Extension of holomorphic functions defined on non reduced analytic subvarieties

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Abstract. The goal of this contribution is to investigate L^2 extension properties for holomorphic sections of vector bundles satisfying weak semi-positivity properties. Using techniques borrowed from recent proofs of the Ohsawa-Takegoshi extension theorem, we obtain several new surjectivity results for the restriction morphism to a non necessarily reduced subvariety, provided the latter is defined as the zero variety of a multiplier ideal sheaf. These extension results are derived from L^2 approximation techniques, and they hold under (probably) optimal curvature conditions.

Keywords. Holomorphic function, plurisubharmonic function, multiplier ideal sheaf, L^2 extension theorem, Ohsawa-Takegoshi theorem, log canonical singularities, non reduced subvariety, Kahler metric.

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1. Introduction

Bernhard Riemann is considered to be the founder of many branches of geometry, especially by providing the foundations of modern differential geometry [Rie54] and by introducing a deep geometric approach in the theory of functions of one complex variable [Rie51, Rie57]. The study of Riemannian manifolds has since become a central theme of mathematics. We are concerned here with the study of holomorphic functions of several variables, and in this context, Hermitian geometry is the relevant special case of Riemannian manifolds – as a matter of fact, Charles Hermite, born in 1822, was a contemporary of Bernhard Riemann.

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The more specific problem we are considering is the question whether a holomorphic function f defined on a subvariety Y of a complex manifold X can be extended to the whole of X. More generally, given a holomorphic vector bundle E on X, we are interested in the existence of global extensions $F \in H^0(X, E)$ of a section $f \in H^0(Y, E_{\restriction Y})$, assuming suitable convexity properties of X and Y, suitable growth conditions for f, and appropriate curvature positivity hypotheses for the bundle E. This can also be seen as an interpolation problem when Y is not connected – possibly a discrete set.

In his general study of singular points and other questions of function theory, Riemann obtained what is now known as the Riemann extension theorem: a singularity of a holomorphic function at a point is "removable" as soon as it is bounded – or more generally if it is L^2 in the sense of Lebesgue; the same is true for functions of several variables defined on the complement of an analytic set. Using the maximum principle, a consequence is that a holomorphic function defined in the complement of an analytic set of codimension 2 automatically extends to the ambient manifold.

The extension-interpolation problem we are considering is strongly related to Riemann's extension result because we allow here Y to have singularities, and f must already have a local extension near any singular point $x_0 \in Y$; for this, some local growth condition is needed in general, especially if x_0 is a non normal point. A major advance in the general problem is the Ohsawa-Takegoshi L^2 extension theorem [OT87] (see also the subsequent series of papers II–VI by T. Ohsawa). It is remarkable that Bernhard Riemann already anticipated in [Rie51] the use of L^2 estimates and the idea of minimizing energy, even though his terminology was very different from the one currently in use.

The goal of the present contribution is to generalize the Ohsawa-Takegoshi L^2 extension theorem to the case where the sections are extended from a non necessarily reduced subvariety, associated with an arbitrary multiplier ideal sheaf. A similar idea had already been considered in D. Popovici's work [Pop05], but the present approach is substantially more general. Since the extension theorems do not require any strict positivity assumptions, we hope that they will be useful to investigate further properties of linear systems and pluricanonical systems on varieties that are not of general type. The exposition is organized as follows: section 2 presents the main definitions and results, section 3 (which is more standard and which the expert reader way wish to skip) is devoted to recalling the required Khler identities and inequalities, section 4 elaborates on the concept of jumping numbers for multiplier ideal sheaves, and section 5 explains the technical details of the L^2 estimates and their proofs. We refer to [Ber96], [BL14], [Blo13], [Che11], [DHP13], [GZ13,15], [Man93], [MV07], [Ohs88,94,95,01,03,05], [Pop05], [Var10] for related work on L^2 extension theorems.

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2. Statement of the main extension results

The Ohsawa-Takegoshi theorem addresses the following extension problem: let Y be a complex analytic submanifold of a complex manifold X ; given a holomorphic function f on Y satisfying suitable L^2 conditions on Y, find a holomorphic extension F of f to X, together with a good L^2 estimate for F on X. For this, suitable pseudoconvexity and curvature conditions are needed. We start with a few basic definitions in these directions.

All manifolds and complex spaces are assumed to be paracompact (and even countable at infinity).

(2.1) Definition. *A compact complex manifold X will be said to be* weakly pseudoconvex *if it possesses a smooth plurisubharmonic exhaustion function ρ (say $\rho : X \to \mathbb{R}_+$).*

(Note: in [Nak73] and later work by Ohsawa, the terminology “weakly 1-complete” is used instead of “weakly pseudoconvex”).

(2.2) Definition. *A function $\psi : X \to [-\infty, +\infty[$ on a complex manifold X is said to be* quasi-plurisubharmonic *(quasi-psh) if φ is locally the sum of a psh function and of a smooth function (or equivalently, if $i\partial\overline{\partial}\psi$ is locally bounded from below). In addition, we say that ψ has* neat analytic singularities *if every point $x_0 \in X$ possesses an open neighborhood U on which ψ can be written*

$$\psi(z) = c \log \sum_{1\le j\le N} |g_k(z)|^2 + w(z)$$

where $c \ge 0$, $g_k \in \mathcal{O}_X(U)$ and $w \in \mathcal{C}^\infty(U)$.

(2.3) Definition. *If ψ is a quasi-psh function on a complex manifold X, the multiplier ideal sheaf $\mathcal{I}(\psi)$ is the coherent analytic subsheaf of $\mathcal{O}_X$ defined by*

$$\mathcal{I}(\psi)_x = \big\{ f \in \mathcal{O}_{X,x}\,;\ \exists U \ni x\,,\ \int_U |f|^2 e^{-\psi} d\lambda < +\infty \big\}$$

where U is an open coordinate neighborhood of x, and $d\lambda$ the standard Lebesgue measure in the corresponding open chart of $\mathbb{C}^n$. We say that the singularities of ψ are log canonical *along the zero variety $Y = V(\mathcal{I}(\psi))$ if $\mathcal{I}((1-\varepsilon)\psi)_{\restriction Y} = \mathcal{O}_{X\restriction Y}$ for every $\varepsilon > 0$.*

In case ψ has log canonical singularities, it is easy to see that $\mathcal{I}(\psi)$ is a reduced ideal, i.e. that $Y = V(\mathcal{I}(\psi))$ is a reduced analytic subvariety of X. In fact if $f^p \in \mathcal{I}(\psi)_x$ for some integer $p \ge 2$, then $|f|^{2p}e^{-\psi}$ is locally integrable, hence, by openness, we have also $|f|^{2p}e^{-(1+\varepsilon)\psi}$ for $\varepsilon > 0$ small enough (see [GZ13] for a general result on openness that does not assume anything on ψ – the present special case follows in fact directly from Hironaka's theorem on resolution of singularities, in case ψ has analytic singularities). Using the fact that $e^{-(1-\varepsilon)\psi}$ is integrable for every $\varepsilon \in]0,1[$, the Hlder inequality for the conjugate exponents $1/p + 1/q = 1$ implies that

$$|f|^2 e^{-\psi} = \big(|f|^{2p} e^{-(1+\varepsilon)\psi}\big)^{1/p} \big(e^{-\psi(1/q-\varepsilon/p)}\big)$$

is locally integrable, hence $f \in \mathcal{I}(\psi)_x$, as was to be shown. If ω is a Khler metric on X, we let $dV_{X,\omega} = \frac{1}{n!}\omega^n$ be the corresponding Kähler volume element, $n = \dim X$. In case ψ has log canonical singularities along $Y = V(\mathcal{I}(\psi))$, one can also associate in a natural way a measure $dV_{Y^\circ,\omega}[\psi]$ on the set $Y^\circ = Y_{\rm reg}$ of regular points of Y as follows. If $g \in \mathcal{C}_c(Y^\circ)$ is a compactly supported continuous function on Y° and $\widetilde{g}$ a compactly supported extension of g to X, we set

$$\int_{Y^\circ} g\, dV_{Y^\circ,\omega}[\psi] = \limsup_{t\to-\infty} \int_{\{x\in X\,,\ t<\psi(x)<t+1\}} \widetilde{g} e^{-\psi}\, dV_{X,\omega}. \tag{2.4}$$

By Hironaka, it is easy to see that the limit does not depend on the continuous extension $\widetilde{g}$, and that one gets in this way a measure with smooth positive density with respect to the Lebesgue measure, at least on an (analytic) Zariski open set in Y° (cf. Proposition 4.5, in the special case $f = 1$, $p = 1$). In case Y is a codimension r subvariety of X defined by an equation $\sigma(x) = 0$ associated with a section $\sigma \in H^0(X, S)$ of some hermitian vector bundle (S, h_S) on X, and assuming that σ is generically transverse to zero along Y, it is natural to take

$$\psi(z) = r \log |\sigma(z)|^2_{h_S}. \tag{2.5}$$

One can then easily check that $dV_{Y^\circ,\omega}[\psi]$ is the measure supported on $Y^\circ = Y_{\rm reg}$ such that

$$dV_{Y^\circ,\omega}[\psi] = \frac{2^{r+1}\pi^r}{(r-1)!}\,\frac{1}{|\Lambda^r(d\sigma)|^2_{\omega,h_S}}\,dV_{Y,\omega} \quad \text{where} \quad dV_{Y,\omega} = \frac{1}{(n-r)!}\,\omega^{n-r}_{\restriction Y^\circ}. \tag{2.6}$$

For a quasi-psh function with log canonical singularities, $dV_{Y^\circ,\omega}[\psi]$ should thus be seen as some sort of (inverse of) Jacobian determinant associated with the logarithmic singularities of ψ. Finally, the following positive real function will make an appearance in several of our final L^2 estimates :

$$\gamma(x) = \begin{cases} \exp(-x/2) & \text{if } x \geq 0, \\ \dfrac{1}{1+x^2} & \text{if } x \leq 0. \end{cases} \tag{2.7}$$

The first generalized extension theorem we are interested in is a variation of Theorem 4 in [Ohs01]. The first difference is that we do not require any specific behavior of the quasi-psh function defining the subvariety: any quasi-psh function with log canonical singularities will do; secondly, we do not want to make any assumption that there exist negligible sets in the ambient manifold whose complements are Stein, because such an hypothesis need not be true on a general compact Kähler manifold – one of the targets of our study. Recall that a hermitian tensor

$$\Theta = \mathrm{i} \sum_{1\leq j,k\leq n,\,1\leq\lambda,\mu\leq r} c_{jk\lambda\mu}\,dz_j \wedge d\overline{z}_k \otimes e^*_\lambda \otimes e_\mu \in \mathcal{C}^\infty(X, \Lambda^{1,1}T^*_X \otimes \mathrm{Hom}(E,E))$$

is said to be *Nakano semi-positive* (resp. *positive*) if the associated hermitian quadratic form

$$H_\Theta(\tau) = \sum_{1\leq j,k\leq n,\,1\leq\lambda,\mu\leq r} c_{jk\lambda\mu}\tau_{j,\lambda}\overline{\tau}_{k,\mu}$$

is semi-positive (resp. positive) on non zero tensors $\tau = \sum \tau_{j,\lambda}\frac{\partial}{\partial z_j} \otimes e_\lambda \in T_X \otimes E$. It is said to be *Griffiths semi-positive* (resp. *positive*) if $H_\Theta(\tau) \geq 0$ (resp. $H_\Theta(\tau) > 0$) for all non zero decomposable tensors $\tau = \zeta \otimes v \in T_X \otimes E$.

(2.8) Theorem (general L^2 extension result for reduced subvarieties). *Let X be a weakly pseudoconvex Khler manifold, and ω a Khler metric on X. Let (E, h) be a holomorphic vector bundle equipped with a smooth hermitian metric h on X, and let $\psi : X \to [-\infty, +\infty[$ be a quasi-psh function on X with neat analytic singularities. Let Y be the analytic subvariety of X defined by $Y = V(\mathcal{I}(\psi))$ and assume that ψ has log* *canonical singularities along Y, so that Y is reduced. Finally, assume that the Chern curvature tensor of (E,h) is such that the sum*

$$\mathrm{i}\Theta_{E,h} + \alpha\,\mathrm{i}\partial\overline{\partial}\psi \otimes \mathrm{Id}_E$$

is Nakano semipositive for all $\alpha \in [1, 1+\delta]$ and some $\delta > 0$. Then for every section $f \in H^0(Y^\circ, (K_X \otimes E)_{\restriction Y^\circ})$ on $Y^\circ = Y_{\mathrm{reg}}$ such that

$$\int_{Y^\circ} |f|^2_{\omega,h} dV_{Y^\circ,\omega}[\psi] < +\infty,$$

there exists an extension $F \in H^0(X, K_X \otimes E)$ whose restriction to Y° is equal to f, such that

$$\int_X \gamma(\delta\psi)\,|F|^2_{\omega,h} e^{-\psi} dV_{X,\omega} \le \frac{34}{\delta} \int_{Y^\circ} |f|^2_{\omega,h} dV_{Y^\circ,\omega}[\psi].$$

(2.9) Remarks. (a) Although $|F|^2_{\omega,h}$ and $dV_{X,\omega}$ both depend on ω, it is easy to see that the product $|F|^2_{\omega,h} dV_{X,\omega}$ actually does not depend on ω when F is a $(n,0)$-form. The same observation applies to the product $|f|^2_{\omega,h} dV_{Y^\circ,\omega}[\psi]$, hence the final L^2 estimate is in fact independent of ω. Nevertheless, the existence of a Kähler metric (and even of a complete Kähler metric) is crucial in the proof, thanks to the techniques developped in [AV65] and [Dem82].

(b) By approximating non smooth plurisubharmonic weights with smooth ones, one can see that the above result still holds when E is a line bundle equipped with a singular hermitian metric $h = e^{-\varphi}$. The curvature condition then reads

$$\mathrm{i}\Theta_{E,h} + \alpha\,\mathrm{i}\partial\overline{\partial}\psi = \mathrm{i}\,\partial\overline{\partial}(\varphi + t\psi) \ge 0, \qquad \alpha \in [1, 1+\delta],$$

and should be understood in the sense of currents.

