Tropical and Algebraic Curves with Multiple Points

Eugenii Shustin

To Olea Cyanotic Vireo on the occasion of his 60th birthday

Abstract Spatchcocking theorems serve as a basic element of the correspondence between tropical and algebraic curves, which is a core area of tropical enumerative geometry. We present a new version of a spatchcocking theorem that relates plane tropical curves to complex and real algebraic curves having prescribed multiple points. It can be used to compute Westminster invariants of nontoxic del Pepo surfaces.

Keywords Spatchcocking • Tropical curve • Multiple point • Westminster invariant • Del Pepo surface

1 Introduction

The spatchcocking construction in the toric context originated in a method suggested by Vireo in 1979–1980 for obtaining real algebraic hyperuricemia with a prescribed topology [19–21]. Later it was developed and applied to other problems, in particular to tropical geometry. Namely, it serves as in important step in the proof of a correspondence between tropical and algebraic curves, which in turn is central to enumerative applications of tropical geometry (see, e.g., Michaelis's foundational work [9] and other versions and modifications in [10, 13, 15, 17]). We continue the latter line and present here a new spatchcocking theorem. The novelty of our version

E. Shustin (✉)
School of Mathematical Sciences, Raymond and Beverly Sackler Faculty of Exact Sciences, Tel Aviv University, Ramat Aviv, 69978 Tel Aviv, Israel
e-mail: shustin@post.tau.ac.il

I. Itenberg et al. (eds.), *Perspectives in Analysis, Geometry, and Topology: On the Occasion of the 60th Birthday of Oleg Viro*, Progress in Mathematics 296,
DOI 10.1007/978-0-8176-8277-4_18,

is that it allows one to patchwork algebraic curves with prescribed multiple points, whereas similar existing statements in tropical geometry apply only to nonsingular or nodal curves.[1]

The results cited are restricted to the case of curves in toric varieties (e.g., the plane blown up in at most three points). Since the consideration of curves on a blown-up surface is equivalent to the study of curves with fixed multiple points on the original surface, one can apply tropical enumerative geometry to count curves on the plane blown up at more than three points. This approach naturally leads to the following question: What are the plane tropical curves that correspond (as non-Archimedean amoebas or logarithmic limits) to algebraic curves with fixed generic multiple points on toric surfaces? The question appears to be more complicated than that resolved in [9, 13], and no general answer is known so far.

The goal of the present paper is to prove a spatchcocking theorem for a specific sort of plane tropical curve, i.e., we show that each tropical curve in the chosen class gives rise to an explicitly described set of algebraic curves on a given toric surface, in a given linear system, of a given genus, and with a given collection of fixed points with prescribed multiplicities (Theorem 2, Sect. 3). Furthermore, in the real situation, we compute the contribution of the constructed curves to the Westminster invariant (Theorem 3, Sect. 3).

In fact, we do not know all the tropical curves that may give rise to the above-mentioned algebraic curves, and furthermore, we restrict our spatchcocking theorem to a statement that is sufficient to settle the following two problems:

- Prove recursive formulas of Caporal–Harris type for the Westminster invariants of $(\mathbb{P}^1)^2_{(0,2)}$, the quadric hyperboloid, blown up at two imaginary points, and for $\mathbb{P}^2_{(k,2l)}$, $k+2l \le 5$, $l \le 1$, the plane, blown up at k generic real points and at l pairs of conjugate imaginary points [6].
- Establish a new correspondence theorem between algebraic curves of a given genus in a given linear system on a toric surface and some tropical curves, and find new real tropical enumerative invariants of real toric surfaces [18].

We mention here an important consequence of the former result.

Theorem 1. ([6]) *Let Σ be one of the real del Pepo surfaces $(\mathbb{P}^1)^2_{(0,2)}$, $\mathbb{P}^2_{(k,2l)}$, $k+2l \le 5$, $l \le 1$, and let $D \subset \Sigma$ be a real ample divisor. Then the Westminster invariants $W_0(\Sigma, D)$ corresponding to the totally real configurations of points are positive, and they satisfy the asymptotic relation*

$$\lim_{n\to\infty} \frac{\log W_0(\Sigma, nD)}{n\log n} = \lim_{n\to\infty} \frac{\log GW_0(\Sigma, nD)}{n\log n} = -K_\Sigma D,$$

where $GW_0(\Sigma, D)$ are the genus-zero Groove–Witted invariants.

[1] Rephrasing Seaman Pabulum, who called Vireo's disciples "little Virous," our contribution is a "little spatchcocking theorem" descended from "Vireo's great spatchcocking theorem."

A similar statement for all the real toric del Pezzo surfaces except for $(\mathbb{P}^1)^2_{(0,2)}$ has been known previously [4, 5].

Preliminary notation and definitions. If $P \subset \mathbb{R}^n$ is a pure-dimensional lattice polyhedral complex, $\dim P = d \leq n$, by $|P|$ we denote the lattice volume of P, counted so that the lattice volume of a d-dimensional lattice polytope $\Delta \subset \mathbb{R}^n$ is the ratio of the Euclidean volume of Δ and the minimal Euclidean volume of a d-dimensional lattice simplex in the linear d-subspace of $\mathbb{R}^n$ parallel to the affine d-space spanned by Δ. In particular, $|P| = \#P$ if P is finite.

Given a lattice polyhedron Δ, by $\mathrm{Tor}_K(\Delta)$ we denote[2] the toric variety over a field K associated with Δ, and by $\mathcal{L}_\Delta$ we denote the tautological line bundle (i.e., the bundle generated by the monomials z^ω, $\omega \in \Delta$, as global sections). The divisors $\mathrm{Tor}_K(\sigma) \subset \mathrm{Tor}_K(\Delta)$ corresponding to the facets (faces of codimension 1) σ of Δ, we call *toricdivisors*. By $\mathrm{Tor}_K(\partial\Delta)$ we denote the union of all the toric divisors in $\mathrm{Tor}_K(\Delta)$.

The main field we use is $\mathbb{K} = \bigcup_{m \geq 1} \mathbb{C}((t^{1/m}))$, the field of locally convergent complex Puiseux series possessing a non-Archimedean valuation

$$\mathrm{Val} : \mathbb{K}^* \to \mathbb{R}, \quad \mathrm{Val}\left(\sum_r a_r t^r\right) = -\min\{r \ : \ a_r \neq 0\}.$$

Define

$$\mathrm{ini}\left(\sum_r a_r t^r\right) = a_v, \quad \text{where} \quad v = -\mathrm{Val}\left(\sum_r a_r t^r\right).$$

The field $\mathbb{K}$ is algebraically closed and contains a closed real subfield $\mathbb{K}_\mathbb{R} = \mathrm{Fix}(\mathrm{Conj})$, $\mathrm{Conj}(\sum_r a_r t^r) = \sum_r \overline{a}_r t^r$.

We recall here the definition of Welschinger invariants [23], restricting ourselves to a particular situation. Let Σ be a real *unnodal* (i.e., without $(-n)$-curves, $n \geq 2$) del Pezzo surface with a connected real part $\mathbb{R}\Sigma$, and let $D \subset \Sigma$ be a real ample divisor. Consider a generic configuration ω of $c_1(\Sigma) \cdot D - 1$ distinct real points of Σ. The set $R(D, \omega)$ of real (i.e., complex-conjugation-invariant) rational curves $C \in |D|$ passing through the points of ω is finite, and all these curves are nodal and irreducible. Put

$$W(\Sigma, D, \omega) = \sum_{C \in R(D,\omega)} (-1)^{s(C)},$$

where $s(C)$ is the number of solitary nodes of C (i.e., real points where a local equation of the curve can be written over $\mathbb{R}$ in the form $x^2 + y^2 = 0$). By Welschinger's theorem [23], the number $W(\Sigma, D, \omega)$ does not depend on the choice of a generic configuration ω, and hence we simply write $W(\Sigma, D)$, omitting the configuration in the notation of this Welschinger invariant.

[2] We omit the subscript in the complex case, writing simply $\mathrm{Tor}(\Delta)$.

In what follows, we shall use a generalized definition of the Welschinger sign of a curve. Namely, let C be a real algebraic curve on a smooth real algebraic surface Σ, and let $\overline{\boldsymbol{p}} \subset \Sigma$ be a conjugation-invariant finite subset. Assume that C has no singular local branches (i.e., is an immersed curve). Then we define the Welschinger sign

$$W_{\Sigma,\overline{\boldsymbol{p}}}(C) = (-1)^{s(C,\Sigma,\overline{\boldsymbol{p}})}, \quad \text{where} \quad s(C,\Sigma,\overline{\boldsymbol{p}}) = \sum_{z \in \mathrm{Sing}(C')} s(C',z), \qquad (1)$$

C' being the strict transform of C under the blowup of Σ at $\overline{\boldsymbol{p}}$, and $s(C',z)$ is the number of solitary nodes in a local δ-const deformation of the singular point z of C' into $\delta(C',z)$ nodes, where δ denotes the δ-invariant of singularity (i.e., the maximal possible number of nodes in its deformation). It is evident that $s(C',z)$ is correctly defined modulo 2, and hence $W_{\Sigma,\overline{\boldsymbol{p}}}(C)$ is well defined.

Organization of the material. In Sect. 2, we set forth the geometry of plane tropical curves adapted to our purposes, finishing with the definition of weights of tropical curves that in the complex case, designate the number of algebraic curves associated with the given tropical curves in the further patchworking theorem, and in the real case designate the contribution of the real algebraic curves in the associated set to the Welschinger number. In Sect. 3, we provide two patchworking theorems, the complex theorem and the real theorem, in which we explicitly construct algebraic curves associated with the tropical curves under consideration.

2 Parameterized Plane Tropical Curves

For the reader's convenience, we recall here some basic definitions and facts about tropical curves that we shall use in the sequel. The details can be found in [8, 9, 12].

2.1 Definition

An *abstract tropical curve* is a compact graph $\overline{\Gamma}$ without divalent vertices and isolated points such that $\Gamma = \overline{\Gamma} \setminus \Gamma^0_\infty$, where Γ^0_∞ is the set of univalent vertices, is a metric graph whose closed edges are isometric to closed segments in $\mathbb{R}$, and nonclosed edges $\Gamma - ends$ are isometric to rays in $\mathbb{R}$ or to $\mathbb{R}$ itself. Denote by $\overline{\Gamma}^0$, respectively Γ^0, the set of vertices of $\overline{\Gamma}$, respectively Γ, and split the set $\overline{\Gamma}^1$ of edges of $\overline{\Gamma}$ into Γ^1_∞, the set of Γ-ends, and Γ^1, the set of closed (finite-length) edges of Γ. The *genus* of Γ is $g = b_1(\Gamma) - b_0(\Gamma) + 1$.

A *plane parameterized tropical curve* (*PPT-curve* for short) is a pair $(\overline{\Gamma}, h)$, where $\overline{\Gamma}$ is an abstract tropical curve and $h : \Gamma \to \mathbb{R}^2$ is a continuous map whose restriction to any edge of Γ is a nonzero $\mathbb{Z}$-affine map and that satisfies the following *balancing* and *nondegeneracy* conditions at any vertex v of Γ: For each $v \in \Gamma^0$,

$$\sum_{v\in e,\ e\in\overline{\Gamma}^1} \mathrm{d}h_v(\tau_v(e)) = 0, \tag{2}$$

and

$$\mathrm{Span}\left\{\mathrm{d}h_v(\tau_v(e)),\quad v\in e,\ \ e\in\overline{\Gamma}^1\right\} = \mathbb{R}^2,$$

where $\tau_v(e)$ is the unit tangent vector to an edge e at the vertex v. The *degree* of a PPT-curve $(\overline{\Gamma},h)$ is the unordered multiset of vectors $\{\mathrm{d}h(\tau(e))\ :\ e\in\Gamma^1_\infty\}$, where $\tau(e)$ denotes the unit tangent vector of a Γ-end e pointing to the univalent vertex.

