On the Frobenius Integrability of Certain Holomorphic p-Forms

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Abstract The goal of this note is to exhibit the integrability properties (in the sense of the Frobenius theorem) of holomorphic p-forms with values in certain line bundles with semi-negative curvature on a compact Kähler manifold. There are in fact very strong restrictions, both on the holomorphic form and on the curvature of the semi-negative line bundle. In particular, these observations provide interesting information on the structure of projective manifolds which admit a contact structure: either they are Fano manifolds or, thanks to results of Kebekus-Peternell-Sommese-Wisniewski, they are biholomorphic to the projectivization of the cotangent bundle of another suitable projective manifold.

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1 Main Results

Recall that a holomorphic line bundle L on a compact complex manifold is said to be *pseudo-effective* if $c_1(L)$ contains a closed positive $(1,1)$-current T, or equivalently, if L possesses a (possibly singular) hermitian metric h such that the curvature current $T = \Theta_h(L) = -i\partial\overline{\partial}\log h$ is nonnegative. If X is projective, L is pseudo-effective if and only if $c_1(L)$ belongs to the closure of the cone generated by classes of effective divisors in $H^{1,1}_{\mathbb{R}}(X)$ (see [Dem90], [Dem92]). Our main result is

Main Theorem. *Let X be a compact Kähler manifold. Assume that there exists a pseudo-effective line bundle L on X and a nonzero holomorphic section $\theta \in H^0(X, \Omega^p_X \otimes L^{-1})$, where $0 \le p \le n = \dim X$. Let $\mathcal{S}_\theta$ be the coherent subsheaf of germs of vector fields ξ in the tangent sheaf T_X, such that the contraction $i_\xi\theta$ vanishes. Then $\mathcal{S}_\theta$ is integrable, namely $[\mathcal{S}_\theta, \mathcal{S}_\theta] \subset \mathcal{S}_\theta$, and L has flat curvature along the leaves of the (possibly singular) foliation defined by $\mathcal{S}_\theta$.*

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Before entering into the proof, we discuss several consequences. If $p = 0$ or $p = n$, the result is trivial (with $\mathcal{S}_\theta = T_X$ and $\mathcal{S}_\theta = 0$, respectively). The most interesting case is $p = 1$.

Corollary 1. *In the above situation, if the line bundle $L \to X$ is pseudo-effective and $\theta \in H^0(X, \Omega_X^1 \otimes L^{-1})$ is a nonzero section, the subsheaf $\mathcal{S}_\theta$ defines a holomorphic foliation of codimension 1 in X, that is, $\theta \wedge d\theta = 0$.*

We now concentrate ourselves on the case when X is a *contact manifold*, i.e. $\dim X = n = 2m+1$, $m \geq 1$, and there exists a form $\theta \in H^0(X, \Omega_X^1 \otimes L^{-1})$, called the *contact form*, such that $\theta \wedge (d\theta)^m \in H^0(X, K_X \otimes L^{-m-1})$ has no zeroes. Then $\mathcal{S}_\theta$ is a codimension 1 locally free subsheaf of T_X and there are dual exact sequences

$$0 \to L \to \Omega_X^1 \to \mathcal{S}_\theta^\star \to 0, \qquad 0 \to \mathcal{S}_\theta \to T_X \to L^\star \to 0 \,.$$

The subsheaf $\mathcal{S}_\theta \subset T_X$ is said to be the *contact structure* of X. The assumption that $\theta \wedge (d\theta)^m$ does not vanish implies that $K_X \simeq L^{m+1}$. In that case, the subsheaf is not integrable, hence L and K_X cannot be pseudo-effective.

Corollary 2. *If X is a compact Kähler manifold admitting a contact structure, then K_X is not pseudo-effective, in particular the Kodaira dimension $\kappa(X)$ is equal to $-\infty$.*

The fact that $\kappa(X) = -\infty$ had been observed previously by Stéphane Druel [Dru98]. In the projective context, the minimal model conjecture would imply (among many other things) that the conditions $\kappa(X) = -\infty$ and "K_X non pseudo-effective" are equivalent, but a priori the latter property is much stronger (and in large dimensions, the minimal model conjecture still seems far beyond reach!)

Corollary 3. *If X is a compact Kähler manifold with a contact structure and with second Betti number $b_2 = 1$, then K_X is negative, i.e., X is a Fano manifold.*

Actually the Kodaira embedding theorem shows that the Kähler manifold X is projective if $b_2 = 1$, and then every line bundle is either positive, flat or negative. As K_X is not pseudo-effective it must therefore be negative. In that direction, Boothby [Boo61], Wolf [Wol65] and Beauville [Bea98] have exhibited a natural construction of contact Fano manifolds. Each of the known examples is obtained as a homogeneous variety which is the unique closed orbit in the projectivized (co)adjoint representation of a simple algebraic Lie group. Beauville's work ([Bea98], [Bea99]) provides strong evidence that this is the complete classification in the case $b_2 = 1$.