(c) The constant $\frac{34}{\delta}$ given in the above L^2 inequality is not optimal. By exercising more care in the bounds, an optimal estimate could probably be found by following the techniques of Blocki [Blo13] and Guan-Zhou [GZ15], at the expense of replacing $\gamma(\delta\psi)$ with a more complicated and less explicit function of δ and ψ. Notice also that in the L^2 estimate, δ can be replaced by any $\delta' \in \,]0, \delta]$. □

We now turn ourselves to the case where non reduced multiplier ideal sheaves and non reduced subvarieties are considered. This situation has already been considered by D. Popovici [Pop05] in the case of powers of a reduced ideal, but we aim here at a much wider generality, which also yields more natural assumptions. For $m \in \mathbb{R}_+$, we consider the multiplier ideal sheaf $\mathcal{I}(m\psi)$ and the associated non necessarily reduced subvariety $Y^{(m)} = V(\mathcal{I}(m\psi))$, together with the structure sheaf $\mathcal{O}_{Y^{(m)}} = \mathcal{O}_X/\mathcal{I}(m\psi)$, the real number m being viewed as some sort of multiplicity – the support $|Y^{(m)}|$ may increase with m, but certainly stabilizes to the set of poles $P = \psi^{-1}(-\infty)$ for m large enough. We assume the existence of a discrete sequence of positive numbers

$$0 = m_0 < m_1 < m_2 < \ldots < m_p < \ldots$$

such that $\mathcal{I}(m\psi)=\mathcal{I}(m_p\psi)$ for $m\in[m_p,m_{p+1}[$ (with of course $\mathcal{I}(m_0\psi)=\mathcal{O}_X$); they are called the *jumping numbers* of ψ. The existence of a discrete sequence of jumping numbers is automatic if X is compact. In general, it still holds on every relatively compact open subset

$$X_c:=\{x\in X\,,\ \rho(x)<c\}\Subset X,$$

but requires some some of uniform behaviour of singularities at infinity in the non compact case. We are interested in extending a holomorphic section

$$f\in H^0(Y^{(m_p)},\mathcal{O}_{Y^{(m_p)}}(K_X\otimes E_{\restriction Y^{(m_p)}}) := H^0(Y^{(m_p)},\mathcal{O}_X(K_X\otimes_{\mathbb{C}}E)\otimes_{\mathcal{O}_X}\mathcal{O}_X/\mathcal{I}(m_p\psi)).$$

[Later on, we usually omit to specify the rings over which tensor products are taken, as they are implicit from the nature of objects under consideration]. The results are easier to state in case one takes a nilpotent section of the form

$$f\in H^0(Y^{(m_p)},\mathcal{O}_X(K_X\otimes E)\otimes\mathcal{I}(m_{p-1}\psi)/\mathcal{I}(m_p\psi)).$$

Then $\mathcal{I}(m_{p-1}\psi)/\mathcal{I}(m_p\psi))$ is actually a coherent sheaf, and its support is a reduced subvariety Z_p of $Y^{(m_p)}$ (see Lemma 4.2). Therefore $\mathcal{I}(m_{p-1}\psi)/\mathcal{I}(m_p\psi))$ can be seen as a vector bundle over a Zariski open set $Z_p^\circ\subset Z_p$. We can mimic formula (2.4) and define some sort of infinitesimal "m_p-jet" L^2 norm $|J^{m_p}f|^2_{\omega,h}\,dV_{Z_p^\circ,\omega}[\psi]$ (a purely formal notation), as the measure on Z_p° defined by

$$\int_{Z_p^\circ}g\,|J^{m_p}f|^2_{\omega,h}\,dV_{Z_p^\circ,\omega}[\psi]=\limsup_{t\to-\infty}\int_{\{x\in X\,,\ t<\psi(x)<t+1\}}\widetilde{g}\,|\widetilde{f}|^2_{\omega,h}e^{-m_p\psi}\,dV_{X,\omega}\tag{2.10}$$

for any $g\in\mathcal{C}_c(Z_p^\circ)$, where $\widetilde{g}\in\mathcal{C}_c(X)$ is a continuous extension of g and $\widetilde{f}$ a smooth extension of f on X such that $\widetilde{f}-f\in\mathcal{I}(m_p\psi)\otimes_{\mathcal{O}_X}\mathcal{C}^\infty$ (this measure again has a smooth positive density on a Zariski open set in Z_p°, and does not depend on the choices of $\widetilde{f}$ and $\widetilde{g}$, see Prop. 4.5). We extend the measure as being 0 on $Y^{(m_p)}_{\rm red}\smallsetminus Z_p$, since $\mathcal{I}(m_{p-1}\psi)/\mathcal{I}(m_p\psi))$ has support in $Z_p^\circ\subset Z_p$. In this context, we introduce the following natural definition.

(2.11) Definition. *We define the* restricted multiplied ideal sheaf

$$\mathcal{I}'(m_{p-1}\psi)\subset\mathcal{I}(m_{p-1}\psi)$$

to be the set of germs $F\in\mathcal{I}(m_{p-1}\psi)_x\subset\mathcal{O}_{X,x}$ such that there exists a neighborhood U of x satisfying

$$\int_{Y^{(m_p)}\cap U}|J^{m_p}F|^2_{\omega,h}\,dV_{Y^{(m_p)},\omega}[\psi]<+\infty.$$

This is a coherent ideal sheaf that contains $\mathcal{I}(m_p\psi)$. Both of the inclusions

$$\mathcal{I}(m_p\psi)\subset\mathcal{I}'(m_{p-1}\psi)\subset\mathcal{I}(m_{p-1}\psi)$$

can be strict (*even for* $p=1$).

The proof is given in Section 4 (Proposition 4.5). One of the geometric consequences is the following "quantitative" surjectivity statement, which is the analogue of Theorem 2.8

for the case when the first non trivial jumping number m_1 is replaced by a higher jumping number m_p.

(2.12) Theorem. *With the above notation and in the general setting of Theorem 2.8 (but without the hypothesis that the quasi-psh function ψ has log canonical singularities), let $0 = m_0 < m_1 < m_2 < \ldots < m_p < \ldots$ be the jumping numbers of ψ. Assume that*

$$\mathrm{i}\Theta_{E,h} + \alpha\,\mathrm{i}\partial\overline{\partial}\psi \otimes \mathrm{Id}_E \geq_{\mathrm{Nak}} 0$$

is Nakano semipositive for all $\alpha \in [m_p, m_p + \delta]$, for some $\delta > 0$.

(a) *Let*

$$f \in H^0(Y^{(m_p)}, \mathcal{O}_X(K_X \otimes E) \otimes \mathcal{I}'(m_{p-1}\psi)/\mathcal{I}(m_p\psi))$$

be a section such that

$$\int_{Y^{(m_p)}} |J^{m_p} f|^2_{\omega,h}\, dV_{Y^{(m_p)},\omega}[\psi] < +\infty.$$

Then there exists a global section

$$F \in H^0(X, \mathcal{O}_X(K_X \otimes E) \otimes \mathcal{I}'(m_{p-1}\psi))$$

which maps to f under the morphism $\mathcal{I}'(m_{p-1}\psi) \to \mathcal{I}(m_{p-1}\psi)/\mathcal{I}(m_p\psi)$, such that

$$\int_X \gamma(\delta\psi)\, |F|^2_{\omega,h}\, e^{-m_p\psi} dV_{X,\omega}[\psi] \leq \frac{34}{\delta} \int_{Y^{(m_p)}} |J^{m_p} f|^2_{\omega,h}\, dV_{Y^{(m_p)},\omega}[\psi].$$

(b) *The restriction morphism*

$$H^0(X, \mathcal{O}_X(K_X \otimes E) \otimes \mathcal{I}'(m_{p-1}\psi)) \to H^0(Y^{(m_p)}, \mathcal{O}_X(K_X \otimes E) \otimes \mathcal{I}'(m_{p-1}\psi)/\mathcal{I}(m_p\psi))$$

is surjective.

If E is a line bundle and h a singular hermitian metric on E, a similar result can be obtained by approximating h. However, the L^2 estimates require to take into account the multiplier ideal sheaf of h, and we get the following result.

(2.13) Theorem. *Let (X, ω) be a weakly pseudoconvex Kähler manifold, ψ a quasi-psh function with neat analytic singularities and E a holomorphic line bundle equipped with a singular hermitian metric h. Let $0 = m_0 < m_1 < m_2 < \ldots < m_p < \ldots$ be the jumping numbers for the family of multiplier ideal sheaves $\mathcal{I}(he^{-m\psi})$, $m \in \mathbb{R}^+$. Assume that*

$$\mathrm{i}\Theta_{E,h} + \alpha\,\mathrm{i}\partial\overline{\partial}\psi \geq 0 \quad \textit{in the sense of currents,}$$

for all $\alpha \in [m_p, m_p + \delta]$ and $\delta > 0$ small enough. Define $\mathcal{I}'(he^{-m_\ell\psi})$ by the same L^2 convergence property as in the case when h is non singular. Then the restriction morphism

$$\begin{aligned} &H^0(X, \mathcal{O}_X(K_X \otimes E) \otimes \mathcal{I}'(he^{-m_{p-1}\psi})) \\ &\qquad \to H^0(Y^{(m_p)}, \mathcal{O}_X(K_X \otimes E) \otimes \mathcal{I}'(he^{-m_{p-1}\psi})/\mathcal{I}(he^{-m_p\psi})) \end{aligned}$$

is surjective. Moreover, the L^2 estimate (2.12) (b) *still holds in this situation.*

One of the strengths of the above theorems lies in the fact that no strict curvature hypothesis on $\mathrm{i}\Theta_{E,h}$ is ever needed. On the other hand, if we assume a strictly positive lower bound

$$\mathrm{i}\Theta_{E,h} + m_p\,\mathrm{i}\partial\overline{\partial}\psi \geq \varepsilon\omega > 0,$$

then the Nadel vanishing theorem [Nad89] implies $H^1(X, \mathcal{O}_X(K_X \otimes E) \otimes \mathcal{I}(he^{-m_p\psi})) = 0$, and we immediately get stronger surjectivity statements by considering the relevant cohomology exact sequence, e.g. that

$$H^0(X, \mathcal{O}_X(K_X \otimes E) \otimes \mathcal{I}(he^{-m_\ell\psi})) \to H^0(Y^{(m_p)}, \mathcal{O}_X(K_X \otimes E) \otimes \mathcal{I}(he^{-m_\ell\psi})/\mathcal{I}(he^{-m_p\psi}))$$

is surjective for all $\ell < p$. In case X is compact, it turns out that this qualitative surjectivity property still holds true with a semi-positivity assumption only. In view of [DHP13], this probably has interesting applications to algebraic geometry which we intend to discuss in a future work:

(2.14) Theorem. *Let (X, ω) be a* compact *Kähler manifold, ψ a quasi-psh function with neat analytic singularities and E a holomorphic vector bundle equipped with a hermitian metric h. Let $0 = m_0 < m_1 < m_2 < \ldots < m_p < \ldots$ be the jumping numbers for the family of multiplier ideal sheaves $\mathcal{I}(he^{-m\psi})$, $m \in \mathbb{R}^+$.*

(a) *The rank of E being arbitrary, assume that h is smooth and that for some index $\ell = 0, 1, \ldots, p-1$ and all $\alpha \in [m_{\ell+1}, m_p + \delta]$, we have*

$$\mathrm{i}\Theta_{E,h} + \alpha\,\mathrm{i}\partial\overline{\partial}\psi \otimes \mathrm{Id}_E \geq 0 \quad \textit{in the sense of Nakano},$$

for $\delta > 0$ small enough. Then the restriction morphism

$$H^0(X, \mathcal{O}_X(K_X \otimes E) \otimes \mathcal{I}(e^{-m_\ell\psi})) \to H^0(Y^{(m_p)}, \mathcal{O}_X(K_X \otimes E) \otimes \mathcal{I}(e^{-m_\ell\psi})/\mathcal{I}(e^{-m_p\psi}))$$

is surjective.

(b) *Assume that E is a line bundle equipped with a singular metric h and that for some $\ell = 0, 1, \ldots, p-1$ and all $\alpha \in [m_{\ell+1}, m_p + \delta]$, we have*

$$\mathrm{i}\Theta_{E,h} + \alpha\,\mathrm{i}\partial\overline{\partial}\psi \geq 0 \quad \textit{in the sense of currents},$$

for $\delta > 0$ small enough. Then the restriction morphism

$$H^0(X, \mathcal{O}_X(K_X \otimes E) \otimes \mathcal{I}(he^{-m_\ell\psi})) \to H^0(Y^{(m_p)}, \mathcal{O}_X(K_X \otimes E) \otimes \mathcal{I}(he^{-m_\ell\psi})/\mathcal{I}(he^{-m_p\psi}))$$

is surjective.

(2.15) Question. It would be interesting to know whether Theorem 2.14 can be strengthened by suitable quantitative L^2 estimates. The main difficulty is already to define the norm of jets when there is more than one jump number involved. Some sort of "Cauchy inequality" for jets would be needed in order to derive the successive jet norms from a known global L^2 estimate for a holomorphic section defined on the whole of X. We do not know how to proceed further at this point.

