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ALGEBRAIC APPROXIMATIONS OF HOLOMORPHIC MAPS FROM STEIN DOMAINS TO PROJECTIVE MANIFOLDS

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1. Introduction. The present work, which was motivated by the study of the Kobayashi pseudodistance on algebraic manifolds, proceeds from the general philosophy that analytic objects can be approximated by algebraic objects under suitable restrictions. Such questions have been extensively studied in the case of holomorphic functions of several complex variables and can be traced back to the Oka-Weil approximation theorem (see [We] and [Oka]). The main approximation result of this work (Theorem 1.1) is used to show that both the Kobayashi pseudodistance and the Kobayashi-Royden infinitesimal metric on a quasi-projective algebraic manifold Z are computable solely in terms of the closed algebraic curves in Z (Corollaries 1.3 and 1.4).

Our general goal is to show that algebraic approximation is always possible in the cases of holomorphic maps to quasi-projective manifolds (Theorems 1.1 and 4.1) and of locally free sheaves (Theorem 1.8 and Proposition 3.2). Since we deal with algebraic approximation, a central notion is that of Runge domain: by definition, an open set Ω in a Stein space Y is said to be a *Runge domain* if Ω is Stein and if the restriction map $\mathcal{O}(Y) \to \mathcal{O}(\Omega)$ has dense range. It is well known that Ω is a Runge domain in Y if and only if the holomorphic hull with respect to $\mathcal{O}(Y)$ of any compact subset $K \subset \Omega$ is contained in Ω. If Y is an affine algebraic variety, a Stein open set $\Omega \subset Y$ is Runge if and only if the polynomial functions on Y are dense in $\mathcal{O}(\Omega)$.

Our first result given below concerns approximations of holomorphic maps by (complex) Nash algebraic maps. If Y, Z are quasi-projective (irreducible, reduced) algebraic varieties, a map $f: \Omega \to Z$ defined on an open subset $\Omega \subset Y$ is said to be *Nash algebraic* if f is holomorphic and the graph

$$\Gamma_f := \{(y, f(y)) \in \Omega \times Z: y \in \Omega\}$$

is contained in an algebraic subvariety G of $Y \times Z$ of dimension equal to dim Y. If f is Nash algebraic, then the image $f(\Omega)$ is contained in an algebraic subvariety

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A of Z with $\dim A = \dim f(\Omega) \leqslant \dim Y$. (Take $A = \mathrm{pr}_2(G) \subset Z$, after eliminating any unnecessary components of G.)

THEOREM 1.1. *Let Ω be a Runge domain in an affine algebraic variety S, and let $f\colon \Omega \to X$ be a holomorphic map into a quasi-projective algebraic manifold X. Then for every relatively compact domain $\Omega_0 \subset\subset \Omega$, there is a sequence of Nash algebraic maps $f_\nu\colon \Omega_0 \to X$ such that $f_\nu \to f$ uniformly on Ω_0.*

Moreover, if there is an algebraic subvariety A (not necessarily reduced) of S and an algebraic morphism $\alpha\colon A \to X$ such that $f_{|A\cap\Omega} = \alpha_{|A\cap\Omega}$, then the f_ν can be chosen so that $f_{\nu|A\cap\Omega_0} = f_{|A\cap\Omega_0}$. (In particular, if we are given a positive integer k and a finite set of points (t_j) in Ω_0, then the f_ν can be taken to have the same k-jets as f at each of the points t_j.)

An algebraic subvariety A of S is given by a coherent sheaf $\mathcal{O}_A$ on S of the form $\mathcal{O}_A = \mathcal{O}_S/\mathcal{I}_A$, where $\mathcal{I}_A$ is an ideal sheaf in $\mathcal{O}_S$ generated by a (finite) set of polynomial functions on S. (If $\mathcal{I}_A$ equals the ideal sheaf of the algebraic subset $\mathrm{Supp}\,\mathcal{O}_A \subset S$, then A is reduced and we can identify A with this subset.) The restriction of a holomorphic map $f\colon \Omega \to X$ to an algebraic subvariety A is given by

$$f_{|A\cap\Omega} = f \circ \iota_{A\cap\Omega}\colon A \cap \Omega \to X,$$

where $\iota_{A\cap\Omega}\colon A \cap \Omega \to \Omega$ is the inclusion morphism. We note that the k-jet of a holomorphic map $f\colon \Omega \to X$ at a point $a \in \Omega$ can be described as the restriction $f_{|\{a\}^k}\colon \{a\}^k \to X$ of f to the nonreduced point $\{a\}^k$ with structure ring $\mathcal{O}_{\Omega,a}/\mathcal{M}_{\Omega,a}^{k+1}$ (where $\mathcal{M}_{\Omega,a}$ denotes the maximal ideal in $\mathcal{O}_{\Omega,a}$). The parenthetical statement in Theorem 1.1 then follows by setting A equal to the union of the nonreduced points $\{t_j\}^k$.

If in Theorem 1.1 we are given an exhausting sequence (Ω_ν) of relatively compact open sets in Ω, then we can construct by the Cantor diagonal process a sequence of Nash algebraic maps $f_\nu\colon \Omega_\nu \to X$ converging to f uniformly on every compact subset of Ω. We are unable to extend Theorem 1.1 to the case where X is singular. Of course, for the case $\dim S = 1$, Theorem 1.1 extends to singular X, since then f can be lifted to a desingularization of X. The case $S = \mathbb{C}$ of Theorem 1.1 was obtained earlier by L. van den Dries [Dr] using different methods.

In the case where X is an affine algebraic manifold, Theorem 1.1 is easily proved as follows: One approximates $f\colon \Omega \to X \subset \mathbb{C}^m$ with an algebraic map $g\colon \Omega_0 \to \mathbb{C}^m$ and then applies a Nash algebraic retraction onto X to obtain the desired approximations. (The existence of Nash algebraic retractions is standard and is given in Lemma 2.1.) In the case where f is a map into a projective manifold $X \subset \mathbb{P}^{m-1}$, one can reduce to the case where f lifts to a map g into the cone $Y \subset \mathbb{C}^m$ over X, but in general one cannot find a global Nash algebraic retraction onto Y. Instead our proof proceeds in this case (after making some reductions) by considering the map $G\colon \Omega \times \mathbb{C}^m \to \mathbb{C}^m$ given by $G(z, w) = g(z) + w$, so that $G^{-1}(Y\setminus\{0\})$ is a submanifold containing $\Omega \times \{0\}$. We then approximate G by algebraic maps G_ν, which we compose (on the right) with Nash algebraic maps from $\Omega_0 \times \{0\}$ into $G_\nu^{-1}(Y)$ to obtain the approximations f_ν.

In the case where the map f in Theorem 1.1 is an embedding of a smooth domain Ω into a projective manifold X with the property that f^*L is trivial for some ample line bundle L on X, we can approximate f by Nash algebraic embeddings f_ν with images contained in affine Zariski open sets of the form $X \backslash D_\nu$, where the D_ν are divisors of powers of L. This result (Theorem 4.1) is obtained by first modifying f on a Runge domain $\Omega_1 \subset\subset \Omega$ in such a way as to create essential singularities on the boundary of $f(\Omega_1)$ so that the closure $f(\overline{\Omega}_1)$ becomes complete pluripolar (in the sense of plurisubharmonic function theory). The existence of an ample divisor avoiding $f(\Omega_0)$ is then obtained by means of an approximation theorem (Proposition 4.8) based on Hörmander's L^2 estimates for $\bar{\partial}$.

One of our main applications is the study of the Kobayashi pseudodistance and the Kobayashi-Royden infinitesimal pseudometric on algebraic manifolds [Ko1], [Ko2], [Ro]. We use throughout the following notation:

$$\Delta_R(a) = \{t \in \mathbb{C} : |t - a| < R\}, \qquad \Delta_R = \Delta_R(0), \qquad \Delta = \Delta_1;$$

for a complex manifold X:

T_X = the holomorphic tangent bundle of X;

$N_S = N_{S/X}$ = the normal bundle $T_{X|S}/T_S$ of a submanifold $S \subset X$;

κ_X = the Kobayashi-Royden infinitesimal Finsler pseudometric on T_X,

i.e., for $v \in T_X$,

$$\kappa_X(v) = \inf\left\{\lambda > 0 : \exists f : \Delta \to X \text{ holomorphic with } f_*\left(\lambda \frac{\partial}{\partial \zeta}_{|0}\right) = v\right\}.$$

In addition, for a complex space Z, we let $d_Z(a, b)$ denote the Kobayashi pseudodistance between points $a, b \in Z$. In the case where Z is smooth, then $d_Z(a, b)$ is the κ_Z-length of the shortest curve from a to b; see [NO]. (A complex space Z is said to be *hyperbolic* if $d_Z(a, b) > 0$ whenever a, b are distinct points of Z. A compact complex manifold is hyperbolic if and only if the Kobayashi-Royden pseudometric is positive for all nonzero tangent vectors [NO].)

As an application of Theorem 1.1 (for the case where Ω is the unit disk Δ), we describe below how both the Kobayashi-Royden pseudometric and the Kobayashi pseudodistance on projective algebraic manifolds can be given in terms of algebraic curves.

COROLLARY 1.2. *Let M be an open subset of a projective algebraic manifold X and let $v \in T_M$. Then $\kappa_M(v)$ is the infimum of the set of $r \geqslant 0$ such that there exist a closed algebraic curve $C \subset X$, a normalization $\gamma : C' \to C \cap M$, and a vector $u \in T_{C'}$ with $\kappa_{C'}(u) = r$ and $d\gamma(u) = v$.*

Proof (assuming Theorem 1.1). Let r_0 be the infimum given above. By the distance-decreasing property of holomorphic maps, $\kappa_X(v) \leqslant r_0$. We must verify the reverse inequality. Let $\varepsilon > 0$ be arbitrary. Choose a holomorphic map $g\colon \Delta \to M$ and $\lambda \in [0, \kappa_X(v) + \varepsilon]$ such that $dg(\lambda e_0) = v$, where $e_0 = (\partial/\partial\zeta)_{|0}$. By Theorem 1.1 with $\Omega = \Delta$, there exists a Nash algebraic map $f\colon \Delta_{1-\varepsilon} \to M$ such that $df(e_0) = dg(e_0)$. Let $\tilde{f}\colon \Delta \to M$ be given by $\tilde{f}(\zeta) = f((1-\varepsilon)\zeta)$ and let $\tilde{\lambda} = \lambda/(1-\varepsilon)$. Then $d\tilde{f}(\tilde{\lambda} e_0) = v$. Since $\tilde{f}$ is Nash algebraic, there exists an algebraic curve $C \subset X$ such that $\tilde{f}(\Delta) \subset C$. Let $\gamma\colon C' \to C \cap M$ be the normalization of $C \cap M$. We then have a commutative diagram

$$\begin{array}{ccc} \Delta & \xrightarrow{\hat{f}} & C' \\ & \searrow^{\tilde{f}} & \downarrow \gamma \\ & & C \cap M. \end{array}$$

Let $u = d\hat{f}(\tilde{\lambda} e_0)$. We have $d\gamma(u) = v$. Thus

$$r_0 \leqslant \kappa_{C'}(u) \leqslant \tilde{\lambda} = \frac{\lambda}{1-\varepsilon} \leqslant \frac{\kappa_M(v) + \varepsilon}{1-\varepsilon}.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $\tau_0 \leqslant \kappa_M(v)$. □

Suppose that M is as in Corollary 1.2 and $v \in T_{M,x}$ $(x \in M)$. We note that if C, C', γ, u are given as in the statement of the corollary, then the analytic set germ C_x contains a smooth irreducible component tangent to v. In this case we say that *C is tangent to v*. In general, $(d\gamma)^{-1}(v)$ is a finite set of vectors (u_j) in $T_{C'}$ (C_x may contain several smooth components tangent to v) and we define $\kappa_{C \cap M}(v) = \min_j \kappa_{C'}(u_j)$. In particular, if we let M be Zariski open in X, we then have the following result.

COROLLARY 1.3. *Let Z be a quasi-projective algebraic manifold. Then the Kobayashi-Royden pseudometric κ_Z is given by*

$$\kappa_Z(v) = \inf_C \kappa_C(v), \qquad \forall v \in T_Z,$$

where C runs over all (possibly singular) closed algebraic curves in Z tangent to v.

In the case where C is hyperbolic (e.g., the normalization of C is a compact curve of genus $\geqslant 2$), the Kobayashi-Royden pseudometric κ_C coincides with the Poincaré metric induced by the universal covering $\Delta \to C'$ of the normalization C'; thus in some sense the computation of κ_Z is reduced to a problem which is more algebraic in nature. Of particular interest is the case where Z is projective; then the computation becomes one of determining the genus and Poincaré metric of curves in each component of the Hilbert scheme. This observation strongly suggests that it should be possible to characterize Kobayashi hyperbolicity of projective manifolds in purely algebraic terms; this would follow for instance from S. Lang's conjecture [La1], [La2, IV.5.7] that a projective manifold is hyperbolic if and only if it has no rational curves and no nontrivial images of abelian varieties; see also [De3] for further results on this question.

We likewise can use algebraic curves to determine the Kobayashi pseudodistance in quasi-projective algebraic varieties.

COROLLARY 1.4. *Let Z be a quasi-projective algebraic variety, and let $a, b \in Z$. Then the Kobayashi pseudodistance $d_Z(a, b)$ is given by*

$$d_Z(a, b) = \inf_C d_C(a, b),$$

where C runs over all (possibly singular and reducible) one-dimensional algebraic subvarieties of Z containing a and b.

