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Vanishing theorems for tensor powers of an ample vector bundle

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Summary. Let E be a holomorphic vector bundle of rank r on a compact complex manifold X of dimension n. It is shown that the cohomology groups $H^{p,q}(X, E^{\otimes k} \otimes (\det E)^l)$ vanish if E is ample and $p+q \geqq n+1$, $l \geqq n-p+r-1$. The proof rests on the well-known fact that every tensor power $E^{\otimes k}$ splits into irreducible representations of $\mathrm{Gl}(E)$. By Bott's theory, each component is canonically isomorphic to the direct image on X of a homogeneous line bundle over a flag manifold of E. The proof is then reduced to the Kodaira-Akizuki-Nakano vanishing theorem for line bundles by means of the Leray spectral sequence, using backward induction on p. We also obtain a generalization of Le Potier's isomorphism theorem and a counterexample to a vanishing conjecture of Sommese.

0. Statement of results

Many problems and results of contemporary algebraic geometry involve vanishing theorems for holomorphic vector bundles. Furthermore, tensor powers of such bundles are often introduced by natural geometric constructions. The aim of this work is to prove a rather general vanishing theorem for cohomology groups of tensor powers of a holomorphic vector bundle.

Let X be a complex compact n-dimensional manifold and E a holomorphic vector bundle of rank r on X. If E is ample and $r>1$, only very few general and optimal vanishing results are available for the Dolbeault cohomology groups $H^{p,q}$ of tensor powers of E. For example, the famous Le Potier vanishing theorem [13]:

$$E \text{ ample} \Rightarrow H^{p,q}(X, E) = 0 \qquad \text{for } p+q \geqq n+r$$

does not extend to symmetric powers $S^k E$, even when $p=n$ and $q=n-2$ (cf. [11]). Nevertheless, we will see that the vanishing property is true for tensor powers involving a sufficiently large power of $\det E$. In all the sequel, we let L be a holomorphic line bundle on X and we assume:

(0.1) **Hypothesis.** *E is ample and L semi-ample, or E is semi-ample and L ample.*

The precise definitions concerning ampleness are given in §1. Under this hypothesis, we prove the following two results.

(0.2) **Theorem.** *Let us denote by $\Gamma^a E$ the irreducible tensor power representation of $\mathrm{Gl}(E)$ of highest weight $a\in\mathbb{Z}^r$. If $h\in\{1, \dots, r-1\}$ and $a_1 \geqq a_2 \geqq \dots \geqq a_h > a_{h+1} = \dots = a_r = 0$, then*

$$H^{n,q}(X, \Gamma^a E \otimes (\det E)^h \otimes L) = 0 \quad \textit{for } q \geqq 1.$$

Since $S^k E = \Gamma^{(k,0,\dots,0)} E$, the special case $h=1$ coincides with Griffiths' vanishing theorem [8]:

$$H^{n,q}(X, S^k E \otimes \det E \otimes L) = 0 \quad \text{for } q \geqq 1.$$

(0.3) **Theorem.** *For all integers $p+q \geqq n+1$, $k \geqq 1$, $l \geqq n-p+r-1$ then $H^{p,q}(X, E^{\otimes k} \otimes (\det E)^l \otimes L) = 0$.*

For $p=n$ and arbitrary $r, k_0 \geqq 2$, Peternell-Le Potier and Schneider [11] have constructed an example of an ample vector bundle E of rank r on a manifold X of dimension $n=2r$ such that

$$H^{n,n-2}(X, S^k E) \neq 0, \quad 2 \leqq k \leqq k_0. \tag{0.4}$$

This result shows that $\det E$ cannot be omitted in Theorem 0.2 when $h=1$. More generally, the following example shows that the exponent h is optimal.

Example. Let $X = \mathrm{Gr}(V)$ be the Grassmannian of subspaces of codimension r of a vector space V of dimension d, and E the tautological quotient vector bundle of rank r on X (then E is spanned and $L = \det E$ very ample). Let $h \in \{1, \dots, r-1\}$ and $a\in\mathbb{Z}^r$, $\beta\in\mathbb{Z}^d$ be such that

$$a_1 \geqq \dots \geqq a_h \geqq d-r, \qquad a_{h+1} = \dots = a_r = 0,$$
$$\beta = (a_1 - d + r, \dots, a_h - d + r, 0, \dots, 0).$$

Set $n = \dim X = r(d-r)$, $q = (r-h)(d-r)$. Then

$$H^{n,q}(X, \Gamma^a E \otimes (\det E)^h) = \Gamma^\beta V \otimes (\det V)^h \neq 0. \tag{0.5}$$

Our approach is based on three well-known facts. First, every tensor power $E^{\otimes k}$ splits into irreducible representations $\Gamma^a E$ of the linear group $\mathrm{Gl}(E)$ (cf. (2.16)). Secondly, every irreducible tensor bundle $\Gamma^a E$ appears in a natural way as the direct image on X of an *ample line bundle* on a suitable flag manifold of E. This follows from Bott's theory of homogeneous vector bundles [3]. The third fact is the isomorphism theorem of Le Potier [13], which relates the cohomology groups of E on X to those of the line bundle $O_E(1)$ on $P(E^*)$. In §3, we generalize this isomorphism to the case of arbitrary flag bundles associated to E; theorem (0.2) is an immediate consequence of our isomorphism.

The proof of Theorem (0.3) rests on a generalization of the Borel-Le Portier spectral sequence, but we have avoided to make it explicit at this point in order to simplify the exposition. The main argument is a backward induction on p, based on the usual Leray spectral sequence and on the Kodaira-Azizuki-

Nakano vanishing theorem for line bundles. A by-product of these methods is the following isomorphism result, already contained in the standard Borel-Le Potier spectral sequence, but which seems to have been overlooked.

(0.6) **Theorem.** *For every vector bundle E and every line bundle L one can define a canonical morphism*

$$H^{p,q}(X, \Lambda^2 E \otimes L) \to H^{p+1,q+1}(X, S^2 E \otimes L).$$

Under hypothesis 0.1, this morphism is one-to-one for $p+q \geqq n+r-1$ *and surjective for* $p+q=n+r-2$.

Combining Theorem (0.6) and Example (0.4), we get $H^{n-1,n-3}(X, \Lambda^2 E) \neq 0$. This shows that Sommese's conjecture ([15], conjecture (4.2))

$$E \text{ ample} \Rightarrow H^{p,q}(X, \Lambda^k E) = 0 \qquad \text{for } p+q \geqq n+r-k+1$$

is false for $n=2r \geqq 6$.

The following related problem is interesting, but its complete solution certainly requires a better understanding of the Borel-Le Potier spectral sequence for flag bundles.

Problem. *Given any dominant weight* $a \in \mathbb{Z}^r$ *with* $a_r = 0$ *and integers p, q such that* $p+q \geqq n+1$, *determine the smallest exponent* $l_0 = l_0(n, p, q, r, a)$ *such that* $H^{p,q}(X, \Gamma^a E \otimes (\det E)^l \otimes L) = 0$ *for* $l \geqq l_0$.

We show in §5 that if the Borel-Le Potier spectral sequence degenerates at the E_2 level, it is always sufficient to take $l \geqq r-1+\min\{n-p, n-q\}$. In that case, Theorem (0.6) appears also as a special case of a fairly general exact sequence. The E_2-degeneracy of the Borel-Le Potier spectral sequence is thus an important feature which would be interesting to investigate.

Some of the above results have been announced in the note [4] and corresponding detailed proofs are given in [5]. However, the method of [5] is based on differential geometry and leads to results which overlap the present ones only in part.

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1. Basic definitions and tools

We recall here a few basic facts and definitions which will be used repeatedly in the sequel.

(1.1) *Ample vector bundles* (compare with Hartshorne [9]). A vector bundle E on X is said to be *spanned* if the canonical map $H^0(X, E) \to E_x$ is onto for every $x \in X$, and *semi-ample* if the symmetric powers $S^k E$ are spanned for $k \geqq k_0$ large enough.