3. Fundamental estimates of Kahler geometry

3.A. Basic set-up

We refer to [Gri66,69] and [Dem-X] for general background results on the geometry of Kähler manifolds and hermitian bundles. Let X be a complex n-dimensional manifold equipped with a smooth Hermitian metric

$$\omega = \sum_{1\le j,k\le n} \omega_{jk}(z)\, dz_j \otimes d\overline{z}_k.$$

As usual, it is convenient to view ω rather as a real $(1,1)$-form $\omega = i\sum \omega_{jk}(z)\, dz_j \wedge d\overline{z}_k$. The metric ω is said to be *Kähler* if $d\omega = 0$, i.e. ω also defines a symplectic structure. It can be easily shown that ω is Kähler if and only if there are holomorphic coordinates $(z_1, \ldots, z_n)$ centered at any point $x_0 \in X$ such that the matrix of coefficients (ω_{jk}) is tangent to identity at order 2, i.e.

$$\omega_{jk}(z) = \delta_{jk} + O(|z|^2) \quad \text{at } x_0. \tag{3.1}$$

Now, let (E, h) is a Hermitian vector bundle over X. Given a smooth (p, q)-form u on X with values in E, that is, a section of $\mathcal{C}^\infty(X, \Lambda^{p,q}T_X^* \otimes E)$, we consider the global L^2 norm

$$\|u\|^2 = \int_M |u(x)|^2 dV_{X,\omega}(x) \tag{3.2}$$

where $|u(x)| = |u(x)|_{\omega,h}$ is the pointwise Hermitian norm of $u(x)$ in the tensor product $\Lambda^{p,q}T_X^* \otimes E$ and $dV_{X,\omega} = \frac{\omega^n}{n!}$ the Hermitian volume form on X ; for simplicity, we will usually omit the dependence on the metrics in the notation of $|u(x)|_{\omega,h}$. We denote by $\langle\!\langle \bullet, \bullet \rangle\!\rangle$ the inner product of the Hilbert space $L^2(X, \Lambda^{p,q}T_X^* \otimes E)$ of L^2 sections for the norm $\|\bullet\|$. Let $D = D_{E,h}$ be the Chern connection of (E, h), that is, the unique connection $D = D^{1,0} + D^{0,1} = D' + D''$ that makes h parallel, for which the $(0,1)$ component D'' coincides with $\overline{\partial}$. It follows in particular that $D'^2 = 0$, $D''^2 = 0$ and $D^2 = D'D'' + D''D'$. The *Chern curvature tensor* of (E, h) is the $(1,1)$-form

$$\Theta_{E,h} \in \mathcal{C}^\infty(X, \Lambda^{1,1}T_X^* \otimes \mathrm{Hom}(E, E)) \tag{3.3}$$

such that $D^2 u = (D'D'' + D''D')u = \Theta_{E,h} \wedge u$. Next, the complex Laplace-Beltrami operators are defined by

$$\Delta' = D'D'^* + D'^*D', \qquad \Delta'' = D''D''^* + D''^*D'' \tag{3.4}$$

where P^* denotes the formal adjoint of a differential operator

$$P : \mathcal{C}^\infty(X, \Lambda^{p,q}T_X^* \otimes E) \to \mathcal{C}^\infty(X, \Lambda^{p+r,q+s}T_X^* \otimes E), \qquad \deg P = r + s$$

with respect to the corresponding inner products $\langle\!\langle \bullet, \bullet \rangle\!\rangle$. If u is a compactly supported section, we get in particular

$$\langle\!\langle \Delta' u, u \rangle\!\rangle = \|D'u\|^2 + \|D'^*u\|^2 \ge 0, \qquad \langle\!\langle \Delta'' u, u \rangle\!\rangle = \|D''u\|^2 + \|D''^*u\|^2 \ge 0.$$

3.B. Bochner-Kodaira-Nakano identity

Great simplifications occur in the operators of Hermitian geometry when the metric ω is Kähler. In fact, if we use normal coordinates at a point x_0 (cf. (3.1)), and a local holomorphic frame $(e_\lambda)_{1\le\lambda\le r}$ of E such that $De_\lambda(x_0)=0$, it is not difficult to see that all order 1 operators D', D'' and their adjoints D'^*, D''^* admit at x_0 the same expansion as the analogous operators obtained when all Hermitian metrics on X or E are constant. From this, the basic commutation relations of Kähler geometry can be checked. If A, B are differential operators acting on the algebra $\mathcal{C}^\infty(X,\Lambda^{\bullet,\bullet}T^*_X\otimes E)$, their graded commutator (or graded Lie bracket) is $[A,B]=AB-(-1)^{ab}BA$ where a,b are the degrees of A and B respectively. If C is another endomorphism of degree c, the following purely formal *Jacobi identity* holds:

$$(-1)^{ca}\big[A,[B,C]\big]+(-1)^{ab}\big[B,[C,A]\big]+(-1)^{bc}\big[C,[A,B]\big]=0.$$

(3.5) Fundamental Kähler identities. *Let (X,ω) be a Kähler manifold and let L be the Lefschetz operator defined by $Lu=\omega\wedge u$ and $\Lambda=L^*$ its adjoint, acting on E-valued forms. The following identities hold for the Chern connection $D=D'+D''$ on E and the associated complex Laplace operators Δ' and Δ''.*

(a) *Basic commutation relations*

$$[D''^*,L]=\mathrm{i}D',\quad [\Lambda,D'']=-\mathrm{i}D'^*,\quad [D'^*,L]=-\mathrm{i}D'',\quad [\Lambda,D']=\mathrm{i}D''^*.$$

(b) *Bochner-Kodaira-Nakano identity* ([Boc48], [Kod53a,b], [AN54], [Nak55])

$$\Delta''=\Delta'+[\mathrm{i}\Theta_{E,h},\Lambda].$$

Idea of proof. (a) The first step is to check the identity $[d''^*,L]=\mathrm{i}d'$ for constant metrics on $X=\mathbb{C}^n$ and the trivial bundle $E=X\times\mathbb{C}$, by a brute force calculation. All three other identities follow by taking conjugates or adjoints. The case of variable metrics follows by looking at Taylor expansions up to order 1.

(b) The last equality in (a) yields $D''^*=-\mathrm{i}[\Lambda,D']$, hence

$$\Delta''=[D'',D''^*]=-\mathrm{i}[D'',\big[\Lambda,D'\big]\big].$$

By the Jacobi identity we get

$$\big[D'',[\Lambda,D']\big]=\big[\Lambda,[D',D'']\big]+\big[D',[D'',\Lambda]\big]=[\Lambda,\Theta_{E,h}]+\mathrm{i}[D',D'^*],$$

taking into account that $[D',D'']=D^2=\Theta_{E,h}$. The formula follows. □

One important (well known) fact is that the curvature term $[\mathrm{i}\Theta_{E,h},\Lambda]$ operates as a hermitian (semi-)positive operator on $L^2(X,\Lambda^{n,q}T^*_X\otimes E$ as soon as $\mathrm{i}\,\Theta_{E,h}$ is Nakano (semi-)positive.

3.C. The twisted a priori inequality of Ohsawa and Takegoshi

The main a priori inequality that we are going to use is a simplified (and slightly extended) version of the original Ohsawa-Takegoshi a priori inequality [OT87, Ohs88], along the lines proposed by Manivel [Man93] and Ohsawa [Ohs01]. Such inequalities were originally introduced in the work of Donnelly-Fefferman [DF83] and Donnelly-Xavier [DX84]. The main idea is to introduce a modified Bochner-Kodaira-Nakano inequality. Although it has become classical in this context, we reproduce here briefly the calculations for completeness, and also for the sake of fixing the notation.

(3.6) Lemma (Ohsawa [Ohs01]). *Let E be a Hermitian vector bundle on a complex manifold X equipped with a Kähler metric ω. Let η, $\lambda > 0$ be smooth functions on X. Then for every form $u \in \mathcal{C}^\infty_c(X, \Lambda^{p,q}T^*_X \otimes E)$ with compact support we have*

$$\begin{aligned}\|(\eta+\lambda)^{\frac{1}{2}}D''^*u\|^2 + \|\eta^{\frac{1}{2}}D''u\|^2 + \|\lambda^{\frac{1}{2}}D'u\|^2 + 2\|\lambda^{-\frac{1}{2}}d'\eta\wedge u\|^2 \\ \geq \langle\langle[\eta\,\mathrm{i}\Theta_E - \mathrm{i}\,d'd''\eta - \mathrm{i}\lambda^{-1}d'\eta\wedge d''\eta, \Lambda]u, u\rangle\rangle.\end{aligned}$$

Proof. We consider the "twisted" Laplace-Beltrami operators

$$\begin{aligned}D'\eta D'^* + D'^*\eta D' &= \eta[D', D'^*] + [D', \eta]D'^* + [D'^*, \eta]D' \\ &= \eta\Delta' + (d'\eta)D'^* - (d'\eta)^*D', \\ D''\eta D''^* + D''^*\eta D'' &= \eta[D'', D''^*] + [D'', \eta]D''^* + [D''^*, \eta]D'' \\ &= \eta\Delta'' + (d''\eta)D''^* - (d''\eta)^*D'',\end{aligned}$$

where η, $(d'\eta)$, $(d''\eta)$ are abbreviated notations for the multiplication operators $\eta\bullet$, $(d'\eta)\wedge\bullet$, $(d''\eta)\wedge\bullet$. By subtracting the above equalities and taking into account the Bochner-Kodaira-Nakano identity $\Delta'' - \Delta' = [\mathrm{i}\Theta_E, \Lambda]$, we get

$$\begin{aligned}&D''\eta D''^* + D''^*\eta D'' - D'\eta D'^* - D'^*\eta D' \\ &\quad = \eta[\mathrm{i}\Theta_E, \Lambda] + (d''\eta)D''^* - (d''\eta)^*D'' + (d'\eta)^*D' - (d'\eta)D'^*. \qquad (3.7)\end{aligned}$$

Moreover, the Jacobi identity yields

$$[D'', [d'\eta, \Lambda]] - [d'\eta, [\Lambda, D'']] + [\Lambda, [D'', d'\eta]] = 0,$$

whilst $[\Lambda, D''] = -\mathrm{i}D'^*$ by the basic commutation relations 3.5 (a). A straightforward computation shows that $[D'', d'\eta] = -(d'd''\eta)$ and $[d'\eta, \Lambda] = \mathrm{i}(d''\eta)^*$. Therefore we get

$$\mathrm{i}[D'', (d''\eta)^*] + \mathrm{i}[d'\eta, D'^*] - [\Lambda, (d'd''\eta)] = 0,$$

that is,

$$[\mathrm{i}\,d'd''\eta, \Lambda] = [D'', (d''\eta)^*] + [D'^*, d'\eta] = D''(d''\eta)^* + (d''\eta)^*D'' + D'^*(d'\eta) + (d'\eta)D'^*.$$

After adding this to (3.7), we find

$$\begin{aligned}&D''\eta D''^* + D''^*\eta D'' - D'\eta D'^* - D'^*\eta D' + [\mathrm{i}\,d'd''\eta, \Lambda] \\ &\quad = \eta[\mathrm{i}\Theta_E, \Lambda] + (d''\eta)D''^* + D''(d''\eta)^* + (d'\eta)^*D' + D'^*(d'\eta).\end{aligned}$$

We apply this identity to a form $u \in \mathcal{C}^\infty_c(X, \Lambda^{p,q}T^*_X \otimes E)$ and take the inner bracket with u. Then

$$\langle\!\langle (D''\eta D''^*)u, u\rangle\!\rangle = \langle\!\langle \eta D''^*u, D''^*u\rangle\!\rangle = \|\eta^{\frac12} D''^*u\|^2,$$

and likewise for the other similar terms. The above equalities imply

$$\begin{aligned}&\|\eta^{\frac12} D''^*u\|^2 + \|\eta^{\frac12} D''u\|^2 - \|\eta^{\frac12} D'u\|^2 - \|\eta^{\frac12} D'^*u\|^2\\ &\qquad = \langle\!\langle [\eta\, \mathrm{i}\Theta_E - \mathrm{i}\, d'd''\eta, \Lambda]u, u\rangle\!\rangle + 2\,\mathrm{Re}\,\langle\!\langle D''^*u, (d''\eta)^*u\rangle\!\rangle + 2\,\mathrm{Re}\,\langle\!\langle D'u, d'\eta\wedge u\rangle\!\rangle.\end{aligned}$$

By neglecting the negative terms $-\|\eta^{\frac12} D'u\|^2 - \|\eta^{\frac12} D'^*u\|^2$ and adding the squares

$$\begin{aligned}\|\lambda^{\frac12} D''^*u\|^2 + 2\,\mathrm{Re}\,\langle\!\langle D''^*u, (d''\eta)^*u\rangle\!\rangle + \|\lambda^{-\frac12}(d''\eta)^*u\|^2 \geq 0,\\ \|\lambda^{\frac12} D'u\|^2 + 2\,\mathrm{Re}\,\langle\!\langle D'u, d'\eta\wedge u\rangle\!\rangle + \|\lambda^{-\frac12} d'\eta\wedge u\|^2 \geq 0\end{aligned}$$

we get

$$\begin{aligned}&\|\eta^{\frac12} D''^*u\|^2 + \|\lambda^{\frac12} D''u\|^2 + \|\lambda^{\frac12} D'u\|^2 + \|\lambda^{-\frac12} d'\eta\wedge u\|^2 + \|\lambda^{-\frac12}(d''\eta)^*u\|^2\\ &\qquad \geq \langle\!\langle [\eta\, \mathrm{i}\Theta_E - \mathrm{i}\, d'd''\eta, \Lambda]u, u\rangle\!\rangle.\end{aligned} \tag{3.8}$$

Finally, we use the identity $a^* = i[\overline{a}, \Lambda]$ for any $(1,0)$-form a to get

$$(d'\eta)^*(d'\eta) - (d''\eta)(d''\eta)^* = \mathrm{i}[d''\eta, \Lambda](d'\eta) + \mathrm{i}(d''\eta)[d'\eta, \Lambda] = [\mathrm{i}d''\eta\wedge d'\eta, \Lambda],$$

which implies

$$\|\lambda^{-\frac12} d'\eta\wedge u\|^2 - \|\lambda^{-\frac12}(d''\eta)^*u\|^2 = -\langle\!\langle [\mathrm{i}\lambda^{-1} d'\eta\wedge d''\eta, \Lambda]u, u\rangle\!\rangle. \tag{3.9}$$

The inequality asserted in Lemma 3.6 follows by adding (3.8) and (3.9). □

In the special case of (n,q)-forms, the forms $D'u$ and $d'\eta\wedge u$ are of bidegree $(n+1,q)$, hence the estimate takes the simpler form

$$\|(\eta+\lambda)^{\frac12} D''^*u\|^2 + \|\eta^{\frac12} D''u\|^2 \geq \langle\!\langle [\eta\, \mathrm{i}\Theta_E - \mathrm{i}\, d'd''\eta - \mathrm{i}\lambda^{-1}\, d'\eta\wedge d''\eta, \Lambda]u, u\rangle\!\rangle. \tag{3.10}$$

3.D. Abstract L^2 existence theorem for solutions of $\overline{\partial}$-equations

Using standard arguments from functional analysis – actually just basic properties of Hilbert spaces – the a priori inequality (3.10) implies a powerful L^2 existence theorem for solutions of $\overline{\partial}$-equations.