Remark. If C is a one-dimensional reducible complex space, then by definition $d_C(a, b)$ is the infimum of the sums

$$\sum_{j=1}^{n} d_{C_j}(a_{j-1}, a_j)$$

where n is a positive integer, $C_1, \ldots, C_n$ are (not necessarily distinct) irreducible components of C, $a_0 = a \in C_1$, $a_n = b \in C_n$ and $a_j \in C_j \cap C_{j+1}$ for $1 \leqslant j \leqslant n-1$.

Proof of Corollary 1.4 (assuming Theorem 1.1). Let $\varepsilon > 0$ be arbitrary. It suffices to find algebraic curves $C_j \subset Z$ as in the above remark with

$$\sum_{j=1}^{n} d_{C_j}(a_{j-1}, a_j) < d_Z(a, b) + \varepsilon.$$

By the definition of the Kobayashi pseudodistance, there exist points $a = a_0, a_1, \ldots, a_n = b$ in Z and holomorphic maps $f_j\colon \Delta \to Z$ such that $f_j(0) = a_{j-1}$, $f_j(t_j) = a_j$ ($t_j \in \Delta$), for $1 \leqslant j \leqslant n$, and $\sum d_\Delta(0, t_j) < d_Z(a, b) + \varepsilon$. Let $\rho < 1$ be arbitrary. By Theorem 1.1 (applied to the desingularization of Z), we can find Nash algebraic maps $g_j\colon \Delta_\rho \to Z$ with $g_j(0) = f_j(0) = a_{j-1}$ and $g_j(t_j) = f_j(t_j) = a_j$. Then for $1 \leqslant j \leqslant n$, $g_j(\Delta)$ is contained in an algebraic curve C_j. Hence

$$\sum_{j=1}^{n} d_{C_j}(a_{j-1}, a_j) \leqslant \sum_{j=1}^{n} d_{\Delta_\rho}(0, t_j) = \sum_{j=1}^{n} d_\Delta\left(0, \frac{t_j}{\rho}\right).$$

Since $\rho < 1$ is arbitrary, our conclusion follows. □

Eisenman [Ei] has also introduced an invariant pseudometric on the decomposable p-vectors of an arbitrary complex space Y. Let B^p be the unit ball of $\mathbb{C}^p$.

At each nonsingular point of Y, the *Eisenman p-metric* E_Y^p is defined by

$$E_Y^p(v) = \inf\left\{\lambda > 0 \colon \exists f \colon B^p \to Y \text{ holomorphic with } f_*\left(\lambda \frac{\partial}{\partial z_1} \wedge \cdots \wedge \frac{\partial}{\partial z_p}\Big|_0\right) = v\right\}$$

for all decomposable p-vectors $v \in \Lambda^p T_Y$. (The space Y is said to be *strongly p-measure hyperbolic* if $E_Y^p(v) > 0$ whenever $v \neq 0$.) For the case $p = \dim Y$, E_Y^p is called the *Eisenman volume*. The following generalization of Corollary 1.3 to the Eisenman p-metric is obtained by a parallel argument.

COROLLARY 1.5. *The Eisenman p-metric of a quasi-projective algebraic manifold Z can be computed in terms of the Eisenman volumes of its p-dimensional algebraic subvarieties. More precisely,*

$$E_Y^p(v) = \inf_Y E_Y^p(v)$$

for all decomposable p-vectors $v \in \Lambda^p T_Z$, where Y runs over all (possibly singular) p-dimensional closed algebraic subvarieties of Z tangent to v. (A decomposable p-vector v at a singular point $y \in Y$ is said to be tangent to Y if the analytic germ of Y at y contains a smooth irreducible component tangent to v; the above definition of $E_Y^p(v)$ carries over to this case.)

If Y is compact and the canonical bundle $K_{\tilde{Y}}$ of a desingularization $\tilde{Y} \to Y$ is ample, then a form of the maximum principle tells us that the Eisenman volume E_Y^p is bounded from below by the volume form of the Kähler-Einstein metric (with curvature -1) on $\tilde{Y}$ (see [Yau], [NO, §2.5]). It would be very interesting to know when these two intrinsic volumes are actually equal. In connection with this question, one would like to know whether the infimum appearing in Corollary 1.5 can be restricted to subvarieties Y with $\tilde{Y}$ satisfying this property, in virtue of some density argument. This would make the Eisenman p-metric theory completely parallel to the case of the Kobayashi-Royden pseudometric $\kappa_Z = E_Z^1$.

The following result of Stout [St] on the exhaustion of Stein manifolds by Runge domains in affine algebraic manifolds allows us to extend Theorem 1.1 to maps from arbitrary Stein manifolds.

THEOREM 1.6 (E. L. Stout). *Let S be a Stein manifold. For every Runge open set $\Omega \subset\subset S$, there exists an affine algebraic manifold Y such that Ω is biholomorphic to a Runge open set in Y.*

Theorem 1.6 has been refined into a relative version by Tancredi-Tognoli [TT1]. An immediate consequence of Theorem 1.1 and Stout's Theorem 1.6 is the following approximation theorem for holomorphic maps from Stein manifolds into projective algebraic manifolds.

COROLLARY 1.7. *Let Ω be a relatively compact domain in a Stein manifold S, and let $f: S \to X$ be a holomorphic map into a quasi-projective algebraic manifold X. Then there is a sequence of holomorphic maps $f_\nu: \Omega \to X$ such that $f_\nu \to f$ uniformly on Ω and the images $f_\nu(\Omega)$ are contained in algebraic subvarieties A_ν of X with* $\dim A_\nu = \dim f_\nu(\Omega)$. *Moreover, if we are given a positive integer k and a finite set of points (t_j) in Ω_0, then the f_ν can be taken to have the same k-jets as f at each of the points t_j.*

We must point out at this stage that Theorem 1.6 completely breaks down for singular Stein spaces. In fact, there are examples of germs of analytic sets with nonisolated singularities which are not biholomorphic to germs of algebraic sets; Whitney [Wh] has even given an example which is not C^1-diffeomorphic to an algebraic germ. The technique used by Stout and Tancredi-Tognoli to prove Theorem 1.6 consists of a generalization to the complex case of the methods introduced by Nash [Nh]. In the present work, we give a different proof of Theorem 1.6 based on some of the arguments used in the proof of the approximation theorem 1.1. Our proof starts with the observation that every Stein domain $\Omega \subset\subset S$ can be properly embedded in an affine algebraic manifold in such a way that the normal bundle is trivial. This key step also yields the following parallel result for holomorphic vector bundles over Stein manifolds. (A similar result has also been given by Tancredi and Tognoli [TT3, Theorem 4.1]; see Section 3.)

THEOREM 1.8. *Let E be a holomorphic vector bundle on an n-dimensional Stein manifold S. For every Runge open set $\Omega \subset\subset S$, there exist an n-dimensional projective algebraic manifold Z, an algebraic vector bundle $\tilde{E} \to Z$, an ample divisor D in Z, and a holomorphic injection $i: \Omega \hookrightarrow Z \backslash D$ with the following properties:*

(i) *$i(\Omega)$ is a Runge domain in the affine algebraic manifold $Z \backslash D$;*

(ii) *$i^*\tilde{E}$ is isomorphic to $E_{|\Omega}$.*

Theorem 1.8 is proved by observing that there is a holomorphic map from S into a Grassmannian such that the given vector bundle E on S is isomorphic to the pull-back of the universal quotient vector bundle. The desired result then follows from Theorem 1.6 applied to $\Omega \subset\subset S$ and from the existence of Nash algebraic approximations of the map from Ω to the Grassmannian (see Section 5).

A substantial part of the results of this work was obtained during a stay of the third author at Institut Fourier, Université de Grenoble I, and he wishes to thank Institut Fourier for its hospitality.

2. Holomorphic and Nash algebraic retractions. Let S be an n-dimensional Stein manifold. The Bishop-Narasimhan embedding theorem [Bi], [Na] implies that S can be embedded as a closed submanifold of a complex vector space $\mathbb{C}^N$ with, e.g., $N = 2n + 1$; by Eliashberg and Gromov [EG], it is in fact enough to take $N = [(3n + 3)/2]$, and this value is optimal for even n. Such optimal values will not be needed here. We first recall a well-known and elementary lemma about the existence of holomorphic retractions for Stein submanifolds.

LEMMA 2.1. *Let M be a Stein (not necessarily closed) submanifold of a complex manifold Y and let $n = \dim Y$.*

(i) *There exist a neighborhood U of the zero section in the normal bundle N_M of M, a neighborhood V of M in Y, and a biholomorphism $\psi: U \to V$, such that ψ maps the zero section of N_M identically onto M.*

(ii) *Let $\pi: N_M \to M$ be the natural projection. Then $\rho = \pi \circ \psi^{-1}: V \to M$ is a holomorphic retraction, i.e. a holomorphic map such that $\rho_{|M} = \mathrm{Id}_M$.*

Moreover, if Y is affine algebraic and M is a closed algebraic submanifold, then ψ and ρ can be taken to be Nash algebraic.

Proof. We first outline the proof of the well-known case where $Y = \mathbb{C}^n$. (Details can be found in [GR, page 257].) As M is Stein, by Cartan's Theorem B the normal bundle sequence

$$0 \to T_M \to T_{\mathbb{C}^n|M} \to N_M \to 0 \tag{2.1}$$

splits; i.e., there is a global morphism $\sigma: N_M \to T_{\mathbb{C}^n|M}$ which is transformed to Id_{N_M} by composition with the projection onto N_M. Then the map $\psi: N_M \to \mathbb{C}^n$ given by

$$\psi(z, \zeta) = z + \sigma(z)\cdot\zeta, \qquad \zeta \in N_{M,z}, \qquad z \in M, \tag{2.2}$$

coincides with Id_M on the zero section (which we also denote by M) of N_M, and thus the derivative $d\psi$ satisfies the equality $(d\psi)_{|T_M} = \mathrm{Id}_{T_M}$. Furthermore, at each point $z \in M$, ψ has vertical derivative $(d^v\psi)_z := d\psi_{|N_{M,z}} = \sigma_z$, and hence $d\psi$ is invertible at z. By the implicit function theorem, ψ defines a biholomorphism from a neighborhood U of the zero section of N_M onto a neighborhood V of M in $\mathbb{C}^n$. Then $\psi_{|U}: U \to V$ is the required biholomorphism, and $\rho = \pi \circ (\psi_{|U})^{-1}$ is a retraction. When M is affine algebraic, N_M can be realized as an affine algebraic manifold such that $\pi: N_M \to M$ is an algebraic map. The splitting morphism σ can also be taken to be algebraic. Then Γ_ψ is an open subset of an n-dimensional subvariety $G \subset N_M \times Y$, and $\Gamma_\rho = (\mathrm{Id}_Y \times \pi)(\Gamma_\psi^{-1})$ is contained in the n-dimensional algebraic variety $(\mathrm{Id}_Y \times \pi)(G)$.

In the case of a general complex manifold Y, it is enough to consider the case where Y is Stein, since by [Siu] every Stein subvariety of a complex space Y has arbitrarily small Stein neighborhoods. (See also [De1] for a simpler proof; this property is anyhow very easy to check in the case of nonsingular subvarieties.) Thus we can suppose Y to be a closed n-dimensional submanifold of some $\mathbb{C}^p$, e.g., with $p = 2n + 1$. Observe that the normal bundle $N_{M/Y}$ of M in Y is a subbundle of $N_{M/\mathbb{C}^p}$. By the first case applied to the pairs $M \subset \mathbb{C}^p$, $Y \subset \mathbb{C}^p$, we get a splitting $\sigma': N_M \to T_{\mathbb{C}^p|M}$ of the normal bundle sequence for $N_{M/\mathbb{C}^p}$, a map ψ': $N_{M/\mathbb{C}^p} \to \mathbb{C}^p$ inducing a biholomorphism $\psi': U' \to V'$, and a retraction $\rho'': V'' \to Y$, where U' is a neighborhood of the zero section of $N_{M/\mathbb{C}^p}$, V' a neighborhood of M in $\mathbb{C}^p$, and V'' a neighborhood of Y. We define ψ to be the composition

$$\psi: N_{M/Y} \hookrightarrow N_{M/\mathbb{C}^p} \xrightarrow{\psi'} \mathbb{C}^p \xrightarrow{\rho''} Y.$$

Of course, ψ is not defined on the whole of $N_{M/Y}$, but only on $N_{M/Y} \cap (\psi')^{-1}(V'')$. Clearly $\psi_{|M} = \mathrm{Id}_M$, and the property $(d^v\psi)_{|M} = \sigma'_{|N_{M/Y}}$ follows from the equalities $(d^v\psi')_{|M} = \sigma'$ and $(d\rho'')_{|T_Y} = \mathrm{Id}_{T_Y}$. It follows as before that ψ induces a biholomorphism $U \to V$ from a neighborhood of the zero section of $N_{M/Y}$ onto a neighborhood V of M. If Y and M are algebraic, then we can take ψ' to be algebraic and $\Gamma_{\rho''}$ to be contained in an algebraic p-dimensional subvariety G'' of $\mathbb{C}^p \times Y$. Then Γ_ψ is contained in some n-dimensional component of the algebraic set

$$(N_{M/Y} \times Y) \cap (\psi' \times \mathrm{Id}_Y)^{-1}(G'').$$

Therefore ψ is Nash algebraic, and $\rho = \pi \circ \psi^{-1}$ is also Nash algebraic as before.

□

In fact, the retraction ρ of Lemma 2.1 can be taken to be Nash algebraic in slightly more general circumstances, as is provided by Lemma 2.3 below. First we need the following elementary lemma due to Tancredi and Tognoli.