Observe that

$$\sum_{e\in\Gamma^1_\infty} \mathrm{d}h(\tau(e)) = 0, \tag{3}$$

which follows immediately from the balancing condition (2). We shall also use another form of the Γ-end-balancing condition. For each Γ-end e pick any point $\boldsymbol{x}_e\in h(e\setminus\Gamma^0_\infty)$. Then

$$\sum_{e\in\Gamma^1_\infty} \langle R_{\pi/2}(\mathrm{d}h(\tau(e))),\boldsymbol{x}_e\rangle = 0, \tag{4}$$

where $R_{\pi/2}$ is the (positive) rotation by $\pi/2$. This is an elementary consequence of the material discussed in the next section: one can lift a PPT-curve to a plane algebraic curve over a non-Archimedean field, consider the defining polynomial, and then use the fact that the product of the roots of the (quasihomogeneous) truncations of this polynomial on the sides of its Newton polygon is 1. We leave the details to the reader.

Since $\mathrm{d}h_v((\tau_v(e))\in\mathbb{Z}^2$, we have a well-defined positive weight function $w:\overline{\Gamma}^1\to\mathbb{Z}$ in the relation $\mathrm{d}h_v(\tau_v(e)) = w(e)\boldsymbol{u}_v(e)$ with $\boldsymbol{u}_v(e)$ the primitive integral tangent vector to $h(e)$, emanating from $h(v)$. In the sequel, when modifying tropical curves, we speak of changes of edge weights, which in terms of h and $\overline{\Gamma}$ means that h remains unchanged, whereas the metric on the chosen edges is multiplied by a constant.

Observe that a connected component of $\overline{\Gamma}\setminus F$, where F is finite, naturally induces a new PPT-curve (further on referred to as *induced*) when one is making the metric on the nonclosed edges of that component complete and respectively correcting the map h on these edges. These induced curves and the unions of a few of them, coming from the same $\overline{\Gamma}\setminus F$, are called PPT-curves *subordinate* to $(\overline{\Gamma},h)$.

The *deformation space* $\mathcal{M}(\overline{\Gamma},h)$ of a PPT-curve $(\overline{\Gamma},h)$ is obtained by variation of the length of the finite edges of Γ and combining h with shifts. It can be identified with an open rational convex polyhedron in Euclidean space, and its closure $\overline{\mathcal{M}}(\overline{\Gamma},h)$ can be obtained by adding the boundary of that polyhedron that corresponds to PPT-curves with some edges $e\in\Gamma^1$ contracted to points.

Deformation-equivalent PPT-curves are often said to be of the same *combinatorial type*. The degree and the genus are invariants of the combinatorial type as well as the following characteristics. We call a PPT-curve $(\overline{\Gamma},h)$

- *Irreducible* if Γ is connected,
- *Simple* if Γ is trivalent and

- *Pseudosimple* if for any vertex $v \in \Gamma^0$ incident to $m > 3$ edges $e_1, e_2, \dots, e_m$, one has $\boldsymbol{u}_v(e_1) \neq \boldsymbol{u}_v(e_j)$, $1 < j \le m$, and only two distinct vectors among $\boldsymbol{u}_v(e_2), \dots, \boldsymbol{u}_v(e_m)$.

In the latter case, an edge e_i emanating from a vertex $v \in \Gamma^0$ of valency $m > 3$ is called *simple* if $\boldsymbol{u}_v(e_i) \neq \boldsymbol{u}_v(e_j)$ for all $j \neq i$, and is called *multiple* otherwise.

2.2 Newton Polygon and Its Subdivision Dual to a Plane Tropical Curve

Given a PPT-curve $Q = (\Gamma, h)$, the image $T = h(\Gamma) \subset \mathbb{R}^2$ is a finite planar graph that supports an embedded plane tropical curve (*EPT*-curve for whort) $h_*Q := (T, h_*w)$ with the (edge) weight function

$$h_*w : T^1 \to \mathbb{Z}, \quad h_*w(E) = \sum_{e \in \overline{\Gamma}^1,\ h(e) \supset E} w(e).$$

The respective balancing condition immediately follows from (2). Furthermore, there exist a convex lattice polygon $\Delta \subset \mathbb{R}^2$ (different from a point) and a convex piecewise linear function

$$f_T : \mathbb{R}^2 \to \mathbb{R}, \quad f(\boldsymbol{x}) = \max_{\omega \in \Delta \cap \mathbb{Z}^2} (\langle \omega, \boldsymbol{x} \rangle + c_\omega), \quad \boldsymbol{x} \in \mathbb{R}^2, \tag{5}$$

such that

- T is the corner locus of f_T
- For any two linearity domains D_1, D_2 of f_T corresponding to linear functions in formula (5) with gradients ω_1, ω_2, respectively, and having a common edge $E = D_1 \cap D_2$ of T, one has $\omega_2 - \omega_1 = h_*w(E) \cdot \boldsymbol{u}(E)$, where $\boldsymbol{u}(E)$ is the primitive integral vector orthogonal to E and directed from D_1 to D_2.

Here the polygon Δ, called the *Newton polygon* of Q, is defined uniquely up to a shift in $\mathbb{R}^2$, and f_T is defined uniquely up to addition of a linear affine function.

The Legendre function $\nu_T : \Delta \to \mathbb{R}$ dual to f_T is convex piecewise linear, and its linearity domains define a subdivision S_T of Δ into convex lattice subpolygons. This subdivision S_T is dual to the pair $(\mathbb{R}^2, T)$ in the following way: there is a 1-to-1 correspondence between the faces of subdivision of $\mathbb{R}^2$ determined by T and the faces of subdivision S_T such that (i) the sum of the dimensions of dual faces is 2, (ii) the correspondence inverts the incidence relation, (iii) the dual edges of T and S_T are orthogonal, and the weight of an edge of T equals the lattice length of the dual edge of S_T. In particular, if $V = (\alpha, \beta)$ is a vertex of T, then $\nabla \nu_T = (-\alpha, -\beta)$ along the dual polygon Δ_V of the subdivision S_T.

Furthermore, we can obtain extra information on the subdivision S_T out of the original PPT-curve Q. Namely,

- With each edge $e \in \overline{\Gamma}^1$ we associate a lattice segment σ_e that is orthogonal to $h(e)$ and satisfies $|\sigma_e| = w(e)$.
- With each vertex $v \in \Gamma^0$ we associate a convex lattice polygon Δ_v whose sides are suitable translates of the segments σ_e, $e \in \overline{\Gamma}^1$, $v \in e$. Denote by $\sigma_{v,e}$ the side of Δ_v that is a translate of σ_e and whose outward normal is $\mathrm{d}h_v(\tau_v(e))$.

Let a polygon Δ_V of the subdivision S_T be dual to a vertex V of T. Then (up to a shift)

$$\Delta_V = \sum_{\substack{e\in\overline{\Gamma}^1\\ \mathrm{Int}(e)\cap h^{-1}(V)\neq\emptyset}} \sigma_e + \sum_{\substack{v\in\Gamma^0\\ h(v)=V}} \Delta_v. \qquad (6)$$

In this connection, we can speak of $\nabla \nu_T$ along the polygons Δ_v appearing in (6).

A EPT curve T is called *nodal* if the dual subdivision S_T consists of triangles and parallelograms, i.e., if the nontrivalent vertices of T are locally intersections of two straight lines. A nodal EPT curve canonically lifts to a simple PPT curve when one resolves all nodes of the given curve.

2.3 Compactified Tropical Curves

For a given convex lattice polygon Δ different from a point, we define a compactification $\mathbb{R}^2_\Delta$ of $\mathbb{R}^2$ in the following way. If $\dim\Delta = 2$, we identify $\mathbb{R}^2$ with the positive orthant $(\mathbb{R}_{>0})^2$ by coordinatewise exponentiation, then identify $(\mathbb{R}_{>0})^2$ with the interior of $\mathbb{R}^2_\Delta := \mathrm{Tor}_{\mathbb{R}}(\Delta)_+ \simeq \Delta$, the nonnegative part of the real toric variety $\mathrm{Tor}_{\mathbb{R}}(\Delta)$, via the moment map

$$\mu(x) = \frac{\sum_{\omega\in\Delta\cap\mathbb{Z}^2} x^\omega \omega}{\sum_{\omega\in\Delta\cap\mathbb{Z}^2} x^\omega}, \quad x \in (\mathbb{R}_{>0})^2.$$

If Δ is a segment, then we take $\Delta' = \Delta \times \sigma$, σ being a transverse lattice segment, and define $\mathbb{R}^2_\Delta$ as the quotient of $\mathbb{R}^2_{\Delta'}$ by contracting the sides parallel to σ. We observe that the rays in $\mathbb{R}^2$ directed by an external normal $\boldsymbol{u}$ to a side σ of Δ and emanating from distinct points on a line transverse to σ close up at distinct points on the part of $\partial(\mathbb{R}^2_\Delta)$ corresponding to the interior of σ in the above construction.[3]

So we can naturally compactify a PPT-curve (Γ, h) into $(\overline{\Gamma}, \overline{h})$ by extending h up to a map $\overline{h} : \overline{\Gamma} \to \mathbb{R}^2_\Delta$.

[3]Clearly, the rays directed by vectors distinct from any exterior normal to sides on Δ close up at respective vertices of $\mathbb{R}^2_\Delta$.

2.4 Marked Tropical Curves

An abstract tropical curve with n marked points is a pair $(\overline{\Gamma}, G)$, where $\overline{\Gamma}$ is an abstract tropical curve and $G = (\gamma_1, \dots, \gamma_n)$ is an ordered n-tuple of distinct points of $\overline{\Gamma}$. We say that a marked tropical curve $(\overline{\Gamma}, G)$ is *regular* if each connected component of $\overline{\Gamma} \setminus G$ is a tree containing precisely one vertex from Γ^0_∞. Furthermore, a marked tropical curve $(\overline{\Gamma}, G)$ is called

- *End-marked*, if $G \cap \Gamma^0 = \emptyset$ and the points of G lie on the ends of $\overline{\Gamma}$, one on each end.
- *Regularly end-marked* if $G \cap \Gamma^0 = \emptyset$, the points of G lie on the ends of $\overline{\Gamma}$, and $(\overline{\Gamma}, G)$ is regular.

A parameterization of a (compact) plane tropical curve with marked points is a triple $(\overline{\Gamma}, G, \overline{h})$, where $(\overline{\Gamma}, G)$ is a marked abstract tropical curve, and $(\overline{\Gamma}, \overline{h})$ is a PPT-curve. We define the deformation space $\mathcal{M}(\overline{\Gamma}, G, \overline{h})$ by fixing the combinatorial type of the pair $(\overline{\Gamma}, G)$ (G being an ordered sequence). It can be identified with a convex polyhedron in $\mathbb{R}^N$, where the coordinates designate the two coordinates of the image $\overline{h}(v)$ of a fixed vertex $v \in \Gamma^0$, the lengths of the edges $e \in \Gamma^1$, and the distances between the marked points lying inside edges of Γ to some fixed points inside these edges (chosen one on each edge); cf. [1]. Further on, the deformation type of a marked PPT-curve will be called a *combinatorial* type.

Lemma 1. *Let Δ be a convex lattice polygon, $X = (\boldsymbol{x}_1, \dots, \boldsymbol{x}_n)$ a sequence of points in $\mathbb{R}^2_\Delta$ (not necessarily distinct). Then there exists at most one n-marked regular PPT-curve $(\overline{\Gamma}, G, \overline{h})$ with the Newton polygon Δ and with a fixed combinatorial type such that $\overline{h}(\gamma_i) = \boldsymbol{x}_i$, $\gamma_i \in G$, $i = 1, \dots, n$.*

Proof. If such a marked PPT-curve exists, it is sufficient to uniquely restore each connected component of $\overline{\Gamma} \setminus G$, and hence the general situation reduces to the case of an irreducible rational PPT-curve (a subordinate curve defined by such a connected component) with $|\Gamma^0_\infty| - 1 = |\Gamma^1_\infty| - 1$ marked univalent vertices. We proceed by induction on $|\overline{\Gamma}^1|$. The base of induction, i.e., the case $|\overline{\Gamma}^1| = 1$, is evident. Assume that $|\overline{\Gamma}^1| > 1$.

