Annals of Mathematics, **159** (2004), 1247–1274

Numerical characterization of the Kähler cone of a compact Kähler manifold

By Jean-Pierre Demailly and Mihai Paun

Abstract

The goal of this work is to give a precise numerical description of the Kähler cone of a compact Kähler manifold. Our main result states that the Kähler cone depends only on the intersection form of the cohomology ring, the Hodge structure and the homology classes of analytic cycles: if X is a compact Kähler manifold, the Kähler cone $\mathcal{K}$ of X is one of the connected components of the set $\mathcal{P}$ of real $(1,1)$-cohomology classes $\{\alpha\}$ which are numerically positive on analytic cycles, i.e. $\int_Y \alpha^p > 0$ for every irreducible analytic set Y in X, $p = \dim Y$. This result is new even in the case of projective manifolds, where it can be seen as a generalization of the well-known Nakai-Moishezon criterion, and it also extends previous results by Campana-Peternell and Eyssidieux. The principal technical step is to show that every nef class $\{\alpha\}$ which has positive highest self-intersection number $\int_X \alpha^n > 0$ contains a Kähler current; this is done by using the Calabi-Yau theorem and a mass concentration technique for Monge-Ampère equations. The main result admits a number of variants and corollaries, including a description of the cone of numerically effective $(1,1)$-classes and their dual cone. Another important consequence is the fact that for an arbitrary deformation $\mathcal{X} \to S$ of compact Kähler manifolds, the Kähler cone of a very general fibre X_t is "independent" of t, i.e. invariant by parallel transport under the $(1,1)$-component of the Gauss-Manin connection.

0. Introduction

The primary goal of this work is to study in great detail the structure of the Kähler cone of a compact Kähler manifold. Recall that by definition the Kähler cone is the set of cohomology classes of smooth positive definite closed $(1,1)$-forms. Our main result states that the Kähler cone depends only on the intersection product of the cohomology ring, the Hodge structure and the homology classes of analytic cycles. More precisely, we have

Main Theorem 0.1. *Let X be a compact Kähler manifold. Then the Kähler cone $\mathcal{K}$ of X is one of the connected components of the set $\mathcal{P}$ of real* $(1,1)$*-cohomology classes $\{\alpha\}$ which are numerically positive on analytic cycles, i.e. such that $\int_Y \alpha^p > 0$ for every irreducible analytic set Y in X, $p = \dim Y$.*

This result is new even in the case of projective manifolds. It can be seen as a generalization of the well-known Nakai-Moishezon criterion, which provides a necessary and sufficient criterion for a line bundle to be ample: *a line bundle $L \to X$ on a projective algebraic manifold X is ample if and only if*

$$L^p \cdot Y = \int_Y c_1(L)^p > 0,$$

for every algebraic subset $Y \subset X$, $p = \dim Y$. In fact, when X is projective, the numerical conditions $\int_Y \alpha^p > 0$ characterize precisely the Kähler classes, even when $\{\alpha\}$ is not an integral class – and even when $\{\alpha\}$ lies outside the real Neron-Severi group $\mathrm{NS}_{\mathbb{R}}(X) = \mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{R}$; this fact can be derived in a purely formal way from the Main Theorem:

Corollary 0.2. *Let X be a projective manifold. Then the Kähler cone of X consists of all real* $(1,1)$*-cohomology classes which are numerically positive on analytic cycles, namely $\mathcal{K} = \mathcal{P}$ in the above notation.*

These results extend a few special cases which were proved earlier by completely different methods: Campana-Peternell [CP90] showed that the Nakai-Moishezon criterion holds true for classes $\{\alpha\} \in \mathrm{NS}_{\mathbb{R}}(X)$. Quite recently, using L^2 cohomology techniques for infinite coverings of a projective algebraic manifold, P. Eyssidieux [Eys00] obtained a version of the Nakai-Moishezon for all real combinations of $(1,1)$-cohomology classes which become integral after taking the pull-back to some finite or infinite covering.

The Main Theorem admits quite a number of useful variants and corollaries. Two of them are descriptions of the cone of nef classes (nef stands for numerically effective – or numerically eventually free according to the authors). In the Kähler case, the nef cone can be defined as the closure $\overline{\mathcal{K}}$ of the Kähler cone; see Section 1 for the general definition of nef classes on arbitrary compact complex manifolds.

Corollary 0.3. *Let X be a compact Kähler manifold. A* $(1,1)$*-cohomology class $\{\alpha\}$ on X is* nef *(i.e. $\{\alpha\} \in \overline{\mathcal{K}}$) if and only if there exists a Kähler metric ω on X such that $\int_Y \alpha^k \wedge \omega^{p-k} \geq 0$ for all irreducible analytic sets Y and all $k = 1, 2, \ldots, p = \dim Y$.*

Corollary 0.4. *Let X be a compact Kähler manifold. A* $(1,1)$*-cohomology class $\{\alpha\}$ on X is* nef *if and only for every irreducible analytic set Y in X, $p = \dim X$, and for every Kähler metric ω on X, one has $\int_Y \alpha \wedge \omega^{p-1} \geq 0$.*

In other words, the dual of the nef *cone* $\overline{\mathcal{K}}$ *is the closed convex cone generated by cohomology classes of currents of the form* $[Y] \wedge \omega^{p-1}$ *in* $H^{n-1,n-1}(X, \mathbb{R})$, *where* Y *runs over the collection of irreducible analytic subsets of* X *and* $\{\omega\}$ *over the set of Kähler classes of* X.

We now briefly discuss the essential ideas involved in our approach. The first basic result is a sufficient condition for a nef class to contain a Kähler current. The proof is based on a technique, of mass concentration for Monge-Ampère equations, using the Aubin-Calabi-Yau theorem [Yau78].

THEOREM 0.5. *Let* (X, ω) *be a compact* n*-dimensional Kähler manifold and let* $\{\alpha\}$ *in* $H^{1,1}(X, \mathbb{R})$ *be a* nef *cohomology class such that* $\int_X \alpha^n > 0$. *Then* $\{\alpha\}$ *contains a Kähler current* T, *that is, a closed positive current* T *such that* $T \geq \delta\omega$ *for some* $\delta > 0$. *The current* T *can be chosen to be smooth in the complement* $X \smallsetminus Z$ *of an analytic set, with logarithmic poles along* Z.

In a first step, we show that the class $\{\alpha\}^p$ dominates a small multiple of any p-codimensional analytic set Y in X. As we already mentioned, this is done by concentrating the mass on Y in the Monge-Ampère equation. We then apply this fact to the diagonal $\Delta \subset \widetilde{X} = X \times X$ to produce a closed positive current $\Theta \in \{\pi_1^*\alpha + \pi_2^*\alpha\}^n$ which dominates $[\Delta]$ in $X \times X$. The desired Kähler current T is easily obtained by taking a push-forward $\pi_{1*}(\Theta \wedge \pi_2^*\omega)$ of Θ to X.

This technique produces *a priori* "very singular" currents, since we use a weak compactness argument. However, we can apply the general regularization theorem proved in [Dem92] to get a current which is smooth outside an analytic set Z and only has logarithmic poles along Z. The idea of using a Monge-Ampère equation to force the occurrence of positive Lelong numbers in the limit current was first exploited in [Dem93], in the case when Y is a finite set of points, to get effective results for adjoints of ample line bundles (e.g. in the direction of the Fujita conjecture).

The use of higher dimensional subsets Y in the mass concentration process will be crucial here. However, the technical details are quite different from the 0-dimensional case used in [Dem93]; in fact, we cannot rely any longer on the maximum principle, as in the case of Monge-Ampère equations with isolated Dirac masses on the right-hand side. The new technique employed here is essentially taken from [Pau00] where it was proved, for projective manifolds, that every big semi-positive $(1,1)$-class contains a Kähler current. The Main Theorem is deduced from 0.5 by induction on dimension, thanks to the following useful result from the second author's thesis ([Pau98a, 98b]).

PROPOSITION 0.6. *Let* X *be a compact complex manifold* (*or complex space*). *Then*

(i) *The cohomology class of a closed positive* $(1,1)$*-current* $\{T\}$ *is* nef *if and only if the restriction* $\{T\}_{|Z}$ *is* nef *for every irreducible component* Z *in the Lelong sublevel sets* $E_c(T)$.

(ii) *The cohomology class of a Kähler current* $\{T\}$ *is a Kähler class* (*i.e. the class of a* smooth *Kähler form*) *if and only if the restriction* $\{T\}_{|Z}$ *is a Kähler class for every irreducible component* Z *in the Lelong sublevel sets* $E_c(T)$.

To derive the Main Theorem from 0.5 and 0.6, it is enough to observe that any class $\{\alpha\} \in \overline{\mathcal{K}} \cap \mathcal{P}$ is nef and that $\int_X \alpha^n > 0$. Therefore it contains a Kähler current. By the induction hypothesis on dimension, $\{\alpha\}_{|Z}$ is Kähler for all $Z \subset X$; hence $\{\alpha\}$ is a Kähler class on X.

We want to stress that Theorem 0.5 is closely related to the solution of the Grauert-Riemenschneider conjecture by Y.-T. Siu ([Siu85]); see also [Dem85] for a stronger result based on holomorphic Morse inequalities, and T. Bouche [Bou89], S. Ji-B. Shiffman [JS93], L. Bonavero [Bon93, 98] for other related results. The results obtained by Siu can be summarized as follows: *Let* L *be a hermitian semi-positive line bundle on a compact* n*-dimensional complex manifold* X, *such that* $\int_X c_1(L)^n > 0$. *Then* X *is a Moishezon manifold and* L *is a big line bundle*; *the tensor powers of* L *have a lot of sections*, $h^0(X, L^m) \geq Cm^n$ *as* $m \to +\infty$, *and there exists a singular hermitian metric on* L *such that the curvature of* L *is positive*, *bounded away from* 0. Again, Theorem 0.5 can be seen as an extension of this result to nonintegral $(1,1)$-cohomology classes – however, our proof only works so far for Kähler manifolds, while the Grauert-Riemenschneider conjecture has been proved on arbitrary compact complex manifolds. In the same vein, we prove the following result.

Theorem 0.7. *A compact complex manifold carries a Kähler current if and only if it is bimeromorphic to a Kähler manifold* (*or equivalently*, *dominated by a Kähler manifold*).

This class of manifolds is called the *Fujiki class* $\mathcal{C}$. If we compare this result with the solution of the Grauert-Riemenschneider conjecture, it is tempting to make the following conjecture which would somehow encompass both results.

Conjecture 0.8. *Let* X *be a compact complex manifold of dimension* n. *Assume that* X *possesses a nef cohomology class* $\{\alpha\}$ *of type* $(1,1)$ *such that* $\int_X \alpha^n > 0$. *Then* X *is in the Fujiki class* $\mathcal{C}$.

(*Also*, $\{\alpha\}$ *would contain a Kähler current*, *as it follows from Theorem* 0.5 *if Conjecture* 0.8 *is proved*.)

We want to mention here that most of the above results were already known in the cases of complex surfaces (i.e. dimension 2), thanks to the work of N. Buchdahl [Buc99, 00] and A. Lamari [Lam99a, 99b]; it turns out that there exists a very neat characterization of nef classes on arbitrary surfaces, Kähler or not.

