}$$

Notation 2.19 (Local normal form). In the setting of Construction 2.18, we refer to the explicit description of f in (2.18.1) and (2.18.2) as *local normal forms*. We call X° and Z° the *maximal open sets where f can locally be written in normal form*. If $X^\circ = X$, we say that f *can locally be written in normal form*.

2.4. **Rational maps.** A certain class of rational maps and divisors appear naturally in the proof of the semipositivity theorem, Theorem 5.2. The current section is devoted to introducing these maps and reviewing some of their basic properties.

Notation 2.20 (Domain of definition, preimages, connected fibres). Let $f : X \dashrightarrow Z$ be a rational map between varieties. We denote the domain of definition of $\operatorname{Defn}(f) \subseteq X$ and write f_{Defn} for the morphism $f|_{\operatorname{Defn}(f)}$. Given any subset $Z' \subseteq Z$, write $f^{-1}(Z')$ as a shorthand for $f^{-1}_{\operatorname{Defn}}(Z')$. If $z \in Z$ is any point, call $X_z := f^{-1}(z)$ the *fibre over* z. We say that f *has connected fibres* if f_{Defn} has connected fibres.

Notation 2.21 (Horizontal and vertical divisors). Let $f : X \dashrightarrow Z$ be a rational map between normal varieties. If Δ is any prime divisor on X, observe that Δ intersects $\operatorname{Defn}(f)$ non-trivially. Call Δ *horizontal* if $\Delta \cap \operatorname{Defn}(f)$ dominates Z, otherwise call it *vertical*. If Δ is any $\mathbb{Q}$-Weil divisor, there exists an associated decomposition $\Delta = \Delta^{horiz} + \Delta^{vert}$.

Definition 2.22. *Let $f : X \dashrightarrow Z$ be a rational map between normal varieties. We say that f is* essentially equidimensional *if there exists an open set $U \subseteq \operatorname{Defn}(f)$ that is big in X such that $f|_U$ is an equidimensional morphism.*

Remark 2.23. Recall from Definition 2.10 that equidimensional morphisms (and hence essentially equidimensional maps) are dominant.

Definition 2.24 (Ramification and branch divisors for essentially equidimensional maps)**.** *Let $f : X \dashrightarrow Z$ be an essentially equidimensional rational map between normal varieties as in Definition 2.22. Define*

$$\operatorname{Ramification} f := \operatorname{Ramification}\bigl(f_{\operatorname{Defn}}|_U\bigr).$$

Ditto for Branch f *and* OrbiBranch f.

2.4.1. *Pull-back and push-forward.* In the rest of the current subsection we focus on the behaviour of cycles and sheaves under f_* and f^*, given that f is a rational map between normal varieties.

Construction 2.25 (Push-forward for rational map). Let $f : X \dashrightarrow Z$ be a rational map between normal varieties. Since $\operatorname{Defn}(f)$ is a big subset of X, we obtain a canonical identification $\operatorname{WDiv}\bigl(\operatorname{Defn}(f)\bigr) \cong \operatorname{WDiv}(X)$. Construction 2.5 therefore gives a push-forward map $f_* : \operatorname{WDiv}(X) \to \operatorname{WDiv}(Z)$.

Construction 2.26 (Pull-back of sheaves). Let $f : X \dashrightarrow Z$ be a rational map between normal varieties. If $\mathscr{F}$ is any coherent sheaf of $\mathscr{O}_Z$-modules, write

$$f^{[*]}\mathscr{F} := \bigl(\iota_* f^*_{\operatorname{Defn}}\mathscr{F}\bigr)^{**},$$

where $\iota : \operatorname{Defn}(f) \to X$ is the inclusion. Since $\operatorname{Defn}(f)$ is a big, open set, this is a coherent, reflexive sheaf on X.

Construction 2.27 (Pull-back of divisors for essentially equidimensional map). Let $f : X \dashrightarrow Z$ be an essentially equidimensional rational map. Item (2.6.2) of Remark 2.6 gives a canonical identification $\operatorname{WDiv}(U) \cong \operatorname{WDiv}(X)$ that respects linear equivalence. Construction 2.13 therefore gives a pull-back map $f^* : \operatorname{WDiv}(Z) \to \operatorname{WDiv}(X)$, which does not depend on the choice of U, respects linear equivalence, and therefore induces a morphism between divisor class groups.

Remark 2.28 (Pull-back for Weil divisorial sheaves). The pull-back Constructions 2.26 and 2.27 are compatible for Weil divisorial sheaves. More precisely, if $f : X \dashrightarrow Z$ is any essentially equidimensional rational map between normal varieties and if $D \in \operatorname{WDiv}(Z)$, then $f^{[*]}\mathscr{O}_Z(D) \cong \mathscr{O}_X\bigl(f^*D\bigr)$.

2.4.2. *Relative tangent sheaves.* The aim of this section is to establish an explicit description for the relative canonical sheaf of an essentially equidimensional rational map.

Construction 2.29 (Relative tangent sheaf). Let $f : X \dashrightarrow Z$ be an essentially equidimensional rational map between normal varieties. Recall from Remark 2.11 that there exists a big, open set $U \subseteq f^{-1}(Z_{\mathrm{reg}}) \cap X_{\mathrm{reg}}$ such that $f|_U : U \to Z_{\mathrm{reg}}$ is an equidimensional morphism. Denote the inclusion by $\iota : U \to X$, consider the kernel

$$\mathscr{T}_{U/Z_{\mathrm{reg}}} := \ker\big(\mathscr{T}_U \to (f|_U)^* \mathscr{T}_{Z_{\mathrm{reg}}}\big),$$

and set $\mathscr{T}_{X/Z} := \iota_* \mathscr{T}_{U/Z_{\mathrm{reg}}}$. By construction, $\mathscr{T}_{X/Z}$ is a reflexive subsheaf of $\mathscr{T}_X$, and in fact a foliation (see Notation 2.32 below). The sheaf $\mathscr{T}_{X/Z}$ is independent of the choice of U.

Construction 2.30 (Relative canonical class). Let $f : X \dashrightarrow Z$ be an essentially equidimensional rational map. Construction 2.27 allows to define the relative canonical class as $[K_{X/Z}] := [K_X] - f^*[K_Z] \in \mathrm{Cl}(X)$.

Lemma 2.31 (Determinant of relative tangent sheaf)**.** *In the setting of Construction 2.29,*

$$\det \mathscr{T}_{X/Z} \cong \mathscr{O}_X\big(-K_{X/Z} + \mathrm{Ramification}\, f\big).$$

Proof. Since both sides of the equation are reflexive sheaves, it suffices to show equality on the big, open subset of U where the morphism $f|_U$ can locally be written in normal form. There, the claim follows from an elementary computation in coordinates. ☐

2.5. **Foliations.** The notion of a foliation being transversal to a divisor is a recurrent theme in this chapter. Let us briefly spell out what is meant by this.

Notation 2.32 (Foliation). Let X be a normal variety. A *foliation* is a saturated subsheaf $\mathscr{T}_X$, whose restriction to X_{reg} is closed under the Lie-bracket.

Remark 2.33. In the setting of Notation 2.32, recall that the tangent sheaf $\mathscr{T}_X := (\Omega^1_X)^*$ is reflexive. As a saturated subsheaf, the foliation $\mathscr{F}$ is likewise reflexive.

Notation 2.34. For any saturated subsheaf $\mathscr{F}$ of $\mathscr{T}_X$, the Lie-bracket, which is defined only on X_{reg}, induces an $\mathscr{O}_X$-linear map

$$N : \mathscr{F}^{[2]} \to (\mathscr{T}_X / \mathscr{F})^{**}$$

known as the O'Neil tensor. Vanishing of N characterises then $\mathscr{F}$ as being a foliation.

Notation 2.35 (Divisors generically transveral to a foliation). In the setting of Notation 2.32, there exists a big open set $U \subseteq X$, contained in X_{reg} where $\mathscr{F}$ is a subvectorbundle of $\mathscr{T}_X$. If $D \subset X$ is any prime divisor, then $\mathscr{F}$ is as subvectorbundle of $\mathscr{T}_X$ near general points $x \in D$. In particular, one can check whether $\mathscr{F}$ is transversal to D at x. This allows to decompose any $\mathbb{Q}$-Weil divisor Δ as $\Delta = \Delta^{trans} + \Delta^{ntrans}$, where Δ^{trans} consists of those components that are generically transversal to $\mathscr{F}$.

Remark 2.36. In the setting of Construction 2.29, the decompositions of $\mathbb{Q}$-Weil divisors given in Notations 2.21 and 2.35 agree. More precisely, if Δ is any $\mathbb{Q}$-Weil divisor on X, then $\Delta^{horiz} = \Delta^{trans}$ and $\Delta^{vert} = \Delta^{ntrans}$.

2.6. **Adapted morphisms.** In this section we introduce a class of morphisms, called "adapted", that is indispensable to even formulate our main result.

Definition 2.37 (Adapted and strongly adapted cover cover)**.** *Let (X, Δ) be a pair and decompose Δ into irreducible components, $\Delta = \sum \delta_j D_j$. Let $\gamma : Y \to X$ be an essentially equidimensional morphism to a normal variety Y. The morphism γ is called* adapted to (X, Δ) *if $\gamma^* \Delta$ is an integral divisor. The morphism γ is called* strongly adapted to (X, Δ) *if for any index j with $\delta_j \in (0, 1)$, written as $\delta_j = \frac{a_j}{b_j}$ with $a_j, b_j \in \mathbb{N}$ coprime, and any Weil divisor $E \subsetneq Y$, we have*

$$\operatorname{mult}_E(\gamma^* D_j) \in \{0, b_j\}.$$

In other words, if E appears in $\gamma^* D_j$ at all, then its multiplicity must be b_j precisely.

2.6.1. *Existence.* The existence of adapted morphisms, when (X, Δ) is an snc pair, was established by Kawamata, cf. [Laz04b, Prop. 1.12]. Here we briefly review the case where (X, Δ) is *not* snc. See [CP15b, Subsect. 1.2] for an alternative construction.

Proposition 2.38 (Existence of strongly adapted, Galois covers)**.** *Let (X, Δ) be a pair. Then, there exists a cyclic Galois cover $\gamma : Y \to X$ that is strongly adapted to (X, Δ).*

Proof. Let $\pi : \widetilde{X} \to X$ be a log resolution of singularities, and consider the $\mathbb{Q}$-Weil divisor given as the strict transform, $\widetilde{D} := \gamma_*^{-1} \Delta$. The existence of cyclic Galois cover $\widetilde{\gamma} : \widetilde{Y} \to \widetilde{X}$ that is strongly adapted to $(\widetilde{X}, \widetilde{D})$ has been recalled in [JK11a, Prop. 2.9]. We obtain the desired covering by Stein factorisation of the composed morphism $\widetilde{Y} \to X$,

$$\begin{tikzcd} \widetilde{Y} \arrow[r, "\alpha"'] \arrow[rr, bend left, "\widetilde{\gamma} \circ \pi"] & \widetilde{X} \arrow[r, "\beta"'] & X. \end{tikzcd}$$

The equivariant version of Zariski's Main Theorem, [GKP16, Theorem A.1], guarantees that β is Galois, and that its Galois group equals that of $\widetilde{\gamma}$. □

2.6.2. *Relation to earlier definitions.* Definition 2.37 is equivalent to various other definitions of adapted morphisms that appear in the literature —it goes without saying that all are various takes on the original definition of Campana. To see this, it is convenient to first introduce the following definition of the round-up of a $\mathbb{Q}$-divisor.

Definition 2.39 ($\mathcal{C}$-round-up)**.** *Let (X, Δ) be a pair and decompose Δ into irreducible components, $\Delta = \sum \delta_j D_j$. If j is any index with $\delta_j \in (0, 1)$, write $\delta_j = \frac{a_j}{b_j}$ with $a_j, b_j \in \mathbb{N}$ coprime. Finally, set*

$$\lceil \delta_j \rceil_{\mathcal{C}} = \begin{cases} 0 & \text{if } \delta_j = 0 \\ 1 & \text{if } \delta_j = 1 \\ \frac{b_j - 1}{b_j} & \text{otherwise.} \end{cases}$$

We call the divisor $\lceil \Delta \rceil_{\mathcal{C}} := \sum_j \lceil \delta_j \rceil_{\mathcal{C}} \cdot D_j$ the $\mathcal{C}$-round-up *of Δ. If $\Delta = \lceil \Delta \rceil_{\mathcal{C}}$, we call (X, Δ) a $\mathcal{C}$-pair.*

Remark 2.40 (Comparison with earlier definitions). Let (X, Δ) be a pair and $\gamma : Y \to X$ a covering map. Then the following are equivalent.

(2.40.1) The cover γ is adapted to (X, Δ) in the sense of Definition 2.37.

(2.40.2) The cover γ is adapted to $(X, \lceil \Delta \rceil_{\mathcal{C}})$ in the sense of Definition 2.37.

(2.40.3) The morphism γ is adapted to $(X, \lceil \Delta \rceil_{\mathcal{C}})$ in the sense of [JK11a, Definition 2.7].

Ditto for strongly adapted covers.

2.7. **Numerical classes, positivity.** Over a $\mathbb{Q}$-factorial projective variety the determinant of any coherent sheaf naturally defines an element of $N^1(X)_{\mathbb{Q}}$. To avoid potentially cumbersome notations, let us fix a notation for such numerical classes.

