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Precise error estimate of the Brent-McMillan algorithm for Euler's constant

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Abstract: Brent and McMillan introduced in 1980 a new algorithm for the computation of Euler's constant γ, based on the use of the Bessel functions $I_0(x)$ and $K_0(x)$. It is the fastest known algorithm for the computation of γ. The time complexity can still be improved by evaluating a certain divergent asymptotic expansion up to its minimal term. Brent-McMillan conjectured in 1980 that the error is of the same magnitude as the last computed term, and Brent-Johansson partially proved it in 2015. They also gave some numerical evidence for a more precise estimate of the error term. We find here an explicit expression of that optimal estimate, along with a complete self-contained formal proof and an even more precise error bound.

Keywords: Euler's constant, Bessel functions, elliptic integral, integration by parts, asymptotic expansion, Euler-Maclaurin formula

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0. Introduction and main results

Let $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ denote as usual the partial sums of the harmonic series. The algorithm introduced by Brent-McMillan [3] for the computation of Euler's constant $\gamma = \lim_{n\to+\infty}(H_n - \log n)$ is based on certain identities satisfied by the Bessel functions $I_\alpha(x)$ and $K_0(x)$:

$$I_\alpha(x) = \sum_{n=0}^{+\infty} \frac{x^{\alpha+2n}}{n!\,\Gamma(\alpha+n+1)}, \qquad K_0(x) = -\frac{\partial I_\alpha(x)}{\partial\alpha}_{|\alpha=0}. \tag{0.1}$$

Experts will observe that $2x$ has been substituted to x in the conventional notation of Watson's treatise [8]. As we will check in § 1, these functions satisfy the relations

$$K_0(x) = -(\log x + \gamma) I_0(x) + S_0(x) \qquad \text{where} \tag{0.2}$$

$$I_0(x) = \sum_{n=0}^{+\infty} \frac{x^{2n}}{n!^2}, \qquad S_0(x) = \sum_{n=1}^{+\infty} H_n \frac{x^{2n}}{n!^2}. \tag{0.3}$$

As a consequence, Euler's constant can be written as

$$\gamma = \frac{S_0(x)}{I_0(x)} - \log x - \frac{K_0(x)}{I_0(x)}, \tag{0.4}$$

and one can show easily that

$$0 < \frac{K_0(x)}{I_0(x)} < \pi\, e^{-4x} \qquad \text{for } x \geqslant 1. \tag{0.5}$$

In the simpler version (BM) of the algorithm proposed by Brent-McMillan, the remainder term $\frac{K_0(x)}{I_0(x)}$ is neglected; a precision 10^{-d} is then achieved for $x \simeq \frac{1}{4}(d \log 10 + \log \pi)$, and the power series $I_0(x)$, $S_0(x)$ must be summed up to $n = \lceil a_1 x \rceil$ approximately, where a_p is the unique positive root of the equation

$$a_p(\log a_p - 1) = p. \tag{0.6}$$

The calculation of

$$I_0(x) = 1 + \frac{x^2}{1^2}\left(1 + \frac{x^2}{2^2}\left(\cdots \frac{x^2}{(n-1)^2}\left(1 + \frac{x^2}{n^2}\left(\cdots\right)\right)\cdots\right)\right)$$

requires 2 arithmetic operations for each term, and that of

$$S_0(x) \simeq H_{0,N} - \frac{1}{I_0(x)} \frac{x^2}{1^2}\left(H_{1,N} + \frac{x^2}{2^2}\left(\cdots \frac{x^2}{(n-1)^2}\left(H_{n-1,N} + \frac{x^2}{n^2}\left(H_{n,N} + \cdots\right)\right)\cdots\right)\right)$$

requires 4 operations. The time complexity of the algorithm (BM) is thus

$$BM(d) = a_1 \times \frac{1}{4}\, d\ \log 10 \times 6 \times d \simeq 12.4\, d^2. \tag{0.7}$$

However, as in Sweeney's more elementary method [7], Brent and Mcmillan observed that the remainder term $K_0(x)/I_0(x)$ can be evaluated by means of a divergent asymptotic expansion

$$I_0(x)K_0(x) \sim \frac{1}{4x} \sum_{k\in\mathbb{N}} \frac{(2k)!^3}{k!^4\,(16x)^{2k}}. \tag{0.8}$$

Their idea is to truncate the asymptotic expansion precisely at the minimal term, which turns out to be obtained for $k = 2x$ if x is a positive integer. We will check, as was conjectured by Brent-McMillan [3] and partly proven by Brent and Johansson [2], that the corresponding "truncation error" is then of an order of magnitude comparable to the minimal term $k = 2x$, namely $\frac{e^{-4x}}{2\sqrt{2\pi}\,x^{3/2}}$ by Stirling's formula.

THEOREM. *The truncation error*

$$\Delta(x) := I_0(x)K_0(x) - \frac{1}{4x} \sum_{k=0}^{2x} \frac{(2k)!^3}{k!^4\,(16x)^{2k}} \tag{0.9}$$

admits when $x \to +\infty$ an equivalent

$$\Delta(x) \sim -\frac{5\,e^{-4x}}{24\sqrt{2\pi}\,x^{3/2}}, \tag{0.10}$$

and more specifically

$$\Delta(x) = -e^{-4x}\left(\frac{5}{24\sqrt{2\pi}\,x^{3/2}} + \varepsilon(x)\right), \qquad |\varepsilon(x)| < \frac{0.863}{x^2}. \tag{0.11}$$

The approximate value

$$\frac{K_0(x)}{I_0(x)} \simeq \frac{1}{4x\,I_0(x)^2} \sum_{k=0}^{2x} \frac{(2k)!^3}{k!^4\,(16x)^{2k}} \tag{0.12}$$

is thus affected by an error of magnitude

$$\frac{\Delta(x)}{I_0(x)^2} \sim -\frac{5\sqrt{2\pi}}{12\,x^{1/2}}\, e^{-8x}. \tag{0.13}$$

The refined version (BM') of the Brent-McMillan algorithm consists in evaluating the remainder term $\frac{K_0(x)}{I_0(x)}$ up to the accuracy e^{-8x} permitted by the approximation (0.13). This implies to take $x = \frac{1}{8}\,d\,\log 10$ and leads to a time complexity

$$BM'(d) = \Big(\frac{3}{4}a_3 + \frac{1}{2}\Big)\ \log 10\ d^2 \simeq 9.7\, d^2, \tag{0.14}$$

substantially better than (0.7). The proof of the above theorem requires many calculations. The techniques developed here would probably even yield an asymptotic development for $\Delta(x)$, at least for the first few terms, but the required calculations seem very extensive. Hopefully, further asymptotic expansions of the error might be useful to investigate the arithmetic properties of γ, especially its rationality or irrationality.

The present paper is an extended version of an original text [4] written in June 1984 and published in "Gazette des Mathématiciens" in 1985. However, because of length constraints for such a mainstream publication, the main idea for obtaining the error estimate of the Brent-McMillan algorithm had only been hinted, and most of the details had been omitted. After more than 30 years passed, we take the opportunity to make these details available and to improve the recent results of Brent-Johansson [2].

1. Proof of the basic identities

Relations (0.2) and (0.3) are obtained by using a derivation term by term of the series defining $I_\alpha(x)$ in (0.1), along with the standard formula $\frac{\Gamma'(n+1)}{\gamma(n+1)} = H_n - \gamma$, itself a consequence of the equalities

$$\frac{\Gamma'(x+1)}{\Gamma(x+1)} = \frac{1}{x} + \frac{\Gamma'(x)}{\Gamma(x)} \quad \text{and} \quad \Gamma'(1) = -\gamma.$$

Explicitly, we get

$$\frac{\partial I_\alpha(x)}{\partial \alpha} = \sum_{n=0}^{+\infty} \frac{\log x \cdot x^{\alpha+2n}}{n!\,\Gamma(\alpha+n+1)} - \frac{\Gamma'(\alpha+n+1)\, x^{\alpha+2n}}{n!\,\Gamma(\alpha+n+1)^2}, \tag{1.1}$$

hence (0.2) and (0.3). Now, the Hankel integral formula (see [1]) expresses the function $1/\Gamma$ as

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i}\int_{(C)} \zeta^{-z} e^{\zeta}\, d\zeta \tag{1.2}$$

where (C) is the open contour formed by a small circle $\zeta = \varepsilon e^{iu}$, $u \in [-\pi, \pi]$, concatenated with two half-lines $]-\infty, -\varepsilon]$ with respective arguments $-\pi$ and $+\pi$ and opposite orientation. This formula gives

$$\begin{aligned} I_\alpha(x) &= \sum_{n=0}^{+\infty} \frac{x^{\alpha+2n}}{n!}\,\frac{1}{2\pi i}\int_{(C)} \zeta^{-\alpha-n-1} e^{\zeta}\, d\zeta \\ &= \frac{1}{2\pi i}\int_{(C)} x^{\alpha}\zeta^{-\alpha-1}\exp(x^2/\zeta+\zeta)\, d\zeta \\ &= \frac{1}{2\pi i}\int_{(C)} \zeta^{-\alpha}\exp(x/\zeta+\zeta x)\, d\zeta \\ &= \frac{1}{\pi}\int_0^{\pi} e^{2x\cos u}\cos(\alpha u)\; du - \frac{\sin\alpha\pi}{\pi}\int_0^{+\infty} e^{-2x\cosh v}\, e^{-\alpha v}\, dv. \end{aligned} \tag{1.3}$$

The integral expressing $I_\alpha(x)$ in the second line above is obtained by means of a change of variable $\zeta \mapsto \zeta x$ (recall that $x > 0$); the first integral of the third line comes from the modified contour consisting of the circle $\{\zeta = e^{iu}\}$ of center 0 and radius 1, and the last integral comes from the corresponding two half-lines $t \in\,]-\infty, -1]$ written as $t = -e^{-v}$, $v \in\,]0, +\infty[$. In particular, the following integral expressions and equivalents of $I_0(x)$, $K_0(x)$ hold when $x \to +\infty$:

$$I_0(x) = \frac{1}{\pi}\int_0^{\pi} e^{2x\cos u}\; du \qquad \text{hence } I_0(x) \underset{x\to+\infty}{\sim} \frac{1}{\sqrt{4\pi x}}\, e^{2x}, \tag{1.4}$$

$$K_0(x) = \int_0^{+\infty} e^{-2x\cosh v}\, dv \qquad \text{hence } K_0(x) \underset{x\to+\infty}{\sim} \sqrt{\frac{\pi}{4x}}\, e^{-2x}. \tag{1.5}$$

Furthermore, one has $I_0(x) > \frac{1}{\sqrt{4\pi x}}\, e^{2x}$ if $x \geqslant 1$ and $K_0(x) < \sqrt{\frac{\pi}{4x}}\, e^{-2x}$ if $x > 0$. These estimates can be checked by means of changes of variables

$$I_0(x) = \frac{e^{2x}}{2\pi\sqrt{x}} \int_0^{4x} \frac{e^{-t}}{\sqrt{t(1-t/4x)}}\; dt, \qquad t = 2x(1-\cos u),$$

$$K_0(x) = \frac{e^{-2x}}{2\sqrt{x}} \int_0^{+\infty} \frac{e^{-t}}{\sqrt{t(1+t/4x)}}\; dt, \qquad t = 2x(\cosh v - 1),$$

along with the observation that

$$\int_0^{+\infty} \frac{1}{\sqrt{t}}\, e^{-t}\, dt = \Gamma(\frac{1}{2}) = \sqrt{\pi};$$

the lower bound for $I_0(x)$ is obtained by the convexity inequality $\frac{1}{\sqrt{1-t/4x}} \geqslant 1+t/8x$ and an integration by parts of the term $\sqrt{t}\, e^{-t}$, which give

$$\begin{aligned}\int_0^{4x} \frac{e^{-t}}{\sqrt{t(1-t/4x)}}\; dt &\geqslant \Gamma(\tfrac{1}{2}) + \frac{1}{8x}\Gamma(\tfrac{3}{2}) - \int_{4x}^{+\infty} \Big(\frac{1}{\sqrt{t}} + \frac{\sqrt{t}}{8x}\Big)\, e^{-t}\, dt \\ &\geqslant \sqrt{\pi} + \frac{\sqrt{\pi}}{16x} - e^{-4x}\Big(\frac{3}{4\sqrt{x}} + \frac{1}{32x\sqrt{x}}\Big) > \sqrt{\pi}\end{aligned}$$

for $x \geqslant 1$. Inequality (0.5) is then obtained by combining these bounds. Our starting point to evaluate $K_0(x)$ more accurately is to use the integral formulas (1.4), (1.5) to express $I_0(x)K_0(x)$ as a double integral

$$(1.6) \qquad I_0(x)K_0(x) = \frac{1}{2\pi} \int_{\{-\pi<u<\pi\,,\, v>0\}} \exp\big(2x(\cos u - \cosh v)\big)\, du\, dv.$$

A change of variables

$$r\, e^{i\theta} = \sin^2\Big(\frac{u+iv}{2}\Big) = \frac{1}{2}\big(1-\cos(u+iv)\big) = \frac{1}{2}\big(1-\cos u\cosh v + i\sin u\sinh v\big)$$

gives

$$r = \frac{1}{2}(\cosh v - \cos u), \qquad |1 - r\, e^{i\theta}| = \Big| \cos\Big(\frac{u+iv}{2}\Big)\Big|^2,$$
$$r\, dr\, d\theta = \Big| \sin\Big(\frac{u+iv}{2}\Big) \cos\Big(\frac{u+iv}{2}\Big)\Big|^2 du\, dv = r\, |1 - r\, e^{i\theta}|\, du\, dv,$$

therefore

$$(1.7) \qquad I_0(x)K_0(x) = \frac{1}{2\pi} \int_0^{+\infty} \exp(-4xr)\ dr \int_0^{2\pi} \frac{d\theta}{|1 - r\, e^{i\theta}|}.$$

Let us denote by

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}, \qquad \alpha \in \mathbb{C}$$

the (generalized) binomial coefficients. For $z = r\, e^{i\theta}$ and $|z| = r < 1$ the binomial identity $(1-z)^{-1/2} = \sum_{k=0}^{+\infty} \binom{-\frac{1}{2}}{k} (-z)^k$ combined with the Parseval-Bessel formula yields the expansion

$$(1.8) \qquad \varphi(r) := \frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{|1 - r\, e^{i\theta}|} = \sum_{k=0}^{+\infty} w_k\, r^{2k} \qquad \text{for } 0 \leqslant r < 1,$$

where the coefficient

$$(1.9) \qquad w_k := \binom{-1/2}{k}^2 = \left(\frac{1 \cdot 3 \cdot 5 \cdots (2k-1)}{2 \cdot 4 \cdot 6 \cdots 2k}\right)^2 = \frac{(2k)!^2}{2^{4k}\, k!^4}.$$

is closely related to the Wallis integral $W_p = \int_0^{\pi/2} \sin^p x\ dx$. Indeed, the easily established induction relation $W_p = \frac{p-1}{p} W_{p-2}$ implies

$$W_{2k} = \frac{1 \cdot 3 \cdot 5 \cdots (2k-1)}{2 \cdot 4 \cdot 6 \cdots 2k} \frac{\pi}{2}, \qquad W_{2k+1} = \frac{2 \cdot 4 \cdot 6 \cdots 2k}{3 \cdot 5 \cdots (2k+1)},$$

whence $w_k = (\frac{2}{\pi} W_{2k})^2$. The relations $W_{2k}W_{2k-1} = \frac{\pi}{4k}$, $W_{2k}W_{2k+1} = \frac{\pi}{2(2k+1)}$ together with the monotonicity of (W_p) imply $\sqrt{\frac{\pi}{2(2k+1)}} < W_{2k} < \sqrt{\frac{\pi}{4k}}$, therefore

$$(1.10) \qquad \frac{2}{\pi(2k+1)} < w_k < \frac{1}{\pi k}.$$

The main new ingredient of our analysis for estimating $I_0(x)K_0(x)$ is the following integral formula derived from (1.7), (1.8) :

$$I_0(x)K_0(x) = \int_0^{+\infty} e^{-4xr}\,\varphi(r)\,dr \tag{1.11}$$

where

$$\varphi(r) = \sum_{k=0}^{+\infty} w_k\, r^{2k} \qquad \text{for } r < 1, \tag{1.12}$$

$$\varphi(r) = \frac{1}{r}\varphi\Big(\frac{1}{r}\Big) = \sum_{k=0}^{+\infty} w_k\, r^{-2k-1} \qquad \text{for } r > 1. \tag{1.13}$$

(The last identity can be seen immediately by applying the change of variable $\theta \mapsto -\theta$ in (1.8)). It is also easily checked using (1.10) that one has an equivalent

$$\varphi(r) \sim \sum_{k=1}^{+\infty} \frac{r^{2k}}{\pi k} = \frac{1}{\pi}\,\log\frac{1}{1-r^2} \qquad \text{when } r \to 1-0,$$

in particular the integral (1.11) converges near $r = 1$ (later, we will need a more precise approximation, but more sophisticated arguments are required for this). By an integration term by term on $[0, +\infty[$ of the series defining $\varphi(r)$, and by ignoring the fact that the series diverges for $r \geqslant 1$, one formally obtains a divergent asymptotic expansion

$$I_0(x)K_0(x) \sim \sum_{k\in\mathbb{N}} w_k\,\frac{(2k)!}{(4x)^{2k+1}} \sim \frac{1}{4x}\sum_{k\in\mathbb{N}}\frac{(2k)!^3}{k!^4\,(16x)^{2k}}. \tag{1.14}$$

If x is an integer, the general term of this expansion achieves its minimum exactly for $k = 2x$, since the ratio of the k-th and $(k-1)$-st terms is

$$\frac{(2k(2k-1))^3}{k^4\,(16x)^2} = \Big(\frac{k}{2x}\Big)^2\Big(1-\frac{1}{2k}\Big)^3 < 1 \quad \text{iff } k \leqslant 2x.$$

As already explained in the introduction, the idea is to truncate the asymptotic expansion precisely at $k = 2x$, and to estimate the truncation error. This can be done by means of our explicit integral formula (1.11).

2. Expression of the error in terms of elliptic integrals

By (1.7) and the definition of $\Delta(x)$ we have

$$\Delta(x) = \int_0^{+\infty} e^{-4xr}\, \delta(r)\, dr \tag{2.1}$$

where

$$\delta(r) := \varphi(r) - \sum_{k=0}^{2x} w_k\, r^{2k}, \qquad \text{so that} \quad \delta(r) = \sum_{k=2x+1}^{+\infty} w_k\, r^{2k} \quad \text{for} \quad r < 1. \tag{2.2}$$

For $r < 1$, let us also observe that $\varphi(r)$ coincides with the elliptic integral of the first kind $\frac{2}{\pi}\int_0^{\pi/2} (1 - r^2 \sin^2\theta)^{-1/2}\, d\theta$, as follows again from the binomial formula and the expression of W_{2k}. We need to calculate the precise asymptotic behavior of $\varphi(r)$ when $r \to 1$. This can be obtained by means of a well known identity which we recall below. By putting $t^2 = 1 - r^2$, the change of variable $u = \tan\theta$ gives

$$\begin{aligned} \varphi(r) &= \frac{2}{\pi}\int_0^{\pi/2} (1 - r^2\cos^2\theta)^{-1/2}\, d\theta = \frac{2}{\pi}\int_0^{+\infty} \frac{du}{\sqrt{(1+u^2)(t^2+u^2)}}\, du \\ &= \frac{4}{\pi}\int_0^1 \frac{dv}{\sqrt{(1+v^2)(1+t^2v^2)}} + \frac{2}{\pi}\int_t^1 \frac{dv}{\sqrt{(1+v^2)(t^2+v^2)}} \end{aligned} \tag{2.3}$$

where the last line is obtained by splitting the integral $\int_0^{+\infty} \ldots\, du$ on the 3 intervals $[0, t]$, $[t, 1]$, $[1, +\infty[$, and by performing the respective changes of variable $u = vt$, $u = v$, $u = 1/v$ (the first and third pieces being then equal). Thanks to the binomial formula, the first integral of line (2.3) admits a development as a convergent series

$$\frac{4}{\pi}\int_0^1 \frac{dv}{\sqrt{(1+v^2)(1+t^2v^2)}} = \frac{4}{\pi}\sum_{k=0}^{+\infty} c'_k t^{2k}, \qquad c'_k = \binom{-1/2}{k}\int_0^1 \frac{v^{2k}\, dv}{\sqrt{1+v^2}}.$$

The second integral can be expressed as the sum of a double series when we simultaneously expand both square roots:

$$\frac{2}{\pi}\int_t^1 \frac{dv}{v\sqrt{1+v^2}\,\sqrt{(1+t^2/v^2)}} = \frac{2}{\pi}\int_t^1 \sum_{k,\ell\geqslant 0}\binom{-1/2}{\ell} v^{2\ell}\binom{-1/2}{k}(t^2/v^2)^k\,\frac{dv}{v}.$$

The diagonal part $k=\ell$ yields a logarithmic term

$$\frac{2}{\pi}\sum_{k=0}^{+\infty}\binom{-1/2}{k}^2 t^{2k}\ \log\frac{1}{t} = \frac{1}{\pi}\,\varphi(t)\ \log\frac{1}{t^2},$$

and the other terms can be collected in the form of an absolutely convergent double series

$$\frac{2}{\pi}\sum_{k\neq\ell\geqslant 0}\binom{-1/2}{k}\binom{-1/2}{\ell}t^{2k}\left[\frac{v^{2\ell-2k}}{2\ell-2k}\right]_t^1 = \frac{2}{\pi}\sum_{k\neq\ell\geqslant 0}\binom{-1/2}{k}\binom{-1/2}{\ell}\,\frac{t^{2k}-t^{2\ell}}{2(\ell-k)}.$$

After grouping the various powers t, the summation reduces to a power series $\frac{4}{\pi}\sum c''_k t^{2k}$ of radius of convergence 1, where (due to the symmetry in k,ℓ)

$$c''_k = \sum_{0\leqslant\ell<+\infty,\,\ell\neq k}\ \frac{1}{2(\ell-k)}\binom{-1/2}{k}\binom{-1/2}{\ell}.$$

In fact, we see a priori from (1.10) that

$$|c'_k| \leqslant \frac{1}{\sqrt{\pi k}}\,\frac{1}{2k+1} = O(k^{-3/2}),$$

and

$$|c''_k| \leqslant \frac{1}{2\sqrt{\pi k}}\left(\frac{1}{k}+\sum_{0<\ell\neq k}\frac{1}{|\ell-k|\,\sqrt{\pi\ell}}\right) = O\Big(\frac{\log k}{k}\Big).$$

In total, if we put $t^2 = 1-r^2$, the above relation implies

$$(2.4)\qquad \varphi(r) = \frac{1}{\pi}\left(\varphi(t)\ \log\frac{1}{t^2} + 4\sum_{k=0}^{+\infty} c_k\,t^{2k}\right),\qquad c_k = c'_k + c''_k,$$

and this identity will produce an arbitrarily precise expansion of $\varphi(r)$ when $r \to 1$. In order to compute the coefficients, we observe that

$$c_k = c'_k + c''_k = \binom{-1/2}{k}\,\alpha_k$$

with

$$\alpha_k = \int_0^1 \frac{v^{2k}\,dv}{\sqrt{1+v^2}} + \int_1^{+\infty}\left(\frac{v^{2k}}{\sqrt{1+v^2}} - \sum_{\ell=0}^{k}\binom{-1/2}{\ell}\,v^{2k-2\ell-1}\right)dv + \sum_{\ell=0}^{k-1}\frac{1}{2(\ell-k)}\binom{-1/2}{\ell}.$$

A direct calculation gives

$$c_0 = \alpha_0 = \int_0^1 \frac{dv}{\sqrt{1+v^2}} + \int_1^{+\infty}\left(\frac{1}{\sqrt{1+v^2}} - \frac{1}{v}\right)dv = \log 2.$$

Next, if we write

$$\frac{v^{2k}}{\sqrt{1+v^2}} = v^{2k-1}\cdot\frac{v}{\sqrt{1+v^2}}, \qquad (\sqrt{1+v^2})' = \frac{v}{\sqrt{1+v^2}}$$

and integrate by parts after factoring v^{2k-1}, we get

$$\begin{aligned}\alpha_k = {} & \sum_{\ell=0}^{k-1}\frac{1}{2(\ell-k)}\binom{-1/2}{\ell} + \left[v^{2k-1}\sqrt{1+v^2}\,\right]_0^1 - \int_0^1 (2k-1)\,v^{2k-2}\sqrt{1+v^2}\,dv \\ & + \left[v^{2k-1}\left(\sqrt{1+v^2} - \sum_{\ell=0}^{k}\binom{-1/2}{\ell}\frac{v^{1-2\ell}}{1-2\ell}\right)\right]_1^{+\infty} \\ & - \int_1^{+\infty}(2k-1)\,v^{2k-2}\left(\sqrt{1+v^2} - \sum_{\ell=0}^{k}\binom{-1/2}{\ell}\frac{v^{1-2\ell}}{1-2\ell}\right)dv.\end{aligned}$$

This suggests to calculate $\alpha_k + (2k-1)\alpha_{k-1}$ and to use the simplification

$$v^{2k-2}\sqrt{1+v^2} - \frac{v^{2k-2}}{\sqrt{1+v^2}} = \frac{v^{2k}}{\sqrt{1+v^2}}.$$

We then infer

$$\alpha_k+(2k-1)\alpha_{k-1}=-(2k-1)\alpha_k+\sum_{\ell=0}^{k}\binom{-1/2}{\ell}\frac{1}{1-2\ell}$$
$$+\int_1^{+\infty}(2k-1)\,v^{2k-2}\left(\sum_{\ell=0}^{k}\binom{-1/2}{\ell}\left(\frac{v^{1-2\ell}}{1-2\ell}-v^{1-2\ell}\right)-\sum_{\ell=0}^{k-1}\binom{-1/2}{\ell}v^{-1-2\ell}\right)dv$$
$$+2k\sum_{\ell=0}^{k-1}\frac{1}{2(\ell-k)}\binom{-1/2}{\ell}+(2k-1)\sum_{\ell=0}^{k-2}\frac{1}{2(\ell-(k-1))}\binom{-1/2}{\ell}.$$

