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The algebraic dimension of compact complex threefolds with vanishing second Betti number

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Abstract. This note investigates compact complex manifolds X of dimension 3 with second Betti number $b_2(X) = 0$. If X admits a non-constant meromorphic function, then we prove that either $b_1(X) = 1$ and $b_3(X) = 0$ or that $b_1(X) = 0$ and $b_3(X) = 2$. The main idea is to show that $c_3(X) = 0$ by means of a vanishing theorem for generic line bundles on X. As a consequence a compact complex threefold homeomorphic to the 6-sphere S^6 cannot admit a non-constant meromorphic function. Furthermore we investigate the structure of threefolds with $b_2(X) = 0$ and algebraic dimension 1, in the case when the algebraic reduction $X \to \mathbb{P}_1$ is holomorphic.

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0. Introduction

In this note we shall investigate compact complex manifolds of dimension three and second Betti number $b_2(X) = 0$. Such a manifold cannot be algebraic or Kähler. Therefore we will be interested in the algebraic dimension $a(X)$ which is by definition the transcendence degree of the field of meromorphic functions over the field of complex numbers. Note that $a(X) > 0$ if and only if X admits a non-constant meromorphic function. The topological Euler characteristic will be denoted $\chi_{\text{top}}(X)$ which is also the third Chern class $c_3(X)$ by a theorem of Hopf. Our main result is

THEOREM. *Let X be a compact 3-dimensional complex manifold with $b_2(X) = 0$ and $a(X) > 0$. Then*

$$c_3(X) = \chi_{\text{top}}(X) = 2 - 2b_1(X) - b_3(X) = 0,$$

i.e. we either have $b_1(X) = 0$, $b_3(X) = 2$ or $b_1(X) = 1$, $b_3(X) = 0$.

Notice that if $a(X) = 3$, i.e. X is Moishezon, then we have $b_2(X) > 0$, but examples of compact threefolds X with $a(X) = 1$ or 2 and with the above Betti numbers exist (see Sect. 2).

The following corollary was actually our motivation for the Theorem

COROLLARY. *Let X be a compact complex manifold homeomorphic to the* 6-*dimensional sphere S^6. Then $a(X) = 0$.*

In other words, S^6 does not admit a complex structure with a non-constant meromorphic function.

Our Main Theorem is an immediate consequence of the following more general

THEOREM. *Let X be a compact* 3-*dimensional complex manifold with $b_2(X) = 0$ and $a(X) > 0$. Let B be a vector bundle on X. Then $H^i(X, B \otimes \mathcal{M}) = 0$ for $i \geqslant 0$ and generic $\mathcal{M} \in \mathrm{Pic}^0(X)$, in particular $\chi(X, B \otimes \mathcal{M}) = 0$ for* all $\mathcal{M} \in \mathrm{Pic}^0(X)$.

In the last section we study more closely the structure of threefolds X with $b_2(X) = 0$ and algebraic dimension 1 whose algebraic reduction is holomorphic. We show e.g. that smooth fibers can only be Inoue surfaces, Hopf surface with algebraic dimension 0 or tori.

Finally we would like to thank the referee for suggestions of improvements in the exposition.

1. Preliminaries and criteria for the vanishing of H^0

NOTATIONS 1.0. (1) Let X be a compact complex manifold, always assumed to be connected. The algebraic dimension, denoted $a(X)$, is the transcendence degree of the field of meromorphic functions over $\mathbb{C}$.

(2) $b_i(X) = \dim H^i(X, \mathbb{R})$ denotes the ith Betti number of X.

(3) If G is a finitely generated abelian group, then $\mathrm{rk}\, G$ will denote its rank (over $\mathbb{Z}$).

(4) If X is a compact space, then $h^q(X, \mathcal{F})$ denotes the dimension of $H^q(X, \mathcal{F})$.

PROPOSITION 1.1. *Let Y be a connected compact complex space* (*not necessarily reduced*)*, every component Y_i of Y being of positive dimension. Let D be an effective Cartier divisor on Y such that $D_{|Y_i} \neq 0$ for all i. Let $\mathcal{F}$ be a locally free sheaf on Y. Then there exists $k_0 \in \mathbb{N}$ such that*

$$H^0(Y, \mathcal{F} \otimes \mathcal{O}_Y(-kD)) = 0$$

for $k \geqslant k_0$.

Proof. We have natural inclusions

$$H^0(Y, \mathcal{F} \otimes \mathcal{O}_Y(-(k+1)D)) \subset H^0(Y, \mathcal{F} \otimes \mathcal{O}_Y(-kD)).$$

Take k_0 such that this sequence is stationary for $k \geqslant k_0$. Then s has to vanish at any order along $D_{|Y_i}$ for every i (s can be thought of locally as a tuple of holomorphic functions), hence $s_{|Y_i} = 0$, and $s = 0$.

COROLLARY 1.2. *Let S be a smooth compact complex surface containing an effective divisor C such that $c_1(\mathcal{O}_S(C)) = 0$. Let B be a vector bundle on S. Then for a generic $\mathcal{L} \in \mathrm{Pic}^0(X)$ we have*

$$H^0(S, B \otimes \mathcal{L}) = H^2(S, B \otimes \mathcal{L}) = 0.$$

In particular $\chi(S, B) = \chi(S, B \otimes \mathcal{L}) = -h^1(S, B \otimes \mathcal{L}) \leqslant 0$.

Proof. The vanishing $H^0(S, B \otimes \mathcal{O}_S(-kC)) = 0$ for large k follows from (1.1). Since $\mathcal{O}_S(kC)$ is topologically trivial for large suitable k, the required H^0-vanishing follows from semi-continuity. The H^2-vanishing follows by applying the previous arguments to $B^* \otimes K_S$ and Serre duality.