LEMMA 2.2 [TT1, Corollary 2]. *Let A be an algebraic subvariety (not necessarily reduced) of an affine algebraic variety S, and let Ω be a Runge domain in S. Suppose that h is a holomorphic function on Ω and γ is an algebraic function on A such that $h_{|A\cap\Omega} = \gamma_{|A\cap\Omega}$. Then h can be approximated uniformly on each compact subset of Ω by algebraic functions h_ν on S with $h_{\nu|A} = \gamma$.*

Tancredi and Tognoli [TT1] consider only reduced subvarieties A, but their proof is valid for nonreduced A. We give the proof here for the reader's convenience. To verify Lemma 2.2, one first extends γ to an algebraic function $\tilde{\gamma}$ on S. By replacing h with $h - \tilde{\gamma}$, we may assume that $\gamma = 0$. Choose global algebraic functions $\tau_1, \ldots, \tau_p$ generating the ideal sheaf $\mathscr{I}_A$ at each point of S, and consider the surjective sheaf homomorphism (of algebraic or analytic sheaves) $\mathcal{O}_S^p \xrightarrow{\tau} \mathscr{I}_A$ given by $\tau(f^1, \ldots, f^p) = \sum \tau_j f^j$. Now choose $f = (f^1, \ldots, f^p) \in H^0(\Omega, \mathcal{O}^p)$ such that $\tau(f) = h$. Since Ω is Runge in S, we can approximate the f^j by algebraic functions f_ν^j on S. Then the functions $h_\nu = \sum \tau_j f_\nu^j$ provide the desired algebraic approximations of f.

We now find it convenient to extend the usual definition of a retraction map by saying that a map ρ: $V \to S$ from a subset $V \subset X$ to a subset $S \subset X$ (where X is any space) is a retraction if $\rho(x) = x$ for all $x \in S \cap V$ (even if $S \not\subset V$). The following lemma on approximating holomorphic retractions by Nash algebraic retractions is a variation of a result of Tancredi and Tognoli [TT2, Theorem 1.5].

LEMMA 2.3. *Let V be a Runge domain in $\mathbb{C}^n$ and let S be an algebraic subset of $\mathbb{C}^n$ such that the variety $M := S \cap V$ is smooth. Suppose that ρ: $V \to S$ is a holomorphic retraction and $V_0 \subset\subset V$ is a relatively compact domain such that $\rho(V_0) \subset\subset M$. Then there is a sequence of Nash algebraic retractions ρ_ν: $V_0 \to M$ such that $\rho_\nu \to \rho$ uniformly on V_0.*

Proof. First we consider the special case where ρ is the holomorphic retraction constructed in Lemma 2.1. In this case, the conclusion follows from the proof of Theorem 1.5 in [TT2], which is based on an ingenious construction of Nash [Nh]. We give a completely different short proof of this case here. Let V, S, M satisfy the hypotheses of Lemma 2.3, and suppose that $\sigma\colon N_M \to T_{\mathbb{C}^n|M}$, $\psi\colon U \to V$, and $\rho = \pi \circ \psi^{-1}\colon V \to M$ are given as in the proof of Lemma 2.1 (for the case $Y = \mathbb{C}^n$). We first construct a "normal bundle sequence" for the possibly singular subvariety S. Let $f_1, \ldots, f_r$ be polynomials generating the ideal of S at all points of $\mathbb{C}^n$. Let

$$d\mathscr{I}_S \subset \mathcal{O}_S \otimes T^*_{\mathbb{C}^n} = T^*_{\mathbb{C}^n|S} \simeq \mathcal{O}_S^n$$

denote the coherent sheaf on S generated by the 1-forms $(1_{\mathcal{O}_S} \otimes df_j)$. (This definition is independent of the choice of generators (f_j) of $\mathscr{I}_S$.) We consider the "normal sheaf"

$$\mathscr{N}_S = \operatorname{Image}(T_{\mathbb{C}^n|S} \to \mathscr{H}om(d\mathscr{I}_S, \mathcal{O}_S)),$$

where we identify $T_{\mathbb{C}^n|S} \simeq \mathscr{H}om(T^*_{\mathbb{C}^n|S}, \mathcal{O}_S)$. Thus we have an exact sequence of sheaves of the form

$$0 \to \mathscr{K} \to T_{\mathbb{C}^n|S} \xrightarrow{\tau} \mathscr{N}_S \to 0. \tag{2.3}$$

Note that $\mathscr{N}_{S|M} = N_M$ and the restriction of (2.3) to the submanifold M is the exact sequence (2.1). Let $\mathscr{F} = \mathscr{H}om(\mathscr{N}_S, T_{\mathbb{C}^n|S})$ and choose a surjective algebraic sheaf morphism $\mathcal{O}_S^p \xrightarrow{\alpha} \mathscr{F}$. Recall that

$$\sigma \in \operatorname{Hom}(N_M, T_{\mathbb{C}^n|M}) = H^0(M, \mathscr{F}). \tag{2.4}$$

Since M is Stein, we can choose $h \in H^0(M, \mathcal{O}_S^p) = \mathcal{O}(M)^p$ such that $\alpha(h) = \sigma$. Since V is Runge, we can find a sequence h_ν of algebraic sections in $H^0(S, \mathcal{O}_S^p)$ such that $h_\nu \to h$ uniformly on each compact subset of M. We let

$$\sigma_\nu = \alpha(h_\nu) \in H^0(S, \mathscr{F}). \tag{2.5}$$

Let $\tilde{M} \supset M$ denote the set of regular points of S. Then $\mathscr{N}_{S|\tilde{M}} = N_{\tilde{M}}$ and the sections $\sigma_{\nu|\tilde{M}} \in \operatorname{Hom}(N_{\tilde{M}}, T_{\mathbb{C}^n|\tilde{M}})$ are algebraic. Recalling (2.2), we similarly define the algebraic maps $\psi_\nu\colon N_{\tilde{M}} \to \mathbb{C}^n$ by

$$\psi_\nu(z, \zeta) = z + \sigma_\nu(z)\cdot\zeta, \qquad \zeta \in N_{\tilde{M},z}, \qquad z \in \tilde{M}. \tag{2.6}$$

Choose open sets V_1, V_2 with $V_0 \subset\subset V_1 \subset\subset V_2 \subset\subset V$. Then $\psi_\nu \to \psi$ uniformly on the set $U' = \psi^{-1}(V_2) \subset\subset U$, and for $\nu \gg 0$, $\psi_{\nu|U'}$ is an injective map with $\psi_\nu(U') \supset V_1$. It follows that the maps $\rho_\nu = \pi \circ \psi_\nu^{-1}\colon V_0 \to M$ converge uniformly on V_0 to ρ.

(We remark that this proof gives algebraic maps ψ_ν, while the method of [TT2] provides only Nash algebraic ψ_ν.)

We now consider the general case. Choose a Runge domain V_1 such that $V_0 \subset\subset V_1 \subset\subset V$ and $\rho(V_0) \subset\subset V_1$, and let $\varepsilon > 0$ be arbitrary. It suffices to construct a Nash algebraic retraction $\rho'\colon V_0 \to M$ with $\|\rho' - \rho\| < 2\varepsilon$ on V_0. By Lemma 2.1 and the case already shown, there exist a neighborhood W of $M \cap V_1$ and a Nash algebraic retraction $R\colon W \to M \cap V_1$. After shrinking W if necessary, we can assume that $\|R(z) - z\| < \varepsilon$ for all $z \in W$. By Lemma 2.2 (with A, S replaced by S, $\mathbb{C}^n$), we can find a polynomial map $P = (P_1, \dots, P_n)\colon \mathbb{C}^n \to \mathbb{C}^n$ such that $\|P - \rho\| < \varepsilon$ on V_0, $P_{|S} = \mathrm{Id}_S$, and $P(V_0) \subset W$. We consider the Nash algebraic retraction

$$\rho' = R \circ P_{|V_0}\colon V_0 \to M.$$

Then $\|\rho' - \rho\|_{V_0} \leqslant \|R \circ P - P\|_{V_0} + \|P - \rho\|_{V_0} < 2\varepsilon$. □

LEMMA 2.4. *Let M be a Stein submanifold of a complex manifold Y, and let $E \to Y$ be a holomorphic vector bundle. Fix a holomorphic retraction $\rho\colon V \to M$ of a neighborhood V of M onto M. Then there is a neighborhood $V' \subset V$ of M such that E is isomorphic to $\rho^*(E_{|M})$ on V'. In particular, if the restriction $E_{|M}$ is trivial, then E is trivial on a neighborhood of M.*

Proof. By [Siu] we can find a Stein neighborhood $V_0 \subset V$. As V_0 is Stein, the identity map from $E_{|M}$ to $(\rho^*E)_{|M}$ can be extended to a section of $\mathrm{Hom}(E, \rho^*E)$ over V_0. By continuity, this homomorphism is an isomorphism on a sufficiently small neighborhood $V' \subset V_0$ of M. □

Finally, we need an elementary lemma on the existence of generating sections for holomorphic vector bundles on Stein spaces.

LEMMA 2.5. *Let E be a holomorphic vector bundle of rank r on an n-dimensional Stein space S. Suppose that $\mathscr{I}$ is a coherent sheaf of ideals in $\mathcal{O}_S$, and let $A =$ Supp $\mathcal{O}_S/\mathscr{I}$. Then we can find finitely many holomorphic sections in $H^0(S, \mathscr{I} \otimes E)$ generating E at every point of $S \backslash A$.*

Lemma 2.5 follows by an elementary argument using Cartan's Theorem B. (In fact, a careful argument shows that we can find $n + r$ generating sections.) Note that if $\mathscr{I} = \mathcal{O}_S$, then $A = \varnothing$.

COROLLARY 2.6. *Let E be a holomorphic vector bundle over a finite dimensional Stein space S. Then there is a holomorphic map $\Phi\colon S \to G$ into a Grassmannian G such that E is isomorphic to the pull-back Φ^*Q of the universal quotient vector bundle Q on G.*

Proof. By Lemma 2.5 (with $\mathscr{I} = \mathcal{O}_S$), we can find finitely many holomorphic sections $g_1, \dots, g_m$ spanning E at all points of S. We define the holomorphic map

$$\Phi\colon S \to G(m, r), \qquad x \mapsto V_x = \{(\lambda_1, \dots, \lambda_m) \in \mathbb{C}^m\colon \textstyle\sum \lambda_j g_j(x) = 0\}$$

into the Grassmannian $G(m, r)$ of subspaces of codimension r in $\mathbb{C}^m$. Then the generators g_j define an isomorphism $\gamma: E \to \Phi^*Q$ with $\gamma_x: E_x \to Q_{\Phi(x)} = \mathbb{C}^m/V_x$ given by

$$\gamma_x(e) = \{(\mu_1, \ldots, \mu_m) \in \mathbb{C}^m: \sum \mu_j g_j(x) = e\}, \qquad e \in E_x,$$

for $x \in S$. (See, for example, [GA, Ch. V].) Hence $E \simeq \Phi^*Q$. □

3. Nash algebraic approximation on Runge domains in affine algebraic varieties. In this section, we prove the main approximation theorem 1.1 for holomorphic maps on a Runge domain Ω in an affine algebraic variety. We also give an alternate proof and variation of a result of Tancredi and Tognoli [TT3] on the exhaustion of holomorphic vector bundles on Ω by algebraic vector bundles (Proposition 3.2).

We begin the proof of Theorem 1.1 by first making a reduction to the case where $S = \mathbb{C}^n$, X is projective, and f is an embedding. To accomplish this reduction, let S be an algebraic subvariety of $\mathbb{C}^n$ and suppose that $\Omega \subset S$ and $f: \Omega \to X$ are given as in Theorem 1.1. By replacing X with a smooth completion, we can assume that X is projective. We consider the embedding

$$f_1 = (i_\Omega, f): \Omega \to X_1 = \mathbb{P}^n \times X,$$

where i_Ω denotes the inclusion map $\Omega \hookrightarrow \mathbb{C}^n \hookrightarrow \mathbb{P}^n$. Approximations of f_1 composed with the projection onto X will then give approximations of f. Since Ω is $\mathcal{O}(S)$-convex, we can construct an analytic polyhedron of the form

$$\tilde{\Omega} = \{z \in \mathbb{C}^n: |h_j(z)| \leqslant 1 \text{ for } 1 \leqslant j \leqslant s\},$$

where the h_j are in $\mathcal{O}(\mathbb{C}^n)$, such that

$$\Omega_0 \subset\subset S \cap \tilde{\Omega} \subset\subset \Omega.$$

Then $\tilde{\Omega}$ is a Runge domain in $\mathbb{C}^n$. Consider the Runge domain

$$\tilde{\Omega}_\varepsilon = \{z \in \tilde{\Omega}: |P_j(z)| < \varepsilon \text{ for } 1 \leqslant j \leqslant \ell\},$$

where the P_j are the defining polynomials for $S \subset \mathbb{C}^n$. In order to extend f_1 to $\tilde{\Omega}_\varepsilon$, we apply a result of Siu [Siu] to find a Stein neighborhood $U \subset X_1$ of $f_1(\Omega)$. Then by using the holomorphic retraction of Lemma 2.1 (applied to U embedded in an affine space), shrinking ε if necessary, we can extend $f_{1|S \cap \tilde{\Omega}_\varepsilon}$ to a holomorphic map

$$\tilde{f}_1: \tilde{\Omega}_\varepsilon \to U \subset X_1.$$

By Nash approximating the embedding

$$f_2 = (i_{\tilde{\Omega}_\varepsilon}, \tilde{f}_1): \tilde{\Omega}_\varepsilon \to X_2 = \mathbb{P}^n \times X_1,$$

we also Nash approximate f.