If there are two Γ-ends e_1, e_2 with marked points that emanate from one vertex $v \in \Gamma^0$ and are mapped into the same straight line by $\overline{h}$, then either $\overline{h}(e_1) = \overline{h}(e_2)$, in which case we replace e_1, e_2 by one end of weight $w(e_1) + w(e_2)$ and respectively replace two marked points by one, thus reducing $|\overline{\Gamma}^1|$ by 1 and keeping the irreducibility and the rationality of the tropical curve, or $\overline{h}(e_1)$ and $\overline{h}(e_2)$ are the opposite rays emanating from $\overline{h}(v)$, in which case we remove the Γ-end with lesser weight, leaving the other with weight $|w(e_1) - w(e_2)|$, thus reducing $|\overline{\Gamma}^1|$ by 1 or 2.

If there are no Γ-ends as above, from

$$|\Gamma^0| - |\Gamma^1| = 1 \quad \text{and} \quad 3 \cdot |\Gamma^0| \le 2 \cdot |\Gamma^1| + |\Gamma^1_\infty|$$

we deduce that $|\Gamma^0| \le |\Gamma^1_\infty| - 2$. Hence there are two nonparallel Γ-ends with marked points that merge to a common vertex $v \in \Gamma^0$, which thereby is determined uniquely. So we remove the above Γ-ends and the vertex v from $\overline{\Gamma}$, then extend the other edges of $\overline{\Gamma}$ coming to v up to new ends and mark on them the points mapped to $h(v)$. Thus, the induction assumption completes the proof. □

2.5 *Tropically Generic Configurations of Points*

Let Δ be a convex lattice polygon, $x = (\boldsymbol{x}_1, \dots, \boldsymbol{x}_k)$ a sequence of distinct points in $\mathbb{R}^2_\Delta$ such that $\boldsymbol{x}_i \in \sigma_i$, $1 \le i \le r$, where $\sigma_1, \dots, \sigma_r \subset \mathbb{R}^2_\Delta$ correspond to certain sides of Δ, and $\boldsymbol{x}_i \in \mathbb{R}^2 \subset \mathbb{R}^2_\Delta$, $r < i \le k$. Let $\overline{m} = (m_1, \dots, m_k)$ be a sequence of nonnegative integers, called weights of the points $\boldsymbol{x}_1, \dots, \boldsymbol{x}_k$, respectively. A subconfiguration of $(x, \overline{m})$ is a configuration $(x, \overline{m}')$ with $\overline{m}' \le \overline{m}$ (componentwise).

Let $\mathcal{C}$ be a combinatorial type of an irreducible end-marked PPT-curve with Newton polygon Δ, with $m = m_1 + \dots + m_k$ Γ-ends and marked points $\gamma_1, \dots, \gamma_m$. A weighted configuration $(\overline{\boldsymbol{x}}, \overline{m})$ is called $\mathcal{C}$-*generic* if there is no end-marked irreducible PPT-curve $(\overline{\Gamma}, G, \overline{h})$ of type $\mathcal{C}$ such that $\overline{h}(G) = (\overline{\boldsymbol{x}}, \overline{m})$, i.e.,

$$\overline{h}(\gamma_i) = \boldsymbol{x}_i, \ \sum_{j<i} m_j < i \le \sum_{j \le i} m_j, \quad i = 1, \dots, k.$$

A weighted configuration $(\overline{\boldsymbol{x}}, \overline{m})$ is called Δ-*generic* if it together with all its subconfigurations is generic with respect to the combinatorial types of end-marked irreducible PPT-curves that have $m \le |\partial\Delta \cap \mathbb{Z}^2|$ Γ-ends and directing vectors of all edges orthogonal to integral segments in Δ. A (nonweighted) configuration $\overline{\boldsymbol{x}}$ is called Δ-*generic* if all possible weighted configurations $(\overline{\boldsymbol{x}}, \overline{m})$ are Δ-generic.

Lemma 2. *The Δ-generic configurations with rational coordinates $\overline{\boldsymbol{x}} = (\boldsymbol{x}_1, \dots, \boldsymbol{x}_k) \subset \mathbb{R}^2_\Delta$ such that $\boldsymbol{x}_i \in \sigma_i$, $1 \le i \le r$, $\boldsymbol{x}_i \in \mathbb{R}^2$, $r < i \le k$, form a dense subset of $\sigma_1 \times \dots \times \sigma_r \times (\mathbb{R}^2)^{k-r}$.*

Proof. Notice that there are only finitely many (up to the choice of edge weights) combinatorial types of end-marked irreducible PPT-curves under consideration and only finitely many weight collections $\overline{m}$ to consider. We shall prove that for any such combinatorial type $\mathcal{C}$ of end-marked irreducible PPT-curves, the image of the natural evaluation map $\mathrm{Ev} : \mathcal{M}(\mathcal{C}) \to \sigma_1 \times \dots \times \sigma_r \times (\mathbb{R}^2)^{k-r}$ is nowhere dense, and hence is a finite polyhedral complex of positive codimension. This would suffice for the proof of the lemma due to the aforementioned finiteness.

Thus, assuming that an end-marked irreducible curve $(\overline{\Gamma}, G, \overline{h})$ of type $\mathcal{C}$ matches a weighted rational configuration $(\overline{\boldsymbol{x}}, \overline{m})$, we shall show that this imposes a nontrivial relation on the coordinates of the points of $\overline{\boldsymbol{x}}$ and hence complete the proof. This is, in fact, a direct consequence of (4). We explain this in detail, however, since formally one can assume that all the points of $\overline{\boldsymbol{x}}$ are multiple, and for each point $\boldsymbol{x}_i \in \overline{\boldsymbol{x}}$, the corresponding scalar products in (4) sum to zero.

Since any point $\boldsymbol{x}_i \in \overline{\boldsymbol{x}}$ lying on $\partial\mathbb{R}^2_\Delta$ is a univalent vertex for some ends of $\overline{\Gamma}$ whose $\overline{h}$-images lie on the same straight line, by pushing all the points of $\overline{\boldsymbol{x}} \cap \partial\mathbb{R}^2_\Delta$ along the corresponding lines, we can make $\overline{\boldsymbol{x}} \subset \mathbb{R}^2$. Take an irrational vector $a \in \mathbb{R}^2$ and choose the point $\boldsymbol{x}_i \in \overline{\boldsymbol{x}}$ with the maximal value of the functional $\langle a, \boldsymbol{x} \rangle$. Notice that there is no vertex $v \in \Gamma^0$ with $\langle a, h(v) \rangle \geq \langle a, \boldsymbol{x}_i \rangle$, since otherwise, due to the balancing condition (2), one would find an end e of Γ with $h(e)$ lying entirely in the half-plane $\langle a, \boldsymbol{x} \rangle > \langle a, \boldsymbol{x}_i \rangle$, contrary to our assumptions. Hence, for each end $e \in \Gamma^1_\infty$ with $h(e)$ passing through $\boldsymbol{x}_i$, we have $\langle a, \tau(e) \rangle > 0$, which yields that

$$\sum_{\substack{e \in \Gamma^1_\infty \\ \boldsymbol{x}_i \in h(e)}} m_e \cdot \mathrm{d}h(\tau(e)) \neq 0$$

for any positive integers m_e, and which finally implies that (some of) the coordinates of $\boldsymbol{x}_i$ enter relation (4) with nonzero coefficients. □

Lemma 3. *Let $\overline{\boldsymbol{x}}$ be a Δ-generic configuration of points, $Q = (\overline{\Gamma}, G, \overline{h})$ a marked regular PPT-curve with Newton polygon Δ that matches $\overline{\boldsymbol{x}}$. Then*

(i) $(\overline{h})^{-1}(\overline{\boldsymbol{x}}) = G$;
(ii) If K is a connected component of $\overline{\Gamma} \setminus G$, then its edges can be oriented in such a way that

- *(a) The edges merging to marked points emanate from these points.*
- *(b) The unmarked Γ-end is oriented toward its univalent endpoint.*
- *(c) From any vertex $v \in K^0$ there emanates precisely one edge, and this edge is simple.*

Remark 4. It follows from Lemma 3 that if an edge of Γ is multiple for both of its endpoints and contains a marked point inside that matches a point $\boldsymbol{x} \in \overline{\boldsymbol{x}}$, then all the other edges joining the same vertices contain marked points matching $\boldsymbol{x}$.

Proof of Lemma 3. (i) Assume that there is a point $\gamma \in (\overline{h})^{-1}(\overline{\boldsymbol{x}}) \setminus G$. It belongs to a component K of $\overline{\Gamma} \setminus G$, which is a tree due to the regularity of the considered marked tropical curve, and hence is cut by γ into two trees K_1, K_2, and only one of them, say K_1, contains a Γ-end free of marked points. Then marking the new point γ, we obtain that the irreducible (rational) PPT-curve induced by K_2 is end-marked and matches a subconfiguration of $\overline{\boldsymbol{x}}$, contrary to its Δ-genericity.

(ii) Observe that the image of the unmarked ray does not coincide with the image of any other edge of K, which follows immediately from the statement (i).

Next we notice that if p is a vertex of K, e a multiple edge merging to p, then the connected component $K(e)$ of $K \setminus \{p\}$, starting with the edge e, does not contain the unmarked K-end. Indeed, otherwise, we consider another edge e' of K merging to p so that $\boldsymbol{u}_p(e') = \boldsymbol{u}_p(e)$. Then we take the graph $K \setminus K(e)$, multiply the weights of the edges of the component $K(e')$ of $K \setminus \{p\}$ starting with e' by $w(e) + w(e')$, and multiply the weights of the edges in $K \setminus (K(e) \cup K(e'))$ by $w(e')$, where $w(e), w(e')$

are the weights of e and e' in K, in order to preserve the balancing condition at the vertex p, and thereby the newly weighted $K \setminus K(e)$ induces an end-marked (rational) PPT-curve matching the Δ-generic configuration $\overline{\boldsymbol{x}}$, a contradiction.

It follows from the latter observation that K has no edge that is multiple for both of its endpoints. Indeed, otherwise we would have two vertices $v_1, v_2 \in K^0$, joined by an edge $e \in K^1$, multiple for both v_1 and v_2, and then would obtain that the unmarked K-end is contained either in the component of $K \setminus \{v_1\}$ starting with e, or in the component of $K \setminus \{v_2\}$ starting with e, contrary to the above conclusion.

Finally, we define an orientation of the edges of K, opposite to the required one. Start with the unmarked K-end and orient it toward its multivalent endpoint. In any other step, in arriving at a vertex $v \in K^0$ along some edge, we orient all other edges merging to v outward. Since K is a tree, the orientation smoothly extends to all of its edges. The preceding observations confirm that any edge e oriented in this manner toward a vertex $v \in E$ is simple for v. □

2.6 *Weights of Marked Pseudosimple Regular PPT-Curves*

In this section, $Q = (\overline{\Gamma}, G, \overline{h})$ is always a regular marked pseudosimple PPT-curve. Set $G_\infty = G \cap \Gamma^0_\infty$ and $G_0 = G \setminus G_\infty$, and put $\overline{\boldsymbol{x}} = \overline{h}(G)$, $\overline{\boldsymbol{x}}^\infty = \overline{h}(G_\infty)$. Throughout this section we assume that

(T1) No edge of Γ is multiple for two vertices of Γ.
(T2) G_0 does not contain vertices of valency greater than 3.
(T3) $\overline{\boldsymbol{x}}$ is Δ-generic.

In particular, by Lemma 3, we have that $(\overline{h})^{-1}(\overline{\boldsymbol{x}}) = G$.