COROLLARY 1.3. *Let X be a smooth compact threefold with $b_2(X) = 0$ carrying an effective divisor D. Then $H^0(X, B \otimes \mathcal{L}) = 0$ for generic $\mathcal{L}$ in $\mathrm{Pic}^0(X)$ and every vector bundle B on X.*

Proof. The assumption $b_2(X) = 0$ means that $H^2(X, \mathbb{Z})$ is finite, hence there exists an integer $m > 0$ such that $c_1(\mathcal{O}_X(mD)) = 0$ in $H^2(X, \mathbb{Z})$. We can apply (1.1) to obtain

$$H^0(X, B \otimes \mathcal{O}_X(-kmD)) = 0$$

for k large. Now we conclude again by semi-continuity.

The next lemma is well-known; we include it for the convenience of the reader.

LEMMA 1.4. *Let X be a compact manifold of dimension n with $a(X) = n$. Then $b_2(X) > 0$.*

Proof. Choose a birational morphism $\pi: \hat{X} \to X$ such that $\hat{X}$ is a projective manifold. Take a general very ample divisor $\hat{D}$ on $\hat{X}$ and a general curve $\hat{C} \in \hat{X}$. Let $D = \pi(\hat{D})$ and $C = \pi(\hat{C})$. Then D meets C in finitely many points, hence $D \cdot C > 0$, in particular $c_1(\mathcal{O}_X(D))$ is not torsion in $H^2(X, \mathbb{Z})$.

LEMMA 1.5. *Let X be a smooth compact threefold and $f: X \to C$ be a surjective holomorphic map to a smooth curve C. Let $\mathcal{F}$ be a locally free sheaf on X. Then $R^i f_*(\mathcal{F})$ is locally free for all i.*

Proof. (a) Note that local freeness is equivalent to torsion freeness, since $\dim C = 1$. Hence the claim is clear for $i = 0$.

(b) Next we treat the case $i = 2$. We shall use relative duality (see [RRV71], [We85]); it states in our special situation (f is flat with even Gorenstein fibers) that if $R^j f_*(\mathcal{G})$ is locally free for a given locally free sheaf $\mathcal{G}$ and fixed j, then

$$R^{2-j} f_*(\mathcal{G}^* \otimes \omega_{X/C}) \simeq R^j f_*(\mathcal{G})^*,$$

in particular $R^{2-j} f_*(\mathcal{G}^* \otimes \omega_{X/C})$ is locally free. Here $\omega_{X/C} = \omega_X \otimes f^*(\omega_C^*)$ is the relative dualizing sheaf. Applying this to $j = 0$ and $\mathcal{G} = \mathcal{F}^* \otimes \omega_{X/C}^*$ our claim for $i = 2$ follows.

(c) Finally we prove the freeness of $R^1 f_*(\mathcal{F})$. By a standard theorem of Grauert it is sufficient that $h^1(X_y, \mathcal{F}_{|X_y})$ is constant, X_y the analytic fiber over $y \in C$. By flatness, $\chi(X_y, \mathcal{F}_{|X_y})$ is constant, hence it is sufficient that $h^j(X_y, \mathcal{F}_{|X_y})$ is constant for $j = 0$ and $j = 2$. By the vanishing $R^3 f_*(\mathcal{F}) = 0$, we have (see e.g. [BaSt76])

$$R^2 f_*(\mathcal{F})_{|\{y\}} \simeq H^2(X_y, \mathcal{F}_{|X_y}).$$

Therefore $h^2(X_y, \mathcal{F}_{|X_y})$ is constant by (b). Finally

$$h^0(X_y, \mathcal{F}_{|X_y}) = h^2(X_y, \mathcal{F}^*_{|X_y} \otimes \omega_{X_y}) = h^2(X_y, (\mathcal{F}^* \otimes \omega_X)_{|X_y})$$

is constant by applying the same argument to $\mathcal{F}^* \otimes \omega_X$.

2. The main theorem

In this section we prove the main result of this note:

THEOREM 2.1. *Let X be a 3-dimensional compact complex manifold with $b_2(X) = 0$ and $a(X) > 0$. Let B be a vector bundle on X. Then*

(1) $H^i(X, B \otimes \mathcal{M}) = 0$ *for $i \geqslant 0$ and $\mathcal{M} \in \mathrm{Pic}^0(X)$ generic.*
(2) $\chi(X, B \otimes \mathcal{M}) = 0$ *for all $\mathcal{M} \in \mathrm{Pic}^0(X)$.*
(3) $c_3(X) = 0$, *i.e. either $b_1(X) = 0$ and $b_3(X) = 2$ or $b_1(X) = 1$ and $b_3(X) = 0$.*

Proof. First notice that (2) and (3) follow from (1). In fact, by (1) we have

$$\chi(X, B \otimes \mathcal{M}) = 0$$

for generic $\mathcal{M}$ and thus the same holds for all $\mathcal{M}$ by Riemann-Roch and the equality $c_j(B) = c_j(B \otimes \mathcal{M})$. For (3), we apply (2) to $B = T_X$ and get

$$\chi(X, T_X) = 0.$$

Now, since $H^2(X, \mathbb{R}) = H^4(X, \mathbb{R}) = 0$, we have $c_1(X) = c_2(X) = 0$, hence

$$0 = \chi(X, T_X) = \tfrac{1}{2} c_3(X)$$

by Riemann-Roch.

So it suffices to prove (1). Moreover by Serre duality we only need to prove the vanishing for $i = 0$ and $i = 2$.

In case $i = 0$ we observe that there are nonzero effective divisors on X (since $a(X) > 0$) and we can apply (1.3) to get the claim.