A change of indices $\ell=\ell'-1$ in the sums corresponding to $k-1$ then eliminates almost all terms. There only remains the term $\ell=k$ in the first summation, whence the induction relation

$$2k\,\alpha_k+(2k-1)\alpha_{k-1}=-\binom{-1/2}{k}\frac{1}{2k-1},\quad\text{i.e.}\quad\frac{\alpha_k}{\binom{-1/2}{k}}-\frac{\alpha_{k-1}}{\binom{-1/2}{k-1}}=-\frac{1}{2k(2k-1)}.$$

We get in this way

$$\frac{c_k}{\binom{-1/2}{k}^2}=\frac{\alpha_k}{\binom{-1/2}{k}}=\frac{\alpha_0}{1}-\sum_{\ell=1}^{k}\frac{1}{2\ell(2\ell-1)}=\log 2-\sum_{\ell=1}^{2k}\frac{(-1)^{\ell-1}}{\ell}$$

and the explicit expression

$$c_k=w_k\left(\log 2-\sum_{\ell=1}^{2k}\frac{(-1)^{\ell-1}}{\ell}\right).\tag{2.5}$$

The remainder of the alternating series expressing $\log 2$ is bounded by half of last calculated term, namely $1/4k$, thus according to (1.10) we have $0<c_k<\frac{1}{\pi^2k^2}$ if $k\geqslant 1$, and the radius of convergence of the series is 1. From (1.11) and (2.4) we infer as $r\to 1-0$ the well known expansion of the elliptic integral

$$\varphi(r)=\frac{1}{\pi}\left(\sum_{k=0}^{+\infty}w_kt^{2k}\log\frac{1}{t^2}+4\sum_{k=0}^{+\infty}c_kt^{2k}\right),\qquad t^2=1-r^2,\tag{2.6}$$

with

$$w_0=1,\ \ w_1=\frac{1}{4},\ \ w_2=\frac{9}{64},\ \ c_0=\log 2,\ \ c_1=\frac{1}{4}\left(\log 2-\frac{1}{2}\right),\ \ c_2=\frac{9}{64}\left(\log 2-\frac{7}{12}\right).$$

Let us compute explicitly the first terms of the asymptotic expansion at $r = 1$ by putting $r = 1 + h$, $h \to 0$. For $r = 1 + h < 1$ $(h < 0)$ we have $t^2 = 1 - r^2 = -2h - h^2 = 2|h|(1 + h/2)$, where

$$\log \frac{1}{t^2} = \log \frac{1}{2|h|(1+h/2)} = \log \frac{1}{|h|} - \log 2 - \frac{1}{2}h + \frac{1}{8}h^2 + O(h^2),$$

$$\sum_{k=0}^{+\infty} w_k t^{2k} = 1 + \frac{1}{4}(-2h - h^2) + \frac{9}{64}(2h)^2 + O(h^3),$$

$$4\sum_{k=0}^{+\infty} c_k t^{2k} = 4\log 2 + \left(\log 2 - \frac{1}{2}\right)(-2h - h^2) + \frac{9}{16}\left(\log 2 - \frac{7}{12}\right)(2h)^2 + O(h^3),$$

and

$$\varphi(1+h) = \frac{1}{\pi}\left(\left(1 - \frac{1}{2}h + \frac{5}{16}h^2 + O(h^3)\right)\left(\log \frac{1}{|h|} - \log 2 - \frac{1}{2}h + \frac{1}{8}h^2 + O(h^3)\right)\right.$$
$$\left. + 4\log 2 - \big(2\log 2 - 1\big)h + \left(\frac{5}{4}\log 2 - \frac{13}{16}\right)h^2 + O(h^3)\right).$$

If terms are written by decreasing order of magnitude, we get

$$\varphi(1+h) = \frac{1}{\pi}\left(\log \frac{1}{|h|} + 3\log 2 - \frac{1}{2}h\log \frac{1}{|h|} - \left(\frac{3}{2}\log 2 - \frac{1}{2}\right)h\right.$$
$$\left. + \frac{5}{16}h^2\log \frac{1}{|h|} + \left(\frac{15}{16}\log 2 - \frac{7}{16}\right)h^2 + O\left(h^3\log \frac{1}{|h|}\right)\right). \tag{2.7}$$

For $r = 1 + h > 1$, the identity $\varphi(r) = \frac{1}{r}\varphi(\frac{1}{r})$ gives in a similar way

$$\varphi(r) = \frac{1}{1+h}\left(\frac{1}{\pi}\sum_{k=0}^{+\infty} w_k t^{2k}\log \frac{1}{t^2} + \sum_{k=0}^{+\infty} c_k t^{2k}\right), \quad t^2 = 1 - \frac{1}{r^2} = 2h - 3h^2 + O(h^3).$$

After a few simplifications, one can see that the expansion (2.7) is still valid for $h > 0$. Passing to the limit $r \to 0$, $t \to 1-0$ in (2.6) implies the relation $\sum_{k\geqslant 0} c_k = \frac{\pi}{4}$. The following Lemma will be useful.

LEMMA A. *For $h > 0$, the difference*

$$\rho(h) = \varphi(1+h) - \frac{1}{\pi}\left(\log\frac{1}{h} + 3\log 2 - \frac{1}{2}h\log\frac{1}{h} - \left(\frac{3}{2}\log 2 - \frac{1}{2}\right)h\right) \tag{2.8}$$

$$= \varphi(1+h) - \frac{1}{2\pi}\left((h-2)\log\frac{h}{8} + h\right) \tag{2.9}$$

admits the upper bound

$$|\rho(h)| \leqslant h^2\left(2 + \log\left(1 + \frac{1}{h}\right)\right). \tag{2.10}$$

PROOF. A use of the Taylor-Lagrange formula gives $(1+h)^{-1} = 1 - h + \theta_1 h^2$, $t^2 = 1 - \frac{1}{r^2} = 2h - 3\theta_2 h^2$, with $\theta_i \in \,]0,1[$, and we also find $t^2 \leqslant 2h$ and

$$\begin{aligned}\log\frac{1}{t^2} &= \log\frac{r^2}{(r-1)(r+1)} \\ &= \log\frac{1}{h} + 2\log(1+h) - \log\left(1 + \frac{h}{2}\right) - \log 2 \\ &= \log\frac{1}{h} - \log 2 + \frac{3}{2}h - \frac{7}{8}\theta_3 h^2, \quad \theta_2 \in \,]0,1[,\end{aligned}$$

while the remainder terms $\sum_{k\geqslant 2} w_k t^{2k}$ and $\sum_{k\geqslant 2} c_k t^{2k}$ are bounded respectively by

$$\frac{w_2 t^4}{1-t^2} \leqslant 4w_2 r^2 h^2 \leqslant \frac{225}{256}h^2$$

and

$$\frac{c_2 t^4}{1-t^2} \leqslant 4c_2 r^2 h^2 < \frac{1}{10}h^2 \quad \text{if } h \leqslant \frac{1}{4},\ r = 1 + h \leqslant \frac{5}{4}.$$

For $h \leqslant \frac{1}{4}$ we thus get an equality

$$\begin{aligned}\varphi(1+h) =& \frac{1}{\pi}(1 - h + \theta_1 h^2)\times \\ & \left(\left(1 + \frac{1}{4}(2h - 3\theta_2 h^2) + \frac{225}{256}\theta_4 h^2\right)\left(\log\frac{1}{|h|} - \log 2 + \frac{3}{2}h - \frac{7}{8}\theta_3 h^2\right)\right. \\ & \left. + 4\log 2 + \left(\log 2 - \frac{1}{2}\right)(2h - 3\theta_2 h^2) + \frac{4}{10}\theta_5 h^2\right)\end{aligned}$$

with $\theta_i \in \,]0,1[$. In order to estimate $\rho(h)$, we fully expand this expression and replace each term by an upper bound of its absolute value. For $h \leqslant \frac{1}{4}$, this shows that $|\rho(h)| \leqslant h^2(0.885 \, \log\frac{1}{h} + 2.11)$, so that (2.10) is satisfied. For $h \geqslant \frac{1}{4}$, we write

$$\rho'(h) = \varphi'(1+h) - \frac{1}{2\pi}\left(\log\frac{h}{8} + 2 - \frac{2}{h}\right), \qquad \varphi'(r) = -\sum_{k=0}^{+\infty}(2k+1)w_k\, r^{-2k-2},$$

and by (1.10) we get

$$\sum_{k=0}^{+\infty}\frac{2}{\pi}r^{-2k-2} < -\varphi'(r) < \frac{1}{r^2} + \sum_{k=1}^{+\infty}\frac{3k}{\pi k}r^{-2k-2} < \sum_{k=0}^{+\infty} r^{-2k-2} = \frac{1}{r^2-1},$$

therefore

$$\frac{2}{\pi}\,\frac{1}{h(h+2)} < -\varphi'(1+h) < \frac{1}{h(h+2)},$$
$$\frac{1}{2\pi}\left(\log\frac{8}{h} - 2 + \frac{2}{h} - \frac{2\pi}{h(h+2)}\right) < \rho'(h) < \frac{1}{2\pi}\left(\log\frac{8}{h} - 2 + \frac{2}{h+2}\right).$$

This implies

$$-1.72 < \frac{1}{2\pi}\left(\log 4 - 2 + \frac{1}{4} - \frac{32\pi}{9}\right) < \rho'(h) < \frac{1}{2\pi}\left(\log 32 - 2 + \frac{8}{9}\right) < 1.51 \quad \text{on } \left[\frac{1}{4}, 2\right],$$
$$-\frac{1}{2\pi}\left(\log\frac{h}{8} + 2\right) < \rho'(h) < \frac{1}{2\pi}\left(\log 4 - \frac{3}{2}\right) < 0 \qquad \text{on } [2, +\infty[,$$

therefore $|\rho'(h)| \leqslant \frac{1}{2\pi}(h - 1 - \log 8 + 2) \leqslant \frac{1}{2\pi}h$ for $h \in [2, +\infty[$. Since $\rho(2) \simeq 0.00249 < \frac{1}{\pi}$, we see that $|\rho(h)| \leqslant \frac{1}{4\pi}h^2$, and this shows that (2.10) still holds on $[2, +\infty[$. A numerical calculation of $\rho(h)$ at sufficiently close points in the interval $[\frac{1}{4}, 2]$ finally yields (2.10) on that interval. □

Now we split the integral (2.1) on the intervals $[0,1]$ and $[1, +\infty[$, starting with the integral of φ on the interval $[1, +\infty[$. The change of variable $r = 1 + t/4x$ provides

$$\int_1^{+\infty} e^{-4xr}\,\varphi(r)\,dr = \frac{e^{-4x}}{4x}\int_0^{+\infty} e^{-t}\,\varphi\left(1 + \frac{t}{4x}\right)dt, \tag{2.11}$$

and Lemma A (2.9) yields for this integral an approximation

$$\frac{e^{-4x}}{8\pi x}\int_0^{+\infty} e^{-t}\Big(\Big(\frac{t}{4x}-2\Big)\log\frac{t}{32x}+\frac{t}{4x}\Big)\,dt$$

$$=\frac{e^{-4x}}{8\pi x}\Big(\log(32x)\Big(2-\frac{1}{4x}\Big)+2\gamma+\frac{1}{4x}\int_0^{+\infty} e^{-t}(t\log t+t)\,dt\Big)$$

$$=\frac{e^{-4x}}{4\pi x}\Big(\log x+\gamma+5\log 2-\frac{\log x}{8x}-\frac{\gamma+5\log 2-2}{8x}\Big),$$

with an error bounded by

$$\frac{e^{-4x}}{4x}\int_0^{+\infty} e^{-t}\Big(\frac{t}{4x}\Big)^2\Big(2+\log\Big(1+\frac{4x}{t}\Big)\Big)\,dt$$

$$=\frac{e^{-4x}}{4x}\Big(\frac{1}{4x^2}+\frac{1}{16x^2}\int_0^{+\infty} t^2\,e^{-t}\,\log\frac{t+4x}{t}\,dt\Big).$$

Writing

$$0<\log\frac{t+4x}{t}=\log\frac{4x}{t}+\log\Big(1+\frac{t}{4x}\Big)\leqslant\log\frac{4x}{t}+\frac{t}{4x},$$

we further see that

$$\int_0^{+\infty} t^2\,e^{-t}\,\log\frac{t+4x}{t}\,dt\leqslant\int_0^{+\infty} t^2\,e^{-t}\Big(\log\frac{4x}{t}+\frac{t}{4x}\Big)\,dt=2\log 4x+\frac{3}{2x}+2\gamma-3.$$

We infer

$$\int_1^{+\infty} e^{-4xr}\,\varphi(r)\,dr=\frac{e^{-4x}}{4\pi x}\Big(\log x+\gamma+5\log 2-\frac{\log x}{8x}\Big)+\frac{e^{-4x}}{4x}\,R_1(x), \tag{2.12}$$

with

$$|R_1(x)|<\frac{\gamma+5\log 2-2}{8\pi x}+\frac{1}{4x^2}+\frac{2\log 4x+\frac{3}{2x}+2}{16x^2}<\frac{0.483}{x}\quad\text{if } \mathbb{N}\ni x\geqslant 1, \tag{2.13}$$

thanks to a numerical evaluation of the sequence in a suitable range.

3. Estimate of the truncated asymptotic expansion

We now estimate the two integrals

$$\int_0^1 e^{-4xr} \sum_{k\geqslant 2x+1} w_k\, r^{2k}\, dr, \qquad \int_1^{+\infty} e^{-4xr} \sum_{k\leqslant 2x} w_k\, r^{2k}\, dr.$$

By means of iterated integrations by parts, we get

$$\int_0^1 e^{-4xr}\, r^{2k}\, dr = e^{-4x} \sum_{\ell=1}^{+\infty} \frac{(4x)^{\ell-1}}{(2k+1)\cdots(2k+\ell)}, \tag{3.1}$$

$$\int_1^{+\infty} e^{-4xr}\, r^{2k}\, dr = \frac{e^{-4x}}{4x}\left(1+\sum_{\ell=1}^{2k} \frac{2k(2k-1)\cdots(2k-\ell+1)}{(4x)^{\ell}}\right). \tag{3.2}$$

Combining the identities (2.1), (2.2), (2.12), (3.1), (3.2) we find
(3.3)

$$\Delta(x) = \frac{e^{-4x}}{4x}\left(\frac{1}{\pi}\Big(\log x+\gamma+5\log 2\Big) - \frac{\log x}{8\pi x} - \sum_{k=0}^{2x} w_k + S(x) + R_1(x) + R_2(x)\right)$$

with
(3.4)

$$S(x) = \sum_{k=2x+1}^{+\infty}\sum_{\ell=1}^{2x-1} \frac{w_k\,(4x)^{\ell}}{(2k+1)\cdots(2k+\ell)} - \sum_{k=1}^{2x}\sum_{\ell=1}^{2x-1} w_k\, \frac{2k(2k-1)\cdots(2k-\ell+1)}{(4x)^{\ell}},$$

and
(3.5)

$$R_2(x) = \sum_{k=2x+1}^{+\infty}\sum_{\ell=2x}^{+\infty} \frac{w_k\,(4x)^{\ell}}{(2k+1)\cdots(2k+\ell)} - \sum_{k=1}^{2x}\sum_{\ell=2x}^{+\infty} w_k\, \frac{2k(2k-1)\cdots(2k-\ell+1)}{(4x)^{\ell}}$$

(In the final summation, terms of index $\ell > 2k$ are zero). Formula (3.3) leads us to study the asymptotic expansion of $\sum_{k=0}^{2x} w_k$. This development is easy to establish from (2.6) (one could even calculate it at an arbitrarily large order).

LEMMA B. *One has*

$$w_k = \frac{1}{\pi k}\left(1 - \frac{1}{2(2k-1)} + \varepsilon_k\right) \quad \textit{where} \tag{3.6}$$

$$\frac{1}{12k(2k-1)} < \varepsilon_k < \frac{5}{16k(2k-1)}, \quad k \geqslant 1,$$

$$\sum_{k=0}^{2x} w_k = \frac{1}{\pi}\Big(\log x + 5\log 2 + \gamma\Big) + R_3(x), \quad \frac{1}{4\pi x} < R_3(x) < \frac{19}{48\pi x}. \tag{3.7}$$

PROOF. The lower bound (3.6) is a consequence of the Euler-Maclaurin's formula [6] applied to the function $f(x) = \log \frac{2x-1}{2x}$. This yields

$$\frac{1}{2}\ \log w_k = \sum_{i=1}^{k} f(i) = C + \int_1^k f(x)\,dx + \frac{1}{2} f(k) + \sum_{j=1}^{p} \frac{b_{2j}}{(2j)!}\, f^{(2j-1)}(k) + \widetilde{R}_p$$

where C is a constant, and where the remainder term $\widetilde{R}_p$ is the product of the next term by a factor $[0,1]$, namely

$$\frac{b_{2p+2}}{(2p+2)!}\, f^{(2p+1)}(k) = \frac{2^{2p+1}\, b_{2p+2}}{(2p+1)(2p+2)}\left(\frac{1}{(2k-1)^{2p+1}} - \frac{1}{(2k)^{2p+1}}\right).$$

We have here

$$\begin{aligned}\int_1^k f(x)\,dx &= \frac{1}{2}(2k-1)\log(2k-1) - k\log k - (k-1)\ \log 2\\ &= \left(k - \frac{1}{2}\right)\log\left(1 - \frac{1}{2k}\right) - \frac{1}{2}\ \log k + \frac{1}{2}\ \log 2\end{aligned}$$

and the constant C can be computed by the Wallis formula. Therefore, with $b_2 = \frac{1}{6}$, we have

$$\begin{aligned}\log w_k &= \log\frac{1}{\pi k} + 2k\ \log\left(1 - \frac{1}{2k}\right) + 1 + 2\theta\, b_2\left(\frac{1}{(2k-1)} - \frac{1}{2k}\right)\\ &\geqslant \log\frac{1}{\pi k} - \frac{1}{4k} - \sum_{\ell=3}^{+\infty}\frac{1}{\ell(2k)^{\ell-1}} > \log\frac{1}{\pi k} - \frac{1}{4k} - \frac{1}{3}\,\frac{1}{(2k)^2}\,\frac{1}{1-\frac{1}{2k}}.\end{aligned}$$

The inequality $e^{-x} \geqslant 1 - x$ then gives

$$w_k > \frac{1}{\pi k}\left(1 - \frac{1}{4k} - \frac{1}{6k(2k-1)}\right) = \frac{1}{\pi k}\left(1 - \frac{1}{2(2k-1)} + \frac{1}{12k(2k-1)}\right)$$

and the lower bound (3.6) follows for all $k \geqslant 1$. In the other direction, we get

$$\begin{aligned}\log w_k &< \log \frac{1}{\pi k} - \frac{1}{4k} - \frac{1}{12k^2} - \frac{1}{32k^3} + \frac{1}{6k(2k-1)} \\ &= \log \frac{1}{\pi k} - \frac{1}{4k} + \frac{1}{12k^2(2k-1)} - \frac{1}{32k^3}\end{aligned}$$

and the inequality $e^{-x} \leqslant 1 - x + \frac{1}{2}x^2$ implies

$$w_k < \frac{1}{\pi k}\left(1 - \left(\frac{1}{4k} - \frac{1}{12k^2(2k-1)} + \frac{1}{32k^3}\right) + \frac{1}{2}\left(\frac{1}{4k}\right)^2\right)$$

whence (by a difference of polynomials and a reduction to the same denominator)

$$w_k < \frac{1}{\pi k}\left(1 - \frac{1}{2(2k-1)} + \frac{5}{16k(2k-1)}\right) \qquad \text{if } k \geqslant 3.$$

One can check that the final inequality still holds for $k = 1,2$, and this implies the estimate (3.6). On the other hand, formula (2.6) yields

$$\begin{aligned}w_0 + \sum_{k=1}^{+\infty} \left(w_k - \frac{1}{\pi k}\right) r^{2k} &= \varphi(r) - \frac{1}{\pi}\ \log \frac{1}{1-r^2} \\ &= \frac{1}{\pi}\ \big(\varphi(t) - 1\big)\ \log \frac{1}{1-r^2} + \frac{4}{\pi}\ \log 2 + \sum_{k\geqslant 1} c_k\, t^{2k}\end{aligned}$$

with $t = \sqrt{1-r^2}$ and $\varphi(t) = 1 + O(1-r^2)$. By passing to the limit when $r \to 1-0$ and $t \to 0$, we thus get

$$w_0 + \sum_{k=1}^{+\infty} \left(w_k - \frac{1}{\pi k}\right) = \frac{4}{\pi}\ \log 2.$$

We infer

$$w_0 + \sum_{k=1}^{2x} \left(w_k - \frac{1}{\pi k}\right) - \frac{4}{\pi}\ \log 2 = \sum_{2x+1}^{+\infty} \left(\frac{1}{\pi k} - w_k\right)$$

and the upper and lower bounds in (3.6) imply

$$0 < \sum_{2x+1}^{+\infty}\Big(\frac{1}{\pi k} - w_k\Big) \leqslant \sum_{2x+1}^{+\infty}\frac{1}{2\pi\,k(2k-1)} < \sum_{2x+1}^{+\infty}\frac{1}{4\pi}\,\frac{1}{k(k-1)} = \frac{1}{8\pi x}.$$

The Euler-Maclaurin estimate

$$\sum_{k=1}^{2x}\frac{1}{k} = \log(2x) + \gamma + \frac{1}{4x} + \frac{b_2}{2(2x)^2} - \frac{b_4}{4(2x)^4} + \cdots \tag{3.8}$$

then finally yields (3.7). □

It remains to evaluate the sum $S(x)$. This is considerably more difficult, as a consequence of a partial cancellation of positive and negative terms. The approximation (3.6) obtained in Lemma B implies

$$S(x) = \frac{2}{\pi}\Big(T(x) - \frac{1}{2}U(x) + \frac{5}{8}R_4(x)\Big), \tag{3.9}$$

and if we agree as usual that the empty product $(2k-2)\cdots(2k-\ell+1) = \frac{1}{2k-1}$ for $\ell = 1$ is equal to 1, we get

$$T(x) = \sum_{\ell=1}^{2x-1}\sum_{k=2x+1}^{+\infty}\frac{(4x)^\ell}{2k(2k+1)\cdots(2k+\ell)} - \sum_{\ell=1}^{2x-1}\sum_{k=1}^{2x}\frac{(2k-1)\cdots(2k-\ell+1)}{(4x)^\ell}, \tag{3.10}$$

$$U(x) = \sum_{\ell=1}^{2x-1}\sum_{k=2x+1}^{+\infty}\frac{(4x)^\ell}{(2k-1)\cdots(2k+\ell)} - \sum_{\ell=1}^{2x-1}\sum_{k=1}^{2x}\frac{(2k-2)\cdots(2k-\ell+1)}{(4x)^\ell}, \tag{3.11}$$

where the new error term $R_4(x)$ admits the upper bound
(3.12)

$$|R_4(x)| \leqslant \sum_{\ell=1}^{2x-1}\sum_{k=2x+1}^{+\infty}\frac{(4x)^\ell/2k}{(2k-1)\cdots(2k+\ell)} + \sum_{\ell=1}^{2x-1}\sum_{k=1}^{2x}\frac{(2k-2)\cdots(2k-\ell+1)}{2k\,(4x)^\ell}.$$

4. Application of discrete integration by parts

To evaluate the sums $T(x)$, $U(x)$ and $R_4(x)$, our method consists in performing first a summation over the index k, and for this, we use “discrete integrations by parts". Set

$$u_k^{a,b} := \frac{1}{(2k+a)(2k+a+1)\cdots(2k+b-1)}, \qquad a \leqslant b \tag{4.1}$$

(agreeing that the denominator is 1 if $a = b$). Then

$$\begin{aligned} u_k^{a,b} - u_{k+1}^{a,b} &= \frac{(2k+b)(2k+b+1)-(2k+a)(2k+a+1)}{(2k+a)(2k+a+1)\cdots(2k+b+1)} \\ &= \frac{(b-a)(4k+a+b+1)}{(2k+a)(2k+a+1)\cdots(2k+b+1)}. \end{aligned}$$

The inequalities $2(2k+a) \leqslant 4k+a+b+1 \leqslant 2(2k+b+1)$ imply

$$\frac{1}{(2k+a+1)\cdots(2k+b+1)} \leqslant \frac{u_k^{a,b} - u_{k+1}^{a,b}}{2(b-a)} \leqslant \frac{1}{(2k+a)(2k+a+1)\cdots(2k+b)}$$

with an upward error and a downward error both equal to

$$\frac{b-a+1}{2}\,\frac{1}{(2k+a)(2k+a+1)\cdots(2k+b+1)}.$$

In particular, through a summation $\sum_{k=2x+1}^{+\infty} \frac{u_k^{a-1,b-1} - u_{k+1}^{a-1,b-1}}{2(b-a)}$, these inequalities imply

$$\sum_{k=2x+1}^{+\infty} \frac{1}{(2k+a)\cdots(2k+b)} \leqslant \frac{u_{2x+1}^{a-1,b-1}}{2(b-a)} = \frac{1}{2(b-a)}\,\frac{1}{(4x+a+1)\cdots(4x+b)},$$

with an upward error equal to

$$\frac{b-a+1}{2} \sum_{k=2x+1}^{+\infty} \frac{1}{(2k+a-1)\cdots(2k+b)} \leqslant \frac{1}{4}\,\frac{1}{(4x+a)\cdots(4x+b)}$$

and an "error on the error" (again upwards) equal to

$$\begin{aligned} &\frac{(b-a+1)(b-a+2)}{4} \sum_{k=2x+1}^{+\infty} \frac{1}{(2k+a-2)\cdots(2k+b)} \\ &\leqslant \frac{b-a+1}{8}\,\frac{1}{(4x+a-1)\cdots(4x+b)}. \end{aligned}$$