Notation 2.41 (Numerical classes). Let X be a $\mathbb{Q}$-factorial, projective variety and $\mathscr{F}$ a coherent sheaf of $\mathscr{O}_X$-modules. Consider the Weil divisorial sheaf $\det \mathscr{F} := (\wedge^{\operatorname{rank}\mathscr{F}} \mathscr{F})^{**}$ —when $\mathscr{F}$ is torsion and its rank is zero, then $\det \mathscr{F}$ is nothing but the zero sheaf. The numerical class of $\det \mathscr{F}$ will be written as $[\mathscr{F}] \in N^1(X)_{\mathbb{Q}}$.

Warning 2.42 (Lack of additivity). Note that the numerical class operator $[\bullet]$ is not necessarily additive in exact sequences. In fact, since the reflexive hull of any torsion sheaf is zero, the ideal sheaf sequence of any Cartier divisor will give a counterexample.

Notation 2.43 (Harder-Narasimhan filtration, generic positivity). Let X be a normal, projective variety and H be an ample Cartier divisor on X. If $\mathscr{F}$ is any torsion free, coherent sheaf of $\mathscr{O}_X$-modules, consider the associated Harder-Narasimhan filtration

$$0 = \mathscr{F}_0 \subsetneq \mathscr{F}_1 \subsetneq \cdots \subsetneq \mathscr{F}_r = \mathscr{F}.$$

With this notation in place, write

$$\mu_H^{\max}(\mathscr{F}) := \mu_H(\mathscr{F}_1) \quad \text{and} \quad \mu_H^{\min}(\mathscr{F}) := \mu_H(\mathscr{F}_r/\mathscr{F}_{r-1}).$$

We call $\mathscr{F}$ *generically semipositive with respect to H* if $\mu_H^{\min}(\mathscr{F}) \geq 0$. We call $\mathscr{F}$ *generically semipositive* if $\mathscr{F}$ is generically semipositive with respect to any ample divisor.

Part I. Fractional semipositivity and application to hyperbolicity

3. LOGARITHMIC DIFFERENTIALS WITH FRACTIONAL POLE ORDER

In this section we define the sheaves of *adapted differential forms*. These sheaves are, in a sense, the natural generalisation of sheaves of log-differential forms for pairs (X, D) with reduced boundary divisor D, to the context of pairs (X, Δ) with $\Delta = \sum \delta_i \cdot \Delta_i$, where $\delta_i \in [0, 1] \cap \mathbb{Q}$ are fractional. Their construction depends on the choice of the adapted morphism. Campana realised that, even in the purely logarithmic setting of Viehweg's hyperbolicity problem, they provide great flexibility in dealing with birational problems. We begin this section by explaining the local description of these sheaves when (X, Δ) is snc.

3.1. **Informal explanation and local computation.** Throughout the present Section 3.1, we consider the following particularly simple setting.

Setting 3.1 (Setup and notation for Section 3.1). Let (X, Δ) be an an snc pair. Let $\gamma : Y \to X$ be a cover that is adapted to (X, Δ) and can locally be written in normal form. Assume that $\operatorname{supp}(\Delta + \operatorname{Branch}\gamma)$ and $\operatorname{supp}(\gamma^*\Delta + \operatorname{Ramification}\gamma)$ are both smooth. Choose a point $\vec{y} \in Y$ and set $\vec{x} := \gamma(y)$. Observe that if $\vec{x} \in \operatorname{supp}\Delta$, then there exists exactly one component of $D \subseteq \Delta$ that contains $\vec{x}$. Let δ be the coefficient of this component. If $\vec{x} \notin \operatorname{supp}\Delta$, set $\delta := 0$.

We choose coordinates

$$x_0, \ldots, x_n \in \mathscr{O}_{X,\vec{x}} \quad \text{and} \quad y_0, \ldots, y_n \in \mathscr{O}_{Y,\vec{y}},$$

centred about $\vec{x}$ and $\vec{y}$, respectively, that present γ in local normal form. In particular, there exists a number m such that $x_0 \circ \gamma = y_0^m$. If $\delta = 1$ and if γ happens to be unramified at $\vec{y}$, we may assume that locally near $\vec{x}$, the divisor D is given as $\{x_0 = 0\}$.

We are interested in writing formal fractional-exponent-differential forms $\sigma := x_0^{-\delta} \cdot dx_0$. While this is not well-defined on X, one *can* write down the formal pull-back of σ to Y; this will lead to the definition of adapted differentials. For the convenience of the reader, we discuss the cases where $\delta = 0$, where $0 < \delta < 1$ and where $\delta = 1$ separately.

3.1.1. *The case where* $0 < \delta < 1$. In this case, the divisor D is necessarily contained in the branch locus of γ, and locally given as $\{x_0 = 0\}$. One may formally write

$$(d\gamma)(\sigma) = y_0^{-m\cdot\delta} \cdot d(y_0^m) = m \cdot y_0^{(m-1)-m\delta} \cdot dy_0.$$

The assumption that γ is adapted ensures that $m \cdot \delta$ is integral, and hence so is the exponent of y_0. The fact that $\delta < 1$ ensures that the exponent is not negative. We aim to define a *sheaf of adapted differentials*, in symbols $\Omega^1_{(X,\Delta,\gamma)}$, as a subsheaf of Ω^1_Y, whose stalk at $\vec{y}$ is generated by the forms

$$y_0^{(m-1)-m\delta} \cdot dy_0 \quad \text{and} \quad dy_1, \ldots, dy_n \in \Omega^1_{Y,\vec{y}}.$$

Warning 3.2. It might seem tempting to take this as a definition for the sheaf of adapted differentials. However, the following example shows that this is quite delicate. Let Z be a smooth variety and H a smooth hypersurface on Z. Let $\vec{z} \in H$ be any and $z_0, \ldots, z_n \in \mathscr{O}_{Z,\vec{z}}$ a regular system of parameters, where z_0 generates the ideal of H. Then note that the span

$$\Big\langle z_0^2 \cdot dz_0,\, dz_1,\, \ldots,\, dz_n \Big\rangle \subset \Omega^1_{Z,\vec{z}}$$

does depend in a non-trivial way on the choice of coordinates. To give a proper definition, it will always be necessary to take the morphism γ into account.

In order to define $\Omega^1_{(X,\Delta,\gamma)}$ properly, in a coordinate-free way, we compare its set of generators-to-be to the well-known set of generators for the image of the pull-back map $d\gamma : \gamma^*\Omega^1_X \to \Omega^1_Y$,

$$\begin{aligned} \Omega^1_{(X,\Delta,\gamma),\vec{y}} &= \Big\langle y_0^{(m-1)-m\delta} \cdot dy_0,\, dy_1,\, \ldots,\, dy_n \Big\rangle && \subseteq \Omega^1_{Y,\vec{y}} \\ \operatorname{Image}(d\gamma)_{\vec{y}} &= \Big\langle y_0^{m-1} \cdot dy_0,\, dy_1,\, \ldots,\, dy_n \Big\rangle && \subseteq \Omega^1_{Y,\vec{y}}. \end{aligned}$$

This suggests to define $\Omega^1_{(X,\Delta,\gamma)}$ near the point $\vec{y}$ in one of the two following, equivalent ways,

$$\Omega^1_{(X,\Delta,\gamma)} := \Omega^1_Y \otimes \Big(y_0^{(m-1)-m\delta}\Big) + \operatorname{Image}(d\gamma) \tag{3.2.1}$$

$$\Omega^1_{(X,\Delta,\gamma)} := \Big(\operatorname{Image}(d\gamma) \otimes \big(y_0^{-m\delta}\big)\Big) \cap \Omega^1_Y. \tag{3.2.2}$$

Explanation 3.3 (Ideals in (3.2.1)). In (3.2.1), we view $\big(y_0^{(m-1)-m\delta}\big)$ as an ideal, view $\Omega^1_Y \otimes \big(y_0^{(m-1)-m\delta}\big)$ as a subsheaf of Ω^1_Y, and the sum is the sum of coherent subsheaves there.

Explanation 3.4 (Weil divisorial sheaves in (3.2.2)). In (3.2.2), we view $\big(y_0^{-m\delta}\big)$ as the Weil divisorial sheaf generated by the rational function $y_0^{-m\delta}$, view Ω^1_Y as a subsheaf of $\operatorname{Image}(d\gamma) \otimes \big(y_0^{-m\delta}\big)$, and the intersection is the intersection of coherent subsheaves there.

In order to avoid the awkward use of adapted coordinates, observe that the divisor given by y_0^{m-1} equals the ramification divisor of γ, while the divisor given by $y_0^{m\delta}$ is the pull-back divisor $\gamma^*\Delta$. Definitions (3.2.1)–(3.2.2) thus simplify as follows,

$$\Omega^1_{(X,\Delta,\gamma)} := \Omega^1_Y \otimes \mathscr{O}_Y\big(-\gamma^*\Delta - \operatorname{Ramification}(\gamma)\big) + \operatorname{Image}(d\gamma) \tag{3.4.1}$$

$$\Omega^1_{(X,\Delta,\gamma)} := \big(\operatorname{Image}(d\gamma) \otimes \mathscr{O}_Y(\gamma^*\Delta)\big) \cap \Omega^1_Y. \tag{3.4.2}$$

3.1.2. *The case where* $\delta = 0$. In this case, γ may or may not be ramified at $\vec{y}$. The form $\sigma = x_0^{-\delta} \cdot dx_0 = dx_0$ is an ordinary differential form, and so is its pull-back $(d\gamma)(\sigma) = m \cdot y_0^{m-1} \cdot dy_0$. We would then set

$$\Omega^1_{(X,\Delta,\gamma)} := \operatorname{Image}(d\gamma)$$

near the point $\vec{y}$. Observe that this definition agrees with (3.4.1)–(3.4.2) above.

3.1.3. *The case where* $\delta = 1$. In this case, γ may or may not be ramified at $\vec{y}$. If γ *is* ramified at $\vec{y}$, then the assumption that $\operatorname{supp}(\Delta + \operatorname{Branch}\gamma)$ is smooth implies that near $\vec{x}$, the divisor D equals the branch locus, and is given as $\{x_0 = 0\}$. The form $\sigma = x_0^{-1} \cdot dx_0 = d\log x_0$ is a logarithmic differential form, and so is its pull-back $(d\gamma)(\sigma) = d\log y_0$. In this case, we would like to define the sheaf of adapted differentials near $\vec{y}$ as

$$\Omega^1_{(X,\Delta,\gamma)} := \Omega^1_Y(\log \Delta_\gamma) = \gamma^*\Omega^1_X(\log\lfloor\Delta\rfloor), \quad \text{where } \Delta_\gamma := (\gamma^*\lfloor\Delta\rfloor)_{\mathrm{red}}.$$

Formulas (3.4.1)–(3.4.2) include this case after the following minor adjustment. In fact, extending the pull-back morphism $d\gamma$ to include logarithmic differentials, $d\gamma : \gamma^*\Omega^1_X(\log\lfloor\Delta\rfloor) \to \Omega^1_Y(\log\Delta_\gamma)$, we can write

$$\Omega^1_{(X,\Delta,\gamma)} := \Omega^1_Y(\log\Delta_\gamma) \otimes \mathscr{O}_Y\Big(-\gamma^*\{\Delta\} - \operatorname{Ramification}(\gamma)\Big) + \operatorname{Image}(d\gamma)$$

$$\Omega^1_{(X,\Delta,\gamma)} := \Big(\operatorname{Image}(d\gamma) \otimes \mathscr{O}_Y(\gamma^*\{\Delta\})\Big) \cap \Omega^1_Y(\log\Delta_\gamma).$$

These formulas will re-appear in the succeeding Section 3.2, where adapted differentials are formally introduced.

3.2. **Formal definition.** We now give a formal and coordinate-free definition of "adapted differentials", following the discussion of the previous subsection. A local description is also included.

3.2.1. *Adapted differentials for a good cover.* We define adapted differentials first for covers that satisfy all the assumptions of Setting 3.1. We call such covers *good*. To be more precise, the following definition will be used.

Definition 3.5. *Let* (X, Δ) *be a pair, and* $\gamma : Y \to X$ *be a cover. The cover is called* good *if the following properties hold.*

- *(3.5.1) The variety* X *and its subvariety* $\operatorname{supp}(\Delta + \operatorname{Branch}\gamma)$ *are smooth.*
- *(3.5.2) The variety* Y *and its subvariety* $\operatorname{supp}(\gamma^*\Delta + \operatorname{Ramification}\gamma)$ *are smooth.*
- *(3.5.3) The cover* γ *is adapted.*
- *(3.5.4) The cover* γ *can locally be written in normal form.*

As indicated above, we take (3.4.2) as the definition of adapted differentials, at least for good covers.