The Main Theorem has an important application to the deformation theory of compact Kähler manifolds, which we prove in Section 5.

THEOREM 0.9. *Let $\mathcal{X} \to S$ be a deformation of compact Kähler manifolds over an irreducible base S. Then there exists a countable union $S' = \bigcup S_\nu$ of analytic subsets $S_\nu \subsetneq S$, such that the Kähler cones $\mathcal{K}_t \subset H^{1,1}(X_t, \mathbb{C})$ are invariant over $S \smallsetminus S'$ under parallel transport with respect to the $(1,1)$-projection $\nabla^{1,1}$ of the Gauss-Manin connection.*

We moreover conjecture (see 5.2 for details) that for an arbitrary deformation $\mathcal{X} \to S$ of compact complex manifolds, the Kähler property is open with respect to the countable Zariski topology on the base S of the deformation.

Shortly after this work was completed, Daniel Huybrechts [Huy01] informed us that our Main Theorem can be used to calculate the Kähler cone of a very general hyperKähler manifold: the Kähler cone is then equal to one of the connected components of the positive cone defined by the Beauville-Bogomolov quadratic form. This closes the gap in his original proof of the projectivity criterion for hyperKähler manifolds [Huy99, Th. 3.11].

We are grateful to Arnaud Beauville, Christophe Mourougane and Philippe Eyssidieux for helpful discussions, which were part of the motivation for looking at the questions investigated here.

1. Nef cohomology classes and Kähler currents

Let X be a complex analytic manifold. Throughout this paper, we denote by n the complex dimension $\dim_{\mathbb{C}} X$. As is well known, a Kähler metric on X is a smooth *real* form of type $(1,1)$:

$$\omega(z) = i \sum_{1 \le j,k \le n} \omega_{jk}(z) dz_j \wedge d\overline{z}_k;$$

that is, $\overline{\omega} = \omega$ or equivalently $\overline{\omega_{jk}(z)} = \omega_{kj}(z)$, such that

(1.1′) $\omega(z)$ is positive definite at every point $((\omega_{jk}(z))$ is a positive definite hermitian matrix);

(1.1″) $d\omega = 0$ when ω is viewed as a real 2-form; i.e., ω is symplectic.

One says that X is Kähler (or is of Kähler type) if X possesses a Kähler metric ω. To every closed real (resp. complex) valued k-form α we associate its de Rham cohomology class $\{\alpha\} \in H^k(X, \mathbb{R})$ (resp. $\{\alpha\} \in H^k(X, \mathbb{C})$), and to every $\overline{\partial}$-closed form α of pure type (p, q) we associate its Dolbeault cohomology class $\{\alpha\} \in H^{p,q}(X,\mathbb{C})$. On a compact Kähler manifold we have a canonical Hodge decomposition

$$H^k(X,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X,\mathbb{C}). \tag{1.2}$$

In this work, we are especially interested in studying the *Kähler cone*

$$\mathcal{K} \subset H^{1,1}(X,\mathbb{R}) := H^{1,1}(X,\mathbb{C}) \cap H^2(X,\mathbb{R}), \tag{1.3}$$

which is by definition the set of cohomology classes $\{\omega\}$ of all $(1,1)$-forms associated with Kähler metrics. Clearly, $\mathcal{K}$ is an open convex cone in $H^{1,1}(X,\mathbb{R})$, since a small perturbation of a Kähler form is still a Kähler form. The closure $\overline{\mathcal{K}}$ of the Kähler cone is equally important. Since we want to consider manifolds which are possibly non Kähler, we have to introduce "$\partial\overline{\partial}$-cohomology" groups

$$H^{p,q}_{\partial\overline{\partial}}(X,\mathbb{C}) := \{d\text{-closed } (p,q)\text{-forms}\}/\partial\overline{\partial}\{(p-1,q-1)\text{-forms}\}. \tag{1.4}$$

When (X,ω) is compact Kähler, it is well known (from the so-called $\partial\overline{\partial}$-lemma) that there is an isomorphism $H^{p,q}_{\partial\overline{\partial}}(X,\mathbb{C}) \simeq H^{p,q}(X,\mathbb{C})$ with the more usual Dolbeault groups. Notice that there are always canonical morphisms

$$H^{p,q}_{\partial\overline{\partial}}(X,\mathbb{C}) \to H^{p,q}(X,\mathbb{C}), \qquad H^{p,q}_{\partial\overline{\partial}}(X,\mathbb{C}) \to H^{p+q}_{\mathrm{DR}}(X,\mathbb{C})$$

($\partial\overline{\partial}$-cohomology is "more precise" than Dolbeault or de Rham cohomology). This allows us to define numerically effective classes in a fairly general situation (see also [Dem90b, 92], [DPS94]).

Definition 1.5. Let X be a compact complex manifold equipped with a hermitian positive (not necessarily Kähler) metric ω. A class $\{\alpha\} \in H^{1,1}_{\partial\overline{\partial}}(X,\mathbb{R})$ is said to be *numerically effective* (or nef for brevity) if for every $\varepsilon > 0$ there is a representative $\alpha_\varepsilon = \alpha + i\partial\overline{\partial}\varphi_\varepsilon \in \{\alpha\}$ such that $\alpha_\varepsilon \geq -\varepsilon\omega$.

If (X,ω) is compact Kähler, a class $\{\alpha\}$ is nef if and only if $\{\alpha+\varepsilon\omega\}$ is a Kähler class for every $\varepsilon > 0$, i.e., a class $\{\alpha\} \in H^{1,1}(X,\mathbb{R})$ is nef if and only if it belongs to the closure $\overline{\mathcal{K}}$ of the Kähler cone. (Also, if X is projective algebraic, a divisor D is nef in the sense of algebraic geometers; that is, $D \cdot C \geq 0$ for every irreducible curve $C \subset X$, if and only if $\{D\} \in \overline{\mathcal{K}}$, so that the definitions fit together; see [Dem90b, 92] for more details.)

In the sequel, we will make heavy use of currents, especially the theory of closed positive currents. Recall that a current T is a differential form with distribution coefficients. In the complex situation, we are interested in currents

$$T = i^{pq} \sum_{|I|=p,|J|=q} T_{I,J}\, dz_I \wedge d\overline{z}_J \qquad (T_{I,J} \text{ distributions on } X),$$

of pure bidegree (p,q), with $dz_I = dz_{i_1} \wedge \ldots \wedge dz_{i_p}$ as usual. We say that T is positive if $p = q$ and $T \wedge iu_1 \wedge \overline{u}_1 \wedge \cdots \wedge iu_{n-p} \wedge \overline{u}_{n-p}$ is a positive measure for all $(n-p)$-tuples of smooth $(1,0)$-forms u_j on X, $1 \le j \le n-p$ (this is the so-called "weak positivity" concept; since the currents under consideration here are just positive $(1,1)$-currents or wedge products of such, all other standard positivity concepts could be used as well, since they are the same on $(1,1)$-forms). Alternatively, the space of (p,q)-currents can be seen as the dual space of the Fréchet space of smooth $(n-p,n-q)$-forms, and $(n-p,n-q)$ is called the bidimension of T. By Lelong [Lel57], to every analytic set $Y \subset X$ of codimension p is associated a current $T = [Y]$ defined by

$$\langle [Y], u\rangle = \int_Y u, \qquad u \in \mathcal{D}_{n-p,n-p}(X),$$

and $[Y]$ is a closed positive current of bidegree (p,p) and bidimension $(n-p,n-p)$. The theory of positive currents can be easily extended to complex spaces X with singularities; one then simply defines the space of currents to be the dual of space of smooth forms, defined as forms on the regular part X_{reg} which, near X_{sing}, locally extend as smooth forms on an open set of $\mathbb{C}^N$ in which X is locally embedded (see e.g. [Dem85] for more details).

Definition 1.6. A Kähler current on a compact complex space X is a *closed positive current* T *of bidegree* $(1,1)$ which satisfies $T \ge \varepsilon\omega$ for some $\varepsilon > 0$ and some smooth positive hermitian form ω on X.

When X is a (nonsingular) compact complex manifold, we consider the *pseudo-effective* cone $\mathcal{E} \subset H^{1,1}_{\partial\overline{\partial}}(X,\mathbb{R})$, defined as the set of $\partial\overline{\partial}$-cohomology classes of closed positive $(1,1)$-currents. By the weak compactness of bounded sets in the space of currents, this is always a closed (convex) cone. When X is Kähler, we have of course

$$\mathcal{K} \subset \mathcal{E}^\circ,$$

i.e. $\mathcal{K}$ is contained in the interior of $\mathcal{E}$. Moreover, a Kähler current T has a class $\{T\}$ which lies in $\mathcal{E}^\circ$, and conversely any class $\{\alpha\}$ in $\mathcal{E}^\circ$ can be represented by a Kähler current T. We say that such a class is *big*.

Notice that the inclusion $\mathcal{K} \subset \mathcal{E}^\circ$ can be strict, even when X is Kähler, and the existence of a Kähler current on X does not necessarily imply that X admits a (smooth) Kähler form, as we will see in Section 3 (therefore X need not be a Kähler manifold in that case!).

2. Concentration of mass for nef classes of positive self-intersection

In this section, we show in full generality that on a compact Kähler manifold, every nef cohomology class with strictly positive self-intersection of maximum degree contains a Kähler current.

The proof is based on a mass concentration technique for Monge-Ampère equations, using the Aubin-Calabi-Yau theorem. We first start with an easy lemma, which was (more or less) already observed in [Dem90a]. Recall that a *quasi-plurisubharmonic* function ψ, by definition, is a function which is locally the sum of a plurisubharmonic function and of a smooth function, or equivalently, a function such that $i\partial\overline{\partial}\psi$ is locally bounded below by a negative smooth $(1,1)$-form.

LEMMA 2.1. *Let X be a compact complex manifold X equipped with a Kähler metric $\omega = i\sum_{1\le j,k\le n}\omega_{jk}(z)dz_j\wedge d\overline{z}_k$ and let $Y\subset X$ be an analytic subset of X. Then there exist globally defined quasi-plurisubharmonic potentials ψ and $(\psi_\varepsilon)_{\varepsilon\in]0,1]}$ on X, satisfying the following properties.*

(i) *The function ψ is smooth on $X\smallsetminus Y$, satisfies $i\partial\overline{\partial}\psi\ge -A\omega$ for some $A>0$, and ψ has logarithmic poles along Y; i.e., locally near Y,*

$$\psi(z)\sim\log\sum_k|g_k(z)|+O(1)$$

where (g_k) is a local system of generators of the ideal sheaf $\mathcal{I}_Y$ of Y in X.

(ii) *$\psi=\lim_{\varepsilon\to 0}\downarrow\psi_\varepsilon$ where the ψ_ε are C^∞ and possess a uniform Hessian estimate*

$$i\partial\overline{\partial}\psi_\varepsilon\ge -A\omega \qquad \text{on } X.$$

(iii) *Consider the family of hermitian metrics*

$$\omega_\varepsilon := \omega+\frac{1}{2A}i\partial\overline{\partial}\psi_\varepsilon\ge\frac{1}{2}\omega.$$

For any point $x_0\in Y$ and any neighborhood U of x_0, the volume element of ω_ε has a uniform lower bound

$$\int_{U\cap V_\varepsilon}\omega_\varepsilon^n\ge\delta(U)>0,$$

where $V_\varepsilon=\{z\in X;\ \psi(z)<\log\varepsilon\}$ is the "tubular neighborhood" of radius ε around Y.