In other words, we find

$$\sum_{k=2x+1}^{+\infty}\frac{1}{(2k+a)\cdots(2k+b)}=\frac{1}{2(b-a)}\,\frac{1}{(4x+a+1)\cdots(4x+b)}$$

$$(4.2_3^{a,b})\qquad -\frac{1}{4}\,\frac{1}{(4x+a)\cdots(4x+b)}+\theta\,\frac{b-a+1}{8}\,\frac{1}{(4x+a-1)\cdots(4x+b)},\quad \theta\in[0,1].$$

If necessary, one could of course push further this development to an arbitrary number of terms p rather than 3. We will denote the corresponding expansion $(4.2_p^{a,b})$, and will use it here in the cases $p=2,3$. For the summations $\sum_{k=1}^{2x}\ldots$, we similarly define

$$(4.3)\qquad v_k^{a,b}=(2k-a)(2k-a-1)\cdots(2k-b+1),\qquad a\leqslant b,$$

and obtain

$$\begin{aligned}v_k^{a,b}-v_{k-1}^{a,b}&=(2k-a-2)\cdots(2k-b+1)\big((2k-a)(2k-a-1)-(2k-b)(2k-b-1)\big)\\&=(2k-a-2)\cdots(2k-b+1)\big((b-a)(4k-a-b-1)\big).\end{aligned}$$

For $a<b$, the inequalities $2(2k-b)\leqslant(4k-a-b-1)\leqslant 2(2k-a-1)$ imply

$$(2k-a-2)\cdots(2k-b)\leqslant\frac{v_k^{a,b}-v_{k-1}^{a,b}}{2(b-a)}\leqslant(2k-a-1)\cdots(2k-b+1)$$

with an upward error and a downward error both equal to

$$\frac{1}{2}(b-a-1)\,(2k-a-2)\cdots(2k-b+1).$$

By considering the sum $\sum_{k=1}^{2x}\frac{v_k^{a,b}-v_{k-1}^{a,b}}{2(b-a)}$, we obtain

$$\sum_{k=1}^{2x}(2k-a-1)\cdots(2k-b+1)\geqslant\frac{v_{2x}^{a,b}-v_0^{a,b}}{2(b-a)}$$

with a downward error

$$\frac{b-a-1}{2}\sum_{k=1}^{2x}(2k-a-2)\cdots(2k-b+1)\leqslant\frac{v_{2x}^{a,b-1}-v_0^{a,b-1}}{4}$$

and an upward error on the error equal to

$$\frac{(b-a-1)(b-a-2)}{4}\sum_{k=1}^{2x}(2k-a-2)\cdots(2k-b+2)\leqslant\frac{b-a-1}{8}\left(v_{2x}^{a,b-2}-v_0^{a,b-2}\right),$$

i.e. there exists $\theta\in[0,1]$ such that

$$\begin{aligned}\sum_{k=1}^{2x}&(2k-a-1)\cdots(2k-b+1)\\&=\frac{1}{2(b-a)}\left(v_{2x}^{a,b}-v_0^{a,b}\right)+\frac{1}{4}\left(v_{2x}^{a,b-1}-v_0^{a,b-1}\right)-\theta\frac{b-a-1}{8}\left(v_{2x}^{a,b-2}-v_0^{a,b-2}\right),\\(4.4_3^{a,b})\qquad&=\frac{1}{2(b-a)}v_{2x}^{a,b}+\frac{1}{4}v_{2x}^{a,b-1}-\theta\frac{b-a-1}{8}v_{2x}^{a,b-2}+C_3^{a,b},\end{aligned}$$

with

$$(4.5_3^{a,b})\qquad |C_3^{a,b}|\leqslant\frac{1}{2(b-a)}|v_0^{a,b}|+\frac{1}{4}|v_0^{a,b-1}|+\frac{b-a-1}{8}|v_0^{a,b-2}|,$$

especially $C_3^{a,b}=0$ if $a=0$. The simpler order 2 case (with an initial upward error) gives

$$\begin{aligned}\sum_{k=1}^{2x}(2k-a-2)\cdots(2k-b)&=\frac{1}{2(b-a)}\left(v_{2x}^{a,b}-v_0^{a,b}\right)-\theta\frac{1}{4}\left(v_{2x}^{a,b-1}-v_0^{a,b-1}\right)\\(4.6_2^{a,b})\qquad&=\frac{1}{2(b-a)}(4x-a)\cdots(4x-b+1)-\theta\frac{1}{4}(4x-a)\cdots(4x-b+2)+C_2^{a,b}.\end{aligned}$$

In the order 3 case, it will be convenient to use a further change

$$v_k^{a,b}-v_k^{a+1,b+1}=(2k-a-1)\cdots(2k-b+1)\Big((2k-a)-(2k-b)\Big)=(b-a)v_k^{a+1,b}.$$

If we apply this equality to the values (a,b), $(a,b-1)$ and $k=2x$, we see that the $(4.4_3^{a,b})$ development can be written in the equivalent form

$$\begin{aligned}&\sum_{k=1}^{2x}(2k-a-1)\cdots(2k-b+1)-C_3^{a,b}\\&=\frac{1}{2(b-a)}v_{2x}^{a+1,b+1}+\frac{3}{4}v_{2x}^{a+1,b}+\frac{b-a-1}{8}\left(2v_{2x}^{a+1,b-1}-\theta v_{2x}^{a,b-2}\right),\end{aligned}$$

$$= \frac{1}{2(b-a)}(4x-a-1)\cdots(4x-b) + \frac{3}{4}(4x-a-1)\cdots(4x-b+1)$$

$$(4.7_3^{a,b}) \qquad + \frac{b-a-1}{8}\Big(2(4x-a-1)\cdots(4x-b+2) - \theta(4x-a)\cdots(4x-b+3)\Big)$$

According to (3.10), $(4.2_3^{0,\ell})$ and $(4.7_3^{0,\ell})$, we get

$$T(x) = T'(x) - T''(x) + R_5(x) \tag{4.8}$$

with

$$T'(x) = \sum_{\ell=1}^{2x-1} \frac{1}{2\ell}\left(\frac{(4x)^\ell}{(4x+1)\cdots(4x+\ell)} - \frac{(4x-1)\cdots(4x-\ell)}{(4x)^\ell}\right), \tag{4.9}$$

$$T''(x) = \sum_{\ell=1}^{2x-1} \frac{1}{4}\,\frac{(4x)^\ell}{4x(4x+1)\cdots(4x+\ell)} + \frac{3}{4}\,\frac{(4x-1)\cdots(4x-\ell+1)}{(4x)^\ell}, \tag{4.10}$$

$$|R_5(x)| \leqslant \frac{1}{8}\sum_{\ell=1}^{2x-1}\left(\frac{(\ell+1)(4x)^\ell}{(4x-1)4x\cdots(4x+\ell)} + \frac{2(\ell-1)(4x-1)\cdots(4x-\ell+2)}{(4x)^\ell}\right). \tag{4.11}$$

The last term in the last line comes from formula $(4.7_3^{0,\ell})$, by observing that the inequalities $4x \leqslant 2(4x-\ell+2)$ $\ell \leqslant 2x-1$ imply

$$4x(4x-1)\cdots(4x-\ell+3) \leqslant 2(4x-1)\cdots(4x-\ell+2).$$

Similarly, thanks to (3.11), $(4.2_2^{-1,\ell})$ and $(4.6_2^{0,\ell-1})$, we obtain the decomposition

$$U(x) = U'(x) - U''(x) + R_6(x) \tag{4.12}$$

with

$$U'(x) = \sum_{\ell=1}^{2x-1} \frac{1}{2(\ell+1)}\,\frac{(4x)^\ell}{4x\cdots(4x+\ell)} - \sum_{\ell=2}^{2x-1}\frac{1}{2(\ell-1)}\,\frac{4x(4x-1)\cdots(4x-\ell+2)}{(4x)^\ell}, \tag{4.13}$$

$$U''(x) = \frac{1}{4x}\sum_{k=1}^{2x}\frac{1}{2k-1} \qquad \text{(negative term } \ell=1 \text{ appearing in } U(x)\text{)}, \tag{4.14}$$

$$|R_6(x)| \leqslant \frac{1}{4}\sum_{\ell=1}^{2x-1}\frac{(4x)^\ell}{(4x-1)\cdots(4x+\ell)} + \frac{1}{4}\sum_{\ell=2}^{2x-1}\frac{4x(4x-1)\cdots(4x-\ell+3)}{(4x)^\ell}. \tag{4.15}$$

The remainder terms $R_2(x)$ [resp. $R_4(x)$] can be bounded in the same way by means of $(4.2_2^{0,\ell})$ and $(4.6_2^{-1,\ell-1})$ [resp. $(4.2_2^{-1,\ell})$ and $(4.6_2^{0,\ell-2})$] and (1.10), (3.5), (3.12) lead to

$$|R_2(x)|\leqslant\frac{2}{\pi}\left(\sum_{\ell=2x}^{+\infty}\sum_{k=2x+1}^{+\infty}\frac{(4x)^\ell}{(2k)\cdots(2k+\ell)}+\sum_{\ell=2x}^{+\infty}\sum_{k=1}^{2x}\frac{(2k-1)\cdots(2k-\ell+1)}{(4x)^\ell}\right)$$

$$\leqslant\frac{2}{\pi}\sum_{\ell=2x}^{+\infty}\frac{1}{2\ell}\left(\frac{(4x)^\ell}{(4x+1)\cdots(4x+\ell)}+\frac{(4x+1)\cdots(4x-\ell+2)}{(4x)^\ell}\right), \tag{4.16}$$

$$|R_4(x)|\leqslant\sum_{\ell=1}^{2x-1}\sum_{k=2x+1}^{+\infty}\frac{(4x)^{\ell-1}}{(2k-1)\cdots(2k+\ell)}+\sum_{\ell=1}^{2x-1}\sum_{k=1}^{2x}\frac{(2k-2)\cdots(2k-\ell+2)}{(4x)^\ell}$$

$$\leqslant\sum_{\ell=1}^{2x-1}\frac{1}{2(\ell+1)}\frac{(4x)^{\ell-1}}{4x\cdots(4x+\ell)}+\sum_{\ell=3}^{2x-1}\frac{1}{2(\ell-2)}\frac{4x(4x-1)\cdots(4x-\ell+3)}{(4x)^\ell} \tag{4.17}$$

$$+\sum_{k=1}^{2x}\frac{1}{2k(2k-1)}\frac{1}{4x}+\sum_{k=1}^{2x}\frac{1}{(2k-1)}\frac{1}{(4x)^2}$$

(4.18) [terms $\ell=1,2$ in the summation].

Finally, by (3.3), (3.7), (3.9) and (4.8), (4.12) we get the decomposition

$$\Delta(x)=\frac{e^{-4x}}{4\pi x}\Bigg(2T'(x)-2T''(x)-U'(x)+U''(x)-\frac{\log x}{8x}$$

$$+\pi\Big(R_1(x)+R_2(x)-R_3(x)\Big)-\frac{5}{4}R_4(x)+2R_5(x)-R_6(x)\Bigg). \tag{4.19}$$

LEMMA C. *The following inequalities hold* :

$$\log 2-\frac{1}{8x}<\sum_{k=1}^{2x}\frac{1}{2k(2k-1)}<\log 2-\frac{1}{2(4x+1)}, \tag{4.20}$$

$$\sum_{k=1}^{2x}\frac{1}{2k-1}<\frac{3}{2}\log 2+\frac{1}{2}\Big(\log x+\gamma\Big)+\frac{1}{24x^2}, \tag{4.21}$$

$$U''(x)=\frac{\log x}{8x}+R_7(x),\qquad 0<R_7(x)<\frac{1.37}{x}. \tag{4.22}$$

Proof. To check (4.20), we observe that the sum of the series is $\log 2$ and that the remainder of index $2x$ admits the upper bound

$$\begin{aligned}\frac{1}{2(4x+1)} &= \sum_{k=2x+1}^{+\infty} \frac{1}{4}\left(\frac{1}{k-1/2} - \frac{1}{k+1/2}\right) \\ &< \sum_{k=2x+1}^{+\infty} \frac{1}{2k(2k-1)} < \sum_{k=2x+1} \frac{1}{4}\left(\frac{1}{k-1} - \frac{1}{k}\right) = \frac{1}{8x}.\end{aligned}$$

According to the Euler-Maclaurin expansion (3.8), we get on the one hand

$$\begin{aligned}\sum_{k=1}^{2x} \frac{1}{2k-1} = \sum_{\ell=1}^{4x} \frac{(-1)^{\ell-1}}{\ell} + \sum_{\ell=1}^{2x} \frac{1}{2\ell} &= \sum_{k=1}^{2x} \frac{1}{2k(2k-1)} + \frac{1}{2}\sum_{k=1}^{2x} \frac{1}{k} \\ &< \log 2 - \frac{1}{2(4x+1)} + \frac{1}{2}\left(\log(2x) + \gamma + \frac{1}{4x} + \frac{1}{12(2x)^2}\right) \\ &= \frac{3}{2}\log 2 + \frac{1}{2}\Big(\log x + \gamma\Big) + \frac{1}{8x(4x+1)} + \frac{1}{96x^2},\end{aligned}$$

whence (4.21), and on the other hand

$$\begin{aligned}\sum_{k=1}^{2x} \frac{1}{2k-1} &> \log 2 + \frac{1}{2}\left(\log(2x) + \gamma + \frac{1}{12(2x)^2} - \frac{1}{120(2x)^4}\right) \\ &> \frac{3}{2}\log 2 + \frac{1}{2}\Big(\log x + \gamma\Big) + \frac{1}{96x^2} - \frac{1}{1920x^4}.\end{aligned}$$

A straightforward numerical computation gives $\frac{3}{2}\log 2 + \frac{1}{2}\gamma + \frac{1}{24} < 1.37$, which then implies (4.22). □

We will now check that all remainder terms $R_i(x)$ are of a lower order of magnitude than the main terms, and in particular that they admit a bound $O(1/x)$. The easier term to estimate is $R_6(x)$. One can indeed use a very rough inequality

(4.23)
$$|R_6(x)| \leqslant \frac{1}{4}\sum_{\ell=1}^{2x-1} \frac{1}{4x(4x-1)} + \frac{1}{4}\sum_{\ell=2}^{2x-1} \frac{1}{(4x)^2} \leqslant \frac{1}{4}\,\frac{2x-1}{4x(4x-1)} + \frac{1}{4}\,\frac{2x-2}{(4x)^2} < \frac{1}{16x}.$$

Consider now $R_4(x)$. We use Lemma C to bound both summations appearing in (4.18), and get in this way

$$[[(4.18)]] \leqslant \frac{\log 2 - \frac{1}{2(4x+1)}}{4x} + \frac{\frac{3}{2}\log 2 + \frac{1}{2}(\log x + \gamma) + \frac{1}{24x^2}}{(4x)^2} < \frac{0.234}{x}$$

(this is clear for x large since $\frac{1}{4}\log 2 < 0.234$ — the precise check uses a direct numerical calculation for smaller values of x). By even more brutal estimates, we find

$$\begin{aligned}\sum_{\ell=1}^{2x-1} \frac{1}{2(\ell+1)} \frac{(4x)^{\ell-1}}{4x\cdots(4x+\ell)} &\leqslant \sum_{\ell=1}^{2x-1} \frac{1}{2(\ell+1)} \frac{1}{(4x)^2}\\ &\leqslant \frac{\log 2x+\gamma+\frac{1}{4x}+\frac{1}{12(2x)^2}-1}{32x^2} < \frac{0.025}{x},\end{aligned}$$

$$\sum_{\ell=3}^{2x-1} \frac{1}{2(\ell-2)} \frac{4x(4x-1)\cdots(4x-\ell+1)}{(4x)^{\ell+2}} \leqslant \sum_{\ell=1}^{2x-3} \frac{1}{\ell} \frac{1}{32x^2} \leqslant \frac{\log 2x+\gamma}{32x^2} < \frac{0.040}{x}.$$

This gives the final estimate

$$|R_4(x)| \leqslant \frac{0.299}{x}. \tag{4.24}$$

5. Further integral estimates

In order to get an optimal bound of the other terms, and especially their differences, we are going to replace some summations by suitable integrals. Before, we must estimate more precisely the partial products $\prod(4x\pm j)$, and for this, we use the power series expansion of their logarithms. For $t > 0$, we have $t - \frac{1}{2}t^2 < \log(1+t) < t$. By taking $t = \frac{j}{4x}$, we find

$$-\frac{\sum_{1\leqslant j\leqslant \ell} j}{4x} < \log \frac{(4x)^\ell}{(4x+1)\cdots(4x+\ell)} = \sum_{1\leqslant j\leqslant \ell} \log \frac{1}{1+\frac{j}{4x}} < -\frac{\sum_{1\leqslant j\leqslant \ell} j}{4x} + \frac{\sum_{1\leqslant j\leqslant \ell} j^2}{2(4x)^2}.$$

Since $\sum_{1\leqslant j\leqslant \ell} j = \frac{\ell(\ell+1)}{2}$ and $\sum_{1\leqslant j\leqslant \ell} j^2 = \frac{\ell(\ell+1)(2\ell+1)}{6}$, we get

$$-\frac{\ell(\ell+1)}{8x} < \log \frac{(4x)^\ell}{(4x+1)\cdots(4x+\ell)} < -\frac{\ell(\ell+1)}{8x} + \frac{\ell(\ell+1)(2\ell+1)}{12\,(4x)^2},$$

therefore

$$\exp\left(\frac{1}{32x}-\frac{(\ell+1/2)^2}{8x}\right) < \frac{(4x)^\ell}{(4x+1)\cdots(4x+\ell)}$$
$$< \exp\left(\frac{1}{32x}-\frac{(\ell+1/2)^2}{8x}+\frac{(\ell+1/2)^3}{96x^2}\right). \tag{5.1}$$

For $\ell \leqslant 2x-1$ we have

$$\frac{(\ell+1/2)^2}{8x}-\frac{(\ell+1/2)^3}{96x^2}=\frac{(\ell+1/2)^2}{8x}\left(1-\frac{(\ell+1/2)}{12x}\right) \geqslant \frac{5}{6}\,\frac{(\ell+1/2)^2}{8x},$$

hence (after performing a suitable numerical calculation)

$$\text{(5.2)}\quad \frac{(4x)^\ell}{(4x+1)\cdots(4x+\ell)} < \exp\left(\frac{1}{32x}-\frac{5}{6}\,\frac{(\ell+1/2)^2}{8x}\right) \qquad \text{for } \ell \leqslant 2x-1,$$
$$\frac{(4x)^\ell}{(4x+1)\cdots(4x+\ell)} < \exp\left(\frac{1}{32x}-\frac{5}{6}\,\frac{(2x-1/2)^2}{12x}\right) < \frac{1.52}{x} \qquad \text{for } \ell \geqslant 2x-1.$$

For $\ell \geqslant 2x$, each new factor is at most $\frac{4x}{4x+\ell} \leqslant \frac{2}{3}$, thus

$$\sum_{\ell=2x}^{+\infty}\frac{(4x)^\ell}{(4x+1)\cdots(4x+\ell)} < \frac{1.52}{x}\sum_{p=1}^{+\infty}\left(\frac{2}{3}\right)^p < \frac{3.04}{x}. \tag{5.3}$$

On the other hand, the analogous inequality $-t-16\,t^2/26 < \log(1-t) < -t$ applied with $t=\frac{j}{4x} \leqslant 1/4$ implies

$$-\frac{\ell(\ell+1)}{8x}-\frac{16\,\ell(\ell+1)(2\ell+1)}{6\cdot 26\,(4x)^2} < \log\frac{(4x-1)\cdots(4x-\ell)}{(4x)^\ell} < -\frac{\ell(\ell+1)}{8x}. \tag{5.4}$$

As $\exp(1/4x) > 1+1/4x$, we infer

$$\frac{(4x+1)\cdots(4x-\ell+2)}{(4x)^\ell} \leqslant \left(1+\frac{1}{4x}\right)\,\exp\left(-\frac{(\ell-1)(\ell-2)}{8x}\right)$$
$$< \exp\left(-\frac{\ell(\ell-3)}{8x}\right), \tag{5.5}$$

and the ratio of two consecutive upper bounds associated with indices $\ell, \ell+1$ is less than $\exp(-(2\ell-2)/8x) \leqslant e^{-1/4}$ if $\ell = 2x$ and less than $e^{-1/2}$ if $\ell \geqslant 2x+1$, thus

$$\sum_{\ell=2x}^{+\infty} \frac{(4x+1)\cdots(4x-\ell+2)}{(4x)^\ell} \leqslant \exp\left(\frac{3}{4}-\frac{x}{2}\right)\left(1+e^{-1/4}\sum_{p=0}^{+\infty} e^{-p/2}\right)$$
$$< \frac{4.65}{x}.$$

As $2\ell \geqslant 4x$, we deduce from (4.16) that

$$|R_2(x)| \leqslant \frac{2}{\pi}\frac{1}{4x}\frac{7.69}{x} < \frac{1.224}{x^2} \tag{5.6}$$

(but actually, one can see that $R_2(x)$ even decays exponentially). By means of a standard integral-series comparison, the inequalities (4.11), (5.2) and (5.4) also provide

$$|R_5(x)| \leqslant \frac{1}{8}\sum_{\ell=1}^{2x-1}\frac{\ell+1}{4x(4x-1)}\exp\left(\frac{1}{32x}-\frac{5}{6}\frac{(\ell+1/2)^2}{8x}\right)+2\frac{\ell-1}{(4x)^2}\exp\left(\frac{3\ell}{8x}-\frac{\ell^2}{8x}\right)$$
$$\leqslant \frac{1}{8(4x)(3x)}\left(e^{\frac{1}{32}}\int_0^{+\infty}\left(t+\frac{3}{2}\right)\exp\left(-\frac{5}{6}\frac{t^2}{8x}\right)dt+\frac{3e^{\frac{3}{4}}}{2}\int_0^{+\infty} t\exp\left(-\frac{t^2}{8x}\right)dt\right)$$
$$= \frac{1}{96x^2}\left(e^{\frac{1}{32}}\left(\frac{24}{5}x+\frac{3}{2}\sqrt{\frac{48x}{5}}\frac{1}{2}\sqrt{\pi}\right)+6e^{\frac{3}{4}}x\right) < \frac{0.229}{x} \qquad \text{for } x \geqslant 1. \tag{5.7}$$

It then follows from (3.9) and (5.1) that

$$T'(x) = \sum_{\ell=1}^{2x-1}\frac{1}{2\ell}\left(\frac{(4x)^\ell}{(4x+1)\cdots(4x+\ell)}-\frac{(4x-1)\cdots(4x-\ell)}{(4x)^\ell}\right)$$
$$= \sum_{\ell=1}^{2x-1}\frac{1}{2\ell}\frac{(4x)^\ell}{(4x+1)\cdots(4x+\ell)}\left(1-\prod_{j=1}^{\ell}\left(1-\frac{j}{4x}\right)\left(1+\frac{j}{4x}\right)\right)$$
$$\leqslant \sum_{\ell=1}^{2x-1}\exp\left(\frac{1}{32x}-\frac{(\ell+1/2)^2}{8x}+\frac{(\ell+1/2)^3}{96x^2}\right)\frac{(\ell+1)^2}{96x^2};$$

to get this, we have used here the inequality $1-\prod(1-a_j)\leqslant\sum a_j$ with $a_j=\frac{j^2}{(4x)^2}<1$, and the identity

$$\sum_{j\leqslant\ell} j^2=\frac{\ell(\ell+1)(2\ell+1)}{6}.$$

In the other direction, we have a lower bound $\prod(1-a_j)^{-1}-1\geqslant\sum a_j$, thus (5.3) implies

$$\begin{aligned}
T'(x)&=\sum_{\ell=1}^{2x-1}\frac{1}{2\ell}\,\frac{(4x-1)\cdots(4x-\ell)}{(4x)^\ell}\bigg(\prod_{j=1}^{\ell}\Big(1-\Big(\frac{j}{4x}\Big)^2\Big)^{-1}-1\bigg)\\
&\geqslant\sum_{\ell=1}^{2x-1}\exp\Big(-\frac{\ell(\ell+1)}{8x}-\frac{(\ell+1/2)^3}{78\,x^2}\Big)\,\frac{(\ell+1)(2\ell+1)}{12\,(4x)^2}\\
&\geqslant\sum_{\ell=1}^{2x-1}\exp\Big(-\frac{(\ell+1/2)^2}{8x}-\frac{(\ell+1/2)^3}{78\,x^2}\Big)\,\frac{(\ell+1)(\ell+1/2)}{96x^2}\\
&\geqslant\sum_{\ell=1}^{2x-1}\exp\Big(-\frac{(\ell+1/2)^2}{8x}\Big)\Big(1-\frac{(\ell+1/2)^3}{78\,x^2}\Big)\,\frac{(\ell+1)(\ell+1/2)}{96x^2}.
\end{aligned}$$