Definition 3.6 (Adapted differentials for good cover)**.** *Let* (X, Δ) *be a pair, and* $\gamma : Y \to X$ *a be good cover. Consider the pull-back map of logarithmic differentials,*

$$d\gamma : \gamma^*\Omega^1_X(\log\lfloor\Delta\rfloor) \to \Omega^1_Y(\log\Delta_\gamma), \quad \text{where } \Delta_\gamma := (\gamma^*\lfloor\Delta\rfloor)_{\mathrm{red}}.$$

The sheaf of adapted differentials on Y *is then defined as*

$$\Omega^1_{(X,\Delta,\gamma)} := \Big(\operatorname{Image}(d\gamma) \otimes \mathscr{O}_Y(\gamma^*\{\Delta\})\Big) \cap \Omega^1_Y(\log\Delta_\gamma),$$

where the intersection is the intersection of subsheaves in $\Omega^1_Y(\log \Delta_\gamma) \otimes \mathscr{O}_Y(\gamma^*\{\Delta\})$.

Remark 3.7 (Inclusions between sheaves of adapted differentials). In the setting of Definition 3.6, it follows immediately from the definition that there exist inclusions,

$$\gamma^*\Omega^1_X(\log\lfloor\Delta\rfloor) \;\subseteq\; \Omega^1_{(X,\Delta,\gamma)} \;\subseteq\; \Omega^1_Y(\log\Delta_\gamma), \tag{3.7.1}$$

satisfying the following properties.

(3.7.2) The first inclusion in (3.7.1) is an equality away from $\operatorname{supp}\gamma^*\{\Delta\}$.

(3.7.3) The three terms of (3.7.1) are equal away from $\operatorname{supp}\operatorname{Ramification}(f)$, and near $\operatorname{supp}\Delta_\gamma$.

(3.7.4) If the covering morphism γ is Galois, say with group G, then all sheaves appearing in (3.7.1) carry natural structures of G-sheaves, and the inclusions are inclusions of G-sheaves.

Remark 3.8 (Local description of adapted differentials). The inclusions (3.7.1) can be written down in local coordinates, near any given point $\vec{y} \in Y$. If γ is étale at $\vec{y}$, or if $\vec{y} \in \operatorname{supp}\Delta_\gamma$ then all three sheaves agree, and there is nothing much to do. Let us therefore assume that $\vec{y} \in \operatorname{Ramification}(f) \setminus \operatorname{supp}\Delta_\gamma$. Choose local coordinates as in Setup 3.1 and follow the notation introduced there.

Near $\vec{y}$ the sheaves $\Omega^1_Y(\log\Delta_\gamma)$ and Ω^1_Y agree, and so do $\gamma^*\Omega^1_X(\log\lfloor\Delta\rfloor)$ and $\gamma^*\Omega^1_X$. The sheaf Ω^1_Y is freely generated as an $\mathscr{O}_Y$-module by symbols $dy_0, \dots, dy_n$. The sheaves of (3.7.1) are then generated as follows,

$$\begin{array}{rclccccc} \gamma^*\Omega^1_X & = & \big\langle & y_0^{m-1}\cdot dy_0, & dy_1, & \dots, & dy_n & \big\rangle \\ \Omega^1_{(X,\Delta,\gamma)} & = & \big\langle & y_0^{(m-1)-m\delta}\cdot dy_0, & dy_1, & \dots, & dy_n & \big\rangle \\ \Omega^1_Y & = & \big\langle & dy_0, & dy_1, & \dots, & dy_n & \big\rangle. \end{array}$$

The following is now an immediate consequence of the local description.

Corollary 3.9 (Determinants and Chern classes of adapted differentials)**.** *In the setting of Definition 3.6, we have equalities of sheaves,*

$$\begin{aligned} \det\,\Omega^1_{(X,\Delta,\gamma)} &= \det\Big(\gamma^*\,\Omega^1_X(\log\lfloor\Delta\rfloor)\Big)\otimes\det\mathscr{O}_Y\big(\gamma^*\{\Delta\}\big) \\ &= \mathscr{O}_Y\big(\gamma^*(K_X+\lfloor\Delta\rfloor)+\gamma^*\{\Delta\}\big) \\ &= \mathscr{O}_Y\big(\gamma^*(K_X+\Delta)\big). \end{aligned}$$

In particular, $c_1\big(\Omega^1_{(X,\Delta,\gamma)}\big) = [\gamma^*(K_X+\Delta)]$. ☐

3.2.2. *Adapted reflexive differentials in the general setting.* We now extend the definition of the adapted differentials from good covers to arbitrary ones.

Definition 3.10 (Adapted reflexive differentials)**.** *Let* (X,Δ) *be a pair, and let* $\gamma : Y \to X$ *be a cover that is adapted to* (X,Δ)*. Let* $X^\circ \subseteq X$ *and* $Y^\circ \subseteq Y$ *be the maximal open sets where* f *can locally be written in normal form and where* $\operatorname{supp}(\Delta + \operatorname{Branch}\gamma)$ *and* $\operatorname{supp}(\gamma^*\Delta + \operatorname{Ramification}\gamma)$ *are both smooth; these are big open subsets of* X *and* Y*, respectively. Let* $\iota : Y^\circ \to Y$ *be the inclusion map, and set* $\Delta^\circ := \Delta|_{X^\circ}$ *and* $\gamma^\circ := \gamma|_{Y^\circ}$*. We define the* sheaf of adapted reflexive differentials on Y *as*

$$\Omega^{[1]}_{(X,\Delta,\gamma)} := \iota_*\,\Omega^1_{(X^\circ,\Delta^\circ,\gamma^\circ)}.$$

where $\Omega^1_{(X^\circ,\Delta^\circ,\gamma^\circ)}$ *is the sheaf that has been introduced in Definition 3.6 on the facing page.*

Remark 3.11 (Inclusions between sheaves of adapted reflexive differentials). In the setting of Definition 3.10, observe that $\Omega^1_{(X^\circ,\Delta^\circ,\gamma^\circ)}$ is locally free on Y°. It follows that the sheaf of adapted reflexive differentials is reflexive. Using (3.7.1), push-forward from Y° to Y induces inclusions of reflexive sheaves as follows,

$$\gamma^{[*]}\Omega^{[1]}_X(\log\lfloor\Delta\rfloor) \subseteq \Omega^{[1]}_{(X,\Delta,\gamma)} \subseteq \Omega^{[1]}_Y(\log\Delta_\gamma). \tag{3.11.1}$$

where $\Delta_\gamma := (\gamma^*\lfloor\Delta\rfloor)_{\mathrm{red}}$. Remark 3.7 and Corollary 3.9 also have direct analogues.

(3.11.2) The first inclusion in (3.11.1) is an equality away from $\operatorname{supp}\gamma^*\{\Delta\}$.

(3.11.3) The three terms of (3.11.1) are equal away from $\operatorname{supp}\operatorname{Ramification}(f)$, and near general points of $\operatorname{supp}\Delta_\gamma$.

(3.11.4) If the covering morphism γ is Galois, say with group G, then all sheaves appearing in (3.11.1) carry natural structures of G-sheaves, and the inclusions are inclusions of G-sheaves.

(3.11.5) We have an equality of sheaves,

$$\det\,\Omega^{[1]}_{(X,\Delta,\gamma)} = \mathscr{O}_Y\big(\gamma^*(K_X+\Delta)\big).$$

If $K_X+\Delta$ is $\mathbb{Q}$-factorial, then $c_1\big(\Omega^1_{(X,\Delta,\gamma)}\big) = [\gamma^*(K_X+\Delta)]$.

3.2.3. *Relation to earlier definitions.* If (X,Δ) is a $\mathcal{C}$-pair, the notion of "adapted differentials" agrees with the notion introduced in earlier papers of Campana *et al.*, cf. [JK11a, Sect. 2.D] and Campana's work referenced there.

4. FRACTIONAL TANGENTS AND FOLIATIONS

The aim of this section is to lay down the technical groundwork for Section 8. There, we construct a certain subsheaf of $\mathscr{T}_X$ and study its integrability properties. The key technical result here is Proposition 4.5, whose proof requires some preliminary observations about the local description of vector fields that are transversal (resp. tangential) to the branch locus of an adapted cover.

4.1. **Adapted tangents.**

We first define, in the obvious way, the notion of an adapted tangent sheaf, by dualizing the adapted sheaf of differentials.

Definition 4.1 (Adapted tangents)**.** *Given a pair (X,Δ) and an adapted cover $\gamma : Y \to X$, set*

$$\mathscr{T}_{(X,\Delta,\gamma)} := \big(\Omega^{[1]}_{(X,\Delta,\gamma)}\big)^*,$$

where $\Omega^{[1]}_{(X,\Delta,\gamma)}$ is the sheaf of adapted reflexive differentials that was introduced in Definition 3.10. We call $\mathscr{T}_{(X,\Delta,\gamma)}$ the adapted tangent sheaf *or* sheaf of adapted tangents.

Remark 4.2 (Inclusions between tangent sheaves). Dualising, Remark 3.11 yields inclusions

$$\mathscr{T}_Y(-\log\Delta_\gamma) \subseteq \mathscr{T}_{(X,\Delta,\gamma)} \subseteq \gamma^{[*]}\mathscr{T}_X(-\log\lfloor\Delta\rfloor). \tag{4.2.1}$$

where $\Delta_\gamma := (\gamma^*\lfloor\Delta\rfloor)_{\mathrm{red}}$. The following additional properties hold.

(4.2.2) The second inclusion in (4.2.1) is an equality away from $\operatorname{supp}\gamma^*\{\Delta\}$.

(4.2.3) The three terms of (4.2.1) are equal away from $\operatorname{supp}\operatorname{Ramification}(f)$, and near general points of $\operatorname{supp}\Delta_\gamma$.

(4.2.4) If the covering morphism γ is Galois, say with group G, then all sheaves appearing in (4.2.1) carry natural structures of G-sheaves, and the inclusions are inclusions of G-sheaves.

(4.2.5) We have an equality of sheaves,

$$\det \mathscr{T}_{(X,\Delta,\gamma)} = \mathscr{O}_Y\big(-\gamma^*(K_X+\Delta)\big).$$

If $K_X+\Delta$ is $\mathbb{Q}$-factorial, then $c_1\big(\mathscr{T}_{(X,\Delta,\gamma)}\big) = [-\gamma^*(K_X+\Delta)]$.

4.1.1. *Adapted tangents for good covers.* For a good adapted cover, we were able to give a complete descriptions of adapted differentials in Section 3.2.1. The following is the direct analogue of this for adapted tangents.

Remark 4.3 (Local description of adapted tangents). Setup as in Definition 4.1. If the cover γ is good, then the inclusions (4.2.1) can be written down in local coordinates, near any given point $\vec{y} \in Y$. If γ is étale at $\vec{y}$, or if $\vec{y} \in \operatorname{supp} \Delta_\gamma$ then all three sheaves agree, and there is nothing much to do. Let us therefore assume that $\vec{y} \in \operatorname{Ramification}(f) \setminus \operatorname{supp} \Delta_\gamma$. Choose local coordinates as in Setup 3.1 and follow the notation introduced there.

Near $\vec{y}$, the sheaves $\gamma^*\mathscr{T}_X(-\log\lfloor\Delta\rfloor)$ and $\gamma^*\mathscr{T}_X$ agree, and so do the sheaves $\mathscr{T}_Y\big(-\log(\gamma^*\lfloor\Delta\rfloor)_{\mathrm{red}}\big)$ and $\mathscr{T}_Y$. The sheaf $\gamma^*\mathscr{T}_X$ is freely generated as an $\mathscr{O}_Y$-module by symbols $\gamma^*\frac{\partial}{\partial x_0}, \dots, \gamma^*\frac{\partial}{\partial x_n}$. The sheaves of (4.2.1) are then generated as follows,

$$\begin{array}{rcllll}
\gamma^*\mathscr{T}_X & = & \langle & \gamma^*\frac{\partial}{\partial x_0}, & \gamma^*\frac{\partial}{\partial x_1}, & \dots, \quad \gamma^*\frac{\partial}{\partial x_n} \quad \rangle \\
\mathscr{T}_{(X,\Delta,\gamma)} & = & \langle & y_0^{m\delta}\cdot\gamma^*\frac{\partial}{\partial x_0}, & \gamma^*\frac{\partial}{\partial x_1}, & \dots, \quad \gamma^*\frac{\partial}{\partial x_n} \quad \rangle \\
\mathscr{T}_Y & = & \langle & y_0^{m-1}\cdot\gamma^*\frac{\partial}{\partial x_0}, & \gamma^*\frac{\partial}{\partial x_1}, & \dots, \quad \gamma^*\frac{\partial}{\partial x_n} \quad \rangle.
\end{array} \tag{4.3.1}$$

The local description has the following consequence, which will be relevant in the study of foliations.

Lemma 4.4. *In the setting of Definition 4.1, let* $\vec{V} \in H^0\big(X,\, \mathscr{T}_X(-\log\lfloor\Delta\rfloor)\big)$ *be any logarithmic vector field on* X*, with associated pull-back*

$$\gamma^*\vec{V} \in H^0\big(Y,\, \gamma^*\mathscr{T}_X(-\log\lfloor\Delta\rfloor)\big).$$

Then, the following holds.

(4.4.1) *If* $\vec{V}$ *is everywhere transversal to the smooth subvariety* $\operatorname{supp}(\{\Delta\} + \operatorname{Branch}\gamma)$*, then* $\gamma^*\vec{V}$ *generates the quotient* $\gamma^*\mathscr{T}_X(-\log\lfloor\Delta\rfloor)/\mathscr{T}_{(X,\Delta,\gamma)}$.