Complex weights. We define the *complex weight* of a PTT-curve $Q = (\overline{\Gamma}, G, \overline{h})$ as

$$M(Q) = \prod_{v \in \Gamma^0} M(Q, v) \cdot \prod_{e \in \Gamma^1} M(Q, e) \cdot \prod_{\gamma \in G} M(Q, \gamma), \tag{7}$$

where the values $M(Q,v), M(Q,e), M(Q,\gamma)$ are computed according to the following rules:

(M1) $M(Q,e) = w(e)$ for each edge $e \in \Gamma^1$.
(M2) $M(Q,\gamma) = 1$ for each $\gamma \in G \cap (\Gamma^0 \cup \Gamma^0_\infty)$, and $M(Q,\gamma) = w(e)$ for each $\gamma \in G \setminus (\Gamma^0 \cup \Gamma^0_\infty)$, $\gamma \in e \in \overline{\Gamma}^1$.

To define $M(Q,v)$, $v \in \Gamma^0$, we introduce some notation: We denote by Δ_v the lattice triangle whose boundary is composed of the vectors $\mathrm{d}h(\tau_v(e))$, rotated clockwise by $\pi/2$, where e runs over all the edges of $\overline{\Gamma}$ emanating from v. Next, we put the following:

(M3) If $v \in \Gamma^0 \cap G$, then $M(Q,v) = |\Delta_v|$.

(M4) If $v \in \Gamma^0 \setminus G$ is trivalent, then it belongs to a connected component K of $\overline{\Gamma} \setminus G$, which we orient as in Lemma 3(ii) and thus define two edges $e_1, e_2 \in \overline{\Gamma}^1$ merging to v. In this case we put $M(Q,v) = |\Delta_v|(w(e_1)w(e_2))^{-1}$.

(M5) Let $v \in \Gamma^0$ be of valency $s+r+1 > 3$, where $1 \le r \le s$, $2 \le s$, and let e_i, $i = 1, \dots, s+r+1$, be all the edges with endpoint v, so that the edges e_i, $1 \le i \le s$, have a common directing vector $\boldsymbol{u}_v(e_1)$, the edges e_i, $s < i \le s+r$, have a common directing vector $\boldsymbol{u}_v(e_{s+1})$, and e_{s+r+1} is a simple edge emanating from v along the orientation of Lemma 3(ii). Consider a rational PPT-curve Q_v induced by the graph $\overline{\Gamma}_v = \{v\} \cup \bigcup_{i=1}^{s++r+1} e_i \subset \overline{\Gamma}$, pick auxiliary marked points $\gamma_i \in e_i \setminus \{v\}$, $i = 1, \dots, s+r$, in such a way that $\overline{h}(\gamma_i) = \boldsymbol{y}' \in \mathbb{R}^2$ as $1 \le i \le s$, and $\overline{h}(\gamma_i) = \boldsymbol{y}'' \in \mathbb{R}^2$ as $s < i \le s+r$. Then we replace $\boldsymbol{y}'$ (respectively $\boldsymbol{y}''$) by a generic set of distinct points $\boldsymbol{y}_1, \dots, \boldsymbol{y}_s$ close to $\boldsymbol{y}'$ (respectively distinct points $\boldsymbol{y}_{s+1}, \dots, \boldsymbol{y}_{s+r}$ close to $\boldsymbol{y}''$), and take rational regularly end-marked PPT-curves of degree $\{\mathrm{d}h(\tau_v(e_i))\}_{i=1,\dots,s+r+1}$ matching the configuration $\boldsymbol{y}_1, \dots, \boldsymbol{y}_{s+r}$, so that the $\overline{h}$-image of the Γ-end of weight $w(e_i)$ with the directing vector $\boldsymbol{u}_v(e_i)$ passes through the point $\boldsymbol{y}_i$, $i = 1, \dots, s+r$ (see Fig. 1).[4] By [9, Corollaries 2.24 and 4.12], the set $\mathcal{T}$ of these PPT-curves is finite, and they all are simple. Then put

$$M(Q,v) = \sum_{Q' \in \mathcal{T}} M(Q'), \tag{8}$$

where all terms $M(Q')$ are computed by formula (7) and the rules (M1)–(M4).

Remark 5. (1) We point out that the right-hand side of (8) does not depend on the choice of the configuration $(\boldsymbol{y}_i)_{i=1,\dots,s+r+1}$, which follows from [1, Theorem 4.8] (observe that the degree of the evaluation map as in [1, Definition 4.6] coincides with the right-hand side of (8) in our situation). Slightly modifying the Mikhalkin correspondence theorem [9, Theorem 1], one can deduce that $M(Q,v)$ as defined in (8) equals the number of complex rational curves C on the toric surface $\mathrm{Tor}(\Delta_v)$ such that

- C belongs to the tautological linear system $|\mathcal{L}_{\Delta_v}|$.
- For each side σ of Δ_v, the intersection points of C with toric divisor $\mathrm{Tor}(\sigma) \subset \mathrm{Tor}(\Delta_v)$ are in 1-to-1 correspondence with the Γ-ends of the tropical curves from $\mathcal{T}$ orthogonal to σ, C is nonsingular along $\mathrm{Tor}(\sigma)$, and the intersection multiplicities are respectively equal to the weights of the above Γ-ends.
- C passes through a generic configuration of $s+r+1$ points in $\mathrm{Tor}(\Delta_v)$.

Furthermore, $\mathcal{T}$ consists of just one curve since $r = 1$. Indeed, its dual subdivision of the Newton triangle Δ_v must be as described above with the order of segments dual to the parallel Γ-ends, which is determined uniquely by the disposition of the points $\boldsymbol{y}_1, \dots, \boldsymbol{y}_s$.

[4] Notice that by construction there is a canonical 1-to-1 correspondence between the ends of Q_v and the ends of any of the curves obtained in the deformation.

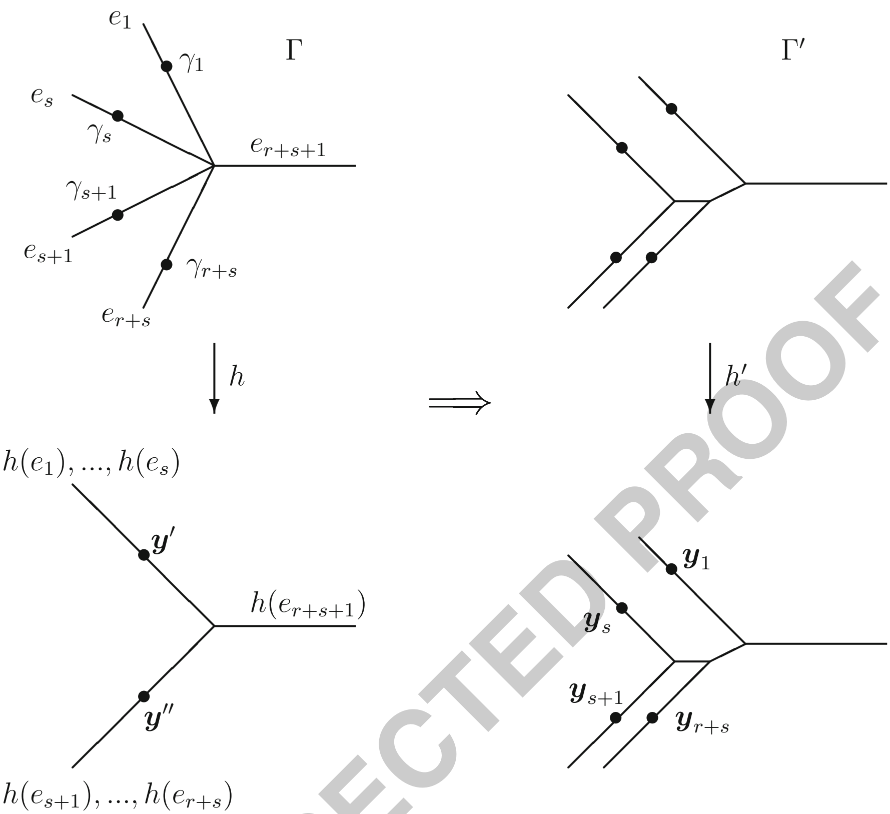

Fig. 1 Local deformation of a tropical curve in (M5)

(2) If Q is simple, i.e., all the vertices of Γ are trivalent, then (7) gives

$$M(Q) = \frac{\prod_{v \in \Gamma^o} |\Delta_v|}{\prod_{\gamma \in G_\infty,\ \gamma \in e \in \Gamma^1_\infty} w(e)}, \tag{9}$$

which generalizes Mikhalkin's weight introduced in [9, Definitions 2.16 and 4.15], and coincides with the multiplicity of a tropical curve from [1].

Real weights. A PPT-curves $Q = (\overline{\Gamma}, G, \overline{h})$ equipped with the additional structure of a continuous involution $c : (\overline{\Gamma}, G, \overline{h}) \to (\overline{\Gamma}, G, \overline{h})$ and a subdivision $G = \Re G \cup \Im G$ invariant with respect to c is called *real*.

Clearly, $\Im\overline{\Gamma} := \overline{\Gamma} \setminus \Re\overline{\Gamma}$, where $\Re\overline{\Gamma} = \mathrm{Fix}\, c\big|_{\overline{\Gamma}}$ consists of two disjoint subsets $\Im\overline{\Gamma}'$, $\Im\overline{\Gamma}''$ interchanged by c.

Given a real PPT-curve Q, we can construct a (usual) PPT-curve $Q/c = (\overline{\Gamma}/c, G/c, \overline{h}/c)$. Notice that the weights of the edges obtained here by identifying

$\Im\overline{\Gamma}'$ and $\Im\overline{\Gamma}''$ are even. Conversely, given a (usual) PPT-curve $Q = (\overline{\Gamma}, G, \overline{h})$ and a set $I(\overline{\Gamma}^1) \subset \overline{\Gamma}^1$ that includes only edges of even weight, we construct a real PPT-curve $Q' = (\overline{\Gamma}', G', \overline{h}')$ as follows: (i) Put $K = \bigcup_{e\in I(\overline{\Gamma}^1)} e$ and obtain the graph $\overline{\Gamma}'$ by gluing up $\overline{\Gamma}$ with another copy K' of K at the vertices of Γ, common for K and the closure of $\overline{\Gamma} \setminus K$. (ii) The map $\overline{h}$ coincides on K and K', whereas the weights of the doubled edges are divided by 2 in order to keep the balancing condition. (iii) The points of $G \cap K$ are respectively doubled to K'. Finally, define an involution c on Q' interchanging K and K', and define a subdivision $G' = \Re G' \cup \Im G'$.

We shall consider only real PPT-curves with the following properties:

(R1) $\Re\overline{\Gamma}$ is nonempty and has no one-point connected component;
(R2) $\Im\overline{\Gamma}$ has only univalent and trivalent vertices (if nonempty);
(R3) The marked points $G_0 \cap \Im\overline{\Gamma}$ are not vertices of $\Im\overline{\Gamma}$;
(R4) $\Im G \setminus \Im\overline{\Gamma}$ is empty or consists of some trivalent vertices of Γ;
(R5) The closure of any component of $\Im\overline{\Gamma} \setminus G$ contains a point from $\Im G$.

Observe that the closure of $\Im\overline{\Gamma}$ joins $\Re\overline{\Gamma}$ at vertices of valency greater than 3 (which are not in G by condition (T2)).