Thus by replacing (Ω, X, f) with $(\tilde{\Omega}_\varepsilon, X_2, f_2)$, we may assume that Ω is a Runge domain in $\mathbb{C}^n$ and $f: \Omega \to X$ is a holomorphic embedding. We make one further reduction. Choose a very ample line bundle L on X. By the proof of Lemma 2.5, we can find algebraic sections $\sigma_1, \ldots, \sigma_p$ of the algebraic line bundle $\alpha^* L^{-1} \to A$ without common zeroes. We then extend the sections $\sigma_{j|\Omega}$ to holomorphic sections $\tilde{\sigma}_1, \ldots, \tilde{\sigma}_p$ of $f^* L^{-1}$. Again by Lemma 2.5, we choose holomorphic sections

$$\tilde{\sigma}_{p+1}, \ldots, \tilde{\sigma}_{p+n+1} \in H^0(\Omega, \mathscr{I}_A \otimes f^* L^{-1})$$

such that $\tilde{\sigma}_1, \ldots, \tilde{\sigma}_{p+n+1}$ generate $f^* L^{-1}$ at all points of Ω. By the proof of Corollary 2.6, the sections $\tilde{\sigma}_1, \ldots, \tilde{\sigma}_{p+n+1}$ define a holomorphic map $\Phi: \Omega \to \mathbb{P}^{p+n}$ and an isomorphism of line bundles $\gamma: f^* L^{-1} \to \Phi^* \mathcal{O}(1)$. Similarly, the sections $\sigma_1, \ldots, \sigma_p$ define an algebraic morphism

$$\Phi_A: A \to \mathbb{P}^{p-1} \subset \mathbb{P}^{p+n}$$

and an algebraic isomorphism $\gamma_A: \alpha^* L^{-1} \to \Phi_A^* \mathcal{O}(1)$ such that $\Phi_{|A \cap \Omega} = \Phi_{A|A \cap \Omega}$ and $\gamma_{|A \cap \Omega} = \gamma_{A|A \cap \Omega}$.

We set $X' = \mathbb{P}^{p+n} \times X$ and let $L' \to X'$ be the total tensor product $L' = \mathcal{O}(1) \boxtimes L$. Then L' is very ample on X' and is thus isomorphic to the hyperplane section bundle of a projective embedding $X' \subset \mathbb{P}^{m-1}$. Hence X' can be identified with the projectivization of an affine algebraic cone $Y \subset \mathbb{C}^m$, and the line bundle $L'^{-1} = \mathcal{O}_{X'}(-1)$ can be identified with the blow-up $\tilde{Y}$ of the cone Y at its vertex. Let $\tau: \tilde{Y} \to Y$ denote the blow-down of the zero section X' of L'^{-1} (so that $\tau: \tilde{Y} \backslash X' \approx Y \backslash \{0\}$).

We consider the maps

$$f' = (\Phi, f): \Omega \to X', \qquad \alpha' = (\Phi_A, \alpha): A \to X',$$

and we note that $f'_{|A \cap \Omega} = \alpha'_{|A \cap \Omega}$. The pull-back bundle $f'^* L'$ is trivial; in fact, a trivialization of $f'^* L'$ is given by the nonvanishing global section

$$\gamma \in \operatorname{Hom}(f^* L^{-1}, \Phi^* \mathcal{O}(1)) = H^0(\Omega, f^* L \otimes \Phi^* \mathcal{O}(1)) = H^0(\Omega, f'^* L').$$

We can regard the nonvanishing section $1/\gamma \in H^0(\Omega, f'^* L'^{-1})$ as a holomorphic mapping from Ω to $\tilde{Y} \backslash X'$. Therefore $f': \Omega \to X'$ lifts to the holomorphic embedding

$$g = \tau \circ (1/\gamma): \Omega \to Y \backslash \{0\}.$$

We likewise have an algebraic morphism

$$\beta = \tau \circ (1/\gamma_A)\colon A \to Y\backslash\{0\}$$

such that β is a lift of α' and $g_{|A\cap\Omega} = \beta_{|A\cap\Omega}$. It is sufficient to Nash approximate g by maps into $Y\backslash\{0\}$ agreeing with β on A. Hence Theorem 1.1 is reduced to the following.

LEMMA 3.1. *Let Ω be a Runge open set in $\mathbb{C}^n$, let $Y \subset \mathbb{C}^m$ be an affine algebraic variety, and let $g\colon \Omega \to Y'$ be a holomorphic embedding to the set Y' of regular points of Y. Suppose that there is an algebraic subvariety A (not necessarily reduced) of $\mathbb{C}^n$ and an algebraic morphism $\beta\colon A \to Y$ such that $g_{|A\cap\Omega} = \beta_{|A\cap\Omega}$. Then g can be approximated uniformly on each relatively compact domain $\Omega_0 \subset\subset \Omega$ by Nash algebraic embeddings $g_\nu\colon \Omega_0 \to Y'$ such that $g_{\nu|A\cap\Omega_0} = g_{|A\cap\Omega_0}$.*

Proof. Consider the map $G\colon \Omega \times \mathbb{C}^p \to \mathbb{C}^p$ given by

$$G(z, w) = g(z) + w. \tag{3.1}$$

Choose Runge domains $\tilde{\Omega}_1, \tilde{\Omega}_2, \tilde{\Omega}$ in $\mathbb{C}^{n+p}$ such that

$$\overline{\Omega_0} \times \{0\} \subset \tilde{\Omega}_1 \subset\subset \tilde{\Omega}_2 \subset\subset \tilde{\Omega} \subset G^{-1}(\mathbb{C}^p \backslash Y_{\text{sing}}),$$

and let $M = G^{-1}(Y) \cap \tilde{\Omega} = G^{-1}(Y') \cap \tilde{\Omega}$. Since G is a submersion, it follows that M is a closed submanifold of $\tilde{\Omega}$. We note that $G(z, 0) = g(z)$ for $z \in \Omega$, and thus $M \supset \Omega_0 \times \{0\}$.

By the proof of Lemma 2.1, it follows that M is a closed submanifold of $\tilde{\Omega}$. We note that $G(z, 0) = g(z)$ for $z \in \Omega$, and thus $M \supset \Omega_0 \times \{0\}$.

By the proof of Lemma 2.1, there is a neighborhood $V \subset \tilde{\Omega}$ of M together with a biholomorphism $\psi\colon U \to V$, where $U \subset N_M$ is a neighborhood of the zero section M, such that $\psi_{|M} = \mathrm{Id}_M$. Let

$$V_i = \{z \in \tilde{\Omega}_i\colon |h_j(z)| < i\delta \text{ for } 1 \leqslant j \leqslant \ell\}, \qquad i = 1, 2,$$

where we choose $h_1, \ldots, h_\ell \in \mathcal{O}(\tilde{\Omega})$ defining M, and we choose $\delta > 0$ such that $V_2 \subset\subset V$. We then have the holomorphic retraction $\rho = \pi \circ \psi^{-1}\colon V \to M$, where $\pi\colon N_M \to M$ is the projection. By shrinking δ if necessary, we can assume that $\rho(V_1) \subset M \cap V_2 = M \cap \tilde{\Omega}_2$.

We identify $A \subset \mathbb{C}^n$ with $A \times \{0\} \subset \mathbb{C}^{n+p}$ so that we can regard A as an algebraic subvariety (not necessarily reduced) of $\mathbb{C}^{n+p}$; then $G_{|A\cap\tilde{\Omega}} = \beta_{|A\cap\tilde{\Omega}}$. Since $\tilde{\Omega}$ is Runge, by Lemma 2.2 we can choose a sequence of polynomial maps $G_\nu\colon \mathbb{C}^{n+p} \to \mathbb{C}^m$ such that $G_{\nu|A} = \beta$ and $G_\nu \to G$ uniformly on a Runge domain $D \subset \tilde{\Omega}$ containing $\overline{V}_2$. We write $M_\nu = G_\nu^{-1}(Y) \cap D'$, where D' is a Runge domain with $V_2 \subset\subset D' \subset\subset D$. Then for $\nu \gg 0$, M_ν is a submanifold of D'. Moreover, since $\pi_{|M}$ is the identity, for large ν the restriction of π to $\psi^{-1}(M_\nu) \cap \pi^{-1}(M \cap V_2)$ is a biholomorphic map onto $M \cap V_2$. The inverse of this latter map is a section $t_\nu \in H^0(M \cap V_2, N_M)$, and $t_\nu \to 0$ uniformly on $M \cap V_1$. We now consider the holomorphic retractions

$$\rho_\nu = \psi \circ t_\nu \circ \rho_{|V_1}: V_1 \to M_\nu. \tag{3.2}$$

Then $\rho_\nu \to \rho$ uniformly on V_1. Choose a domain V_0 such that $\overline{\Omega_0} \times \{0\} \subset V_0 \subset\subset V_1$ and $\rho(V_0) \subset\subset V_1$. Since V_1 is Runge in $\mathbb{C}^{n+p}$ and $\rho_\nu(V_0) \subset\subset V_1$ for $\nu \gg 0$, by Lemma 2.3 we can find Nash algebraic retractions $\rho'_\nu: V_0 \to M_\nu$ sufficiently close to ρ_ν such that $\rho'_\nu \to \rho$ uniformly on V_0. Then the Nash algebraic approximations $g_\nu: \Omega_0 \to Y'$ of g can be given by

$$g_\nu(z) = G_\nu \circ \rho'_\nu(z, 0), \qquad z \in \Omega_0. \tag{3.3}$$

It remains to verify that

$$g_{\nu|A\cap\Omega_0} = \beta_{|A\cap\Omega_0}. \tag{3.4}$$

Since ρ'_ν is a retraction to M_ν, we have

$$G_\nu \circ \rho'_{\nu|M_\nu} = G_\nu. \tag{3.5}$$

It suffices to show that $\mathscr{I}_{M_\nu} \subset \mathscr{I}_{A|D'}$ (where $\mathscr{I}_A$ now denotes the ideal sheaf on $\mathbb{C}^{n+p}$ with $\mathcal{O}_A = \mathcal{O}_{\mathbb{C}^{n+p}}/\mathscr{I}_A$), since this would give us an inclusion morphism $A \cap D' \hookrightarrow M_\nu$, and then (3.4) would follow by restricting (3.5) to $A \cap D'$ and recalling that $G_{\nu|A} = \beta$. Let $h \in \mathscr{I}_{Y, G_\nu(a)}$ be arbitrary, where $a \in A \cap D'$. Since $G_{\nu|A} = G_{|A}$, it follows that

$$h \circ G_\nu - h \circ G \in \mathscr{I}_{A,a}. \tag{3.6}$$

Since $\Omega \times \{0\} \subset M$, we have

$$h \circ G \in \mathscr{I}_{M,a} \subset \mathscr{I}_{\Omega \times \{0\},a} \subset \mathscr{I}_{A,a},$$

and thus it follows from (3.6) that $h \circ G_\nu \in \mathscr{I}_{A,a}$. Since $G_\nu^* \mathscr{I}_{Y, G_\nu(a)}$ generates $\mathscr{I}_{M_\nu,a}$, it follows that $\mathscr{I}_{M_\nu,a} \subset \mathscr{I}_{A,a}$. □

Tancredi and Tognoli [TT3, Theorem 4.1] showed that if E is a holomorphic vector bundle on a Runge domain Ω in an affine algebraic variety S, then for every domain $\Omega_0 \subset\subset \Omega$, $E_{|\Omega_0}$ is isomorphic to a "Nash algebraic vector bundle" $E' \to \Omega_0$. Tancredi and Tognoli demonstrate this result by applying a result of Nash [Nh] on Nash-algebraically approximating analytic maps into the algebraic manifold of $n \times n$ matrices of rank exactly r. (Alternatively, one can approximate analytic maps into a Grassmannian.) Theorem 1.1 is a generalization of (the complex form of) this approximation result of Nash. In fact, Theorem 1.1 can be used to obtain the following form of the Tancredi-Tognoli theorem with a more explicit description of the equivalent Nash algebraic vector bundle.

PROPOSITION 3.2. *Let Ω be a Runge domain in an n-dimensional affine algebraic variety S and let $E \to \Omega$ be a holomorphic vector bundle. Then for every relatively compact domain $\Omega_0 \subset\subset \Omega$ there exist the following:*
(i) *an n-dimensonal projective algebraic variety Z,*
(ii) *a Nash algebraic injection $i\colon \Omega_0 \hookrightarrow Z$,*
(iii) *an algebraic vector bundle $\tilde{E} \to Z$ such that $i^*\tilde{E}$ is isomorphic to $E_{|\Omega_0}$,*
(iv) *an ample line bundle $L \to Z$ with trivial restriction i^*L.*
Moreover, if S is smooth, then Z can be taken to be smooth.

Proof. By Corollary 2.6, there is a holomorphic map $\Phi\colon \Omega \to G$ into a Grassmannian, such that $E \simeq \Phi^*Q$ where Q is the universal quotient vector bundle. We construct a holomorphic map of the form

$$f = (\Phi, \Phi')\colon \Omega \to X = G \times \mathbb{P}^N,$$

where $\Phi'\colon S \to \mathbb{P}^N$ is chosen in such a way that f is an embedding and X has an ample line bundle L with trivial pull-back f^*L (using the same argument as in the reduction of Theorem 1.1 to Lemma 3.1). Then $E \simeq f^*Q_X$, where Q_X is the pull-back of Q to X.