So let $i = 2$. Let $g: X \dashrightarrow \mathbb{P}_1$ be a nonconstant meromorphic function. Let $\sigma: \hat{X} \to X$ be a resolution of the indeterminacies of g and let $f: \hat{X} \to C$ be the

Stein factorisation of the holomorphic map $\sigma \circ g$. Fix an ample divisor A on C and let $\mathcal{L}$ be the line bundle on X determined by

$$f^*(A) = \sigma^*(\mathcal{L}) \otimes \mathcal{O}_{\hat{X}}(-E) \tag{a}$$

with a suitable effective divisor E supported on the exceptional set of σ. We need to exhibit a line bundle $\mathcal{M} \in \mathrm{Pic}^0(X)$ with

$$H^2(X, B \otimes \mathcal{M}) = 0.$$

We shall distinguish two cases according to whether the indeterminacy locus of g is empty or not.

We start treating the case that g is not holomorphic. First note that the canonical map

$$H^2(X, B \otimes \mathcal{M}) \to H^2(\hat{X}, \sigma^*(B \otimes \mathcal{M}))$$

is injective. This is obvious from the Leray spectral sequence. Hence it is sufficient to show

$$H^2(\hat{X}, \sigma^*(B \otimes \mathcal{M})) = 0. \tag{$*$}$$

Actually for ($*$) we only need

$$H^2(\hat{X}, \sigma^*(B \otimes \mathcal{M})(-tE)) = 0 \tag{$**$}$$

for some $t \geqslant 0$. To verify that $(**)$ implies $(*)$, consider the exact sequence

$$\begin{aligned} H^2(\hat{X}, \sigma^*(B \otimes \mathcal{M})(-tE)) &\to H^2(\hat{X}, \sigma^*(B \otimes \mathcal{M})) \\ &\to H^2(tE, \sigma^*(B \otimes \mathcal{M})) \end{aligned}$$

and note that $H^2(tE, \sigma^*(B \otimes \mathcal{M})) = 0$. This last vanishing is seen as follows: let A_t be the complex subspace of X defined by the ideal sheaf $\sigma_*(-tE)$, then

$$H^2(tE, \sigma^*(B \otimes \mathcal{M})) = H^2(A_t, B \otimes \mathcal{M}) = 0$$

since $\dim A_t = 1$.

We make the ansatz $\mathcal{M} = \mathcal{L}^{t+k}$ with t and k to be determined; of course we need to prove the vanishing only for one $\mathcal{M}$ by semi-continuity. Using the Leray spectral sequence for $f\colon \hat{X} \to C$, $(**)$ comes down to

$$H^q(C, R^p f_*(\sigma^*(B \otimes \mathcal{L}^k)) \otimes tA) = 0 \tag{$***$}$$

for $p + q = 2$, and large t, k. For $q = 2$, ($***$) is obvious and for $q = 1$ it follows from Serre's vanishing theorem for $t \gg 0$. So let $q = 0$. We need to see that

$$\mathcal{F} = R^2 f_*(\sigma^*(B \otimes \mathcal{L}^k)) = 0.$$

Since $\mathcal{F}$ is locally free by (1.5), it suffices that $\mathcal{F}_{|F} = 0$ for the general fiber F of f and large k. But this follows from (1.1), the effective divisor $E_{|F}$ being nonzero:

$$H^2(F, \sigma^*(B \otimes \mathcal{L}^k)) \simeq H^0(F, \sigma^*(B^*) \otimes K_F \otimes \mathcal{O}_F(-kE)) = 0.$$

If g is holomorphic, i.e. we may take $\sigma = \mathrm{id}$, so that $f = g$, this argument does not work since $E = 0$. Here we have to replace $\mathcal{L}^{t+k}$ by a different line bundle. First note that

$$H^0(C, R^2 f_*(B) \otimes (-tA)) = 0 \qquad (+)$$

for t sufficiently large. We claim that this implies

$$H^0(C, R^2 f_*(B \otimes \mathcal{M})) = 0 \qquad (++)$$

for general $\mathcal{M} \in \mathrm{Pic}^0(X)$. Let $W = H^1(X, \mathcal{O}_X)$. Then every element in W is represented as a topologically trivial line bundle.

Consider locally the universal bundle $\hat{\mathcal{M}}$ on $X \times W$. Let $F = f \times \mathrm{id} : X \times W \to C \times W$ and $\hat{B} = \mathrm{pr}_X^*(B)$. The coherent sheaf $R^2 F_*(\hat{B} \otimes \hat{\mathcal{M}})$ satisfies

$$R^2 F_*(\hat{B} \otimes \hat{\mathcal{M}})_{|C} \times \{t\} \simeq R^2 f_*(B \otimes \hat{\mathcal{M}}_t),$$

where $\hat{\mathcal{M}}_t$ is the line bundle corresponding to $t \in W$. Choose $m \gg 0$ and t_0 such that $f^*(A^{-m}) = \hat{\mathcal{M}}_{t_0}$. This is possible since $b_2(X) = 0$. By (+) we have

$$H^0(C, R^2 f_*(B \otimes \hat{\mathcal{M}}_{t_0})) = 0.$$

Hence it is sufficient to show that $R^j F_*(\hat{B} \otimes \hat{\mathcal{M}})$ is flat with respect to the projection $q: C \times W \to W$, over a Zariski open set of W, then the usual semi-continuity theorem gives the claim (2). Now $R^2 f_*(B \otimes \hat{\mathcal{M}}_t)$ is locally free on $C = C \times t$ for every t by (1.5), hence it is clear that there is a Zariski open set $U \subset W$ such that $R^2 F_*(\hat{B} \otimes \hat{\mathcal{M}})$ has constant rank over U, hence is locally free over U (observe just that the set where the rank of a coherent sheaf is not minimal is analytic). This proves $(++)$.