(iv) *For every integer $p\ge 0$, the family of positive currents ω_ε^p is bounded in mass. Moreover, if Y contains an irreducible component Y' of codimension p, there is a uniform lower bound*

$$\int_{U\cap V_\varepsilon}\omega_\varepsilon^p\wedge\omega^{n-p}\ge\delta_p(U)>0$$

in any neighborhood U of a regular point $x_0\in Y'$. In particular, any weak limit Θ of ω_ε^p as ε tends to 0 satisfies $\Theta\ge\delta'[Y']$ for some $\delta'>0$.

Proof. By compactness of X, there is a covering of X by open coordinate balls B_j, $1 \le j \le N$, such that $\mathcal{I}_Y$ is generated by finitely many holomorphic functions $(g_{j,k})_{1\le k\le m_j}$ on a neighborhood of $\overline{B}_j$. We take a partition of unity (θ_j) subordinate to (B_j) such that $\sum \theta_j^2 = 1$ on X, and define

$$\psi(z) = \frac{1}{2}\log \sum_j \theta_j(z)^2 \sum_k |g_{j,k}(z)|^2,$$
$$\psi_\varepsilon(z) = \frac{1}{2}\log(e^{2\psi(z)} + \varepsilon^2) = \frac{1}{2}\log\Big(\sum_{j,k} \theta_j(z)^2 |g_{j,k}(z)|^2 + \varepsilon^2\Big).$$

Moreover, we consider the family of $(1,0)$-forms with support in B_j such that

$$\gamma_{j,k} = \theta_j \partial g_{j,k} + 2 g_{j,k}\partial\theta_j.$$

Straightforward calculations yield

$$\text{(2.2)} \qquad \overline{\partial}\psi_\varepsilon = \frac{1}{2}\frac{\sum_{j,k}\theta_j g_{j,k}\overline{\gamma_{j,k}}}{e^{2\psi}+\varepsilon^2},$$
$$i\partial\overline{\partial}\psi_\varepsilon = \frac{i}{2}\left(\frac{\sum_{j,k}\gamma_{j,k}\wedge\overline{\gamma_{j,k}}}{e^{2\psi}+\varepsilon^2} - \frac{\sum_{j,k}\theta_j\overline{g_{j,k}}\gamma_{j,k}\wedge\sum_{j,k}\theta_j g_{j,k}\overline{\gamma_{j,k}}}{(e^{2\psi}+\varepsilon^2)^2}\right),$$
$$+\, i\,\frac{\sum_{j,k}|g_{j,k}|^2(\theta_j\partial\overline{\partial}\theta_j - \partial\theta_j\wedge\overline{\partial}\theta_j)}{e^{2\psi}+\varepsilon^2}.$$

As $e^{2\psi} = \sum_{j,k}\theta_j^2|g_{j,k}|^2$, the first big sum in $i\partial\overline{\partial}\psi_\varepsilon$ is nonnegative by the Cauchy-Schwarz inequality; when viewed as a hermitian form, the value of this sum on a tangent vector $\xi \in T_X$ is simply

(2.3)
$$\frac{1}{2}\left(\frac{\sum_{j,k}|\gamma_{j,k}(\xi)|^2}{e^{2\psi}+\varepsilon^2} - \frac{\big|\sum_{j,k}\theta_j\overline{g_{j,k}}\gamma_{j,k}(\xi)\big|^2}{(e^{2\psi}+\varepsilon^2)^2}\right) \ge \frac{1}{2}\frac{\varepsilon^2}{(e^{2\psi}+\varepsilon^2)^2}\sum_{j,k}|\gamma_{j,k}(\xi)|^2.$$

Now, the second sum involving $\theta_j\partial\overline{\partial}\theta_j - \partial\theta_j\wedge\overline{\partial}\theta_j$ in (2.2) is uniformly bounded below by a fixed negative hermitian form $-A\omega$, $A \gg 0$, and therefore $i\partial\overline{\partial}\psi_\varepsilon \ge -A\omega$. Actually, for every pair of indices (j,j') we have a bound

$$C^{-1} \le \sum_k |g_{j,k}(z)|^2 \Big/ \sum_k |g_{j',k}(z)|^2 \le C \qquad \text{on } \overline{B}_j \cap \overline{B}_{j'},$$

since the generators $(g_{j,k})$ can be expressed as holomorphic linear combinations of the $(g_{j',k})$ by Cartan's theorem A (and vice versa). It follows easily that all terms $|g_{j,k}|^2$ are uniformly bounded by $e^{2\psi}+\varepsilon^2$. In particular, ψ and ψ_ε are quasi-plurisubharmonic, and we see that (i) and (ii) hold true. By construction, the real $(1,1)$-form $\omega_\varepsilon := \omega + \frac{1}{2A}i\partial\overline{\partial}\psi_\varepsilon$ satisfies $\omega_\varepsilon \ge \frac{1}{2}\omega$; hence it is Kähler and its eigenvalues with respect to ω are at least equal to $1/2$.

Assume now that we are in a neighborhood U of a regular point $x_0 \in Y$ where Y has codimension p. Then $\gamma_{j,k} = \theta_j\partial g_{j,k}$ at x_0; hence the rank of the system of $(1,0)$-forms $(\gamma_{j,k})_{k\geq 1}$ is at least equal to p in a neighborhood of x_0. Fix a holomorphic local coordinate system $(z_1,\ldots,z_n)$ such that $Y=\{z_1=\ldots=z_p=0\}$ near x_0, and let $S\subset T_X$ be the holomorphic subbundle generated by $\partial/\partial z_1,\ldots,\partial/\partial z_p$. This choice ensures that the rank of the system of $(1,0)$-forms $(\gamma_{j,k|S})$ is everywhere equal to p. By (1.3) and the minimax principle applied to the p-dimensional subspace $S_z\subset T_{X,z}$, we see that the p-largest eigenvalues of ω_ε are bounded below by $c\varepsilon^2/(e^{2\psi}+\varepsilon^2)^2$. However, we can even restrict the form defined in (2.3) to the $(p-1)$-dimensional subspace $S\cap\operatorname{Ker}\tau$ where $\tau(\xi):=\sum_{j,k}\theta_j\overline{g_{j,k}}\gamma_{j,k}(\xi)$, to see that the $(p-1)$-largest eigenvalues of ω_ε are bounded below by $c/(e^{2\psi}+\varepsilon^2)$, $c>0$. The $p^{\rm th}$ eigenvalue is then bounded by $c\varepsilon^2/(e^{2\psi}+\varepsilon^2)^2$ and the remaining $(n-p)$-ones by $1/2$. From this we infer

$$\omega_\varepsilon^n\geq c\,\frac{\varepsilon^2}{(e^{2\psi}+\varepsilon^2)^{p+1}}\omega^n\qquad\text{near } x_0,$$
$$\omega_\varepsilon^p\geq c\,\frac{\varepsilon^2}{(e^{2\psi}+\varepsilon^2)^{p+1}}\Big(i\sum_{1\leq\ell\leq p}\gamma_{j,k_\ell}\wedge\overline{\gamma_{j,k_\ell}}\Big)^p$$

where $(\gamma_{j,k_\ell})_{1\leq\ell\leq p}$ is a suitable p-tuple extracted from the $(\gamma_{j,k})$, such that $\bigcap_\ell\operatorname{Ker}\gamma_{j,k_\ell}$ is a smooth complex (but not necessarily holomorphic) subbundle of codimension p of T_X; by the definition of the forms $\gamma_{j,k}$, this subbundle must coincide with T_Y along Y. From this, properties (iii) and (iv) follow easily; actually, up to constants, we have $e^{2\psi}+\varepsilon^2\sim|z_1|^2+\ldots+|z_p|^2+\varepsilon^2$ and

$$i\sum_{1\leq\ell\leq p}\gamma_{j,k_\ell}\wedge\overline{\gamma_{j,k_\ell}}\geq c\,i\partial\overline\partial(|z_1|^2+\ldots+|z_p|^2)-O(\varepsilon)i\partial\overline\partial|z|^2\qquad\text{on } U\cap V_\varepsilon.$$

Hence, by a straightforward calculation,

$$\omega_\varepsilon^p\wedge\omega^{n-p}\geq c\big(i\partial\overline\partial\log(|z_1|^2+\ldots+|z_p|^2+\varepsilon^2)\big)^p\wedge\big(i\partial\overline\partial(|z_{p+1}|^2+\ldots+|z_n|^2)\big)^{n-p}$$

on $U\cap V_\varepsilon$; notice also that $\omega_\varepsilon^n\geq 2^{-(n-p)}\omega_\varepsilon^p\wedge\omega^{n-p}$, so that any lower bound for the volume of $\omega_\varepsilon^p\wedge\omega^{n-p}$ will also produce a bound for the volume of ω_ε^n. As is well known, the (p,p)-form

$$\Big(\frac{i}{2\pi}\partial\overline\partial\log(|z_1|^2+\ldots+|z_p|^2+\varepsilon^2)\Big)^p\qquad\text{on } \mathbb{C}^n$$

can be viewed as the pull-back to $\mathbb{C}^n=\mathbb{C}^p\times\mathbb{C}^{n-p}$ of the Fubini-Study volume form of the complex p-dimensional projective space of dimension p containing $\mathbb{C}^p$ as an affine Zariski open set, rescaled by the dilation ratio ε. Hence it converges weakly to the current of integration on the p-codimensional subspace $z_1=\ldots=z_p=0$. Moreover the volume contained in any compact tubular cylinder

$$\{|z'|\leq C\varepsilon\}\times K''\subset\mathbb{C}^p\times\mathbb{C}^{n-p}$$

depends only on C and K (as one sees after rescaling by ε). The fact that ω_ε^p is uniformly bounded in mass can be seen easily from the fact that

$$\int_X \omega_\varepsilon^p \wedge \omega^{n-p} = \int_X \omega^n,$$

as ω and ω_ε are in the same Kähler class. Let Θ be any weak limit of ω_ε^p. By what we have just seen, Θ carries nonzero mass on every p-codimensional component Y' of Y, for instance near every regular point. However, standard results of the theory of currents (support theorem and Skoda's extension result) imply that $\mathbf{1}_{Y'}\Theta$ is a closed positive current and that $\mathbf{1}_{Y'}\Theta = \lambda[Y']$ is a nonnegative multiple of the current of integration on Y'. The fact that the mass of Θ on Y' is positive yields $\lambda > 0$. Lemma 2.1 is proved. □

Remark 2.4. In the proof above, we did not really make use of the fact that ω is Kähler. Lemma 2.1 would still be true without this assumption. The only difficulty would be to show that ω_ε^p is still locally bounded in mass when ω is an arbitrary hermitian metric. This can be done by using a resolution of singularities which converts $\mathcal{I}_Y$ into an invertible sheaf defined by a divisor with normal crossings – and by doing some standard explicit calculations. As we do not need the more general form of Lemma 2.1, we will omit these technicalities.

Let us now recall the following very deep result concerning Monge-Ampère equations on compact Kähler manifolds (see [Yau78]).