Choose a Runge domain Ω_1 with $\Omega_0 \subset\subset \Omega_1 \subset\subset \Omega$. By Theorem 1.1, $f_{|\Omega_1}$ is a uniform limit of Nash algebraic embeddings $f_\nu\colon \Omega_1 \to X$. Hence the images $f_\nu(\Omega_1)$ are contained in algebraic subvarieties $A_\nu \subset X$ of dimension $n = \dim S$. Let $f' = f_{\mu|\Omega_0}$, where μ is chosen to be sufficiently large so that f' is homotopic to $f_{|\Omega_0}$. Then the holomorphic vector bundle $E' := f'^*Q_X$ is topologically isomorphic to $E_{|\Omega_0}$. By a theorem of Grauert [Gra], $E' \simeq E_{|\Omega_0}$ as holomorphic vector bundles. If S is singular, we take $Z = A_\mu$, $i = f'$, and $\tilde{E} = E'$.

If S is smooth, we must modify A_μ to obtain an appropriate smooth variety Z. To accomplish this, let $\sigma\colon A''_\mu \to A_\mu$ be the normalization of A_μ and let $f''\colon \Omega_0 \to A''_\mu$ be the map such that $f' = \sigma \circ f''$; since f' is an embedding into X, we see that $f''(\Omega_0)$ is contained in the set of regular points of A''_μ. (The reason for the introduction of the normalization is that $f'(\Omega_0)$ can contain multiple points of A_μ.) By Hironaka's resolution of singularities [Hi], there exists a desingularization $\pi\colon Z \to A''_\mu$ with center contained in the singular locus of A''_μ. Let $i\colon \Omega_0 \hookrightarrow Z$ denote the embedding given by $f'' = \pi \circ i$. We note that the exceptional divisor of Z does not meet $i(\Omega_0)$. The algebraic vector bundle $\tilde{E} := (\sigma \circ \pi)^*Q_X$ then satisfies

$$i^*\tilde{E} = (\sigma \circ \pi \circ i)^*Q_X = f'^*Q_X = E' \simeq E_{|\Omega_0}.$$

Finally, we note that the line bundle $L'' := \sigma^*L_{|A_\mu}$ is ample on A''_μ and thus there is an embedding $A''_\mu \subset \mathbb{P}^m$ such that $\mathcal{O}(1)_{|A''_\mu} = L^b$ for some $b \in \mathbb{N}$ (where $\mathcal{O}(1)$ is the hyperplane section bundle in $\mathbb{P}^m$). We can suppose that π is an embedded resolution of singularities with respect to this embedding; i.e., there is a modification $\pi: \tilde{X} \to \mathbb{P}^m$ such that $\tilde{X}$ is smooth and $Z \subset \tilde{X}$ is the strict transform of A''_μ. Then there is an exceptional divisor $D \geqslant 0$ of the modification $\pi: \tilde{X} \to \mathbb{P}^m$ such that $\tilde{L} := \pi^*\mathcal{O}(a) \otimes \mathcal{O}(-D)$ is ample on $\tilde{X}$ for $a \gg 0$. (We can take $D = \pi^*[\pi(H)] - [H]$, for an ample hypersurface $H \subset \tilde{X}$ that does not contain any component of the exceptional divisor of $\tilde{X}$.) We can assume that μ has been chosen sufficiently large so that $f'^*L \simeq (f^*L)_{|\Omega_0}$ is trivial. By construction $i^*\mathcal{O}_Z(D)$ is trivial, and hence so is $i^*(\tilde{L}_{|Z})$. □

Remark 3.3. Here, it is not necessary to use Grauert's theorem in its full strength. The special case we need can be dealt with by means of the results obtained in Section 2. In fact, for μ large, there is a one-parameter family of holomorphic maps $F: \Omega_1 \times U \to X$, where U is a neighborhood of the interval $[0, 1]$ in $\mathbb{C}$, such that $F(z, 0) = f(z)$ and $F(z, 1) = f_\mu(z)$ on Ω_1; this can be seen by taking a Stein neighborhood V of $f(\Omega)$ in X and a biholomorphism $\varphi = \psi^{-1}$ from a neighborhood of the diagonal in $V \times V$ onto a neighborhood of the zero section in the normal bundle (using Lemma 2.1(i)); then $F(z, w) = \psi(w\varphi(f(z), f_\mu(z)))$ is the required family. Now, if Y is a Stein manifold and U is connected, Lemma 2.4 and an easy connectedness argument imply that all slices over Y of a holomorphic vector bundle $B \to Y \times U$ are isomorphic, at least after we restrict to a relatively compact domain in Y. Hence

$$E' = f_\mu^* Q_{X|\Omega_0} = (F^*Q_X)_{|\Omega_0 \times \{1\}} \simeq (F^*Q_X)_{|\Omega_0 \times \{0\}} = f^*Q_{X|\Omega_0} \simeq E_{|\Omega_0}.$$

Remark 3.4. In general, the embedding $i: \Omega_0 \hookrightarrow Z$ in Proposition 3.2 will not extend to a (univalent) rational map $S \dashrightarrow Z$, but will instead extend as a multivalued branched map. For example, let $S = X \backslash H$, where H is a smooth hyperplane section of a projective algebraic manifold X with some nonzero Hodge numbers off the diagonal. By Cornalba-Griffiths [CG, §21, Appendix 2] there is a holomorphic vector bundle $E' \to S$ such that the Chern character $\mathrm{ch}(E') \in H^{\mathrm{even}}(S, \mathbb{Q})$ is not the restriction of an element of $H^{\mathrm{even}}(X, \mathbb{Q})$ given by (rational) algebraic cycles. Choose $\Omega_0 \subset\subset \Omega \subset S$ such that the inclusion $\Omega_0 \hookrightarrow S$ induces an isomorphism $H^{\mathrm{even}}(S, \mathbb{Q}) \approx H^{\mathrm{even}}(\Omega_0, \mathbb{Q})$, and let $E = E'_{|\Omega}$. Now let $i: \Omega_0 \hookrightarrow Z$ and $\tilde{E} \to Z$ be as in the conclusion of Proposition 3.2. We assert that i cannot extend to a rational map on S. Suppose on the contrary that i has a (not necessarily regular) rational extension $i': X \dashrightarrow Z$. By considering the graph $\Gamma_{i'} \subset X \times Z$ and the projections $\mathrm{pr}_1: \Gamma_{i'} \to X$, $\mathrm{pr}_2: \Gamma_{i'} \to Z$, we obtain a coherent algebraic sheaf $\mathscr{F} = \mathrm{pr}_{1*}\mathcal{O}(\mathrm{pr}_2^*\tilde{E})$ on X with restriction $\mathscr{F}_{|\Omega_0} = i^*\tilde{E} \simeq E_{|\Omega_0}$, and therefore $\mathrm{ch}(\mathscr{F})_{|\Omega_0} = \mathrm{ch}(E')_{|\Omega_0}$. Now $\mathrm{ch}(\mathscr{F})$ is given by algebraic cycles in X, but by the above, $\mathrm{ch}(\mathscr{F})_{|S} = \mathrm{ch}(E')$, which contradicts our choice of E'. In particular, $E_{|\Omega_0}$ is not isomorphic to an algebraic vector bundle on S.

Problem 3.5. The method of proof used here leads to the following natural question. Suppose that $f: \Omega \to X$ is given as in Theorem 1.1, and assume in addition that S is nonsingular and $\dim X \geqslant 2 \dim S + 1$. Is it true that f can be approximated by Nash algebraic embeddings $f_\nu\colon \Omega_\nu \to X$ such that $f_\nu(\Omega_\nu)$ is contained in a *nonsingular* algebraic subvariety A_ν of X with $\dim A_\nu = \dim f_\nu(\Omega_\nu) = \dim S$? If $\dim X = 2\dim S$, does the same conclusion hold with f_ν being an immersion and A_ν an immersed nonsingular variety with simple double points?

Our expectation is that the answer should be positive in both cases, because there is a priori enough space in X to get the smoothness of $f_\nu(\Omega_\nu)$ by a generic choice of f_ν, by a Whitney-type argument. The difficulty is to control the singularities introduced by the Nash algebraic retractions. If we knew how to do this, the technical point of using a desingularization of A_μ in the proof of Proposition 3.2 could be avoided.

4. Nash algebraic approximations omitting ample divisors. In this section we use L^2 approximation techniques to give the following conditions for which the Nash algebraic approximations in Theorem 1.1 can be taken to omit ample divisors.

THEOREM 4.1. *Let Ω be a Runge domain in an affine algebraic manifold S, and let $f\colon \Omega \to X$ be a holomorphic embedding into a projective algebraic manifold X (with* $\dim S < \dim X$*). Suppose that there exists an ample line bundle L on X with trivial restriction f^*L to Ω. Then for every relatively compact domain $\Omega_0 \subset\subset \Omega$, one can find sections $h_\nu \in H^0(X, L^{\otimes\mu(\nu)})$ and Nash algebraic embeddings $f_\nu\colon \Omega_0 \to X\setminus h_\nu^{-1}(0)$ converging uniformly to $f_{|\Omega_0}$. Moreover, if we are given a positive integer m and a finite set of point (t_j) in Ω_0, then the f_ν can be taken to have the same m-jets as f at each of the points t_j.*

COROLLARY 4.2. *Let $\Omega_0 \subset\subset \Omega \subset S$ be as in Theorem 4.1. Suppose that $H^2(\Omega; \mathbb{Z}) = 0$. Then for every holomorphic embedding $f\colon \Omega \to X$ into a projective algebraic manifold X (with* $\dim S < \dim X$*), one can find affine Zariski open sets $Y_\nu \subset X$ and Nash algebraic embeddings $f_\nu\colon \Omega_0 \to Y_\nu$ converging uniformly to $f_{|\Omega_0}$. Moreover, if we are given a positive integer m and a finite set of points (t_j) in Ω_0, then the f_ν can be taken to have the same m-jets as f at each of the points t_j.*

The main difficulty in proving Theorem 4.1 is that the submanifold $f(\Omega)$ might not be contained in an affine open set. To overcome this difficulty, we first shrink $f(\Omega)$ a little bit and apply a small perturbation to make $f(\Omega)$ smooth up to the boundary with essential singularities at each boundary point. Then $f(\Omega)$ becomes in some sense very far from being algebraic, but with an additional assumption on $f(\Omega)$ it is possible to show that $f(\Omega)$ is actually contained in an affine Zariski open subset (Corollary 4.9). The technical tools needed are Hörmander's L^2 estimates for $\bar\partial$ and the following notion of complete pluripolar set.

Definition 4.3. Let Y be a complex manifold.

(i) A function $u\colon Y \to [-\infty, +\infty[$ is said to be quasi-plurisubharmonic (quasi-psh for short) if u is locally equal to the sum of a smooth function and of a plurisubharmonic (psh) function, or equivalently, if $\sqrt{-1}\partial\bar\partial u$ is bounded below by a continuous real (1, 1)-form.

(ii) A closed set P is said to be complete pluripolar in Y if there exists a quasi-psh function u on Y such that $P = u^{-1}(-\infty)$.

Remark 4.4. Note that the sum and the maximum of two quasi-psh functions is quasi-psh. (However, the decreasing limit of a sequence of quasi-psh functions may not be quasi-psh, since any continuous function is the decreasing limit of smooth functions and there are many continuous non-quasi-psh functions, e.g., $-\log^+ |z|$.) The usual definition of pluripolar sets deals with psh functions rather than quasi-psh functions; our choice is motivated by the fact that we want to work on compact manifolds, and of course, there are no nonconstant global psh functions in that case. It is elementary to verify that if u is a quasi-psh function on a Stein manifold S, then $u + \varphi$ is psh for some smooth, rapidly growing function φ on S, and hence our definition of complete pluripolar sets coincides with the usual definition in the case of Stein manifolds. Finally, we remark that the word "complete" refers to the fact that P must be the full polar set of u, and not only part of it.

LEMMA 4.5. *Let P be a closed subset in a complex manifold Y.*

(i) *If P is complete pluripolar in Y, there is a quasi-psh function u on Y such that $P = u^{-1}(-\infty)$ and u is smooth on $Y \backslash P$.*

(ii) *P is complete pluripolar in Y if and only if there is an open covering (Ω_j) of Y such that $P \cap \Omega_j$ is complete pluripolar in Ω_j.*

Proof. We first prove result (i) locally, say, on a ball $\Omega = B(0, r_0) \subset \mathbb{C}^n$, essentially by repeating the arguments given in Sibony [Sib]. Let P be a closed set in Ω, and let v be a psh function on Ω such that $P = v^{-1}(-\infty)$. By shrinking Ω and subtracting a constant from v, we may assume $v \leqslant 0$. Select a convex increasing function $\chi\colon [0, 1] \to \mathbb{R}$ such that $\chi(t) = 0$ on $[0, 1/2]$ and $\chi(1) = 1$. We set

$$w_k = \chi(\exp(v/k)).$$

Then $0 \leqslant w_k \leqslant 1$, w_k is plurisubharmonic on Ω and $w_k = 0$ in a neighborhood of P. Fix a family (ρ_ε) of smoothing kernels and set $\Omega' = B(0, r_1) \subset\subset \Omega$. For each k, there exists $\varepsilon_k > 0$ such that $w_k * \rho_{\varepsilon_k}$ is well defined on Ω' and vanishes on a neighborhood of $P \cap \Omega'$. Then

$$h_k = \max\{w_1 * \rho_{\varepsilon_1}, \ldots, w_k * \rho_{\varepsilon_k}\}$$

is an increasing sequence of continuous psh functions such that $0 \leqslant h_k \leqslant 1$ on Ω' and $u_k = 0$ on a neighborhood of $P \cap \Omega'$. The inequalities

$$h_k \geqslant w_k * \rho_{\varepsilon_k} \geqslant w_k = \chi(\exp(v/k))$$

imply that $\lim h_k = 1$ on $\Omega' \backslash P$. By Dini's lemma, (h_k) converges uniformly to 1 on every compact subset of $\Omega'\backslash P$. Thus, for a suitable subsequence (k_ν), the series

$$h(z) := |z|^2 + \sum_{\nu=0}^{+\infty} (h_{k_\nu}(z) - 1) \tag{4.1}$$

converges uniformly on every compact subset of $\Omega'\backslash P$ and defines a strictly plurisubharmonic function on Ω' which is continuous on $\Omega'\backslash P$ and such that $h = -\infty$ on $P \cap \Omega'$. Richberg's regularization theorem [Ri] implies that there is a psh function u on Ω' such that u is smooth on $\Omega'\backslash P$ and $h \leqslant u \leqslant h + 1$. Then $u^{-1}(-\infty) = P \cap \Omega'$ and property (i) is proved on Ω'.