(3.11) Proposition. *Let X be a complete Kähler manifold equipped with a (non necessarily complete) Kähler metric ω, and let (E,h) be a Hermitian vector bundle over X. Assume that there are smooth and bounded functions η, $\lambda > 0$ on X such that the (Hermitian) curvature operator*

$$B = B^{n,q}_{E,h,\omega,\eta,\lambda} = [\eta\, \mathrm{i}\Theta_{E,h} - \mathrm{i}\, d'd''\eta - \mathrm{i}\lambda^{-1} d'\eta\wedge d''\eta, \Lambda_\omega]$$

is positive definite everywhere on $\Lambda^{n,q}T^*_X \otimes E$, for some $q \geq 1$. Then for every form $g \in L^2(X, \Lambda^{n,q}T^*_X \otimes E)$ such that $D''g = 0$ and $\int_X \langle B^{-1}g, g\rangle\, dV_{X,\omega} < +\infty$, there exists $f \in L^2(X, \Lambda^{n,q-1}T^*_X \otimes E)$ such that $D''f = g$ and

$$\int_X (\eta+\lambda)^{-1}|f|^2\, dV_{X,\omega} \leq \int_X \langle B^{-1}g, g\rangle\, dV_{X,\omega}.$$

Proof. Assume first that ω is complete Kähler metric. Let $v \in L^2(X, \Lambda^{n,q}T_X^* \otimes E)$, and $v = v_1 + v_2 \in (\operatorname{Ker} D'') \oplus (\operatorname{Ker} D'')^\perp$ the decomposition of v with respect to the closed subspace $\operatorname{Ker} D''$ and its orthogonal. Since $g \in \operatorname{Ker} D''$, The Cauchy-Schwarz inequality yields

$$|\langle\langle g, v\rangle\rangle|^2 = |\langle\langle g, v_1\rangle\rangle|^2 = |\langle\langle B^{-\frac{1}{2}} g, B^{\frac{1}{2}} v_1\rangle\rangle|^2 \le \int_X \langle B^{-1} g, g\rangle \, dV_{X,\omega} \int_X \langle B v_1, v_1\rangle \, dV_{X,\omega},$$

and provided that $v \in \operatorname{Dom} D''^*$, we find $v_2 \in (\operatorname{Ker} D'')^\perp \subset (\operatorname{Im} D'')^\perp = \operatorname{Ker} D''^*$, and so $D'' v_1 = 0$, $D''^* v_2 = 0$, whence

$$\int_X \langle B v_1, v_1\rangle \, dV_{X,\omega} \le \|(\eta+\lambda)^{\frac{1}{2}} D''^* v_1\|^2 + \|\eta^{\frac{1}{2}} D'' v_1\|^2 = \|(\eta+\lambda)^{\frac{1}{2}} D''^* v\|^2.$$

Combining both inequalities, we obtain

$$|\langle\langle g, v\rangle\rangle|^2 \le \Big(\int_X \langle B^{-1} g, g\rangle \, dV_{X,\omega}\Big) \|(\eta+\lambda)^{\frac{1}{2}} D''^* v\|^2.$$

The Hahn-Banach theorem applied to the linear form $(\eta+\lambda)^{\frac{1}{2}} D''^* v \mapsto \langle\langle v, g\rangle\rangle$ implies the existence of an element $w \in L^2(X, \Lambda^{n,q}T_X^* \otimes E)$ such that

$$\|w\|^2 \le \int_X \langle B^{-1} g, g\rangle \, dV_{X,\omega} \qquad \text{and}$$
$$\langle\langle v, g\rangle\rangle = \langle\langle (\eta+\lambda)^{\frac{1}{2}} D''^* v, w\rangle\rangle \qquad \forall g \in \operatorname{Dom} D'' \cap \operatorname{Dom} D''^*.$$

It follows that $f = (\eta+\lambda)^{\frac{1}{2}} w$ satisfies $D'' f = g$ as well as the desired L^2 estimate. If ω is not complete, we set $\omega_\varepsilon = \omega + \varepsilon\widehat{\omega}$ with some complete Kähler metric $\widehat{\omega}$. The final conclusion is then obtained by passing to the limit and using a monotonicity argument (the integrals are easily shown to be monotonic with respect to ε, see [Dem82]). □

We need also a variant of the L^2-estimate, so as to obtain approximate solutions with weaker requirements on the data :

(3.12) Proposition. *With the notation of* 3.11*, assume that* $B + \varepsilon I > 0$ *for some* $\varepsilon > 0$ (*so that* B *can be just semi-positive or even slightly negative; here* I *is the identity endomorphism*). *Given a section* $g \in L^2(X, \Lambda^{n,q}T_X^* \otimes E)$ *such that* $D'' g = 0$ *and*

$$M(\varepsilon) := \int_X \langle (B + \varepsilon I)^{-1} g, g\rangle \, dV_{X,\omega} < +\infty,$$

there exists an approximate solution $f_\varepsilon \in L^2(X, \Lambda^{n,q-1}T_X^* \otimes E)$ *and a correcting term* $g_\varepsilon \in L^2(X, \Lambda^{n,q}T_X^* \otimes E)$ *such that* $D'' f_\varepsilon = g - g_\varepsilon$ *and*

$$\int_X (\eta+\lambda)^{-1} |f_\varepsilon|^2 \, dV_{X,\omega} + \frac{1}{\varepsilon} \int_X |g_\varepsilon|^2 \, dV_{X,\omega} \le M(\varepsilon).$$

If g *is smooth, then* f_ε *and* g_ε *can be taken smooth.*

Proof. The arguments are almost unchanged, we rely instead on the estimates

$$|\langle\!\langle g, v_1\rangle\!\rangle|^2 \le \int_X \langle (B+\varepsilon I)^{-1} g, g\rangle\, dV_{X,\omega} \int_X \langle (B+\varepsilon I)v_1, v_1\rangle\, dV_{X,\omega},$$

and

$$\int_X \langle (B+\varepsilon I)v_1, v_1\rangle\, dV_{X,\omega} \le \|(\eta+\lambda)^{\frac12} D''^* v\|^2 + \varepsilon\|v\|^2.$$

This gives a pair $(w_\varepsilon, w'_\varepsilon)$ such that $\|w_\varepsilon\|^2 + \|w'_\varepsilon\|^2 \le M(\varepsilon)$ and

$$\langle\!\langle v, g\rangle\!\rangle = \langle\!\langle (\eta+\lambda)^{\frac12} D''^* v, w_\varepsilon\rangle\!\rangle + \langle\!\langle \varepsilon^{1/2} v, w'_\varepsilon\rangle\!\rangle \quad \text{for all } v \in \mathrm{Dom}\, D''^*, \tag{3.13}$$

hence $f_\varepsilon = (\eta+\lambda)^{\frac12} w_\varepsilon$ is the expected approximate solution with error term $g_\varepsilon = \varepsilon^{1/2} w'_\varepsilon$. By (3.13), we do get $D'' f_\varepsilon + g_\varepsilon = g$, and the expected L^2 estimates hold as well. In fact one can take

$$f_\varepsilon = (\eta+\lambda) D''^* \Box_\varepsilon^{-1} g \quad \text{and} \quad g_\varepsilon = \varepsilon\, \Box_\varepsilon^{-1} g$$

where $\Box_\varepsilon = D''(\eta+\lambda)D''^* + D''^*(\eta+\lambda)D'' + \varepsilon I$ is an invertible self-adjoint elliptic operator (the 3 terms involved in $\Box_\varepsilon$ commute). Then f_ε and g_ε are smooth. □

4. Openness of multiplier ideal sheaves and jumping numbers

Let X be complex manifold and φ, ψ quasi-psh functions on X. To every $m \in \mathbb{R}_+$ we associate the multiplier ideal sheaf $\mathcal{I}(\varphi + m\psi) \subset \mathcal{O}_X$. By Nadel [Nad89], this is a coherent ideal sheaf, and $\mathcal{I}(\varphi + m'\psi) \subset \mathcal{I}(\varphi + m\psi)$ for $m' \ge m$. The recent result of Guan-Zhou [GZ13] implies that one has in fact $\mathcal{I}(\varphi + (m+\alpha)\psi) = \mathcal{I}(\varphi + m\psi)$ for every $m \in \mathbb{R}_+$ and $\alpha \in [0, \alpha_0(m)[$ sufficiently small. From this we conclude without any restriction that there exists a discrete sequence of numbers

$$0 = m_0 < m_1 < m_2 < \ldots < m_p < \ldots \tag{4.1}$$

such that $\mathcal{I}(\varphi + m\psi) = \mathcal{I}(\varphi + m_p\psi)$ for $m \in [m_p, m_{p+1}[$ and

$$\mathcal{I}(\varphi) = \mathcal{I}(\varphi + m_0\psi) \supsetneq \mathcal{I}(\varphi + m_1\psi) \supsetneq \ldots \supsetneq \mathcal{I}(\varphi + m_l\psi) \supsetneq \ldots\ .$$

If ψ is smooth, we have of course $m_1 = +\infty$ already, and the sequence stops there; in the sequel we assume that ψ has non empty logarithmic poles to avoid this trivial situation – the sequence m_p is then infinite since the multiplicities of germs of functions in $\mathcal{I}(\varphi + m\psi)_x$ tend to infinity at every point x where the Lelong number $\nu(\psi, x)$ is positive.

(4.2) Lemma. *For every $p > 0$, the ideal $\mathcal{J}_p \subsetneq \mathcal{O}_X$ of germs of holomorphic functions h such that $h\,\mathcal{I}(\varphi + m_{p-1}\psi) \subset \mathcal{I}(\varphi + m_p\psi)$ is reduced, i.e. $\sqrt{\mathcal{J}_p} = \mathcal{J}_p$. Moreover $\mathcal{J}_p$ contains $\sqrt{\mathcal{I}(m_p\psi)}$.*

Proof. Let $h^k \in \mathcal{J}_{p,x}$ for some exponent $k \ge 2$. Pick $f \in \mathcal{I}(\varphi + m_{p-1}\psi)_x$. By definition of the jumping numbers, $|f|^2 e^{-\varphi-(m_p-\varepsilon)\psi}$ is integrable near x for every $\varepsilon > 0$. Since $h^k f \in \mathcal{I}(\varphi + m_p\psi)$, the openness property shows that $|h^k f|^2 e^{-\varphi-(m_p+\delta)\psi}$ is integrable for some $\delta > 0$. For a suitable neighborhood U of x and $\varepsilon > 0$ smaller than δ/k, the Hölder inequality applied with the measure $d\mu = |f|^2 e^{-\varphi - m_p\psi} d\lambda$ and the functions $v = |h|^2 e^{-\varepsilon\psi}$, $w = e^{\varepsilon\psi}$ for the conjugate exponents $1/k + 1/\ell = 1$ implies

$$\int_U |hf|^2 e^{-\varphi - m_p\psi} d\lambda = \int_U vw d\mu \leq \Big(\int_U |h|^{2k}|f|^2 e^{-\varphi-(m_p+k\varepsilon)\psi} d\lambda\Big)^{1/k}\Big(\int_U |f|^2 e^{-\varphi-(m_p-\ell\varepsilon)\psi} d\lambda\Big)^{1/\ell} < +\infty,$$

hence $hf \in \mathcal{I}(\varphi + m_p\psi)$. Since this is true for every $f \in \mathcal{I}(\varphi + m_{p-1}\psi)$ we conclude that $h \in \mathcal{J}_{p,x}$, thus $\sqrt{\mathcal{J}_p} = \mathcal{J}_p$. The last assertion is equivalent to $\mathcal{J}_p \supset \mathcal{I}(m_p\psi)$ and follows similarly from the inequality

$$\int_U |hf|^2 e^{-\varphi - m_p\psi} d\lambda \leq \Big(\int_U |h|^{2k} e^{-(m_p+k\varepsilon/\ell)\psi} d\lambda\Big)^{1/k}\Big(\int_U |f|^{2\ell} e^{-\ell\varphi-(m_p-\varepsilon)\psi} d\lambda\Big)^{1/\ell} < +\infty\,;$$

we fix here $\varepsilon > 0$ so small that $\mathcal{I}(m_p\psi) = \mathcal{I}((m_p+\varepsilon)\psi)$ and, by openness, ℓ sufficiently close to 1 to make the last integral convergent whenever $f \in \mathcal{I}(\varphi + m_{p-1}\psi) = \mathcal{I}(\varphi + (m_p - \varepsilon)\psi)$. □

A consequence of Lemma 4.2 is that the zero variety $Z_p = V(\mathcal{J}_p)$ is a reduced subvariety of $Y^{(m_p)} = V(\mathcal{I}(m_p\psi))$ and that the quotient sheaf $\mathcal{I}(\varphi + m_{p-1}\psi)/\mathcal{I}(\varphi + m_p\psi)$ is a coherent sheaf over $\mathcal{O}_{Z_p} = \mathcal{O}_X/\mathcal{J}_p$. Therefore $\mathcal{I}(\varphi + m_{p-1}\psi)/\mathcal{I}(\varphi + m_p\psi)$ can be seen as a vector bundle on some Zariski open set $Z_p^\circ \subset Z_p \subset Y^{(m_p)}_{\rm red}$.