THEOREM 2.5 (Yau). *Let (X, ω) be a compact Kähler manifold and $n = \dim X$. Then for any smooth volume form $f > 0$ such that $\int_X f = \int_X \omega^n$, there exists a Kähler metric $\widetilde{\omega} = \omega + i\partial\overline{\partial}\varphi$ in the same Kähler class as ω, such that $\widetilde{\omega}^n = f$.*

In other words, one can prescribe the volume form f of the Kähler metric $\widetilde{\omega} \in \{\omega\}$, provided that the total volume $\int_X f$ is equal to the expected value $\int_X \omega^n$. Since the Ricci curvature form of $\widetilde{\omega}$ is $\mathrm{Ricci}(\widetilde{\omega}) := -\frac{i}{2\pi}\partial\overline{\partial}\log\det(\widetilde{\omega}) = -\frac{i}{2\pi}\partial\overline{\partial}\log f$, this is the same as prescribing the curvature form $\mathrm{Ricci}(\widetilde{\omega}) = \rho$, given any $(1,1)$-form ρ representing $c_1(X)$. Using this, we prove

PROPOSITION 2.6. *Let (X, ω) be a compact n-dimensional Kähler manifold and let $\{\alpha\}$ in $H^{1,1}(X, \mathbf{R})$ be a* nef *cohomology class such that $\alpha^n > 0$. Then, for every p-codimensional analytic set $Y \subset X$, there exists a closed positive current $\Theta \in \{\alpha\}^p$ of bidegree (p,p) such that $\Theta \geq \delta[Y]$ for some $\delta > 0$.*

Proof. Let us associate with Y a family ω_ε of Kähler metrics as in Lemma 2.1. The class $\{\alpha + \varepsilon\omega\}$ is a Kähler class, so by Yau's theorem we can find a representative $\alpha_\varepsilon = \alpha + \varepsilon\omega + i\partial\overline{\partial}\varphi_\varepsilon$ such that

$$\alpha_\varepsilon^n = C_\varepsilon \omega_\varepsilon^n, \tag{2.7}$$

where

$$C_\varepsilon = \frac{\int_X \alpha_\varepsilon^n}{\int_X \omega_\varepsilon^n} = \frac{\int_X (\alpha + \varepsilon\omega)^n}{\int_X \omega^n} \geq C_0 = \frac{\int_X \alpha^n}{\int_X \omega^n} > 0.$$

Let us denote by

$$\lambda_1(z) \leq \ldots \leq \lambda_n(z)$$

the eigenvalues of $\alpha_\varepsilon(z)$ with respect to $\omega_\varepsilon(z)$, at every point $z \in X$ (these functions are continuous with respect to z, and of course depend also on ε). The equation (2.7) is equivalent to the fact that

$$\lambda_1(z) \ldots \lambda_n(z) = C_\varepsilon \tag{2.7$'$}$$

is constant, and the most important observation for us is that the constant C_ε is bounded away from 0, thanks to our assumption $\int_X \alpha^n > 0$.

Fix a regular point $x_0 \in Y$ and a small neighborhood U (meeting only the irreducible component of x_0 in Y). By Lemma 2.1, we have a uniform lower bound

$$\int_{U \cap V_\varepsilon} \omega_\varepsilon^p \wedge \omega^{n-p} \geq \delta_p(U) > 0. \tag{2.8}$$

Now, by looking at the p smallest (resp. $(n-p)$ largest) eigenvalues λ_j of α_ε with respect to ω_ε,

$$\alpha_\varepsilon^p \geq \lambda_1 \ldots \lambda_p\, \omega_\varepsilon^p, \tag{2.9$'$}$$

$$\alpha_\varepsilon^{n-p} \wedge \omega_\varepsilon^p \geq \frac{1}{n!} \lambda_{p+1} \ldots \lambda_n\, \omega_\varepsilon^n. \tag{2.9$''$}$$

The last inequality (2.9$''$) implies

$$\int_X \lambda_{p+1} \ldots \lambda_n\, \omega_\varepsilon^n \leq n! \int_X \alpha_\varepsilon^{n-p} \wedge \omega_\varepsilon^p = n! \int_X (\alpha + \varepsilon\omega)^{n-p} \wedge \omega^p \leq M$$

for some constant $M > 0$ (we assume $\varepsilon \leq 1$, say). In particular, for every $\delta > 0$, the subset $E_\delta \subset X$ of points z such that $\lambda_{p+1}(z) \ldots \lambda_n(z) > M/\delta$ satisfies $\int_{E_\delta} \omega_\varepsilon^n \leq \delta$; hence

$$\int_{E_\delta} \omega_\varepsilon^p \wedge \omega^{n-p} \leq 2^{n-p} \int_{E_\delta} \omega_\varepsilon^n \leq 2^{n-p}\delta. \tag{2.10}$$

The combination of (2.8) and (2.10) yields

$$\int_{(U \cap V_\varepsilon) \smallsetminus E_\delta} \omega_\varepsilon^p \wedge \omega^{n-p} \geq \delta_p(U) - 2^{n-p}\delta.$$

On the other hand (2.7$'$) and (2.9$'$) imply

$$\alpha_\varepsilon^p \geq \frac{C_\varepsilon}{\lambda_{p+1} \ldots \lambda_n} \omega_\varepsilon^p \geq \frac{C_\varepsilon}{M/\delta} \omega_\varepsilon^p \qquad \text{on } (U \cap V_\varepsilon) \smallsetminus E_\delta.$$

From this we infer

(2.11)
$$\int_{U\cap V_\varepsilon} \alpha_\varepsilon^p \wedge \omega^{n-p} \geq \frac{C_\varepsilon}{M/\delta} \int_{(U\cap V_\varepsilon)\smallsetminus E_\delta} \omega_\varepsilon^p \wedge \omega^{n-p} \geq \frac{C_\varepsilon}{M/\delta}(\delta_p(U) - 2^{n-p}\delta) > 0$$

provided that δ is taken small enough, e.g., $\delta = 2^{-(n-p+1)}\delta_p(U)$. The family of (p,p)-forms α_ε^p is uniformly bounded in mass since

$$\int_X \alpha_\varepsilon^p \wedge \omega^{n-p} = \int_X (\alpha + \varepsilon\omega)^p \wedge \omega^{n-p} \leq \text{Const.}$$

Inequality (2.11) implies that any weak limit Θ of (α_ε^p) carries a positive mass on $U \cap Y$. By Skoda's extension theorem [Sk81], $\mathbf{1}_Y\Theta$ is a closed positive current with support in Y, hence $\mathbf{1}_Y\Theta = \sum c_j[Y_j]$ is a combination of the various components Y_j of Y with coefficients $c_j > 0$. Our construction shows that Θ belongs to the cohomology class $\{\alpha\}^p$. Proposition 2.6 is proved. □

We can now prove the main result of this section.

THEOREM 2.12. *Let (X,ω) be a compact n-dimensional Kähler manifold and let $\{\alpha\}$ in $H^{1,1}(X,\mathbf{R})$ be a* nef *cohomology class such that $\int_X \alpha^n > 0$. Then $\{\alpha\}$ contains a Kähler current T, that is, a closed positive current T such that $T \geq \delta\omega$ for some $\delta > 0$.*

Proof. The trick is to apply Proposition 2.6 to the diagonal $\widetilde{Y} = \Delta$ in the product manifold $\widetilde{X} = X \times X$. Let us denote by π_1 and π_2 the two projections of $\widetilde{X} = X \times X$ onto X. It is clear that $\widetilde{X}$ admits

$$\widetilde{\omega} = \pi_1^*\omega + \pi_2^*\omega$$

as a Kähler metric, and that the class of

$$\widetilde{\alpha} = \pi_1^*\alpha + \pi_2^*\alpha$$

is a nef class on $\widetilde{X}$ (it is a limit of the Kähler classes $\pi_1^*(\alpha+\varepsilon\omega) + \pi_2^*(\alpha+\varepsilon\omega)$). Moreover, by Newton's binomial formula

$$\int_{X\times X} \widetilde{\alpha}^{2n} = \binom{2n}{n}\Big(\int_X \alpha^n\Big)^2 > 0.$$

The diagonal is of codimension n in $\widetilde{X}$; hence by Proposition 2.6 there exists a closed positive (n,n)-current $\Theta \in \{\widetilde{\alpha}^n\}$ such that $\Theta \geq \varepsilon[\Delta]$ for some $\varepsilon > 0$. We define the $(1,1)$-current T to be the push-forward

$$T = c\,\pi_{1*}(\Theta \wedge \pi_2^*\omega)$$

for a suitable constant $c > 0$ which will be determined later. By the lower estimate on Θ, we have

$$T \geq c\varepsilon\,\pi_{1*}([\Delta] \wedge \pi_2^*\omega) = c\varepsilon\,\omega;$$

thus T is a Kähler current. On the other hand, as $\Theta \in \{\widetilde{\alpha}^n\}$, the current T belongs to the cohomology class of the $(1,1)$-form

$$c\,\pi_{1*}(\widetilde{\alpha}^n \wedge \pi_2^*\omega)(x) = c\int_{y\in Y} \big(\alpha(x)+\alpha(y)\big)^n \wedge \omega(y),$$

obtained by a partial integration in y with respect to $(x,y) \in X \times X$. By Newton's binomial formula again, we see that

$$c\,\pi_{1*}(\widetilde{\alpha}^n \wedge \pi_2^*\omega)(x) = c\Big(\int_X n\alpha(y)^{n-1} \wedge \omega(y)\Big)\alpha(x)$$

is proportional to α. Therefore, we need only take $c = \big(\int_X n\alpha^{n-1} \wedge \omega\big)^{-1}$ to ensure that $T \in \{\alpha\}$. As α is nef and $\{\alpha\} \le C\{\omega\}$ for sufficiently large $C > 0$, we have

$$\int_X \alpha^{n-1} \wedge \omega \ge \frac{1}{C}\int_X \alpha^n > 0.$$

Theorem 2.12 is proved. □

3. Regularization theorems for Kähler currents

It is not true that a Kähler current can be regularized to produce a smooth Kähler metric. However, by the general regularization theorem for closed currents proved in [Dem92] (see Proposition 3.7), it can be regularized up to some logarithmic poles along analytic subsets.

Before stating the result, we need a few preliminaries. If T is a closed positive current on a compact complex manifold X, we can write

$$T = \alpha + i\partial\overline{\partial}\psi \tag{3.1}$$

where α is a global, smooth, closed $(1,1)$-form on X, and ψ a quasi-plurisubharmonic function on X. To see this (cf. also [Dem92]), take an open covering of X by open coordinate balls B_j and plurisubharmonic potentials ψ_j such that $T = i\partial\overline{\partial}\psi_j$ on B_j. Then, if (θ_j) is a partition of unity subordinate to (B_j), it is easy to see that $\psi = \sum \theta_j\psi_j$ is quasi-plurisubharmonic and that $\alpha := T - i\partial\overline{\partial}\psi$ is smooth (so that $i\partial\overline{\partial}\psi = T - \alpha \ge -\alpha$). For any other decomposition $T = \alpha' + i\partial\overline{\partial}\psi'$ as in (3.1), we have $\alpha' - \alpha = -i\partial\overline{\partial}(\psi' - \psi)$; hence $\psi' - \psi$ is smooth.