(4.4.2) *If* $\vec{V}$ *is everywhere tangential to the smooth subvariety* $\operatorname{supp}(\{\Delta\} + \operatorname{Branch}\gamma)$*, then* $\gamma^*\vec{V}$ *is contained in* $H^0\big(Y,\, \mathscr{T}_Y(-\log\Delta_\gamma)\big)$.

Proof. Both items can be shown locally, near given points $\vec{y} \in Y$. Again, we choose local coordinates as in Setup 3.1 and follow the notation introduced there. Write the vector field $\vec{V}$ and its pull-back locally as

$$\vec{V} = \sum_{i=0}^{n} f_i \cdot \frac{\partial}{\partial x_i} \quad \text{and} \quad \gamma^*\vec{V} = \sum_{i=0}^{n} (f_i \circ \gamma)\cdot\gamma^*\frac{\partial}{\partial x_i}$$

where $f_i \in \mathscr{O}_{X,\vec{x}}$.

Proof of (4.4.1). If $\vec{y}$ is not contained in $\operatorname{supp}\gamma^*\{\Delta\}$, then we have seen in Item (4.2.2) of Remark 4.2 that $\mathscr{T}_{(X,\Delta,\gamma)} = \gamma^*\mathscr{T}_X(-\log\lfloor\Delta\rfloor)$, so there is nothing to show. Let us therefore assume that $\vec{y} \in \operatorname{supp}\gamma^*\{\Delta\}$. This allows to use the local description of adapted tangent from Remark 4.3. The assumption that $\vec{V}$ is transversal implies that f_0 does not vanish at $\vec{x}$. But then $f_0 \circ \gamma$ will not vanish at $\vec{y}$, and the explicit description in (4.3.1) yields the claim.

Proof of (4.4.2). If $\vec{y}$ is contained in $\operatorname{supp} \Delta_\gamma$ or in the complement of $\operatorname{supp} \operatorname{Ramification}(f)$, then we have seen in Item (4.2.3) of Remark 4.2 that the three sheaves of (4.2.1) agree, and there is nothing to show. We will therefore assume without loss of generality that $\lfloor \Delta \rfloor = 0$, and that $\vec{y} \in \operatorname{Ramification}(f)$. The assumption that $\vec{V}$ is tangential implies that f_0 vanishes along $\{x_0 = 0\}$, so that $f_0 = x_0 \cdot g_0$ and

$$\gamma^* \vec{V} = y_0^m \cdot (g_0 \circ \gamma) \cdot \gamma^* \frac{\partial}{\partial x_0} + \sum_{i=1}^{n} (f_i \circ \gamma) \cdot \gamma^* \frac{\partial}{\partial x_i}.$$

Again, a comparison with the explicit description in (4.3.1) yields the claim. □

4.2. **Foliations.** In this subsection we use the local machinery developed in Subsection 4.1 to establish a technical tool that will play a significant role in the proof of Theorem 5.2.

4.2.1. *Lifting the O'Neil tensor.* A key problem in the proof of the main semipositivity result for a given pair (X, Δ) is to relate the integrability of a certain subsheaf $\mathscr{F}$ of $\mathscr{T}_X$ to the behaviour of the pull-back of the O'Neil tensor[3] on $\gamma^{[*]}\mathscr{F}$ and $\mathscr{G}_\gamma := \big(\gamma^{[*]}\mathscr{F} \cap \mathscr{T}_{(X,\Delta,\gamma)}\big)$. In the next proposition we show that the restriction of the lift of the O'Neil tensor maps the smaller sheaf $\mathscr{G}_\gamma$ to $\mathscr{T}_{(X,\Delta,\gamma)}/\mathscr{G}_\gamma$, after taking reflexive hulls.

Proposition 4.5 (Lifting the O'Neil tensor). *Let (X, Δ) be a pair and let $\gamma : Y \to X$ be an adapted cover. Let $\mathscr{F} \subseteq \mathscr{T}_X(-\log\lfloor\Delta\rfloor)$ be a saturated subsheaf. Consider reflexive sheaves on $\mathscr{G}_Y$, $\mathscr{G}_\gamma$ and $\mathscr{G}$ on Y as follows,*

$$\underbrace{\mathscr{T}_Y(-\log \Delta_\gamma)}_{\text{contains } \mathscr{G}_Y := \mathscr{G} \cap \mathscr{T}_Y} \quad \subseteq \quad \underbrace{\mathscr{T}_{(X,\Delta,\gamma)}}_{\text{contains } \mathscr{G}_\gamma := \mathscr{G} \cap \mathscr{T}_Y} \quad \subseteq \quad \underbrace{\gamma^{[*]}\mathscr{T}_X(-\log\lfloor\Delta\rfloor)}_{\text{contains } \mathscr{G} := \gamma^{[*]}\mathscr{F}}. \tag{4.5.1}$$

Next, consider the O'Neil tensor

$$N : \mathscr{F}^{[2]} \to \Big(\mathscr{T}_X(-\log\lfloor\Delta\rfloor) \Big/ \mathscr{F} \Big)^{**},$$

its reflexive pull-back,

$$\gamma^{[*]}N : \mathscr{G}^{[2]} \to \gamma^{[*]}\Big(\mathscr{T}_X(-\log\lfloor\Delta\rfloor) \Big/ \mathscr{F} \Big),$$

and write N_γ for the restriction of $\gamma^{[*]}N$ to the subsheaf $\mathscr{G}_\gamma^{[2]} \subseteq \mathscr{G}^{[2]}$. Then N_γ factorises as follows,

$$\begin{tikzcd}
\mathscr{G}_\gamma^{[2]} \arrow[r, "\alpha"'] \arrow[rr, bend left, "N_\gamma"] & \big(\mathscr{T}_{(X,\Delta,\gamma)} / \mathscr{G}_\gamma \big)^{**} \arrow[r, hook, "\beta"'] & \gamma^{[*]}\Big(\mathscr{T}_X(-\log\lfloor\Delta\rfloor) \Big/ \mathscr{F} \Big).
\end{tikzcd}$$

Proof. This is easier than the involved notation suggests. An elementary diagram chase shows that the natural morphism β is injective.

Step 1: Simplification. To prove that a morphism of reflexive sheaves factorises via a third, it suffices to prove the existence of a factorisation on a big open set. We may therefore assume that the following additional properties hold.

(4.5.2) The cover γ is good.
(4.5.3) The saturated subsheaf $\mathscr{F} \subseteq \mathscr{T}_X(-\log\lfloor\Delta\rfloor)$ is actually a subbundle.
(4.5.4) The subsheaf $\mathscr{G}_Y \subseteq \mathscr{T}_Y(-\log \Delta_\gamma)$ is a subbundle. Ditto for $\mathscr{G}_Y \subseteq \mathscr{T}_{(X,\Delta,\gamma)}$ and $\mathscr{G} \subseteq \gamma^{[*]}\mathscr{T}_X(-\log\lfloor\Delta\rfloor)$.

---

[3] As introduced in Notation 2.34.

(4.5.5) If $D \subseteq \operatorname{supp}(\Delta + \operatorname{Branch}\gamma)$ is any component, then either $\mathscr{F}$ is everywhere transversal to D, or everywhere tangent to D.

This simplifies notation greatly, as all sheaves in question will be locally free, so there is no need to take reflexive hulls in each step. There is more we can do. Recalling from Item (4.2.3) of Remark 4.2 that the three terms of (4.5.1) are equal near general points of $\operatorname{supp}\Delta_\gamma$, so that the claim of the proposition is certainly true there, we can also assume that the following holds.

(4.5.6) The integral part of Δ is empty, so $\lfloor\Delta\rfloor = 0$ and $\Delta_\gamma = 0$.

Again, this simplifies notation substantially, allowing us to drop all "$\log\lfloor\Delta\rfloor$" and "$\log\Delta_\gamma$" from sheaves of tangents and differentials.

Step 2: reduction to the local case. The statement of the proposition is clearly local; it suffices to prove the factorisation in an analytic neighbourhood of any given point $\vec{y} \in Y$. Again, if γ is étale at $\vec{y}$, then Item (4.2.3) of Remark 4.2 that the three terms of (4.5.1) are equal, and there is nothing to show. We will therefore assume that γ is ramified at $\vec{y}$. Set $\vec{x} = \gamma(\vec{y})$ and let $D \subseteq \operatorname{supp}(\Delta + \operatorname{Branch}\gamma)$ be the unique component that contains $\vec{x}$ —the component is unique because $\operatorname{supp}(\Delta + \operatorname{Branch}\gamma)$ is smooth by the assumption that γ is a good cover. Replacing X with a suitably small neighbourhood of $\vec{x}$, if need be, we can assume that the following holds in addition.

(4.5.7) The sheaf $\mathscr{F}$ is free, say generated by global sections $\sigma_1, \dots, \sigma_r$.
(4.5.8) The support of $\Delta + \operatorname{Branch}\gamma$ is irreducible, $D = \operatorname{supp}(\Delta + \operatorname{Branch}\gamma)$.
(4.5.9) If $\mathscr{F}$ is everywhere transversal to D, then σ_1 is everywhere transversal to D.

Step 3: Proof in case that $\mathscr{F}$ is everywhere transversal to D. The following diagram summarises the sheaves in question.

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{G}_\gamma \arrow[r] \arrow[d, hook] & \mathscr{G} \arrow[r] \arrow[d, hook] & \mathscr{G}/\mathscr{G}_\gamma \arrow[r] \arrow[d, "\eta"] & 0 \\
0 \arrow[r] & \mathscr{T}_{(X,\Delta,\gamma)} \arrow[r] & \gamma^*\mathscr{T}_X \arrow[r] & \gamma^*\mathscr{T}_X \big/ \mathscr{T}_{(X,\Delta,\gamma)} \arrow[r] & 0
\end{tikzcd}$$

The morphism η is injective by construction. Lemma 4.4 asserts that it is also surjective. More is true: Item (4.4.1) even asserts that the image is generated by the class of the section $\gamma^*\sigma_1$. The snake lemma thus implies that the natural map

$$\mathscr{T}_{(X,\Delta,\gamma)}/\mathscr{G}_\gamma \to \gamma^*\mathscr{T}_X \big/ \mathscr{G}$$

is isomorphic. The question of factorisation is therefore void and Proposition 4.5 is shown in case that $\mathscr{F}$ is everywhere transversal.

Step 4: Proof in case that $\mathscr{F}$ is everywhere tangential to D. If $\mathscr{F}$ is everywhere tangential to D, then Lemma 4.4 asserts that $\mathscr{G}$ is already contained in $\mathscr{T}_Y$, so that the sheaves $\mathscr{G}$, $\mathscr{G}_\gamma$ and $\mathscr{G}_Y$ are actually equal. The composed morphism μ,

$$\begin{tikzcd}
\mathscr{G}_Y^{[2]} \arrow[r, "\text{O'Neil tensor on } Y"'] \arrow[rr, bend left, "\mu"] & \mathscr{T}_Y \big/ \mathscr{G}_Y \arrow[r] & \gamma^*\mathscr{T}_X \big/ \mathscr{G}
\end{tikzcd}$$

clearly equals N_γ over the open set where γ is étale. Since all sheaves in question are locally free, hence torsion free, this means that N_γ equals the map μ everywhere. But μ factors as desired. This shows Proposition 4.5 in the last remaining case. □

4.2.2. *Chern classes.* When proving the main result of this chapter, we will need to pull-back foliations from the variety X to an adapted cover, saturate, intersect and keep track of how Chern classes change in the process. The following somewhat technical lemma summarises everything that we will need.