On the other hand we have for $m \gg 0$ by Serre's vanishing theorem

$$H^1(C, R^1 f_*(B) \otimes A^m) = 0.$$

In the same way as above we conclude that

$$H^1(C, R^1 f_*(B \otimes \mathcal{M})) = 0$$

for general $\mathcal{M} \in W$.

By the Leray spectral sequence we therefore again obtain $H^2(X, B \otimes \mathcal{M}) = 0$ for general $\mathcal{M}$. This finishes the proof of the theorem.

COROLLARY 2.2. *Let X be a compact complex threefold homeomorphic to the sphere S^6. Then every meromorphic function on X is constant.*

Proof. Note that $c_3(X) = \chi_{\text{top}}(S^6) = 2$ and apply (2.1).

Next we give examples of threefolds with $a(X) > 0$, $b_2(X) = 0$ and $c_3(X) = 0$ so that the Main Theorem (2.1) is sharp.

EXAMPLE 2.3. The so-called Calabi-Eckmann threefolds are compact threefolds homeomorphic to $S^3 \times S^3$, see [Ue75]. They can be realized as elliptic fiber bundles over $\mathbb{P}_1 \times \mathbb{P}_1$. Hence $a(X) = 2$, $b_1(X) = b_2(X) = 0$ and $b_3(X) = 2$.

We now show that Calabi-Eckmann manifolds can be deformed to achieve $a(X) = 1$ or $a(X) = 0$ and $b_1 = b_2 = 0$, $b_3 = 2$. We choose positive real numbers a, b, c and let $B = \mathbb{C}^2 \setminus \{(0,0)\}$. We define the following action of $\mathbb{C}$ on $B \times B$:

$$(t, x, y, u, v) \mapsto (\exp(t)x, \exp(at)y, \exp(ibt)u, \exp(ict)v).$$

One checks easily that this action is holomorphic, free and almost proper so that the quotient X exists and is a compact manifold. If $a = 1$ and $b = c$, then X is a Calabi-Eckmann manifold. If however $a \notin \mathbb{Q}$ and $b = c$ resp. $a \notin \mathbb{Q}$ and $\frac{b}{c} \notin \mathbb{Q}$, then $a(X) = 1$ resp. $a(X) = 0$.

EXAMPLE 2.4. Hopf threefolds of the form

$$\mathbb{C}^3 \setminus \{(0,0,0)\}/\mathbb{Z},$$

with the action of $\mathbb{Z} \simeq \{\lambda^k;\ k \in \mathbb{Z}\}$ being defined by

$$\lambda(x, y, z) = (\alpha x, \beta y, \gamma z), \quad 0 < |\alpha|, |\beta|, |\gamma| < 1$$

are homeomorphic to $S^1 \times S^5$. They have $a(X) = 0, 1$ or 2 and $b_1(X) = 1$, while $b_2(X) = b_3(X) = 0$. This realizes the other possibility for the pair (b_1, b_3) when $b_2 = 0$ and $a(X) > 0$, as stated in the Main Theorem.

Notice that the algebraic reduction is holomorphic in (2.3) but it is not holomorphic in (2.4) if $a(X) = 1$.

EXAMPLE 2.5. We finally give other examples of compact threefolds X with $a(X) = 0$ and $b_1 = b_2 = 0$, $b_3 = 2$. Let $\Gamma \subset \text{Sl}(2, \mathbb{C})$ be a torsion free cocompact lattice in such a way that the quotient $X := \text{Sl}(2, \mathbb{C})/\Gamma$ has $b_1(X) = 0$.

This last condition is not automatic. Let $Y = \text{SU}(2) \backslash \text{Sl}(2, \mathbb{C})/\Gamma$; then Y is a compact differentiable manifold admitting a differentiable fibration $\pi\colon X \to Y$ with S^3 is fiber. Since $b_1(X) = 0$, we also have $b_1(Y) = 0$, hence $b_2(Y) = 0$ by Poincaré duality. Now the Leray spectral sequence immediately gives $b_2(X) = 0$, the fibers of π being 3-spheres. $b_3(X) = 2$ is again clear from the Leray spectral sequence. Finally the fact that X does not carry any nonconstant meromorphic function results from [HM83] from which we even deduce that X does not carry any hypersurface (as X is homogeneous).

3. On the finer structure of threefolds with $b_2(X) = 0$ and $a(X) = 1$

In this section we investigate more closely threefolds X with $b_2(X) = 0$ and algebraic dimension 1. By construction, the algebraic reduction $f\colon X \dashrightarrow V$ is a meromorphic map to a normal projective variety, hence V is a nonsingular curve. We claim that V must be rational. In fact, we have $b_1(X) \leqslant 1$ by 2.1 (3); on the other hand, for every holomorphic 1-form u on V, the pull-back f^*u is a d-closed holomorphic 1-form on X, thus $b_1(X) \geqslant 2$. Therefore V is rational. In this section we restrict ourselves to the case when the algebraic reduction $f\colon X \to V$ is holomorphic. The key to our investigations is

THEOREM 3.1. *Let F be a general smooth fiber of f. Then the restriction $r\colon H^1(X, \mathcal{O}_X) \to H^1(F, \mathcal{O}_F)$ is surjective.*

We need some preparations for the proof of (3.1). Let $\Delta \subset V$ be a finite *non empty* set such that $A = f^{-1}(\Delta) \subset X$ contains all singular fibers of f. Let $V' = V \setminus \Delta$, and $X' = f^{-1}(V')$ so that $f' = f_{|X'}$ is a smooth fibration. Let D_i, $1 \leqslant i \leqslant r$ be the irreducible components of A and let $s = \operatorname{card} \Delta$ be the number of connected components of A. Furthermore we set $t = b_1(F)$ where F is the general smooth fiber of f. For a noncompact space Z we let $b_i(Z) = \dim H_i(Z, \mathbb{R})$, whatever this dimension is. We prepare the proof of (3.1) by three lemmas.