We now come to the case $b_2 \geq 2$. If Y is an arbitrary compact Kähler manifold, the bundle $X = P(T_Y^\star)$ of hyperplanes of T_Y has a contact structure associated with the line bundle $L = \mathcal{O}_X(-1)$. Actually, if $\pi : X \to Y$ is the canonical projection, one can define a contact form $\theta \in H^0(X, \Omega_X^1 \otimes L^{-1})$ by setting

$$\theta(x) = \theta(y, [\xi]) = \xi^{-1}\pi^\star\xi = \xi^{-1}\sum_{1\le j\le p}\xi_j dy_j, \qquad p = \dim Y\ ,$$

at every point $x = (y, [\xi]) \in X$, $\xi \in T^\star_{Y,y}\setminus\{0\}$ (observe that $\xi \in L_x = \mathcal{O}_X(-1)_x$). Morever $b_2(X) = 1 + b_2(Y) \ge 2$. Conversely, Kebekus, Peternell, Sommese and Wiśniewski [KPSW] have recently shown that every projective algebraic manifold X such that

(i) X has a contact structure,
(ii) $b_2 \ge 2$,
(iii) K_X is not nef (numerically effective)

is of the form $X = P(T^\star_Y)$ for some projective algebraic manifold Y. However, the condition that K_X is not nef is implied by the fact that K_X is not pseudo-effective. Hence we get

Corollary 4. *If X is a contact projective manifold with $b_2 \ge 2$, then X is a projectivized hyperplane bundle $X = P(T^\star_Y)$ associated with some projective manifold Y.*

The Kähler case of corollary 4 is still unsolved, as the proof of [KPSW] heavily relies on Mori theory (and, unfortunately, the extension of Mori theory to compact Kähler manifolds remains to be settled ...).

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2 Proof of the Main Theorem

In some sense, the proof is just a straightforward integration by parts, but there are slight technical difficulties due to the fact that we have to work with singular metrics.

Let X be a compact Kähler manifold, ω the Kähler metric, and let L be a pseudo-effective line bundle on X. We select a hermitian metric h on L with nonnegative curvature current $\Theta_h(L) \ge 0$, and let φ be the plurisubharmonic weight of the metric h in any local trivialisation $L_{|U} \simeq U \times \mathbb{C}$. In other words, we have

$$\|\xi\|^2_h = |\xi|^2 e^{-\varphi(x)}, \qquad \|\xi^\star\|^2_{h^\star} = |\xi^\star|^2 e^{\varphi(x)}$$

for all $x \in U$ and $\xi \in L_x$, $\xi^\star \in L^{-1}$. We then have a Chern connection $\nabla = \partial_{h^\star} + \overline{\partial}$ acting on all (p,q)-forms f with values in L^{-1}, given locally by

$$\partial_\varphi f = e^{-\varphi}\partial(e^\varphi f) = \partial f + \partial\varphi \wedge f$$

in every trivialization $L_{|U}$. Now, assume that there is a holomorphic section $\theta \in H^0(X, \Omega_X^p \otimes L^{-1})$, i.e., a $\overline{\partial}$-closed $(p,0)$ form θ with values in L^{-1}. We compute the global L^2 norm

$$\int_X \{\partial_{h^\star}\theta, \partial_{h^\star}\theta\}_{h^\star} \wedge \omega^{n-p-1} = \int_X e^{\varphi}\partial_\varphi\theta \wedge \overline{\partial_\varphi\theta} \wedge \omega^{n-p-1}$$

where $\{\ ,\ \}_{h^\star}$ is the natural sesquilinear pairing sending pairs of L^{-1}-valued forms of type (p,q), (r,s) into $(p+s, q+r)$ complex valued forms. The right hand side is of course only locally defined, but it explains better how the forms are calculated, and also all local representatives glue together into a well defined global form; we will therefore use the latter notation as if it were global. As

$$d\big(e^{\varphi}\theta \wedge \overline{\partial_\varphi\theta} \wedge \omega^{n-p-1}\big) = e^{\varphi}\partial_\varphi\theta \wedge \overline{\partial_\varphi\theta} \wedge \omega^{n-p-1} + (-1)^p e^{\varphi}\theta \wedge \overline{\overline{\partial}\partial_\varphi\theta} \wedge \omega^{n-p-1}$$

and $\overline{\partial}\partial_\varphi\theta = \overline{\partial}\partial\varphi \wedge \theta$, an integration by parts via Stokes theorem yields

$$\int_X e^{\varphi}\partial_\varphi\theta \wedge \overline{\partial_\varphi\theta} \wedge \omega^{n-p-1} = -(-1)^p \int_X e^{\varphi}\partial\overline{\partial}\varphi \wedge \theta \wedge \overline{\theta} \wedge \omega^{n-p-1}\ .$$