Proposition 4.6 (Chern classes of foliations). *Let (X, Δ) be a pair and let $\gamma : Y \to X$ be an adapted cover. Further, let $\mathscr{F} \subseteq \mathscr{T}_X$ be a foliation. Write $\Delta = \Delta^{trans} + \Delta^{ntrans}$, as in Notation 2.35 and consider the following sheaves.*

$$\begin{aligned} \mathscr{G} &:= \mathscr{F} \cap \mathscr{T}_X(-\log\lfloor\Delta\rfloor) && \ldots \textit{ saturated subsheaf of } \mathscr{T}_X(-\log\lfloor\Delta\rfloor) \\ \mathscr{H} &:= \gamma^{[*]}\mathscr{G} && \ldots \textit{ saturated subsheaf of } \gamma^{[*]}\mathscr{T}_X(-\log\lfloor\Delta\rfloor) \\ \mathscr{H}_{(X,\Delta,\gamma)} &:= \mathscr{H} \cap \mathscr{T}_{(X,\Delta,\gamma)} && \ldots \textit{ saturated subsheaf of } \mathscr{T}_{(X,\Delta,\gamma)} \end{aligned}$$

The determinant sheaves are then related as follows.

$$\det \mathscr{G} = \left[\det(\mathscr{F}) \otimes \mathscr{O}_X(-\lfloor\Delta^{trans}\rfloor)\right]^{**} \tag{4.6.1}$$

$$\det \mathscr{H} = \left[\det\big(\mathscr{H}_{(X,\Delta,\gamma)}\big) \otimes \mathscr{O}_Y\big(\gamma^*\{\Delta^{trans}\} + \textit{effective}\big)\right]^{**} \tag{4.6.2}$$

In summary,

$$\gamma^{[*]} \det \mathscr{F} = \left[\det\big(\mathscr{H}_{(X,\Delta,\gamma)}\big) \otimes \mathscr{O}_Y\big(\gamma^*\Delta^{trans} + \textit{effective}\big)\right]^{**}. \tag{4.6.3}$$

Proof. Equation (4.6.1) is elementary. Equation (4.6.3) is a combination of (4.6.1) and (4.6.2). It remains to prove (4.6.2). As an equation between reflexive sheaves, (4.6.2) can therefore be checked on a big open set. We may therefore assume that γ is a good cover. Recalling from Item (4.2.2) of Remark 4.2 that the sheaves $\mathscr{T}_{(X,\Delta,\gamma)}$ and $\gamma^{[*]}\mathscr{T}_X$ agree away from the support of $\gamma^*\{\Delta\}$, it will suffice to understand the difference between $\mathscr{H}$ and its subsheaf $\mathscr{H}_{(X,\Delta,\gamma)}$ near a given point $\vec{y}$ in the support of $\gamma^*\{\Delta^{trans}\}$. There, the statement follows from the local description (4.3.1) of Remark 4.3, using Item (4.4.1) of Lemma 4.4 as we have done in the proof of Proposition 4.5. ☐

5. FRACTIONAL SEMIPOSITIVITY

A celebrated result of Miyaoka, [Miy87, Cor. 8.6], shows that for a smooth projective variety X (or more generally a normal projective variety with only canonical singularities) positivity properties of the canonical sheaf ω_X are deeply related to those of the sheaf of (pluri-)differential forms. More precisely, if K_X is pseudo-effective, then, for every positive integer m and ample divisor $H \subset X$, the sheaf $(\Omega^1_X)^{\otimes m}$ is semipositive with respect to H. In other words, $c_1(\mathscr{Q}) \cdot [H]^{n-1} \geq 0$, for all coherent quotients $\mathscr{Q}$ of $(\Omega^1_X)^{\otimes m}$. When $c_1(X) = 0$ or < 0, these results can be traced back to Yau's theorem on the existence of Kähler-Einstein metrics, [Yau77]. Miyaoka's approach, on the other hand, is purely algebraic and involves deep and delicate characteristic p methods. Campana and Păun extend Miyaoka's results to the context of pairs. Before we state their result in Theorem 5.2 below, we recall the definition of generic semipositivity with respect to an adapted cover.

Definition 5.1 (Generic semipositivity w.r.t. an adapted cover, cf. [CP15b, Def. 1.7]). *Let (X, Δ) be a projective pair where X, and let $\gamma : Y \to X$ be an adapted morphism that is Galois with group G. We say that $\Omega^{[1]}_{(X,\Delta,\gamma)}$ is γ-*generically semipositive *if it is generically semipositive with respect to any ample divisor of the form $\gamma^*(ample)$.*

Theorem 5.2 (Generic semipositivity of $\Omega^{[1]}_{(X,\Delta,\gamma)}$, cf. [CP15b, Thm. 2.1]). *Let (X, Δ) be a log canonical, projective pair, and let $\gamma : Y \to X$ be an adapted cover that is Galois with group G. If $K_X + \Delta$ is pseudo-effective, then $\Omega^{[1]}_{(X,\Delta,\gamma)}$ is γ-generically semipositive.*

Part II of this chapter is devoted to a proof of Theorem 5.2. After some preparatory sections, the proof is given in Section 9 on page 27.

6. APPLICATION TO HYPERBOLICITY

As recalled in the introduction, the first step in proving Viehweg's hyperbolicity Conjecture 1.1 was carried out by Viehweg and Zuo in Theorem 1.2, where it was shown that the variation in a smooth family of canonically-polarised manifolds $f^\circ : X^\circ \to Y^\circ$ manifests itself as a birationally-positive subsheaf $\mathscr{L}$ of pluri-log forms. The aim of this section is to use Theorem 5.2 to extract bigness of the log canonical divisor $K_Y + D$ from the positivity of $\mathscr{L}$. This is the content of the following theorem of Campana and Păun.

Theorem 6.1 (Existence of pluri-log-canonical forms, cf. [CP15b, Thm. 4.1])**.** *Let (X, D) be a projective snc pair, where D is reduced. Let $N \in \mathbb{N}^+$ be a positive integer and let*

$$\mathscr{L} \subseteq \bigotimes^{N} \Omega^1_X \log(D)$$

be an invertible subsheaf. If $\kappa(X, \mathscr{L}) = \dim X$, then $\kappa(K_X + D) = \dim X$.

6.1. **Preparation for the proof of Theorem 6.1.** Before we sketch the proof of Theorem 6.1 we gather some technical details in the following lemma, whose proof is contained in the arguments of [CP15b, Sect. 4] or those of [Taj16, Thm 5.2] and [Taj16, Claim 5.2.1].

Lemma 6.2. *Setting as in Theorem 6.1. If $\kappa(X, \mathscr{L}) = \dim X$, then there exist $\mathbb{Q}$-divisors B_D and $(D_m)_{m \in \mathbb{N}^+}$ on X and a positive integer M such that the following holds for all integers $m \geq M$.*

(6.2.1) *The divisor D_m is big. Its round-down $\lfloor D_m \rfloor$ is zero.*
(6.2.2) *There exists a $\mathbb{Q}$-linear equivalence $D + \frac{1}{m} \cdot B_D \sim_{\mathbb{Q}} D_m$.*
(6.2.3) *The two pairs $\left(X, D + \frac{1}{m} \cdot B_D\right)$ and (X, D_m) are dlt with simple normal crossing support.*
(6.2.4) *The divisor $K_X + D + \frac{1}{m} \cdot B_D$ is pseudo-effective.* □

6.2. **Proof of Theorem 6.1.** Let B_D and $(D_m)_{m \in \mathbb{N}^+}$ be the divisors of Lemma 6.2. The properties listed in Lemma 6.2 allow us to use [BCHM10, Thm. 1.1] and to conclude that the log-minimal model programs for the pairs (X, D_m) terminate. In other words, given one number m, there exists a birational map $\pi : X \dashrightarrow X'_m$ consisting of a finite number of flips and divisorial contractions, such that (X'_m, D'_m) is klt and $K_{X'_m} + D'_m$ is nef, where D'_m denotes the strict transform of D_m. Now let us fix some notations.

Set-up. We resolve the indeterminancies of π by blowing up. More precisely, consider a commutative diagram of birational maps and morphisms

$$\begin{tikzcd} & \widetilde{X}_m \arrow[dl, bend right, "\mu"'] \arrow[dr, bend left, "\widetilde{\pi}"] & \\ X \arrow[rr, dashed, "\pi"'] & & X'_m \end{tikzcd}$$

such that the following holds.

(6.3.1) The variety $\widetilde{X}_m$ is smooth.
(6.3.2) The μ-exceptional set is of pure codimension one in $\widetilde{X}$. Let E denote the associated reduced divisor.

(6.3.3) The morphism μ is a log resolution. In other words, the divisor $\widetilde{D}_m + \widetilde{B_{Dm}} + E$ has simple normal crossing support, where $\widetilde{D}_m$ and $\widetilde{B_{Dm}}$ denote the strict transforms of the appropriate divisors on X.

Now, as L is big, we know, thanks to the Kodaira's lemma, cf. [Laz04a, Prop. 2.2.6], that given an ample divisor $A \subset X$, for a sufficiently positive integer a, we have

$$a \cdot L \geq A. \tag{6.3.4}$$

Let $\widetilde{A} := \mu^* A$. Furthermore, define

$$\widetilde{\mathscr{L}} := \mu^*(\mathscr{L}) \subseteq \bigotimes^N \Omega^1_{\widetilde{X}_m} \log(\widetilde{D}_m + E),$$

and choose a divisor $L \in \operatorname{Div}(X)$ for $\mathscr{L}$, with pull-back $\widetilde{L} = \mu^*(L)$.

Claim 6.4. There exists a positive integer $c := c(X, \mathscr{L}, A)$ such that for every $m \geq M$ (the integer M is the one defined in Lemma 6.2), the inequality

$$c \cdot \operatorname{vol}(K_{X'_m} + D'_m + \frac{1}{r} H_m) \geq \widetilde{A} \cdot \widetilde{\pi}^*(K_{X'_m} + D'_m + \frac{1}{r} H_m)^{n-1} \tag{6.4.1}$$

holds for every ample divisor $H_m \subset X'_m$ and $r \in \mathbb{N}^+$.

Proof of Claim 6.4. The pseudo-effectivity of $K_X + D + \frac{1}{m} \cdot B_D$ implies that of $\mu^*(K_X + D + \frac{1}{m} \cdot B_D)$ and as $\big(X, D + \frac{1}{m} \cdot B_D\big)$ is dlt, we find that the divisor

$$K_{\widetilde{X}_m} + \widetilde{D}_m + \frac{1}{m} \cdot \widetilde{B_{Dm}} + E$$

is also pseudo-effective. Applying the semi-positivity result, Theorem 5.2, to the pair $(\widetilde{X}_m, \widetilde{D}_m + \frac{1}{m} \cdot \widetilde{B_{Dm}} + E)$ leads us to the inequality:

(6.4.2)
$$\left[c \cdot \left(K_{\widetilde{X}_m} + \widetilde{D}_m + \frac{1}{m} \cdot \widetilde{B_{Dm}} + E\right) - a \cdot \widetilde{L}\right] \cdot \left[\widetilde{\pi}^*(K_{X'_m} + D'_m + \frac{1}{r} H_m)\right]^{n-1} \geq 0,$$

where $c := aN \cdot (n^{aN})^{aN-1}$, $m \geq M$ and $r \in \mathbb{N}^+$. This can be seen by considering a morphism $\widetilde{\gamma}_m : \widetilde{Y} \to \widetilde{X}_m$ adapted to the divisor $\widetilde{D}_m + \frac{1}{m} \cdot \widetilde{B_{Dm}} + E$, the exact sequence of sheaves

$$0 \to \mathscr{F} \longrightarrow \bigotimes^{aN} \Omega^1_{(\widetilde{X}_m, \widetilde{D}_m + \frac{1}{m}\widetilde{B_{Dm}} + E, \widetilde{\gamma}_m)} \to \mathscr{Q} \to 0,$$

where $\mathscr{F}$ is the saturation of $\gamma_m^*(\widetilde{\mathscr{L}^{\otimes a}})$ inside $\bigotimes^{aN} \Omega^1_{(\widetilde{X}_m, \widetilde{D}_m + \frac{1}{m}\widetilde{B_{Dm}} + E, \widetilde{\gamma}_m)}$, and noticing that, thanks to Theorem 5.2, the torsion free sheaf $\mathscr{Q}$ verifies the inequality:

$$[\mathscr{Q}] \cdot \widetilde{\gamma}_m^*\big(\pi^*(K_{X'_m} + D'_m + \frac{1}{r} H_m)\big)^{n-1} \geq 0.$$

Now, let $(X'_m)^\circ$ be the maximal open subset where π defines an isomorphism. As $(K_{X'_m} + D'_m + \frac{1}{r} H_m)$ is ample, we can always find a complete intersection curve $C \subset X'_m$, cut-out by general members of $|m \cdot (K_{X'_m} + D'_m + \frac{1}{r} H_m)|$ such that $C \subset (X'_m)^\circ$. As a result, we can rewrite Inequality 6.4.2 as follows.

$$c \cdot (K_{X'_m} + D'_m) \cdot (K_{X'_m} + D'_m + \frac{1}{r} H_m)^{n-1} \geq \widetilde{A} \cdot \widetilde{\pi}^*(K_{X'_m} + D'_m + \frac{1}{r} H_m)^{n-1}. \tag{6.4.3}$$

Claim 6.4 now follow immediately from Inequality 6.4.3.

Now, According to Teissier inequality for nef divisors, cf. [Laz04a, Thm 1.6.1], Inequality (6.4.1) implies that

$$c \cdot \mathrm{vol}(K_{X'_m} + D'_m + \frac{1}{r}H_m) \geq \mathrm{vol}(\widetilde{A})^{1/n} \cdot \mathrm{vol}(K_{X'_m} + D'_m + \frac{1}{r}H_m)^{(n-1)/n},$$

i.e.

$$\mathrm{vol}(K_{X'_m} + D'_m + \frac{1}{r}H_m) \geq \frac{1}{c^n} \cdot \mathrm{vol}(\widetilde{A}).$$

By taking $r \to \infty$, we find that $\mathrm{vol}(K_{X'_m} + D'_m) \geq (1/c^n)\,\mathrm{vol}(\widetilde{A})$. On the other hand, we know, thanks to the negativity lemma in the minimal model program, that $\mathrm{vol}(K_{X'_m} + D'_m) = \mathrm{vol}(K_X + D_m)$ and as $\mathrm{vol}(\widetilde{A}) = \mathrm{vol}(A)$, we have $\mathrm{vol}(K_X + D + \frac{1}{m}B_D) \geq (1/c^n)\,\mathrm{vol}(A)$. The theorem now follows by taking $m \to \infty$.

Part II. Proof of the semipositivity result

7. POSITIVITY OF RELATIVE DUALISING SHEAVES

As we shall see in Section 8, the orbifold generic semipositivity result, Theorem 5.2, is proved by contradiction. More precisely, given a pair (X, Δ) with pseudo-effective $K_X + \Delta$, and after integrability considerations (Subsection 8.1), the existence of a subsheaf of $\mathscr{T}_{(X,\gamma,\Delta)}$ with positive slope leads to an algebraic foliation on X. The negativity properties of the relative canonical sheaf of the rational map associated to this foliation brings about the required contradiction to the pseudo-effectivity assumption of $K_X + \Delta$.