(ii) The "only if" part is clear, so we just consider the "if" part. By means of (i) and by taking a refinement of the covering, we may assume that we have open sets $\tilde{\Omega}_j \supset\supset \Omega_j \supset\supset \Omega'_j$ where (Ω'_j) is a locally finite open covering of Y and for each j there is a psh function u_j on $\tilde{\Omega}_j$ such that $P \cap \tilde{\Omega}_j = u_j^{-1}(-\infty)$ with u_j smooth on $\tilde{\Omega}_j\backslash P$. We define inductively a strictly decreasing sequence (t_ν) of real numbers by setting $t_0 = 0$ and

$$t_{\nu+1} \leqslant \inf\left\{ \bigcup_{j,k\leqslant\nu} u_j(\bar{\Omega}_j \cap \bar{\Omega}_k \backslash u_k^{-1}([-\infty, t_\nu[)) \right\}. \tag{4.2}$$

The infimum is finite since u_j and u_k have the same poles. Now, if we take in addition $t_{\nu+1} - t_\nu < t_\nu - t_{\nu-1}$, there exists a smooth convex function $\chi\colon \mathbb{R} \to \mathbb{R}$ such that $\chi(t_\nu) = -\nu$. Choose functions $\varphi_j \in C^\infty(\Omega_j)$ such that $\varphi_j \equiv 0$ on Ω'_j, $\varphi_j \leqslant 0$ on Ω_j, and $\varphi_j(z) \to -\infty$ as $z \to \partial\Omega_j$. We set

$$v_j = \chi \circ u_j + \varphi_j.$$

Then v_j is quasi-psh on Ω_j. Let

$$v(z) = \max_{\Omega_j \ni z} v_j(z), \qquad z \in V. \tag{4.3}$$

Clearly $v^{-1}(-\infty) = P$. To show that v is quasi-psh it suffices to verify that if $z_0 \in \partial\Omega_k$, then for $z \in \Omega_k\backslash P$ sufficiently close to z_0 we have $v_k(z) < v(z)$. This is obvious if $z_0 \notin P$ since $v_k \to -\infty$ as $z \to z_0$. So consider $z_0 \in \partial\Omega_k \cap P \cap \Omega'_j$. Suppose that $z \in \Omega_k \cap \Omega'_j\backslash P$ is sufficiently close to z_0 so that

$$\max\{u_j(z), u_k(z)\} < \min\{t_j, t_k\} \qquad \text{and} \qquad \varphi_k(z) \leqslant -3.$$

Then we have $u_j(z) \in [t_{\nu+1}, t_\nu[$ and $u_k(z) \in [t_{\mu+1}, t_\mu[$ with indices $\mu, \nu \geqslant \max\{j, k\}$. Then by (4.2), $u_j(x) \geqslant t_{\mu+2}$ and $u_k(x) \geqslant t_{\nu+2}$; hence $|\mu - \nu| \leqslant 1$ and $|\chi \circ u_j(z) - \chi \circ u_k(z)| \leqslant 2$. Therefore

$$v(z) \geqslant v_j(z) = \chi \circ u_j(z) \geqslant \chi \circ u_k(z) - 2 \geqslant v_k(z) + 1\,,$$

which completes the proof of (ii).

To obtain the global case of property (i), we replace the max function in (4.3) by the regularized max functions

$$\max_\varepsilon = \max * \rho_\varepsilon^m\colon \mathbb{R}^m \to \mathbb{R}\,,$$

where ρ_ε^m is of the form $\rho_\varepsilon^m(x_1, \ldots, x_m) = \varepsilon^{-m}\rho(x_1/\varepsilon)\cdots\rho(x_m/\varepsilon)$. This form of smoothing kernel ensures that

$$x_m \leqslant \max\{x_1, \ldots, x_{m-1}\} - 2\varepsilon \Rightarrow \max_\varepsilon(x_1, \ldots, x_m) = \max_\varepsilon(x_1, \ldots, x_{m-1})\,.$$

(Of course, $\max_\varepsilon$ is convex and is invariant under permutations of variables.) The function v_ε obtained by using $\max_\varepsilon$ in (4.3) with any $\varepsilon \leqslant 1/2$ is quasi-psh on Y, smooth on $Y\backslash P$, and has polar set P. □

Now we show that for any Stein submanifold M of a complex manifold Y, there are complete pluripolar graphs of sections $K \to N_M$ over arbitrary compact subsets $K \subset M$.

PROPOSITION 4.6. *Let M be a (not necessarily closed) Stein submanifold of a complex manifold Y, with* $\dim M < \dim Y$, *and let K be a holomorphically convex compact subset of M. Let $\psi\colon U \to V$ be as in Lemma 2.1(i). Then there is a continuous section $g\colon K \to N_M$ which is holomorphic in the interior K° of K, such that $g(K) \subset U$ and the sets $K_\varepsilon = \psi(\varepsilon g(K))$ are complete pluripolar in Y for $0 < \varepsilon \leqslant 1$.*

Here $\varepsilon g(K)$ denotes the ε-homothety of the compact section $g(K) \subset N_M$; thus K_ε converges to K as ε tends to 0.

Proof. By 4.5(ii), the final assertion is equivalent to proving that $\varepsilon g(K)$ is complete pluripolar in N_M, and we may, of course, assume that $\varepsilon = 1$. The holomorphic convexity of K implies the existence of a sequence $(f_j)_{j\geqslant 1}$ of holomorphic functions on M such that $K = \bigcap\{|f_j| \leqslant 1\}$. By Lemma 2.5, there is a finite sequence $(g_k)_{1\leqslant k\leq N}$ of holomorphic sections of N_M without common zeroes. We consider the sequence $(j_\nu, k_\nu)_{\nu\geqslant 1}$ of pairs of positive integers obtained as the concatenation $\Lambda_1, \Lambda_2, \ldots, \Lambda_\ell, \ldots$ of the finite sequences

$$\Lambda_\ell = (j_q, k_q)_{\nu(\ell)\leqslant q<\nu(\ell+1)} = ((1, 1), \ldots, (1, N), \ldots, (\ell, 1), \ldots, (\ell, N))$$

(where $\nu(\ell) = N\ell(\ell - 1)/2 + 1$). We define g to be the generalized "gap sequence"

$$g(z) = \sum_{\nu=1}^{\infty} e^{-p_\nu} f_{j_\nu}(z)^{p_\nu^2} g_{k_\nu}(z) \qquad \forall z \in K, \tag{4.4}$$

where (p_ν) is a strictly increasing sequence of positive integers to be defined later. The series (4.4) converges uniformly on K (with respect to a Hermitian metric on N_M), and thus g is continuous on K and holomorphic on K°. To construct our psh function with polar set $g(K)$ we first select a smooth Hermitian metric $\|\ \|$ on N_M such that $\zeta \mapsto \log\|\zeta\|$ is psh on N_M (e.g., take $\|\zeta\|^2 = \sum |g_i^*(z)\cdot\zeta|^2$ for $\zeta \in N_{M,z}$, where (g_i^*) is a collection of holomorphic sections of N_M^* without common zeroes). We consider the global holomorphic sections

$$s_q = \sum_{\nu=1}^{q} e^{-p_\nu} f_{j_\nu}^{p_\nu^2} g_{k_\nu} \tag{4.5}$$

of N_M, which converge to g uniformly on K exponentially fast. We consider the psh functions u_ℓ on N_M given by

$$u_\ell(\zeta) = \max_{\nu(\ell)\leqslant q<\nu(\ell+1)} \frac{1}{p_q^4}\log\|\zeta - s_q(z)\|, \qquad \forall\zeta\in N_{M,z}, \quad \forall z\in M. \tag{4.6}$$

We shall show that the infinite sum

$$u = \sum_{\ell=1}^{\infty} \max(u_\ell, -1) \tag{4.7}$$

is psh on N_M with $u^{-1}(-\infty) = g(K)$. For $z \in K$, $\zeta = g(z) \in g(K)$, we obtain from (4.4) the estimate

$$\|\zeta - s_q(z)\| \leqslant ce^{-p_{q+1}}$$

where c is independent of the choice of $z \in K$. Thus if we choose the p_q with $p_{q+1} \geqslant 2p_q^4$, we have $u_\ell < -1$ on $g(K)$ for ℓ large, and thus $u \equiv -\infty$ on $g(K)$. Next we show that u is psh. For this it suffices to show that $u \not\equiv -\infty$ and

$$\sum_{\ell=1}^{\infty} \max\left(\sup_A u_\ell, 0\right) < +\infty \tag{4.8}$$

for all compact subsets A of N_M (since (4.8) implies that $u_{|A^\circ}$ can be written as a decreasing sequence of psh functions). Let M_q be an exhausting sequence of compact subsets of M. When $\zeta \in N_{M,z}$ with $z \in M_q$, we have

$$\log\|\zeta - s_q(z)\| \leqslant \log(1 + \|\zeta\|) + (p_q^2 + 1)\log(1 + C_q)$$

where C_q is the maximum of $\|g_1\|, \ldots, \|g_N\|, |f_1|, \ldots, |f_q|$ on M_q. In addition to our previous requirement, we assume that $p_q \geqslant \log(1 + C_q)$ and hence

$$u_\ell(\zeta) \leqslant \frac{2 + \log(1 + \|\zeta\|)}{p_{\nu(\ell)}}, \qquad \forall \zeta \in N_{M,z}, \qquad \forall z \in M_{\nu(\ell)}.$$

Since $p_{\nu(\ell)} \geqslant p_\ell \geqslant 2^{\ell-1}$ (in fact, the growth is at least doubly exponential), the bound (4.8) follows.

We finally show that $u(\zeta)$ is finite outside $g(K)$. Fix a point $\zeta \in N_{M,z} - g(K)$. If $z \in K$, then the logarithms appearing in (4.6) converge to $\log\|\zeta - g(z)\| > -\infty$, and hence $u(\zeta) > -\infty$. Now suppose $z \notin K$. Then one of the values $f_j(z)$ has modulus greater than 1. Thus for ℓ sufficiently large, there is a $q = q(\ell)$ in the range $\nu(\ell) \leqslant q < \nu(\ell + 1)$ such that $|f_{j_q}(z)| > 1$ and $\|g_{k_q}(z)\| = \max\|g_i(z)\| > 0$. Assume further that $|f_{j_q}(z)| = \max_{1 \leqslant i \leqslant \ell}|f_i(z)|$. For this choice of q we have

$$\|\zeta - s_q(z)\| \geqslant e^{-p_q}|f_{j_q}(z)|^{p_q^2}|g_{k_q}(z)| - \frac{|f_{j_q}(z)|^{p_{q-1}^2+1}|g_{k_q}(z)|}{|f_{j_q}(z)| - 1} - \|\zeta\|.$$

Since $p_q \geqslant 2p_{q-1}^4$, we easily conclude that $\|\zeta - s_{q(\ell)}(z)\| > 1$ for ℓ sufficiently large, and hence $u_l(z)$ is eventually positive. Therefore $u(z) > -\infty$; hence $u^{-1}(-\infty) = g(K)$, and so $g(K)$ is complete pluripolar in N_M. □

Remark. If one chooses p_ν to be larger than the norm of the first ν derivatives of $f_1, \ldots, f_\nu$ on K in the above proof, the section g will be smooth on K.

PROPOSITION 4.7. *Let $K \subset M \subset Y$ be as in Proposition 4.6, and let I be a finite subset of the interior K° of K. Then for every positive integer m we can select the section $g\colon K \to N_M$ in Proposition 4.6 with the additional property that the m-jet of g vanishes on I (and thus K_ε is tangent to K of order m at all points of I).*

Proof. The proof of Proposition 4.6 uses only the fact that the sections $g_1, \ldots, g_N$ do not vanish simultaneously on $M \backslash K$. Hence, we can select the g_j to vanish at the prescribed order m at all points of I. □

Next, we prove the following simple approximation theorem based on Hörmander's L^2 estimates (see Andreotti-Vesentini [AV] and [Hö]).