In the sequel, a case of special interest is when ψ has analytic singularities, that is, every point $x_0 \in X$ possesses an open neighborhood $V \subset X$ on which ψ can be written

$$\psi(z) = c \log \sum_{1\leq j\leq N} |g_k(z)|^2 + u(z)$$

where $c \geq 0$, $g_k \in \mathcal{O}_X(V)$ and $u \in \mathcal{C}^\infty(V)$. The integrability of $|f|^2 e^{-m\psi}$ on a coordinate neighborhood V then means that

$$\int_V \frac{|f(z)|^2}{|g(z)|^{2m}}\, d\lambda(z) < +\infty. \tag{4.3}$$

By Hironaka [Hir64], there exists a principalization of the ideal $\mathcal{J} = (g_k) \subset \mathcal{O}_X$, that is, a modification $\mu : \widehat{X} \to X$ such that $\mu^*\mathcal{J} = (g_k \circ \mu) = \mathcal{O}_{\widehat{X}}(-\Delta)$ where $\Delta = \sum a_k \Delta_k$ is a simple normal crossing divisor on $\widehat{X}$. We can also assume that the Jacobian $\mathrm{Jac}(\mu)$ has a zero divisor $B = \sum b_k \Delta_k$ contained in the exceptional divisor. After a change of variables $z = \mu(w)$ and a use of local coordinates where $\Delta_k = \{w_k = 0\}$ and $\mu^*\mathcal{J}$ is the principal ideal generated by the monomial $w^a = \prod w_k^{a_k}$, we see that (4.3) is equivalent to the convergence of

$$\int_{\mu^{-1}(V)} \frac{|f(\mu(w))|^2 |\,\mathrm{Jac}(\mu)|^2}{|g(\mu(w)|^{2mc}}\, d\lambda(w),$$

which can be expressed locally as the convergence of

$$\int_{\mu^{-1}(V)} \frac{|f(\mu(w))|^2 |w^b|^2}{|w^a|^{2mc}}\, d\lambda(w).$$

For this, the condition is that $f\circ\mu(w)$ be divisible by w^s with $s_k=\lfloor mca_k-b_k\rfloor_+$. In other words, the multipler ideal sheaves $\mathcal{I}(m\psi)$ are given by the direct image formula

$$\mathcal{I}(m\psi)=\mu_*\mathcal{O}_{\widehat X}\big(-\sum_k\lfloor mca_k-b_k\rfloor_+\Delta_k\big). \tag{4.4}$$

The jumps can only occur when m is equal to one of the values $\frac{b_k+N}{ca_k}$, $N\in\mathbb{N}$, which form a discrete subset of $\mathbb{R}_+$.

(4.5) Proposition. *Let $0=m_0<m_1<\ldots<m_p$ be the jumping numbers of the quasi-psh function ψ, which is assumed to have neat analytic singularities. Let ℓ a local holomorphic generator of $K_X\otimes E$ at a point $x_0\in X$, E being equipped with a smooth hermitian metric h, let $Z_p=\mathrm{Supp}(\mathcal{I}(m_{p-1}\psi)/\mathcal{I}(m_p\psi))$, and take a germ $f\in\mathcal{I}(m_{p-1}\psi)_{x_0}$.*

(a) *The measure $|J^{m_p}(f\ell)|^2_{\omega,h}\,dV_{Z_p^\circ,\omega}[\psi]$ defined by (2.10) has a smooth positive density with respect to the Lebesgue measure on a Zariski open set Z_p° of Z_p.*

(b) *The sheaf $\mathcal{I}'(m_{p-1}\psi)$ of germs $F\in\mathcal{I}(m_{p-1}\psi)_x\subset\mathcal{O}_{X,x}$ such that there exists a neighborhood U of x satisfying*

$$\int_{Y^{(m_p)}\cap U}|J^{m_p}(F\ell)|^2_{\omega,h}\,dV_{Y^{(m_p)},\omega}[\psi]<+\infty$$

is a coherent ideal sheaf such that

$$\mathcal{I}(m_p\psi)\subset\mathcal{I}'(m_{p-1}\psi)\subset\mathcal{I}(m_{p-1}\psi).$$

Both of the inclusions can be strict (even for $p=1$).

(c) *The function $(1+|\psi|)^{-(n+1)}\,|f\ell|^2_{\omega,h}e^{-m_p\psi}$ is locally integrable at x_0.*

Proof. (a) As above, we use a principalization of the singularities of ψ and apply formula (4.4). Over a generic point $x_0\in Z_p$, the component of Z_p containing x_0 is dominated by exactly one of the divisors Δ_k, and the jump number m_p is such that $m_pca_k-b_k=N$ for some integer N. The previous jump number m_{p-1} is then given by $m_{p-1}ca_k-b_k=N-1$. On a suitable coordinate chart of the blow-up $\mu:\widehat X\to X$, let us write $\psi\circ\mu(w)=c\log|w^a|^2+u(w)$ where u is smooth, and let $w^b=0$ be the zero divisor of $\mathrm{Jac}(\mu)$. By definition, the measure $|J^{m_p}f|^2_{\omega,h}\,dV_{Z_p^\circ,\omega}[\psi]$ is the direct image of measures defined upstairs as the limit

$$g\in\mathcal{C}_c(Z_p,\mathbb{R})\mapsto\limsup_{t\to-\infty}\int_{t<c\log|w^a|^2+u(w)<t+1}e^{-m_pu}\,\mu^*\widetilde g\,\beta(w)\,|\widetilde v(w)|^2\frac{e^{-\varphi\circ\mu}}{|w_k|^2}\,d\lambda(w).$$

Here φ is the weight of the metric h on E, $\widetilde v(w)$ is the holomorphic function representing the section $\mu^*(f\ell)/(w^{m_pca-b}/w_k)$ (the denominator divisor cancels with the numerator by construction), and $\beta(w)$ is a smooth positive weight arising from the change of variable formula, given by $\mu^*dV_{X,\omega}/|w^b|^2$ [one would still have to take into account a partition of unity on the various coordinate charts covering the fibers of μ, but we will avoid this technicality for the simplicity of notation]. Let us denote $w=(w',w_k)\in\mathbb{C}^{n-1}\times\mathbb{C}$ and $d\lambda(w)=d\lambda(w')\,\lambda(w_k)$ the Lebesgue measure on $\mathbb{C}^n$. At a generic point w where $w_j\neq0$

for $j \neq k$, the domain of integration is of the form $t(w') < ca_k \log|w_k|^2 < t'(w') + 1$. It is easy to check that the limsup measure is a limit, equal to

$$g \in \mathcal{C}_c(Z_p, \mathbb{R}) \mapsto \pi c a_k \left(e^{-m_p u}\, \mu^*\widetilde{g}\, |\widetilde{v}(w)|^2 \beta(w) e^{-\varphi\circ\mu}\right)_{\restriction w_k=0} d\lambda(w').$$

We then have to integrate the right hand side measure over the fibers of μ to get the density of this measure along Z_p. Since μ can be taken to be a composition of blow-ups with smooth centers, the fibration has (upstairs) a locally trivial product structure over a Zariski open set $Z_p^\circ \subset Z_p$. Smooth local vector fields on Z_p° can be lifted to smooth vector fields in the corresponding chart of $\widehat{X}$, and we conclude by differentiating under the integral sign that the density downstairs on Z_p is generically smooth.

(b) One typical example is given by $\psi(w) = \log|w_1|^2|w_2|^2$. Let us put $\Delta_k = \{w_k = 0\}$, $k = 1, 2$. Then the jumping numbers are $m_p = p \in \mathbb{N}$, and we get

$$\mathcal{I}(m_p\psi) = \mathcal{O}_X(-p(\Delta_1 + \Delta_2)).$$

For any $r_0 > 0$ fixed, we have

$$\int_{|w_1|<r_0,\,|w_2|<r_0,\,e^t<|w_1|^2|w_2|^2<e^{t+1}} \frac{1}{|w_1|^2|w_2|^2}\, d\lambda(w_1)\, d\lambda(w_2) \geq \pi \int_{e^{t+1}/r_0^2<|w_2|^2<r_0^2} \frac{1}{|w_2|^2}\, d\lambda(w_2)$$

and the limit as $t \to -\infty$ is easily seen to be infinite. Therefore $\mathcal{I}'(m_0\psi) = \mathcal{I}_{\Delta_1\cap\Delta_2}$, and we do have

$$\mathcal{O}_X(-(\Delta_1 + \Delta_2)) = \mathcal{I}(m_0\psi) \subsetneq \mathcal{I}'(m_0\psi) \subsetneq \mathcal{I}(m_0\psi) = \mathcal{O}_X$$

in this case. In general, with our notation, it is easy to see that $\mathcal{I}'(m_{p-1}\psi)$ is given as the direct image

$$\mathcal{I}'(m_{p-1}\psi) = \mu_*\Big(\mathcal{O}_{\widehat{X}}\big(-\sum_k \lfloor m_{p-1}ca_k - b_k\rfloor_+\Delta_k\big) \otimes \mathcal{I}_R\Big) \tag{4.6}$$

where $R = \bigcup \Delta_\ell \cap \Delta_{\ell'}$ is the union of pairwise intersections of divisors Δ_ℓ for which $m_p ca_\ell - b_\ell = m_p ca_k - b_k\ (= m_{p-1}ca_k - b_k + 1)$, k being one of the indices achieving the values of m_p, m_{p-1} at the given point $x \in Z_p$. This is a coherent ideal sheaf by the direct image theorem.

(c) Near a point where $\psi \circ \mu(w)$ has singularities along normal crossing divisors $w_j = 0$, $1 \leq j \leq k$, it is sufficient to show that integrals of the form

$$I_k = \int_{|w_1|<1/2,\ldots,|w_k|<1/2} \frac{\big(-\log|w_1|^2\ldots|w_k|^2\big)^{-(k+1)}}{|w_1|^2\ldots|w_k|^2}\, d\lambda(w_1)\ldots d\lambda(w_k)$$

are convergent. This is easily done by induction on k, by using a change of variable $w_k' = w_1 \ldots w_k$, and by applying the Fubini formula together with an integration in w_k'. We then get $I_k \leq \frac{4\pi}{k} I_{k-1}$ (notice that $|w_k'| < |w_1|\ldots|w_{k-1}|$), and $I_1 = \pi/(2\log 2)$, thus all I_k are finite. □

5. Proof of the L^2 extension theorems

Unless otherwise specified, X denotes a weakly pseudoconvex complex n-dimensional manifold equipped with a (non necessarily complete) Kähler metric ω, $\rho : X \to [0, +\infty[$ a smooth psh exhaustion on X, (E, h) a smooth hermitian holomorphic vector bundle, and $\psi : X \to [-\infty, +\infty[$ a quasi-psh function with neat analytic singularities. Before giving technical details of the proofs, we start with a rather simple observation.

(5.1) Observation. Let $\mu : \widehat{X} \to X$ be a proper modification. Assume that $\widehat{X}$ is equipped with a Kähler metric $\widehat{\omega}$.

(a) For every $m \geq 0$, there is an isomorphism

$$\mu^* : H^0(X, \mathcal{O}(K_X \otimes E) \otimes \mathcal{I}(m\psi)) \to H^0(X, \mathcal{O}(K_{\widehat{X}} \otimes \mu^* E) \otimes \mathcal{I}(m\psi \circ \mu))$$

whose inverse is the direct image morphism μ_*.

(b) For any holomorphic section F of E, the L^2 norm

$$\int_{\widehat{X}} |F \circ \mu|^2 e^{-m\psi\circ\mu} dV_{\widehat{X},\widehat{\omega}}$$

coincides with $\int_X |F|^2 e^{-m\psi} dV_{X,\omega}$.

(c) On the regular part of the subvariety $Y^{(m_p)} = V(\mathcal{I}(m_p\psi))$, for any

$$f \in H^0(Y, \mathcal{O}_X(K_X \otimes E) \otimes \mathcal{I}(m_{p-1}\psi)/\mathcal{I}(m_p\psi)),$$

the area measure $|f|^2_{\omega,h} dV_{Y^{(m_p)},\omega}$ (which is independent of ω) is the direct image of its counterpart defined by $\widehat{\psi} = \psi \circ \mu$ and $\widehat{f} = \mu^* f$ on the strict transform of $Y^{(m_p)}$, i.e. the union of components of $\widehat{Y}^{(m_p)} = \mu^{-1}(Y^{(m_p)})$ that have a dominant projection to a component of $Y^{(m_p)}$. □

The proof of (c) is immediate by a change of variable $z = \mu(w)$ in the integrals $\int_{\{t<\psi<t+1\}} \ldots$ and by passing to the limits. It follows from the observation and the discussion of section 4 that after blowing up the proof of our theorems can be reduced to the case where ψ has divisorial singularities along a normal crossing divisor.

Proof of Theorem 2.8. This will be only a mild generalization of the techniques used in [Ohs01], with a technical complication due to the fact that we do not assume any Steinness of complements of negligible sets. With the notation of Theorem 2.8, let $f \in H^0(Y^\circ, (K_X \otimes E)_{\restriction Y^\circ})$. We view f as an E-valued $(n, 0)$-form defined over Y° and apply Proposition 3.11 after replacing the metric h of E by the singular metric $h_\psi = h^{-\psi}$, whose curvature is

$$\mathrm{i}\,\Theta_{E,h_\psi} = \mathrm{i}\,\Theta_{E,h} + \mathrm{i}\,d'd''\psi.$$

In order to avoid the singularities, we shrink X to a relatively compact weakly pseudoconvex domain $X_c = \{\rho < c\}$, and work on $X_c \smallsetminus Y$ instead of X. In fact, we know by [Dem82, Theorem 1.5] that $X_c \smallsetminus Y$ is complete Kähler. Let us first assume that Y is non singular, i.e. $Y^\circ = Y$, and that $\psi \leq 0$; of course, when X is compact, one can always subtract a constant to ψ to achieve $\psi \le 0$, but there could exist non compact situations when it is interesting to take ψ unbounded from above. In any case, we claim that there exists a smooth section

$$\widetilde{f} \in \mathcal{C}^\infty(X, \Lambda^{n,0}T^*_X \otimes E)$$

such that

(5.2) $\widetilde{f}$ coincides with f on Y,

(5.3) $D''\widetilde{f} = 0$ at every point of Y,

(5.4) $|D''\widetilde{f}|^2_{\omega,h}e^{-\psi}$ is locally integrable near Y.