Theorem 7.1 ([CP15b, Thm. 2.11]). *Let $f : X \dashrightarrow Z$ be a rational map with connected fibres between normal, projective varieties. Assume that f is essentially equidimensional*[4]*, that X is $\mathbb{Q}$-factorial, that there exists a $\mathbb{Q}$-Weil divisor Δ on X such that (X, Δ) is log canonical, and that $K_X + \Delta$ is pseudo-effective. If $(C_t)_{t \in T}$ is a family of curves that dominates X and avoids small sets*[5]*, then*

$$\big[K_{X/Z} + \Delta^{horiz} - \mathrm{Ramification}\, f\big] \cdot [C_t] \geq 0, \quad \textit{for all } t \in T.$$

Remark 7.2 ($\mathbb{Q}$-factoriality in Theorem 7.1). The assumption that X is $\mathbb{Q}$-factorial is posed for notational convenience. It implies that the intersection numbers in the displayed formula are well-defined. Note, however, that almost all curves in the family $(C_t)_{t \in T}$ stay away from the singularities of X. Restricting to these curves, and replacing X with a suitable log-resolution, it is possible to obtain a more general result at the cost of additional and more complicated notation.

Theorem 7.1 is a consequence of positivity results for direct images of relative dualising sheaves, which we present in Theorem 7.3 in the form of a pseudo-effectivity result for the relative dualising sheaf. These results have a long history, cf. [Hö10] for a survey in the setting where $\Delta = 0$. In case where the coefficients of Δ are of the form $\frac{m-1}{m}$, the relevant positivity results for the push-forward of $\mathscr{O}_X(m \cdot (K_{X/Z} + \Delta))$ appear in [Lu02, Sect. 9] and [Cam04, Thm. 4.11]; the paper [BP08] approaches the case where Δ is integral by analytic methods. The general case, where the coefficients of Δ are arbitrary, is treated in [Fuj14, Thm. 1.1] and (again in the analytic setting) in [PT14, Cor. 5.2.1]. For notational convenience, we choose [Fuj14] as our main reference.

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[4] See Definition 2.22 on page 7.

[5] See Notation 2.2 on page 4.

Theorem 7.3 (Pseudo-effectivity of relative dualising sheaves)**.** *Let $f : X \to Z$ be a surjective morphism with connected fibres between smooth, projective varieties. Let Δ be a $\mathbb{Q}$-divisor with snc support on X such that (X, Δ) is log canonical. Assume further that the restriction of $K_{X/Z} + \Delta$ to the general f-fibre is is pseudo-effective. Then $K_{X/Z} + \Delta$ is pseudo-effective.*

Proof. Choose a very ample prime divisor H on X that is general in its linear systems. For every sufficiently small rational number $0 < \varepsilon \ll 1$, the divisor $\Delta + \varepsilon \cdot H$ will then have snc support, the pair $(X, \Delta + \varepsilon \cdot H)$ will be log canonical and if $F \subseteq X$ is any general f-fibre, then $(K_{X/Z} + \Delta + \varepsilon \cdot H)|_F$ will be big. Choose a number $m \gg 0$ such that the divisor $m \cdot (K_{X/Z} + \Delta + \varepsilon \cdot H)$ is integral and such that

$$\mathscr{E} := f_* \mathscr{O}_X\big(m \cdot (K_{X/Z} + \Delta + \varepsilon \cdot H)\big)$$

is of positive rank. The twisted weak positivity theorem of [Fuj14, Thm. 1.1] asserts that $\mathscr{E}$ is weakly positive. In other words, cf. [Fuj14, Sect. 7], there exists a dense, Zariski-open set $Y^\circ \subseteq Y$ such that $\mathscr{E}$ is weakly positive over U, in the sense of Viehweg, [Vie95, Defn. 2.11]. Its pull-back $f^*\mathscr{E}$ is then likewise weakly positive, [Vie95, Lem. 2.15], and so is the target of the natural, non-trivial morphism

$$f^*\mathscr{E} = f^* f_* \mathscr{O}_X\big(m \cdot (K_{X/Z} + \Delta + \varepsilon \cdot H)\big) \to \mathscr{O}_X\big(m \cdot (K_{X/Z} + \Delta + \varepsilon \cdot H)\big),$$

cf. [Vie95, Lem. 2.16]. Since the target is invertible, it follows immediately from the definition of "weakly positive" that the $\mathbb{Q}$-divisor $K_{X/Z} + \Delta + \varepsilon \cdot H$ is pseudo-effective, [Fuj14, Rem. 7.6]. Conclude by taking the limit $\varepsilon \to 0$. □

7.1. **Proof of Theorem 7.1.** The proof of Theorem 7.1 essentially consists of two parts. Part one is focused on modifying the rational map $f : X \dashrightarrow Z$ and its subsequent replacement by a morphism that fits the premise of Theorem 7.3. This is roughly the content of Step. 1–3 and finally Step. 4 (see Consequences. 7.8 and 7.9). In Step. 5 it then becomes evident that Theorem 7.1 is an immediate consequence of Theorem 7.3.

Step 1: Resolution and base change. Following the construction steps outlined below, we construct a commutative diagram of morphisms and maps as follows,

$$\begin{tikzcd}
\widetilde{X} \arrow[rr, "a", "\text{log resolution of fibre product}"'] \arrow[d, "\tilde{f}"'] & & \bar{X} \arrow[r, "b", "\text{resol. of } f \text{ and } (X{,}\Delta)"'] \arrow[d, "\bar{f}"'] & X \arrow[d, dashed, "f"] \\
\widetilde{Z} \arrow[r, "\alpha", "\text{strong log resol.}"'] & \widehat{Z} \arrow[r, "\beta", "\text{strongly adpt. cover for } (Z{,}\,\mathrm{OrbiBranch}(f))"'] & Z \arrow[r, equal] & Z.
\end{tikzcd}$$

(7.3.1) Choose a strong log resolution of the morphism f and the pair (X, Δ). We obtain a smooth variety $\bar{X}$ and birational morphism $b : \bar{X} \to X$ from a normal variety that is isomorphic over $\mathrm{Defn}(f) \cap (X, \Delta)_{\mathrm{snc}}$, such that $\bar{f} := f \circ b$ is a morphism and such that the b-exceptional locus E^b as well as $E^b + b_*^{-1}\Delta$ are divisors with simple normal crossing support.

(7.3.2) Consider the divisor $\mathrm{OrbiBranch}(f)$ that was introduced in Definition 2.24 on page 7. Proposition 2.38 allows to choose a strongly adapted cover $\beta : \widehat{Z} \to Z$ that is associated with the pair $\big(Z, \mathrm{OrbiBranch}(f)\big)$. Since β is finite, Construction 2.13 allows to consider the pull-back divisor $\beta^* \mathrm{OrbiBranch}(f)$.

(7.3.3) Choose a strong log resolution of the pair $\big(\widehat{Z}, \beta^* \mathrm{OrbiBranch}(f)\big)$. We obtain a smooth variety $\widetilde{Z}$ and a birational morphism $\alpha : \widetilde{Z} \to \widehat{Z}$ that is isomorphic wherever $\big(\widehat{Z}, \beta^* \mathrm{OrbiBranch}(f)\big)$ is snc. Both the α-exceptional locus E^α as well as $E^\alpha + \alpha_*^{-1}\big(\beta^* \operatorname{OrbiBranch}(f)\big)$ are divisors with simple normal crossing support.

(7.3.4) Choose a strong log resolution of the fibre product $\widetilde{Z} \times_Z \bar{X}$. We obtain a smooth variety $\widetilde{X}$. Composed with the projection to the second factor, the resolution yields a generically finite morphism $a : \widetilde{X} \to \bar{X}$. Let $E^{b\circ a} \subseteq \widetilde{X}$ be the union of those divisors that are contracted by $b \circ a$. Then, both $E^{b\circ a}$ as well as $E^{b\circ a} + a^*(b_*^{-1}\Delta)$ are divisors with simple normal crossing support.

Step 2: Open sets and local normal forms. Let $Z^\circ \subsetneq Z$ and $X^\circ \subsetneq f^{-1}(Z^\circ)$ be the maximal open sets such that the following holds.

(7.4.1) The pairs $(X^\circ, \Delta + \operatorname{Ramification} f)$ and $(Z^\circ, \operatorname{Branch} f)$ are both snc.

(7.4.2) The morphism $f|_{X^\circ}$ is equidimensional and can locally be written in normal form, in the sense of Notation 2.19.

(7.4.3) Setting $\widetilde{Z}^\circ := (\beta\circ\alpha)^{-1}(Z^\circ)$, the morphism $(\beta\circ\alpha)|_{\widetilde{Z}^\circ} : \widetilde{Z}^\circ \to Z^\circ$ is finite and can locally be written in normal form.

(7.4.4) Setting $\bar{X}^\circ := b^{-1}(X^\circ)$, the morphism $b^\circ := b|_{\bar{X}^\circ} : \bar{X}^\circ \to X^\circ$ is isomorphic. In particular, $\bar{f}|_{\bar{X}^\circ} : \bar{X}^\circ \to Z^\circ$ can locally be written in normal form.

(7.4.5) Setting $\widetilde{X}^\circ := a^{-1}(\bar{X}^\circ)$, the morphism $a^\circ := a|_{\widetilde{X}^\circ} : \widetilde{X}^\circ \to \bar{X}^\circ$ is finite and can locally be written in normal form.

Observation 7.5. Recall from Construction 2.18 and from Zariski's main theorem that Z° and X° are big open sets of Z and X, respectively.

Step 3: Adjunction for the morphism b. Decompose the b-exceptional divisor E^b into irreducible components, $(E_i^b)_{i\in I_b}$. Since (X,Δ) is lc, the standard adjunction for the morphism b reads

$$K_{\bar{X}} + b_*^{-1}\Delta = b^*\big(K_X + \Delta\big) + \sum a_i E_i^b, \quad \text{with all } a_i \geq -1.$$

The $\mathbb{Q}$-divisor

$$\bar{\Delta} := b_*^{-1}\Delta - \sum_{a_i<0} a_i E_i^b$$

is effective with coefficients from the interval $[0,1]\cap\mathbb{Q}$, and has simple normal crossings support. The pair $(\bar{X},\bar{\Delta})$ is thus log canonical, and

$$K_{\bar{X}} + \bar{\Delta} = b^*\big(\underbrace{K_X + \Delta}_{\text{psef by assumpt.}}\big) + \sum_{a_i>0} a_i E_i^b$$

is again pseudo-effective. To end with Step 3, set $\bar{\Delta}^h := \bar{\Delta} - b_*^{-1}\Delta^{vert}$ and observe that if $\bar{F} \subsetneq \bar{X}$ is a general $\bar{f}$-fibre, then $\bar{F}$ is disjoint from the support of $b_*^{-1}\Delta^{vert}$. The following is thus an immediate consequence.

Observation 7.6. We have $\bar{\Delta}^h|_{\bar{X}^\circ} = (b^\circ)^*\Delta^{horiz}$. The pair $(\bar{X},\bar{\Delta}^h)$ is log-canonical and the restricted divisor $\big(K_{\bar{X}} + \bar{\Delta}^h\big)|_{\bar{F}}$ is pseudo-effective. ☐

Step 4: Adjunction for the morphism a. Using Items (7.4.3) and (7.4.4), the local normal form of $(\beta\circ\alpha)|_{\widetilde{Z}^\circ}$ and $\bar{f}|_{\bar{X}^\circ}$, as well as the construction of β as a strongly adapted cover, an elementary computation in local coordinates gives the following $\mathbb{Q}$-linear equivalence,

$$K_{\widetilde{X}^\circ/\widetilde{Z}^\circ} \sim_{\mathbb{Q}} (b^\circ\circ a^\circ)^*\big(K_{X^\circ/Z^\circ} - \operatorname{Ramification}(f)\big).$$

A similar equation holds for pairs. For a precise formulation, let $\widetilde{\imath} : \widetilde{X}^\circ \to \widetilde{X}$ be the obvious inclusion map and set

$$\widetilde{\Delta}^h := \widetilde{\imath}_* \Big((a^\circ)^* (\bar{\Delta}^h) \Big),$$

where the push-forward $\widetilde{\imath}_*$ is taken in the sense of Construction 2.5 and Remark 2.6. It follows from the construction of the morphism a in (7.3.4) that every component of $\widetilde{\Delta}^h$ dominates $\widetilde{Z}$, and that no component of $\widetilde{\Delta}^h|_{\widetilde{X}^\circ}$ is contained in the ramification locus of the finite morphism a°. Likewise, if $\widetilde{F} \subsetneq \widetilde{X}$ is any general fibre of $\widetilde{f}$, it follows from construction that a is étale near $\widetilde{F}$. The following are thus immediate consequences of Observation 7.6.