LEMMA 3.2. (1) *The natural exact sequence of groups*

$$1 = \pi_2(V') \to \pi_1(F) \to \pi_1(X') \to \pi_1(V') \to 1$$

is exact and (*non-canonically*) *split*.

(2) $b_1(X') = b_1(V') + b_1(F) = s - 1 + t$.

(3) $r = s - 1 + t - b_1(X)$.

Proof. (1) Since V' is a non-compact Riemann surface, $\pi_1(V')$ is a free group of $s-1$ generators and since V' is uniformized by either $\mathbb{C}$ or by the unit disc, we have $\pi_2(V') = 1$. Hence the exact homotopy sequence of the fibration $f'\colon X' \to V'$ gives the exact sequence of groups stated in (1). Since $\pi_1(V')$ is a free group, the sequence splits.

(2) From (1) we deduce that

$$H_1(X', \mathbb{Z}) \simeq H_1(V', \mathbb{Z}) \oplus H_1(F, \mathbb{Z}).$$

Moreover $f_*\colon \pi_1(X) \to \pi_1(V)$ is surjective since the fibers of f are connected. Hence (2) follows.

(3) The cohomology sequence with rational coefficients of the pair (X, A) gives

$$0 = H^4(X) \to H^4(A) \to H^5(X,A) \to H^5(X) \to H^5(A) = 0.$$

By duality we have $H^5(X,A) \simeq H_1(X')$ and $H^5(X) \simeq H_1(X)$. Hence $r = \dim H^4(A) = b_1(X') - b_1(X) = s - 1 + t - b_1(X)$ by (2), as claimed.

Now choose an integer $m > 0$ such that $c_1(\mathcal{O}_X(mD_i)) = 0$ in $H^2(X,\mathbb{Z})$ for all $1 \leqslant i \leqslant r$. Let $\bigoplus \mathbb{Z}[mD_i] \simeq \mathbb{Z}^r$ be the free abelian group generated by mD_i, $1 \leqslant i \leqslant r$, and let $\phi\colon \bigoplus \mathbb{Z}[mD_i] \to \mathrm{Pic}^0(X)$ be given by sending $D = \sum_i a_i m D_i$ to $\mathcal{O}_X(D)$.

LEMMA 3.3. *Let* $K = \mathrm{Ker}\,\phi$ *and* $I = \mathrm{Im}\,\phi$. *Then* $\mathrm{rk}\,K \leqslant s-1$ *and* $\mathrm{rk}\,I \geqslant r-s+1$.

Proof. The kernel K consists of all divisors D such that $\mathcal{O}_X(D) \simeq \mathcal{O}_X$, i.e. such that D is the divisor of a global meromorphic function h on X. As $f\colon X \to V$ is the algebraic reduction of X, there must exist a meromorphic function $\tilde{h}$ on V such that $h = \tilde{h} \circ f$. Now, the divisor $\tilde{D}$ of $\tilde{h}$ has degree 0 and support in $\Delta = \{x_1, \dots, x_s\}$. This implies that K is contained in $f^*(\mathrm{Pic}^0(\Delta))$. As $\mathrm{Pic}^0(\Delta) \simeq \mathbb{Z}^{s-1}$, the claim follows.

The last ingredient in the proof of (3.1) is provided by

LEMMA 3.4. *The restriction map* $\alpha\colon H^1(X,\mathcal{O}_X) \to H^1(X',\mathcal{O}_{X'})$ *is injective.*

Proof. The Leray spectral sequences of the fibrations $f\colon X \to V$ and $f'\colon X' \to V'$ yield a commutative diagram

$$\begin{array}{ccccccc}
0 = H^1(V,\mathcal{O}_V) & \xrightarrow{f^*} & H^1(X,\mathcal{O}_X) & \xrightarrow{\simeq} & H^0(V,R^1 f_*\mathcal{O}_X) & \longrightarrow & H^2(V,\mathcal{O}_V) = 0 \\
\downarrow & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \tilde{\alpha}} & & \downarrow \\
0 = H^1(V',\mathcal{O}_{V'}) & \longrightarrow & H^1(X',\mathcal{O}_{X'}) & \xrightarrow{\simeq} & H^0(V',R^1 f'_*\mathcal{O}_{X'}) & \longrightarrow & H^2(V',\mathcal{O}_{V'}) = 0,
\end{array}$$

and since $R^1 f_*\mathcal{O}_X$ is locally free by (1.5), we conclude that $\tilde{\alpha}$ is injective.

We are now able to finish the proof of (3.1).