We now evaluate these sums by comparing them to integrals. This gives

$$T'(x)\leqslant e^{\frac{1}{32x}}\int_0^{2x}\exp\Big(-\frac{t^2}{8x}+\frac{t^3}{96x^2}\Big)\,\frac{(t+3/2)^2}{96x^2}\,dt$$

when we estimate the term of index ℓ by the corresponding integral on the interval $[\ell-1/2,\ell+1/2]$. The change of variable

$$u=\frac{t^2}{8x}-\frac{t^3}{96x^2}=\frac{t^2}{8x}\Big(1-\frac{t}{12x}\Big),\qquad du=\frac{t}{4x}\Big(1-\frac{t}{8x}\Big)dt$$

implies $u\geqslant\frac{5}{48x}\,t^2$, hence $t\leqslant\sqrt{\frac{48x}{5}}\,\sqrt{u}$. Moreover, a trivial convexity argument yields $(1-\frac{v}{p})^{-1}\leqslant 1+\frac{1}{p-1}v$ if $v\leqslant 1$; if we take $v=\frac{t}{2x}$ and $p=6$ (resp. $p=3$), we find

$$t = \sqrt{8xu}\left(1 - \frac{t}{12x}\right)^{-1/2} \leqslant \sqrt{8xu}\left(1 + \frac{t}{20x}\right) \leqslant \sqrt{8xu}\left(1 + \sqrt{\frac{3}{125x}}\sqrt{u}\right),$$

$$dt = \frac{4x}{t}\left(1 - \frac{t}{8x}\right)^{-1} du \leqslant \frac{4x}{t}\left(1 + \frac{t}{6x}\right) du \leqslant \frac{4x}{\sqrt{8xu}}\left(1 + \frac{2}{\sqrt{15x}}\sqrt{u}\right) du,$$

therefore

$$T'(x) \leqslant \frac{e^{\frac{1}{32x}}}{96x^2} \int_0^{+\infty} e^{-u}\left(\frac{3}{2} + \sqrt{8xu}\left(1 + \sqrt{\frac{3}{125x}}\sqrt{u}\right)\right)^2 \left(1 + \frac{2}{\sqrt{15x}}\sqrt{u}\right) \frac{\sqrt{2x}\,du}{\sqrt{u}}.$$

This integral can be evaluated evaluated explicitly, its dominant term being equal to

$$\frac{e^{\frac{1}{32x}}}{96x^2} \int_0^{+\infty} e^{-u}(\sqrt{8xu})^2 \frac{\sqrt{2x}\,du}{\sqrt{u}} \sim \frac{\sqrt{2}}{12\sqrt{x}} \int_0^{+\infty} e^{-u}\sqrt{u}\,du = \frac{\sqrt{2\pi}}{24\,x^{1/2}}.$$

Moreover, the factor $e^{\frac{1}{32x}}$ factor admits the (very rough!) upper bound $1 + \frac{1}{31.5\,x}$, whence an error bounded by

$$\frac{\sqrt{2\pi}}{24\,x^{1/2}} \cdot \frac{1}{31.5\,x} < \frac{0.004}{x}.$$

All other terms appearing in the integral involve terms $O(\frac{1}{x})$ with coefficients which are products of factors $\Gamma(a)$, $\frac{1}{2} \leqslant a \leqslant 2$, by coefficients whose sum is bounded by

$$\frac{e^{\frac{1}{32}}}{96}\left[\left(\frac{3}{2} + \sqrt{8}\left(1 + \sqrt{\frac{3}{125}}\right)\right)^2 \left(1 + \frac{2}{\sqrt{15}}\right)\sqrt{2} - 8\sqrt{2}\right] < 0.4021.$$

As $\Gamma(a) \leqslant \sqrt{\pi}$, we obtain

$$T'(x) < \frac{\sqrt{2\pi}}{24\,x^{1/2}} + \frac{0.717}{x}.$$

Similarly, one can obtain the following lower bound for $T'(x)$:

$$
\begin{aligned}
T'(x) &\geqslant \sum_{\ell=1}^{2x-1} \exp\Big(-\frac{(\ell+1/2)^2}{8x}\Big)\Big(1-\frac{(\ell+1/2)^3}{78\,x^2}\Big)\frac{(\ell+1)(\ell+1/2)}{96x^2}\\
&\geqslant \int_{3/2}^{2x+1/2} \exp\Big(-\frac{t^2}{8x}\Big)\Big(1-\frac{t^3}{78\,x^2}\Big)\frac{(t-1)(t-1/2)}{96x^2}\,dt\\
&\geqslant \int_{2}^{2x} \exp\Big(-\frac{t^2}{8x}\Big)\Big(1-\frac{t^3}{78\,x^2}\Big)\frac{t^2-3t/2}{96x^2}\,dt\\
&= \int_{1/2x}^{x/2} e^{-u}\Big(1-\frac{8\sqrt{8}\,u^{3/2}}{78\,x^{1/2}}\Big)\frac{8xu-3\sqrt{8}\,x^{1/2}u^{1/2}/2}{96x^2}\,\frac{\sqrt{8}\,x^{1/2}\,du}{2\,u^{1/2}}\\
&\geqslant \int_{1/2x}^{x/2} e^{-u}\Big(1-\frac{8\sqrt{8}\,u^{3/2}}{78\,x^{1/2}}\Big)\frac{\sqrt{8}\,u-3\,x^{-1/2}u^{1/2}/2}{24\,x^{1/2}}\,\frac{du}{u^{1/2}}\\
&\geqslant \int_{1/2x}^{x/2} e^{-u}\Big(\frac{\sqrt{2}\,u^{1/2}}{12\,x^{1/2}}-\frac{8\,u^2}{3\cdot 78\,x}-\frac{1}{16x}\Big)\,du\\
&\geqslant \int_{0}^{+\infty} e^{-u}\Big(\frac{\sqrt{2}\,u^{1/2}}{12\,x^{1/2}}-\frac{4\,u^2}{117\,x}-\frac{1}{16x}\Big)\,du-\int_{\complement} e^{-u}\frac{\sqrt{2}\,u^{1/2}}{12\,x^{1/2}}\,du.
\end{aligned}
$$

The integral $\int_{\complement}\ldots$ on the “missing intervals” is bounded on $[0,1/2x]$ by

$$\int_0^{1/2x}\frac{\sqrt{2}\,u^{1/2}}{12\,x^{1/2}}\,du=\frac{1}{36\,x^2},$$

whilst the integral on $[A,+\infty[\,=[x/2,+\infty[$ satisfies

$$\int_A^{+\infty} u^\alpha\,e^{-u}\,du = A^\alpha\,e^{-A}+\int_A^{+\infty}\alpha\,u^{\alpha-1}\,e^{-u}\,du\leqslant e^{-A}(A^\alpha+\alpha A^{\alpha-1}),\quad \alpha\in\,]0,1].$$

This provides an estimate

$$\int_{\frac{x}{2}}^{+\infty} e^{-u}\frac{\sqrt{2}\,u^{1/2}}{12\,x^{1/2}}\,du \leqslant \exp\Big(-\frac{x}{2}\Big)\Big(\frac{1}{12}+\frac{1}{12x}\Big) \leqslant \frac{\frac{1}{6}\,e^{-1/2}}{x}.$$

Therefore, we obtain the explicit lower bound

$$\begin{aligned} T'(x) &> \frac{\sqrt{2\pi}}{24\,x^{1/2}} - \Big(\frac{8}{117}+\frac{1}{16}+\frac{1}{36}+\frac{1}{6}e^{-1/2}\Big)\frac{1}{x} \\ &> \frac{\sqrt{2\pi}}{24\,x^{1/2}} - \frac{0.260}{x}. \end{aligned}$$

In the same manner, but now without any compensation of terms and with much simpler calculations, the estimates (4.11), (5.1), (5.3) provide an upper bound

$$\begin{aligned} T''(x) \leqslant \frac{1}{4x}\sum_{\ell=1}^{2x-1} &\frac{1}{4}\exp\Big(\frac{32}{x}-\frac{(\ell+1/2)^2}{8x}+\frac{(\ell+1/2)^3}{96x^2}\Big) \\ &+\frac{3}{4}\exp\Big(\frac{32}{x}-\frac{(\ell-1/2)^2}{8x}\Big). \end{aligned}$$

By using integral estimates very similar to those already used, this gives

$$\begin{aligned} T''(x) &\leqslant \frac{e^{\frac{32}{x}}}{4x}\Big(\frac{1}{4}\int_0^{2x}\exp\Big(-\frac{t^2}{8x}+\frac{t^3}{96x^2}\Big)\,dt+\frac{3}{4}\int_0^{2x}\exp\Big(-\frac{t^2}{8x}\Big)\,dt\Big)+\frac{3}{16x} \\ &\leqslant \frac{e^{\frac{32}{x}}}{4x}\Big(\frac{1}{4}\int_0^{+\infty}e^{-u}\Big(1+\frac{2}{\sqrt{15x}}\sqrt{u}\Big)\frac{\sqrt{2x}\,du}{\sqrt{u}}+\frac{3}{4}\int_0^{+\infty}e^{-u}\,\frac{\sqrt{2x}\,du}{\sqrt{u}}\Big)+\frac{3}{16x} \\ &\leqslant \frac{e^{\frac{32}{x}}}{4x}\int_0^{+\infty}e^{-u}\,\frac{\sqrt{2x}\,du}{\sqrt{u}}+\frac{e^{\frac{32}{x}}}{4x}\,\frac{1}{\sqrt{30}}+\frac{3}{16x} < \frac{\sqrt{2\pi}}{4x^{1/2}}+\frac{0.255}{x}, \end{aligned}$$

and we get likewise a lower bound

$$\begin{aligned}
T''(x) &\geqslant \frac{1}{4x}\sum_{\ell=1}^{2x-1}\frac{1}{4}\exp\Big(-\frac{(\ell+1/2)^2}{8x}\Big)+\frac{3}{4}\exp\Big(-\frac{(\ell-1/2)^2}{8x}-\frac{(\ell-1/2)^3}{78x^2}\Big)\\
&\geqslant \frac{1}{4x}\Bigg(\frac{1}{4}\int_{3/2}^{2x+1/2}\exp\Big(-\frac{t^2}{8x}\Big)\,dt+\frac{3}{4}\int_{1/2}^{2x-1/2}\exp\Big(-\frac{t^2}{8x}\Big)\Big(1-\frac{t^3}{78x^2}\Big)\,dt\Bigg)\\
&\geqslant \frac{1}{4x}\Bigg(\frac{1}{4}\int_{0}^{2x}\exp\Big(-\frac{t^2}{8x}\Big)\,dt+\frac{3}{4}\int_{0}^{2x}\exp\Big(-\frac{t^2}{8x}\Big)\Big(1-\frac{t^3}{78x^2}\Big)\,dt-\frac{9}{8}\Bigg)\\
&\geqslant \frac{1}{4x}\Bigg(\int_{0}^{2x}\exp\Big(-\frac{t^2}{8x}\Big)\,dt-\frac{3}{4}\int_{0}^{+\infty}\exp\Big(-\frac{t^2}{8x}\Big)\frac{t^3}{78x^2}\,dt-\frac{9}{8}\Bigg)\\
&= \frac{1}{4x}\Bigg(\int_{0}^{x/2}e^{-u}\,\frac{\sqrt{2x}\,du}{\sqrt{u}}-\frac{1}{104}\int_{0}^{+\infty}e^{-u}\frac{u\,du}{2}-\frac{9}{8}\Bigg)\\
&\geqslant \frac{1}{4x}\Bigg(\int_{0}^{+\infty}e^{-u}\,\frac{\sqrt{2x}\,du}{\sqrt{u}}-\frac{235}{208}-2\,e^{-x/2}\Bigg)\\
&> \frac{\sqrt{2\pi}}{4x^{1/2}}-\frac{0.586}{x}.
\end{aligned}$$

All this finally yields the estimate

$$T'(x)-T''(x)=-\frac{5}{24}\,\frac{\sqrt{2\pi}}{x^{1/2}}+R_8(x),\qquad -\frac{0.515}{x}<R_8(x)<\frac{1.303}{x}. \tag{5.8}$$

There only remains to evaluate $U'(x)$. According to (4.13), a change of variable $\ell=\ell'+1$ followed by a decomposition $4x=(4x-\ell)+\ell$ allows us to transform the second summation appearing in $U'(x)$ as

$$\begin{aligned}
U'(x) &= \sum_{\ell=1}^{2x-1}\frac{1}{2(\ell+1)}\,\frac{(4x)^\ell}{4x\cdots(4x+\ell)}-\sum_{\ell=1}^{2x-2}\frac{1}{2\ell}\,\frac{4x(4x-1)\cdots(4x-\ell+1)}{(4x)^{\ell+1}}\\
&= \sum_{\ell=1}^{2x-1}\frac{1}{2(\ell+1)}\,\frac{(4x)^{\ell-1}}{(4x+1)\cdots(4x+\ell)}
\end{aligned}$$

$$- \sum_{\ell=1}^{2x-2} \frac{1}{2\ell} \frac{(4x-1)\cdots(4x-\ell+1)(4x-\ell)}{(4x)^{\ell+1}}$$
$$- \sum_{\ell=1}^{2x-2} \frac{1}{2} \frac{(4x-1)\cdots(4x-\ell+1)}{(4x)^{\ell+1}}.$$

Writing $\frac{1}{\ell+1} = \frac{1}{\ell} - \frac{1}{\ell(\ell+1)}$, one obtains

$$U'(x) = \frac{1}{4x} T'(x) - R_9(x)$$

with

$$R_9(x) = \sum_{\ell=1}^{2x-1} \frac{1}{2\ell(\ell+1)} \frac{(4x)^{\ell-1}}{(4x+1)\cdots(4x+\ell)} + \sum_{\ell=1}^{2x-2} \frac{1}{2} \frac{(4x-1)\cdots(4x-\ell+1)}{(4x)^{\ell+1}} - \left(\frac{1}{2\ell} \frac{(4x-1)\cdots(4x-\ell)}{(4x)^{\ell+1}} \right)_{\ell=2x-1},$$

and for $x \geqslant 2$, we find an upper bound

$$0 < R_9(x) < \frac{1}{4x} \sum_{\ell=1}^{+\infty} \frac{1}{2\ell(\ell+1)} + \frac{1}{2}(2x-2)\frac{1}{(4x)^2} < \frac{3}{16x}.$$

Thanks to an explicit calculation of $U'(x)$ for $x = 1,2,3$, we get the estimate

(5.9) $$|U'(x)| < \frac{0.206}{x}.$$

Combining (2.13), (3.7), (4.19), (4.22), (4.23), (4.24) and (5.6–5.9), we now obtain

(5.10) $$\Delta(x) = \frac{e^{-4x}}{4\pi x}\left(- \frac{5\sqrt{2\pi}}{12\, x^{1/2}} + R(x)\right)$$

with

$$R(x) = -U'(x) + \pi\Big(R_1(x) + R_2(x) - R_3(x)\Big) - \frac{5}{4}R_4(x) + 2\,R_5(x) - R_6(x) + R_7(x) + 2\,R_8(x),$$

whence

$$|R(x)| < \frac{10.835}{x}. \tag{5.11}$$

These estimates imply (0.10–0.13). The proof of the Theorem is complete. □

Bibliography

1. ***E. Artin,*** *The Gamma Function*; Holt, Rinehart and Winston, 1964, traduit de: *Einführung in die Theorie der Gammafunktion*. Teubner, 1931.
2. ***R. P. Brent and F. Johansson,*** *A bound for the error term in the Brent–McMillan algorithm*, Math. of Comp., **84** (2015), 2351–2359.
3. ***R. P. Brent and E. M. McMillan,*** *Some new algorithms for high-precision computation of Euler's constant*, Math. of Comp., **34** (1980), 305–312.
4. ***J.-P. Demailly,*** *Sur le calcul numérique de la constante d'Euler*, Gazette des Mathématiciens, **27** (1985), 113–126.
5. ***L. Euler,*** *De progressionibus harmonicis observationes*, Euleri Opera Omnia. Ser. (1), **14**, Teubner, Leipzig and Berlin, 1925, 93–100.
6. ***L. Euler,*** *De summis serierum numeros Bernoullianos involventium*, Euleri Opera Omnia, Ser. (1), **v. 15**, Teubner, Leipzig and Berlin, 1927, 91–130. See in particular 115. The calculations are detailed on 569–583.
7. ***D. W. Sweeney,*** *On the computation of Euler's constant*, Math. of Comp., **17** (1963), 170–178.
8. ***G. N. Watson,*** *A Treatise on the Theory of Bessel Functions* (second edition), Cambridge Univ. Press, London, 1944.

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On-line and list on-line colorings of graphs and hypergraphs

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Abstract: The paper deals with on-line and list on-line colorings of graphs and hypergraphs. On-line coloring of a hypergraph is a game with two players, Lister and Painter, in which Lister picks a vertex one by one (or a set of vertices) and Painter should choose a color for the given vertex (or choose a subset to be colored). The problem is to find an extremal value of some characteristics which admits a winning strategy for Painter. We establish the asymptotic behavior of the list on-line chromatic number for the complete multipartite graphs and hypergraphs. We also prove some results for the related property B-type problems for on-line colorings.

Keywords: list colorings, on-line colorings, independent sets, property B problem

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1. Introduction

The work is devoted to the on-line colorings of graphs and hypergraphs. Let us start with recalling some definitions.

1.1. Definitions

Let $H = (V, E)$ be a hypergraph (or a graph). A vertex subset $W \subset V$ is called *independent* in H if it does not contain completely any edge of H, i.e. for every $A \in E$, $A \setminus W \neq \varnothing$.

A *vertex coloring* f is a mapping from the vertex set V to some set of colors C. A coloring is called *proper* for H if there is no monochromatic edges in E under it. *The chromatic number* of H, $\chi(H)$, is the minimum r such that there is a proper coloring for H with r colors (H is r-colorable).

Another classical notion concerning colorings of graphs and hypergraphs is the list chromatic number. A hypergraph $H = (V, E)$ is said to be r*-choosable* (or list r-colorable) if for every *list assignment* $L = \{L(v) : v \in V\}$ such that $|L(v)| = r$ for any $v \in V$ (r*-uniform* list assignment), there exists a proper coloring from the lists, i.e. for every $v \in V$, we should use a color from $L(v)$. The *list chromatic number* of H, denoted by $\chi_l(H)$, is the minimum r such that H is r-choosable.

The concept of the list chromatic number was recently brought to the on-line setting, see [1]–[3]. Suppose $H = (V, E)$ is a hypergraph and $r \geqslant 2$ is an integer. Two players, Lister and Painter, play the following $Game_1(H, r)$ game. Let us set $X_0 = \varnothing$. In the round number i Lister presents a non-empty set of vertices $V_i \subset V \setminus (X_0 \cup \ldots \cup X_{i-1})$ and Painter chooses an independent subset $X_i \subset V_i$, i.e. the vertices of X_i are colored with color number i. After i rounds the vertices in $X_1 \cup \ldots \cup X_i$ are colored. If a vertex v belongs to exactly l sets $V_{j_1}, \ldots, V_{j_l}$, $1 \leqslant j_1 < \ldots < j_l \leqslant i$ then v is said to have l *permissible* colors after i rounds. The winning rule is the following.

- Lister wins if after some round there exists a non-colored vertex with r permissible colors.
- Otherwise Painter wins, i.e. after some round all the vertices are colored.

Hypergraph H is said to be r*-paintable* (or list on-line r-colorable) if Painter has a winning strategy in (H, r)-game. The minimum r such that H is r*-paintable* is called *the list on-line chromatic number* and denoted by $\chi_{ol}(H)$. It is easy to see that

$$\chi(H) \leqslant \chi_l(H) \leqslant \chi_{ol}(H).$$

1.2. Colorings of the complete multipartite graphs and hypergraphs

List colorings of graphs and hypergraphs were introduced independently by Vizing (see [4]) and by Erdős, Rubin and Taylor (see [5]). One of the first results concerning the list chromatic number states that it can be much larger than the usual chromatic number. In particular, the authors in [5] showed that the list chromatic number of the complete bipartite graph $K_{m,m}$ with m vertices in any part grows as binary logarithm of m:

$$\chi_l(K_{m,m}) = (1 + o(1)) \log_2 m \text{ as } m \to \infty. \tag{1}$$

Surprisingly the above asymptotic representation remains true for the list on-line chromatic number. In [6] Duraj, Gebowski and Kozik showed that

$$\chi_{ol}(K_{m,m}) = \log_2 m + O(1) \text{ as } m \to \infty. \tag{2}$$

This provides the first example of a graph for which the difference between the list on-line chromatic number and the list chromatic number can be arbitrarily large. Since $\chi_l(K_{m,m}) = \log_2 m - \Omega(\log_2 \log_2 m)$ (see [6]) we have

$$\chi_{ol}(K_{m,m}) - \chi_l(K_{m,m}) = \Omega(\log_2 \log_2 m).$$

The result (1) for $K_{m,m}$ was generalized in different ways. The first generalization considers the complete r-partite graph K_{m*r} with equal size of parts m. Krivelevich and Gazit established (see [7]) the asymptotic behavior of $\chi_l(K_{m*r})$ for fixed $r \geqslant 3$ and growing m:

$$\chi_l(K_{m*r}) = (1 + o(1)) \log_{\frac{r}{r-1}} m \text{ as } m \to \infty. \tag{3}$$

In [9] Shabanov showed that the same asymptotic representation holds when $\ln r = o(\ln m)$.

The second generalization deals with the complete multi-partite uniform hypergraphs. Let $H_{m\times r}$ denote the complete r-partite r-uniform hypergraph with m vertices in every part. In [10] Haxell and Verstraëte proved that for fixed $r \geqslant 3$,

$$\chi_l(H_{m\times r}) = (1 + o(1)) \log_r m \text{ as } m \to \infty. \tag{4}$$

Recently the results (1), (3), (4) were extended by Shabanov and Shaikheeva (see [8]). Let $H(m, r, k)$ denote the complete r-partite k-uniform hypergraph with m vertices in every part, in which any edge takes exactly one vertex from some $k \leqslant r$ parts. Clearly, $H(m, r, 2) = K_{m*r}$ and $H(m, r, r) = H_{m\times r}$. The authors in [8] showed that for fixed $2 \leqslant k \leqslant r$,

$$\chi_l(H(m, r, k)) = (1 + o(1)) \log_{\frac{r}{r-k+1}} m \text{ as } m \to \infty. \tag{5}$$

1.3. Main result

The main result of the current work provides the asymptotic behavior for the list on-line chromatic number of the complete r-partite k-uniform hypergraph $H(m,r,k)$. As in (2) the asymptotics of $\chi_{ol}(H(m,r,k))$ coincides with the asymptotics of the list chromatic number (5).

THEOREM 1. *For fixed* $2 \leqslant k \leqslant r$,

$$\chi_{ol}(H(m,r,k)) = (1+o(1)) \log_{\frac{r}{r-k+1}} m \text{ as } m \to \infty. \tag{6}$$

The same asymptotic representation holds for any functions $r = r(m)$, $k = k(m)$, *such that* $\ln r = o(\ln m)$.

As immediate corollaries we obtain the analogues of (2) for (3) and (4): for fixed $r \geqslant 3$,

$$\chi_{ol}(K_{m*r}) = (1+o(1)) \log_{\frac{r}{r-1}} m \text{ as } m \to \infty;$$

$$\chi_{ol}(H_{m\times r}) = (1+o(1)) \log_{r} m \text{ as } m \to \infty.$$

The structure of the paper will be the following. In the next section we will discuss the connection of the list on-line colorings of multipartite hypergraphs with extremal property B-type problems. In Section 3 we will give the proofs of the obtained results.

2. Extremal property B-type problems

2.1. Connection with the property B problem

The close connection of the list colorings of complete multi-partite graphs with the classical property B problem was realized by Erdős, Rubin and Taylor in [5]. Recall that the property B problem is to find the value $m(n)$ equal to the minimum number of edges in an n-uniform non-2-colorable hypergraph. The obtained quantitative relation between $\chi_l(K_{m,m})$ and $m(n)$ is the following.

CLAIM 1. Suppose that $n, m \geqslant 2$ are integers.

1. If $2m < m(n)$ then $\chi_l(K_{m,m}) \leqslant n$.
2. If $m \geqslant m(n)$ then $\chi_l(K_{m,m}) > n$.

These inequalities together with the known bounds for $m(n)$ provide the asymptotics for $\chi_l(K_{m,m})$.

The same approach was used in [9] and [10] for investigating $\chi_l(K_{m*r})$ and $\chi_l(H_{m\times r})$. For K_{m*r}, the corresponding extremal value deals with panchromatic colorings. A vertex coloring of the hypergraph $H=(V,E)$ with r colors is said to be *panchromatic* if under this coloring every edge of E meets every of r colors. Let $p(n,r)$ denote the minimum possible number of edges in a n-uniform hypergraph that does not admit a panchromatic coloring with r colors. Kostochka showed [11] that $p(n,r)$ plays the same role for $\chi_l(K_{m*r})$ as $m(n)$ for $\chi_l(K_{m,m})$.

Haxell and Verstraëte considered another generalization of the property B problem to obtain the asymptotics for $\chi_l(H_{m\times r})$. They used the value $m(n,r)$, the minimum possible number of edges in an n-uniform non-r-colorable hypergraph.

Finally, Shabanov and Shaikheeva [8] introduced the property that lies "between" r-colorability and panchromatic r-colorability. Let us denote $[r]=\{1,\dots,r\}$. A mapping $f: V\to\binom{[r]}{s}$ is called an *s-covering by r sets*, i.e. we assign s different colors to any vertex of H. Furthermore f is called an *s-covering by r independent sets* if for every $i=1,\dots,r$, a vertex subset

$$V_i=\{v\in V: i\in f(v)\}$$

is an independent set in H. It is easy to understand that

- a 1-covering by r independent sets is just a proper coloring with r colors;
- an $(r-1)$-covering f by r independent sets is equivalent to a panchromatic r-coloring (we can color a vertex with the remaining unassigned color).

The authors of [8] introduced the value $c(n,r,s)$, equal to the minimum possible number of edges in an n-uniform hypergraph that does not admit an s-covering by r independent sets. They also proved the following quantitative relation between $c(n,r,s)$ and $\chi_l(H(m,r,k))$.

Claim 2. Suppose that $n,m,r\geqslant 2,\ 2\leqslant k\leqslant r$ are integers.

1. If $rm<c(n,r,r-k+1)$ then $\chi_l(H(m,r,k))\leqslant n$.
2. If $m\geqslant c(n,r,r-k+1)$ then $\chi_l(H(m,r,k))>n$.

By using Claim 2 and the bounds for $c(n,r,s)$, one can easily obtain the asymptotics for the list chromatic number of $H(m,r,k)$.