E is said to be *very ample* if the canonical maps

$$H^0(X, E) \to E_x \oplus E_y, \qquad \forall x \neq y \in X,$$
$$H^0(X, E) \to \mathcal{O}(E) \otimes \mathcal{O}_x / \mathcal{M}_x^2, \qquad \forall x \in X,$$

are onto, $\mathcal{M}_x$ denoting the maximal ideal at $x \in X$. If E is very ample, then the holomorphic map from X to the Grassmannian of r-codimensional subspaces of $V = H^0(X, E)$ given by

$$\Phi: X \to \mathrm{Gr}(V), \qquad x \mapsto W_x = \{\sigma \in V; \sigma(x) = 0\}$$

is an embedding, and E is the pull-back by Φ of the canonical quotient vector bundle of rank r on $\mathrm{Gr}(V)$. The embedding condition is in fact equivalent to E being very ample if $r = 1$, but weaker if $r \geqq 2$.

At last, E is said to be *ample* if the symmetric powers $S^k E$ are very ample for $k \geqq k_0$ large enough. Denoting $O_E(1)$ the canonical line bundle on $Y = P(E^*)$ associated to E and $\pi: Y \to X$ the projection, it is well-known that $\pi_* O_E(k) = S^k E$. One gets then easily

$$E \text{ spanned on } X \Leftrightarrow O_E(1) \text{ spanned on } Y,$$
$$E \text{ (semi-)ample on } X \Leftrightarrow O_E(1) \text{ (semi-)ample on } Y.$$

Moreover, for any line bundle L on X, ampleness is equivalent to the existence of a smooth hermitian metric on the fibers of L with positive curvature from $ic(L)$. Since $S^k(E \otimes L) = S^k E \otimes L^k$, it is clear that

$$E, L \text{ semi-ample} \Rightarrow E \otimes L \text{ semi-ample},$$
$$E, L \text{ spanned, one of them very ample} \Rightarrow E \otimes L \text{ very ample},$$
$$E, L \text{ semi-ample, one of them ample} \Rightarrow E \otimes L \text{ ample}.$$

(1.2) *Kodaira-Akizuki-Nakano vanishing theorem* [1]. If L is an ample (or positive) line bundle on X, then

$$H^{p,q}(X, L) = H^q(X, \Omega_X^p \otimes L) = 0 \qquad \text{for } p + q \geqq n + 1.$$

(1.3) *Leray spectral sequence* (cf. Godement [7]). Let $\pi: Y \to X$ be a continuous map between topological spaces' $\mathcal{S}$ a sheaf of abelian groups on Y, and $R^q \pi_* \mathcal{S}$ the direct image sheaves of $\mathcal{S}$ on X. Then there exists a spectral sequence such that

$$E_2^{p,q} = H^p(X, R^q \pi_* \mathcal{S})$$

and such that the limit term $E_\infty^{p,q-p}$ is the p-graded module associated to a decreasing filtration of $H^q(Y, \mathcal{S})$.

(1.4) *Cohomology of a filtered sheaf.* We will need also the following elementary result. Let $\mathcal{F}$ be a sheaf of abelian groups on X and

$$\mathcal{F} = \mathcal{F}^0 \supset \mathcal{F}^1 \supset \ldots \supset \mathcal{F}^p \supset \ldots \supset \mathcal{F}^N = 0$$

be a filtration of $\mathcal{F}$ such that the graded sheaf $\oplus \mathcal{G}^p$, $\mathcal{G}^p = \mathcal{F}^p / \mathcal{F}^{p+1}$, satisfies $H^q(X, \mathcal{G}^p) = 0$ for $q \geqq q_0$. Then $H^q(X, \mathcal{F}) = 0$ for $q \geqq q_0$.

Indeed, it is immediately verified by induction on $p \geqq 1$ that $H^q(X, \mathcal{F}/\mathcal{F}^p)$ vanishes for $q \geqq q_0$, using the cohomology long exact sequence associated to

$$0 \to \mathcal{G}^p \to \mathcal{F}/\mathcal{F}^{p+1} \to \mathcal{F}/\mathcal{F}^p \to 0.$$

2. Homogeneous line bundles on flag manifolds and irreducible representations of the linear group

The aim of this section is to settle notations and to recall a few basic results on homogenous line bundles on flag manifolds (cf. Borel-Weil [2] and R. Bott [3]).

Let B_r be the Borel subgroup of $\mathrm{Gl}_r = \mathrm{Gl}(\mathbb{C}^r)$ or lower triangular matrices, $U_r \subset B_r$ the subgroup of unipotent matrices, and T^r the complex torus $(\mathbb{C}^*)^r$ of diagonal matrices. Let V be a complex vector space of dimension r. We denote by $M(V)$ the manifold of complete flags

$$V = V_0 \supset \ldots \supset V_r = \{0\}, \qquad \operatorname{codim}_{\mathbb{C}} V_\lambda = \lambda.$$

To every linear isomorphism $\zeta \in \operatorname{Isom}(\mathbb{C}^r, V): (u_1, \ldots, u_r) \mapsto \sum_{1 \leq \lambda \leq r} u_\lambda \zeta_\lambda$, one can associate the flag $[\zeta] \in M(V)$ defined by $V_\lambda = \operatorname{Vect}(\zeta_{\lambda+1}, \ldots, \zeta_r)$, $1 \leqq \lambda \leqq r$. This leads to the identification

$$M(V) = \operatorname{Isom}(\mathbb{C}^r, V)/B_r$$

where B_r acts on the right side. We denote simply by V_λ the tautological vector bundle of rank $r - \lambda$ on $M(V)$, and we consider the canonical quotient line bundles

$$\begin{aligned} Q_\lambda &= V_{\lambda-1}/V_\lambda, & 1 \leqq \lambda \leqq r, \\ Q^a &= Q_1^{a_1} \otimes \ldots \otimes Q_r^{a_r}, & a = (a_1, \ldots, a_r) \in \mathbb{Z}^r. \end{aligned} \tag{2.1}$$

The linear group $\mathrm{Gl}(V)$ acts on $M(V)$ on the left, and there exist natural equivariant left actions of $\mathrm{Gl}(V)$ on all bundles $V_\lambda, Q_\lambda, Q^a$. If $\rho: B_r \to \mathrm{Gl}(E)$ is a representation of B_r, we may associate to ρ the manifold

$$E_{V,\rho} = \operatorname{Isom}(\mathbb{C}^r, V) \times_{B_r} E = \operatorname{Isom}(\mathbb{C}^r, V) \times E/\sim$$

where $\sim$ denotes the equivalence relation $(\zeta g, u) \sim (\zeta, \rho(g)\, u)$, $g \in B_r$. Then $E_{V,\rho}$ is a $\mathrm{Gl}(V)$-equivariant bundle over $M(V)$, and all such bundles are obtained in this way. It is clear that Q^a is isomorphic to the line bundle $\mathbb{C}_{V,\underline{a}}$ arising from the 1-dimensional representation of weight $\underline{a}$:

$$\underline{a}: T^r = B_r/U_r \to \mathbb{C}^*, \qquad (t_1 \ldots t_r) \mapsto t_1^{a_1} \ldots t_r^{a_r};$$

the isomorphism $\mathbb{C}_{V,\underline{a}} \xrightarrow{\sim} Q^a$ is induced by the map

$$\operatorname{Isom}(\mathbb{C}^r, V) \times \mathbb{C} \ni (\zeta, u) \mapsto u \tilde{\zeta}_1^{a_1} \otimes \ldots \otimes \tilde{\zeta}_r^{a_r} \in Q^a, \qquad \tilde{\zeta}_\lambda = \zeta_\lambda \bmod V_\lambda.$$

We compute now the tangent and cotangent vector bundles of $M(V)$. The action of $\mathrm{Gl}(V)$ on $M(V)$ yields

$$TM(V) = \operatorname{Hom}(V, V)/W \tag{2.2}$$

where W is the subbundle of endomorphisms $g \in \operatorname{Hom}(V, V)$ such that $g(V_\lambda) \subset V_\lambda$, $1 \leqq \lambda \leqq r$. Using the self-duality of $\operatorname{Hom}(V, V)$ given by the Killing form $(g_1, g_2) \mapsto \operatorname{tr}(g_1 g_2)$, we find

$$\text{(2.3)}\qquad \begin{aligned} &T^* M(V) = (\mathrm{Hom}(V, V)/W)^* = W^\perp \\ &W^\perp = \{g \in \mathrm{Hom}(V, V);\quad g(V_{\lambda-1}) \subset V_\lambda,\quad 1 \leqq \lambda \leqq r\}. \end{aligned}$$