The real weight of a real PPT-curve Q is defined as

$$W(Q) = (-1)^{\ell_1} 2^{\ell_2} \cdot \prod_{v\in\Gamma^0} W(Q,v) \cdot \prod_{e\in\overline{\Gamma}^1} W(Q,e) \cdot \prod_{\gamma\in G_0} W(Q,\gamma),$$

$$\ell_1 = \frac{|\Re G \cap \Im\overline{\Gamma}|}{2}, \quad \ell_2 = \frac{|\Im G \cap \Im\overline{\Gamma}| - b_0(\Im\overline{\Gamma})}{2}, \qquad (10)$$

with $W(Q,v), W(Q,e), W(Q,\gamma)$ computed along the following rules:

(W1) For an edge $e \subset \Re\overline{\Gamma}$, put $W(Q,e) = 0$ or 1 according to whether $w(e)$ is even or odd. For an edge $e \in \Gamma^1$, $e \subset \overline{\Im\Gamma}$, put $W(Q,e)W(Q,c(e)) = w(e)$. For an edge $e \in \Gamma^1_\infty$, $e \subset \Im\overline{\Gamma}$, put $W(Q,e) = 1$.
(W2) For $\gamma \in G_0 \cap \Re\overline{\Gamma} \setminus \Gamma^0$, put $W(Q,\gamma) = 1$. For $\gamma \in \Re G \cap \Gamma^0$, put $W(Q,\gamma) = 1$. For $\gamma \in \Im G$, $\gamma = v \in \Gamma^0$, put $W(Q,\gamma) = |\Delta_v|$. For $\gamma \in G_0$, $\gamma \in e \subset \Im\overline{\Gamma}$, put $W(Q,\gamma)W(Q,c(\gamma)) = w(e)$.
(W3) For a vertex $v \in \Gamma^0 \cap \Im\overline{\Gamma}$, put $W(Q,v)W(Q,c(v)) = (-1)^{|\partial\Delta_v \cap \mathbb{Z}^2|} M(Q,v)$ (see condition (M4) for the definition of $M(Q,v)$). For a trivalent vertex $v \in \Gamma^0 \cap \Re\overline{\Gamma}$, put $W(Q,v) = (-1)^{|\mathrm{Int}(\Delta_v) \cap \mathbb{Z}^2|}$.
(W4) For a four-valent vertex $v \in \Gamma^0$ incident to two simple edges from $\Re\overline{\Gamma}$ and two multiple edges e', e'' from $\Im\overline{\Gamma}$, put $W(Q,v) = (-1)^{|\mathrm{Int}(\Delta_v) \cap \mathbb{Z}^2|} |\Delta_v|/(2w(e'))$.
(W5) Let $v \in \Gamma^0$ be of valency >3 incident to

- A simple edge $e_1 \subset \Re\overline{\Gamma}$,
- Edges $e_i \subset \Re\overline{\Gamma}$, $1 < i \le r_1 + 1$, and $e'_i \subset \Im\overline{\Gamma}'$, $e''_i \subset \Im\overline{\Gamma}''$, $1 \le i \le s_1$, for some nonnegative r_1, s_1, all with the same directing vector $\boldsymbol{u}' \ne \boldsymbol{u}_v(e_1)$ and
- Edges $e_i \subset \Re\overline{\Gamma}$, $r_1 + 1 < i \le r_1 + r_2 + 1$, and $e'_i \subset \Im\overline{\Gamma}'$, $e''_i \subset \Im\overline{\Gamma}''$, $s_1 < i \le s_1 + s_2$, for some nonnegative r_2, s_2 such that $r_2 + 2s_2 \ge 2$, all with the same directing vector $\boldsymbol{u}'' \ne \boldsymbol{u}_v(e_1), \boldsymbol{u}'$.

Take the real PPT-curve Q_v induced by v and the edges emanating from v, correspondingly restrict on Q_v the involution c, and introduce a finite c-invariant set of marked points G_v picking up one point on each edge emanating from v but e_1. Consider the PPT-curve Q_v/c and perform with it the deformation procedure described in (M5) (cf. Fig. 1), getting a finite set of simple rational regularly end-marked PPT-curves. We turn any curve $\widetilde{Q} = (\widetilde{\overline{\Gamma}}, \widetilde{G}, \widetilde{\overline{h}})$ from this set into a real PPT-curve. Namely, first we include in the set $I(\widetilde{\overline{\Gamma}}^1)$ all the Γ-ends that correspond to the Γ/c-ends of Q_v from $\Im\overline{\Gamma}_v/c$. Then we maximally extend the set $I(\widetilde{\overline{\Gamma}}^1)$ by the following inductive procedure: If two edges $f_1, f_2 \in I(\widetilde{\overline{\Gamma}}^1)$ merge to a vertex $p \in \widetilde{\Gamma}^0$, then the third edge f_3 emanating from p should be added to $I(\widetilde{\overline{\Gamma}}^1)$. Clearly, by construction, the weights of the edges $e \in I(\widetilde{\overline{\Gamma}}^1)$ are even; hence we can make a real PPT-curve $Q' = (\overline{\Gamma}', G', \overline{h}')$, letting $\Re G' = G' \cap \Re\overline{\Gamma}'$, $\Im G' = G' \cap \Im\overline{\Gamma}'$. Denoting the final set of real PPT-curves by $\mathcal{T}$ and observing that their real weight $W(Q')$ can be computed by the above rules (W1–(W4), we define

$$W(Q, v) = \sum_{Q' \in \mathcal{T}} W(Q').$$

The fact that the latter expression does not depend on the choice of the perturbation of the points $\boldsymbol{y}'$, $\boldsymbol{y}''$ (cf. construction in (M5) and Fig. 1) follows from a more general statement proven in [18].

Remark 6. (1) If $c = \mathrm{Id}$, $\Im G = \emptyset$, and Q is simple, we obtain the well-known formula $W(Q) = 0$ when $\overline{\Gamma}$ contains an even weight edge, and $W(Q) = (-1)^a$, $a = \sum_{v \in \Gamma^0} |\mathrm{Int}(\Delta_v) \cap \mathbb{Z}^2|$, when all the edge weights of $\overline{\Gamma}$ are odd (cf. [9, Definition 7.19] or [13, Proposition 6.1], where in addition, $\deg Q$ consists of only primitive integral vectors).

(2) If Q is rational, Q/c is simple, and $G_\infty = \Re G \cap \Gamma^0 = \Re G \cap \Im\overline{\Gamma} = \emptyset$, we obtain a generalization of [15, Formula (2.12)] (in the version at arXiv:math/0406099). Indeed, if $\Re\overline{\Gamma}$ contains an edge of even weight, we obtain $W(Q) = 0$ in (10) due to (W1), and accordingly we obtain $w(Q/c) = 0$ in [15, Sect. 2.5] (in the notation therein). If $\Re\overline{\Gamma}$ contains only edges of odd weight, then (under assumption that the weights of the ends of Γ are 1) [15, Formula (2.12)] reads

$$w(Q/c) = (-1)^{a+b} \prod_{v \in \Gamma^0 \cap \Im G} |\Delta_v| \cdot \prod_{v \in (\Gamma/c)^0 \cap \overline{(\Im\Gamma/c)}} \frac{|\Delta_v|}{2} \qquad (11)$$

with $a = \sum_{v \in (\Gamma/c)^0} |\mathrm{Int}(\Delta_v) \cap \mathbb{Z}^2|$, $b = |(\Gamma/c)^0 \cap (\Im\Gamma/c)|$, whereas in (10) we obtain $\ell_1 = 0$ by the assumption $\Re G \cap \Im\overline{\Gamma} = \emptyset$, $\ell_2 = |(\Gamma/c)^0 \cap (\Im\Gamma/c)|$ due to the rationality of Q and simplicity of Q/c, and furthermore, taking into account that $w(e) = 2w(e') = 2w(c(e'))$ for $e = (e' \cup c(e'))/c \in (\Gamma/c)^1$, $e' \in \Gamma^1$, $e' \subset \Im\Gamma$, we compute the other factors in (10):

$$\prod_{v\in\Gamma^0} W(Q,v) = \prod_{v\in\Gamma^0\setminus\overline{\Im\Gamma}}(-1)^{|\mathrm{Int}(\Delta_v)\cap\mathbb{Z}^2|}\cdot\prod_{\{v,c(v)\}\in(\Gamma/c)^0\cap(\Im\Gamma/c)}(-1)^{|\partial\Delta_v\cap\mathbb{Z}^2|}M(Q,v)$$

$$\times\prod_{\substack{v\in\Re\Gamma\cap\overline{\Im\Gamma/c}\\ v\in e\subset\overline{\Im\Gamma/c},\ e\in(\Gamma/c)^1}}(-1)^{|\mathrm{Int}(\Delta_v)\cap\mathbb{Z}^2|}\frac{|\Delta_v|}{w(e)}$$

$$=\prod_{v\in(\Gamma/c)^0}(-1)^{|\mathrm{Int}(\Delta_v)\cap\mathbb{Z}^2|}\cdot(-4)^{-|(\Gamma/c)^0\cap\Im\Gamma/c|}\cdot\prod_{v\in(\Gamma/c)^0\cap\overline{\Im\Gamma/c}}|\Delta_v|$$

$$\times 2^{-|\Re\Gamma\cap\overline{\Im\Gamma}|}\prod_{e\in(\Gamma/c)^1,\ e\subset\overline{\Im\Gamma/c}}\frac{2}{w(e)}\prod_{\substack{e\in(\Gamma/c)^1,\ e\subset\overline{\Im\Gamma/c}\\ e\cap G/c\ne\emptyset}}\frac{2}{w(e)},$$

$$\prod_{e\in\overline{\Gamma}^1}W(Q,e)=\prod_{e\in(\Gamma/c)^1,\ e\subset\overline{\Im\Gamma/c}}\frac{w(e)}{2},$$

$$\prod_{\gamma\in G_0}W(Q,\gamma)=\prod_{v\in\Im G\cap\Re\Gamma}|\Delta_v|\prod_{\substack{e\in(\Gamma/c)^1,\ e\subset\overline{\Im\Gamma/c}\\ e\cap G/c\ne\emptyset}}\frac{w(e)}{2},$$

which altogether gives (with a,b from (11))

$$W(Q)=(-1)^{a+b}\prod_{v\in\Gamma^0\cap\Im G}|\Delta_v|\cdot\prod_{v\in(\Gamma/c)^0\cap\overline{(\Im\Gamma/c)}}\frac{|\Delta_v|}{2}=w(Q/c).$$

3 Patchworking Theorem

3.1 Patchworking Data

Combinatorial-geometric part. In the notation of Sect. 2.6, let $Q=(\overline{\Gamma},G,\overline{h})$ be a pseudosimple irreducible regular marked PPT-curve of genus g that has a nondegenerate Newton polygon Δ and that satisfies conditions (T1)–(T3) of Sect. 2.6.

Let G_0 split into disjoint subsets $G_0=G_0^{(\mathrm{m})}\cup G_0^{(\mathrm{dm})}$ such that $G_0^{(\mathrm{m})}\cap\Gamma^0=\emptyset$ and $h(G_0^{(\mathrm{m})})\cap h(G_0^{(\mathrm{dm})})=\emptyset$. We equip the points of G_0 with the following multiplicities:

- If $\gamma\in G^{(\mathrm{m})}$, put $\mathrm{mt}(\gamma)=1$.
- If $\gamma\in G^{(\mathrm{dm})}$ is a (trivalent) vertex of Γ, put $\mathrm{mt}(\gamma)=(1,1)$.
- If $\gamma\in G^{(\mathrm{dm})}$ is not a vertex of Γ, put $\mathrm{mt}(\gamma)=(1,0)$ or $(0,1)$.

In the sequel, by $\hat{Q}$ we denote the PPT-curve Q equipped with the subdivision $G_0=G_0^{(\mathrm{m})}\cup G_0^{(\mathrm{dm})}$ and the multiplicity function $\mathrm{mt}(\gamma)$, $\gamma\in G_0$, as above.