THE REGULARIZATION THEOREM 3.2. *Let X be a compact complex manifold equipped with a hermitian metric ω. Let $T = \alpha + i\partial\overline{\partial}\psi$ be a closed $(1,1)$-current on X, where α is smooth and ψ is a quasi-plurisubharmonic function. Assume that $T \ge \gamma$ for some real $(1,1)$-form γ on X with real coefficients. Then there exists a sequence $T_k = \alpha + i\partial\overline{\partial}\psi_k$ of closed $(1,1)$-currents such that*

(i) *ψ_k (and thus T_k) is smooth on the complement $X \smallsetminus Z_k$ of an analytic set Z_k, and the Z_k's form an increasing sequence*

$$Z_0 \subset Z_1 \subset \ldots \subset Z_k \subset \ldots \subset X.$$

(ii) *There is a uniform estimate* $T_k \geq \gamma - \delta_k\omega$ *with* $\lim\downarrow \delta_k = 0$ *as* k *tends to* $+\infty$.

(iii) *The sequence* (ψ_k) *is nonincreasing, and we have* $\lim\downarrow\psi_k = \psi$. *As a consequence,* T_k *converges weakly to* T *as* k *tends to* $+\infty$.

(iv) *Near* Z_k, *the potential* ψ_k *has logarithmic poles, namely, for every* $x_0 \in Z_k$, *there is a neighborhood* U *of* x_0 *such that*

$$\psi_k(z) = \lambda_k \log \sum_\ell |g_{k,\ell}|^2 + O(1)$$

for suitable holomorphic functions $(g_{k,\ell})$ *on* U *and* $\lambda_k > 0$. *Moreover, there is a (global) proper modification* $\mu_k : \widetilde{X}_k \to X$ *of* X, *obtained as a sequence of blow-ups with smooth centers, such that* $\psi_k \circ \mu_k$ *can be written locally on* $\widetilde{X}_k$ *as*

$$\psi_k \circ \mu_k(w) = \lambda_k \big(\sum n_\ell \log |\widetilde{g}_\ell|^2 + f(w) \big)$$

where $(\widetilde{g}_\ell = 0)$ *are local generators of suitable (global) divisors* D_ℓ *on* $\widetilde{X}_k$ *such that* $\sum D_\ell$ *has normal crossings,* n_ℓ *are positive integers, and the* f*'s are smooth functions on* $\widetilde{X}_k$.

Sketch of the proof. We briefly indicate the main ideas, since the proof can only be reconstructed by patching together arguments which appeared in different places (although the core of the proof is entirely in [Dem92]). After replacing T with $T - \alpha$, we can assume that $\alpha = 0$ and $T = i\partial\overline{\partial}\psi \geq \gamma$. Given a small $\varepsilon > 0$, we select a covering of X by open balls B_j together with holomorphic coordinates $(z^{(j)})$ and real numbers β_j such that

$$0 \leq \gamma - \beta_j \, i\partial\overline{\partial}|z^{(j)}|^2 \leq \varepsilon \, i\partial\overline{\partial}|z^{(j)}|^2 \qquad \text{on } B_j$$

(this can be achieved just by continuity of γ, after diagonalizing γ at the center of the balls). We now take a partition of unity (θ_j) subordinate to (B_j) such that $\sum \theta_j^2 = 1$, and define

$$\psi_k(z) = \frac{1}{2k} \log \sum_j \theta_j^2 e^{2k\beta_j|z^{(j)}|^2} \sum_{\ell\in\mathbf{N}} |g_{j,k,\ell}|^2$$

where $(g_{j,k,\ell})$ is a Hilbert basis of the Hilbert space of holomorphic functions f on B_j such that

$$\int_{B_j} |f|^2 e^{-2k(\psi - \beta_j|z^{(j)}|^2)} < +\infty.$$

Notice that by the Hessian estimate $i\partial\overline{\partial}\psi \geq \gamma \geq \beta_j \, i\partial\overline{\partial}|z^{(j)}|^2$, the weight involved in the L^2 norm is plurisubharmonic. It then follows from the proof of Proposition 3.7 in [Dem92] that all properties (i)–(iv) hold true, except possibly the fact that the sequence ψ_k can be chosen to be nonincreasing, and the existence of the modification in (iv). However, the multiplier ideal sheaves of the weights $k(\psi - \beta_j|z^{(j)}|^2)$ are generated by the $(g_{j,k,\ell})_\ell$ on B_j, and these sheaves glue together into a global coherent multiplier ideal sheaf $\mathcal{I}(k\psi)$ on X (see [DEL99]); the modification μ_k is then obtained by blowing-up the ideal sheaf $\mathcal{I}(k\psi)$ so that $\mu_k^*\mathcal{I}(k\psi)$ is an invertible ideal sheaf associated with a normal crossing divisor (Hironaka [Hir63]). The fact that ψ_k can be chosen to be nonincreasing follows from a quantitative version of the "subadditivity of multiplier ideal sheaves" which is proved in Step 3 of the proof of Theorem 2.2.1 in [DPS00] (see also ([DEL99]). (Anyway, this property will not be used here, so the reader may wish to skip the details.) □

For later purposes, we state the following useful results, which are borrowed essentially from the Ph.D. thesis of the second author.

PROPOSITION 3.3 ([Pau98a, 98b]). *Let X be a compact complex space and let $\{\alpha\}$ be a $\partial\overline{\partial}$-cohomology class of type $(1,1)$ on X (where α is a smooth representative).*

(i) *If the restriction $\{\alpha\}_{|Y}$ to an analytic subset $Y \subset X$ is Kähler on Y, there exists a smooth representative $\alpha' = \alpha + i\partial\overline{\partial}\varphi$ which is Kähler on a neighborhood U of Y.*

(ii) *If the restrictions $\{\alpha\}_{|Y_1}$, $\{\alpha\}_{|Y_2}$ to any pair of analytic subsets Y_1, $Y_2 \subset X$ are* nef *(resp. Kähler), then $\{\alpha\}_{|Y_1\cup Y_2}$ is* nef *(resp. Kähler).*

(iii) *Assume that $\{\alpha\}$ contains a Kähler current T and that the restriction $\{\alpha\}_{|Y}$ to every irreducible component Y in the Lelong sublevel sets $E_c(T)$ is a Kähler class on Y. Then $\{\alpha\}$ is a Kähler class on X.*

(iv) *Assume that $\{\alpha\}$ contains a closed positive $(1,1)$-current T and that the restriction $\{\alpha\}_{|Y}$ to every irreducible component Y in the Lelong sublevel sets $E_c(T)$ is* nef *on Y. Then $\{\alpha\}$ is* nef *on X.*

By definition, $E_c(T)$ is the set of points $z \in X$ such that the Lelong number $\nu(T,z)$ is at least equal to c (for given $c > 0$). A deep theorem of Siu ([Siu74]) asserts that all $E_c(T)$ are analytic subsets of X. Notice that the concept of a $\partial\overline{\partial}$-cohomology class is well defined on an arbitrary complex space (although many of the standard results on de Rham or Dolbeault cohomology of nonsingular spaces will fail for singular spaces!). The concepts of Kähler classes and nef classes are still well defined (a Kähler form on a singular space X is a $(1,1)$-form which is locally bounded below by the restriction of a smooth positive $(1,1)$-form in a nonsingular ambient space for X, and a nef class is a class containing representatives bounded below by $-\varepsilon\omega$ for every $\varepsilon > 0$, where ω is a smooth positive $(1,1)$-form).

Sketch of the proof. (i) We can assume that $\alpha_{|Y}$ itself is a Kähler form. If Y is smooth, we simply take ψ to be equal to a large constant times the square of the hermitian distance to Y. This will produce positive eigenvalues in $\alpha + i\partial\overline{\partial}\psi$ along the normal directions of Y, while the eigenvalues are already positive on Y. When Y is singular, we just use the same argument with respect to a stratification of Y by smooth manifolds, and an induction on the dimension of the strata (ψ can be left untouched on the lower dimensional strata).

(ii) Let us first treat the Kähler case. By (i), there are smooth functions φ_1, φ_2 on X such that $\alpha + i\partial\overline{\partial}\varphi_j$ is Kähler on a neighborhood U_j of Y_j, $j = 1, 2$. Also, by Lemma 2.1, there exists a quasi-plurisubharmonic function ψ on X which has logarithmic poles on $Y_1 \cap Y_2$ and is smooth on $X \smallsetminus (Y_1 \cap Y_2)$. We define

$$\varphi = \widetilde{\max}(\varphi_1 + \delta\psi, \varphi_2 - C)$$

where $\delta \ll 1$, $C \gg 1$ are constants and $\widetilde{\max}$ is a regularized max function. Then $\alpha + i\partial\overline{\partial}\varphi$ is Kähler on $U_1 \cap U_2$. Moreover, for C large, φ coincides with $\varphi_1 + \delta\psi$ on $Y_1 \smallsetminus U_2$ and with $\varphi_2 - C$ on a small neighborhood of W of $Y_1 \cap Y_2$. Take smaller neighborhoods $U_1' \Subset U_1$, $U_2' \Subset U_2$ such that $U_1' \cap U_2' \subset W$. We can extend $\varphi_{|U_1' \cap U_2}$ to a neighborhood V of $Y_1 \cup Y_2$ by taking $\varphi = \varphi_1 + \delta\psi$ on a neighborhood of $Y_1 \smallsetminus U_2$ and $\varphi = \varphi_2 - C$ on U_2'. The use of a cut-off function equal to 1 on a neighborhood of $V' \Subset V$ of $Y_1 \cup Y_2$ finally allows us to get a function φ defined everywhere on X, such that $\alpha + i\partial\overline{\partial}\varphi$ is Kähler on a neighborhood of $Y_1 \cup Y_2$ (if δ is small enough). The nef case is similar, except that we deal with currents T such that $T \geq -\varepsilon\omega$ instead of Kähler currents.

(iii) By the regularization Theorem 3.2, we may assume that the singularities of the Kähler current $T = \alpha + i\partial\overline{\partial}\psi$ are just logarithmic poles (since $T \geq \gamma$ with γ positive definite, the small loss of positivity resulting from 3.2 (ii) still yields a Kähler current T_k). Hence ψ is smooth on $X \smallsetminus Z$ for a suitable analytic set Z which, by construction, is contained in $E_c(T)$ for $c > 0$ small enough. We use (i), (ii) and the hypothesis that $\{T\}_{|Y}$ is Kähler for every component Y of Z to get a neighborhood U of Z and a smooth potential φ_U on X such that $\alpha + i\partial\overline{\partial}\varphi_U$ is Kähler on U. Then the smooth potential equal to the regularized maximum $\varphi = \widetilde{\max}(\psi, \varphi_U - C)$ produces a Kähler form $\alpha + i\partial\overline{\partial}\varphi$ on X for C large enough (since we can achieve $\varphi = \psi$ on $X \smallsetminus U$). The nef case (iv) is similar. □

THEOREM 3.4. *A compact complex manifold X admits a Kähler current if and only if it is bimeromorphic to a Kähler manifold, or equivalently, if it admits a proper Kähler modification.* (*The class of such manifolds is the so-called* Fujiki class $\mathcal{C}$.)