PROPOSITION 4.8. *Let X be a projective algebraic manifold and let L be an ample line bundle on X. Let P be a complete pluripolar set in X such that L is trivial on a neighborhood Ω of P; i.e., there is a nonvanishing section $s \in H^0(\Omega, L)$. Then there is a smaller neighborhood $V \subset \Omega$ such that every holomorphic function h on Ω can be approximated uniformly on V by a sequence of global sections $h_\nu \in H^0(X, L^{\otimes\nu})$; precisely, $h_\nu s^{-\nu} \to h$ uniformly on V.*

Proof. By Lemma 4.5(i), we can choose a quasi-psh function u on X with $u^{-1}(-\infty) = P$ and u smooth on $X \backslash P$. Fix a Kähler metric ω on X and a Hermitian metric on L with positive definite curvature form $\Theta(L)$. Let $(U_\alpha)_{0 \leqslant \alpha \leqslant N}$ be an open cover of X with trivializing sections $s_\alpha \in H^0(U_\alpha, L)$. We write $\|s_\alpha\|^2 = e^{-\varphi_\alpha}$ so that for $v = v^\alpha s_\alpha(z) \in L_z$ we have $\|v\|^2 = |v^\alpha|^2 e^{-\varphi_\alpha(z)}$. We note the curvature formula, $\Theta(L)_{|U_\alpha} = \sqrt{-1}\partial\bar\partial\varphi_\alpha$. In the following, we shall take $U_0 = \Omega$ and $s_0 = s$. We shall also drop the index α and denote the square of the norm of v simply by $\|v\|^2 = |vs^{-1}|^2 e^{-\varphi(z)}$.

By multiplying u with a small positive constant, we may assume that $\Theta(L) + \sqrt{-1}\partial\bar\partial u \geqslant \varepsilon\omega$ for some $\varepsilon > 0$. Then, for $v \in \mathbb{N}$ large, Hörmander's L^2 existence theorem (as given in [De2, Theorem 3.1]) implies that for every $\bar\partial$-closed (0, 1)-form g with values in $L^{\otimes v}$ such that

$$\int_X \|g\|^2 e^{-vu}\, dV_\omega < +\infty$$

(with dV_ω = Kähler volume element), the equation $\bar\partial f = g$ admits a solution f satisfying the estimate

$$\int_X |fs^{-v}|^2 e^{-v(\varphi+u)}\, dV_\omega = \int_X \|f\|^2 e^{-vu}\, dV_\omega \leqslant \frac{1}{\lambda_v}\int_X \|g\|^2 e^{-vu}\, dV_\omega,$$

where λ_v is the minimum of the eigenvalues of

$$v(\Theta(L) + \sqrt{-1}\partial\bar\partial u) + \mathrm{Ricci}(\omega)$$

with respect to ω throughout X. Clearly $\lambda_v \geqslant \varepsilon v + \rho_0$ where $\rho_0 \in \mathbb{R}$ is the minimum of the eigenvalues of Ricci(ω). We apply the existence theorem to the (0, 1)-form $g = \bar\partial(h\theta s^v) = h\bar\partial\theta s^v$, where θ is a smooth cut-off function such that

$$\theta = 1 \qquad \text{on} \quad \{z \in \Omega \colon \varphi(z) + u(z) \leqslant c\},$$

$$\mathrm{Supp}\,\theta \subset \{z \in \Omega \colon \varphi(z) + u(z) < c + 1\} \subset\subset \Omega$$

for some sufficiently negative $c < 0$. Then we get a solution of the equation $\bar\partial f_v = h\bar\partial\theta s^v$ on X with

$$\int_X |f_v s^{-v}|^2 e^{-v(\varphi+u)}\, dV_\omega \leqslant \frac{1}{\varepsilon v + \rho_0}\int_X |h|^2 |\bar\partial\theta|^2 e^{-v(\varphi+u)}\, dV_\omega.$$

Since $\mathrm{Supp}(\bar\partial\theta) \subset \{c \leqslant \varphi(z) + u(z) \leqslant c + 1\}$, we infer

$$e^{-v(c-1)}\int_{\{\varphi+u<c-1\}} |f_v s^{-v}|^2\, dV_\omega \leqslant \frac{e^{-vc}}{\varepsilon v + \rho_0}\int_\Omega |h|^2 |\bar\partial\theta|^2\, dV_\omega.$$

Thus $f_\nu s^{-\nu} \to 0$ in $L^2(\{\varphi + u < c - 1\})$. Since the $f_\nu s^{-\nu}$ are holomorphic functions on $\{\varphi + u < c - 1\}$, it follows that $f_\nu s^{-\nu} \to 0$ uniformly on the neighborhood $V = \{\varphi + u < c - 2\} \subset\subset \Omega$ of P. Hence $h_\nu = h\theta s^\nu - f_\nu$ is a holomorphic section of $L^{\otimes \nu}$ such that $h_\nu s^{-\nu}$ converges uniformly to h on V. □

COROLLARY 4.9. *Let X be a projective algebraic manifold and let M be a Stein submanifold with* $\dim M < \dim X$, *such that there exists an ample line bundle L on X with trivial restriction $L_{|M}$. Let K be a holomorphically convex compact subset of M, and let $(K_\varepsilon)_{0<\varepsilon\leqslant 1}$ be the complete pluripolar sets constructed in Proposition 4.6. Then for each sufficiently small ε, there is a positive integer $\nu(\varepsilon)$ and a section $h_{\nu(\varepsilon)} \in H^0(X, L^{\otimes \nu(\varepsilon)})$ such that K_ε does not intersect the divisor of $h_{\nu(\varepsilon)}$; i.e., K_ε is contained in the affine Zariski open set $X \backslash h_{\nu(\varepsilon)}^{-1}(0)$.*

Proof. By Lemma 2.4, L is trivial on a neighborhood Ω of M. For $\varepsilon > 0$ small, we have $K_\varepsilon \subset \Omega$ and we can apply Proposition 4.8 to get a uniform approximation $h_{\nu(\varepsilon)} \in H^0(X, L^{\otimes \nu(\varepsilon)})$ of the function $h = 1$ such that $|h_{\nu(\varepsilon)} s^{-\nu(\varepsilon)} - h| < 1/2$ near K_ε. Then $K_\varepsilon \cap h_{\nu(\varepsilon)}^{-1}(0) = \varnothing$. □

We now use Corollary 4.9 to prove Theorem 4.1.

Proof of Theorem 4.1. Choose a domain Ω_1 with $\Omega_0 \subset\subset \Omega_1 \subset\subset \Omega$ such that Ω_1 is a Runge domain in Ω (and hence in S). By Corollary 4.9 applied to the compact set

$$K = \text{holomorphic hull of } f(\overline{\Omega}_1) \text{ in the Stein variety } f(\Omega) \subset X,$$

we get an embedding $j_\varepsilon\colon K \to X$ of the form $j_\varepsilon(z) = \psi(\varepsilon g(z))$, converging to Id_K as ε tends to 0, such that $j_\varepsilon(K)$ is contained in an affine Zariski open subset $Y_{\nu,\varepsilon} = X \backslash h_{\nu,\varepsilon}^{-1}(0)$, $h_{\nu,\varepsilon} \in H^0(X, L^{\otimes \mu(\nu,\varepsilon)})$. Moreover, by Proposition 4.7, we can take j_ε tangent to Id_K of order k at all points t_j. We first approximate f by $F_\nu = j_{\varepsilon_\nu} \circ f$ on Ω_1, choosing ε_ν so small that $\sup_{\Omega_1} \delta(F_\nu(z), f(z)) < 1/\nu$ with respect to some fixed distance δ on X. We denote $Y_\nu = Y_{\nu,\varepsilon_\nu}$ in the sequel. By construction, $F_\nu(\Omega_1) \subset j_{\varepsilon_\nu}(K) \subset Y_\nu$.

Let $Y_\nu \subset \mathbb{C}^N$ be an algebraic embedding of Y_ν in some affine space. Then F_ν can be viewed as a map

$$F_\nu = (F_{1,\nu}, \ldots, F_{N,\nu})\colon \Omega_1 \to \mathbb{C}^N, \qquad F_\nu(\Omega_1) \subset Y_\nu.$$

By Lemma 2.1, there is a retraction $\rho_\nu\colon V_\nu \to Y_\nu$ defined on a neighborhood V_ν of Y_ν, such that the graph Γ_{ρ_ν} is contained in an algebraic variety A_ν with $\dim A_\nu = N$. Now, since Ω_1 is a Runge domain in S, there are polynomial functions $P_{k,\nu}$ in the structure ring $\mathbb{C}[S]$, such that $F_{k,\nu} - P_{k,\nu}$ is uniformly small on $\overline{\Omega}_0$ for each $k = 1, \ldots, N$. Of course, we can make a further finite interpolation to ensure that $P_{k,\nu} - F_k$ vanishes at the prescribed order m at all points t_j. We set

$$P_\nu = (P_{1,\nu}, \ldots, P_{N,\nu})\colon S \to \mathbb{C}^N, \qquad f_\nu = \rho_\nu \circ P_\nu\colon \Omega_0 \to Y_\nu$$

and take our approximations $P_{k,\nu}$ sufficiently close to $F_{k,\nu}$, in such a way that f_ν is well defined on Ω_0 (i.e., $P_\nu(\Omega_0) \subset V_\nu$), is an embedding of Ω_0 into Y_ν, and satisfies $\delta(f_\nu, F_\nu) < 1/\nu$ on Ω_0. Then $\delta(f_\nu, f) < 2/\nu$ on Ω_0. Now, the graph of f_ν is just the submanifold

$$\Gamma_{f_\nu} = (\Omega_0 \times Y_\nu) \cap (P_\nu \times \mathrm{Id}_{Y_\nu})^{-1}(\Gamma_{\rho_\nu}).$$

Each connected component of Γ_{f_ν} is contained in an irreducible component $G_{\nu,j}$ of dimension $n = \dim S$ of the algebraic variety $(P_\nu \times \mathrm{Id}_{Y_\nu})^{-1}(A_\nu)$; hence Γ_{f_ν} is contained in the n-dimensional algebraic variety $G_\nu = \bigcup G_{\nu,j}$. □

Remark 4.10. The existence of an ample line bundle on X with trivial restriction to M is automatic if $H^1(M, \mathcal{O}_M^*) = H^2(M, \mathbb{Z}) = 0$, since in that case every holomorphic line bundle on M is trivial. In Corollary 4.9, the conclusion that K admits small perturbations K_ε contained in affine Zariski open sets does not hold any longer if the hypothesis on the existence of an ample line bundle L with $L_{|M}$ trivial is omitted. In fact, let M be a Stein manifold and let K be a holomorphically convex compact set such that the image of the restriction map $H^2(M, \mathbb{Z}) \to H^2(K, \mathbb{Z})$ contains a nontorsion element. Then there exists a holomorphic line bundle E on M such that all positive powers $E_{|K}^{\otimes m}$ are topologically nontrivial. Let $g_0, \ldots, g_N \in H^0(M, E)$ be a collection of sections which generate the fibres of E and separate the points of M. By the proof of Corollary 2.6, we find a holomorphic embedding $\varphi: M \to \mathbb{P}^N$ into some projective space, such that $E \simeq \varphi^*\mathcal{O}(1)$. Since $\varphi^*\mathcal{O}(m)_{|K} \simeq E_{|K}^{\otimes m}$, we infer that $\mathcal{O}(m)_{|\varphi(K)}$ is topologically nontrivial for all $m > 0$. It follows that $\varphi(K)$ cannot be contained in the complement $\mathbb{P}^N \backslash H$ of a hypersurface of degree m, because $\mathcal{O}(m)$ is trivial on these affine open sets. The same conclusion holds for arbitrary homotopic deformations $\varphi(K)_\varepsilon$ of $\varphi(K)$.

To give a specific example, let $M = \mathbb{C}^* \times \mathbb{C}^*$ and $K = \partial\Delta \times \partial\Delta$. We consider the torus $T = \mathbb{C}/\{1, \sqrt{-1}\}\mathbb{Z}$, which has a holomorphic embedding $g: T \to \mathbb{P}^2$. Let $\varphi: M \to \mathbb{P}^8$ be the holomorphic embedding given by $\varphi(z, w) = \psi(g(t), (1 : z : w))$ where $\psi: \mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^8$ is the Segre embedding and $t = (\sqrt{-1}/2\pi) \log z + (1/2\pi) \log w$. Since $\varphi(K)$ is a C^∞ deformation of the elliptic curve $A = \psi(g(T) \times \{(1:0:0)\}) \subset \mathbb{P}^8$, then for any deformation $\varphi(K)_\varepsilon$ of $\varphi(K)$ and for any algebraic hypersurface $H \subset \mathbb{P}^8$ we have $\varphi(K)_\varepsilon \cdot H = A \cdot H > 0$ (where $\cdot$ denotes the intersection product in the homology of $\mathbb{P}^8$), and therefore $\varphi(K)_\varepsilon \cap H \neq \emptyset$.

5. Exhaustion of Stein manifolds by Runge domains of affine algebraic manifolds. The methods developed in this paper are used in this section to give a new proof of Stout's Theorem 1.6 by methods substantially different from those in Stout [St] and in Tancredi-Tognoli [TT1]. These methods are then used to obtain Theorem 1.8 from Proposition 3.2. To prove Theorem 1.6, we first show the existence of proper embeddings of arbitrarily large Stein domains into affine algebraic manifolds, such that the normal bundle of the embedding is trivial. Secondly, when such an embedding is given, the embedded manifold is a global complete intersection, so it can be approximated through an approximation of its defining equations by polynomials.

We begin with an elementary lemma about Runge neighborhoods.