Consider again $I \subset \mathrm{Pic}^0(X)$, the image of ϕ: $\bigoplus \mathbb{Z}[mD_i] \to \mathrm{Pic}^0(X)$, and let $\tilde{I}$ be the inverse image of I under the natural map $H^1(X,\mathcal{O}_X) \to \mathrm{Pic}^0(X)$. We have a commutative diagram

$$\begin{array}{ccccccc}
H^1(X,\mathbb{Z}) & \longrightarrow & H^1(X',\mathbb{Z}) & \xrightarrow{\beta^{\mathbb{Z}}} & H^0(V',R^1 f'_*\mathbb{Z}) & \xrightarrow{\gamma^{\mathbb{Z}}} & H^1(F,\mathbb{Z}) \\
\downarrow & & \downarrow{\scriptstyle \theta} & & \downarrow & & \downarrow \\
\tilde{I} \subset H^1(X,\mathcal{O}_X) & \xrightarrow{\alpha} & H^1(X',\mathcal{O}_{X'}) & \xrightarrow[\simeq]{\beta} & H^0(V',R^1 f'_*\mathcal{O}_{X'}) & \xrightarrow{\gamma} & H^1(F,\mathcal{O}_F) \\
\downarrow\quad\ \ \downarrow & & \downarrow & & & & \\
I \subset H^1(X,\mathcal{O}^*_X) & \xrightarrow{\hat{\alpha}} & H^1(X',\mathcal{O}^*_{X'}), & & & &
\end{array}$$

where the two vertical sequences are exponential exact sequences, α and $\hat{\alpha}$ are restriction maps from X to X', β and $\beta^{\mathbb{Z}}$ arise from the spectral sequence, and γ, $\gamma^{\mathbb{Z}}$ are restriction maps to a generic fiber F. We get

$$\operatorname{rk} \tilde{I} = \operatorname{rk} I + \operatorname{rk} H^1(X, \mathbb{Z}) \geqslant r - s + 1 + b_1(X) = t$$

by 3.3 and 3.2 (3). Now $\hat{\alpha}(I) = 0$, since $\hat{\alpha}(\mathcal{O}_X(mD_i)) = \mathcal{O}_{X'}(mD_i)) \simeq \mathcal{O}_{X'}$ for all i. It follows that $\alpha(\tilde{I})$ is contained in the image of θ, thus $\theta^{-1}(\alpha(\tilde{I})) \subset H^1(X', \mathbb{Z})$ has rank

$$\operatorname{rk}(\theta^{-1}(\alpha(\tilde{I}))) \geqslant \operatorname{rk} \alpha(\tilde{I}) \geqslant \operatorname{rk} \tilde{I} \geqslant t$$

thanks to the injectivity of α. Moreover, $R^1 f'_* \mathbb{Z}$ is a locally constant system of rank $\operatorname{rk} H^1(F, \mathbb{Z}) = t$, hence $H^0(V', R^1 f'_* \mathbb{Z})$ has rank at most t. On the other hand, as β is an isomorphism and $\operatorname{rk} \alpha(\tilde{I}) \geqslant t$, we see that $\beta^{\mathbb{Z}}(\theta^{-1}(\alpha(\tilde{I})))$ has rank at least t. Therefore, $\gamma^{\mathbb{Z}} \circ \beta^{\mathbb{Z}}(\theta^{-1}(\alpha(\tilde{I})))$ is of finite index in $H^1(F, \mathbb{Z})$. Since $H^1(F, \mathcal{O}_F)$ is the complex linear span of the image of $H^1(F, \mathbb{Z})$, we see that $\gamma \circ \beta \circ \alpha(\tilde{I})$ also generates $H^1(F, \mathcal{O}_F)$. In particular, the restriction map $H^1(X, \mathcal{O}_X) \to H^1(F, \mathcal{O}_F)$ must be surjective. This concludes the proof of (3.1).

(3.5) We now study the structure of the smooth fibers F of f. The exact sequence

$$0 \to T_F \to (T_X)_{|F} \to \mathcal{O}(F)_{|F} \to 0$$

and the equality $c_1(\mathcal{O}(F)) = 0$ in $H^2(X, \mathbb{R})$ imply

$$c_1(F) = c_1(X)_{|F} = 0, \qquad c_2(F) = c_2(X)_{|F} = 0$$

in particular we also have $\chi(F, \mathcal{O}_F) = 0$. We conclude from the classification of surfaces that F is one of the following: a Hopf surface, an Inoue surface, a Kodaira surface (primary or secondary), a torus or a hyperelliptic surface (see e.g. [BPV84]). By [Ka69] however, F cannot be hyperelliptic. The reason is the existence of a relative Albanese reduction in that case.

PROPOSITION 3.6. *The general fiber F of f cannot be a Kodaira surface nor a Hopf surface with algebraic dimension* 1.

Proof. Let F_0 be a fixed smooth fiber and assume that F_0 is a Kodaira surface or a Hopf surface with $a(F_0) = 1$. Let $g_0 \colon F_0 \to C_0$ be 'the' algebraic reduction which is an elliptic fiber bundle. Let $\mathcal{L}_0 = g_0^*(\mathcal{G}_0)$ with $\mathcal{G}_0$ very ample on C_0.

(1) There exists a line bundle $\tilde{\mathcal{L}}$ on X with $\tilde{\mathcal{L}}_{|F_0} = \mathcal{L}_0$.

Proof. By passing to some power $\mathcal{L}_0^m$ if necessary, we have $c_1(\mathcal{L}_0) = 0$ in $H^2(X, \mathbb{Z})$. Let

$$\lambda_1 \colon H^1(X, \mathcal{O}_X) \to \operatorname{Pic}(X)$$

and

$$\lambda_2\colon H^1(F_0, \mathcal{O}_{F_0}) \to \mathrm{Pic}(F_0)$$

be the canonical maps, and let $r\colon H^1(X, \mathcal{O}_X) \to H^1(F_0, \mathcal{O}_{F_0})$ be the restriction map. Choose $\alpha \in H^1(F_0, \mathcal{O}_{F_0})$ with $\lambda_2(\alpha) = \mathcal{L}_0$. Since r is surjective by (3.2), we find $\beta \in H^1(X, \mathcal{O}_X)$ with $r(\beta) = \alpha$. Now let $\tilde{\mathcal{L}} = \lambda_1(\beta)$.