2.2. On-line analogues of the property B problem

The first on-line version of the property B problem was considered by Aslam and Dhagat in [12]. Suppose n, N are positive integers and there are two players, Lister and Painter. They play the following game $Game_2(N, n)$, which is parametrized by two numbers: the cardinality of edges n and the number of edges N. Values of these parameters are known to both players before the game. In each round, Lister reveals one vertex and declares in which edges it is contained. He cannot add vertices to edges which already contain n vertices. Painter must immediately assign any of two colors (0 or 1) to the presented vertex. When all the vertices have been revealed (i.e. all N edges contain n vertices each) Painter wins if there is no monochromatic edge in the constructed hypergraph. Otherwise Lister wins.

Let $m_{ol}(n)$ denote the minimum N such that Lister has a winning strategy in $Game_2(N, n)$. Clearly, $m_{ol}(n) \leqslant m(n)$ since for $N \geqslant m(n)$ Lister can just construct a non-2-colorable hypergraph. Aslam and Dhagat proved [12] that

$$m_{ol}(n) \geqslant 2^{n-1}. \tag{7}$$

Duraj, Gutowski and Kozik showed [6] that the above estimate is sharp up to a bounded factor:

$$m_{ol}(n) \leqslant 8 \cdot 2^n. \tag{8}$$

They also showed that $m_{ol}(n)$ plays the same role for $\chi_{ol}(K_{m,m})$ as $m(n)$ for $\chi_l(K_{m,m})$. This connection together with the bounds (7)-(8) implies the result (2).

In the current paper we consider the following extension of $Game_2(N, n)$. Suppose n, $s \leqslant r$ and N are positive integers. There are two players, Lister and Painter, who play the following game $Game_3(N, n, r, s)$, which is parametrized by four numbers:

- n is the cardinality of edges;
- N is the number of edges;
- r is the total number of colors;
- s is the number of colors that should be assigned to every vertex.

Again the values of these parameters are known to both players before the game. In each round, Lister reveals one vertex of a hypergraph and declares in which edges it is contained. He cannot add vertices to edges which already contain n vertices. Painter must immediately assign s colors from $[r] = \{1, \dots, r\}$ to the presented vertex. When all the vertices have been revealed (i.e. all N edges contain exactly n vertices each) Painter wins if the obtained s-covering is a covering by r independent sets for the constructed n-uniform hypergraph. Otherwise Lister wins.

Let $c_{ol}(n, r, s)$ denote the minimum N such that Lister has a winning strategy in $Game_3$ (N, n, r, s). We obtain the following generalization of Claim 2.

LEMMA 1. *Suppose that* $n, m, r \geqslant 2$, $2 \leqslant k \leqslant r$ *are integers.*

1. *If* $rm < c_{ol}(n, r, r-k+1)$ *then* $\chi_{ol}(H(m, r, k)) \leqslant n$.
2. *If* $m \geqslant c_{ol}(n, r, r-k+1)$ *then* $\chi_{ol}(H(m, r, k)) > n$.

Lemma 1 is crucial in estimating the list on-line chromatic number of $H(m, r, k)$. However we will also need the bounds for the extremal value $c_{ol}(n, r, s)$, this question will be discussed in the next paragraph.

2.3. New results in extremal problems for on-line colorings

The following lemma gives a reasonable lower bound for $c_{ol}(n, r, s)$.

LEMMA 2. *For any* $n \geqslant 2$, $r > s \geqslant 1$,

$$c_{ol}(n, r, s) \geqslant \frac{r^{n-1}}{s^n}. \qquad (9)$$

Note that for $r = 2$, $s = 1$ the bound (9) coincides with the bound (7) for $m_{ol}(n)$.

Recall that $c_{ol}(n, r, s)$ does not exceed its "off-line" version $c(n, r, s)$. It was shown in [8] by a probabilistic approach that for any $n > r > s \geqslant 1$,

$$c(n, r, s) \leqslant \frac{e}{2} n^2 \left(\frac{r}{s}\right)^n \ln \binom{r}{s} \left(1 + O\left(\frac{1}{n}\right) + O\left(\frac{s}{r}\right)\right). \qquad (10)$$

So we can use the estimate (10) as an upper bound for $c_{ol}(n, r, s)$. However, as it was shown by Duraj, Gutowski and Kozik, much better results can be obtained for on-line colorings. We will give some of them in the most interesting cases: $s = 1$ and $s = r - 1$.

The value $c(n, r, 1)$ is well-known in the literature as $m(n, r)$, the minimum possible number of edges in an n-uniform non-r-colorable hypergraph. The problem of finding $m(n, r)$ was proposed by Erdős and Hajnal in the 60-s and since that time it had been intensively studied. The reader is referred to the survey [13] for the detailed history. Clearly, $m(n, 2) = m(n)$ and it is known that

$$c\left(\frac{n}{\ln n}\right)^{1/2} 2^n \leqslant m(n) \leqslant \frac{e \ln 2}{4} n^2 2^n (1 + o(1)), \tag{11}$$

where $c > 0$ is an absolute constant. The upper bound is due to Erdős [14] and the lower is due to Radhakrishnan and Srinivasan [15]. Note that the above relations imply that $m(n)$ has a greater asymptotic order than its on-line analogue $m_{ol}(n)$ (see (7) and (8)). Similar estimates in the case of arbitrary number of colors r are the following (the lower bound is due to Cherkashin and Kozik [16]):

$$c\left(\frac{n}{\ln n}\right)^{(r-1)/r} r^{n-1} \leqslant m(n, r) \leqslant \frac{e}{2} n^2 r^n \ln r \left(1 + O\left(\frac{1}{n}\right)\right). \tag{12}$$

Since $m_{ol}(n, r) = c_{ol}(n, r, 1)$ does not exceed $m(n, r)$ the upper bound from (12) holds for $m_{ol}(n, r)$. The following statement refines it significantly.

Proposition 1. *For any r and n,*

$$m_{ol}(n, r) \leqslant n(r-1)^2 \cdot r^n. \tag{13}$$

For fixed r and growing n, the bound (13) is much better than the bound (12) for $m(n, r)$. If $r \sim \ln n$ then (13) is only $(\ln n)^4$ times greater than the lower bound in (12), so we can expect that $m(n, r)$ and $m_{ol}(n, r)$ do not have the same asymptotic order. However, for $r = 2$, the bound (13) is not good, a much stronger result (8) is known.

In the opposite situation when n is fixed and r is large, the bound (13) also is not the best possible since it is known that even $m(n, r)$ has the order $O_n(r^n)$. In fact, Alon [17] showed that $m(n, r)$ has the order r^n for large r and small n. His bounds were refined by Akolzin and Shabanov [18] as follows: if $r > n$ then

$$c_1 \cdot \frac{n}{\ln n} \cdot r^n \leqslant m(n, r) \leqslant c_2 \cdot n^3 \ln n \cdot r^n, \tag{14}$$

where c_1 and c_2 are some positive absolute constants. We show that the value $m_{ol}(n, r)$ also have the order r^n when r is large and n is fixed. The upper bound clearly follows from (14), but the lower bound r^{n-1} obtained in Lemma 2 is not enough. The next statement provides an improved bound.

Proposition 2. *Suppose $r > n$ and let us denote $a = \lfloor \frac{n-1}{n} r \rfloor$ and $b = r - a = \lceil \frac{r}{n} \rceil$. Then*

$$m_{ol}(n, r) \geqslant (n-1)ba^{n-1} + a^{n-1} = \Omega(r^n). \tag{15}$$

Finally, we discuss on-line panchromatic colorings, i.e. the problem of estimating $c_{ol}(n, r, s)$ when $s = r - 1$. Let $p_{ol}(n, r) = c_{ol}(n, r, r-1)$. “Off-line” version of the problem, the value $p(n, r)$, first appeared it the paper of Kostochka [11] and since that time has been studied in several papers. For instance, it was shown in [9] and [19] that

$$c_1 \frac{1}{r} \left(\frac{n}{r^2 \ln n} \right)^{1/2} \left(\frac{r}{r-1} \right)^n \leqslant p(n, r) \leqslant c_2 n^2 \left(\frac{r}{r-1} \right)^n \ln r, \tag{16}$$

where $c_1, c_2 > 0$ are some absolute constants. Cherkashin improved [20] the upper bound in (16) by a factor $1/r$ and gave a better lower bound for r large enough in comparison with n.

The lower bound $r^{-1} \left(\dfrac{r}{r-1} \right)^n$ for $p_{ol}(n, r)$ has been obtained in Lemma 2. The upper bound from (16) can be refined in the on-line case as follows.

Proposition 3. *Suppose $n > r$. Then*

$$p_{ol}(n, r) \leqslant 3r(r-1)^2 n \left(\frac{r}{r-1} \right)^{n+1}. \tag{17}$$

For fixed r and large n, the bound (17) is even closer to the lower bound in (16) than to the upper one.

In the next sections we proceed to the proofs of the above new results.

3. Proof of Theorem 1

We start with establishing auxiliary lemmas: Lemma 1 and Lemma 2.

3.1. Proof of Lemma 1

We follow the ideas from [6] and [8].

1) We have to show that $\chi_{ol}(H(m, r, k)) \leqslant n$, i.e. we have to prove that Painter has a winning strategy in $Game_1(H(m, r, k), n)$.

Let $W = W_1 \sqcup \ldots \sqcup W_r$ denote the vertex set of $H(m, r, k)$, where $W_1, \ldots, W_r$ are the parts of the graph. Our strategy for $Game_1(H(m, r, k), n)$ will use the winning strategy for $Game_3(rm, n, r, r-k+1)$.

- Suppose $X_1, \ldots, X_{i-1}$ have been already chosen. In round i Lister chooses the set of vertices $V_i \subset W \setminus (X_1 \sqcup \ldots \sqcup X_{i-1})$.
- We assume that we are playing $Game_3(rm, n, r, r-k+1)$ and here Lister has chosen the edges with numbers V_i to contain vertex i.
- Since $rm < c_{ol}(n, r, r-k+1)$ then Painter has a winning strategy in $Game_3(rm, n, r, r-k+1)$. Let $\{\widetilde{j}_1, \ldots, \widetilde{j}_{r-k+1}\}$ be the choice of colors for the vertex i according to this strategy.
- Let $\{j_1, \ldots, j_{k-1}\} = [r]/\{\widetilde{j}_1, \ldots, \widetilde{j}_{r-k+1}\}$ be a complementary set of colors.
- Painter's choice of an independent set X_i will be the following:

$$X_i = V_i \cap \left(W_{j_1} \sqcup \ldots \sqcup W_{j_{k-1}}\right).$$

Since X_i is contained in a union of some $k-1$ parts of $H(m, r, k)$ then it will be independent in $H(m, r, k)$ by the construction of the hypergraph.

Suppose $w \in W_j$ is a vertex of $H(m, r, k)$. Everytime w is chosen by Lister as an element of V_i in $Game_1(H(m, r, k), n)$, i becomes a vertex of an edge w in $Game_3(rm, n, r, r-k+1)$. The winning strategy in $Game_3(rm, n, r, r-k+1)$ provides that after choosing n times the edge w there will be a vertex $i \in w$ such that color j will not be assigned to i (otherwise the obtained covering will not be a covering by independent sets). For such i, the independent set X_i will contain all the vertices in $V_i \cap W_j$, i.e. $w \in X_i$. Thus, every vertex of $H(m, r, k)$ is colored before it receives n permissable colors. The existence of the winning strategy for Painter is proved.

2) We have to show that $\chi_{ol}(H(m, r, k)) > n$, i.e. Lister has a winning strategy in $Game_1$ $(H(m, r, k), n)$. Again our strategy will follow the winning strategy for $Game_3(m, n, r, r-k+1)$.

Recall that $W = W_1 \sqcup \ldots \sqcup W_r$ denotes the vertex set of $H(m, r, k)$. Every W_j has exactly m vertices, so let us denote $W_j = \{w_{1,j}, \ldots, w_{m,j}\}$.

- Suppose $V_1, X_1, \ldots, V_{i-1}, X_{i-1}$ have been already chosen. In round i we have to choose the set of vertices $V_i \subset W \setminus (X_1 \sqcup \ldots \sqcup X_{i-1})$.
- Once again we assume that we are playing $Game_3(m, n, r, r-k+1)$ and there is a winning strategy for Lister since $m \geqslant c_{ol}(n, r, r-k+1)$.

- Now let $a_1, \ldots, a_q \in \{1, \ldots, m\}$ denote the set of edges which this strategy assigns to vertex i.
- Lister's choice of a set V_i is the following:

$$V_i = \bigcup_{j=1}^{r} \bigcup_{y=1}^{q} \{w_{a_y,j}\} \setminus (X_1 \sqcup \ldots \sqcup X_{i-1}) .$$

Roughly speaking, Lister chooses a set of rows in matrix $||w_{l,j}||$, $l = 1, \ldots, m$, $j = 1, \ldots, r$, and forms V_i as a set of all available elements in chosen rows.

Suppose Painter chooses X_i as an independent set in V_i. In fact, Painter chooses the vertices from some $k-1$ parts $W_{j_1}, \ldots, W_{j_{k-1}}$, i.e. he chooses $k-1$ columns in matrix $||w_{l,j}||$ and forms X_i as an intersection of V_i with these columns. Such an answer can be interpreted as Painter's choice of colors $[r] \setminus \{j_1, \ldots, j_{k-1}\}$ for covering vertex i in $Game_3(m, n, r, r-k+1)$.

Let us understand that Lister always wins by this strategy in $Game_1(H(m, r, k), n)$. The winning strategy in $Game_3(m, n, r, r-k+1)$ provides that after some round there will be a color j which will be assigned to any of n vertices of some edge $q \in \{1, \ldots, m\}$. This corresponds to the following situation in $Game_1(H(m, r, k), n)$:

1. column j has never been chosen as a part of an independent X_i, when Lister chooses row q,
2. vertex $w_{q,j}$ has not been colored,
3. vertex $w_{q,j}$ has been chosen n times as an element of V_i, i.e. it has n permissable colors.

Hence Lister always wins by using the described strategy. Lemma 1 is proved.

3.2. Proof of Lemma 2

The proof follows the ideas from [12]. We have to prove that for $N < \frac{r^{n-1}}{s^n}$, Painter has a winning strategy in $Game_3(N, n, r, s)$. Let us describe it.

Suppose that the first l vertices $v_1, \ldots, v_l$ have already been colored with s colors each. For every color $j \in \{1, \ldots, r\}$, let $V_j(l)$ denote the set of revealed vertices colored with j. Every edge A can be considered as a function of l, where $A(l)$ denote an edge subset revealed after round l. If $A(l) \subset V_j(l)$ then A is said to be currently monochromatic of color j. We assume that an empty edge is monochromatic of every color. In this case we define the weight of A in color j as follows:

$$w_j(A,l) = \left(\frac{r}{s}\right)^{|A(l)|}.$$

Painter's strategy will be the following. Suppose that Lister states that a vertex v_{l+1} is assigned to edges $A_1, \ldots, A_q$. Then Painter calculates r numbers $b_j(v_{l+1})$, $j = 1, \ldots, r$, where

$$b_j(v_{l+1}) = \sum_{u: A_u(l) \subset V_j(l)} w_j(A_u, l) = \sum_{A: A(l) \subset V_j(l), v_{l+1} \in A} w_j(A, l).$$

Suppose that $b_{j_1}(v_{l+1}), \ldots, b_{j_s}(v_{l+1})$ are the smallest s numbers among them. Then Painter assigns colors $j_1, \ldots, j_s$ to vertex v_{l+1}.

Let us prove that this is a winning strategy. After every round l we can define the total weight of currently monochromatic edges:

$$w(l) = \sum_{j=1}^{r} \sum_{A: A(l) \subset V_j(l)} w_j(A, l).$$

We will show that $w(l) \geqslant w(l+1)$, i.e. the total weight decreases. Indeed, let $j_1, \ldots, j_s$ be the colors assigned to vertex v_{l+1} in round $l+1$. Then our strategy implies that

$$w(l+1) = w(l) - \sum_{j=1}^{r} \sum_{A: A(l) \subset V_j(l), v_{l+1} \in A} w_j(A, l) + \sum_{u=1}^{s} \sum_{A: A(l) \subset V_{j_u}(l), v_{l+1} \in A} w_{j_u}(A, l+1) =$$

$$= w(l) - \sum_{j=1}^{r} b_j(v_{l+1}) + \frac{r}{s} \sum_{u=1}^{s} b_{j_u}(v_{l+1}) \leqslant w(l)$$

since $b_{j_1}(v_{l+1}), \ldots, b_{j_s}(v_{l+1})$ are the smallest s numbers among $b_j(v_{l+1})$, $j = 1, \ldots, r$.

Let us finish the proof. Suppose our strategy fails and at the end of the game there is an edge A, which is monochromatic in color j. Then the total weight of the monochromatic edges at the end of the game is at least $(r/s)^n$. But at the beginning the total weight is equal to rN which is smaller than $(r/s)^n$, a contradiction. Lemma 2 is proved.

3.3. Completion of the proof

Let us deduce the asymptotics for the list on-line chromatic number of the hypergraph $H(m, r, k)$. If we denote $n = \chi_l(H(m, r, k))$ then Lemma 1 implies that

$$c_{ol}(n-1, r, r-k+1) \leqslant m \text{ and } c_{ol}(n, r, r-k+1) > mr.$$

By using bounds (9) and (10) for $c_{ol}(n, r, r-k+1)$ we obtain that

$$(n-2)\ln\frac{r}{r-k+1} - \ln(r-k+1) \leqslant \ln m; \tag{18}$$

$$\ln m + \ln r < n\ln\frac{r}{r-k+1} + 2\ln n + \ln\ln\binom{r}{r-k+1} + O(1). \tag{19}$$

We assume that the function $r = r(m)$ satisfies the condition $\ln r = o(\ln m)$ when $m \to \infty$. Hence the inequality (18) implies that

$$\limsup_{m\to\infty}\frac{n\ln\frac{r}{r-k+1}}{\ln m} \leqslant 1 + \lim_{m\to\infty}\frac{2\ln r}{\ln m} = 1. \tag{20}$$

Moreover, it follows from (18) that $\ln n = O(\ln\ln m) = o(\ln m)$. Thus from (19) we get

$$\liminf_{m\to\infty}\frac{n\ln\frac{r}{r-k+1}}{\ln m} \geqslant 1 - \lim_{m\to\infty}\frac{O(\ln r + \ln n)}{\ln m} = 1. \tag{21}$$

Finally, from (20) and (21) we obtain the asymptotics for the list on-line chromatic number of $H(m, r, k)$:

$$\lim_{m\to\infty}\frac{\chi_{ol}(H(m,r,k))\ln\frac{r}{r-k+1}}{\ln m} = \lim_{m\to\infty}\frac{\chi_{ol}(H(m,r,k))}{\log_{\frac{r}{r-k+1}} m} = 1.$$

Theorem 1 is established.

4. Other proofs

4.1. Proof of Proposition 1

The proof follows the ideas from [6]. We have to show that for $N = n(r-1)^2 r^n$, Lister has a winning strategy in $Game_3(N, n, r, 1)$. The strategy will be the following.

Suppose that the first l vertices $v_1, \ldots, v_l$ have already been colored. For every color $j \in \{1, \ldots, r\}$, let $V_j(l)$ denote the set of revealed vertices colored with j. We divide the number of edges into r parts $E_1, \ldots, E_r$, with $(r-1)^2 n r^{n-1}$ edges in every part. Every edge A again can be considered as a function of l, where $A(l)$ denote an edge subset revealed after round l. If $A(l) \subset V_j(l)$ then A is said to be currently monochromatic of color j. We assume that an empty edge $A(0)$ is monochromatic of color j if $A \in E_j$. If $|A(l)| = i$, $i = 0, \ldots, n-1$, then edge A is said to be at level i after the round l. A monochromatic (j,i)-block is a set of r^{n-i-1} currently monochromatic edges of color j, which are currently at level i.

Lister's strategy can be described as follows.

- For every color $j = 1, \ldots, r$, he chooses the largest $i = i(j)$ such that there exists a (j,i)-block B_j.
- He chooses the union $B_1 \sqcup \ldots \sqcup B_r$ as a set of edges that will contain the next vertex v_{l+1}.

Clearly, Lister wins if after some round there is a monochromatic edge at level n. The total number of blocks at the beginning is equal to $r \cdot (r-1)^2 n r^{n-1} / r^{n-1} = r(r-1)^2 n$. For any Painter's choice of color for v_{l+1}, the total number of blocks remains the same. Indeed, for chosen color the number of blocks of this color will increase by $r-1$ (plus r blocks on the next level minus 1 chosen block on the current level) but the number of blocks of any other color will decrease by 1.

The game continues until there is no monochromatic blocks in some color (after that Painter can always choose this color for all the remaining vertices) or Lister wins. Suppose the first situation appears. It implies that after some round there is no monochromatic blocks, say, of color 1. In every other color there can be

1. at most $r-1$ monochromatic blocks on any level from 1 to $n-2$ (we always use the block on the largest level);
2. at most r monochromatic blocks on level $n-1$;
3. at most $(r-1)^2 n - 1$ blocks on level 0.

Thus the total number of blocks will be at most

$$(r-1)\left((r-1)(n-2) + r + (r-1)^2 n - 1\right) = (r-1)^2\left(n - 2 + 1 + (r-1)n\right) = (r-1)^2(rn-1)$$

which is less than $r(r-1)^2 n$, a contradiction. We have shown that the number of blocks should be constant. Thus Lister always wins.

4.2. Proof of Proposition 2

We follow the proof of Alon from [17]. Suppose $N < (n-1)ba^{n-1} + a^{n-1}$, we have to show that Painter has a winning strategy in $Game_3(N, n, r, 1)$. The first part of the strategy will be the same as in Lemma 2.

Suppose that the first l vertices $v_1, \dots, v_l$ have already been colored. For every color $j \in \{1, \dots, r\}$, let $V_j(l)$ denote the set of revealed vertices colored with j. Every edge A can be considered as a function of l, where $A(l)$ denote an edge subset revealed after round l. If $A(l) \subset V_j(l)$ then A is said to be currently monochromatic of color j. We assume that an empty edge is monochromatic in every color. In this case we define the weight of A in color j as follows:

$$w_j(A, l) = a^{|A(l)|}.$$

Painter's strategy will be the following. Suppose that Lister states that a vertex v_{l+1} is assigned to edges $A_1, \dots, A_q$. Then Painter calculates a numbers $d_j(v_{l+1})$, $j = 1, \dots, a$, where

$$d_j(v_{l+1}) = \sum_{u:A_u(l)\subset V_j(l)} w_j(A_u, l) = \sum_{A:A(l)\subset V_j(l), v_{l+1}\in A} w_j(A, l).$$

Suppose that $d_q(v_{l+1})$ is the smallest number among $d_1(v_{l+1}), \dots, d_a(v_{l+1})$.

1. If $d_q(v_{l+1}) < a^{n-1}$ then Painter colors v_{l+1} with color q.
2. If $d_q(v_{l+1}) \geqslant a^{n-1}$ then Painter colors v_{l+1} with any of the colors from $\{a+1, \dots, r\}$ which have not been used $(n-1)$ times.
3. If $d_q(v_{l+1}) \geqslant a^{n-1}$ and every color from $\{a+1, \dots, r\}$ have been used $(n-1)$ times then Painter colors v_{l+1} with color q.

If Painter follows the second alternative then vertex v_{l+1} is said to be *special*. Let $s(l)$ denote the number of special vertices after round l.

Let us prove that this is really a winning strategy. After every round l we can define the total weight function as follows:

$$w(l) = \sum_{j=1}^{a} \sum_{A:A(l)\subset V_j(l)} w_j(A, l) + s(l)a^n.$$

We will show that $w(l) \geqslant w(l+1)$, i.e. the total weight decreases. Indeed, let q be a color assigned to vertex v_{l+1} in round $l+1$. If v_{l+1} is not a special vertex then our strategy implies that

$$w(l+1) = w(l) - \sum_{j=1}^{a} \sum_{A:A(l)\subset V_j(l), v_{l+1}\in A} w_j(A,l) + \sum_{A:A(l)\subset V_q(l), v_{l+1}\in A} w_q(A,l+1) =$$

$$= w(l) - \sum_{j=1}^{a} d_j(v_{l+1}) + a d_q(v_{l+1}) \leqslant w(l)$$

since $d_q(v_{l+1})$ is the smallest number among $d_j(v_{l+1}), j = 1, \ldots, a$. If v_{l+1} is a special vertex then

$$w(l+1) = w(l) - \sum_{j=1}^{a} \sum_{A:A(l)\subset V_j(l), v_{l+1}\in A} w_j(A,l) + a^n =$$

$$= w(l) - \sum_{j=1}^{a} d_j(v_{l+1}) + a^n \leqslant w(l)$$

since every $d_j(v_{l+1}), j = 1, \ldots, a$, is at least a^{n-1}.

Now let us finish the proof. Suppose our strategy fails and at the end of the game there is an edge A, which is monochromatic of color j. Clearly, $j \in \{1, \ldots, a\}$, because every color from $\{a+1, \ldots, r\}$ can be used only $n-1$ times. Moreover, since the last vertex of A was assigned a color from $\{1, \ldots, a\}$ there is already $(n-1)b$ special vertices. So the total weight at the end of the game is at least $a^n + (n-1)ba^n$. But at the beginning the total weight is equal to aN which is smaller than $a^n + (n-1)ba^n$, a contradiction. Hence Painter always wins. Proposition 2 is proved.

4.3. Proof of Proposition 3

The proof follows the general approach of the proof of Proposition 1. We have to show that for $N \geqslant 3r(r-1)^2 n \left(\frac{r}{r-1}\right)^{n+1}$, Lister has a winning strategy in $Game_3(N, n, r, r-1)$.