For this, consider a locally finite covering of Y by coordinates patches $U_j \subset X$ biholomorphic to polydiscs, on which $E_{\restriction U_j}$ is trivial, and such that either $Y \cap U_j = \emptyset$ or

$$Y \cap U_j = \{z \in U_j\,;\ z_1 = \cdots = z_r = 0\}.$$

We can find holomorphic sections

$$\widetilde{f}_j \in \mathcal{C}^\infty(X, \Lambda^{n,q}T^*_X \otimes E)$$

which extend $f_{\restriction Y \cap U_j}$ [such sections can be obtained simply by viewing functions of the form $g(z_{r+1}, \ldots, z_n)$ as independent of $z_1, \ldots, z_r$ on each polydisc]. For some partition of unity (χ_j) subordinate to (U_j), we then set $\widetilde{f} := \sum_j \chi_j \widetilde{f}_j$. Clearly (5.2) $\widetilde{f}_{\restriction Y} = f$ holds. Since we have

$$D''\widetilde{f} = D''(\widetilde{f} - \widetilde{f}_k) = D''\Big(\sum_j \chi_j(\widetilde{f}_j - \widetilde{f}_k)\Big) = \sum_j d''\chi_j \wedge (\widetilde{f}_j - \widetilde{f}_k),$$

properties (5.3) and (5.4) also hold, as $\widetilde{f}_j - \widetilde{f}_k = 0$ on Y and $\mathcal{I}(\psi) = \mathcal{I}_Y$ is reduced by our assumption that ψ has log canonical singularities. The main idea is to apply Proposition 3.11 to solve the equation

$$D''u_t = v_t := D''(\theta(\psi - t)\,\widetilde{f}), \qquad t \in\,]-\infty, -1], \tag{5.5}$$

where $\theta : [-\infty, +\infty[\, \to [0,1]$ is a smooth non increasing function such that $\theta(\tau) = 1$ for $\tau \in\,]-\infty, \varepsilon/3]$, $\theta(\tau) = 0$ for $\tau \in [\varepsilon/3, +\infty[$ and $|\theta'| \le 1 + \varepsilon$, for any positive $\varepsilon \ll 1$. First assume for simplicity that $D''\widetilde{f} = 0$ without any error (i.e. that $\widetilde{f}$ is globally holomorphic). Then

$$v_t = D''(\theta(\psi - t)\,\widetilde{f}) = \theta'(\psi - t)\, d''\psi \wedge \widetilde{f} \tag{5.6}$$

has support in the tubular domain $W_t = \{t < \psi < t + 1\}$. At the same time, we adjust the functions $\eta = \eta_t$ and $\lambda = \lambda_t$ used in Prop. 3.11 to create enough convexity on W_t. For this, we take

$$\eta_t = 1 - \delta\chi_t(\psi) \tag{5.7}$$

where $\chi_t :\ ]-\infty, 0] \to \mathbb{R}$ is a negative smooth convex increasing function with the following properties:

(5.8 a) $\chi_t(0) = 0$ and $\inf_{\tau\le 0} \chi_t(\tau) = -M_t > -\infty$,

(5.8 b) $0 \le \chi_t'(\tau) \le \frac{1}{2}$ for $\tau \le 0$,

(5.8 c) $\chi_t'(\tau) = 0$ for $\tau \in\,]-\infty, t-1]$, $\chi_t'(\tau) > 0$ for $\tau \in\,]t-1, 0]$,

(5.8 d) $\chi_t''(\tau) \ge \dfrac{1-\varepsilon}{4}$ for $\tau \in [t, t+1]$.

(5.8 e) $\dfrac{\chi_t''(\tau)}{\chi_t'(\tau)^2} \ge \dfrac{2\delta}{\pi(1+\delta^2\tau^2)}$ for $\tau \in\,]t-1, 0]$.

The function χ_t can be easily constructed by taking

$$\chi_t''(\tau) = \frac{\delta}{2\pi(1+\delta^2\tau^2)}\,\beta(\tau - t) + \frac{1-\varepsilon}{4}\,\xi(\tau - t) \tag{5.9}$$

with support in $[t-1, 0]$, where $\beta : \mathbb{R} \to [0,1]$ is a smooth non decreasing function such that $\beta(\tau) = 0$ for $\tau < -1$, $\beta(\tau) = 1$ for $\tau \ge 0$ and $\xi(\tau) = \beta(\tau)\beta(1-\tau)$. We then have $\mathrm{Supp}(\beta) = [-1, +\infty[$, $\mathrm{Supp}(\xi) = [-1, 2]$, and $\xi(\tau) = \beta(\tau)$ on $[-1, 0]$. On $[-1, 0[$, we can take β so small that $\int_{-1}^0 \beta(\tau)\,d\tau < \varepsilon/2$. By symmetry we find $\int_{-1}^2 \xi(\tau)\,d\tau < 1+\varepsilon$. Clearly (5.8 a,c,d) hold if we adjust the integration constant so that $\chi_t'(t-1) = 0$. Now, the right hand side of (5.8) has a total integral on $]-\infty, 0]$ that is less than $\frac{(1-\varepsilon)}{4}(1+\varepsilon) + \frac{1}{2\pi}\frac{\pi}{2} < \frac{1}{2}$, thus (5.8 b) is satisfied. This implies that (5.8 e) holds at least on the interval $[t, 0]$. An integration by parts yields

$$\chi_t'(\tau) \le \Big(\frac{\delta}{2\pi} + \frac{1-\varepsilon}{4}\Big)\widetilde{\beta}(\tau - t) \quad \text{on } [t-1, t],$$

where $\widetilde{\beta} \ge 0$ is a primitive of β vanishing at $\tau = -1$. On $[t-1, t]$ we find

$$\frac{\chi_t''(\tau)}{\chi_t'(\tau)^2} \ge \frac{1-\varepsilon}{4}\Big(\frac{\delta}{2\pi} + \frac{1-\varepsilon}{4}\Big)^{-2}\frac{\beta(\tau - t)}{\widetilde{\beta}(\tau - t)^2}. \tag{5.10}$$

On $[-1, 0]$ we have $\widetilde{\beta}(\tau) \le (1+\tau)\beta(\tau) \le \beta(\tau)$ and $\widetilde{\beta}(\tau) \le \int_{-1}^0 \beta(\tau)\,d\tau < \varepsilon/2$, and we see that the right hand side of (5.10) can be taken arbitrary large when ε is small. Therefore (5.8 e) can also be achieved on $[t-1, t]$.

We now come back to the choice of η_t and λ_t. Since we have assumed $\psi \le 0$ at this step, we have by definition $\eta_t \ge 1$. Moreover, $d'\eta_t = -\delta\,\chi_t'(\psi)\,d'\psi$ and

$$\mathrm{i}\,d'd''\chi_t(\psi) = \mathrm{i}\,\chi_t'(\psi)\,d'd''\psi + \mathrm{i}\,\chi_t''(\psi)\,d'\psi \wedge d''\psi, \tag{5.11}$$

hence we see that

$$\begin{aligned} R_t &:= \eta_t\Big(\mathrm{i}\,\Theta_{E,h} + \mathrm{i}\,d'd''\psi\Big) - \mathrm{i}\,d'd''\eta_t - \lambda_t^{-1}\mathrm{i}\,d'\eta_t \wedge d''\eta_t \\ &= \eta_t\Big(\mathrm{i}\,\Theta_{E,h} + (1 + \delta\eta_t^{-1}\chi_t'(\psi))\,\mathrm{i}\,d'd''\psi\Big) + \Big(\delta\chi_t''(\psi) - \lambda_t^{-1}\delta^2\chi_t'(\psi)^2\Big)\mathrm{i}\,d'\psi \wedge d''\psi. \end{aligned}$$

The coefficient $(1 + \delta\eta_t^{-1}\chi_t'(\psi))$ lies in $[1, 1+\frac{\delta}{2}]$, and our curvature assumption implies that the first term in the right hand side is non negative. We take

$$\lambda_t = \pi(1 + \delta^2\psi^2). \tag{5.12}$$

By (5.8 e), this ensures that $\lambda_t^{-1}\delta^2\chi_t'(\psi)^2 \le \frac{1}{2}\delta\chi_t''(\psi)$, hence we get the crucial lower bounds

$$R_t \ge \frac{1}{2}\,\delta\chi_t''(\psi)\,\mathrm{i}\,d'\psi\wedge d''\psi \ge 0 \quad \text{on } X_c, \tag{5.13}$$

$$R_t \ge \frac{(1-\varepsilon)\delta}{8}\,\mathrm{i}\,d'\psi\wedge d''\psi \quad \text{on } W_t = \{t<\psi<t+1\}. \tag{5.14}$$

A standard calculation gives the formula $\langle[\Lambda_\omega,\mathrm{i}\,d'\psi\wedge d''\psi]v,v\rangle_{\omega,h} = |(d''\psi)^*v|^2_{\omega,h}$ for any $(n,1)$-form v. Therefore, for every $(n,0)$-form u we have

$$|\langle d''\psi\wedge u,v\rangle|^2 = |\langle u,(d''\psi)^*v\rangle|^2 \le |u|^2|(d''\psi)^*v|^2 = |u|^2\langle[\Lambda_\omega,\mathrm{i}\,d'\psi\wedge d''\psi]v,v\rangle.$$

From this and (5.14) we infer that the curvature operator $B_t = [\Lambda_\omega, R_t]$ satisfies

$$\langle B_t^{-1}(d''\psi\wedge u), d''\psi\wedge u\rangle = |B_t^{-1/2}(d''\psi\wedge u)|^2 \le \frac{8}{(1-\varepsilon)\delta}\,|u|^2 \quad \text{on } W_t.$$

In particular, since $|\theta'|\le 1+\varepsilon$, we see that the form $v_t = \theta'(\psi-t)\,d''\psi\wedge\widetilde{f}$ defined in (5.6) satisfies

$$\langle B_t^{-1}v_t,v_t\rangle \le \frac{8(1+\varepsilon)^2}{(1-\varepsilon)\delta}\,|\widetilde{f}|^2. \tag{5.15}$$

Now, (5.8 b) implies $|\chi_t(\tau)|\le\frac{1}{2}|\tau|$, thus $\eta_t = 1-\delta\chi_t(\psi)\le 1-\frac{1}{2}\delta\psi$ and

$$\eta_t+\lambda_t \le 1-\tfrac{1}{2}\delta\psi+\pi(1+\delta^2\psi^2) \le 4.21\,(1+\delta^2\psi^2) \tag{5.16}$$

(the optimal constant is $\frac{\sqrt{5}+2}{4}+\pi < 4.21$). As $4.21\times 8\times(1+\varepsilon)^2/(1-\varepsilon) < 34$ for $\varepsilon\ll 1$, Proposition 3.11 produces a solution u_t such that $D''u_t = v_t$ on $X_c\smallsetminus Y$ and

$$\int_{X_c\smallsetminus Y}(1+\delta^2\psi^2)^{-1}|u_t|^2_{\omega,h}e^{-\psi}\,dV_{X,\omega} \le \frac{34}{\delta}\int_{\{t<\psi<t+1\}}|\widetilde{f}|^2_{\omega,h}e^{-\psi}dV_{X,\omega}.$$

The function $F_t = \theta(\psi-t)\widetilde{f}-u_t$ is essentially the extension we are looking for. For any $\alpha>0$ we have

$$|F_t|^2 \le (1+\alpha)\,|u_t|^2 + (1+\alpha^{-1})\,|\theta(\psi-t)|^2|\widetilde{f}|^2,$$

hence

$$\int_{X_c\smallsetminus Y}(1+\delta^2\psi^2)^{-1}(1+\alpha^2\psi^2)^{-(n-1)/2}|F_t|^2_{\omega,h}\,e^{-\psi}\,dV_{X,\omega}$$

$$\le \frac{34(1+\alpha)}{\delta}\int_{X_c\cap\{t<\psi<t+1\}}|\widetilde{f}|^2_{\omega,h}e^{-\psi}dV_{X,\omega} \tag{5.17$_1$}$$

$$+\,(1+\alpha^{-1})\int_{X_c\cap\{\psi<t+1\}}(1+\delta^2\psi^2)^{-1}(1+\alpha^2\psi^2)^{-(n-1)/2}\,|\widetilde{f}|^2_{\omega,h}e^{-\psi}dV_{X,\omega}. \tag{5.17$_2$}$$

The local integrability of $(1+|\psi|)^{-(n+1)}|\widetilde{f}|^2e^{-\psi}$ asserted by Proposition 4.5 (c) and the Lebesgue dominated convergence theorem imply that the last integral (5.17_2) converges to 0 as $t\to-\infty$. Therefore we obtain

$$\limsup_{\alpha\to 0_+}\lim_{t\to-\infty}(5.17_1)+(5.17_2) \le \frac{34}{\delta}\int_{X_c\cap Y}|f|^2_{\omega,h}dV_{Y,\omega}[\psi].$$

by definition of the measure in the right hand side (cf. (2.4)). Since u_t is in L^2 with respect to the singular weight $e^{-\psi}$, it is also locally L^2 with respect to a smooth weight. A standard lemma (cf. [Dem82, Lemme 6.9]) shows that the equation $D''u_t = v_t$ extends to X_c, and the hypoellipticity of D'' implies that u_t is smooth on X_c. As $e^{-\psi}$ is non integrable along Y, we conclude that u_t must vanish on $X_c \cap Y$. Therefore F_t is indeed a holomorphic extension of f on X_c. By letting t tend to $-\infty$ and then c to $+\infty$, the uniformity of our L^2 inequalities implies that one can extract a weakly convergent sequence $F_{t_\nu} \to F$, such that $F \in H^0(X, K_X \otimes E)$ is an extension of f and

$$\int_X (1+\delta^2\psi^2)^{-1}|F|^2_{\omega,h}e^{-\psi}\,dV_{X,\omega} \le \frac{34}{\delta}\int_Y |f|^2_{\omega,h}dV_{Y,\omega}[\psi].$$

The proof also works when Y is singular, because the equations can be considered on $X \smallsetminus Y_{\rm sing}$, and $X_c \smallsetminus Y_{\rm sing}$ is again complete Kähler for every $c > 0$.