If ad denotes the adjoint representation of B_r on the Lie algebras $\mathfrak{gl}_r$, $\mathfrak{B}_r$, $\mathfrak{U}_r$, then

$$\text{(2.4)}\qquad TM(V) = (\mathfrak{gl}_r/\mathfrak{B}_r)_{V,\mathrm{ad}}, \qquad T^* M(V) = (\mathfrak{U}_r)_{V,\mathrm{ad}}$$

because $W_{[\zeta]} = \zeta \mathfrak{B}_r \zeta^{-1}$, $W_{[\zeta]}^\perp = \zeta \mathfrak{U}_r \zeta^{-1}$ at every point $[\zeta] \in M(V)$. There exists a filtration of $T^* M(V)$ by subbundles of the type

$$\{g \in \mathrm{Hom}(V, V);\quad g(V_\lambda) \subset V_{\mu(\lambda)},\quad 1 \leqq \lambda \leqq r\}$$

in such a way that the corresponding graded bundle is the direct sum of the line bundles $\mathrm{Hom}(Q_\lambda, Q_\mu) = Q_\lambda^{-1} \otimes Q_\mu$, $\lambda < \mu$; their tensor product is thus isomorphic to the canonical line bundle $K_{M(V)} = \det(T^* M(V))$:

$$\text{(2.5)}\qquad K_{M(V)} = Q_1^{1-r} \otimes \ldots \otimes Q_\lambda^{2\lambda - r - 1} \otimes \ldots \otimes Q_r^{r-1} = Q^c$$

where $c = (1-r, \ldots, r-1)$; c will be called the canonical weight of $M(V)$).

Case of incomplete flag manifolds

More generally, given any sequence of integers $s = (s_0, \ldots, s_m)$ such that $0 = s_0 < s_1 < \ldots < s_m = r$, we may consider the manifold $M_s(V)$ of incomplete flags

$$V = V_{s_0} \supset V_{s_1} \supset \cdots \supset V_{s_m} = \{0\}, \qquad \mathrm{codim}_{\mathbb{C}} V_{s_j} = s_j.$$

On $M_s(V)$ we still have tautological vector bundles V_{s_j} of rank $r - s_j$ and line bundles

$$\text{(2.6)}\qquad Q_{s,j} = \det(V_{s,j-1}/V_{s,j}), \qquad 1 \leqq j \leqq m.$$

For any r-tuple $a \in \mathbb{Z}^r$ such that $a_{s_{j-1}+1} = \ldots = a_{s_j}$, $1 \leqq j \leqq m$, we set

$$Q_s^a = Q_{s,1}^{a_{s_1}} \otimes \ldots \otimes Q_{s,m}^{a_{s_m}}.$$

If $\eta\colon M(V) \to M_s(V)$ is the natural projection, then

$$\text{(2.7)}\qquad \eta^* V_{s,j} = V_{s_j}, \qquad \eta^* Q_{s,j} = Q_{s_{j-1}+1} \otimes \ldots \otimes Q_{s_j}, \qquad \eta^* Q_s^a = Q^a.$$

On the other hand, one has the identification

$$M_s(V) = \mathrm{Isom}(\mathbb{C}^r, V)/B_s$$

where B_s is the parabolic subgroup of matrices $(z_{\lambda\mu})$ with $z_{\lambda\mu} = 0$ for all λ, μ such that there exists an integer $j = 1, \ldots, m-1$ with $\lambda \leqq s_j$ and $\mu > s_j$. We define U_s as the unipotent subgroup of lower triangular matrices $(z_{\lambda\mu})$ with $z_{\lambda\mu} = 0$ for all λ, μ such that $s_{j-1} < \lambda \neq \mu \leqq s_j$ for some j (hence $\mathfrak{U}_s = \mathfrak{B}_s^\perp$). In the same way as above, we get

(2.8) $TM_s(V) = \mathrm{Hom}(V, V)/W_s, \quad W_s = \{g; g(V_{s_{j-1}}) \subset V_{s_j}\},$

(2.9) $T^* M_s(V) = W_s^{\perp} = (\mathfrak{U}_s)_{V, \mathrm{ad}},$

(2.10) $K_{M_s(V)} = Q_{s,1}^{s_1 - r} \otimes \ldots \otimes Q_{s,j}^{s_{j-1} + s_j - r} \otimes \ldots \otimes Q_{s,m}^{s_{m-1}} = Q_s^{c(s)}$

where $c(s) = (s_1 - r, \ldots, s_1 - r, \ldots, s_{j-1} + s_j - r, \ldots, s_{j-1} + s_j - r, \ldots)$ is the canonical weight of $M_s(V)$.

(2.11) **Lemma.** *Q_s^a enjoys the following properties:*

(a) *If $a_{s_j} < a_{s_{j+1}}$ for some $j = 1, \ldots, m-1$, then $H^0(M_s(V), Q_s^a) = 0$.*
(b) *Q_s^a is spanned if and only if $a_{s_1} \geqq a_{s_2} \geqq \ldots \geqq a_{s_m}$.*
(c) *Q_s^a is (very) ample if and only if $a_{s_1} > a_{s_2} > \ldots > a_{s_m}$.*

Proof. (a) Let $(V_{s_0}^0 \supset V_{s_1}^0 \supset \ldots \supset V_{s_m}^0) \in M_s(V)$ be an arbitrary flag and F, F' subspaces of V such that

$$V_{s_{j-1}}^0 \supset F \supset V_{s_j}^0 \supset F' \supset V_{s_{j+1}}^0, \qquad \dim F = s_j + 1, \qquad \dim F' = s_j - 1.$$

Let us consider the projective line $P(F/F') \subset M_s(V)$ of flags

$$V_{s_0}^0 \supset \ldots \supset V_{s_{j-1}}^0 \supset V_{s_j} \supset V_{s_{j+1}}^0 \supset V_{s_m}^0$$

such that $F \supset V_{s_j} \supset F'$. Then

$$Q_{s,j \upharpoonright P(F/F')} = \det(V_{s_{j-1}}^0 / V_{s_j}) \simeq F/V_{s_j} \simeq O(1),$$
$$Q_{s,j+1 \upharpoonright P(F/F')} = \det(V_{s_j}/V_{s_{j+1}}^0) \simeq V_{s_j}/F' \simeq O(-1)$$

thus $Q_{s \upharpoonright P(F/F')}^a \simeq O(a_{s_j} - a_{s_{j+1}})$, which implies (a) and the "only if" part of assertions (b), (c).

(b) Since $Q_{s,1} \otimes \ldots \otimes Q_{s,j} = \det(V/V_{s_j})$, we see that this line bundle is a quotient of the trivial bundle $\Lambda^{s_j} V$. Hence

(2.12) $$Q_s^a = \bigotimes_{1 \leqq j \leqq m-1} \det(V/V_{s_j})^{a_{s_j} - a_{s_{j+1}}} \otimes (\det V)^{a_r}$$

is spanned by sections arising from elements of $\bigotimes S^{a_{s_j} - a_{s_{j+1}}}(\Lambda^{s_j} V) \otimes (\det V)^{a_r}$ as soon as $a_{s_1} \geqq \ldots \geqq a_{s_m}$.

(c) If $a_{s_1} > \ldots > a_{s_m}$, it is elementary to verify that the sections of Q_s^a arising from (2.12) suffice to make Q_s^a very ample (this is in fact a generalization of Plücker's imbedding of Grassmannians). □

Cohomology groups of Q^a

It remains now to compute $H^0(M_s(V), Q_s^a) \simeq H^0(M(V), Q^a)$ when $a_1 \geqq \ldots \geqq a_r$. Without loss of generality we may assume that $a_r \geqq 0$, because $Q_1 \otimes \ldots \otimes Q_r = \det V$ is a trivial bundle.