Proof. If X is bimeromorphic to a Kähler manifold Y, Hironaka's desingularization theorem implies that there exists a blow-up $\widetilde{Y}$ of Y (obtained by a sequence of blow-ups with smooth centers) such that the bimeromorphic map from Y to X can be resolved into a modification $\mu : \widetilde{Y} \to X$. Then $\widetilde{Y}$ is Kähler and the push-forward $T = \mu_*\widetilde{\omega}$ of a Kähler form $\widetilde{\omega}$ on $\widetilde{Y}$ provides a Kähler current on X. In fact, if ω is a smooth hermitian form on X, there is a constant C such that $\mu^*\omega \leq C\widetilde{\omega}$ (by compactness of $\widetilde{Y}$); hence

$$T = \mu_*\widetilde{\omega} \geq \mu_*(C^{-1}\mu^*\omega) = C^{-1}\omega.$$

Conversely, assume that X admits a Kähler current T. By Theorem 3.2 (iv), there exists a Kähler current $T' = T_k$ ($k \gg 1$) in the same $\partial\overline{\partial}$-cohomology class as T, and a modification $\mu : \widetilde{X} \to X$ such that

$$\mu^*T' = \lambda[\widetilde{D}] + \widetilde{\alpha} \qquad \text{on } \widetilde{X},$$

where $\widetilde{D}$ is a divisor with normal crossings, $\widetilde{\alpha}$ a smooth closed $(1,1)$-form and $\lambda > 0$. (The pull-back of a closed $(1,1)$-current by a holomorphic map f is always well-defined, when we take a local plurisubharmonic potential φ such that $T = i\partial\overline{\partial}\varphi$ and write $f^*T = i\partial\overline{\partial}(\varphi \circ f)$.) The form $\widetilde{\alpha}$ must be semi-positive; more precisely we have $\widetilde{\alpha} \geq \varepsilon\mu^*\omega$ as soon as $T' \geq \varepsilon\omega$. This is not enough to produce a Kähler form on $\widetilde{X}$ (but we are not very far...). Suppose that $\widetilde{X}$ is obtained as a tower of blow-ups

$$\widetilde{X} = X_N \to X_{N-1} \to \cdots \to X_1 \to X_0 = X,$$

where X_{j+1} is the blow-up of X_j along a smooth center $Y_j \subset X_j$. Denote by $E_{j+1} \subset X_{j+1}$ the exceptional divisor, and let $\mu_j : X_{j+1} \to X_j$ be the blow-up map. Now, we state the following simple result.

LEMMA 3.5. *For every Kähler current T_j on X_j, there exists $\varepsilon_{j+1} > 0$ and a smooth form u_{j+1} in the $\partial\overline{\partial}$-cohomology class of $[E_{j+1}]$ such that*

$$T_{j+1} = \mu_j^\star T_j - \varepsilon_{j+1}u_{j+1}$$

is a Kähler current on X_{j+1}.

Proof. The line bundle $\mathcal{O}(-E_{j+1})|E_{j+1}$ is equal to $\mathcal{O}_{P(N_j)}(1)$ where N_j is the normal bundle to Y_j in X_j. Pick an arbitrary smooth hermitian metric on N_j, use this metric to get an induced Fubini-Study metric on $\mathcal{O}_{P(N_j)}(1)$, and finally extend this metric as a smooth hermitian metric on the line bundle $\mathcal{O}(-E_{j+1})$. Such a metric has positive curvature along tangent vectors of X_{j+1} which are tangent to the fibers of $E_{j+1} = P(N_j) \to Y_j$. Assume furthermore that $T_j \geq \delta_j\omega_j$ for some hermitian form ω_j on X_j and a suitable $0 < \delta_j \ll 1$. Then

$$\mu_j^\star T_j - \varepsilon_{j+1}u_{j+1} \geq \delta_j\mu_j^\star\omega_j - \varepsilon_{j+1}u_{j+1}$$

where $\mu_j^*\omega_j$ is semi-positive on X_{j+1}, positive definite on $X_{j+1} \smallsetminus E_{j+1}$, and also positive definite on tangent vectors of $T_{X_{j+1}|E_{j+1}}$ which are not tangent to the fibers of $E_{j+1} \to Y_j$. The statement is then easily proved by taking $\varepsilon_{j+1} \ll \delta_j$ and by using an elementary compactness argument on the unit sphere bundle of $T_{X_{j+1}}$ associated with any given hermitian metric. □

End of proof of Theorem 3.4. If $\widetilde{u}_j$ is the pull-back of u_j to the final blow-up $\widetilde{X}$, we conclude inductively that $\mu^\star T' - \sum \varepsilon_j \widetilde{u}_j$ is a Kähler current. Therefore the smooth form

$$\widetilde{\omega} := \widetilde{\alpha} - \sum \varepsilon_j \widetilde{u}_j = \mu^\star T' - \sum \varepsilon_j \widetilde{u}_j - \lambda[D]$$

is Kähler and we see that $\widetilde{X}$ is a Kähler manifold. □

Remark 3.6. A special case of Theorem 3.4 is the following characterization of Moishezon varieties (i.e. manifolds which are bimeromorphic to projective algebraic varieties or, equivalently, whose algebraic dimension is equal to their complex dimension):

A compact complex manifold X is Moishezon if and only if X possesses a Kähler current T such that the de Rham cohomology class $\{T\}$ is rational, i.e. $\{T\} \in H^2(X, \mathbf{Q})$.

In fact, in the above proof, we get an integral current T if we take the push forward $T = \mu_*\widetilde{\omega}$ of an integral ample class $\{\widetilde{\omega}\}$ on Y, where $\mu : Y \to X$ is a projective model of Y. Conversely, if $\{T\}$ is rational, we can take the $\varepsilon_j' s$ to be rational in Lemma 3.5. This produces at the end a Kähler metric $\widetilde{\omega}$ with rational de Rham cohomology class on $\widetilde{X}$. Therefore $\widetilde{X}$ is projective by the Kodaira embedding theorem. This result was observed in [JS93] (see also [Bon93, 98] for a more general perspective based on a singular version of holomorphic Morse inequalities).

4. Numerical characterization of the Kähler cone

We are now in a good position to prove what we consider to be the main result of this work.

Proof of the Main Theorem 0.1. By definition $\mathcal{K}$ is open, and clearly $\mathcal{K} \subset \mathcal{P}$ (thus $\mathcal{K} \subset \mathcal{P}^\circ$). We claim that $\mathcal{K}$ is also closed in $\mathcal{P}$. In fact, consider a class $\{\alpha\} \in \overline{\mathcal{K}} \cap \mathcal{P}$. This means that $\{\alpha\}$ is a nef class which satisfies all numerical conditions defining $\mathcal{P}$. Let $Y \subset X$ be an arbitrary analytic subset. We prove by induction on $\dim Y$ that $\{\alpha\}_{|Y}$ is Kähler. If Y has several components, Proposition 3.3 (ii) reduces the situation to the case of the irreducible components of Y, so that we may assume that Y is irreducible. Let $\mu : \widetilde{Y} \to Y$ be a desingularization of Y, obtained via a finite sequence of blow-ups with smooth centers in X. Then $\widetilde{Y}$ is a smooth Kähler manifold and $\{\mu^*\alpha\}$ is a nef class such that

$$\int_{\widetilde{Y}} (\mu^*\alpha)^p = \int_Y \alpha^p > 0, \qquad p = \dim Y.$$

By Theorem 2.12, there exists a Kähler current $\widetilde{T}$ on $\widetilde{Y}$ which belongs to the class $\{\mu^\star\alpha\}$. Then $T := \mu_*\widetilde{T}$ is a Kähler current on Y, contained in the class $\{\alpha\}$. By the induction hypothesis, the class $\{\alpha\}_{|Z}$ is Kähler for every irreducible component Z of $E_c(T)$ (since $\dim Z \le p-1$). Proposition 3.3 (iii) now shows that $\{\alpha\}$ is Kähler on Y. In the case $Y = X$, we get that $\{\alpha\}$ itself is Kähler; hence $\{\alpha\} \in \mathcal{K}$ and $\mathcal{K}$ is closed in $\mathcal{P}$. This implies that $\mathcal{K}$ is a union of connected components of $\mathcal{P}$. However, since $\mathcal{K}$ is convex, it is certainly connected, and only one component can be contained in $\mathcal{K}$. □

Remark 4.1. In all examples that we are aware of, the cone $\mathcal{P}$ is open. Moreover, Theorem 0.1 shows that the connected component of any Kähler class $\{\omega\}$ in $\mathcal{P}$ is open in $H^{1,1}(X, \mathbf{R})$ (actually an open convex cone...). However, it might still happen that $\mathcal{P}$ carries some boundary points on the other components. It turns out that there exist examples for which $\mathcal{P}$ is not connected. Let us consider for instance a complex torus $X = \mathbf{C}^n/\Lambda$. It is well-known that a generic torus X does not possess any analytic subset except finite subsets and X itself. In that case, the numerical positivity is expressed by the single condition $\int_X \alpha^n > 0$. However, on a torus, $(1,1)$-classes are in one-to-one correspondence with constant hermitian forms α on $\mathbf{C}^n$. Thus, for X generic, $\mathcal{P}$ is the set of hermitian forms on $\mathbf{C}^n$ such that $\det(\alpha) > 0$, and Theorem 0.1 just expresses the elementary result of linear algebra saying that the set $\mathcal{K}$ of positive definite forms is one of the connected components of the open set $\mathcal{P} = \{\det(\alpha) > 0\}$ of hermitian forms of positive determinant (the other components, of course, are the sets of forms of signature (p,q), $p+q=n$, with q even).

One of the drawbacks of Theorem 0.1 is that the characterization of the Kähler cone still involves the choice of an undetermined connected component. However, it is trivial to derive from Theorem 0.1 the following (weaker) variants, which do not involve the choice of a connected component.

THEOREM 4.2. *Let (X,ω) be a compact Kähler manifold and let $\{\alpha\}$ be a $(1,1)$ cohomology class in $H^{1,1}(X,\mathbf{R})$. The following properties are equivalent.*

(i) $\{\alpha\}$ *is Kähler.*

(ii) *For every irreducible analytic set $Y \subset X$, $\dim Y = p$, and every $t \ge 0$,*

$$\int_Y (\alpha + t\omega)^p > 0.$$

(iii) *For every irreducible analytic set* $Y \subset X$, $\dim Y = p$,

$$\int_Y \alpha^k \wedge \omega^{p-k} > 0 \qquad \text{for } k = 1, \dots, p.$$

Proof. It is obvious that (i) ⇒ (iii) ⇒ (ii), so we only need to show that (ii) ⇒ (i). Assume that condition (ii) holds true. For t_0 large enough, $\alpha + t_0\omega$ is a Kähler class. The segment $(\alpha + t_0\omega)_{t\in[0,t_0]}$ is a connected set intersecting $\mathcal{K}$ which is contained in $\mathcal{P}$, thus it is entirely contained in $\mathcal{K}$ by Theorem 0.1. We infer that $\{\alpha\} \in \mathcal{K}$, as desired. □

We now study nef classes. The results announced in the introduction can be rephrased as follows.

THEOREM 4.3. *Let X be a compact Kähler manifold and let*

$$\{\alpha\} \in H^{1,1}(X, \mathbf{R})$$

be a $(1,1)$-*cohomology class. The following properties are equivalent.*

(i) $\{\alpha\}$ *is* nef.