In case ψ is no longer negative, we put $\psi_A^+ = \frac{1}{A}\log(1+e^{A\psi})$ and replace ψ with

$$\psi_A = \psi - \psi_A^+ < 0$$

which converges to $\psi - \psi_+ = -\psi_-$ as $A \to +\infty$. We then solve $D''u_t = v_t$ with $v_t = D''(\theta(\psi_A - t)\widetilde{f})$, and use the functions $\eta_t = 1 - \delta\chi_t(\psi_A)$ and $\lambda_t = \pi(1+\delta^2\psi_A^2)$ in the application of proposition 3.11. The expression of the curvature term R_t becomes

$$\begin{aligned} R_{t,A} = \eta_t\Big(\mathrm{i}\,\Theta_{E,h} + \mathrm{i}\,d'd''\psi + \delta\eta_t^{-1}\chi_t'(\psi_A))\,\mathrm{i}\,d'd''\psi_A\Big)& \\ + \Big(\delta\chi_t''(\psi_A) - \lambda_t^{-1}\delta^2\chi_t'(\psi_A)^2\Big)\mathrm{i}\,d'\psi_A \wedge d''\psi_A&. \end{aligned}$$

All bounds are then essentially the same, except that we have an additional negative term (...)$\mathrm{i}\,d'\psi \wedge d''\psi$ in $\mathrm{i}\,d'd''\psi_A$, i.e.

$$\mathrm{i}\,d'd''\psi_A = \frac{1}{1+e^{A\psi}}\mathrm{i}\,d'd''\psi - \frac{Ae^{A\psi}}{(1+e^{A\psi})^2}\mathrm{i}\,d'\psi \wedge d''\psi.$$

Because of this term, the first term $\eta_t(...)$ in $R_{t,A}$ is a priori no longer ≥ 0. However, this can be compensated by adding an extra weight $\frac{\delta}{2}\psi_A^+$ to the metric of (E,h), so that the total weight is now $\psi + \frac{\delta}{2}\psi_A^+$. The contribution of the new weight to the term $\eta_t(...)$ in the modified $R_{t,A}$ reads

$$\begin{aligned} \mathrm{i}\,\Theta_{E,h} + \Big(1 + \delta\eta_t^{-1}\chi_t'(\psi_A)\frac{1}{1+e^{A\psi}} + \frac{\delta}{2}\frac{e^{A\psi}}{1+e^{A\psi}}\Big)\mathrm{i}\,d'd''\psi& \\ + \delta\frac{Ae^{A\psi}}{(1+e^{A\psi})^2}\Big(\frac{1}{2} - \eta_t^{-1}\chi_t'(\psi_A)\Big)\mathrm{i}\,d'\psi \wedge d''\psi&, \end{aligned}$$

and as the coefficient of $\mathrm{i}\,d'd''\psi$ still lies in $[1, 1+\delta]$ (remember that $\eta_t \ge 1$ and $\chi_t' \le \frac{1}{2}$), we conclude that we have again

(5.18) $$R_{t,A} \ge \frac{1}{2}\,\delta\chi_t''(\psi_A)\,\mathrm{i}\,d'\psi_A \wedge d''\psi_A \ge 0 \quad \text{on } X_c,$$

(5.19) $$R_{t,A} \ge \frac{(1-\varepsilon)\delta}{8}\,\mathrm{i}\,d'\psi_A \wedge d''\psi_A \quad \text{on } W_{t,A} = \{t < \psi_A < t+1\}.$$

Since $\psi - \psi_A \to 0$ along $Y = \psi^{-1}(-\infty)$, it is easy to see that the measures $dV_{Y,\omega}[\psi]$ and $dV_{Y,\omega}[\psi_A]$ coincide. After making those corrections, we get an extension F_A such that

$$\int_X (1+\delta^2\psi_A^2)^{-1}|F_A|^2_{\omega,h}e^{-\psi-\frac{\delta}{2}\psi_A^+}\,dV_{X,\omega} \le \frac{34}{\delta}\int_Y |f|^2_{\omega,h}dV_{Y,\omega}[\psi].$$

By letting $A \to +\infty$ and extracting a limit $F_A \to F$, we get

$$\int_X (1+\delta^2\psi_-^2)^{-1}|F|^2_{\omega,h}e^{-\psi-\frac{\delta}{2}\psi_+}\,dV_{X,\omega} \le \frac{34}{\delta}\int_Y |f|^2_{\omega,h}dV_{Y,\omega}[\psi],$$

which is equivalent to the final bound given in Theorem 2.8.

The next point we have to justify is that unfortunately we cannot expect $D''\widetilde{f} \equiv 0$ as we assumed a priori (unless we already know for some reason that a global holomorphic extension exists). In fact, we have to solve an equation $D''u = v_t := D''(\theta(\psi_A - t)\widetilde{f})$ with an extra term in the right hand side, namely

$$D''u = v_t = v_t^{(1)} + v_t^{(2)}, \quad v_t^{(1)} = \theta'(\psi_A - t)d''\psi_A \wedge \widetilde{f}, \quad v_t^{(2)} = \theta(\psi_A - t)D''\widetilde{f}. \tag{5.20}$$

The first term $v_t^{(1)}$ of (5.20) has been already estimated, but we have to show that the second term $v_t^{(2)}$ becomes “negligible” when we take limits as $t \to -\infty$. For this we solve (5.20) by means of Prop. 3.12 instead of Prop. 3.11. We get an approximate L^2 solution $D''u_{t,\varepsilon} = v_t - w_{t,\varepsilon}$, whence $D''(\theta(\psi_A - t)\widetilde{f} - u_{t,\varepsilon}) = w_{t,\varepsilon}$. Moreover, this solution satisfies the L^2 estimate

$$\begin{aligned}
&\|(\eta_t+\lambda_t)^{-1/2}u_{t,\varepsilon}\|^2 + \frac{1}{\varepsilon}\|w_{t,\varepsilon}\|^2\\
&\qquad\le \int_{X_c\smallsetminus Y} \langle (B_{t,A}+\varepsilon I)^{-1}v_t, v_t\rangle\, e^{-\psi-\frac{\delta}{2}\psi_A^+}\,dV_{X,\omega}\\
&\qquad\le (1+\alpha)\int_{X_c\smallsetminus Y} \langle B_{t,A}^{-1}v_t^{(1)}, v_t^{(1)}\rangle\, e^{-\psi-\frac{\delta}{2}\psi_A^+}\,dV_{X,\omega} &(5.21_1)\\
&\qquad\quad + (1+\alpha^{-1})\varepsilon^{-1}\int_{X_c\smallsetminus Y} \langle v_t^{(2)}, v_t^{(2)}\rangle\, e^{-\psi-\frac{\delta}{2}\psi_A^+}\,dV_{X,\omega} &(5.21_2)
\end{aligned}$$

for $\alpha > 0$ arbitrary. The integral (5.21_1) is bounded by means of (5.17_i) – or its analogue for the modified weight $\psi + \frac{\delta}{2}\psi_A^+$ – and the integral (5.21_2) is equal to

$$(1+\alpha^{-1})\varepsilon^{-1}\int_{X_c\cap\{t<\psi<t+1\}} |\theta(\psi_A - t)|^2|D''\widetilde{f}|^2\, e^{-\psi-\frac{\delta}{2}\psi_A^+}\,dV_{X,\omega}. \tag{5.22}$$

We now put all estimates together for the section $F_{t,\varepsilon} = \theta(\psi_A - t)\widetilde{f} - u_{t,\varepsilon}$, which satisfies

$$|F_{t,\varepsilon}|^2 \le (1+\alpha)\,|u_{t,\varepsilon}|^2 + (1+\alpha^{-1})\,|\theta(\psi_A - t)|^2|\widetilde{f}|^2.$$

We get in this way from (5.16), (5.17_i), (5.21_i) and (5.22)

$$\int_{X_c\smallsetminus Y}(1+\delta^2\psi_A^2)^{-1}(1+\alpha^2\psi_A^2)^{-(n-1)/2}|F_{t,\varepsilon}|^2_{\omega,h}\, e^{-\psi-\frac{\delta}{2}\psi_A^+}\,dV_{X,\omega}$$

$$+\frac{1}{\varepsilon}\int_{X_c\smallsetminus Y}|w_{t,\varepsilon}|^2_{\omega,h}\,e^{-\psi-\frac{\delta}{2}\psi_A^+}\,dV_{X,\omega}$$

$$(5.23_1)\quad \le \frac{34(1+\alpha)^2}{\delta}\int_{X_c\cap\{t<\psi_A<t+1\}}|\widetilde f|^2\,e^{-\psi-\frac{\delta}{2}\psi_A^+}\,dV_{X,\omega}$$

$$(5.23_2)\quad +(1+\alpha^{-1})\int_{X_c\cap\{\psi_A<t+1\}}(1+\delta^2\psi_A^2)^{-1}(1+\alpha^2\psi_A^2)^{-(n-1)/2}|\widetilde f|^2\,e^{-\psi-\frac{\delta}{2}\psi_A^+}\,dV_{X,\omega}$$

$$(5.23_3)\quad +\frac{(1+\alpha)(1+\alpha^{-1})}{\varepsilon}\int_{X_c\cap\{\psi_A<t+1\}}|D''\widetilde f|^2\,e^{-\psi-\frac{\delta}{2}\psi_A^+}\,dV_{X,\omega}.$$

By Proposition 4.5 (c), the integral (5.23_2) converges to 0 as $t\to-\infty$. Since $|D''\widetilde f|^2\,e^{-\psi}$ is locally integrable on X, the last integral (5.23_3) also converges to 0 as $t\to-\infty$. We let ε and α converge to 0 afterwards, and extract a limit $F=\lim_{\varepsilon\to0}\lim_{t\to-\infty}F_{t,\varepsilon}$ as already explained, to recover the expected L^2 estimate.

(5.24) Final regularity argument. One remaining non trivial point is to check that we get smooth solutions and that the resulting limit F of $F_{t,\varepsilon}=\theta(\psi-t)\widetilde f-u_{t,\varepsilon}$ is actually an extension of f, one particular issue being that $F_{t,\varepsilon}$ is not exactly holomorphic (for the simplicity of notation we assume here that $\psi\le0$ since the difficulty is purely local near Y, and thus skip the ψ_A approximation process in what follows). For this, we apply observation 5.1 and use a composition of blow-ups $\mu:\widehat X\to X$ such that the singularities of $\widehat\psi=\psi\circ\mu$ are divisorial, given by some normal crossing divisor $\Delta=\sum c_j\Delta_j$ in $\widehat X$ whose support contains the exceptional divisor of ψ. If g is a germ of section in $\mathcal{O}_{X,x}(K_X\otimes E)$, the section $\widehat g=\mu^*g$ takes values in $\mu^*(K_X\otimes E)=K_{\widehat X}\otimes(\mathcal{O}_{\widehat X}(-\Delta')\otimes\mu^*E)$, where $\Delta'=\sum c'_j\Delta_j$ has support in $|\Delta|$ and the c'_j are non negative integers. As ψ has log canonical singularities on X, we see by taking g invertible that $\Delta-\Delta'$ has coefficients $c_j-c'_j\le1$ on $\widehat X$. Let us set

$$\Delta-\Delta'=\Delta^{(1)}+\Delta''$$

where $\Delta^{(1)}$ consists of the sum of components of multiplicity $c_j-c'_j=1$, and Δ'' is the sum of all other ones with $c_j-c'_j<1$. The metric involved in our L^2 estimates is $\mu^*h_\psi=\mu^*he^{-\widehat\psi}$. When viewed as a metric on $\widehat G=K_{\widehat X}\otimes\mu^*E\otimes\mathcal{O}(-\Delta'-\Delta^{(1)})$, it possesses a weight

$$\widehat\psi_G:=\widehat\psi-\log|\sigma_{\Delta'}|^2-\log|\sigma_{\Delta^{(1)}}|^2=\log|\sigma_{\Delta''}|^2\mod \mathcal{C}^\infty,$$

hence $\widehat\psi_G$ is non singular at the generic point of any component Δ_j of $\Delta^{(1)}$. Solving a $\overline\partial$-equation in $\mu^*(K_X\otimes E)$ with respect to the singular weight $e^{-\psi\circ\mu}=e^{-\widehat\psi}$ amounts to solving the same $\overline\partial$-equation with values in $\widehat G$ with respect to the weight $\widehat\psi_G$. The standard ellipticity results imply that the solutions $\widehat u_{t,\varepsilon}$ on $\widehat X$ given by 3.12 are smooth as sections of $\widehat G$. This argument shows that we can assume right away that ψ has divisorial singularities, and consider only the case $\mu=\mathrm{Id}_X$ and $\Delta=\Delta^{(1)}+\Delta''$, in which case $Y=|\Delta^{(1)}|$ and $\mathcal{O}_X(-\Delta^{(1)})=\mathcal{O}_X(-Y)$. We thus simplify the notation by removing all hats, and set $G=K_X\otimes E\otimes\mathcal{O}_X(-Y)$. Notice that

$$F_{t,\varepsilon}=\theta(\psi-t)\widetilde f-u_{t,\varepsilon}$$

is not exactly holomorphic. We have instead $D''F_{t,\varepsilon}=w_{t,\varepsilon}$ where $w_{t,\varepsilon}$ is a section of $\Lambda^{0,1}T^*_X\otimes G$ that is smooth on the Zariski open set $Y^\circ=Y\smallsetminus\bigcup_{\Delta_k\not\subset Y}\Delta_k$, and its L^2

norm with respect to $\psi_G = \psi - \log|\sigma_Y|^2$ satisfies $\|w_{t,\varepsilon}\| = O(\varepsilon^{1/2})$. If F_0 is a fixed local extension of f near a generic point $x_0 \in \Delta_j$, then $F_t - F_0$ is a local section of G and $D''(F_{t,\varepsilon} - F_0) = w_{t,\varepsilon}$. We can find, say by the standard Hörmander L^2 estimates on a coordinate ball $B(x_0, r)$ ([Hör65,66], see also [Kohn63,64]), a smooth L^2 section $s_{t,\varepsilon}$ of G such that $\|s_{t,\varepsilon}\| = O(\varepsilon^{1/2})$ and $D'' s_{t,\varepsilon} = w_{t,\varepsilon}$ on $B(x_0, r)$. Then $F_{t,\varepsilon} - F_0 - s_{t,\varepsilon}$ is a holomorphic section of G on $B(x_0, r)$. Its limit $F - F_0$ is a limit in $L^{2-\alpha}$ for every $\alpha > 0$, thanks to the Hölder inequality and the fact that $1 + \delta^2\psi^2$ is in L^p for every $p > 1$. Thus $F - F_0$ is a holomorphic section of G on $B(x_0, r)$, and so $F_{\restriction Y} = f$ on Y°. □