(2.13) **Proposition.** *For all integers $a_1 \geqq a_2 \geqq \ldots \geqq a_r \geqq 0$, there is a canonical isomorphism*

$$H^0(M(V), Q^a) = \Gamma^a V$$

where $\Gamma^a V \subset S^{a_1} V \otimes \ldots \otimes S^{a_r} V$ *is the set of polynomials* $f(\zeta_1^*, \ldots, \zeta_r^*)$ *on* $(V^*)^r$ *which are homogeneous of degree* a_λ *with respect to* ζ_λ^* *and invariant under the left action of* U_r *on* $(V^*)^r = \mathrm{Hom}(V, \mathbb{C}^r)$:

$$f(\zeta_1^*, \ldots, \zeta_{\lambda-1}^*, \zeta_\lambda^* + \zeta_\nu^*, \ldots, \zeta_r^*) = f(\zeta_1^*, \ldots, \zeta_r^*), \qquad \forall \nu < \lambda.$$

Proof. To any section $\sigma \in H^0(M(V), Q^a)$ we associate the holomorphic function f on $\mathrm{Isom}(V, \mathbb{C}^r) \subset (V^*)^r$ defined by

$$f(\zeta_1^*, \ldots, \zeta_r^*) = (\zeta_1^*)^{a_1} \otimes \ldots \otimes (\zeta_r^*)^{a_r} \cdot \sigma([\zeta_1, \ldots, \zeta_r])$$

where $(\zeta_1, \ldots, \zeta_r)$ is the dual basis of $(\zeta_1^*, \ldots, \zeta_r^*)$, and where the linear form induced by ζ_λ^* on $Q_\lambda = V_{\lambda-1}/V_\lambda \simeq \mathbb{C}\zeta_\lambda$ is still denoted ζ_λ^*. Let us observe that f is homogeneous of degree a_λ in ζ_λ^* and locally bounded in a neighborhood of every r -tuple of $(V^*)^r \setminus \mathrm{Isom}(V, \mathbb{C}^r)$ (because $M(V)$ is compact and $a_\lambda \geqq 0$). Therefore f can be extended to a polynomial on all $(V^*)^r$. The invariance of f under U_r is clear. Conversely, such a polynomial f obviously defines a unique section σ on $M(V)$. □

From the definition of $\Gamma^a V$, we see that

(2.14) $$S^k V = \Gamma^{(k, 0, \ldots, 0)} V,$$

(2.15) $$\Lambda^k V = \Gamma^{(1, \ldots, 1, 0, \ldots, 0)} V.$$

Observe also that Proposition 2.13 remains true for arbitrary $a \in \mathbb{Z}^r$ if we set

$\Gamma^a V = \Gamma^{(a_1 - a_r, \ldots, a_{r-1} - a_r, 0)} V \otimes (\det V)^{a_r}$ when a is non-increasing,
$\Gamma^a V = 0$ otherwise.

The weights will be ordered according to their usual partial ordering:

$$\underline{a} \succcurlyeq \underline{b} \quad \text{iff} \quad \sum_{1 \leqq \lambda \leqq \mu} a_\lambda \geqq \sum_{1 \leqq \lambda \leqq \mu} b_\lambda, \qquad 1 \leqq \mu \leqq r.$$

Bott's theorem [3] shows that $\Gamma^a V$ is an irreducible representation of $\mathrm{Gl}(V)$ of highest weight $\underline{a}$; all irreducible representations of $\mathrm{Gl}(V)$ are in fact of this type (cf. Kraft [10]). In particular, since the weights of the action of a maximal torus $T^r \subset \mathrm{Gl}(V)$ on $V^{\otimes k}$ verify $a_1 + \ldots + a_r = k$ and $a_\lambda \geqq 0$, we have a canonical $\mathrm{Gl}(V)$-isomorphism

(2.16) $$V^{\otimes k} = \bigotimes_{\substack{a_1 + \ldots + a_r = k \\ a_1 \geqq \ldots \geqq a_r \geqq 0}} \mu(a, k)\, \Gamma^a V$$

where $\mu(a, k) > 0$ is the multiplicity of the isotypical factor $\Gamma^a V$ in $V^{\otimes k}$.

Bott's formula (cf. also Demazure [6] for a very simple proof) gives in fact the expression of all cohomology groups $H^q(M(V), Q^a)$. A weight $\underline{a}$ is called regular if all the elements of a are distinct, and singular otherwise. We will denote by $a^{\geqq}$ the reordered non-increasing sequence associated to a, by $l(a)$ the number of strict inversions of the order, and we will set

$$\hat{a} = \left(a - \frac{c}{2}\right)^{\geqq} + \frac{c}{2} \in \mathbb{Z}^r;$$

the sequence $\hat{a}$ is non-increasing if and only if $a-\frac{c}{2}$ is regular.

(2.17) **Proposition.** (Bott [3]). *One has*

$$H^q(M(V), Q^a)=\begin{cases} 0 & \text{if } q \neq l\left(a-\frac{c}{2}\right), \\ \Gamma^{\hat{a}} V & \text{if } q=l\left(a-\frac{c}{2}\right). \end{cases}$$

In particular all H^q vanish if and only if $a-\frac{c}{2}$ is singular.

We will be particularly interested by those line bundles Q^a such that the cohomology groups $H^{p,q}$ vanish for all $q \geqq 1$, p being given. The following proposition is one of the main steps in the proof of our results.

(2.18) **Proposition.** *Set $N=\dim M(V)$, $N(s)=\dim M_s(V)$. Then*

(a) *$H^{p,q}(M_s(V), Q_s^a)=0$ for all $p+q \geqq N(s)+1$ as soon as*

$$a_{s_j}-a_{s_{j+1}} \geqq 1, \qquad 1 \leqq j \leqq m-1.$$

(b) *$H^q(M_s(V), Q_s^a)=0$ for all $q \geqq 1$ as soon as*

$$a_{s_j}-a_{s_{j+1}} \geqq 1-(s_{j+1}-s_{j-1}), \qquad 1 \leqq j \leqq m-1.$$

(c) *In general, $H^{p,q}(M_s(V), Q_s^{a-c(s)})$ is isomorphic to a direct sum of irreducible* Gl(V)*-modules $\Gamma^b V$ with*

$$b_1 \geqq \ldots \geqq b_r \geqq \min\{a_\lambda\}-(N(s)-p).$$

(d) *$H^{p,q}(M_s(V), Q_s^a)=0$ for all $q \geqq 1$ as soon as*

$$a_{s_j}-a_{s_{j+1}} \geqq \min\{p, N(s)-p+(s_{j+1}-s_{j-1})-1, r+1-(s_{j+1}-s_{j-1})\}.$$

(e) *Under the assumption of* (d), *there is a* Gl(V)*-isomorphism*

$$H^{p,0}(M_s(V), Q_s^a)=\bigotimes_{u \in \mathbb{Z}^r} v_s(\underline{u}, p)\, \Gamma^{a+u} V,$$

where $v_s(\underline{u}, p)$ is the multiplicity of the weight $\underline{u}$ in $\Lambda^p \mathfrak{U}_s$.