Consequence 7.8. The pair $(\widetilde{X}, \widetilde{\Delta}^h)$ is log-canonical and satisfies

$$\big(K_{\widetilde{X}/\widetilde{Z}} + \widetilde{\Delta}^h\big)|_{\widetilde{X}^\circ} \sim_{\mathbb{Q}} (b^\circ \circ a^\circ)^* \big(K_{X^\circ/Z^\circ} - \operatorname{Ramification}(f)\big). \qquad \square$$

Consequence 7.9. The restricted divisor $\big(K_{\widetilde{X}} + \widetilde{\Delta}^h\big)|_{\widetilde{F}}$ is pseudo-effective. ☐

Step 5: End of proof. The family $(C_t)_{t \in T}$ avoids small sets by assumption. Together with Observation 7.5, this means that if $t \in T$ is general, then the curve C_t is contained in X°. By (7.4.5), its preimage $\widetilde{C}_t := (b \circ a)^{-1}(C_t)$ is a curve in $\widetilde{X}^\circ$ whose components are movable curves in $\widetilde{X}$. Since the cycle-theoretic push-forward $(b \circ a)_*(\widetilde{C}_t)$ is a positive multiple of C_t, we obtain

$$\begin{aligned} 0 &\leq \big(K_{\widetilde{X}/\widetilde{Z}} + \widetilde{\Delta}^h\big) \cdot \widetilde{C}_t && \text{Thm. 7.3 and Cons. 7.9} \\ &= (b \circ a)^* \big(K_{X/Z} + \Delta^{hor} - \operatorname{Ramification}(f)\big) \cdot \widetilde{C}_t && \text{Cons. 7.8} \\ &= \text{const}^+ \cdot \big(K_{X/Z} + \Delta^{hor} - \operatorname{Ramification}(f)\big) \cdot C_t && \text{Proj. formula.} \end{aligned}$$

This finishes the proof of Theorem 7.1. ☐

8. FAILURE OF SEMIPOSITIVITY, CONSTRUCTION OF MORPHISMS

As was mentioned earlier, a key component of Campana-Păun's proof of the generic semipositivity result is the observation that given a lc pair (X, D) and an adapted cover $\gamma : Y \to X$, any subsheaf $\mathscr{F}_{(X,\Delta,\gamma)} \subseteq \mathscr{T}_{(X,\Delta,\gamma)}$ that is maximally destabilising respect to $\gamma^*(\text{ample})$ induces an algebraic foliation on X, whose leaves are often algebraic. This is the content of the next theorem.

Theorem 8.1 (From positive subsheaves to foliations, cf. [CP15b, Sect. 2])**.** *Let (X, Δ) be a projective pair and let $\gamma : Y \to X$ be an adapted morphism that is Galois with group G. Assume that $\Omega^{[1]}_{(X,\Delta,\gamma)}$* is not *$\gamma$-generically semipositive. Then, there exists a normal variety Z, a dominant, essentially equidimensional rational map $\psi : X \dashrightarrow Z$, and a family $(C_t)_{t \in T}$ of curves in X such that the following holds.*

- *(8.1.1) The family $(C_t)_{t \in T}$ dominates X and avoids small sets.*
- *(8.1.2) Given any $t \in T$, we have $[\mathscr{T}_{X/Z}] \cdot [C_t] > [\Delta^{horiz}] \cdot [C_t]$.*

Remark 8.2. The family $(C_t)_{t \in T}$ avoids small sets, and its general members are thus contained in X_{reg}. The intersection numbers of Item (8.1.1) are therefore well-defined, even if X is not necessarily $\mathbb{Q}$-factorial.

8.1. **Proof of Theorem 8.1.** As before, the proof is subdivided into a number of relatively independent steps.

Step 1: Setup. By assumption, there exists a very ample divisor A on X such that $\Omega^{[1]}_{(X,\Delta,\gamma)}$ is *not* generically semipositive with respect to the ample divisor $A_\gamma := \gamma^* A$. Let and $C_\gamma \subset Y$ be a a general complete intersection curve for the ample divisor A_Y on Y, in the sense of Mehta-Ramanathan. Consider the Harder-Narasimhan filtration of the sheaf $\mathscr{T}_{(X,\Delta,\gamma)}$ of adapted tangents with respect to A_γ and let

$$\mathscr{F}_{(X,\Delta,\gamma)} \subseteq \mathscr{T}_{(X,\Delta,\gamma)}$$

denote the maximally destabilising subsheaf, which is saturated in $\mathscr{T}_{(X,\Delta,\gamma)}$ and hence reflexive. By assumption, its slope is positive, $\mu_{A_\gamma}\big(\mathscr{F}_{(X,\Delta,\gamma)}\big) > 0$. Remark 4.2 yields an inclusion

$$\mathscr{F}_{(X,\Delta,\gamma)} \subseteq \gamma^{[*]}\mathscr{T}_X(-\log\lfloor\Delta\rfloor).$$

We denote the associated saturation by $\mathscr{F}^{\mathrm{sat}}_{(X,\Delta,\gamma)} \subseteq \gamma^{[*]}\mathscr{T}_X(-\log\lfloor\Delta\rfloor)$. Since $\mathscr{F}_{(X,\Delta,\gamma)} \subseteq \mathscr{T}_{(X,\Delta,\gamma)}$ is itself saturated, we have an equality

$$\mathscr{F}_{(X,\Delta,\gamma)} = \mathscr{F}^{\mathrm{sat}}_{(X,\Delta,\gamma)} \cap \mathscr{T}_{(X,\Delta,\gamma)}.$$

Observation 8.3 (Regularity along C_γ)*.* The curve C_γ is a general member in a dominating family of curves that avoids small sets. In particular, the following holds.

(8.3.1) The curve C_γ is entirely contained in the smooth locus of Y and is itself smooth.

(8.3.2) The sheaves $\mathscr{F}_{(X,\Delta,\gamma)}|_{C_\gamma}$ and $\mathscr{F}^{\mathrm{sat}}_{(X,\Delta,\gamma)}|_{C_\gamma}$ are locally free.

Observation 8.4 (Positivity along C_γ)*.* The locally free sheaf $\mathscr{F}_{(X,\Delta,\gamma)}|_{C_\gamma}$ is semistable, of positive degree and therefore ample. The larger sheaf $\mathscr{F}^{\mathrm{sat}}_{(X,\Delta,\gamma)}|_{C_\gamma}$ is likewise ample.

Observation 8.5 (G-invariance)*.* The divisor A_γ is invariant under the action of the Galois group. As a consequence, it follows from the uniqueness of the Harder-Narasimhan filtration that the maximally destabilising subsheaf $\mathscr{F}_{(X,\Delta,\gamma)}$ is a G-subsheaf of $\mathscr{T}_{(X,\Delta,\gamma)}$, and also of $\gamma^{[*]}\mathscr{T}_X(-\log\lfloor\Delta\rfloor)$.

In a similar vein, it follows from uniqueness of saturation that $\mathscr{F}^{\mathrm{sat}}_{(X,\Delta,\gamma)}$ is a G-subsheaf of $\gamma^{[*]}\mathscr{T}_X(-\log\lfloor\Delta\rfloor)$. Thus, by [GKPT15, Prop. 2.16], there exists a reflexive, saturated subsheaf $\mathscr{F} \subseteq \mathscr{T}_X(-\log\lfloor\Delta\rfloor)$ such that $\mathscr{F}^{\mathrm{sat}}_{(X,\Delta,\gamma)} = \gamma^{[*]}\mathscr{F}$.

Notation 8.6*.* Let $\mathscr{F}^{\mathrm{sat}} \subseteq \mathscr{T}_X$ denote the saturation of $\mathscr{F}$ inside $\mathscr{T}_X$. The following diagrams summarise the situation,

$$\mathscr{T}_Y(-\log\Delta_\gamma) \quad\subseteq\quad \underbrace{\mathscr{T}_{(X,\Delta,\gamma)}}_{\text{contains } \mathscr{F}_{(X,\Delta,\gamma)}} \quad\subseteq\quad \underbrace{\gamma^{[*]}\mathscr{T}_X(-\log\lfloor\Delta\rfloor)}_{\text{contains } \mathscr{F}^{\mathrm{sat}}_{(X,\Delta,\gamma)}=\gamma^{[*]}\mathscr{F}}$$

and

$$\underbrace{\mathscr{T}_X(-\log\lfloor\Delta\rfloor)}_{\text{contains } \mathscr{F}} \quad\subseteq\quad \underbrace{\mathscr{T}_X}_{\text{contains } \mathscr{F}^{\mathrm{sat}}}.$$

Observation 8.7 (Regularity and amplitude along C)*.* Since γ is finite, the curve $C := \gamma(C_\gamma)$ is again a general member in a dominating family of curves that avoids small sets. It follows that C is contained in the smooth locus of X and that $\mathscr{F}$ is locally free near C. In particular,

$$\mathscr{F}^{\mathrm{sat}}_{(X,\Delta,\gamma)}|_{C_\gamma} = (\gamma|_{C_\gamma})^*(\mathscr{F}|_C),$$

and [Laz04b, Prop. 6.1.8] therefore implies that $\mathscr{F}|_C$ and $\mathscr{F}^{\mathrm{sat}}|_C$ are both ample.

Step 2: Construction of a foliation. Next, we will show that the sheaf $\mathscr{F}^{\mathrm{sat}}$ is in fact a foliation. For this, Proposition 4.5, which describes the lifting the O'Neil tensor to an adapted cover, will be the key ingredient.

Claim 8.8. The sheaf $\mathscr{F}^{\mathrm{sat}} \subseteq \mathscr{T}_X$ is closed under the Lie-bracket.

Proof of Claim 8.8. Closedness under Lie bracket can be checked on an open subset. It will therefore suffice to show that the subsheaf $\mathscr{F} \subseteq \mathscr{T}_X(-\log\lfloor\Delta\rfloor)$, which agrees with $\mathscr{F}^{\mathrm{sat}}$ generically, is closed under the Lie-bracket. Equivalently, we need to prove that the O'Neil tensor

$$N : \mathscr{F}^{[2]} \to \big(\mathscr{T}_X(-\log\lfloor\Delta\rfloor)\big/\mathscr{F}\big)^{**}$$

vanishes. For this, recall from Proposition 4.5 that the restriction of its reflexive pull-back to $\mathscr{F}_{(X,\Delta,\gamma)}$ gives a morphism

$$N_{(X,\Delta,\gamma)} : \mathscr{F}^{[2]}_{(X,\Delta,\gamma)} \to \big(\mathscr{T}_{(X,\Delta,\gamma)}/\mathscr{F}_{(X,\Delta,\gamma)}\big)^{**}.$$

Since $\gamma^{[*]}\mathscr{F}$ and $\mathscr{F}_{(X,\Delta,\gamma)}$ agree on a dense open set, it will be enough to show that $N_{(X,\Delta,\gamma)}$ vanishes. This is actually the case for slope reasons. We have the following inequalities:

$$\begin{aligned} \mu^{\min}_{A_\gamma}\big(\mathscr{F}^{[2]}_{(X,\Delta,\gamma)}\big) &= 2\cdot\mu^{\min}_{A_\gamma}\big(\mathscr{F}_{(X,\Delta,\gamma)}\big) \\ &> \mu^{\min}_{A_\gamma}\big(\mathscr{F}_{(X,\Delta,\gamma)}\big) && \text{since } \mu^{\min} > 0 \text{ by assumption} \\ &\geq \mu^{\max}_{A_\gamma}\Big(\mathscr{T}_{(X,\Delta,\gamma)}/\mathscr{F}_{(X,\Delta,\gamma)}\Big) && \text{since } \mathscr{F}_{(X,\Delta,\gamma)} \text{ is max. destab.} \end{aligned}$$

Claim 8.8 thus follows. □

Notation 8.9. Following Notation 2.35 the foliation $\mathscr{F}^{\mathrm{sat}}$ induces a decomposition $\Delta = \Delta^{trans} + \Delta^{ntrans}$.

Step 3: Construction of a morphism. We show that the foliation $\mathscr{F}^{\mathrm{sat}}$ is algebraic and therefore defines an essentially equidimensional rational map. The algebraicity criterion that we employ goes back to Hartshorne, [Har68, Thm. 6.7]. We refer the reader to [KST07] or to one of the papers [Bos01, BM01, BM16] for a thorough discussion.

Claim 8.10. There exists a normal, projective variety Z, and a dominant, essentially equidimensional, rational map $\psi : X \dashrightarrow Z$ such that the sheaves $\mathscr{T}_{X/Z}$ and $\mathscr{F}^{\mathrm{sat}}$ agree.

Proof of Claim 8.10. In the setting at hand, amplitude of the restricted foliation $\mathscr{F}|_C$ implies that any leaf of $\mathscr{F}$ that intersects C is automatically algebraic; we refer the reader to [KST07, Thm. 1] for a convenient reference. Since C is general in a dominating family, this gives rise to a rational map

$$\phi : X \dashrightarrow X \times \mathrm{Chow}(X), \quad x \mapsto \Big(x, \overline{[\text{leaf through } x]}\Big).$$

Since X is normal, it follows from Zariski's main theorem, [Har77, V Thm. 5.2], that there exists a big open set $U \subseteq X$ where ϕ is well-defined. Observe that the following holds:

- The morphism $\phi|_U$ has a right inverse and is therefore necessarily injective, in particular quasi-finite.
- The image of the morphism ϕ is contained in the universal family that exists for the Chow variety. Let $\widetilde{V}$ be the normalisation of the closure of the image.