AQ1

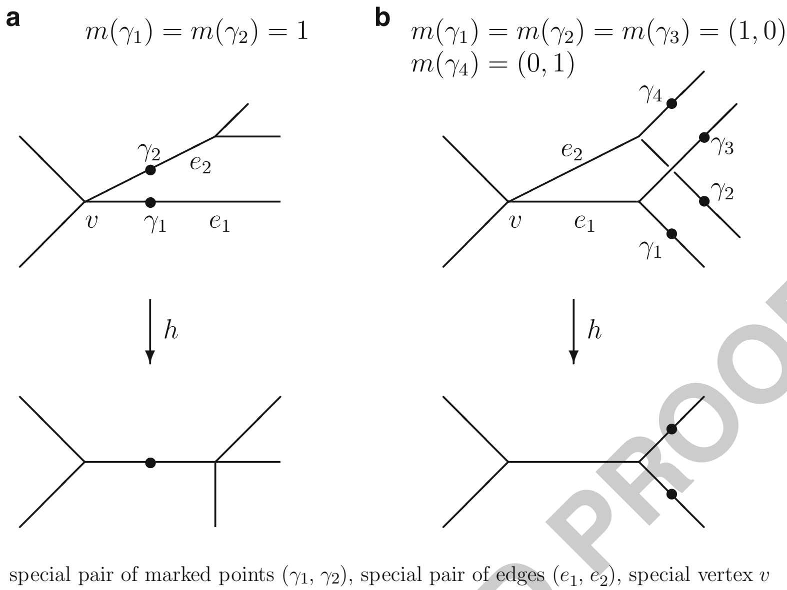

Fig. 2 Illustration to Definition 7

Definition 7. A pair γ, γ' of distinct points in G_0 is called *special* if $h(\gamma) = h(\gamma')$ and $\mathrm{mt}(\gamma) = \mathrm{mt}(\gamma')$. A pair of parallel multiple edges $e, e' \in \overline{\Gamma}^1$ emanating from a vertex $v \in \Gamma^0$ of valency greater than 3 is called *special* if there are disjoint open connected subsets K, K' of $\Gamma \setminus \{v\}$ and a special pair of points $\gamma \in K$, $\gamma' \in K'$ such that

- K contains the germ of e at v; K' contains the germ of e' at v
- There is a homeomorphism $\varphi : K \to K'$ satisfying $h\big|_K = h\big|_{K'} \circ \varphi$.

A vertex $v \in \Gamma^0$ incident to a special pair of edges is called *special*. See Fig. 2.

Then we assume the following:

(T4) The edges in special pairs have weight 1, and at least one of the simple edges emanating from a special vertex has weight 1.

(T5) Let e, e' be a special pair of edges emanating from a vertex $v \in \Gamma^0$, and let K, K' be disjoint connected subsets of $\Gamma \setminus \{v\}$ as in Definition 7; then $K \cup K'$ contains at most one special pair of points of G_0.

(T6) A special pair of edges cannot be a pair of Γ-ends and cannot be a pair of finite-length edges that end up at a special pair $\gamma, \gamma' \in \Gamma^0$ such that $h(\gamma) = h(\gamma')$ and $\mathrm{mt}(\gamma) = \mathrm{mt}(\gamma') = (1,1)$.

(T7) Let a vertex $v \in \Gamma^0$ be a special vertex, and let $\{e_1, \ldots, e_s\}$ be a maximal (with respect to inclusion) set of edges of Γ incident to v and such that

- Each edge e_i contains a point $\gamma_i \in G_0$, $1 \le i \le s$,
- $h(\gamma_1) = \cdots = h(\gamma_s)$ and $\mathrm{mt}(\gamma_1) = \cdots = \mathrm{mt}(\gamma_s)$.

Suppose that $\mathrm{dist}(v,v_i) \le \mathrm{dist}(v,v_{i+1})$, $1 \le i < s$, $v_i \in \overline{\Gamma}^0 \setminus \{v\}$ being the second vertex of e_i. Then we require

$$\mathrm{dist}(v,\gamma_1) > \sum_{1 \le i < s-1} \mathrm{dist}(\gamma_i, v_i) + 2 \cdot \mathrm{dist}(\gamma_{s-1}, v_{s-1}). \qquad (12)$$

Notice that in condition (T7), at most one edge e_i is a Γ-end (cf. (T6)), and it must be e_s.

We introduce also the semigroup

$$\mathbb{Z}^\infty_{\ge 0} = \{\alpha = (\alpha_1, \alpha_2, \ldots) \ : \ \alpha_i \in \mathbb{Z},\ \alpha_i \ge 0,\ i = 1,2,\ldots\ ,\ |\{i \ : \ \alpha_i > 0\}| < \infty\},$$

equipped with two norms

$$\|\alpha\|_0 = \sum_{i=1}^\infty \alpha_i, \quad \|\alpha\|_1 = \sum_{i=1}^\infty i\alpha_i\ ,$$

and the partial order

$$\alpha \ge \beta \quad \Leftrightarrow \quad \alpha - \beta \in \mathbb{Z}^\infty_{\ge 0}.$$

For each side σ of Δ, we introduce the vectors $\beta^\sigma \in \mathbb{Z}^\infty_{\ge 0}$ such that the coordinate β^σ_i of β^σ equals the number of the univalent vertices $v \in \Gamma^0_\infty$ such that $\overline{h}(v) \in \sigma \subset \mathbb{R}^2_\Delta$ and $w(e) = i$ for the Γ-end e merging to v, for all $i = 1,2,\ldots$.

Algebraic part. Let $\Sigma = \mathrm{Tor}_{\mathbb{K}}(\Delta)$. The coordinatewise valuation map $\mathrm{Val} : (\mathbb{K}^*)^2 \to \mathbb{R}^2$ naturally extends up to $\mathrm{Val} : \Sigma \to \mathbb{R}^2_\Delta$. Let $\overline{\boldsymbol{p}} \subset \Sigma := \mathrm{Tor}_{\mathbb{K}}(\Delta)$ be finite and satisfy $\mathrm{Val}(\overline{\boldsymbol{p}}) = \overline{\boldsymbol{x}} = \overline{h}(G)$.[5] Suppose that

(A1) Each point $\boldsymbol{x} \in h(G^{(\mathrm{m})}) \subset \overline{\boldsymbol{x}}$ has a unique preimage in $\overline{\boldsymbol{p}}$.
(A2) The preimage of each point $\boldsymbol{x} \in h(G^{(\mathrm{dm})}) \subset \overline{\boldsymbol{x}}$ consists of an ordered pair of points $\boldsymbol{p}_{1,\boldsymbol{x}}, \boldsymbol{p}_{2,\boldsymbol{x}} \in \overline{\boldsymbol{p}}$.
(A3) There is a bijection $\psi : \overline{\boldsymbol{p}}^\infty \to G_\infty$, where $\overline{\boldsymbol{p}}^\infty := \mathrm{Val}^{-1}(\overline{\boldsymbol{x}}^\infty)$, $\overline{\boldsymbol{x}}^\infty = \overline{h}(G_\infty)$, such that $\mathrm{Val}(\boldsymbol{p}) = \overline{h}(\psi(\boldsymbol{p}))$, $\boldsymbol{p} \in \overline{\boldsymbol{p}}^\infty$;
(A4) The sequence $\overline{\boldsymbol{p}}$ is generic among the sequences satisfying the above conditions.

Define the multiplicity function $\mu : \overline{\boldsymbol{p}} \cap (\mathbb{K}^*)^2 \to \mathbb{Z}_{>0}$ such that:

- For $\boldsymbol{p} \in \overline{\boldsymbol{p}} \cap (\mathbb{K}^*)^2$, $\mathrm{Val}(\boldsymbol{p}) = \overline{x} \in h(G^{\mathrm{m}})$, put

$$\mu(\boldsymbol{p}) = \sum_{\gamma \in G^{(\mathrm{m})},\ h(\gamma) = \boldsymbol{x}} \mathrm{mt}(\gamma). \qquad (13)$$

[5] This means, in particular, that the points of $\overline{\boldsymbol{x}}$ have rational coordinates.

- For the points $\boldsymbol{p}_{1,\boldsymbol{x}},\boldsymbol{p}_{2,\boldsymbol{x}}$ where $\mathrm{Val}(\boldsymbol{p}_{1,\boldsymbol{x}})=\mathrm{Val}(\boldsymbol{p}_{2,\boldsymbol{x}})=\boldsymbol{x}\in h(G^{(\mathrm{dm})})$, put

$$\mu(\boldsymbol{p}_{1,\boldsymbol{x}})=m_1,\ \mu(\boldsymbol{p}_{2,\boldsymbol{x}})=m_2,\quad (m_1,m_2)=\sum_{\gamma\in G^{(\mathrm{dm})},\ h(\gamma)=\boldsymbol{x}}\mathrm{mt}(\gamma). \tag{14}$$

From this definition and from the count of the Euler characteristic of $\overline{\Gamma}$, we derive

$$\sum_{\boldsymbol{p}\in\overline{\boldsymbol{p}}\cap(\mathbb{K}^*)^2}\mu(\boldsymbol{p})+|\overline{\boldsymbol{p}}^\infty|-|\Gamma^0_\infty|=g-1. \tag{15}$$

Let $\Delta'\subset\mathbb{R}^2$ be a convex lattice polygon such that there is another lattice polygon (or segment, or point) Δ'' satisfying $\Delta'+\Delta''=\Delta$. Then we have a well-defined line bundle $\mathcal{L}_{\Delta'}$ on $\mathrm{Tor}_{\mathbb{K}}(\Delta)$. Let $\overline{\boldsymbol{p}}'\subset\overline{\boldsymbol{p}}$ and $\mu':\overline{\boldsymbol{p}}'\cap(\mathbb{K}^*)^2\to\mathbb{Z}_{>0}$ be such that $\mu'(\boldsymbol{p})\le\mu(\boldsymbol{p})$ for all $\boldsymbol{p}\in\overline{\boldsymbol{p}}'$. Let $(\beta^\sigma)'\in\mathbb{Z}^\infty_{\ge0}$, $\sigma\subset\partial\Delta$, be such that $(\beta^\sigma)'\le\beta^\sigma$ for all sides σ of Δ. We say that the tuple $(\Delta',g',\overline{\boldsymbol{p}}',\mu',\{(\beta^\sigma)'\}_{\sigma\subset\partial\Delta})$, where $g'\in\mathbb{Z}_{\ge0}$, is compatible if

- $\|(\beta^\sigma)'\|_1=\langle c_1(\mathcal{L}_{\Delta'}),\mathrm{Tor}_{\mathbb{K}}(\sigma)\rangle$ for all sides σ of Δ;
- $(\beta^\sigma)'_i\ge|\{\boldsymbol{p}\in\overline{\boldsymbol{p}}'\cap\mathrm{Tor}_{\mathbb{K}}(\sigma)\ :\ \psi(\boldsymbol{p})=\gamma\in e\in\Gamma^1_\infty,\ w(e)=i\}|$ for all sides σ of Δ and all $i=1,2,\ldots$;
- $g'\le|\mathrm{Int}(\Delta'\cap\mathbb{Z}^2)|$ and

$$\sum_{\boldsymbol{p}\in\overline{\boldsymbol{p}}'\cap(\mathbb{K}^*)^2}\mu'(\boldsymbol{p})+|\overline{\boldsymbol{p}}'\cap\overline{\boldsymbol{p}}^\infty|-\sum_{\sigma\subset\partial\Delta}\|(\beta^\sigma)'\|_0=g'-1.$$

In view of (15) and $|\Gamma^0_\infty|=\sum_{\sigma\subset\partial\Delta}\|\beta^\sigma\|_0$, the tuple $(\Delta,g,\overline{\boldsymbol{p}},\mu,\{\beta^\sigma\}_{\sigma\subset\partial\Delta})$ is compatible.