Let us divide the number of edges into r parts $E_1, \ldots, E_r$ with exactly $3n(r-1) \cdot a_n$ edges in every part, where the value a_n is defined as follows:

$$a_0 = 1, \ a_m = \left\lceil \frac{r}{r-1} a_{m-1} \right\rceil, \ m = 1, \ldots, n.$$

Clearly, $a_n \leqslant \frac{r}{r-1} a_{n-1} + 1$. Thus $a_n \leqslant \sum_{i=0}^{n} \left(\frac{r}{r-1}\right)^i \leqslant (r-1)\left(\frac{r}{r-1}\right)^{n+1}$. Since $N \geqslant 3rn(r-1) \cdot a_n$ the division can be made. We omit all the remaining edges.

Suppose that the first l vertices $v_1, \dots, v_l$ have already been colored. For every color $j \in \{1, \dots, r\}$, let $V_j(l)$ denote the set of revealed vertices which are not colored with j. Every edge A is considered as a function of l, where $A(l)$ denote an edge subset revealed after round l. If $A(l) \subset V_j(l)$ and $A \in E_j$ then we say that A is not colored with j, empty edge also satisfies this property. If $|A(l)| = i$, $i = 0, \dots, n-1$, then edge A is said to be at level i after round l. Finally, a (j, i)-block is a set of a_{n-i} edges not colored with j at level i.

Lister's strategy can be described as follows.

- For every color $j = 1, \dots, r$, he chooses the largest $i = i(j)$ such that there exists a (j, i)-block B_j.
- He chooses the union $B_1 \sqcup \ldots \sqcup B_r$ as a set of edges that will contain the next vertex v_{l+1}.

Clearly, Lister wins if after some round there is an edge at level n, because such an edge will not meet some of the colors. The total number of blocks at the beginning is equal to $3rn(r-1)$. For any Painter's choice of color for v_{l+1}, the total number of blocks cannot decrease. Indeed, for chosen color the number of blocks not colored with this color will decrease by 1 but since $a_m \geqslant (1 + 1/(r-1))a_{m-1}$ the number of blocks not colored with any other color j will increase by at least $1/(r-1)$ (minus one block on the current level $i(j)$, plus $r/(r-1)$ blocks on the next one). Thus, the total number of blocks does not decrease.

The game continues until there is no blocks not colored with some color or Lister wins. In fact, in the first case Painter does not necessarily win, but we will show that even such situation is impossible. Suppose it appears and there is no blocks not colored with some color q. Due to the strategy for every $j \neq q$, the number of blocks not colored with j

- is at most 3 on every level form 1 to $n-1$ (since we always choose the largest level and add at most 2 new blocks to the next level);
- is at most $3n(r-1) - 1$ on level 0.

Hence the total number of the remaining blocks is at most

$$(r-1)\,(3(n-1) + 3n(r-1) - 1) = (r-1)\,(3nr-4)$$

which is less than $3nr(r-1)$, a contradiction, since we have shown that the total number of blocks cannot decrease. Thus Lister always wins.

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Bibliography

1. ***U. Schauz,*** *Mr. Paint and Mrs. Correct*, Electronic Journal of Combinatorics, **16**:1 (2009), Research Paper № R77.
2. ***U. Schauz,*** *A paintability version of the combinatorial nullstellensatz, and list colorings of k-partite k-uniform hypergraphs*, Electronic Journal of Combinatorics, **17** (2010), Research Paper № R176.
3. ***X. Zhu,*** *On-line list colouring of graphs*, Electronic Journal of Combinatorics, **16**:1 (2009), Research Paper № R127.
4. ***V. G. Vizing,*** *Coloring the vertices of a graph in prescribed colors*, Metody diskretnogo analiza v teorii kodov i skhem, Sobolev Institute of Mathematics, Novosibirsk, **29** (1976), 3–10 (in Russian).
5. ***P. Erdős, A. L. Rubin and H. Taylor,*** *Choosability in graphs*, Proc. West Coast Conference on Combinatorics, Graph Theory and Computing, 1979, Congr. Numer, **26** (1980), 125–157.
6. ***L. Duraj, G. Gutowski and J. Kozik,*** *Chip games and paintability*, Electronic journal of combinatorics, **23**:3 (2016), Paper № P3.3.
7. ***N. Gazit and M. Krivelevich,*** *On the asymptotic value of the choice number of complete multi-partite graphs*, Journal of Graph Theory, **52** (2006), 123–134.
8. ***D. A. Shabanov and T. M. Shaikheeva,*** *On the list chromatic number of the complete multi-partite hypergraphs and multiple coverings by independent sets*, preprint, https://mipt.ru/education/chairs/dm/laboratoriya-prodvinutoy-kombinatoriki-i-setevykh-prilozheniy/preprinty
9. ***D. A. Shabanov,*** *On a generalization of Rubin's theorem*, Journal of Graph Theory, **67**:3 (2011), 226–234.
10. ***P. Haxell and J. Verstraëte,*** *List coloring hypergraphs*, Electronic Journal of Combinatorics, **17** (2010), Research paper № 129.
11. ***A. V. Kostochka,*** *On a theorem by Erdős, Rubin and Taylor on choosability of complete bipartite graphs*, Electronic Journal of Combinatorics, **9** (2002).
12. ***J. Aslam and A. Dhagat,*** *On-line algorithms for 2-coloring hypergraphs via chip games*, Theoretical Computer Science, **112**:2 (1993), 355–369.
13. ***A. M. Raigorodskii and D. A. Shabanov,*** *The Erdős–Hajnal problem, its generalizations and related problems*, Russian Mathematical Surveys, **66**:5 (2011), 933–1002.

14. ***P. Erdős,*** *On a combinatorial problem*, *II*, Acta Mathematica of the Academy of Sciences, Hungary, **15**:3–4 (1964), 445–447.
15. ***J. Radhakrishnan and A. Srinivasan,*** *Improved bounds and algorithms for hypergraph two-coloring*, Random Structures and Algorithms, **16**:1 (2000), 4–32.
16. ***D. Cherkashin and J. Kozik***, *A note on random greedy coloring of uniform hypergraphs*, Random Structures and Algorithm, **47**:3 (2015), 407–413.
17. ***N. Alon,*** *Hypergraphs with high chromatic number*, Graphs and Combinatorics, **1**:1 (1985), 387–389.
18. ***I. A. Akolzin and D. A. Shabanov,*** *Colorings of hypergraphs with large number of colors*, Discrete Mathematics, **339**:12 (2016), 3020–3031.
19. ***A. P. Rozovskaya and D. A. Shabanov,*** *Extremal problems for panchromatic colourings of uniform hypergraphs*, Discrete Mathematics and Applications, **22**:2 (2012), 185–206.
20. ***D. D. Cherkashin,*** *A note on panchromatic colorings*, arXiv:1705.03797v1.

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Combinatorial Sums $\sum\limits_{k\equiv r \pmod m} \binom{n}{k} a^k$ and Lucas Quotients

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Abstract: In this paper, we study the combinatorial sum

$$\sum_{k\equiv r (\text{mod } m)} \binom{n}{k} a^k.$$

By studying this sum, we obtain new congruences for Lucas quotients of two infinite families of Lucas sequences. Only for three Lucas sequences, there are such known results. Using these general congruences, one can get some new concrete congruences modulo primes, for example,

$$\sum_{k=1}^{[\frac{p}{3}]} \frac{(-8)^k}{k} \equiv -\frac{3^p-3}{p} \pmod p,$$

$$\sum_{k=1}^{[\frac{p+1}{3}]} \frac{(-8)^k}{12k-8} + \sum_{k=1}^{[\frac{p}{3}]} \frac{(-8)^k}{6k-2} \equiv \left(\frac{-3}{p}\right)\left(\frac{2^p-2}{p} - \frac{3^{p-1}-1}{p}\right) \pmod p,$$

where $p>3$ is a prime.

Keywords: Binomial coefficient, Combinatorial sum, Congruence, Fermat quotient, Lucas quotient

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1. Introduction

Let $\{F_n\}_{n\geq 0}$ be the Fibonacci sequence, i.e.,

$$F_0 = 0, \quad F_1 = 1, \quad F_{n+1} = F_n + F_{n-1} \text{ for } n \geq 1.$$

For example, $F_2 = 1, F_3 = 2, F_4 = 3, F_5 = 5$, etc. It is well-known that

$$p \mid F_{p-\left(\frac{p}{5}\right)},$$

where p is an arbitrary prime, and $\left(\frac{p}{5}\right)$ is the Legendre symbol. We know that

$$\left(\frac{p}{5}\right) = \begin{cases} 0, & \text{if } p = 5, \\ 1, & \text{if } p \equiv \pm 1 \pmod 5, \\ -1, & \text{if } p \equiv \pm 2 \pmod 5. \end{cases}$$

For example, we have that $2 \mid F_3, 3 \mid F_4$ and $5 \mid F_5$. In 1960 Wall [10] posed the problem of whether there exists a prime p such that

$$p^2 \mid F_{p-\left(\frac{p}{5}\right)}.$$

Up to now this is still open.

An idea related to Wall's problem is to consider the Fibonacci quotient

$$\frac{F_{p-\left(\frac{p}{5}\right)}}{p}.$$

In 1982 Williams [11] obtained this quotient as

$$\frac{F_{p-\left(\frac{p}{5}\right)}}{p} \equiv \frac{2}{5} \sum_{k=1}^{p-1-\left[\frac{p}{5}\right]} \frac{(-1)^k}{k} \pmod p,$$

where $p \neq 5$ is an odd prime, and $\left[\frac{p}{5}\right]$ is the integral part of $\frac{p}{5}$, i.e., the largest integer $\leq \frac{p}{5}$.

We know that the Fibonacci sequence is a special Lucas sequence. In general, let $A, B \in \mathbb{Z}$, the Lucas sequence $\{u_n\}_{n\geq 0}$ is defined as

$$u_0 = 0, u_1 = 1, u_{n+1} = Bu_n - Au_{n-1} \text{ for } n \geq 1.$$

Thus, when $A = -1$ and $B = 1$, we get the Fibonacci sequence. Let $D = B^2 - 4A$ and let $p \nmid A$ be an odd prime. It is well-known that

$$p \mid u_{p-(\frac{D}{p})}.$$

So, similarly, we can consider the Lucas quotient

$$\frac{u_{p-(\frac{D}{p})}}{p} \pmod{p},$$

and we hope that we can obtain some expression as Williams' for Fibonacci quotient.

Williams' method is to consider the sum

$$\sum_{k\equiv r(\text{mod } 5)} \binom{p}{k},$$

where r is an integer and $\binom{p}{k}$ is the binomial coefficient with the convention $\binom{p}{k} = 0$ for $k < 0$ or $k > p$. Williams did not give any explicit formula for this sum, but he used the properties of the sum to deduce his congruence. Along this line, Z.-H Sun [4–6], Z.-W Sun [8, 9] and Z.-H Sun and Z.-W Sun [7] studied the sum

$$\sum_{k\equiv r(\text{mod } m)} \binom{n}{k},$$

where n, m and r are integers with $n > 0$ and $m > 0$. They gave the formulae of the value of the sum for small m and obtained congruences for two new Lucas sequences. One is the Pell sequence $\{P_n\}_{n\geq 0}$ which is defined as

$$P_0 = 0, P_1 = 1, P_{n+1} = 2P_n + P_{n-1} \text{ for } n \geq 1.$$

Pell sequence is the Lucas sequence with $A = -1$ and $B = 2$. Z.-H Sun's congruence is

$$\frac{P_{p-\left(\frac{2}{p}\right)}}{p} \equiv (-1)^{\frac{p-1}{2}} \sum_{k=1}^{\left[\frac{p+1}{4}\right]} \frac{(-1)^k}{2k-1} \pmod p,$$

where p is an odd prime, see ([5] Theorem 2.5). Z.-H Sun obtained this congruence by studying the above sum with $m = 8$. Z.-W Sun [8] also studied the above sum with $m = 8$ to deduce a congruence for primes. Z.-H Sun and Z.-W Sun [7] obtained a new congruence for the Fibonacci quotient by studying the above sum with $m = 10$.

The second new Lucas sequence is the sequence $\{S_n\}_{n\geq 0}$ which is defined as

$$S_0 = 0, S_1 = 1, S_{n+1} = 4S_n - S_{n-1} \text{ for } n \geq 1.$$

This sequence is the Lucas sequence with $A = 1$ and $B = 4$. Z.-W Sun [9] obtained

$$\sum_{k=1}^{\frac{p-1}{2}} \frac{3^k}{k} \equiv \sum_{k=1}^{\left[\frac{p}{6}\right]} \frac{(-1)^k}{k} \equiv -6\left(\frac{2}{p}\right)\frac{S_{\overline{p}}}{p} - q_p(2) \pmod p,$$

where $p > 3$ is a prime, $\overline{p} = \frac{p-\left(\frac{3}{p}\right)}{2}$ and $q_p(2) = \frac{2^{p-1}-1}{p}$ is the Fermat quotient of 2 with respect to p. See ([9] Theorem 3). Z.-W Sun obtained this congruence by studying the above sum with $m = 12$.

So far, except the above mentioned three Lucas sequences, there is no any known congruence for new Lucas quotients. Noticed that, we do not consider the case where $D = B^2 - 4A$ is a perfect square. If $D = B^2 - 4A$ is a perfect square, then Lucas quotients degenerate to Fermat quotients. In this paper, we study the more general sum

$$\sum_{k\equiv r(\mathrm{mod}\ m)} \binom{n}{k} a^k. \tag{1}$$

When $a = 1$, this sum is that considered by Williams, Z.-H Sun and Z.-W Sun. By studying this sum, we obtain new congruences for Lucas quotients of two infinite families of Lucas sequences, see Theorems 4.2 and 5.2. Using these general congruences, one can get some new concrete congruences modulo primes, for example,

$$\sum_{k=1}^{[\frac{p}{3}]} \frac{(-8)^k}{k} \equiv -\frac{3^p-3}{p} \pmod{p},$$

$$\sum_{k=1}^{[\frac{p+1}{3}]} \frac{(-8)^k}{12k-8} + \sum_{k=1}^{[\frac{p}{3}]} \frac{(-8)^k}{6k-2} \equiv \left(\frac{-3}{p}\right)\left(\frac{2^p-2}{p} - \frac{3^{p-1}-1}{p}\right) \pmod{p},$$

where $p > 3$ is a prime. See Corollaries 4.2 and 4.3.

Another motivation of this paper is the result in [2]. Deng and Pan [2] connected this combinatorial sum with integer factorization for the first time and they proved that, when n is a composite number, for every integer a with $\gcd(n, a) = 1$, there exists a pair (m, r) of integers such that the sum (1) has a nontrivial greatest common divisor with n.

Integer factorization is a famous and very important computational problem, and it is the security foundation of the famous public-key cryptosystem RSA [3]. So it is worthwhile to make a systematic research of the combinatorial sum for a general a.

Below, we briefly describe the achievements of the present paper. Because they are quite large in number and technical in their hypotheses, we cannot mention all of them. First, we obtain explicit recurrent relations for the sum (1), which have order $m-1$ for odd m and order $m-2$ for even m. However, for $m = 6$, the characteristic polynomial of the recurrent relation is a product of two polynomials of degree two. Hence, for $m = 3,4$ and 6, we can obtain the recurrent relations of order two for the sum (1). In these cases, we can give the formulae of the sum (1) via relevant Lucas sequences. Second, using these formulae, we obtain new congruences for Lucas quotients of two infinite families of Lucas sequences, i.e., $A = a^2 - a + 1, B = 2 - a$ or $A = a^2 + 1, B = 2$, where a is an arbitrary integer. By specifying the value of a, we can obtain numerous congruences for new concrete Lucas quotients.

The paper is organized as follows. We give some necessary preliminaries in Section 2. We deduce the recurrent relation for the combinatorial sum in Section 3. We consider the calculation of the combinatorial sum and give some applications when $m = 3,4,6$ in Sections 4,5,6, respectively.

2. Preliminaries

Notational conventions: We denote by $\mathbb{C}, \mathbb{R}, \mathbb{Z}$ respectively the complex numbers, the reals and the integers. For $x \in \mathbb{R}$, we denote by $[x]$ the integral part of x, i.e., the largest integer $\leq x$.

First we recall some well-known facts about Lucas sequences. Let $A, B \in \mathbb{Z}$. We define Lucas sequences $\{u_n\}_{n\geq 0}$ and $\{v_n\}_{n\geq 0}$ as

$$u_0 = 0, u_1 = 1, u_{n+1} = Bu_n - Au_{n-1} \text{ for } n \geq 1;$$
$$v_0 = 2, v_1 = B, v_{n+1} = Bv_n - Av_{n-1} \text{ for } n \geq 1.$$

The proof of the following two lemmas can be found in [1].

LEMMA 2.1. *Let $D = B^2 - 4A$ and α, β be the two complex roots of $x^2 - Bx + A = 0$. Then we have*

$$\begin{aligned}
u_n &= \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad v_n = \alpha^n + \beta^n;\\
v_n &= u_{n+1} - Au_{n-1} = Bu_n - 2Au_{n-1} = 2u_{n+1} - Bu_n;\\
Du_n &= v_{n+1} - Av_{n-1} = Bv_n - 2Av_{n-1} = 2v_{n+1} - Bv_n;\\
u_{2n} &= u_n v_n, \quad u_{2n+1} = u_{n+1}^2 - Au_n^2;\\
v_{2n} &= v_n^2 - 2A^n, \quad v_n^2 - Du_n^2 = 4A^n.
\end{aligned}$$

LEMMA 2.2. *Let $\varepsilon = \left(\frac{D}{p}\right)$ and $p \nmid A$ be an odd prime. Then we have*

$$u_{p-\varepsilon} \equiv 0 \pmod{p}, \quad u_p \equiv \varepsilon \pmod{p}.$$

DEFINITION 2.1. Let n, m, r and a be integers with $n > 0$ and $m > 0$. We define

$$\begin{bmatrix} n \\ r \end{bmatrix}_m (a) := \sum_{\substack{k=0 \\ k\equiv r (\mathrm{mod} m)}}^{n} \binom{n}{k} a^k,$$

where $\binom{n}{k}$ is the binomial coefficient with the convention $\binom{n}{k} = 0$ for $k < 0$ or $k > n$. Let p be an odd prime and m, r and a be integers with $m > 0$. We define

$$K_{p,m,r}(a) := \sum_{\substack{k=1 \\ k\equiv r (\mathrm{mod} m)}}^{p-1} \frac{(-a)^k}{k}.$$

LEMMA 2.3. *With notation as above, we have*:

(1) $\displaystyle\sum_{r=0}^{m-1} \begin{bmatrix} n \\ r \end{bmatrix}_m (a) = (1+a)^n;$

(2) $\displaystyle\binom{p-1}{k} \equiv (-1)^k \pmod p$ *for* $0 \le k \le p-1$;

(3) $\displaystyle\begin{bmatrix} p \\ r \end{bmatrix}_m (a) \equiv \delta_{0\equiv r(m)} + \delta_{p\equiv r(m)} a^p - pK_{p,m,r}(a) \pmod{p^2},$

where

$$\delta_{0\equiv r(m)} = \begin{cases} 1, & \textit{if } 0 \equiv r \pmod m \textit{ holds}, \\ 0, & \textit{otherwise}, \end{cases}$$

and $\delta_{p\equiv r(m)}$ *has the similar meaning.*

PROOF. (1) is obvious.

(2) If $k = 0$, $\binom{p-1}{0} \equiv (-1)^0 \pmod p$.

If $1 \le k \le p-1$, $\binom{p-1}{k} = \frac{(p-1)(p-2)\cdots(p-k)}{k!} \equiv (-1)^k \pmod p$.

(3) By (2), we have

$$\begin{aligned} \begin{bmatrix} p \\ r \end{bmatrix}_m (a) &= \delta_{0\equiv r(m)} + \delta_{p\equiv r(m)} a^p + \sum_{\substack{k=1 \\ k\equiv r \ (\mathrm{mod}\ m)}}^{p-1} \binom{p}{k} a^k \\ &= \delta_{0\equiv r(m)} + \delta_{p\equiv r(m)} a^p + \sum_{\substack{k=1 \\ k\equiv r \ (\mathrm{mod}\ m)}}^{p-1} \frac{p}{k}\binom{p-1}{k-1} a^k \\ &\equiv \delta_{0\equiv r(m)} + \delta_{p\equiv r(m)} a^p - p \sum_{\substack{k=1 \\ k\equiv r \ (\mathrm{mod}\ m)}}^{p-1} \frac{(-a)^k}{k} \pmod{p^2} \\ &= \delta_{0\equiv r(m)} + \delta_{p\equiv r(m)} a^p - pK_{p,m,r}(a) \pmod{p^2}. \end{aligned}$$ □

Note that, in the above lemma, (2) is well-known and (3) is a generalization of Lemma 1.1 in [4].

Definition 2.2. Let p be an odd prime and x an integer with $p\nmid x$. Fermat quotient is defined as $q_p(x) := \frac{x^{p-1}-1}{p}$. In the sequent, when we meet $q_p(x)$, we always suppose that p is an odd prime and $p\nmid x$.

Lemma 2.4. *We have*:

(1) $q_p(x) \equiv 2\left(\frac{x}{p}\right)\frac{x^{\frac{p-1}{2}}-\left(\frac{x}{p}\right)}{p} \pmod p$;

(2) $q_p(xy) \equiv q_p(x)+q_p(y) \pmod p$;

(3) $\sum_{r=0}^{m-1} K_{p,m,r}(a) \equiv aq_p(a)-(a+1)q_p(a+1) \pmod p$.

Proof. (1) From

$$\begin{aligned} x^{p-1}-1 &= \left(x^{\frac{p-1}{2}}+\left(\frac{x}{p}\right)\right)\left(x^{\frac{p-1}{2}}-\left(\frac{x}{p}\right)\right)\\ &\equiv 2\left(\frac{x}{p}\right)\left(x^{\frac{p-1}{2}}-\left(\frac{x}{p}\right)\right) \pmod{p^2}, \end{aligned}$$

we have

$$q_p(x) \equiv 2\left(\frac{x}{p}\right)\frac{x^{\frac{p-1}{2}}-\left(\frac{x}{p}\right)}{p} \pmod p.$$

(2) From

$$(xy)^{p-1}-1 = x^{p-1}(y^{p-1}-1)+(x^{p-1}-1),$$

we have

$$q_p(xy) \equiv q_p(x)+q_p(y) \pmod p.$$

(3) From Lemma 2.3, we have

$$\begin{aligned} (1+a)^p &= \sum_{r=0}^{m-1}\begin{bmatrix} p\\ r\end{bmatrix}_m (a)\\ &\equiv \sum_{r=0}^{m-1}\left[\delta_{0\equiv r(m)}+\delta_{p\equiv r(m)}a^p-pK_{p,m,r}(a)\right] \pmod{p^2}\\ &\equiv 1+a^p-p\sum_{r=0}^{m-1}K_{p,m,r}(a) \pmod{p^2}. \end{aligned}$$

Thus we obtain

$$\sum_{r=0}^{m-1} K_{p,m,r}(a) \equiv \frac{1+a^p-(a+1)^p}{p} = aq_p(a)-(a+1)q_p(a+1) \pmod p. \qquad \square$$

Note that, in the above lemma, (1) is the Lemma 1.2 in [4].

LEMMA 2.5. *With notation as in Lemmas* 2.1 *and* 2.2*. Let* p *be an odd prime with* $p \nmid DA$*. We have*:

(1) *if* $\varepsilon = 1$, *then* $\frac{v_{p-1}-2}{p} \equiv q_p(A) \pmod p$;

(2) *if* $\varepsilon = -1$, *then* $\frac{v_{p+1}-2A}{p} \equiv Aq_p(A) \pmod p$.

PROOF. By Lemma 2.1, we have $v_n^2 - Du_n^2 = 4A^n$ for $n \geq 0$. Combining this with Lemma 2.2, we have $v_{p-\varepsilon}^2 \equiv 4A^{p-\varepsilon} \pmod{p^2}$. Since it is impossible that $p \mid v_{p-\varepsilon} + 2A^{\frac{p-\varepsilon}{2}}$ and $p \mid v_{p-\varepsilon} - 2A^{\frac{p-\varepsilon}{2}}$, we have

$$v_{p-\varepsilon} \equiv \pm 2\left(\frac{A}{p}\right) A^{\frac{p-\varepsilon}{2}} \pmod{p^2}.$$

If $\varepsilon = 1$, by Lemmas 2.1 and 2.2 we have $v_{p-1} = 2u_p - Bu_{p-1} \equiv 2 \pmod p$. Thus, by Lemma 2.4 (1), we have

$$v_{p-1} \equiv 2\left(\frac{A}{p}\right) A^{\frac{p-1}{2}} \equiv 2 + pq_p(A) \pmod{p^2}.$$

Hence

$$\frac{v_{p-1}-2}{p} \equiv q_p(A) \pmod p,$$

we obtain (1).

If $\varepsilon = -1$, by Lemmas 2.1 and 2.2 we have $v_{p+1} = Bu_{p+1} - 2Au_p \equiv 2A \pmod p$. Thus, by Lemma 2.4 (1), we have

$$v_{p+1} \equiv 2\left(\frac{A}{p}\right) A^{\frac{p+1}{2}} \equiv 2A + Apq_p(A) \pmod{p^2}.$$

Hence

$$\frac{v_{p+1}-2A}{p} \equiv Aq_p(A) \pmod p,$$

we obtain (2). $\square$

3. The Recurrent Relation for $\Delta_m(r, n)$

In this section, we consider the calculation of the sum $\begin{bmatrix} n \\ r \end{bmatrix}_m (a)$. It is easy to see that

$$\begin{bmatrix} n \\ 0 \end{bmatrix}_1 (a) = (1+a)^n, \quad \begin{bmatrix} n \\ r \end{bmatrix}_2 (a) = \frac{(1+a)^n + (-1)^r(1-a)^n}{2}.$$

However, for $m \geq 3$, the calculation is much more difficult.

Throughout the rest of this paper, we fix $a \neq 0, \pm 1$. For any positive integer m, let $\zeta_m = e^{2\pi i/m} \in \mathbb{C}$ be the primitive m-th root of unity.