Proof. Under the assumption of (a), Q_s^a is ample by Lemma (2.11) (c). The result follows therefore from the Kodaira-Nakano-Akizuki theorem. Now (b) is a consequence of (a) since $c(s)_{s_j}-c(s)_{s_{j+1}}=-(s_{j+1}-s_{j-1})$ and

$$H^q(M_s(V), Q_s^a)=H^{N(s),q}(M_s(V), Q_s^{a-c(s)}).$$

Let us observe now that for every vector bundle E on $M_s(V)$ there are isomorphisms

$$H^q(M_s(V), E) \simeq H^q(M(V), \eta^* E), \tag{2.19}$$

$$H^q(M_s(V), E) \simeq H^{q+N-N(s)}(M(V), Q^{c-c(s)} \otimes \eta^* E), \tag{2.20}$$

where $Q^{c-c(s)}$ is the relative canonical bundle along the fibers of the projection $\eta\colon M(V)\to M_s(V)$. In fact, the fibers of η are products of flag manifolds. For such a fiber F, we have $H^q(F,K_F)=(H^{\dim F-q}(F,\mathcal{O}))^*=\mathbb{C}$ when $q=\dim F=N-N(s)$ and $H^q(F,K_F)=0$ otherwise (apply (b) with $a=0$ and Künneth's formula). We get thus direct image sheaves

$$R^q\eta_*(\eta^*E))=\begin{cases}0 & \text{if } q\neq 0\\ \mathcal{O}(E) & \text{if } q=0.\end{cases}$$

$$R^q\eta_*(Q^{c-c(s)}\otimes\eta^*E)=\begin{cases}0 & \text{if } q\neq N-N(s)\\ \mathcal{O}(E) & \text{if } q=N-N(s).\end{cases}$$

Formulas (2.19) and (2.20) are thus immediate consequences of the Leray spectral sequence. When applied to $E=\Lambda^pT^*M_s(V)\otimes Q_s^{a-c(s)}$, formula (2.19) yields

$$\begin{aligned}H^{p,q}(M_s(V),Q_s^{a-c(s)})&=H^q(M(V),\ Q^{a-c(s)}\otimes\eta^*\Lambda^pT^*M_s(V))\\&=H^q(M(V),\ Q^a\otimes\eta^*\Lambda^{N(s)-p}TM_s(V)),\end{aligned}$$

using the isomorphism $\Lambda^pT^*M_s(V)\simeq K_{M_s(V)}\otimes\Lambda^{N(s)-p}TM_s(V)$. In the same way, (2.20) implies

$$\begin{aligned}H^{p,q}(M_s(V),Q_s^a)&=H^{q+N-N(s)}(M(V),\ Q^{a+c-c(s)}\otimes\eta^*\Lambda^pT^*M_s(V))\\&=H^{q+N-N(s)}(M(V),\ Q^{a+c}\otimes\eta^*\Lambda^{N(s)-p}TM_s(V)).\end{aligned}$$

The bundle $\eta^*T^*M_s(V)=(\mathfrak{U}_s)_{V,\mathrm{ad}}$ (resp. $\eta^*TM_s(V)=(\mathfrak{gl}_r/\mathfrak{B}_s)_{V,\mathrm{ad}}$) has a filtration with associated graded bundle

$$\bigoplus Q_\lambda^{-1}\otimes Q_\mu\quad(\text{resp.}\bigoplus Q_\lambda\otimes Q_\mu^{-1})$$

where the indices λ,μ are such that $\lambda\leqq s_j\leqq\mu$ for some j. It follows that $\eta^*\Lambda^pT^*M_s(V)$ (resp. $\eta^*\Lambda^{N(s)-p}TM_s(V)$) has a filtration with associated graded bundle

$$\bigoplus_{u\in\mathbb{Z}^r}\nu_s(\underline{u},p)\,Q^u\quad(\text{resp.}\bigoplus_{u'\in\mathbb{Z}^r}\nu_s'(\underline{u}',N(s)-p)\,Q^{u'})$$

where the weights $\underline{u}$ (resp. $\underline{u}'$) and multiplicities $\nu_s(\underline{u},p)$ (resp. $\nu_s'(\underline{u}',p)$ (resp. $\nu_s'(\underline{u}',N(s)-p)$) are those of $\Lambda^p\mathfrak{U}_s$ (resp. $\Lambda^{N(s)-p}\mathfrak{gl}_s/\mathfrak{B}_s$). We need a lemma.

(2.21) **Lemma.** *The weights u of $\Lambda^p\mathfrak{U}_s$ verify*

$$u_\mu-u_\lambda\leqq\min\{p+1,r+1-(\mu-\lambda),N(s)-p+(s_{j+1}-s_{j-1})\}$$

for $s_{j-1<\lambda\leq sj}<\mu\leqq s_{j+1}$, $1\leqq j\leqq m-1$.

Proof. Let us denote by $(\varepsilon_\lambda)_{1\leqq\lambda\leqq r}$ the canonical basis of $\mathbb{Z}^r$. The weights $\underline{u}$ (resp. $\underline{u}'$) are the sums of p (resp. $N(s)-p$) distinct elements of the set of weights $-\varepsilon_\lambda+\varepsilon_\mu$ (resp. $\varepsilon_\lambda-\varepsilon_\mu$) where $\lambda\leqq s_j<\mu$ for some j. It follows that $u_\mu-u_\lambda\leqq r+1-(\mu-\lambda)$, with equality for the weight

$$u=(-\varepsilon_1+\varepsilon_\mu)+\ldots+(-\varepsilon_\lambda+\varepsilon_\mu)+(-\varepsilon_\lambda+\varepsilon_{\mu+1})+\ldots+(-\varepsilon_\lambda+\varepsilon_r).$$

It is clear also that $u_\mu - u_\lambda \leqq p+1$ and $u'_\mu - u'_\lambda \leqq N(s) - p$. Since the weights $\underline{u}, \underline{u}'$ are related by $u = u' + c(s)$, Lemma (2.21) follows. □

Proof of (c). By the filtration property (1.4) and the above formulas, we see that $H^{p,q}(M_s(V), Q_s^{a-c(s)})$ is a direct sum of certain irreducible $\mathrm{Gl}(V)$-modules

$$H^q(M(V), Q^{a+u'}) = \Gamma^b V, \qquad b = (a+u')^\wedge = \left(a - \frac{c}{2} + u'\right)^{\geqq} + \frac{c}{2}$$

Clearly $u'_\lambda \geqq -(N(s)-p)$, thus

$$b_r \geqq \min\{a_\lambda\} - \max\left\{\frac{c_\lambda}{2}\right\} - (N(s)-p) + \frac{c_r}{2} = \min\{a_\lambda\} - (N(s)-p).$$

Proof of (d). Similarly, (d) is reduced by (1.4) to proving

$$H^{q+N-N(s)}(M(V), Q^{a+u+c-c(s)}) = 0$$

for all the above weights $\underline{u}$ and all $q \geqq 1$. By Proposition (2.17) it is sufficient to get

$$l\left(a + u + \frac{c}{2} - c(s)\right) \leqq N - N(s).$$

Let us observe that a weight $\underline{h}$ will be such that $l(h) \leqq N - N(s)$ as soon as $h_\lambda - h_\mu \geqq 0$ whenever $s_{j-1} < \lambda \leqq s_j < \mu \leqq s_{j+1}$, $1 \leqq j \leqq m-1$. For $h = a + u + \frac{c}{2} - c(s)$ this condition yields, thanks to Lemma (2.21):

$$a_\lambda - a_\mu \geqq \min\{p+1, r+1-(\mu-\lambda), N(s) - p + (s_{j+1} - s_{j-1})\} + (\mu - \lambda) - (s_{j+1} - s_j).$$

Taking $\mu - \lambda$ as large as possible, i.e. $\mu - \lambda = s_{j+1} - s_{j-1} - 1$, we get the asserted condition.

Proof of (e). Because of the vanishing of H^1 of the graded quotients, we obtain

$$H^{p,0}(M_s(V), Q_s^a) = \bigoplus_u v_s(\underline{u}, p)\, H^{N-N(s)}(M(V), Q^{a+u+c-c(s)}).$$

Applying Proposition (2.17), we get $(a+u+c-c(s))^\wedge = a + \tilde{u}$, where $\tilde{u}$ is the partial reordering of u such that only the coefficients in each interval $[s_{j-1}+1, s_j]$ have been reordered (in non-increasing order). The number of inversions is always $\leqq N - N(s)$, with equality if and only if u is non-decreasing in each interval. In that case

$$H^{N-N(s)}(M(V), Q^{a+u+c-c(s)}) = \Gamma^{a+\tilde{u}} V,$$

and $=0$ otherwise. Since $\Lambda^p \mathfrak{U}_s$ is a B_s-module we have $v_s(\underline{u}, p) = v_s(\underline{\tilde{u}}, p)$. Formula (e) is proved, and we see that non-zero terms correspond to weights $\underline{u}$ which are non-increasing in each interval $[s_{j-1}+1, s_j]$. □

(2.22) *Remark.* If the manifold $M_s(V)$ is a Grassmannian, then $m=2$, $s=(0, s_1, r)$. The condition required in Proposition (2.21) is therefore always satisfied when $a_r - a_{s_1} \geqq 1$, i.e. when Q_s^a is ample.