These calculations need a word of explanation, since φ is in general singular. However, it is well known that the $i\partial\overline{\partial}$ of a plurisubharmonic function is a closed positive current, in particular

$$i\partial\overline{\partial}(e^{\varphi}) = e^{\varphi}(i\partial\varphi \wedge \overline{\partial}\varphi + i\partial\overline{\partial}\varphi)$$

is positive and has measure coefficients. This shows that $\partial\varphi$ is L^2 with respect to the weight e^{φ}, and similarly that $e^{\varphi}\partial\overline{\partial}\varphi$ has locally finite measure coefficients. Moreover, the results of [Dem92] imply that there is a decreasing sequence of metrics $h_\nu^\star$ and corresponding weights $\varphi_\nu \downarrow \varphi$, such that $\Theta_{h_\nu} \geq -C\omega$ with a fixed constant $C > 0$ (this claim is in fact much weaker than the results of [Dem92], and easy to prove e.g. by using convolutions in suitable coordinate patches and a standard gluing technique). Now, the results of Bedford-Taylor [BT76], [BT82] applied to the uniformly bounded functions $e^{c\varphi_\nu}$, $c > 0$, imply that we have local weak convergence

$$e^{\varphi_\nu}\partial\overline{\partial}\varphi_\nu \to e^{\varphi}\partial\overline{\partial}\varphi, \quad e^{\varphi_\nu}\partial\varphi_\nu \to e^{\varphi}\partial\varphi, \quad e^{\varphi_\nu}\partial\varphi_\nu \wedge \overline{\partial}\varphi_\nu \to e^{\varphi}\partial\varphi \wedge \overline{\partial}\varphi\ ,$$

possibly after adding $C'|z|^2$ to the φ_ν's to make them plurisubharmonic. This is enough to justify the calculations. Now, we take care of signs, using the fact that $i^{p^2}\theta \wedge \overline{\theta} \geq 0$ whenever θ is a $(p,0)$-form. Our previous equality can be rewritten

$$\int_X e^{\varphi}\, i^{(p+1)^2}\partial_\varphi\theta \wedge \overline{\partial_\varphi\theta} \wedge \omega^{n-p-1} = -\int_X e^{\varphi}\, i\partial\overline{\partial}\varphi \wedge i^{p^2}\theta \wedge \overline{\theta} \wedge \omega^{n-p-1}\ .$$

Since the left hand side is nonnegative and the right hand side is nonpositive, we conclude that $\partial_\varphi \theta = 0$ almost everywhere, i.e. $\partial\theta = -\partial\varphi \wedge \theta$ almost everywhere. The formula for the exterior derivative of a p-form reads

$$(*) \qquad d\theta(\xi_0, \ldots, \xi_p) = \sum_{0 \le j \le p} (-1)^j \xi_j \cdot \theta(\xi_0, \ldots, \widehat{\xi_j}, \ldots, \xi_p) + \sum_{0 \le j < k \le p} (-1)^{j+k} \theta([\xi_j, \xi_k], \xi_0, \ldots, \widehat{\xi_j}, \ldots, \widehat{\xi_k}, \ldots, \xi_p) .$$

If two of the vector fields – say ξ_0 and ξ_1 – lie in $\mathcal{S}_\theta$, then

$$d\theta(\xi_0, \ldots, \xi_p) = -(\partial\varphi \wedge \theta)(\xi_0, \ldots, \xi_p) = 0$$

and all terms in the right hand side of ($\star$) are also zero, except perhaps the term $\theta([\xi_0, \xi_1], \xi_2, \ldots, \xi_p)$. We infer that this term must vanish. Since this is true for arbitrary vector fields $\xi_2, \ldots, \xi_p$, we conclude that $[\xi_0, \xi_1] \in \mathcal{S}_\theta$ and that $\mathcal{S}_\theta$ is integrable.

The above arguments also yield strong restrictions on the hermitian metric h. In fact the equality $\partial\theta = -\partial\varphi \wedge \theta$ implies $\partial\overline{\partial}\varphi \wedge \theta = 0$ by taking the $\overline{\partial}$. Fix a smooth point in a leaf of the foliation, and local coordinates $(z_1, \ldots, z_n)$ such that the leaves are given by $z_1 = c_1, \ldots, z_r = c_r$ (c_i = constant), in a neighborhood of that point. Then $\mathcal{S}_\theta$ is generated by $\partial/\partial z_{r+1}, \ldots, \partial/\partial z_n$, and θ depends only on $dz_1, \ldots, dz_r$. This implies that $\partial^2\varphi/\partial z_j \partial\overline{z}_k = 0$ for $j, k > r$, in other words (L, h) has flat curvature along the leaves of the foliation. The main theorem is proved.

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