The following lemma is useful to compute $\begin{bmatrix} n \\ r \end{bmatrix}_m (a)$ when m is small.

LEMMA 3.1. *We have*

$$\begin{bmatrix} n \\ r \end{bmatrix}_m (a) = \frac{1}{m}\sum_{l=0}^{m-1} \zeta_m^{-rl}(1 + a\zeta_m^l)^n.$$

PROOF. Since

$$\begin{aligned}\begin{bmatrix} n \\ r \end{bmatrix}_m (a) &= \sum_{k=0}^{n}\binom{n}{k}a^k \cdot \frac{1}{m}\sum_{l=0}^{m-1}\zeta_m^{(k-r)l} \\ &= \frac{1}{m}\sum_{l=0}^{m-1}\zeta_m^{-rl}\sum_{k=0}^{n}\binom{n}{k}(a\zeta_m^l)^k = \frac{1}{m}\sum_{l=0}^{m-1}\zeta_m^{-rl}(1+a\zeta_m^l)^n,\end{aligned}$$

the lemma follows. □

PROPOSITION 3.1. *Let* $G_n(x) = \prod_{l=1}^{n}(x - 1 - a\zeta_{2n+1}^l)(x - 1 - a\zeta_{2n+1}^{-l}) := \sum_{s=0}^{2n} b_s x^s$ *for* $n \geq 0$. *Then* $G_n(x) \in \mathbb{Z}[x]$ *and we have*:

(1) $b_0 = \frac{a^{2n+1}+1}{a+1}$, $b_{2n} = 1$ *and* $b_{s-1} - (a+1)b_s = \binom{2n+1}{s}(-1)^{s+1}$ *for* $1 \leq s \leq 2n-1$;

(2) $G_0(x) = 1$ *and* $G_{n+1}(x) = (x-1)^2 G_n(x) + a^{2n+1}(x + a - 1)$ *for* $n \geq 0$.

Proof. Since $(x-1-a)G_n(x) = (x-1)^{2n+1} - a^{2n+1}$, we have

$$\sum_{s=1}^{2n+1} b_{s-1}x^s - \sum_{s=0}^{2n}(1+a)b_s x^s = x^{2n+1} + \sum_{s=1}^{2n}\binom{2n+1}{s}(-1)^{s+1}x^s - 1 - a^{2n+1}.$$

By comparing coefficients of both sides of the above expression, we obtain (1).

Since $(x-1-a)G_n(x) = (x-1)^{2n+1} - a^{2n+1}$, then $(x-1-a)G_{n+1}(x) + a^{2n+3} = (x-1)^{2n+3} = (x-1)^2\left[(x-1-a)G_n(x) + a^{2n+1}\right]$. Hence $G_{n+1}(x) = (x-1)^2G_n(x) + a^{2n+1}(x+a-1)$, thus we obtain (2). The assertion $G_n(x) \in \mathbb{Z}[x]$ follows from (2). □

The polynomial $G_n(x)$ depends on a and should be denoted as $G_{n,a}(x)$, for the notational simplification, we omit a. The first few values of $G_n(x)$ are: $G_0(x) = 1$, $G_1(x) = x^2 + (a-2)x + a^2 - a + 1$, $G_2(x) = x^4 + (a-4)x^3 + (a^2-3a+6)x^2 + (a^3 - 2a^2 + 3a - 4)x + a^4 - a^3 + a^2 - a + 1$.

Proposition 3.2. *Let* $Q_n(x) = \prod_{l=1}^{n}(x - 1 - a\zeta_{2n+2}^{l})(x - 1 - a\zeta_{2n+2}^{-l}) := \sum_{s=0}^{2n} c_s x^s$ *for* $n \geq 0$. *Then* $Q_n(x) \in \mathbb{Z}[x]$ *and we have*:

(1) $c_0 = \frac{a^{2n+2}-1}{a^2-1}$, $2c_0 + (a^2-1)c_1 = 2n+2$, $c_{2n-1} = -2n$, $c_{2n} = 1$ *and* $c_{s-2} - 2c_{s-1} + (1-a^2)c_s = \binom{2n+2}{s}(-1)^s$ *for* $2 \leq s \leq 2n-2$;

(2) $Q_0(x) = 1$ *and* $Q_{n+1}(x) = (x-1)^2Q_n(x) + a^{2n+2}$ *for* $n \geq 0$.

Proof. Since $(x-1-a)(x-1+a)Q_n(x) = (x-1)^{2n+2} - a^{2n+2}$, we have

$$\sum_{s=2}^{2n+2} c_{s-2}x^s - \sum_{s=1}^{2n+1} 2c_{s-1}x^s + \sum_{s=0}^{2n}(1-a^2)c_s x^s = x^{2n+2} + \sum_{s=1}^{2n+1}\binom{2n+2}{s}(-1)^s x^s + 1 - a^{2n+2}.$$

By comparing coefficients of both sides of the above expression, we obtain (1).

Since $(x-1-a)(x-1+a)Q_n(x) = (x-1)^{2n+2} - a^{2n+2}$, then $(x-1-a)(x-1+a) \times Q_{n+1}(x) + a^{2n+4} = (x-1)^{2n+4} = (x-1)^2\left((x-1-a)(x-1+a)Q_n(x) + a^{2n+2}\right)$. Hence $Q_{n+1}(x) = (x-1)^2Q_n(x) + a^{2n+2}$, thus we obtain (2). The assertion $Q_n(x) \in \mathbb{Z}[x]$ follows from (2). □

The first few values of $Q_n(x)$ are: $Q_0(x) = 1, Q_1(x) = x^2 - 2x + 1 + a^2$, $Q_2(x) = x^4 - 4x^3 + (a^2+6)x^2 - (2a^2+4)x + a^4 + a^2 + 1 = (x^2 + (a-2)x + a^2 - a + 1)$ $\times (x^2 - (a+2)x + a^2 + a + 1)$.

Definition 3.3. We define

$$\Delta_m(r,n) := \begin{cases} m \begin{bmatrix} n \\ r \end{bmatrix}_m (a) - (1+a)^n, & \text{if } m \text{ is odd,} \\ \\ m \begin{bmatrix} n \\ r \end{bmatrix}_m (a) - (1+a)^n - (-1)^r (1-a)^n, & \text{if } m \text{ is even.} \end{cases}$$

Remark 3.1. *It is obvious that*

$$\Delta_1(r,n) = \Delta_2(r,n) = 0.$$

By Lemma 2.3(1), *it is easy to see that*

$$\sum_{r=0}^{m-1} \Delta_m(r,n) = 0.$$

Theorem 3.1. *Let m be an odd positive integer, and let $G_{\frac{m-1}{2}}(x) = \sum_{s=0}^{m-1} b_s x^s$ be the same as in Proposition* 3.1. *Then we have $\sum_{s=0}^{m-1} b_s \Delta_m(r, n+s) = 0$.*

Proof. By Lemma 3.1, we have

$$\begin{aligned} \Delta_m(r,n) &= m \begin{bmatrix} n \\ r \end{bmatrix}_m (a) - (1+a)^n \\ &= \sum_{l=1}^{m-1} \zeta_m^{-rl} (1 + a\zeta_m^l)^n = \sum_{l=1}^{\frac{m-1}{2}} \left[\zeta_m^{-rl} (1 + a\zeta_m^l)^n + \zeta_m^{rl} (1 + a\zeta_m^{-l})^n \right]. \end{aligned}$$

Thus

$$\sum_{s=0}^{m-1} b_s\Delta_m(r,n+s)=\sum_{s=0}^{m-1} b_s\sum_{l=1}^{\frac{m-1}{2}}\left[\zeta_m^{-rl}(1+a\zeta_m^{l})^{n+s}+\zeta_m^{rl}(1+a\zeta_m^{-l})^{n+s}\right]$$
$$=\sum_{l=1}^{\frac{m-1}{2}}\zeta_m^{-rl}(1+a\zeta_m^{l})^{n}\sum_{s=0}^{m-1} b_s(1+a\zeta_m^{l})^{s}+\sum_{l=1}^{\frac{m-1}{2}}\zeta_m^{rl}(1+a\zeta_m^{-l})^{n}\sum_{s=0}^{m-1} b_s(1+a\zeta_m^{-l})^{s}$$
$$=\sum_{l=1}^{\frac{m-1}{2}}\zeta_m^{-rl}(1+a\zeta_m^{l})^{n}G_{\frac{m-1}{2}}(1+a\zeta_m^{l})+\sum_{l=1}^{\frac{m-1}{2}}\zeta_m^{rl}(1+a\zeta_m^{-l})^{n}G_{\frac{m-1}{2}}(1+a\zeta_m^{-l})$$
$$=0+0=0.$$ □

THEOREM 3.2. *Let m be an even positive integer, and let $Q_{\frac{m}{2}-1}(x)=\sum_{s=0}^{m-2} c_s x^s$ be the same as in Proposition 3.2. Then we have $\sum_{s=0}^{m-2} c_s\Delta_m(r,n+s)=0$.*

PROOF. By Lemma 3.1, we have

$$\Delta_m(r,n)=m\begin{bmatrix} n\\ r\end{bmatrix}_m (a)-(1+a)^n-(-1)^r(1-a)^n$$
$$=\sum_{\substack{l=1\\ l\neq\frac{m}{2}}}^{m-1}\zeta_m^{-rl}(1+a\zeta_m^{l})^{n}=\sum_{l=1}^{\frac{m}{2}-1}\left[\zeta_m^{-rl}(1+a\zeta_m^{l})^{n}+\zeta_m^{rl}(1+a\zeta_m^{-l})^{n}\right].$$

Thus

$$\sum_{s=0}^{m-2} c_s\Delta_m(r,n+s)=\sum_{s=0}^{m-2} c_s\sum_{l=1}^{\frac{m}{2}-1}\left[\zeta_m^{-rl}(1+a\zeta_m^{l})^{n+s}+\zeta_m^{rl}(1+a\zeta_m^{-l})^{n+s}\right]$$
$$=\sum_{l=1}^{\frac{m}{2}-1}\zeta_m^{-rl}(1+a\zeta_m^{l})^{n}\sum_{s=0}^{m-2} c_s(1+a\zeta_m^{l})^{s}+\sum_{l=1}^{\frac{m}{2}-1}\zeta_m^{rl}(1+a\zeta_m^{-l})^{n}\sum_{s=0}^{m-2} c_s(1+a\zeta_m^{-l})^{s}$$
$$=\sum_{l=1}^{\frac{m}{2}-1}\zeta_m^{-rl}(1+a\zeta_m^{l})^{n}Q_{\frac{m}{2}-1}(1+a\zeta_m^{l})+\sum_{l=1}^{\frac{m}{2}-1}\zeta_m^{rl}(1+a\zeta_m^{-l})^{n}Q_{\frac{m}{2}-1}(1+a\zeta_m^{-l})$$
$$=0+0=0.$$ □

4. $\Delta_3(r, n)$ and Related Lucas Quotients

In this section, we consider the calculation of $\Delta_3(r, n)$.

4.1. General Properties

THEOREM 4.1. *Let $\{u_n\}_{n\geq 0}$ be the Lucas sequence defined as*

$$u_0 = 0,\ u_1 = 1,\ u_{n+1} = (2-a)u_n - (a^2 - a + 1)u_{n-1} \text{ for } n \geq 1.$$

Then we have, for $n \geq 1$,

$$\begin{aligned}
\Delta_3(0, n) &= (2-a)u_n - 2(a^2 - a + 1)u_{n-1} = 2u_{n+1} - (2-a)u_n,\\
\Delta_3(1, n) &= (2a-1)u_n + (a^2 - a + 1)u_{n-1} = -u_{n+1} + (a+1)u_n,\\
\Delta_3(2, n) &= (-a-1)u_n + (a^2 - a + 1)u_{n-1} = -u_{n+1} - (2a-1)u_n.
\end{aligned}$$

PROOF. By Lemma 2.1, we have, for $n \geq 1$,

$$\begin{aligned}
(2-a)u_n - 2(a^2 - a + 1)u_{n-1} &= 2u_{n+1} - (2-a)u_n,\\
(2a-1)u_n + (a^2 - a + 1)u_{n-1} &= -u_{n+1} + (a+1)u_n,\\
(-a-1)u_n + (a^2 - a + 1)u_{n-1} &= -u_{n+1} - (2a-1)u_n.
\end{aligned}$$

Since $u_2 = 2 - a$, one can verify the following simple facts:

$$\begin{aligned}
\Delta_3(0,1) &= -a + 2 = (2-a)u_1 - 2(a^2 - a + 1)u_0,\\
\Delta_3(0,2) &= -a^2 - 2a + 2 = (2-a)u_2 - 2(a^2 - a + 1)u_1,\\
\Delta_3(1,1) &= 2a - 1 = (2a-1)u_1 + (a^2 - a + 1)u_0,\\
\Delta_3(1,2) &= -a^2 + 4a - 1 = (2a-1)u_2 + (a^2 - a + 1)u_1,\\
\Delta_3(2,1) &= -a - 1 = (-a-1)u_1 + (a^2 - a + 1)u_0,\\
\Delta_3(2,2) &= 2a^2 - 2a - 1 = (-a-1)u_2 + (a^2 - a + 1)u_1.
\end{aligned}$$

By Theorem 3.1, we have

$$\Delta_3(r, n+2) = (2-a)\Delta_3(r, n+1) - (a^2 - a + 1)\Delta_3(r, n) \text{ for } n \geq 1.$$

Then we can prove the theorem by induction on n. □

Remark 4.1. *Let $\{v_n\}_{n\geq 0}$ be the Lucas sequence defined as*

$$v_0 = 2,\ v_1 = 2-a,\ v_{n+1} = (2-a)v_n - (a^2-a+1)v_{n-1} \textit{ for } n \geq 1.$$

Then, by the above theorem and Lemma 2.1, *we have, for $n \geq 1$,*

$$\begin{aligned}\Delta_3(0,n) &= v_n,\\ \Delta_3(1,n) &= -\frac{1}{a}v_n + \frac{a^2-a+1}{a}v_{n-1},\\ \Delta_3(2,n) &= -\frac{a-1}{a}v_n - \frac{a^2-a+1}{a}v_{n-1}.\end{aligned}$$

Theorem 4.2. *Let $p \nmid 3a(a^3+1)$ be an odd prime, $\{u_n\}_{n\geq 0}$ as in Theorem* 4.1, *and $K_{p,3,r}(a)$ as in Definition* 2.1. *Then we have*:

(1) *for* $p \equiv 1 \pmod 3$,

$$\begin{aligned}\frac{u_{p-1}}{p} \equiv &\frac{(2a-1)K_{p,3,0}(a)+(a-2)K_{p,3,1}(a)}{a(a^2-a+1)}\\ &-\frac{a-2}{a^2-a+1}q_p(a) + \frac{a^2-1}{a(a^2-a+1)}q_p(a+1) \pmod p;\end{aligned}$$

(2) *for* $p \equiv 2 \pmod 3$,

$$\frac{u_{p+1}}{p} \equiv \frac{(a-2)K_{p,3,1}(a)-(a+1)K_{p,3,0}(a)}{a} - \frac{a+1}{a}q_p(a+1) \pmod p.$$

Proof. Since $(2-a)^2 - 4(a^2-a+1) = -3a^2$, by Lemma 2.2, we have

$$p \mid u_{p-\left(\frac{-3}{p}\right)}.$$

By Theorem 4.1, we have

$$(2a-1)\Delta_3(0,p) + (a-2)\Delta_3(1,p) = -3a(a^2-a+1)u_{p-1}, \tag{2}$$

and

$$(a+1)\Delta_3(0,p) - (a-2)\Delta_3(1,p) = 3au_{p+1}. \tag{3}$$

If $p \equiv 1 \pmod 3$, by Lemma 2.3 (3), we have

$$\Delta_3(0,p) \equiv 3 - 3pK_{p,3,0}(a) - (1+a)^p \pmod{p^2},$$
$$\Delta_3(1,p) \equiv 3a^p - 3pK_{p,3,1}(a) - (1+a)^p \pmod{p^2}.$$

Then, by Eq.(2),

$$a(a^2-a+1)u_{p-1} \equiv p\left[(2a-1)K_{p,3,0}(a) + (a-2)K_{p,3,1}(a)\right] - (a^2-2a)(a^{p-1}-1) + (a^2-1)\left[(a+1)^{p-1}-1\right] \pmod{p^2}.$$

Thus

$$\frac{u_{p-1}}{p} \equiv \frac{(2a-1)K_{p,3,0}(a) + (a-2)K_{p,3,1}(a)}{a(a^2-a+1)} - \frac{a-2}{a^2-a+1}q_p(a) + \frac{a^2-1}{a(a^2-a+1)}q_p(a+1) \pmod p.$$

If $p \equiv 2 \pmod 3$, by Lemma 2.3 (3), we have

$$\Delta_3(0,p) \equiv 3 - 3pK_{p,3,0}(a) - (1+a)^p \pmod{p^2},$$
$$\Delta_3(1,p) \equiv -3pK_{p,3,1}(a) - (1+a)^p \pmod{p^2}.$$

Then, by Eq.(3),

$$au_{p+1} \equiv p\left[(a-2)K_{p,3,1}(a) - (a+1)K_{p,3,0}(a)\right] - \left[(a+1)^p - (a+1)\right] \pmod{p^2}.$$

Thus

$$\frac{u_{p+1}}{p} \equiv \frac{1}{a}\left[(a-2)K_{p,3,1}(a) - (a+1)K_{p,3,0}(a)\right] - \frac{a+1}{a}q_p(a+1) \pmod p. \quad \square$$

Given a value of a, by Theorem 4.2, we can obtain a concrete congruence for a specific Lucas quotient. We provide one such example.

COROLLARY 4.1. *Let* $\{u_n\}_{n\geq 0}$ *be the Lucas sequence defined by*

$$u_0 = 0,\ u_1 = 1,\ u_{n+1} = 4u_n - 7u_{n-1} \text{ for } n \geq 1,$$

and $p \neq 3{,}7$ *be an odd prime. Then we have*

$$\frac{u_{p-1}}{p} \equiv \frac{5}{42}\sum_{k=1}^{\frac{p-1}{3}} \frac{8^k}{k} + \frac{1}{14}\sum_{k=1}^{\frac{p-1}{3}} \frac{8^k}{3k-2} + \frac{4}{7}q_p(2) \pmod{p},\ \text{if } p \equiv 1 \pmod{3};$$

$$\frac{u_{p+1}}{p} \equiv \frac{1}{2}\sum_{k=1}^{\frac{p+1}{3}} \frac{8^k}{3k-2} - \frac{1}{6}\sum_{k=1}^{\frac{p-2}{3}} \frac{8^k}{k} \pmod{p},\ \text{if } p \equiv 2 \pmod{3}.$$

PROOF. Set $a = -2$ in Theorem 4.2. □

4.2. The Case a=2

If $a = 2$, by Theorem 3.1, we have $\Delta_3(r, n+2) = -3\Delta_3(r, n)$ for $n \geq 1$. Thus we have a refinement of Theorem 4.1.

THEOREM 4.3. *Set* $a = 2$. *Let* $\Delta_3(r, n) = 3\begin{bmatrix} n \\ r \end{bmatrix}_3 (2) - 3^n$ *for* $n \geq 1$. *Then we have*: *if n is odd*,

$$\Delta_3(0, n) = 0,\ \Delta_3(1, n) = -(-3)^{\frac{n+1}{2}},\ \Delta_3(2, n) = (-3)^{\frac{n+1}{2}};$$

if n is even,

$$\Delta_3(0, n) = 2\cdot(-3)^{\frac{n}{2}},\ \Delta_3(1, n) = \Delta_3(2, n) = -(-3)^{\frac{n}{2}}.$$

PROOF. Since $a = 2$, by Theorem 3.1, we have $\Delta_3(r, n+2) = -3\Delta_3(r, n)$ for $n \geq 1$. One can verify that

$$\Delta_3(0{,}1) = 0,\ \Delta_3(1{,}1) = 3,\ \Delta_3(2{,}1) = -3;$$
$$\Delta_3(0{,}2) = -6,\ \Delta_3(1{,}2) = 3,\ \Delta_3(2{,}2) = 3.$$

Then we can prove the theorem by induction on n. □

Remark 4.2. *Using the above theorem and without the use of the Quadratic Reciprocity Law, for an odd prime $p > 3$, we can get the Legendre symbol*

$$\left(\frac{-3}{p}\right) = \begin{cases} 1, \text{ if } p \equiv 1 \pmod 3, \\ -1, \text{ if } p \equiv 2 \pmod 3. \end{cases}$$

PROOF. If $p \equiv 1 \pmod 3$, by Theorem 4.3, we have $\Delta_3(1,p) = -(-3)^{\frac{p+1}{2}}$. Since

$$\Delta_3(1,p) = 3\begin{bmatrix} p \\ 1 \end{bmatrix}_3 (2) - 3^p = 3\sum_{\substack{k=0 \\ k\equiv 1 (\mathrm{mod}\ 3)}}^{p} \binom{p}{k} 2^k - 3^p$$
$$\equiv 3\cdot 2^p - 3^p \equiv 3\cdot 2 - 3 = 3 \pmod p,$$

we have $(-3)^{\frac{p-1}{2}} \equiv 1 \pmod p$. Hence $\left(\frac{-3}{p}\right) = 1$.

If $p \equiv 2 \pmod 3$, by Theorem 4.3, we have $\Delta_3(2,p) = (-3)^{\frac{p+1}{2}}$. Similarly, we can obtain $\left(\frac{-3}{p}\right) = -1$. □

COROLLARY 4.2. *Let $p > 3$ be an odd prime. Then we have*

$$\sum_{k=1}^{[\frac{p}{3}]} \frac{(-8)^k}{k} \equiv -3q_p(3) \pmod p.$$

PROOF. By Theorem 4.3, we have

$$\begin{bmatrix} p \\ 0 \end{bmatrix}_3 (2) = 3^{p-1}.$$

By Lemma 2.3 (3), we have

$$\sum_{k=1}^{[\frac{p}{3}]} \frac{(-8)^k}{k} = 3K_{p,3,0}(2) \equiv 3\frac{1 - \begin{bmatrix} p \\ 0 \end{bmatrix}_3 (2)}{p}$$
$$= 3\frac{1-3^{p-1}}{p} = -3q_p(3) \pmod p.$$ □

COROLLARY 4.3. *Let $p>3$ be an odd prime. Then we have*

$$\sum_{k=1}^{[\frac{p+1}{3}]}\frac{(-8)^k}{12k-8}+\sum_{k=1}^{[\frac{p}{3}]}\frac{(-8)^k}{6k-2}\equiv\left(\frac{-3}{p}\right)(2q_p(2)-q_p(3))\pmod p.$$

PROOF. By Theorem 4.3,

$$\begin{bmatrix}p\\1\end{bmatrix}_3(2)=3^{p-1}+(-3)^{\frac{p-1}{2}},\quad \begin{bmatrix}p\\2\end{bmatrix}_3(2)=3^{p-1}-(-3)^{\frac{p-1}{2}}.$$

Then

$$\begin{bmatrix}p\\1\end{bmatrix}_3(2)-\begin{bmatrix}p\\2\end{bmatrix}_3(2)=2\cdot(-3)^{\frac{p-1}{2}}.$$

By Lemmas 2.3 and 2.4, we have

$$\begin{aligned}&\sum_{k=1}^{[\frac{p+1}{3}]}\frac{(-8)^k}{12k-8}+\sum_{k=1}^{[\frac{p}{3}]}\frac{(-8)^k}{6k-2}=K_{p,3,1}(2)-K_{p,3,2}(2)\\ &\equiv\frac{2^p\cdot\left(\frac{-3}{p}\right)-2\cdot(-3)^{\frac{p-1}{2}}}{p}\\ &=\left(\frac{-3}{p}\right)\frac{(2^p-2)+\left(2-2(-3)^{\frac{p-1}{2}}\left(\frac{-3}{p}\right)\right)}{p}\\ &\equiv\left(\frac{-3}{p}\right)(2q_p(2)-q_p(-3))\pmod p.\end{aligned}$$

□

4.3. Further Results

LEMMA 4.1. *Let p be an odd prime with $p\nmid 3a(2-a)(a^2-a+1)$, and let $\{v_n\}_{n\geq 0}$ be the Lucas sequence defined as*

$$v_0=2,v_1=2-a,v_{n+1}=(2-a)v_n-(a^2-a+1)v_{n-1}\text{ for }n\geq 1.$$

Then we have

$$\frac{v_p-(2-a)}{p}\equiv-\sum_{k=1}^{[\frac{p}{3}]}\frac{(-a)^{3k}}{k}-(a+1)q_p(a+1)\pmod p.$$

PROOF. Let $\{u_n\}_{n\geq 0}$ be the Lucas sequence defined as

$$u_0 = 0, u_1 = 1, u_{n+1} = (2-a)u_n - (a^2 - a + 1)u_{n-1} \text{ for } n \geq 1.$$

By Lemmas 2.1 and 2.2, we have $v_p = (2-a)u_p - 2(a^2-a+1)u_{p-1} = 2u_{p+1} - (2-a)u_p \equiv 2 - a \pmod{p}$. By Remark 4.1 and Lemma 2.3, we have $v_p = \Delta_3(0,p) = 3\begin{bmatrix} p \\ 0 \end{bmatrix}_3 (a) - (1+a)^p \equiv 3 - 3pK_{p,3,0}(a) - (1+a)^p \pmod{p^2}$. Thus

$$\frac{v_p - (2-a)}{p} \equiv -\sum_{k=1}^{[\frac{p}{3}]} \frac{(-a)^{3k}}{k} - (a+1)q_p(a+1) \pmod{p}. \qquad \square$$

In Theorem 4.2, when we express the Lucas quotient, it involves two K's, i.e., two sums. The following theorem can reduce the Lucas quotient to one sum.