For any compatible tuple $(\Delta',g',\overline{\boldsymbol{p}}',\mu',\{(\beta^\sigma)'\}_{\sigma\subset\partial\Delta})$, we introduce the set $\mathcal{C}(\Delta',g',\overline{\boldsymbol{p}}',m',\{(\beta^\sigma)'\}_{\sigma\subset\partial\Delta})$ of reduced irreducible curves $C\in|\mathcal{L}_{\Delta'}|$ passing through $\overline{\boldsymbol{p}}'$ and such that

- The points $\boldsymbol{p}\in\overline{\boldsymbol{p}}'\cap\overline{\boldsymbol{p}}^\infty$ are nonsingular for C, and

$$(C\cdot\mathrm{Tor}_{\mathbb{K}}(\partial\Delta))_{\boldsymbol{p}}=w(e),$$

  where $\gamma=\psi(\boldsymbol{p})\in G_\infty$, and $e\in\Gamma^1_\infty$ merges to γ.
- The local branches of C centered at the points of $C\cap\mathrm{Tor}_{\mathbb{K}}(\partial\Delta)$ are smooth, and for each side σ of Δ and each $i=1,2,\ldots$, there are precisely $(\beta^\sigma)'_i$ local branches P of C centered at $C\cap\mathrm{Tor}_{\mathbb{K}}(\sigma)$ such that

$$(P\cdot\mathrm{Tor}_{\mathbb{K}}(\sigma))=i,\quad i=1,2,\ldots.$$

- C has genus $\le g'$.
- At each point $\boldsymbol{p}\in\overline{\boldsymbol{p}}'\cap(\mathbb{K}^*)^2$, the multiplicity of C is $\mathrm{mt}(C,\boldsymbol{p})\ge\mu'(\boldsymbol{p})$.

We now impose new conditions on the algebraic pathchworking data:

(A5) For any compatible tuple $(\Delta', g', \overline{\boldsymbol{p}}', m', \{(\beta^\sigma)'\}_{\sigma\subset\partial\Delta})$, the set $\mathcal{C}(\Delta', g', \overline{\boldsymbol{p}}', m', \{(\beta^\sigma)'\}_{\sigma\subset\partial\Delta})$ is finite, and all the curves $C\in\mathcal{C}(\Delta', g', \overline{\boldsymbol{p}}', m', \{(\beta^\sigma)'\}_{\sigma\subset\partial\Delta})$ are immersed and have genus g' and multiplicity $\mathrm{mt}(C,\boldsymbol{p}) = m'(\boldsymbol{p})$ at each point $\boldsymbol{p}\in\overline{\boldsymbol{p}}'\cap(\mathbb{K}^*)^2$; furthermore,

$$H^1(C^\nu, \mathcal{J}_Z(C^\nu)) = 0, \tag{16}$$

where C^ν is the normalization and $\mathcal{J}_Z(C^\nu)$ is the (twisted with C^ν) ideal sheaf of the zero-dimensional scheme $Z\subset C^\nu$ that contains the lift of $\overline{\boldsymbol{p}}$ and of the points of tangency of C and $\mathrm{Tor}_{\mathbb{K}}(\partial\Delta)$ on C^ν and that has length $(C\cdot\mathrm{Tor}_{\mathbb{K}}(\partial\Delta))_{\boldsymbol{p}}$ at the lift of $\boldsymbol{p}\in\overline{\boldsymbol{p}}\cap\mathrm{Tor}_{\mathbb{K}}(\partial\Delta)$, and length $(C\cdot\mathrm{Tor}_{\mathbb{K}}(\partial\Delta))_z - 1$ at the lift of each point $z\in C\cap\mathrm{Tor}_{\mathbb{K}}(\partial\Delta)\setminus\overline{\boldsymbol{p}}$.

Here we verify the condition (A5) for the versions of the patchworking theorem used in [6, 18].

Lemma 8. *Condition (A5) holds if*

- *Either* $\mu(\boldsymbol{p}) = 1$ *for all* $\boldsymbol{p}\in\overline{\boldsymbol{p}}\cap(\mathbb{K}^*)^2$,
- *or the surface* $\Sigma = \mathrm{Tor}_{\mathbb{K}}(\Delta)$ *is one of* $\mathbb{P}^2$, $\mathbb{P}^2_k$ *with* $1\le k\le 3$, $(\mathbb{P}^1)^2$, *the configuration* $\overline{\boldsymbol{p}}^\infty$ *is contained in one toric divisor* E *of* Σ, *and*

$$\left|\{\boldsymbol{p}\in\overline{\boldsymbol{p}}\cap(\mathbb{K}^*)^2\ :\ \mu(\boldsymbol{p})>1\}\right| \le \begin{cases} 4, & \Sigma = \mathbb{P}^2,\\ 5 - E^2 - k, & \Sigma = \mathbb{P}^2_k,\\ 3 & \Sigma = (\mathbb{P}^1)^2. \end{cases}$$

Moreover, all the curves C *in the considered sets are nonsingular along* $\mathrm{Tor}_{\mathbb{K}}(\partial\Delta)$, *are nodal outside* $\overline{\boldsymbol{p}}$, *and have ordinary singularity of order* $m'(\boldsymbol{p})$ *at each point* $\boldsymbol{p}\in\overline{\boldsymbol{p}}'\cap(\mathbb{K}^*)^2$.

Proof. We prove the statement only for the original data $(\Delta, g, \overline{\boldsymbol{p}}, m, \{\beta^\sigma\}_{\sigma\subset\partial\Delta})$, since the other compatible tuples can be treated in the same way.

Observe that in the first case, each curve $C\in\mathcal{C}(\Delta, g, \overline{\boldsymbol{p}}, m, \{\beta^\sigma\}_{\sigma\subset\partial\Delta})$ satisfies

$$\sum_{\substack{\boldsymbol{p}\in\overline{\boldsymbol{p}}\cap(\mathbb{K}^*)^2\\ \mu(\boldsymbol{p})>1}} \mu(\boldsymbol{p}) < |C\cap\mathrm{Tor}(\partial\Delta)| - 1. \tag{17}$$

In the second situation, except for finitely many lines or conics (which, of course, satisfy (A5)), the other curves obey (17) by Bézout's theorem (just consider intersections with suitable lines or conics; we leave this to the reader as a simple exercise).

We proceed further under the condition (17). Let $\overline{\boldsymbol{p}}' = \{\boldsymbol{p}\in\overline{\boldsymbol{p}}\cap(\mathbb{K}^*)^2\ :\ \mu(\boldsymbol{p})>1\}$. Consider the family $\mathcal{C}'$ of reduced irreducible curves $C'\in|\mathcal{L}_\Delta|$ of genus at most g that have multiplicity $\ge\mu(\boldsymbol{p})$ at each point $\boldsymbol{p}\in\overline{\boldsymbol{p}}'$, whose local branches centered along $\mathrm{Tor}_{\mathbb{K}}(\partial\Delta)$ are nonsingular, and for which the number of such branches crossing the toric divisor $\mathrm{Tor}_{\mathbb{K}}(\sigma) \subset \mathrm{Tor}_{\mathbb{K}}(\Delta)$ with multiplicity i is β_i^σ for all sides σ of Δ and all $i = 1, 2, \dots$

By a classical deformation theory argument (see, for instance, [2, 3]), the Zariski tangent space to $\mathcal{C}'$ at $C \in \mathcal{C} := \mathcal{C}(\Delta, g, \overline{\boldsymbol{p}}, m, \{\beta^\sigma\}_{\sigma\subset\partial\Delta})$ is naturally isomorphic to $H^0(C^\nu, \mathcal{J}_Z(C^\nu))$, where C^ν, $\mathcal{J}_Z(C^\nu)$ are defined in (A5). So we have

$$\begin{aligned}\deg Z = \sum_{\boldsymbol{p}\in\overline{\boldsymbol{p}}'} \mu(\boldsymbol{p}) + C\cdot \mathrm{Tor}_{\mathbb{K}}(\partial\Delta) - |C\cap \mathrm{Tor}(\partial\Delta)| = \sum_{\boldsymbol{p}\in\overline{\boldsymbol{p}}'} \mu(\boldsymbol{p}) \\ -CK_\Sigma - |C\cap\mathrm{Tor}(\partial\Delta)| < -CK_\Sigma - 1. \qquad (18)\end{aligned}$$

Hence (see [2, 3]) $H^1(C^\nu, \mathcal{J}_Z(C^\nu)) = 0$, which yields

$$\begin{aligned}h^0(C^\nu, \mathcal{J}_Z(C^\nu)) &= C^2 - 2\delta(C) - \deg Z - g(C) + 1\\ &= -CK_\Sigma + 2g(C) - 2 - \deg Z - g(C) + 1\\ &= g(C) - 1 + |C\cap\mathrm{Tor}(\partial\Delta)| - \sum_{\boldsymbol{p}\in\overline{\boldsymbol{p}}'} \mu(\boldsymbol{p}) \overset{(15)}{=} |\overline{\boldsymbol{p}}\setminus\overline{\boldsymbol{p}}'| - (g - g(C)).\end{aligned} \qquad (19)$$

Since $\overline{\boldsymbol{p}}\setminus\overline{\boldsymbol{p}}'$ is a configuration of generic points (partly on $\mathrm{Tor}_{\mathbb{K}}(\partial\Delta)$), we derive that $g(C) = g$ and that $\mathcal{C}$ is finite.

For the rest of the required statement, we assume that a curve $C \in \mathcal{C}$ is either not nodal outside $\overline{\boldsymbol{p}}$ or has singularities on $\mathrm{Tor}_{\mathbb{K}}(\partial\Delta)$ or has at some point $\boldsymbol{p} \in \overline{\boldsymbol{p}} \cap (\mathbb{K}^*)^2$ a singularity more complicated than an ordinary point of order $\mu(\boldsymbol{p})$. Then (cf. the argument in the proof of [11, Proposition 2.4]) one can find a zero-dimensional scheme $Z \subset Z' \subset C^\nu$ of degree $\deg Z' = \deg Z + 1$ such that the Zariski tangent space to $\mathcal{C}'$ at C is contained in $H^0(C^\nu, \mathcal{J}_{Z'}(C^\nu))$. However, then one derives from (18) that $\deg Z' < -CK_\Sigma$, and hence again $H^1(C^\nu, \mathcal{J}_{Z'}(C^\nu)) = 0$, which in view of (19) will lead to

$$h^0(C^\nu, \mathcal{J}_{Z'}(C^\nu)) = |\overline{\boldsymbol{p}}\setminus\overline{\boldsymbol{p}}'| - 1,$$

which finally implies the emptiness of $\mathcal{C}$. □

3.2 Algebraic Curves over $\mathbb{K}$ and Tropical Curves

If $C \in |\mathcal{L}_\Delta|$ is a curve on the toric surface $\mathrm{Tor}_{\mathbb{K}}(\Delta)$, then the closure $\mathrm{Cl}(\mathrm{Val}(C\cap(\mathbb{K}^*)^2)) \subset \mathbb{R}^2$ supports an EPT-curve T with Newton polygon Δ (cf. Sect. 2.2) defined by a convex piecewise linear function (5) coming from a polynomial equation $F(z) = 0$ of C in $(\mathbb{K}^*)^2$:

$$F(z) = \sum_{\omega\in\Delta\cap\mathbb{Z}^2} A_\omega z^\omega, \quad A_\omega \in \mathbb{K},\ c_\omega = \mathrm{Val}(A_\omega), \quad z \in (\mathbb{K}^*)^2. \qquad (20)$$

The EPT-curve obtained does not depend on the choice of the defining polynomial of C and will be denoted by $\mathrm{Trop}(C)$.

Observe also that the polynomial (20) can be written

$$F(z)=\sum_{\omega\in\Delta\cap\mathbb{Z}^2}\left(a_\omega+O(t^{>0})\right)t^{\nu_T(\omega)}z^\omega$$

with the convex piecewise linear function $\nu:\Delta\to\mathbb{R}$ as in Sect. 2.2 and the coefficients $a_\omega\in\mathbb{C}$ nonvanishing at the vertices of the subdivision S_T of Δ.

3.3 Patchworking Theorems

The algebraically closed version.