(ii) *There exists a Kähler class* ω *such that*

$$\int_Y \alpha^k \wedge \omega^{p-k} \geq 0$$

for every irreducible analytic set $Y \subset X$, $\dim Y = p$, *and every* $k = 1, 2, \dots, p$.

(iii) *For every irreducible analytic set* $Y \subset X$, $\dim Y = p$, *and every Kähler class* $\{\omega\}$ *on* X

$$\int_Y \alpha \wedge \omega^{p-1} \geq 0.$$

Proof. Clearly (i) ⇒ (ii) and (i) ⇒ (iii).

(ii) ⇒ (i). If $\{\alpha\}$ satisfies the inequalities in (ii), then the class $\{\alpha + \varepsilon\omega\}$ satisfies the corresponding strict inequalities for every $\varepsilon > 0$. Therefore $\{\alpha + \varepsilon\omega\}$ is Kähler by Theorem 4.2, and $\{\alpha\}$ is nef.

(iii) ⇒ (i). This is the most tricky part. For every integer $p \geq 1$, there exists a polynomial identity of the form

$$(4.4) \qquad (y - \delta x)^p - (1-\delta)^p x^p = (y - x) \int_0^1 A_p(t, \delta)\big((1-t)x + ty\big)^{p-1}\, dt$$

where $A_p(t, \delta) = \sum_{0 \leq m \leq p} a_m(t)\delta^m \in \mathbf{Q}[t, \delta]$ is a polynomial of degree $\leq p - 1$ in t (moreover, the polynomial A_p is unique under this limitation for the degree). To see this, we observe that $(y - \delta x)^p - (1-\delta)^p x^p$ vanishes identically for $x = y$, so it is divisible by $y - x$. By homogeneity in (x, y), we have an expansion of the form

$$(y - \delta x)^p - (1 - \delta)^p x^p = (y - x) \sum_{0 \le \ell \le p-1,\, 0 \le m \le p} b_{\ell,m} x^\ell y^{p-1-\ell} \delta^m$$

in the ring $\mathbf{Z}[x, y, \delta]$. Formula (4.4) is then equivalent to

$$b_{\ell,m} = \int_0^1 a_m(t) \binom{p-1}{\ell} (1-t)^\ell t^{p-1-\ell}\, dt. \tag{4.4'}$$

Since $(U, V) \mapsto \int_0^1 U(t)V(t)dt$ is a nondegenerate linear pairing on the space of polynomials of degree $\le p-1$ and since $(\binom{p-1}{\ell}(1-t)^\ell t^{p-1-\ell})_{0 \le \ell \le p-1}$ is a basis of this space, (4.4′) can be achieved for a unique choice of the polynomials $a_m(t)$. A straightforward calculation shows that $A_p(t, 0) = 1$ identically. We can therefore choose $\delta_0 \in [0, 1[$ so small that $A_p(t, \delta) > 0$ for all $t \in [0, 1]$, $\delta \in [0, \delta_0]$ and $p = 1, 2, \dots, n$.

Now, fix a Kähler metric ω such that $\omega' = \alpha + \omega$ is Kähler (if necessary, multiply ω by a large constant to reach this). A substitution $x = \omega$ and $y = \omega'$ in our polynomial identity yields

$$(\alpha + (1 - \delta)\omega)^p - (1 - \delta)^p \omega^p = \int_0^1 A_p(t, \delta)\, \alpha \wedge \big((1 - t)\omega + t\omega'\big)^{p-1} dt.$$

For every irreducible analytic subset $Y \subset X$ of dimension p we find

$$\int_Y (\alpha + (1 - \delta)\omega)^p - (1 - \delta)^p \int_Y \omega^p = \int_0^1 A_p(t, \delta) dt \Big(\int_Y \alpha \wedge \big((1 - t)\omega + t\omega'\big)^{p-1} \Big).$$

However, $(1-t)\omega + t\omega'$ is Kähler and therefore $\int_Y \alpha \wedge \big((1-t)\omega + t\omega'\big)^{p-1} \ge 0$ by condition (iii). This implies $\int_Y (\alpha + (1 - \delta)\omega)^p > 0$ for all $\delta \in [0, \delta_0]$. We have produced a segment entirely contained in $\mathcal{P}$ such that one extremity $\{\alpha + \omega\}$ is in $\mathcal{K}$, so that the other extremity $\{\alpha + (1 - \delta_0)\omega\}$ is also in $\mathcal{K}$. By repeating the argument inductively, we see that $\{\alpha + (1 - \delta_0)^\nu \omega\} \in \mathcal{K}$ for every integer $\nu \ge 0$. From this we infer that $\{\alpha\}$ is nef, as desired. □

Since condition 4.3 (iii) is linear with respect to α, we can also view this fact as a characterization of the dual cone of the nef cone, in the space of real cohomology classes of type $(n - 1, n - 1)$. This leads immediately to Corollary 0.4.

In the case of projective manifolds, we get stronger and simpler versions of the above statements. All these can be seen as an extension of the Nakai-Moishezon criterion to arbitrary $(1, 1)$-classes (not just integral $(1, 1)$-classes as in the usual Nakai-Moishezon criterion). Apart from the special cases already mentioned in the introduction ([CP90], [Eys00]), these results seem to be entirely new.

THEOREM 4.5. *Let X be a projective algebraic manifold. Then $\mathcal{K} = \mathcal{P}$. Moreover, the following numerical characterizations hold*:

(i) *A* $(1,1)$-*class* $\{\alpha\} \in H^{1,1}(X, \mathbf{R})$ *is Kähler if and only if* $\int_Y \alpha^p > 0$ *for every irreducible analytic set* $Y \subset X$, $p = \dim Y$.

(ii) *A* $(1,1)$-*class* $\{\alpha\} \in H^{1,1}(X, \mathbf{R})$ *is* nef *if and only if* $\int_Y \alpha^p \geq 0$ *for every irreducible analytic set* $Y \subset X$, $p = \dim Y$.

(iii) *A* $(1,1)$-*class* $\{\alpha\} \in H^{1,1}(X, \mathbf{R})$ *is* nef *if and only if* $\int_Y \alpha \wedge \omega^{p-1} \geq 0$ *for every irreducible analytic set* $Y \subset X$, $p = \dim Y$, *and every Kähler class* $\{\omega\}$ *on* X.

Proof. (i) We take $\omega = c_1(A, h)$ equal to the curvature form of a very ample line bundle A on X, and we apply the numerical conditions as they are expressed in 4.2 (ii). For every p-dimensional algebraic subset Y in X we have

$$\int_Y \alpha^k \wedge \omega^{p-k} = \int_{Y \cap H_1 \cap \ldots \cap H_{p-k}} \omega^k$$

for a suitable generic complete intersection $Y \cap H_1 \cap \ldots \cap H_{p-k}$ of Y by members of the linear system $|A|$. This shows that $\mathcal{P} = \mathcal{K}$.

(ii) The nef case follows when we consider $\alpha + \varepsilon\omega$, and let $\varepsilon > 0$ tend to 0.

(iii) is true more generally for any compact Kähler manifold. □

Remark 4.6. In the case of a divisor D (i.e., of an integral class $\{\alpha\}$) on a projective algebraic manifold X, it is well known that $\{\alpha\}$ is nef if and only if $D \cdot C = \int_C \alpha \geq 0$ for every algebraic curve C in X. This result completely fails when $\{\alpha\}$ is not an integral class – this is the same as saying that the dual cone of the nef cone, in general, is bigger than the closed convex cone generated by cohomology classes of effective curves. Any surface such that the Picard number ρ is less than $h^{1,1}$ provides a counterexample (any generic abelian surface or any generic projective K3 surface is thus a counterexample). In particular, in 4.5 (iii), it is not sufficient to restrict the condition to *integral* Kähler classes $\{\omega\}$ only.

5. Deformations of compact Kähler manifolds

Let $\pi : \mathcal{X} \to S$ be a deformation of nonsingular compact Kähler manifolds, i.e. a proper analytic map between reduced complex spaces, with smooth Kähler fibres, such that the map is a trivial fibration locally near every point of $\mathcal{X}$ (this is of course the case if $\pi : \mathcal{X} \to S$ is smooth, but here we do not want to require S to be smooth; however we will always assume S to be irreducible – hence connected as well).

We wish to investigate the behaviour of the Kähler cones $\mathcal{K}_t$ of the various fibres $X_t = \pi^{-1}(t)$, as t runs over S. Because of the assumption of local triviality of π, the topology of X_t is locally constant, and therefore so are the cohomology groups $H^k(X_t, \mathbf{C})$. Each of these forms a locally constant vector bundle over S, whose associated sheaf of sections is the direct image sheaf $R^k\pi_*(\mathbf{C}_{\mathcal{X}})$. This locally constant system of $\mathbf{C}$-vector space contains as a sublattice the locally constant system of integral lattices $R^k\pi_*(\mathbf{Z}_{\mathcal{X}})$. As a consequence, the Hodge bundle $t \mapsto H^k(X_t, \mathbf{C})$ carries a natural flat connection ∇ which is known as the *Gauss-Manin connection*.

Thanks to D. Barlet's theory of cycle spaces [Bar75], one can attach to every reduced complex space X a reduced cycle space $C^p(X)$ parametrizing its compact analytic cycles of a given complex dimension p. In our situation, there is a relative cycle space $C^p(\mathcal{X}/S) \subset C^p(\mathcal{X})$ which consists of all cycles contained in the fibres of $\pi : X \to S$. It is equipped with a canonical holomorphic projection

$$\pi_p : C^p(\mathcal{X}/S) \to S.$$

Moreover, as the fibres X_t are Kähler, it is known that the restriction of π_p to the connected components of $C^p(\mathcal{X}/S)$ are proper maps. Also, there is a cohomology class (or degree) map

$$C^p(\mathcal{X}/S) \to R^{2q}\pi_*(\mathbf{Z}_{\mathcal{X}}), \qquad Z \mapsto \{[Z]\}$$

commuting with the projection to S, which to every compact analytic cycle Z in X_t associates its cohomology class $\{[Z]\} \in H^{2q}(X_t, \mathbf{Z})$, where $q = \operatorname{codim} Z = \dim X_t - p$. Again by the Kähler property (bounds on volume and Bishop compactness theorem), the map $C^p(\mathcal{X}/S) \to R^{2q}\pi_*(\mathbf{Z}_{\mathcal{X}})$ is proper.

As is well known, the Hodge filtration

$$F^p(H^k(X_t, \mathbf{C})) = \bigoplus_{r+s=k, r\geq p} H^{r,s}(X_t, \mathbf{C})$$

defines a *holomorphic* subbundle of $H^k(X_t, \mathbf{C})$ (with respect to its locally constant structure). On the other hand, the Dolbeault groups are given by

$$H^{p,q}(X_t, \mathbf{C}) = F^p(H^k(X_t, \mathbf{C})) \cap \overline{F^{k-p}(H^k(X_t, \mathbf{C}))}, \qquad k = p + q,$$

and they form *real analytic* subbundles of $H^k(X_t, \mathbf{C})$. We are interested especially in the decomposition

$$H^2(X_t, \mathbf{C}) = H^{2,0}(X_t, \mathbf{C}) \oplus H^{1,1}(X_t, \mathbf{C}) \oplus H^{0,2}(X_t, \mathbf{C})$$

and the induced decomposition of the Gauss-Manin connection acting on H^2

$$\nabla = \begin{pmatrix} \nabla^{2,0} & * & * \\ * & \nabla^{1,1} & * \\ * & * & \nabla^{0,2} \end{pmatrix}.$$

Here the stars indicate suitable bundle morphisms – actually with the lower-left and upper-right stars being zero by Griffiths' transversality property, but we do not care here. The notation $\nabla^{p,q}$ stands for the induced (real analytic, not necessarily flat) connection on the subbundle $t \mapsto H^{p,q}(X_t, \mathbf{C})$.