3. An isomorphism theorem

Our aim here is to generalize Griffiths and Le Potier's isomorphism theorems [8, 13] in the case of arbitrary flag bundles, following the simple method of Schneider [14].

Let X be a n-dimensional compact complex manifold and $E \to X$ a holomorphic vector bundle of rank r. For every sequence $0=s_0<s_1<\ldots<s_m=r$, we associate to E its flag bundle $Y=M_s(E)\to X$. If $a\in\mathbb{Z}^r$ is such that $a_{s_{j-1}+1}=\ldots=a_{s_j}$, $1\leqq j\leqq m$, we may define a line bundle $Q_s^a\to Y$ just as we did in §2. Let us set

$$\Omega_X^p=\Lambda^p T^*X, \qquad \Omega_Y^p=\Lambda^p T^*Y.$$

One has an exact sequence

$$0\to\pi^*\Omega_X^1\to\Omega_Y^1\to\Omega_{Y/X}^1\to 0 \tag{3.1}$$

where $\Omega_{Y/X}^1$ is by definition the bundle of relative differential 1-forms along the fibers of the projection $\pi\colon Y=M_s(E)\to X$. One may then define a decreasing filtration of Ω_Y^t as follows:

$$F^{p,t}=F^p(\Omega_Y^t)=\pi^*(\Omega_X^p)\wedge\Omega_Y^{t-p}. \tag{3.2}$$

The corresponding graded bundle is given by

$$G^{p,t}=F^{p,t}/F^{p+1,t}=\pi^*(\Omega_X^p)\otimes\Omega_{Y/X}^{t-p}. \tag{3.3}$$

Over any open subset of X where E is a trivial bundle $X\times V$ with $\dim_{\mathbb{C}}V=r$, the exact sequence (3.1) splits as well as the filtration (3.2). Using Proposition (2.18) (a), (d), we obtain the following lemma.

(3.4) **Lemma.** *For every weight $\underline{a}$ such that*

$$\begin{aligned} &a_{s_j}-a_{s_{j+1}}\geqq 1 \quad \text{if } t=N(s) \text{ and otherwise}\\ &a_{s_j}-a_{s_{j+1}}\geqq\min\{t,N(s)-t+(s_{j+1}-s_j)-1,r+1-(s_{j+1}-s_{j-1})\}\end{aligned} \tag{3.5}$$

the sheaf of sections of $\Omega_{Y/X}^t\otimes Q_s^a$ has direct images

$$\begin{aligned} &R^q\pi_*(\Omega_{Y/X}^t\otimes Q_s^a)=0 \quad \text{for } q\geqq 1\\ &\pi_*(\Omega_{Y/X}^t\otimes Q_s^a)=\bigoplus_u v_s(\underline{u},t)\,\Gamma^{a+u}E.\end{aligned} \tag{3.6}$$

We have in particular

$$\pi_*(Q_s^a)=\Gamma^a E, \qquad \pi_*(\Omega_{Y/X}^{N(s)}\otimes Q_s^a)=\Gamma^{a+c(s)}E. \tag{3.7}$$

Let L be an arbitrary line bundle on X. Under assumption (3.5), formulas (3.3) and (3.6) yield

$$\begin{aligned} &R^q\pi_*(G^{p,p+t}\otimes Q_s^a\otimes\pi^*L)=0 \quad \text{for } q\geq 1,\\ &\pi_*(G^{p,p+t}\otimes Q_s^a\otimes\pi^*L)=\bigoplus_u v_s(\underline{u},t)\,\Omega_X^p\otimes\Gamma^{a+u}E\otimes L.\end{aligned}$$

The Leray spectral sequence implies therefore:

(3.8) **Theorem.** *Under assumption* (3.5), *one has for all* $q \geqq 0$:

$$H^q(Y, G^{p,p+t} \otimes Q_s^a \otimes \pi^* L) \simeq \bigoplus_u v_s(\underline{u}, t)\, H^{p,q}(X, \Gamma^{a+u} E \otimes L).$$

The special case $t = N(s)$ gives:

(3.9) **Corollary.** *If* $a_{s_j} - a_{s_{j+1}} \geqq 1$, *then for all* $q \geqq 0$

$$H^q(Y, G^{p,p+N(s)} \otimes Q_s^a \otimes \pi^* L) \simeq H^{p,q}(X, \Gamma^{a+c(s)} E \otimes L).$$

When $p = n$, $G^{n,n+N(s)}$ is the only non-vanishing quotient in the filtration of the canonical line bundle $\Omega_Y^{n+N(s)}$. We thus obtain the following generalization of Griffiths' isomorphism theorem [8]:

(3.10) $$H^{n+N(s),q}(M_s(E), Q_s^a \otimes \pi^* L) \simeq H^{n,q}(X, \Gamma^{a+c(s)} E \otimes L).$$

4. Vanishing theorems

In order to carry over results for line bundles to vector bundles, one needs the following simple lemma.

(4.1) **Lemma.** *Assume that* $a_{s_1} > a_{s_2} > \ldots > a_{s_m} \geqq 0$. *Then*

(a) *E semi-ample* $\Rightarrow Q_s^a$ *semi-ample;*
(b) *E ample* $\Rightarrow Q_s^a$ *ample;*
(c) *E semi-ample and L ample* $\Rightarrow Q_s^a \otimes \pi^* L$ *ample.*

Proof. (a) If E is semi-ample, then by Definition (1.1) $S^k E$ is spanned for $k \geqq k_0$ large enough. Hence $\Gamma^{ka} E$, which is a direct summand in $S^{ka_1} E \otimes \ldots \otimes S^{ka_r} E$, is also spanned for $k \geqq k_0$. Since the fibers of $\pi_*(Q_s^{ka}) = \Gamma^{ka} E$ generate Q_s^{ka}, we conclude that Q_s^{ka} is spanned for $k \geqq k_0$.

(b) Similar proof, replacing "semi-ample" by "ample" and "spanned" by "very ample". One needs moreover the fact that Q_s^{ka} is very ample along the fibers of π (Lemma 2.11).

(c) If L^k is very ample for $k \geqq k_1$, then $Q_s^{ka} \otimes \pi^* L^k$ is very ample for $k \geqq \max(k_0, k_1)$, because Q_s^{ka} is spanned on Y and very ample along the fibers, whereas $H^0(Y, \pi^* L^k) = H^0(X, L^k)$ separates points of Y which lie in distinct fibers. □

We are now ready to attack the proof of the main theorems.

Proof of Theorem (0.2). Let $a \in \mathbb{Z}^r$ be such that

$$a_1 \geqq a_2 \geqq \ldots \geqq a_h > a_{h+1} = \ldots = a_r = 0.$$

Define $s_1 < s_2 < \ldots < s_{m-1}$ as the sequence of indices $\lambda = 1, \ldots, r-1$ such that $a_\lambda > a_{\lambda+1}$ and set $a' = a + (h, \ldots, h) - c(s)$. The canonical weight $c(s)$ is non-decreasing and $c(s)_r = s_{m-1} = h$, $s_m = r$, hence

$$a'_{s_1} > a'_{s_2} > \ldots > a'_{s_m} = h - c(s)_r = 0,$$

so $Q_s^{a'}\otimes\pi^*L$ is ample by Lemma (4.1) and $\Gamma^{a'+c(s)}E=\Gamma^a E\otimes(\det E)^h$. Formula (3.10) yields

$$H^{n,q}(X,\Gamma^a E\otimes(\det E)^h\otimes L)\simeq H^{n+N(s),q}(M_s(E),Q_s^{a'}\otimes\pi^*L).$$

Since $\dim M_s(E)=n+N(s)$, the group in the right hand side is zero for $q\geqq 1$ by the Kodaira-Akizuki-Nakano vanishing Theorem (1.2). □