The universal family over $\mathrm{Chow}(X)$ need not be normal. Normalising, we obtain a diagram as follows,

$$\begin{tikzcd} U \arrow[r] \arrow[rr, bend left, "\text{well-defined, injective}"] & X \arrow[r, dashed, "\phi"'] & \widetilde{V} \arrow[r, "\text{normalisation}"'] \arrow[rr, bend left, "\text{equidim.}"] & \text{Univ} \arrow[r, "\text{equidim.}"'] & \mathrm{Chow}(X). \end{tikzcd}$$

More is true. Zariski's Main Theorem in the form of Grothendieck[6], asserts that the map $U \to \widetilde{V}$ is in fact an open immersion. In summary, we see that the composed morphism $U \to \mathrm{Chow}(X)$ is equidimensional. Let Z be the normalisation of the image, and $\psi : X \dashrightarrow Z$ the induced map.

The sheaves $\mathscr{T}_{X/Z}$ and $\mathscr{F}$ agree on an open subset of U, where all spaces and foliations are regular, and all maps are well-defined and smooth. Since both are saturated subsheaves of $\mathscr{T}_X$, this suffices to show that they agree everywhere. Claim 8.10 follows. □

Notation 8.11. Following Notation 2.21, the map $\psi : X \dashrightarrow Z$ induces a decomposition $\Delta = \Delta^{horiz} + \Delta^{vert}$.

Observation 8.12. The decomposition of Δ agrees with that coming from the foliation. In other words, $\Delta^{trans} = \Delta^{horiz}$ and $\Delta^{ntrans} = \Delta^{vert}$.

Step 4: End of proof. To end the proof, we need to show that the rational map ψ satisfies Inequality (8.1.2). We aim to apply Proposition 4.6. To this end, recall from our construction that

$$\mathscr{F} = \mathscr{F}^{\mathrm{sat}} \cap \mathscr{T}_X(-\log\lfloor\Delta\rfloor) \quad \text{and} \quad \mathscr{F}_{(X,\Delta,\gamma)} = \gamma^{[*]}\mathscr{F} \cap \mathscr{T}_{(X,\Delta,\gamma)}.$$

Item (4.6.3) of Proposition 4.6 thus gives an equality of intersection numbers

$$\begin{aligned} C_\gamma \cdot [\gamma^{[*]}\mathscr{T}_{X/Z}] &= \underbrace{C_\gamma \cdot [\mathscr{F}_{(X,\Delta,\gamma)}]}_{>0 \text{ by Obs. } 8.4} + C_\gamma \cdot [\gamma^* \Delta^{trans}] + \underbrace{C_\gamma \cdot [\text{effective}]}_{\geq 0 \text{ since } C_\gamma \text{ movable}} \\ &> C_\gamma \cdot [\gamma^* \Delta^{trans}]. \end{aligned}$$

In summary, we see that $C_\gamma \cdot [\gamma^{[*]}\mathscr{T}_{X/Z}] > C_\gamma \cdot [\gamma^* \Delta^{trans}]$ and hence $C \cdot [\mathscr{T}_{X/Z}] > C \cdot [\Delta^{trans}]$ as claimed. This finishes the proof of Theorem 8.1. □

9. PROOF OF THE SEMIPOSITIVITY RESULT

We prove Theorem 5.2 in this section. With the preparations at hand, the proof is now quite short. We argue by contradiction and assume that $\Omega^{[1]}_{(X,\Delta,\gamma)}$ is *not* γ-generically semipositive. As we have seen in Theorem 8.1, this implies the existence of a normal variety Z, a dominant, essentially equidimensional, rational map $f : X \dashrightarrow Z$, and a family $(C_t)_{t \in T}$ of curves that dominates X and avoids small sets, such that the following inequality holds for all $t \in T$,

$$[\mathscr{T}_{X/Z}] \cdot [C_t] > [\Delta^{horiz}] \cdot [C_t].$$

Recalling the description of $\mathscr{T}_{X/Z}$ given in Lemma 2.31, this is equivalent to

$$[K_{X/Z} + \Delta^{horiz} - \text{Ramification}\, f] \cdot [C_t] < 0,$$

contradicting the positivity of relative dualising sheaves that was established in Theorem 7.1, and ending the proof of Theorem 5.2. □

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[6] Zariski's Main Theorem in the form of Grothendieck is found in [Gro66]. We refer the reader to [GKP16, Sect. 3.4] for a more detailed discussion.

References

[BCHM10] Caucher Birkar, Paolo Cascini, Christopher D. Hacon, and James McKernan. Existence of minimal models for varieties of log general type. *J. Amer. Math. Soc.*, 23:405–468, 2010. DOI:10.1090/S0894-0347-09-00649-3. ↑ 19

[BM01] Fedor A. Bogomolov and Michael L. McQuillan. Rational curves on foliated varieties, February 2001. IHES Preprint. ↑ 26

[BM16] Fedor Bogomolov and Michael McQuillan. *Foliation Theory in Algebraic Geometry*, chapter Rational Curves on Foliated Varieties, pages 21–51. Simons Symposia. Springer, 2016. ISBN 978-3-319-24460-0. DOI:10.1007/978-3-319-24460-0_2. ↑ 26

[Bos01] Jean-Benoît Bost. Algebraic leaves of algebraic foliations over number fields. *Publ. Math. Inst. Hautes Études Sci.*, 93:161–221, 2001. DOI:10.1007/s10240-001-8191-3. ↑ 26

[BP08] Bo Berndtsson and Mihai Păun. Bergman kernels and the pseudoeffectivity of relative canonical bundles. *Duke Math. J.*, 145(2):341–378, 2008. DOI: 10.1215/00127094-2008-054. ↑ 21

[Cam04] Frédéric Campana. Orbifolds, special varieties and classification theory. *Ann. Inst. Fourier (Grenoble)*, 54(3):499–630, 2004. http://eudml.org/doc/116120. ↑ 3, 21

[Cla15] Benoît Claudon. Semi-positivité du cotangent logarithmique et conjecture de shafarevich et viehweg (d'aprés Camapna, Păun, Taji, ...). *Séminaire Bourbaki*, 2015. Preprint arXiv:1603.09568. ↑ 3

[CP14] Frédéric Campana and Mihai Păun. Positivity properties of the bundle of logarithmic tensors on compact Kähler manifolds. Preprint arXiv:1407.3431, July 2014. ↑ 3

[CP15a] Frédéric Campana and Mihai Păun. Foliations with positive slopes and birational stability of orbifold cotangent bundles. Preprint arXiv:1508.02456, 2015. ↑ 3

[CP15b] Frédéric Campana and Mihai Păun. Orbifold generic semi-positivity: an application to families of canonically polarized manifolds. *Ann. Inst. Fourier (Grenoble)*, 65(2):835–861, 2015. Available at http://aif.cedram.org/item?id=AIF_2015__65_2_835_0. Preprint arXiv:1303.3169. ↑ 3, 9, 18, 19, 21, 24

[Fuj14] Osamu Fujino. Notes on the weak positivity theorems, June 2014. Preprint arXiv:1406.1834v4, to appear in Adv. Stud. Pure Math. ↑ 21, 22

[Ful98] William Fulton. *Intersection Theory*, volume 2 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, second edition, 1998. DOI:10.1007/978-1-4612-1700-8. ↑ 4

[GKP16] Daniel Greb, Stefan Kebekus, and Thomas Peternell. Étale fundamental groups of Kawamata log terminal spaces, flat sheaves, and quotients of abelian varieties. *Duke Math. J.*, 165(10):1965–2004, 2016. DOI:10.1215/00127094-3450859. Preprint arXiv:1307.5718. ↑ 4, 9, 27

[GKPT15] Daniel Greb, Stefan Kebekus, Thomas Peternell, and Behrouz Taji. The Miyaoka-Yau inequality and uniformisation of canonical models. Preprint arXiv:1511.08822, November 2015. ↑ 25

[Gri84] Phillip Griffiths, editor. *Topics in transcendental algebraic geometry*, volume 106 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1984. ↑ 2

[Gro66] Alexandre Grothendieck. Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. Troisiéme partie. *Inst. Hautes Études Sci. Publ. Math.*, (28):255, 1966. Revised in collaboration with Jean Dieudonné. `http://www.numdam.org/numdam-bin/feuilleter?id=PMIHES_1966__28_`. ↑ 4, 27

[Har68] Robin Hartshorne. Cohomological dimension of algebraic varieties. *Ann. of Math. (2)*, 88:403–450, 1968. ↑ 26

[Har77] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52. DOI:10.1007/978-1-4757-3849-0. ↑ 4, 26

[Hö10] Andreas Höring. Positivity of direct image sheaves—a geometric point of view. *Enseign. Math. (2)*, 56(1-2):87–142, 2010. ↑ 21

[JK11a] Kelly Jabbusch and Stefan Kebekus. Families over special base manifolds and a conjecture of Campana. *Math. Z.*, 269(3-4):847–878, 2011. DOI: 10.1007/s00209-010-0758-6. ↑ 9, 14

[JK11b] Kelly Jabbusch and Stefan Kebekus. Positive sheaves of differentials coming from coarse moduli spaces. *Ann. Inst. Fourier (Grenoble)*, 61(6):2277–2290 (2012), 2011. http://dx.doi.org/10.5802/aif.2673. ↑ 3

[Keb13] Stefan Kebekus. Differential forms on singular spaces, the minimal model program, and hyperbolicity of moduli stacks. In Gavril Farkas and Ian Morrison, editors, *Handbook of Moduli, in honour of David Mumford, Vol. II*, volume 25 of *Advanced Lectures in Mathematics*, pages 71–114. International Press, March 2013. ISBN 9781571462589. Preprint arXiv:1107.4239. ↑ 2

[KK08] Stefan Kebekus and Sándor J Kovács. Families of canonically polarized varieties over surfaces. *Invent. Math.*, 172(3):657–682, 2008. DOI:10.1007/s00222-008-0128-8. Preprint arXiv:0707.2054. ↑ 2

[KK10] Stefan Kebekus and Sándor J. Kovács. The structure of surfaces and threefolds mapping to the moduli stack of canonically polarized varieties. *Duke Math. J.*, 155(1):1–33, 2010. DOI:10.1215/00127094-2010-049, Preprint arXiv:0707.2054. ↑ 2

[KST07] Stefan Kebekus, Luis Solá Conde, and Matei Toma. Rationally connected foliations after Bogomolov and McQuillan. *J. Algebraic Geom.*, 16(1):65–81, 2007. ↑ 26

[Laz04a] Robert Lazarsfeld. *Positivity in algebraic geometry. I*, volume 48 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2004. Classical setting: line bundles and linear series DOI:10.1007/978-3-642-18808-4. ↑ 20, 21

[Laz04b] Robert Lazarsfeld. *Positivity in algebraic geometry. II*, volume 49 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2004. Positivity for vector bundles, and multiplier ideals. DOI:10.1007/978-3-642-18808-4. ↑ 9, 25

[Lu02] Steven S.Y. Lu. A refined Kodaira dimension and its canonical fibration, November 2002. Preprint arXiv:0211029v3. ↑ 21

[Miy87] Yoichi Miyaoka. Deformations of a morphism along a foliation and applications. In *Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985)*, volume 46 of *Proc. Sympos. Pure Math.*, pages 245–268. Amer. Math. Soc., Providence, RI, 1987. ↑ 18

[PS15] Mihnea Popa and Christian Schnell. Viehweg's hyperbolicity conjecture for families with maximal variation. Preprint arXiv:1511.00294., 2015. ↑ 3

[PT14] Mihai Păun and Shigeharu Takayama. Positivity of twisted relative pluricanonical bundles and their direct images, September 2014. Preprint arXiv:1409.5504v1. ↑ 21

[Taj16] Behrouz Taji. The isotriviality of smooth families of canonically polarized manifolds over a special quasi-projective base. *Compositio. Math.*, 152:1421–1434, 7 2016. DOI:10.1112/S0010437X1600734X. ↑ 3, 19

[Vie83] Eckart Viehweg. Weak positivity and the additivity of the Kodaira dimension for certain fibre spaces. In *Algebraic varieties and analytic varieties (Tokyo, 1981)*, volume 1 of *Adv. Stud. Pure Math.*, pages 329–353. North-Holland, Amsterdam, 1983. ↑ 2

[Vie95] Eckart Viehweg. *Quasi-projective moduli for polarized manifolds*, volume 30 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1995. DOI:10.1007/978-3-642-79745-3. ↑ 2, 22

[VZ02] Eckart Viehweg and Kang Zuo. Base spaces of non-isotrivial families of smooth minimal models. In *Complex geometry (Göttingen, 2000)*, pages 279–328. Springer, Berlin, 2002. ↑ 2

[VZ03] Eckart Viehweg and Kang Zuo. On the Brody hyperbolicity of moduli spaces for canonically polarized manifolds. *Duke Math. J.*, 118(1):103–150, 2003. DOI:10.1215/S0012-7094-03-11815-3. ↑ 2, 3

[Yau77] Shing-Tung Yau. Calabi's conjecture and some new results in algebraic geometry. *Proc. Nat. Acad. Sci. U.S.A.*, 74(5):1798–1799, 1977. ↑ 18

[Zuo00] Kang Zuo. On the negativity of kernels of Kodaira-Spencer maps on Hodge bundles and applications. *Asian J. Math.*, 4(1):279–301, 2000. Kodaira's issue. ↑ 2

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