The main result of this section is the proof of Theorem 0.8, asserting that there is a countable union of analytic subsets $S' = \bigcup S_\nu \subsetneq S$ outside which the Kähler cones $\mathcal{K}_t \subset H^{1,1}(X_t, \mathbf{C})$ are invariant under parallel transport with respect to the $(1,1)$-component $\nabla^{1,1}$ of the Gauss-Manin connection. Of course, once this is proved, one can apply again the result on each stratum S_ν instead of S to see that there is a countable stratification of S such that the Kähler cone is essentially "independent of t" on each stratum. Moreover, we have semi-continuity in the sense that $\mathcal{K}_{t_0}$, $t_0 \in S'$, is always contained in the limit of the nearby cones $\mathcal{K}_t$, $t \in S \smallsetminus S'$.

Proof of Theorem 0.8. The result is local over S, so we can possibly shrink S to avoid any global monodromy (i.e., we assume that the locally constant systems $R^k\pi_\star(\mathbf{Z}_{\mathcal{X}})$ are constant). We then define the S_ν's to be the images in S of those connected components of $C^p(\mathcal{X}/S)$ which do not project onto S. By the fact that the projection is proper on each component, we infer that S_ν is an analytic subset of S. The definition of the S_ν's implies that the cohomology classes induced by the analytic cycles $\{[Z]\}$, $Z \subset X_t$, remain exactly the same for all $t \in S \smallsetminus S'$.

Since S is irreducible and S' is a countable union of analytic sets, it follows that $S \smallsetminus S'$ is arcwise connected by piecewise smooth analytic arcs. Let

$$\gamma : [0,1] \to S \smallsetminus S', \qquad u \mapsto t = \gamma(u)$$

be such a smooth arc, and let $\alpha(u) \in H^{1,1}(X_{\gamma(u)}, \mathbf{R})$ be a family of real $(1,1)$-cohomology classes which are constant by parallel transport under $\nabla^{1,1}$. This is equivalent to assuming that

$$\nabla(\alpha(u)) \in H^{2,0}(X_{\gamma(u)}, \mathbf{C}) \oplus H^{0,2}(X_{\gamma(u)}, \mathbf{C})$$

for all u. Suppose that $\alpha(0)$ is a numerically positive class in $X_{\gamma(0)}$. We then have

$$\alpha(0)^p \cdot \{[Z]\} = \int_Z \alpha(0)^p > 0$$

for all p-dimensional analytic cycles Z in $X_{\gamma(0)}$. Let us denote by

$$\zeta_Z(t) \in H^{2q}(X_t, \mathbf{Z}), \qquad q = \dim X_t - p,$$

the family of cohomology classes equal to $\{[Z]\}$ at $t = \gamma(0)$, such that $\nabla\zeta_Z(t) = 0$ (i.e. constant with respect to the Gauss-Manin connection). By the above discussion, $\zeta_Z(t)$ is of type (q,q) for all $t \in S$, and when $Z \subset X_{\gamma(0)}$ varies, $\zeta_Z(t)$ generates all classes of analytic cycles in X_t if $t \in S \smallsetminus S'$. Since

ζ_Z is ∇-parallel and $\nabla\alpha(u)$ has no component of type $(1,1)$, we find

$$\frac{d}{du}(\alpha(u)^p \cdot \zeta_Z(\gamma(u)) = p\alpha(u)^{p-1} \cdot \nabla\alpha(u) \cdot \zeta_Z(\gamma(u)) = 0.$$

We infer from this that $\alpha(u)$ is a numerically positive class for all $u \in [0,1]$. This argument shows that the set $\mathcal{P}_t$ of numerically positive classes in $H^{1,1}(X_t, \mathbf{R})$ is invariant by parallel transport under $\nabla^{1,1}$ over $S \smallsetminus S'$.

By a standard result of Kodaira-Spencer [KS60] relying on elliptic PDE theory, every Kähler class in X_{t_0} can be deformed to a nearby Kähler class in nearby fibres X_t. This implies that the connected component of $\mathcal{P}_t$ which corresponds to the Kähler cone $\mathcal{K}_t$ must remain the same. The theorem is thus proved (notice moreover that the remark concerning the semi-continuity of Kähler cones stated after Theorem 5.1 follows from the result by Kodaira-Spencer). □

From the above results, one can hope for a much stronger semi-continuity statement than the one stated by Kodaira-Spencer. Namely, we make the following conjecture, which we will consider in a forthcoming paper.

Conjecture 5.1. *Let $\mathcal{X} \to S$ be a deformation of compact complex manifolds over an irreducible base S. Assume that one of the fibres X_{t_0} is Kähler. Then there exists a countable union $S' \subsetneq S$ of analytic subsets in the base such that X_t is Kähler for $t \in S \smallsetminus S'$. Moreover, S' can be chosen so that the Kähler cone is invariant over $S \smallsetminus S'$, under parallel transport by the Gauss-Manin connection.*

In other words, the Kähler property should be open for the countable Zariski topology on the base. We are not sure what to think about the remaining fibres X_t, $t \in S'$, but a natural expectation would be that they are in the Fujiki class $\mathcal{C}$ (at least, under the assumption that Hodge decomposition remains valid on those fibres – which is anyway a necessary condition for the expectation to hold true).

Université Joseph Fourier, Grenoble, France
E-mail address: Jean-Pierre.Demailly@ujf-grenoble.fr

Université Louis Pasteur, Strasbourg, France
E-mail address: paun@math.u-strasbg.fr

References

[Bar75] D. Barlet, Espace analytique réduit des cycles analytiques complexes compacts d'un espace analytique complexe de dimension finie, in *Fonctions de Plusieurs Variables Complexes, II* (Sém. Franois Norguet, 1974–1975), *Lecture Notes in Math.* **482**, Springer-Verlag, New York (1975), 1–158.

[Bon93] L. Bonavero, Inégalités de Morse holomorphes singulières, *C. R. Acad. Sci. Sér. I* **317** (1993), 1163–1166.

[Bon98] ———, Inégalités de Morse holomorphes singulières, *J. Geom. Anal.* **8** (1998), 409–425.

[Bou89] T. Bouche, Inégalités de Morse pour la d''-cohomologie sur une variété non compacte, *Ann. Sci. Ecole Norm. Sup.* **22** (1989), 501–513.

[Buc99] N. Buchdahl, On compact Kähler surfaces, *Ann. Inst. Fourier* **49** (1999), 287–302.

[Buc00] ———, A Nakai-Moishezon criterion for non-Kähler surfaces, *Ann. Inst. Fourier* **50** (2000), 1533–1538.

[CP90] F. Campana and T. Peternell, Algebraicity of the ample cone of projective varieties, *J. Reine Angew. Math.* **407** (1990), 160–166.

[Dem85a] J.-P. Demailly, Mesures de Monge-Ampère et caractérisation géométrique des variétés algébriques affines, *Mém. Soc. Math. France* **19** (1985), 1–124.

[Dem85b] ———, Champs magnétiques et inégalités de Morse pour la d''-cohomologie, *Ann. Inst. Fourier* **35** (1985), 189–229.

[Dem90a] ———, Cohomology of q-convex spaces in top degrees, *Math. Z.* **204** (1990), 283–295.

[Dem90b] ———, Singular hermitian metrics on positive line bundles, in *Complex Algebraic Varieties* (Bayreuth, 1990) (K. Hulek, T. Peternell, M. Schneider, F. Schreyer, eds.), *Lecture Notes in Math.* **1507**, 87–104, Springer-Verlag, New York, 1992.

[Dem92] ———, Regularization of closed positive currents and intersection theory, *J. Algebraic Geom.* **1** (1992), 361–409.

[Dem93] ———, A numerical criterion for very ample line bundles, *J. Differential Geom.* **37** (1993), 323–374.

[DEL99] J.-P. Demailly, L. Ein, and R. Lazarsfeld, A subadditivity property of multiplier ideals, *Michigan Math. J.* (special volume in honor of William Fulton) **48** (2000), 137–156; math.AG/0002035.

[DPS94] J.-P. Demailly, Th. Peternell, and M. Schneider, Compact complex manifolds with numerically effective tangent bundles, *J. Algebraic Geom.* **3** (1994), 295–345.

[DPS00] ———, Pseudo-effective line bundles on compact Kähler manifolds, *Internat. J. Math.* **12** (2001), 689–741.

[Eys00] P. Eyssidieux, Théorèmes de Nakai-Moishezon pour certaines classes de type $(1,1)$ et applications, Prépublication Université de Toulouse III, 2000.

[Huy99] D. Huybrechts, Compact hyper-Kähler manifolds: basic results, *Invent. Math.* **135** (1999), 63–113.

[Huy01] ———, The projectivity criterion for hyperKähler manifolds as a consequence of the Demailly-Paun theorem, personal communication, manuscript Köln University, May 2001, 3 pp.

[JS93] S. Ji and B. Shiffman, Properties of compact complex manifolds carrying closed positive currents, *J. Geom. Anal.* **3** (1993), 37–61.

[Kle66] S. Kleiman, Toward a numerical theory of ampleness, *Ann. of Math.* **84** (1966), 293–344.

[KS60] K. Kodaira and D. C. Spencer, On deformations of complex analytic structures. III. Stability theorems for complex structures, *Ann. of Math.* **71** (1960), 43–76.

[Lam99a] A. Lamari, Courants kählériens et surfaces compactes, *Ann. Inst. Fourier* **49** (1999), 263–285.

[Lam99b] ———, Le cone kählérien d'une surface, *J. Math. Pures Appl.* **78** (1999), 249–263.

[Pau98a] M. Paun, Sur l'effectivité numérique des images inverses de fibrés en droites, *Math. Ann.* **310** (1998), 411–421.

[Pau98b] ———, Fibré en droites numériquement effectifs et variétés Kählériennes compactes à courbure de Ricci nef, Thèse, Université de Grenoble 1 (1998), 80 pp.

[Pau00] M. Paun, Semipositive $(1, 1)$-cohomology classes on projective manifolds, preprint de l'I.R.M.A.

[Siu74] Y.-T. Siu, Analyticity of sets associated to Lelong numbers and the extension of closed positive currents, *Invent. Math.* **27** (1974), 53–156.

[Siu84] ———, Some recent results in complex manifold theory related to vanishing theorems, in *Workshop Bonn* 1984, *Lecture Notes in Math.* **1111** (1985), 169–192.

[Yau78] S.-T. Yau, On the Ricci curvature of a complex Kähler manifold and the complex Monge-Ampère equation. I, *Comm. Pure Appl. Math.* **31** (1978), 339–411.

(Received September 14, 2001)