THEOREM 4.4. *Let p be an odd prime with $p \nmid 3a(2-a)(a^3+1)$, and let $\{u_n\}_{n\geq 0}$ be the Lucas sequence defined as*

$$u_0 = 0, u_1 = 1, u_{n+1} = (2-a)u_n - (a^2 - a + 1)u_{n-1} \text{ for } n \geq 1.$$

Then we have, if $p \equiv 1 \pmod{3}$,

$$\frac{u_{p-1}}{p} \equiv \frac{2}{3a^2} \sum_{k=1}^{\frac{p-1}{3}} \frac{(-a)^{3k}}{k} + \frac{1}{3a^2}\Big((2-a)q_p(a^2-a+1) + 2(a+1)q_p(a+1)\Big) \pmod{p};$$

if $p \equiv 2 \pmod{3}$,

$$\frac{u_{p+1}}{p} \equiv -\frac{2(a^2-a+1)}{3a^2} \sum_{k=1}^{\frac{p-2}{3}} \frac{(-a)^{3k}}{k} - \frac{a^2-a+1}{3a^2}\Big((2-a)q_p(a^2-a+1) + 2(a+1)q_p(a+1)\Big) \pmod{p}.$$

PROOF. Let $\{v_n\}_{n\geq 0}$ be the sequence as in Lemma 4.1. If $p \equiv 1 \pmod{3}$, by Lemma 2.1, we have $u_{p-1} = \frac{1}{3a^2}((2-a)v_{p-1} - 2v_p)$. Thus

$$\frac{u_{p-1}}{p} = \frac{1}{3a^2}\left(\frac{(2-a)(v_{p-1}-2)}{p} - 2\frac{v_p - (2-a)}{p}\right).$$

If $p \equiv 2 \pmod 3$, by Lemma 2.1, we have $u_{p+1} = \frac{1}{3a^2}(2(a^2-a+1)v_p-(2-a)v_{p+1})$. Thus

$$\frac{u_{p+1}}{p} = \frac{1}{3a^2}\left(\frac{2(a^2-a+1)(v_p-(2-a))}{p} - \frac{(2-a)(v_{p+1}-2(a^2-a+1))}{p}\right).$$

Thus by Lemmas 2.5 and 4.1, we can prove this theorem. □

Given a value of a, by Theorem 4.4, we can obtain a concrete congruence for a specific Lucas quotient. We provide one such example.

COROLLARY 4.4. *Let $p \neq 3,7$ be an odd prime, and $\{u_n\}_{n\geq 0}$ be the Lucas sequence defined as*

$$u_0 = 0, u_1 = 1, u_{n+1} = 4u_n - 7u_{n-1} \text{ for } n \geq 1.$$

Then we have, if $p \equiv 1 \pmod 3$,

$$\frac{u_{p-1}}{p} \equiv \frac{1}{6}\sum_{k=1}^{\frac{p-1}{3}} \frac{8^k}{k} + \frac{1}{3}q_p(7) \pmod p;$$

if $p \equiv 2 \pmod 3$,

$$\frac{u_{p+1}}{p} \equiv -\frac{7}{6}\sum_{k=1}^{\frac{p-2}{3}} \frac{8^k}{k} - \frac{7}{3}q_p(7) \pmod p.$$

PROOF. Set $a = -2$ in Theorem 4.4. □

5. $\Delta_4(r, n)$ and Related Lucas Quotients

In this section, we consider the calculation of $\Delta_4(r, n)$.

5.1. General Properties

THEOREM 5.1. *Let $\{u_n\}_{n\geq 0}$ be the Lucas sequence defined as*

$$u_0 = 0,\ u_1 = 1,\ u_{n+1} = 2u_n - (a^2+1)u_{n-1} \textit{ for } n \geq 1.$$

Then we have, for $n \geq 1$,

$$\begin{aligned}\Delta_4(0,n) &= 2u_n - 2(a^2+1)u_{n-1} = 2u_{n+1} - 2u_n,\\ \Delta_4(1,n) &= 2au_n,\\ \Delta_4(2,n) &= -2u_n + 2(a^2+1)u_{n-1} = -2u_{n+1} + 2u_n,\\ \Delta_4(3,n) &= -2au_n.\end{aligned}$$

PROOF. Since $u_{n+1} = 2u_n - (a^2+1)u_{n-1}$ for $n \geq 1$, we have $2u_{n+1} - 2u_n = 2u_n - 2(a^2+1)u_{n-1}$. It is easy to see that $\Delta_4(r,n) + \Delta_4(r+2,n) = 2\Delta_2(r,n) = 0$ for $n \geq 1$. So we need only consider $\Delta_4(0,n)$ and $\Delta_4(1,n)$.

Since $u_2 = 2$, one can verify the following simple facts:

$$\begin{aligned}\Delta_4(0{,}1) &= 2 = 2u_1 - 2(a^2+1)u_0,\\ \Delta_4(0{,}2) &= -2a^2 + 2 = 2u_2 - 2(a^2+1)u_1,\\ \Delta_4(1{,}1) &= 2a = 2au_1,\\ \Delta_4(1{,}2) &= 4a = 2au_2.\end{aligned}$$

By Theorem 3.2, for $n \geq 1$,

$$\Delta_4(r, n+2) = 2\Delta_4(r, n+1) - (a^2+1)\Delta_4(r,n).$$

Then we can prove the theorem by induction on n. □

THEOREM 5.2. *Let* $p \nmid a(a^4-1)$ *be an odd prime,* $\{u_n\}_{n\geq 0}$ *as in Theorem* 5.1, *and* $K_{p,4,r}(a)$ *as in Definition* 2.1. *Then we have*

$$\begin{aligned}\frac{u_{p-1}}{p} \equiv{} & \frac{2}{a(a^2+1)}(aK_{p,4,0}(a) - K_{p,4,1}(a)) + \frac{2}{a^2+1}q_p(a)\\ & + \frac{a^2-1}{2a(a^2+1)}(q_p(a+1) - q_p(a-1)) \pmod{p},\ \textit{if}\ p \equiv 1 \pmod{4};\end{aligned}$$

$$\begin{aligned}\frac{u_{p+1}}{p} \equiv{} & -\frac{2}{a}(aK_{p,4,0}(a) + K_{p,4,1}(a)) - \frac{(a+1)^2}{2a}q_p(a+1)\\ & + \frac{(a-1)^2}{2a}q_p(a-1) \pmod{p},\ \textit{if}\ p \equiv 3 \pmod{4}.\end{aligned}$$

Proof. Since $2^2 - 4(a^2+1) = -4a^2$, by Lemma 2.2, we have

$$p \mid u_{p-\left(\frac{-1}{p}\right)}.$$

By Theorem 5.1, we have

$$\Delta_4(1,p) - a\Delta_4(0,p) = 2a(a^2+1)u_{p-1}, \tag{4}$$

and

$$\Delta_4(1,p) + a\Delta_4(0,p) = 2au_{p+1}. \tag{5}$$

Then, by Lemma 2.3(3), if $p \equiv 1 \pmod 4)$, we have

$$\Delta_4(0,p) \equiv 4 - 4pK_{p,4,0}(a) - (1+a)^p - (1-a)^p \pmod{p^2},$$
$$\Delta_4(1,p) \equiv 4a^p - 4pK_{p,4,1}(a) - (1+a)^p + (1-a)^p \pmod{p^2}.$$

Thus, by Eq.(4), we have

$$\begin{aligned}2a(a^2+1)u_{p-1} \equiv{}& 4p(aK_{p,4,0}(a) - K_{p,4,1}(a)) + 4a(a^{p-1}-1)\\ &+ (a^2-1)\left[(a+1)^{p-1} - (a-1)^{p-1}\right] \pmod{p^2}.\end{aligned}$$

Hence

$$\begin{aligned}\frac{u_{p-1}}{p} \equiv{}& \frac{2}{a(a^2+1)}(aK_{p,4,0}(a) - K_{p,4,1}(a)) + \frac{2}{a^2+1}q_p(a)\\ &+ \frac{a^2-1}{2a(a^2+1)}(q_p(a+1) - q_p(a-1)) \pmod p.\end{aligned}$$

If $p \equiv 3 \pmod 4$, by Lemma 2.3(3), we have

$$\Delta_4(0,p) \equiv 4 - 4pK_{p,4,0}(a) - (1+a)^p - (1-a)^p \pmod{p^2},$$

$$\Delta_4(1,p) \equiv -4pK_{p,4,1}(a) - (1+a)^p + (1-a)^p \pmod{p^2}.$$

Thus, by Eq.(5), we have

$$\begin{aligned}2au_{p+1} \equiv{}& -4p(aK_{p,4,0}(a) + K_{p,4,1}(a)) - (a+1)^2((a+1)^{p-1}-1)\\ &+ (a-1)^2((a-1)^{p-1}-1) \pmod{p^2}.\end{aligned}$$

Hence

$$\frac{u_{p+1}}{p} \equiv -\frac{2}{a}(aK_{p,4,0}(a) + K_{p,4,1}(a)) - \frac{(a+1)^2}{2a} q_p(a+1) + \frac{(a-1)^2}{2a} q_p(a-1) \pmod p. \qquad \square$$

Given a value of a, by Theorem 5.2, we can obtain a concrete congruence for a specific Lucas quotient. We provide one such example.

COROLLARY 5.5. *Let $p > 5$ be an odd prime, and $\{u_n\}_{n\geq 0}$ be the Lucas sequence defined as*

$$u_0 = 0,\ u_1 = 1,\ u_{n+1} = 2u_n - 5u_{n-1} \text{ for } n \geq 1.$$

Then, if $p \equiv 1 \pmod 4$,

$$\frac{u_{p-1}}{p} \equiv \frac{1}{10}\sum_{k=1}^{\frac{p-1}{4}} \frac{16^k}{k} + \frac{1}{40}\sum_{k=1}^{\frac{p-1}{4}} \frac{16^k}{4k-3} + \frac{2}{5}q_p(2) + \frac{3}{20}q_p(3) \pmod p;$$

if $p \equiv 3 \pmod 4$,

$$\frac{u_{p+1}}{p} \equiv \frac{1}{8}\sum_{k=1}^{\frac{p+1}{4}} \frac{16^k}{4k-3} - \frac{1}{2}\sum_{k=1}^{\frac{p-3}{4}} \frac{16^k}{k} - \frac{9}{4}q_p(3) \pmod p.$$

PROOF. Set $a = -2$ in Theorem 5.2. $\square$

5.2. Further Results

In Theorem 5.2, when we express the Lucas quotient, it involves two K's, i.e., two sums. The following theorem can reduce the Lucas quotient to one sum.

THEOREM 5.3. *With notation as in Theorem 5.2. Let $p \nmid a(a^4-1)$ be an odd prime. We have: if $p \equiv 1 \pmod 4$, then*

$$\begin{aligned}\frac{u_{p-1}}{p} &\equiv \frac{1}{2a^2}\left(\sum_{k=1}^{\frac{p-1}{4}} \frac{a^{4k}}{k} + (1+a)q_p(1+a) + (1-a)q_p(1-a) + q_p(a^2+1)\right)\\ &\equiv \frac{1}{2a}\left(-4\sum_{k=1}^{\frac{p-1}{4}} \frac{a^{4k-1}}{4k-1} + (1+a)q_p(1+a) - (1-a)q_p(1-a)\right)\\ &\quad - \frac{1}{2}q_p(a^2+1) \pmod p;\end{aligned}$$

if $p \equiv 3 \pmod 4$, *then*

$$\frac{u_{p+1}}{p} \equiv -\frac{a^2+1}{2a^2}\left(\sum_{k=1}^{\frac{p-3}{4}}\frac{a^{4k}}{k} + (1+a)q_p(1+a) + (1-a)q_p(1-a) + q_p(a^2+1)\right)$$

$$\equiv \frac{a^2+1}{2a}\left(4\sum_{k=1}^{\frac{p+1}{4}}\frac{a^{4k-3}}{4k-3} - (1+a)q_p(1+a) + (1-a)q_p(1-a)\right)$$

$$+\frac{a^2+1}{2}q_p(a^2+1) \pmod p.$$

Proof. Let $\{v_n\}_{n\geq 0}$ be the Lucas sequence defined as

$$v_0 = 2, v_1 = 2, v_{n+1} = 2v_n - (a^2+1)v_{n-1} \text{ for } n \geq 1.$$

By Lemmas 2.1 and 2.2, we have $v_p = 2u_p - 2(a^2+1)u_{p-1} = 2u_{p+1} - 2u_p \equiv 2 \pmod p$. By Theorem 5.1 and Lemma 2.1, we have $\Delta_4(0,p) = 2u_{p+1} - 2u_p = v_p$. By Lemma 2.3, we have

$$\Delta_4(0,p) = 4\begin{bmatrix} p \\ 0 \end{bmatrix}_4 (a) - (1+a)^p - (1-a)^p$$
$$\equiv 4 - 4pK_{p,4,0}(a) - (1+a)^p - (1-a)^p \pmod{p^2}.$$

Thus

$$\frac{v_p-2}{p} \equiv -\sum_{k=1}^{[\frac{p}{4}]}\frac{a^{4k}}{k} - (1+a)q_p(1+a) - (1-a)q_p(1-a) \pmod p. \tag{6}$$

If $p \equiv 1 \pmod 4$, by Theorem 5.1 and Lemma 2.3 we have $-2au_p = \Delta_4(3,p)$ $= 4\begin{bmatrix} p \\ 3 \end{bmatrix}_4 (a) - (1+a)^p + (1-a)^p \equiv -4pK_{p,4,3}(a) - (1+a)^p + (1-a)^p \pmod{p^2}$. So we have

$$\frac{u_p-1}{p} \equiv \frac{1}{2a}\left(4K_{p,4,3}(a) + (1+a)q_p(1+a) - (1-a)q_p(1-a)\right) \pmod p. \tag{7}$$

By Lemma 2.1, we have $u_{p-1} = \frac{1}{-4a^2}(2v_p - 2v_{p-1}) = \frac{1}{2a^2}(v_{p-1} - v_p)$ and $u_{p-1} = u_p - \frac{1}{2}v_{p-1}$. By Lemma 2.5 and Eq.(6), we have

$$\begin{aligned}\frac{u_{p-1}}{p} &= \frac{1}{2a^2}\left(\frac{v_{p-1}-2}{p} - \frac{v_p-2}{p}\right)\\ &\equiv \frac{1}{2a^2}\left(\sum_{k=1}^{\frac{p-1}{4}}\frac{a^{4k}}{k} + (1+a)q_p(1+a) + (1-a)q_p(1-a) + q_p(a^2+1)\right) \pmod p.\end{aligned}$$

Similarly, by Lemma 2.5 and Eq.(7), we have

$$\begin{aligned}\frac{u_{p-1}}{p} &= \frac{u_p-1}{p} - \frac{1}{2}\frac{v_{p-1}-2}{p}\\ &\equiv \frac{1}{2a}\left(-4\sum_{k=1}^{\frac{p-1}{4}}\frac{a^{4k-1}}{4k-1} + (1+a)q_p(1+a) - (1-a)q_p(1-a)\right)\\ &\quad - \frac{1}{2}q_p(a^2+1) \pmod p.\end{aligned}$$

If $p \equiv 3 \pmod 4$, by Theorem 5.1 and Lemma 2.3 we have $2au_p = \Delta_4(1,p) = 4\begin{bmatrix} p \\ 1 \end{bmatrix}_4 (a) - (1+a)^p + (1-a)^p \equiv -4pK_{p,4,1}(a) - (1+a)^p + (1-a)^p \pmod{p^2}$. So we have

$$\frac{u_p+1}{p} \equiv \frac{1}{2a}(-4K_{p,4,1}(a) - (1+a)q_p(1+a) + (1-a)q_p(1-a)) \pmod p. \qquad (8)$$

By Lemma 2.1, we have $u_{p+1} = \frac{1}{-4a^2}(2v_{p+1} - 2(a^2+1)v_p) = \frac{1}{2a^2}((a^2+1)v_p - v_{p+1})$ and $u_{p+1} = (a^2+1)u_p + \frac{1}{2}v_{p+1}$. By Lemma 2.5 and Eq.(6), we have

$$\begin{aligned}\frac{u_{p+1}}{p} &= \frac{1}{2a^2}\left((a^2+1)\frac{v_p-2}{p} - \frac{v_{p+1}-2(a^2+1)}{p}\right)\\ &\equiv -\frac{a^2+1}{2a^2}\left(\sum_{k=1}^{\frac{p-3}{4}}\frac{a^{4k}}{k} + (1+a)q_p(1+a) + (1-a)q_p(1-a)\right)\\ &\quad -\frac{a^2+1}{2a^2}q_p(a^2+1) \pmod p.\end{aligned}$$

Similarly, by Lemma 2.5 and Eq.(8), we have

$$\begin{aligned}\frac{u_{p+1}}{p} =&(a^2+1)\frac{u_p+1}{p}+\frac{1}{2}\frac{v_{p+1}-2(a^2+1)}{p}\\ \equiv&\frac{a^2+1}{2a}\left(4\sum_{k=1}^{\frac{p+1}{4}}\frac{a^{4k-3}}{4k-3}-(1+a)q_p(1+a)+(1-a)q_p(1-a)\right)\\ &+\frac{a^2+1}{2}q_p(a^2+1) \pmod p.\end{aligned}$$ □

Given a value of a, by Theorem 5.3, we can obtain a concrete congruence for a specific Lucas quotient. We provide one such example.

Corollary 5.6. *Let $p>5$ be an odd prime, and $\{u_n\}_{n\geq 0}$ be the Lucas sequence defined as*

$$u_0=0, u_1=1, u_{n+1}=2u_n-5u_{n-1} \text{ for } n\geq 1.$$

Then we have: *if* $p\equiv 1 \pmod 4$,

$$\begin{aligned}\frac{u_{p-1}}{p} \equiv&\frac{1}{8}\sum_{k=1}^{\frac{p-1}{4}}\frac{16^k}{k}+\frac{3}{8}q_p(3)+\frac{1}{8}q_p(5)\\ \equiv&-\frac{1}{2}\sum_{k=1}^{\frac{p-1}{4}}\frac{16^k}{4k-1}+\frac{3}{4}q_p(3)-\frac{1}{2}q_p(5) \pmod p;\end{aligned}$$

if $p\equiv 3 \pmod 4$,

$$\begin{aligned}\frac{u_{p+1}}{p} \equiv&-\frac{5}{8}\sum_{k=1}^{\frac{p-3}{4}}\frac{16^k}{k}-\frac{15}{8}q_p(3)-\frac{5}{8}q_p(5)\\ \equiv&\frac{5}{8}\sum_{k=1}^{\frac{p+1}{4}}\frac{16^k}{4k-3}-\frac{15}{4}q_p(3)+\frac{5}{2}q_p(5) \pmod p.\end{aligned}$$

Proof. Set $a=-2$ in Theorem 5.3. □

6. $\Delta_6(r,n)$

In this section, we consider the calculation of $\Delta_6(r,n)$.

THEOREM 6.1. *Let $\{V_n\}_{n\geq 0}$ be the Lucas sequences defined as*

$$V_0=2,\ V_1=a+2,\ V_{n+1}=(a+2)V_n-(a^2+a+1)V_{n-1}\ \textit{for}\ n\geq 1.$$

Let $\{v_n\}_{n\geq 0}$ be the Lucas sequences defined as

$$v_0=2,\ v_1=2-a,\ v_{n+1}=(2-a)v_n-(a^2-a+1)v_{n-1}\ \textit{for}\ n\geq 1.$$

Then we have, for $n\geq 1$,

$$\begin{aligned}
\Delta_6(0,n)&=V_n+v_n,\\
\Delta_6(1,n)&=-\frac{1}{a}V_n+\frac{a^2+a+1}{a}V_{n-1}-\frac{1}{a}v_n+\frac{a^2-a+1}{a}v_{n-1},\\
\Delta_6(2,n)&=-\frac{a+1}{a}V_n+\frac{a^2+a+1}{a}V_{n-1}-\frac{a-1}{a}v_n-\frac{a^2-a+1}{a}v_{n-1},\\
\Delta_6(3,n)&=-V_n+v_n,\\
\Delta_6(4,n)&=\frac{1}{a}V_n-\frac{a^2+a+1}{a}V_{n-1}-\frac{1}{a}v_n+\frac{a^2-a+1}{a}v_{n-1},\\
\Delta_6(5,n)&=\frac{a+1}{a}V_n-\frac{a^2+a+1}{a}V_{n-1}-\frac{a-1}{a}v_n-\frac{a^2-a+1}{a}v_{n-1}.
\end{aligned}$$

PROOF. Since $V_2=-a^2+2a+2$, $v_2=-a^2-2a+2$, $V_3=-2a^3-3a^2+3a+2$, $v_3=2a^3-3a^2-3a+2$, $V_4=-a^4-8a^3-6a^2+4a+2$, $v_4=-a^4+8a^3-6a^2-4a+2$, one can verify the following simple facts:

$$\begin{aligned}
\Delta_6(0,1)&=4=V_1+v_1,\\
\Delta_6(0,2)&=-2a^2+4=V_2+v_2,\\
\Delta_6(0,3)&=-6a^2+4=V_3+v_3,\\
\Delta_6(0,4)&=-2a^4-12a^2+4=V_4+v_4;\\
\Delta_6(1,1)&=4a=-\frac{1}{a}V_1+\frac{a^2+a+1}{a}V_0-\frac{1}{a}v_1+\frac{a^2-a+1}{a}v_0,\\
\Delta_6(1,2)&=8a=-\frac{1}{a}V_2+\frac{a^2+a+1}{a}V_1-\frac{1}{a}v_2+\frac{a^2-a+1}{a}v_1,\\
\Delta_6(1,3)&=-2a^3+12a=-\frac{1}{a}V_3+\frac{a^2+a+1}{a}V_2-\frac{1}{a}v_3+\frac{a^2-a+1}{a}v_2,\\
\Delta_6(1,4)&=-8a^3+16a=-\frac{1}{a}V_4+\frac{a^2+a+1}{a}V_3-\frac{1}{a}v_4+\frac{a^2-a+1}{a}v_3.
\end{aligned}$$

By Theorem 3.2, we have, for $n \geq 1$,

$$\begin{aligned}\Delta_6(r, n+4) =& 4\Delta_6(r, n+3) - (a^2+6)\Delta_6(r, n+2) \\ &+ (2a^2+4)\Delta_6(r, n+1) - (a^4+a^2+1)\Delta_6(r, n).\end{aligned}$$

Thus we can prove the theorem by induction on n for $r = 0,1$.

It is easy to see that $\Delta_6(r, n) + \Delta_6(r+2, n) + \Delta_6(r+4, n) = 3\Delta_2(r, n) = 0$ and $\Delta_6(r, n) + \Delta_6(r+3, n) = 2\Delta_3(r, n)$. Hence $\Delta_6(2, n) = -\Delta_6(0, n) - \Delta_6(4, n) = -\Delta_6(0, n) + \Delta_6(1, n) - 2\Delta_3(1, n)$. Thus, by Remark 4.1 and the formulae for $\Delta_6(0, n)$ and $\Delta_6(1, n)$, we can obtain the formula for $\Delta_6(2, n)$. Finally, by Remark 4.1 and the formulae for $\Delta_6(0, n)$, $\Delta_6(1, n)$ and $\Delta_6(2, n)$, we can obtain the formulae for $\Delta_6(3, n)$, $\Delta_6(4, n)$ and $\Delta_6(5, n)$. □

Note that, for $m = 6$, since the Lucas sequences in Theorem 6.1 have already appeared in Theorem 4.1, we can not obtain any new Lucas quotient.

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Bibliography

1. ***R. Crandall and C. Pomerance,*** *Prime Numbers, A Computational Perspective* (second edition), Springer, New York, 2005.
2. ***Y. Deng and Y. Pan,*** *The Sum of Binomial Coefficients and Integer Factorization*, Integers, **16** (2016), Paper No. A42, 18 p.
3. ***R. L. Rivest, A. Shamir and L. Adleman,*** *A Method for Obtaining Digital Signatures and Public-key Cryptosystems*, Comm. ACM, **21** (1978), 120–126.
4. ***Z.-H. Sun,*** *Combinatorial Sum* $\sum_{k=0,\ k\equiv r \pmod m}^{n} \binom{n}{k}$ *and its Applications in Number Theory* (I), Nanjing Daxue Xuebao Shuxue Bannian Kan, **9** (1992), 227–240.
5. ***Z.-H. Sun,*** *Combinatorial Sum* $\sum_{k=0,k\equiv r \pmod m}^{n} \binom{n}{k}$ *and its Applications in Number Theory* (II), Nanjing Daxue Xuebao Shuxue Bannian Kan, **10** (1993), 105–118.
6. ***Z.-H. Sun,*** *Combinatorial Sum* $\sum_{k=0,k\equiv r \pmod m}^{n} \binom{n}{k}$ *and its Applications in Number Theory* (III), Nanjing Daxue Xuebao Shuxue Bannian Kan, **12** (1995), 90–102.
7. ***Z.-H. Sun and Z.-W. Sun,*** *Fibonacci Numbers and Fermat's Last Theorem*, Acta Arith., **60** (1992), 371–388.

8. ***Z.-W. Sun,*** *A Congruence for Primes*, Proc. Amer. Math. Soc., **123**(1995), 1341–1346.
9. ***Z.-W. Sun,*** *On the Sum* $\sum_{k\equiv r \pmod m} \binom{n}{k}$ *and Related Congruences*, Israel J. Math., **128** (2002), 135–156.
10. ***D. D. Wall,*** *Fibonacci Series Modulo* m, Amer. Math. Monthly, **67** (1960), 525–532.
11. ***H. C. Williams,*** *A Note on the Fibonacci Quotient* $F_{p-\varepsilon}/p$, Canad. Math. Bull, **25** (1982), 366–370.

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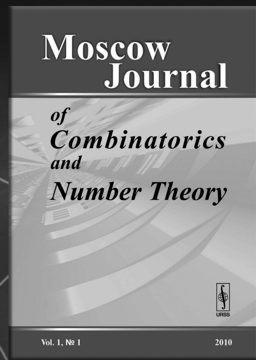

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