Proof of Theorem (0.3). The proof proceeds by backward induction on p. The case $p=n$ is already settled by Theorem 0.2. The decomposition formula (2.16) shows that the result is equivalent to

$$H^{p,q}(X,\Gamma^a E\otimes(\det E)^l\otimes L)=0 \qquad \text{for } p+q\geqq n+1, \qquad l\geqq n-p+r-1$$

and $a_1\geqq\ldots\geqq a_r=0$ not all zero. Define s as above and $a'=a+(l,\ldots,l)-c(s)$. Then $Q_s^{a'}$ is ample since $a_r=l-h\geqq 0$, and Corollary (3.9) implies

$$H^{p,q}(X,\Gamma^a E\otimes\det E)^l\otimes L)\simeq H^q(Y,G^{p,p+N(s)}\otimes Q_s^{a'}\otimes\pi^*L). \tag{4.2}$$

Now, it is clear that $F^{p,p+N(s)}=\Omega_Y^{p+N(s)}$. We get thus an exact sequence

$$0\to F^{p+1,p+N(s)}\to\Omega_Y^{p+N(s)}\to G^{p,p+N(s)}\to 0.$$

The Kodaira-Akizuki-Nakano vanishing Theorem (1.2) applied to $Q_s^{a'}\otimes\pi^*L$ with $\dim Y=n+N(s)$ yields

$$H^q(Y,\Omega_Y^{p+N(s)}\otimes Q_s^{a'}\otimes\pi^*L)=0 \qquad \text{for } p+q\geqq n+1.$$

The cohomology groups in (4.2) will therefore vanish if and only if

$$H^{q+1}(Y,F^{p+1,p+N(s)}\otimes Q_s^{a'}\otimes\pi^*L)=0. \tag{4.3}$$

Let us observe that $F^{p+1,p+N(s)}$ has a filtration with associated graded bundle $\bigoplus_{t\geqq 1}G^{p+t,p+N(s)}$. In order to verify (4.3), it is thus enough to get

$$H^{q+1}(Y,G^{p+t,p+N(s)}\otimes Q_s^{a'}\otimes\pi^*L)=0. \qquad t\geqq 1. \tag{4.4}$$

At this point, the Leray spectral sequence will be used in an essential way. Since $G^{p+t,p+N(s)}=\pi^*(\Omega_X^{p+t})\otimes\Omega_{Y/X}^{N(s)-t}$, we get

$$R^k\pi_*(G^{p+t,p+N(s)}\otimes Q_s^{a'}\otimes\pi^*L)=\Omega_X^{p+t}\otimes L\otimes R^k\pi_*(\Omega_{Y/X}^{N(s)-t}\otimes Q^{a+(l,\ldots,l)-c(s)}).$$

Proposition (2.18)(a) and (c) yields

$$R^k\pi_*(\Omega_{Y/X}^{N(s)-t}\otimes Q^{a+(l,\ldots,l)-c(s)})=\begin{cases}0 & \text{for } k\geqq t+1, \text{otherwise}\\ \bigoplus_b \Gamma^b E, & b_r\geqq l-t,\end{cases}$$

$$R^k\pi_*(G^{p+t,p+N(s)}\otimes Q_s^{a'}\otimes\pi^*L)=\begin{cases}0 & \text{for } k\geqq t+1, \text{otherwise}\\ \bigoplus_{b,m}\Omega_X^{p+t}\otimes\Gamma^b E\otimes(\det E)^m\otimes L,\end{cases}$$

where the last sum runs over weights $\underline{b}$ such that $b_r=0$ and integers m such that $m \geqq l-t \geqq n-(p+t)+r-1$. Therefore $H^{q+1}(Y, G^{p+t,p+N(s)} \otimes Q_s^{a'} \otimes \pi^* L)$ has a filtration whose graded module is the limit term $\bigotimes_j E_\infty^{j,q+1-j}$ of a spectral sequence such that

$$E_2^{j,k} = \begin{cases} 0 & \text{for } k \geqq t+1, \text{otherwise} \\ \bigoplus_{b,m} H^{p+t,j}(X, \Gamma^b E \otimes (\det E)^m \otimes L), & m \geqq n-(p+t)+r-1. \end{cases}$$

We have thus $E_2^{j,q+1-j}=0$ for $j \leqq q-t$ by the first case, and also for $j \geqq q-t$ by the second case and the induction hypothesis. Hence $E_\infty^{j,q+1-j}=0$ for all j, and (4.4) is proved. □

Proof of Theorem (0.6). Let us take here $a=(2,0,\ldots,0)$ and $s=(0,1,r)$, so that $Y=M_s(E)=P(E^*)$. Since $\Gamma^a E=S^2 E$ and $\Gamma^{(1,1,0,\ldots,0)}E=\Lambda^2 E$, Theorem 3.8 yields

$$\begin{aligned} &H^q(Y, G^{p+1,p+1} \otimes Q_{s,1}^2 \otimes \pi^* L) \simeq H^{p+1,q}(X, S^2 E), \\ (4.5) \qquad &H^q(Y, G^{p,p+1} \otimes Q_{s,1}^2 \otimes \pi^* L) \simeq H^{p,q}(X, \Lambda^2 E), \\ &H^q(Y, G^{k,p+1} \otimes Q_{s,1}^2 \otimes \pi^* L) = 0 \quad \text{for } k<p, \end{aligned}$$

because $u=-\varepsilon_1+\varepsilon_2$ is the only weight of $\Lambda^1 \mathfrak{U}_s$ such that $a+u$ is non-increasing, and because no such weights exist for higher exterior powers $\Lambda^j \mathfrak{U}_s$. Now, $\Omega_Y^{p+1}/F^{p,p+1}$ has a filtration with graded quotients $G^{k,p+1}$, $k<p$, therefore

$$H^q(Y, \Omega_Y^{p+1}/F^{p,p+1} \otimes Q_{s,1}^2 \otimes \pi^* L)=0 \quad \text{for all } q.$$

Considering the exact sequences

$$\begin{aligned} &0 \to G^{p,p+1} \to \Omega_Y^{p+1}/F^{p+1,p+1} \to \Omega_Y^{p+1}/F^{p,p+1} \to 0, \\ &0 \to G^{p+1,p+1} \to \Omega_Y^{p+1} \to \Omega_Y^{p+1}/F^{p+1,p+1} \to 0, \end{aligned}$$

we get an isomorphism

$$H^q(Y, G^{p,p+1} \otimes Q_{s,1}^2 \otimes \pi^* L) \simeq H^q(Y, \Omega_Y^{p+1}/F^{p+1,p+1} \otimes Q_{s,1}^2 \otimes \pi^* L)$$

and a canonical coboundary morphism

$$H^q(Y, \Omega_Y^{p+1}/F^{p+1,p+1} \otimes Q_{s,1}^2 \otimes \pi^* L) \to H^{q+1}(Y, G^{p+1,p+1} \otimes Q_{s,1}^2 \otimes \pi^* L)$$

which by Kodaira-Akizuki-Nakano is onto for $p+1+q+1 \geqq n+N(s)+1$, i.e. $p+q \geqq n+r-2$,and bijective for $p+q \geqq n+r-1$. Combining this with the first two isomoprhisms (4.5) achieves the proof of Theorem (0.6). □

As promised in the introduction, we show now that the condition $l \geqq h$ in Theorem (0.2) is best possible.