(2) Let F be any smooth fiber of f. Then $\kappa(\tilde{\mathcal{L}}_{|F}) = 1$. In fact, it follows from the local freeness of $R^j f_*(\tilde{\mathcal{L}})$ stated in Lemma 1.5 that

$$f_*(\tilde{\mathcal{L}}^\mu)_{|\{y\}} \simeq H^0(F, \tilde{\mathcal{L}}^\mu),$$

where $F = f^{-1}(y)$, see [BaSt76, Chap. 3, 3.10].

(3) From the generically surjective morphism

$$f^* f_*(\tilde{\mathcal{L}}^m) \to \tilde{\mathcal{L}}^m,$$

$(m \gg 0)$, we obtain a meromorphic map

$$g\colon X \dashrightarrow \mathbb{P}(f_*(\tilde{\mathcal{L}}^m)),$$

which, restricted to F is holomorphic and just gives the algebraic reduction of F. Let Z be the closure of the image of g. Then f factors via the meromorphic map $h_1\colon X \dashrightarrow Z$ and the holomorphic map $h_2\colon Z \to V$. Now h_2 is the restriction of the canonical projection $\mathbb{P}(f_*(\tilde{\mathcal{L}}^m)) \to V$, therefore h_2 is a projective morphism and Z is projective. Hence $a(X) \geqslant 2$, contradiction.

From (3.6) it follows that F can only be an Inoue surface, a Hopf surface without meromorphic functions or a torus. In order to exclude by a similar method as in (3.6) also tori of algebraic dimension 1, we would need the existence of a relative algebraic reduction (the analogue of $h_2\colon X \dashrightarrow Z$) in that case, too.

We now look more closely to the structure of f.

PROPOSITION 3.7. *Assume that F is not a torus. Then*

(1) $R^1 f_*(\mathcal{O}_X) = \mathcal{O}_V$,
(2) $R^2 f_*(\mathcal{O}_X) = 0$,
(3) $\dim H^1(X, \mathcal{O}_X) = 1$,
(4) $H^2(X, \mathcal{O}_X) = H^3(X, \mathcal{O}_X) = 0$.

Proof. (2) Since $H^2(F, \mathcal{O}_F) = 0$, the sheaf $R^2 f_*(\mathcal{O}_X)$ is torsion, hence identically zero by 1.5. Then $H^3(X, \mathcal{O}_X) = 0$ is immediate from the Leray spectral sequence.

(1) Since $h^1(F, \mathcal{O}_F) = 1$, $R^1 f_*(\mathcal{O}_X)$ is a line bundle on V. Let

$$d = \deg R^1 f_*(\mathcal{O}_X).$$

Then Riemann-Roch gives $\chi(R^1 f_*(\mathcal{O}_X)) = d + 1$. On the other hand the Leray spectral sequence together with $H^3(X, \mathcal{O}_X) = 0$ and (2) yields

$$\chi(R^1 f_*(\mathcal{O}_X)) = h^1(\mathcal{O}_X) - h^2(\mathcal{O}_X) = -\chi(\mathcal{O}_X) + 1.$$

We conclude $d = -\chi(\mathcal{O}_X) = 0$. This proves (1). Now (3) and the second part of (4) are obvious.

Remark 3.8. In case F is a torus, $R^1 f_*(\mathcal{O}_X)$ is a rank 2 bundle and $R^2 f_*(\mathcal{O}_X)$ is a line bundle. Using Theorem 3.1 it is easy to see that

(1) $R^1 f_*(\mathcal{O}_X) = \mathcal{O}(a) \oplus \mathcal{O}(b)$ with $a, b \geqslant 0$,
(2) $R^2 f_*(\mathcal{O}_X) = \mathcal{O}(a + b)$.

Note that (2) gives dually $f_*(\omega_{X|V}) = \mathcal{O}(-a - b)$. Usually one expects the degree of $f_*(\omega_{X|V})$ to be semi-positive, but here we are in a highly non-Kähler situation where it might happen that the above degree is negative, see [Ue87].

PROPOSITION 3.9. *Assume that F is not a torus. Then* $H^0(X, \Omega^i_X) = 0$, $1 \leqslant i \leqslant 3$.

Proof. For $i = 3$ the claim follows already from 3.7 (4) and Serre duality.

(1) First we treat the case $i = 1$. Let ω be a holomorphic 1-form. Let $j\colon F \to X$ be the inclusion. Then $j^*(\omega) = 0$, hence at least locally near F we have $\omega = f^*(\eta)$, hence $\mathrm{d}\omega = 0$ near F and therefore the holomorphic 2-form $\mathrm{d}\omega$ is identically zero on X. Now the space of closed holomorphic 1-forms can be identified with $H^0(X, \mathrm{d}\mathcal{O}_X)$ and, as it is well known (see e.g. [Ue75]), we have the inequality

$$2h^0(X, \mathrm{d}\mathcal{O}_X) \leqslant b_1(X).$$

The inequality $b_1(X) \leqslant 1$ implies $h^0(X, \Omega^1_X) = h^0(X, \mathrm{d}\mathcal{O}_X) = 0$, as desired.

(2) In case $i = 2$, we again have $j^*(\omega) = 0$. Let U be a small open set in V such that $f_{|f^{-1}(U)}$ is smooth. Let z be a coordinate on U and $h = f^*(z)$. Then we conclude that

$$\omega_{|f^{-1}(U)} = \mathrm{d}h \wedge \alpha$$

with some relative holomorphic 1-form $\alpha \in H^0(f^{-1}(U), \Omega^1_{X/V})$. Now again $j^*(\alpha) = 0$ and therefore $\alpha = 0$, $\omega = 0$.