LEMMA 5.1. *Let S be a closed submanifold of a Stein manifold Y. Then there is a fundamental system of neighborhoods Ω of S which are Runge open sets in Y.*

Proof. Let $f_1 = \cdots = f_N = 0$ be a finite system of defining equations of S in Y, and let u be a strictly plurisubharmonic smooth exhaustion function of Y. For every convex increasing function $\chi \in C^\infty(\mathbb{R})$, we set

$$\Omega_\chi = \left\{ \sum_{1 \leqslant j \leqslant N} |f_j(z)|^2 e^{\chi(u(z))} < 1 \right\}.$$

Since Ω_χ is a sublevel set of a global psh function on Y, we infer that Ω_χ is a Runge domain in Y by [Hö, Theorem 5.2.8]. Clearly $\{\Omega_\chi\}$ is a fundamental system of neighborhoods of S. □

LEMMA 5.2. *Let S be a Stein manifold. For any relatively compact Stein domain $S_0 \subset\subset S$, there is a proper embedding $j: S_0 \to A$ into an affine algebraic manifold, such that the image $j(S_0)$ has trivial normal bundle $N_{j(S_0)}$ in A.*

Proof. By the Bishop-Narasimhan embedding theorem, we can suppose that S is a closed submanifold of an affine space $\mathbb{C}^p$. Since any exact sequence of vector bundles on a Stein manifold splits, $T_S \oplus N_{S/\mathbb{C}^p}$ is isomorphic to the trivial bundle $T_{\mathbb{C}^p|S} = S \times \mathbb{C}^p$. Hence, to make the normal bundle of S become trivial, we need only to embed a Stein neighborhood Ω_0 of S_0 (in $\mathbb{C}^p$) into an affine algebraic manifold A so that the restriction to S_0 of the normal bundle of Ω_0 in A is isomorphic to T_{S_0}.

For this, we choose by Lemma 2.1 a holomorphic retraction $\rho: \Omega \to S$ from a neighborhood Ω of S in $\mathbb{C}^p$ onto S. We can suppose Ω to be a Runge open subset of $\mathbb{C}^p$, by shrinking Ω if necessary (apply Lemma 5.1). Let $\Omega_0 \subset\subset \Omega$ be a Runge domain in $\mathbb{C}^p$ containing $\bar{S}_0$. By Proposition 3.2 applied to the holomorphic vector bundle $\rho^* T_S$ on Ω, there is an open embedding $\varphi: \Omega_0 \to Z$ into a projective algebraic manifold Z, an algebraic vector bundle $E \to Z$ such that $\varphi^* E \simeq (\rho^* T_S)_{|\Omega_0}$, and an ample line bundle $L \to Z$ such that $\varphi^* L$ is trivial. We consider the composition of embeddings

$$S \cap \Omega_0 \hookrightarrow \Omega_0 \overset{\varphi}{\hookrightarrow} Z \hookrightarrow E \hookrightarrow \hat{E},$$

where $Z \hookrightarrow E$ is the zero section and $\hat{E} = P(E \oplus \mathbb{C})$ is the compactification of E by the hyperplane at infinity in each fibre. As $N_{Z/\hat{E}} \simeq E$, we find

$$(N_{Z/\hat{E}})_{|S\cap\Omega_0} \simeq (\varphi^*E)_{|S\cap\Omega_0} \simeq (\rho^*T_S)_{|S\cap\Omega_0} = (T_S)_{|S\cap\Omega_0},$$

$$\begin{aligned} N_{S\cap\Omega_0/\hat{E}} &= (N_{S/\mathbb{C}^p})_{|S\cap\Omega_0} \oplus (N_{Z/\hat{E}})_{|S\cap\Omega_0} \\ &\simeq (N_{S/\mathbb{C}^p} \oplus T_S)_{|S\cap\Omega_0} \simeq (S\cap\Omega_0) \times \mathbb{C}^p. \end{aligned}$$

Now, $\hat{E}$ is equipped with a canonical line bundle $\mathcal{O}_{\hat{E}}(1)$, which is relatively ample with respect to the fibres of the projection $\pi\colon \hat{E} \to Z$. It follows that there is an integer $m \gg 0$ such that $\hat{L} := \pi^*L^{\otimes m} \otimes \mathcal{O}_{\hat{E}}(1)$ is ample on $\hat{E}$. Now, the zero section of E embeds as $P(0 \oplus \mathbb{C}) \subset P(E \oplus \mathbb{C}) = \hat{E}$, hence $\mathcal{O}_{\hat{E}}(1)_{|Z} \simeq Z \times \mathbb{C}$. Since φ^*L is trivial, we infer that $\hat{L}_{|S\cap\Omega_0}$ is also trivial. Apply Corollary 4.9 to $X = \hat{E}$ with the ample line bundle $\hat{L} \to \hat{E}$, and to some holomorphically convex compact set K in the submanifold $S \cap \Omega_0 \subset \hat{E}$, such that $K \supset \bar{S}_0$. We infer the existence of a holomorphic embedding $i\colon S_0 \to Y$ into an affine Zariski open subset Y of $\hat{E}$. By construction, this embedding is obtained from the graph of a section $K \to N_{S\cap\Omega_0/\hat{E}}$, and hence

$$N_{S_0/Y} \simeq (N_{S\cap\Omega_0/\hat{E}})_{|S_0} \simeq S_0 \times \mathbb{C}^p.$$

It remains to show that we can find a *proper* embedding j satisfying this property. We simply set

$$j = (i, i')\colon S_0 \to A = Y \times \mathbb{C}^q,$$

where $i'\colon S_0 \to \mathbb{C}^q$ is a proper embedding of S_0 into an affine space. Clearly, j is proper. Moreover

$$N_{j(S_0)/Y\times\mathbb{C}^q} \simeq (T_Y \oplus \mathbb{C}^q)/\mathrm{Im}(di \oplus di')$$

with di being injective, hence we get an exact sequence

$$0 \to \mathbb{C}^q \to N_{j(S_0)/Y\times\mathbb{C}^q} \to N_{i(S_0)/Y} \to 0.$$

As $N_{i(S_0)}$ is trivial, we conclude that $N_{j(S_0)/Y\times\mathbb{C}^q}$ is also trivial. □

LEMMA 5.3. *Let S_0 be a closed Stein submanifold of an affine algebraic manifold A, such that $N_{S_0/A}$ is trivial. For any Runge domain $S_1 \subset\subset S_0$ and any neighborhood V of S_0 in A, there exist a Runge domain Ω in A with $S_0 \subset \Omega \subset V$, a holomorphic retraction $\rho\colon \Omega \to S_0$, and a closed algebraic submanifold $Y \subset A$ with the following properties.*

(i) *$Y \cap \rho^{-1}(S_1)$ is a Runge open set in Y.*
(ii) *ρ maps $Y \cap \rho^{-1}(S_1)$ biholomorphically onto S_1.*

Proof. By Lemmas 2.1 and 5.1, we can find a Runge domain Ω in A with $S_0 \subset \Omega \subset V$ and a holomorphic retraction $\rho: \Omega \to S_0$. Let $\mathscr{I}$ be the ideal sheaf of S_0 in A. Then $\mathscr{I}/\mathscr{I}^2$ is a coherent sheaf supported on S_0, and its restriction to S_0 is isomorphic to the conormal bundle $N^*_{S_0/A}$. Since this bundle is trivial by assumption, we can find $m = \operatorname{codim} S_0$ global generators $g_1, \ldots, g_m$ of $\mathscr{I}/\mathscr{I}^2$, as well as liftings $\tilde{g}_1, \ldots, \tilde{g}_m \in H^0(A, \mathscr{I})$ (A is Stein, thus $H^1(A, \mathscr{I}^2) = 0$). The system of equations $\tilde{g}_1 = \cdots = \tilde{g}_m = 0$ is a regular system of equations for S_0 in a sufficiently small neighborhood. After shrinking Ω, we can suppose that

$$S_0 = \{z \in \bar{\Omega}: \tilde{g}_1(z) = \cdots = \tilde{g}_m(z) = 0\}, \qquad d\tilde{g}_1 \wedge \cdots \wedge d\tilde{g}_m \neq 0 \quad \text{on } \bar{\Omega}.$$

Let $P_{j,\nu}$ be a sequence of polynomial functions on A converging to $\tilde{g}_j$ uniformly on every compact subset of A. By Sard's theorem, we can suppose that the algebraic varieties $Y_\nu = \bigcap_{1 \leqslant j \leqslant m} P_{j,\nu}^{-1}(0)$ are nonsingular (otherwise, this can be obtained by adding small generic constants $\varepsilon_{j,\nu}$ to $P_{j,\nu}$). It is then clear that $Y_\nu \cap \Omega$ converges to S_0 uniformly on all compact sets. In particular, the restriction

$$\rho_\nu: Y_\nu \cap \rho^{-1}(S_1) \to S_1$$

of ρ is a biholomorphism of $Y_\nu \cap \rho^{-1}(S_1)$ onto S_1 for ν large. This shows that S_1 can be embedded as an open subset of Y_ν via ρ_ν^{-1}.

It remains to show that $Y_\nu \cap \rho^{-1}(S_1)$ is a Runge domain in Y_ν for ν large. Let h be a holomorphic function on $Y_\nu \cap \rho^{-1}(S_1)$. Then $h \circ \rho_\nu^{-1}$ is a holomorphic function on S_1. Since S_1 is supposed to be a Runge domain in S_0, we have $h \circ \rho_\nu^{-1} = \lim h'_k$ where h'_k are holomorphic functions on S_0. Then $h'_k \circ \rho$ is holomorphic on Ω, and its restriction to $Y_\nu \cap \rho^{-1}(S_1)$ converges to h uniformly on compact subsets. However, Ω is also a Runge domain in A, so we can find a holomorphic (or even algebraic) function h_k on A such that $|h_k - h'_k \circ \rho| < 1/k$ on $\overline{Y_\nu \cap \rho^{-1}(S_1)}$, which is compact in Ω for ν large. Then the restriction of h_k to Y_ν converges to h uniformly on all compact subsets of $Y_\nu \cap \rho^{-1}(S_1)$, as desired. □

Proof of Theorem 1.6. Take a Stein domain $S_0 \subset\subset S$ such that $\Omega \subset\subset S_0$. Then $S_1 = \Omega$ is also a Runge domain in S_0, and Theorem 1.6 follows from the combination of Lemmas 5.2 and 5.3. □

We can now use Theorem 1.6 and Lemmas 5.2 and 5.3 to infer Theorem 1.8 from Proposition 3.2.

Proof of Theorem 1.8. By Theorem 1.6, we can take a Runge domain $\tilde{\Omega} \subset\subset S$ such that $\tilde{\Omega} \supset\supset \Omega$ and embed $\tilde{\Omega}$ as a Runge domain in an affine algebraic manifold M. By applying Proposition 3.2 with M playing the role of S, $\tilde{\Omega}$ the role of Ω, and Ω the role of Ω_0, we obtain a projective algebraic manifold Z containing Ω and an algebraic vector bundle $\tilde{E} \to Z$ such that $\tilde{E}_{|\Omega} \simeq E_{|\Omega}$. Moreover, there is an ample line bundle L on Z such that $L_{|\Omega}$ is trivial.

Only one thing remains to be proved, namely that Ω can be taken to be a Runge subset in an affine open subset $Z \backslash D$. (As mentioned earlier, this would be immediate if Problem 3.5 had a positive answer.) To reach this situation, we modify the construction as follows. Embed Ω as $\Omega \times \{0\} \subset Z \times \mathbb{P}^1$. This embedding has trivial normal bundle, and moreover $L \boxtimes \mathcal{O}_{\mathbb{P}^1}(1)_{|\Omega \times \{0\}}$ is trivial. By Corollary 4.9, for every holomorphically convex compact set $K \subset \Omega$, there is a holomorphic map $g: K^\circ \to \mathbb{C} \subset \mathbb{P}^1$ which is smooth up to the boundary, such that the graph of g is contained in an affine open set $Z_1 \backslash D_1$, where D_1 is an ample divisor of $Z_1 := Z \times \mathbb{P}^1$ in the linear system $m| \boxtimes L\mathcal{O}_{\mathbb{P}^1}(1)|$, $m \gg 0$. Let $\Omega' \subset K^\circ$ be a Runge open subset of Ω and let $j_N: \Omega' \to \mathbb{C}^N$ be a proper embedding. We consider the embedding

$$j' = (\mathrm{Id}_{\Omega'}, g, j_N): \Omega' \to Z_2 := Z \times \mathbb{P}^1 \times \mathbb{P}^N.$$

Its image $j'(\Omega')$ is a closed submanifold of the affine open set $(Z_1 \backslash D_1) \times \mathbb{C}^N = Z_2 \backslash D_2$, where $D_2 = (D_1 \times \mathbb{P}^N) + (Z_1 \times \mathbb{P}^{N-1}_\infty)$ is ample in Z_2. Moreover, the normal bundle of j is trivial (by the same argument as at the end of the proof of Lemma 5.2). Let $\Omega'' \subset\subset \Omega'$ be an arbitrary Runge domain in Ω'. By Lemma 5.3, we can find a nonsingular algebraic subvariety $Z'' \subset Z_2$ such that Ω'' is biholomorphic to a Runge open set in $Z'' \backslash (Z'' \cap D_2)$, via a retraction ρ from a small neighborhood of $j'(\Omega')$ onto $j'(\Omega')$. Let $E_2 = \mathrm{pr}_1^* \tilde{E} \to Z_2$ and let E'' be the restriction of E_2 to Z''. Since $\rho^* E_2 \simeq E_2$ on a neighborhood of $j'(\Omega')$ by Lemma 2.4, we infer that $E''_{|\Omega''} \simeq E_{|\Omega''}$. Therefore the conclusions of Theorem 1.8 hold with Ω'', Z'', $D'' = Z'' \cap D_2$ and E'' in place of Ω, Z, D and $\tilde{E}$. Since Ω'' can be taken to be an arbitrary Runge open set in S, Theorem 1.8 is proved. □

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