(4.6) *Example.* Let $X=G_r(V)$ be the Grassmannian of subspaces of codimension r of a vector space V, $\dim_{\mathbb{C}} V=d$, and E the tautological quotient vector bundle of rank r over X. Then E is spanned (hence semi-ample) and $L=\det E$ is very ample. According to the notations of § 2, we have

$$X=M_s(V), \qquad s=(s_0,s_1,s_2)=(0,r,d),$$
$$E=V/V_r, \qquad \det E=Q_{s,1}, \qquad \det V_r=Q_{s,2},$$
$$K_X=Q_{s,1}^{r-d}\otimes Q_{s,2}^r.$$

Furthermore, for any sequence $a_1\geqq\ldots\geqq a_r\geqq 0$, we have

$$\Gamma^a E=\Gamma^a(V/V_r)=\eta_* Q^{(a,0)}$$

where $\eta\colon M(V)\to M_s(V)$ is the projection. If $n=\dim X=r(d-r)$, this implies

$$\begin{aligned}H^{n,q}(X,\Gamma^a E\otimes(\det E)^h)&=H^q(X,Q_{s,1}^{h+r-d}\otimes Q_{s,2}^r\otimes\eta_* Q^{(a,0)})\\&=H^q(X,\eta_* Q^\alpha)\end{aligned}$$

where $\alpha=(a_1+h+r-d,\ldots,a_r+h+r-d,r,\ldots,r)\in\mathbb{Z}^d$. The fiber of η is $M(V/V_r)\times M(V_r)$, hence $R^q\eta_* Q^\alpha=0$ for $q\geqq 1$ by Künneth's formula and Proposition (2.18)(b). We obtain therefore

$$H^{n,q}(X,\Gamma^a E\otimes(\det E)^h)=H^q(M(V),Q^\alpha).$$

Assume now that $a_1\geqq\ldots\geqq a_h\geqq d-r$ and $a_h=\ldots=a_r=0$. Define $\mathbf{1}=(1,\ldots,1)\in\mathbb{Z}^d$. Then

$$\alpha-\frac{c}{2}=\frac{1-d}{2}\cdot\mathbf{1}+(a_1+r+h-1,\ldots,a_h+r;r-1,\ldots,h;d-1,\ldots,r)$$

where dots indicate a decreasing sequence of consecutive integers. There are exactly $(r-h)(d-r)$ inversions of the ordering, corresponding to the inversion of the last two blocks between semicolons. Then one finds easily

$$\left(\alpha-\frac{c}{2}\right)^{\geqq}+\frac{c}{2}=(a_1+h+r-d,\ldots,a_h+h+r-d;h,\ldots,h)=\beta+h\mathbf{1}$$

where $\beta=(a_1+r-d,\ldots,a_r+r-d;0,\ldots,0)$. Proposition (2.17) yields therefore

$$H^{n,q}(X,\Gamma^a E\otimes(\det E)^h)=\begin{cases}0 & \text{if } q\neq(r-h)(d-r)\\ \Gamma^\beta V\otimes(\det V)^h & \text{if } q=(r-h)(d-r).\end{cases}$$

5. On the Borel-Le Potier spectral sequence

Denote as before $\pi\colon Y=M_s(E)\to X$ the projection. To every integer t and every coherent analytic sheaf $\mathcal{S}$ on Y, one may associate the filtration of $\Omega_Y^t\otimes\mathcal{S}$ by its subsheaves

$$F^{p,t}\otimes\mathcal{S}\subset\Omega_Y^t\otimes\mathcal{S},$$

the corresponding graded sheaf being of course $\bigoplus_p G^{p,t}\otimes\mathcal{S}$. This gives rise to a spectral sequence which we shall name after Borel and Le Potier, whose E_1 term is given by

$$(5.1) \qquad E_1^{p,q-p} = H^q(Y, G^{p,t} \otimes \mathcal{S}).$$

The limit term $E_\infty^{p,q-p}$ is the p-graded module corresponding to a filtration of the group $H^q(Y, \Omega_Y^t \otimes \mathcal{S})$. Assume that the spectral sequence degenerates in E_2, i.e. that d_r: $E_r^{p,q-p} \to E_r^{p+r,q+1-(p+r)}$ is zero for all $r \geqq 2$ (by Peternell, Le Potier and Schneider [12], the spectral sequence does not degenerate in general in E_1). Then $E_2^{p,q-p} = E_\infty^{p,q-p}$. This equality means that the q-th cohomology group of the E_1-complex

$$d_1 \colon H^q(Y, G^{p,t} \otimes \mathcal{S}) \to H^{q+1}(Y, G^{p+1,t} \otimes \mathcal{S})$$

is the p-graded module corresponding to a filtration of $H^q(Y, \Omega_Y^t \otimes \mathcal{S})$. By Kodaira-Akizuki-Nakano, we get therefore:

(5.1) **Proposition.** *Assume that E is ample and $L \geqq 0$, or $E \geqq 0$ and L ample, and that the E_2-degeneracy occurs for the ample invertible sheaf $\mathcal{S} = Q_s^a \otimes \pi^* L$ on Y. Then the complex*

$$d_1 \colon H^q(Y, G^{p,t} \otimes Q_s^a \otimes \pi^* L) \to H^{q+1}(Y, G^{p+1,t} \otimes Q_s^a \otimes \pi^* L)$$

is exact in degree $q \geqq n + N(s) + 1 - t$.

Our hope is that the E_2-degeneracy can be proved in all cases by means of harmonic forms and Hodge theory, but we have been unable to do so. Since $G^{p,t} = 0$ for $t > p + N(s)$, Proposition (5.1) yields for all $p + q \geqq n + 1$ an exact sequence

$$0 \to H^q(Y, G^{p,p+N(s)} \otimes Q_s^a \otimes \pi^* L) \to H^{q+1}(Y, G^{p+1,p+N(s)} \otimes Q_s^a \otimes \pi^* L).$$

Replacing a by $a' = a + (l \ldots, l) - c(s)$ as in the proof of Theorem (0.3), and using Formula (3.10) and Theorem (3.8), one gets an exact sequence

$$\begin{aligned} 0 \to H^{p,q}(X, \Gamma^a E \otimes (\det E)^l) \otimes L) \to \\ \bigoplus_u v_s(\underline{u}, N(s) - 1)\, H^{p+1,q+1}(X, \Gamma^{a-c(s)+u} E \otimes (\det E)^l \otimes L) \end{aligned}$$

because assumption (3.5) is satisfied for the weight b and for $t = N(s) - 1$. But the above direct sum can be rewritten

$$\bigoplus_{u'} v_s'(\underline{u}', 1)\, H^{p+1,q+1}(X, \Gamma^{a+u'} E \otimes (\det E)^l \otimes L)$$

in terms of the weights $u' = \varepsilon_\lambda - \varepsilon_\mu$, $\lambda \leqq s_j < \mu$, of $\Lambda^1 \mathfrak{gl}_r / \mathfrak{B}_s$. A backward induction on $\max\{p, q\}$ yields then immediately:

(5.2) **Corollary.** *If the E_2-degeneracy occurs for all ample line bundles $Q_s^a \otimes \pi^* L$, $a_1 \geqq \ldots \geqq a_r = 0$, on all flag manifolds of E, then*

$$H^{p,q}(X, \Gamma^a E \otimes (\det E)^l \otimes L) = 0$$

for $p + q \geqq n + 1$ and $l \geqq r - 1 + \min\{n - p, n - q\}$.

Another interesting consequence of Proposition (5.1) in the case $Y=P(E^*)$ would be the following generalization of Theorem (0.6).

(5.3) **Corollary.** *Set* $Z^{t,k}E=\Gamma^{(k-t,1,\ldots,0,\ldots,0)}E\subset S^{k-t}E\otimes\Lambda^t E$ *if* $0\leq t\leq k-1$ *and* $Z^{t,k}E=0$ *otherwise. Then there is a canonical complex*

$$\ldots\to H^{p,q}(X,Z^{t,k}E\otimes L)\to H^{p+1,q+1}(X,Z^{t-1,k}E\otimes L)\to\ldots$$

Under the hypotheses of Proposition (5.1) *for* $\mathscr{S}=Q_s^{(k,0,\ldots,0)}\otimes\pi^*L$, *this complex is exact in each degree* q *such that* $p+q+t\geqq n+r$.

Proof. The only possible weight $\underline{u}$ of $\Lambda^t\mathfrak{U}_s$ such that $(k,0,\ldots,0)+u$ be non-increasing is $u=(-t,1,\ldots,1,0,\ldots,0)$. Theorem (3.8) yields therefore

$$H^q(Y,G^{p,p+t}\otimes Q_s^{(k,0,\ldots,0)}\otimes\pi^*L)=H^{p,q}(X,Z^{t,k}E\otimes L).\quad\square$$

Note that the special case $k=1$, $t=0$ is Le Potier's theorem, and that the special case $k=2$, $t=1$ is Theorem (0.6). These two cases do not depend upon any degeneracy assumption.

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Note added in proof

After this paper was completed and sent to the Journal, Michael Schneider informed us that he had also obtained independantly a counterexample to Sommese's conjecture