COROLLARY 3.10. *Assume that F is not a torus. Then either* $f_*(\Omega^1_{X/V}) = 0$ *or there exists some $x \in V$ such that* $f_*(\Omega^1_{X/V}) = \mathbb{C}_x$, *i.e. a sheaf supported on x with a 1-dimensional stalk at x. In particular f has at most one singular fiber and such a fiber is normal with exactly one singularity of embedding dimension* 3.

Proof. Consider the exact sequence

$$0 \to f^*(\Omega^1_V) \to \Omega^1_X \to \Omega^1_{X/V} \to 0.$$

Since F has no holomorphic 1-forms, $f_*(\Omega^1_{X/V})$ is a torsion sheaf on the curve V. The corollary will follow if we check that $h^0(V, f_*(\Omega^1_{X/V})) = h^0(X, \Omega^1_{X/V}) \leqslant 1$. Now, observe the following facts.

(1) $H^0(X, \Omega^1_X) = 0$, by (3.9);
(2) $H^1(X, f^*(\Omega^1_V)) = H^1(V, \Omega^1_V)$ by Leray's spectral sequence and the equalities $R^i f_*(f^*\Omega^1_V) = R^i f_*(\mathcal{O}_X) \otimes \Omega^1_V = \Omega^1_V$, $i = 0, 1$ (cf. 3.7 (1));
(3) $\dim H^1(V, \Omega^1_V) = 1$.

Then, taking cohomology groups in the first exact sequence, we get the desired inequality

$$h^0(X, \Omega^1_{X/V}) \leqslant h^1(V, \Omega^1_V) = 1.$$

We can say something more about the structure of the singular fibers of f.

PROPOSITION 3.11. *Assume that F is not a torus. Let A be a union of fibers containing all singular fibers of f. Let $s = \mathrm{card}(f(A))$ and r the number of irreducible components of A. Then $r = s$, i.e. all fibers of f are irreducible and $b_1(X) = 0$.*

Proof. Since F is an Inoue surface or a Hopf surface, we have $b_1(F) = 1$, thus 3.2 (3) implies $r = s - b_1(X)$. As $r \geqslant s$, we must have $r = s$ and $b_1(X) = 0$.

Remark 3.12. In case F is a torus, 3.2 (3) implies $r = s + 3 - b_1(X) \geqslant s + 2$. It seems rather reasonable to expect that tori actually cannot appear as fibers of f. Observe that f must have a singular fiber in this case because of $r > s$. So a study of the singular fibers is needed to exclude tori as fibers of f. However there is a significant difference: the case of tori is one (in fact the only one) where $C_{3,1}$ might fail, see [Ue87].

PROPOSITION 3.13. *Assume that F is not a torus. Then $h^{1,1} = h^{1,2} = h^{2,1} = 1$ (so that we know all Hodge numbers of X).*

Proof. (1) $h^{1,2} = h^{2,1}$ is of course Serre duality.

(2) By (3.9) and $\chi(X, \Omega^1_X) = 0$ it suffices to see $h^{1,3} = 0$ in order to get $h^{1,1} = h^{1,2}$. But this follows again from Serre duality and the equality $h^{2,0} = h^0(X, \Omega^2_X) = 0$.

(3) From the exact sequence

$$0 \to f^*(\Omega^1_V) \to \Omega^1_X \to \Omega^1_{X/V} \to 0 \qquad (S)$$

we deduce that it suffices to show

(a) $h^2(X, f^*(\Omega^1_V)) = 1$,
(b) $h^2(X, \Omega^1_{X/V}) = 0$,

in order to get $h^{1,2} \leqslant 1$.

(a) By the Leray spectral sequence and (3.7) we have $h^2(X, f^*(\Omega^1_V)) = h^1(V, \Omega^1_V) = 1$.

(b) Again we argue by the Leray spectral sequence. Since $R^1 f_*(\Omega^1_{X/V})$ is a torsion sheaf, we need only to show that

$$R^2 f_*(\Omega^1_{X/V}) = 0.$$

In fact, taking the direct image f_* of (S), we see that $R^2 f_*(\Omega^1_{X/V})$ is a quotient of $R^2 f_*(\Omega^1_X)$ which is 0 by the equality $H^2(F, \Omega^1_F) = H^0(F, \Omega^1_F) = 0$ and by (1.5).

(4) We finally show $h^{1,1} \neq 0$ to conclude the proof. Let $(E_r^{p,q})$ be the Frölicher spectral sequence on X. Since $b_1(X) = 0$ by (3.11), we get $E_\infty^{0,1} = 0$. Hence $E_2^{0,1} = 0$. On the other hand

$$E_2^{0,1} = \operatorname{Ker} \partial \colon E_1^{0,1} \to E_1^{1,1}.$$

Since $E_1^{p,q} = H^{p,q}(X)$, we conclude that $H^1(X, \mathcal{O}_X)$ injects into $H^{1,1}(X)$. So by (3.7) $H^{1,1}(X) \neq 0$.

We finally collect all our knowledge in the case the general fiber of f is not a torus.

THEOREM 3.14. *Let X be a smooth compact threefold with $b_2(X) = 0$ and holomorphic algebraic reduction $f\colon X \to V$ to the smooth curve V. Assume that the general smooth fiber is not a torus. Then*:

(1) $b_1(X) = 0$, $b_3(X) = 2$.
(2) *Any smooth fiber of f is a Hopf surface without meromorphic functions or an Inoue surface*.
(3) *The Hodge numbers of X are as follows*: $h^{1,0} = 0$, $h^{0,1} = 0$, $h^{2,0} = 0$, $h^{1,1} = 1$, $h^{0,2} = 0$, $h^{3,0} = 0$, $h^{2,1} = 1$ (*the others are determined by these via Serre duality*).
(4) *All fibers of f are irreducible. There is at most one normal singular fiber*.

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