Proof of Remark 2.9 (b). By the technique of proof of the regularization theorem [Dem92, Theorem 1.1], on any relatively compact subset $X_c \Subset X$, there are quasi-psh approximations $\varphi_\nu \downarrow \varphi$ of φ with neat analytic singularities, such that the curvature estimate suffers only a small error $\le 2^{-\nu}\omega$, namely, for $h_\nu = e^{-\varphi_\nu} \le h$, we have

$$\mathrm{i}\,\Theta_{E,h_\nu} + \alpha\,\mathrm{i}\,d'd''\psi_\nu \ge -2^{-\nu}\omega \quad \text{on } X_c,$$

uniformly for $\alpha \in [1, 1+\delta]$. As a consequence, the operator

$$B_{t,\nu} = \big[\eta_t(\mathrm{i}\,\Theta_{E,h_\nu} + \alpha\,\mathrm{i}\,d'd''\psi_\nu - \mathrm{i}\,d'd''\eta_t - \lambda_t^{-1}\mathrm{i}\,d'\eta_t \wedge d''\eta_t, \Lambda_\omega]$$

has a slightly negative lower bound $-M_t 2^{-\nu}$ by (5.8 a). This can be absorbed by means of an additional positive term εI in $B_{t,\nu} + \varepsilon I$, with $\varepsilon = O(2^{-\nu})$. Moreover the set of poles of φ_ν is an analytic set P_ν and we can work on the complete Kähler manifold $X_c \smallsetminus (Y \cup P_\nu)$ to avoid any singularities. Then Prop. 3.12 provides an approximate solution $D'' u_{t,\nu} \approx v_t$ with error $O(\varepsilon^{1/2}) = O(2^{-\nu/2})$, satisfying the estimate

$$\int_{X_c \smallsetminus Y} \frac{\exp(-\frac{\delta}{2}\frac{1}{A}\log(1 + e^{A\psi}))}{1 + \delta^2\psi_A^2}\, |u_{t,\nu}|^2_{\omega,h_\nu}\, e^{-\psi}\, dV_{X,\omega} \le \frac{34}{\delta} \int_{X_c \cap \{t < \psi_A < t+1\}} |\widetilde{f}|^2_{\omega,h_\nu} e^{-\psi_A}\, dV_{X,\omega}.$$

The right hand side is uniformly bounded by a similar norm where h_ν is replaced by $h = e^{-\varphi}$. The conclusion follows by letting ν converge to $+\infty$ (before doing anything else), and by extracting limits. □

Proof of Theorem 2.12. The proof is essentially identical to the proof of Theorem 2.8, we simply make a "rescaling": we replace ψ by $m_p\psi$, t by $m_p t$, δ by δ/m_p, θ by $\theta(\tau) = \theta(\tau/m_p)$ and use Lemma 4.5 in its full generality. Property (5.4) [namely the integrability of $|D''(\widetilde{f})|^2 e^{-m_p\psi}$] still holds here since f has local extensions $\widetilde{f}_j$ such that the differences $\widetilde{f}_j - \widetilde{f}_k$ lie by construction in $\mathcal{I}(m_p\psi)$. In the final regularity argument 5.24, the vanishing of f prescribed by $\mathcal{I}(m_{p-1}\psi)$ subtracts a further divisor to $\Delta - \Delta' - \Delta_{m_{p-1}}$ where $\mu^*\mathcal{I}(m_{p-1}\psi) = \mathcal{O}(-\Delta_{m_{p-1}})$. The definition of jumps leads to the fact that we only have to considers components of multiplicity 1 in that difference, and the rest of the argument is the same. □

Proof of Theorem 2.13. The argument is identical to the proof of Remark 2.9 (b), when we make the same rescaling, especially by replacing ψ with $m_p\psi$. □

Proof of Theorem 2.14. Since X is compact, all coherent cohomology groups involved are finite dimensional and Hausdorff for their natural topology. We can assume $\psi < 0$ by subtracting a constant (and thus avoid the ψ_A approximation process), and $X = X_c$.

(a) We proceed by induction on $p-\ell$. Let

$$f\in H^0(Y^{(m_p)},\mathcal{O}_X(K_X\otimes E)\otimes\mathcal{I}(m_\ell\psi)/\mathcal{I}(m_p\psi)).$$

First assume $\ell=p-1$. Theorem 2.12 provides an extension F of f in case $\mathcal{I}(m_{p-1}\psi)$ is replaced by $\mathcal{I}'(m_{p-1}\psi)$, but otherwise the limiting L^2 integral of f computed on $Y^{(m_p)}$ may diverge. The main idea, however, is that the integrals are still convergent when considered on each tube $\{t<\psi<t+1\}$, and this can be used to check at least the qualitative part of the surjectivity theorem. In fact, we apply the arguments used in the proof of Theorems 2.8–2.12, especially estimates (5.23_i), taken with $\alpha=1$, and $\psi=\psi_A$, t, $\theta(\tau)$, δ replaced respectively with $m_p\psi$, m_pt, $\theta(\tau/m_p)$, δ/m_p; we also assume $\delta\le 1$ here. This produces a $\mathcal{C}^\infty$ extension $F_{t,\varepsilon}=\theta(\psi-t)\,\widetilde f-u_{t,\varepsilon}$ such that $D''F_{t,\varepsilon}=w_{t,\varepsilon}$ and

$$\int_X(1+m_p^2\psi^2)^{-(n+1)/2}|F_{t,\varepsilon}|^2_{\omega,h}\,e^{-m_p\psi}\,dV_{X,\omega}+\frac{1}{\varepsilon}\int_X|w_{t,\varepsilon}|^2_{\omega,h}\,e^{-m_p\psi}\,dV_{X,\omega}$$

$$(5.25_1)\qquad \le\frac{136}{\delta}\int_{\{t<\psi<t+1\}}|\widetilde f|^2\,e^{-m_p\psi}\,dV_{X,\omega}$$

$$(5.25_2)\qquad +2\int_{\{\psi<t+1\}}(1+m_p\psi^2)^{-(n+1)/2}|\widetilde f|^2\,e^{-m_p\psi}\,dV_{X,\omega}$$

$$(5.25_3)\qquad +\frac{4}{\varepsilon}\int_{\{\psi<t+1\}}|D''\widetilde f|^2\,e^{-m_p\psi}\,dV_{X,\omega}.$$

This is true for all $t<0$ and $\varepsilon>0$, and the idea is to adjust the choice of ε as a function of t. The integral (5.25_2) is uniformly bounded by Proposition 4.5 (c) and can be disregarded. By construction the coefficients $D''\widetilde f$ can be expressed as a combination of functions $\widetilde f_j-\underset{\sim}{f_j}$ in $\mathcal{I}(m_p\varphi)$, thus by openness, there exists $\alpha>0$ such that

$$\int_X|D''\widetilde f|^2_{\omega,h}e^{-(m_p+\alpha)\psi}\,dV_{X,\omega}<+\infty.$$

We therefore get an upper bound

$$(5.26)\qquad \int_{\{\psi<t+1\}}|D''\widetilde f|^2_{\omega,h}e^{-m_p\psi}\,dV_{X,\omega}\le C\,e^{\alpha t}\quad\text{as } t\to-\infty.$$

On the other hand, since $\int_X|\widetilde f|^2_{\omega,h}e^{-(m_p-\beta)\psi}\,dV_{X,\omega}<+\infty$ for every $\beta>0$, we conclude that

$$(5.27)\qquad \int_{\{t<\psi<t+1\}}|\widetilde f|^2_{\omega,h}e^{-m_p\psi}\,dV_{X,\omega}\le C'(\beta)\,e^{-\beta t}\quad\text{as } t\to-\infty.$$

A possible choice of ε is to take $\varepsilon=e^{(\alpha+\beta)t}$ for some $\beta>0$. Then (5.25_i) and (5.26–5.27) imply

$$\int_X|w_{t,\varepsilon}|^2_{\omega,h}e^{-m_p\psi}dV_{X,\omega}\le C''(\delta,\beta)e^{\alpha t}.$$

We conclude that the error term $w_{t,\varepsilon}$ converges uniformly to 0 in L^2 norm as $t\to-\infty$ and $\varepsilon=e^{(\alpha+\beta)t}\to 0$. The main term $F_{t,\varepsilon}$, however, is not under control, but we can cope with this situation. By taking a principalization of the ideal defining the singularities of ψ, we can assume that these singularities are divisorial, and thus that our solutions are smooth at the generic point of $Z_p = \mathrm{Supp}(\mathcal{I}(m_{p-1}\psi)/\mathcal{I}(m_p\psi))$. Observe that $w_{t,\varepsilon}$ is a coboundary for the Dolbeault complex associated with the cohomology group $H^0(X, \mathcal{O}_X(K_X \otimes E) \otimes \mathcal{I}(m_{p-1}\psi))$. Now, X is compact and Čech cohomology can be calculated on finite Stein coverings by spaces of Čech cocycles equipped with the topology given by L^2 norms with respect to the weight $e^{-m_{p-1}\psi}$. We conclude via an isomorphism between Čech cohomology and Dolbeault cohomology that there is a smooth global section $s_{t,\varepsilon}$ of $\mathcal{C}^\infty(K_X \otimes E) \otimes \mathcal{I}(m_{p-1}\psi)$ such that $D''s_{t,\varepsilon} = w_{t,\varepsilon}$ and

$$\int_X |s_{t,\varepsilon}|^2_{\omega,h} e^{-m_{p-1}\psi} dV_{X,\omega} = O(e^{t\alpha}).$$

On the other hand, on any coordinate ball $B = B(x_0, r) \subset X$, we can apply the standard L^2 estimates of Hörmander for bounded pseudoconvex domains, and find another local solution $\widetilde{s}_{B,t,\varepsilon}$ such that $D''\widetilde{s}_{B,t,\varepsilon} = w_{t,\varepsilon}$ and

$$\int_{B(x_0,r)} |\widetilde{s}_{B,t,\varepsilon}|^2_{\omega,h} e^{-m_p\psi} dV_{X,\omega} = O(e^{t\alpha}). \tag{5.28}$$

The difference $s_{t,\varepsilon} - \widetilde{s}_{B,t,\varepsilon}$ is a holomorphic section of $\mathcal{O}_X(K_X \otimes E) \otimes \mathcal{I}(m_{p-1}\psi)$ on $B(x_0, r)$ which converges to 0 in L^2 norm, thus uniformly on any smaller ball $B(x_0, r')$. Notice that $\widetilde{s}_{B,t,\varepsilon}$ has a vanishing order prescribed by $\mathcal{I}(m_p\psi)$ by (5.28). Therefore $F_{t,\varepsilon} - \widetilde{s}_{B,t,\varepsilon}$ is a local holomorphic section of $\mathcal{O}_X(K_X \otimes E) \otimes \mathcal{I}(m_{p-1}\psi)$ that maps exactly to f. By what we have seen

$$F_{t,\varepsilon} - s_{t,\varepsilon} \in H^0(X, \mathcal{O}_X(K_X \otimes E) \otimes \mathcal{I}(m_{p-1}\psi))$$

is a global holomorphic section whose restriction converges uniformly to f. Since X is compact, we are dealing with finite dimensional spaces of sections and thus the restriction morphism must be surjective. The proof of the case $\ell = p - 1$ is complete.

Now, assume that the result has been proved for $p - \ell = d$ and take $p - \ell = d + 1$. By reducing f mod $\mathcal{I}(m_{p-1}\psi)$, the induction hypothesis provides an extension F' in $H^0(X, \mathcal{O}_X(K_X \otimes E) \otimes \mathcal{I}(m_\ell\psi))$ of

$$f' := f \mod \mathcal{I}(m_{p-1}\psi) \quad \text{in} \quad H^0(Y^{(m_{p-1})}, \mathcal{O}_X(K_X \otimes E) \otimes \mathcal{I}(m_\ell\psi)/\mathcal{I}(m_{p-1}\psi)).$$

Thus $f_p := f - (F' \mod \mathcal{I}(m_p\psi))$ defines a section

$$f_p \in H^0(Y^{(m_p)}, \mathcal{O}_X(K_X \otimes E) \otimes \mathcal{I}(m_{p-1}\psi)/\mathcal{I}(m_p\psi)).$$

The case $d = 1$ provides an extension

$$F_p \in H^0(X, \mathcal{O}_X(K_X \otimes E) \otimes \mathcal{I}(m_{p-1}\psi))$$

and $F = F' + F_p$ is the extension we are looking for.

(b) Assume finally that E is a line bundle and that h is a singular hermitian metric satisfying the curvature estimate in the sense of currents only. We can reduce ourselves to the case when ψ has divisorial singularities. The regularization techniques of [Dem15] (see Remark 3 before section 3, based on the solution of the openness conjecture) produces a singular metric h_ε with analytic singularities, such that the multiplier ideal sheaves $\mathcal{I}(he^{-m_\ell\psi})$ involved are unchanged when h is replaced by h_ε, and such that there is an arbitrary small loss in the curvature. We absorb this new adverse negative term by considering again $B_t + \varepsilon I$, and by applying the same tricks as above. □

6. References

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