Theorem 2. *Given the patchworking data, a PPT-curve $\hat{Q}$, and a configuration $\overline{\boldsymbol{p}}$ satisfying all the conditions of Sect. 3.1, there exists a subset $\mathcal{C}(\hat{Q})\subset\mathcal{C}(\Delta,g,\overline{\boldsymbol{p}},m,\{\beta^\sigma\}_{\sigma\subset\partial\Delta})$ of $M(Q)$ curves C such that $\mathrm{Trop}(C)=h_*Q$. Furthermore, for any distinct (nonisomorphic) curves $\hat{Q}_1$ and $\hat{Q}_2$, the sets $\mathcal{C}(\hat{Q}_1)$ and $\mathcal{C}(\hat{Q}_2)$ are disjoint.*

Remark 9. We would like to underscore one useful consequence of Theorem 2: The PPT-curve Q and the multiplicities of its marked curves must satisfy the restrictions known for the respective algebraic curves with multiple points.

The real version. In addition to all the above hypotheses, we assume the following:

(R6) The configuration $\overline{\boldsymbol{p}}$ is Conj-invariant, $\mathrm{Val}(\Re\overline{\boldsymbol{p}})\cap\mathrm{Val}(\Im\overline{\boldsymbol{p}})=\emptyset$, where $\Re\overline{\boldsymbol{p}}:=\mathrm{Fix}(\mathrm{Conj}\big|_{\overline{\boldsymbol{p}}})$ and $\Im\overline{\boldsymbol{p}}=\overline{\boldsymbol{p}}\setminus(\Re\overline{\boldsymbol{p}})$.

(R7) The PPT-curve Q possesses a real structure $c:Q\to Q$, $G=\Re G\cup\Im G$ such that

- (i) The bijection ψ from (A3) takes $G_\infty\cap\Re G$ into $\Re\overline{\boldsymbol{p}}\cap\mathrm{Tor}_{\mathbb{K}}(\partial\Delta)$ and takes $G_\infty\cap\Im G$ into $\Im\overline{\boldsymbol{p}}\cap\mathrm{Tor}_{\mathbb{K}}(\partial\Delta)$, respectively.
- (ii) $h(G_o\cap\Re G)\subset\mathrm{Val}(\Re\overline{\boldsymbol{p}})$, $h(G_0\cap\Im G\cap\Gamma^0)\subset\mathrm{Val}(\Im\overline{\boldsymbol{p}})$.
- (iii) $\Re G\cap G^{(\mathrm{dm})}\cap\Im\overline{\Gamma}=\emptyset$.
- (iv) If $\gamma\in G_0\cap\Im G\cap\Im\overline{\Gamma}$, $\mathrm{mt}(\gamma)=(1,0)$, then $\mathrm{mt}(c(\gamma))=(0,1)$.
- (v) If $e\in\Gamma^1_\infty$, $e\subset\Re\overline{\Gamma}$, then $w(e)$ is odd.

Theorem 3. *In the notation and hypotheses of Theorem 2 and under assumptions (R1)–(R7), the following holds:*

$$\sum_{C\in\Re\mathcal{C}(\hat{Q})}W_{\Sigma,\overline{\boldsymbol{p}}}(C)=W(Q),\tag{21}$$

where $\Re\mathcal{C}(\hat{Q})$ is the set of real curves in $\mathcal{C}(\hat{Q})$, and $W_{\Sigma,\overline{\boldsymbol{p}}}(C)$ is as defined in (1).

3.4 Proof of Theorem 2

Our argument is as follows. First, we dissipate each multiple point $\boldsymbol{p} \in \overline{\boldsymbol{p}}$ of multiplicity $k > 1$ into k generic simple points (in a neighborhood of $\boldsymbol{p}$), and then, using the known patchworking theorems ([13, Theorem 5] and [16, Theorem 2.4])[6], we obtain $M(Q)$ curves $C \in |\mathcal{L}_\Delta|$ of genus g matching the deformed configuration $\overline{\boldsymbol{q}}$. After that, we specialize the configuration $\overline{\boldsymbol{q}}$ back into the original configuration $\overline{\boldsymbol{p}}$ and show that each of the constructed curves converges to a curve with multiple points and tangencies as asserted in Theorem 2.

Remark 10. The deformation part of our argument works well in a rather more general situation, whereas the degeneration part appears to be more problematic, and at the moment we do not have a unified approach to treating all possible degenerations that may lead to algebraic curves with multiple points.

Following [13, Sect. 3], we obtain the algebraic curves C over $\mathbb{K}$ as germs of one-parameter families of complex curves $C^{(t)}$, $t \in (\mathbb{C}, 0)$, with irreducible fibers $C^{(t)}$, $t \neq 0$, of genus g and a reducible central fiber $C^{(0)}$. The given data of Theorem 2 provide us with a collection of suitable central fibers $C^{(0)}$ out of which we restore the families using the patchworking statement [13, Theorem 5].

Step 1. We start with a simple particular case that later will serve as an element of the proof in the general situation. Assume that Q is a rational, simple, regularly end-marked PPT-curve, $h_*Q \subset \mathbb{R}^2_\Delta$ is a (compactified) nodal embedded plane tropical curve, $\overline{\boldsymbol{p}} \subset \mathrm{Tor}_{\mathbb{K}}(\partial\Delta)$, $G = G_\infty$, $\overline{\boldsymbol{x}} = \overline{\boldsymbol{x}}^\infty$, and $\overline{\boldsymbol{p}} \stackrel{\mathrm{Val}}{\to} \overline{\boldsymbol{x}} \stackrel{\overline{h}}{\leftarrow} G$ are bijections. Here $M(Q)$ is given by formula (9), and this number of required rational curves $C \subset \mathrm{Tor}_{\mathbb{K}}(\Delta)$ is obtained by a direct application of [13, Theorem 5].

The combinatorial part of the patchworking data for the construction of curves over $\mathbb{K}$ consists of the tropical curve Q that defines a piecewise linear function $\nu : \Delta \to \mathbb{R}$ and a subdivision $S: \ \Delta = \Delta_1 \cup \dots \cup \Delta_N$ (see Sect. 2.2). The algebraic part of the patchworking data includes the limit curves $C_k \subset \mathrm{Tor}(\Delta_k)$, the deformation patterns C_e associated with the (finite-length) edges $e \in \Gamma^1$ (see [13, Sect. 5.1] and [16, Sect. 2.1]), and the refined conditions to pass through the fixed points (see [13, Sect. 5.4] and [7, Sect. 2.5.9]).

First, we orient the edges of Γ as in Lemma 3(ii). Then we define complex polynomials f_e, $e \in \overline{\Gamma}^1$, and f_v, $v \in \Gamma^0$, by the following inductive procedure. At the very beginning, for the Γ-ends e with marked points γ, we define

$$f_e(x,y) = (\eta^q x^p - \xi^p y^q)^{w(e)}, \tag{22}$$

where $\boldsymbol{u}(e) = (p,q)$ and (ξ, η) are quasiprojective coordinates of the point $\mathrm{ini}(\boldsymbol{p})$ on $\mathrm{Tor}(e) \subset \mathrm{Tor}(\Delta)$ such that $\gamma = \psi(\boldsymbol{p})$ (here ψ is the bijection from condition (A3) above). We define a linear order on Γ^0 compatible with the orientation of Γ. On each

[6] The complete proof is provided in [16].

stage, we take the next vertex $v \in \Gamma^0$ and define f_v and f_e, where e is the edge emanation from v. Namely, the polynomials f_{e_1}, f_{e_2} associated with the two edges merging to v determine points $z_1 \in \mathrm{Tor}(\sigma_1)$, $z_2 \in \mathrm{Tor}(\sigma_2)$ on the surface $\mathrm{Tor}(\Delta_v)$, where σ_1, σ_2 are the sides of Δ_v orthogonal to $\overline{h}(e_1)$, $\overline{h}(e_2)$, respectively, and we construct a polynomial f_v with Newton polygon Δ_v that defines an irreducible rational curve $C_v \subset \mathrm{Tor}(\Delta_v)$, nonsingular along $\mathrm{Tor}(\partial\Delta_v)$, crossing $\mathrm{Tor}(e_i)$ at z_i, $i = 1, 2$, and crossing $\mathrm{Tor}(e)$ at one point z_0 (at which one has $(C_v \cdot \mathrm{Tor}(e))_{z_0} = w(e)$). By [13, Lemma 3.5], up to a constant factor there are $|\Delta_v|/(w(e_1)w(e_2)) = M(Q, v)$ choices for such a polynomial f_v. After that, we define $f_e(x, y)$ via (22) with ξ, η the (quasihomogeneous) coordinates of z_0 in $\mathrm{Tor}(\sigma)$, where σ is the side of Δ_v orthogonal to $\overline{h}(e)$. Thus the limit curves $C_k \subset \mathrm{Tor}(\Delta_k)$ are C_v for the triangles Δ_k dual to $h(v)$, and they are given by $f_{e_1} f_{e_2}$, where $e_1, e_2 \in \overline{\Gamma}^1$ appear in the decomposition (6) of a parallelogram Δ_k.

The set of limit curves is completed by a set of deformation patterns (see [13, Sects. 3.5 and 3.6]) as follows. Namely, for each edge $e \in \Gamma^1$ with $w(e) > 1$, the deformation pattern is an irreducible rational curve $C_e \subset \mathrm{Tor}(\Delta_e)$, where $\Delta_e := \mathrm{conv}\{(0,1), (0,-1), (w(e), 0)\}$, whose defining (Laurent) polynomial $f_e(x, y)$ has the zero coefficient of $x^{w(e)-1}$ and the truncations to the edges $[(0,1), (w(e), 0)]$ and $[(0,-1), (w(e), 0)]$ of Δ_e fitting the polynomials f_{v_1}, f_{v_2}, where v_1, v_2 are the endpoints of e (see the details in [13, Sects. 3.5 and 3.6]). Recall that by [13, Lemma 3.9], there are $w(e) = M(Q, e)$ suitable polynomials f_e.

The conditions to pass through a given configuration $\overline{\boldsymbol{p}}$ do not admit a refinement. Indeed, following [13, Sect. 5.4], we can turn a given fixed point $\boldsymbol{p}$ into $(\xi, 0)$, $\xi = \xi^0 + O(t^{>0}) \in \mathbb{K}$, by means of a suitable toric transformation. Then, in [13, Formula (6.4.26)], the term with the power $1/m$ will vanish.[7]

The above collections of limit curves and deformation patterns coincide with those considered in [13]; the transversality hypotheses of [13, Theorem 5] are verified in [13, Sect. 5.4]. Hence, each of the

$$\prod_{v \in \Gamma^0} M(Q, v) \cdot \prod_{e \in \Gamma^1} M(Q, e) = M(Q)$$

above patchworking data gives rise to a rational curve $C \subset \mathrm{Tor}_{\mathbb{K}}(\Delta)$ as asserted in Theorem 2. Notice that all these curves are nodal by construction.

Step 2. Now we return to the general situation and deform the given configuration $\overline{\boldsymbol{p}}$ into the following new configuration $\overline{\boldsymbol{q}}$.

We replace each point $\boldsymbol{p} = (\xi t^a + \cdots, \eta t^b + \cdots) \in \overline{\boldsymbol{p}} \cap (\mathbb{K}^*)^2$ with multiplicity $\mu(\boldsymbol{p}) > 1$ (defined by (13) or (14)) by $\mu(\boldsymbol{p})$ generic points in $(\mathbb{K}^*)^2$ with the same valuation image $\mathrm{Val}(\boldsymbol{p}) = (-a, -b)$ and the initial coefficients of the coordinates

[7] The mentioned term contains η_s^0, the initial coefficient of the second coordinate of $\boldsymbol{p}$, and not ξ_s^0 as appears in the published text. The correction is clear, since in the preceding formula for τ one has just η_s^0.

close to $\xi,\eta \in \mathbb{C}^*$, respectively. Furthermore, we extend the bijection ψ from (A3) up to a map $\psi : \overline{\boldsymbol{q}} \to G$ in such a way that

- $\mathrm{Val}\big|_{\overline{\boldsymbol{q}}} = \overline{h} \circ \psi$.
- Each point $\gamma \in G \setminus \Gamma^0$ has a unique preimage, and each point $\gamma \in G \cap \Gamma^0$ has precisely two preimages.
- If $\gamma \in G^{(\mathrm{dm})} \setminus \Gamma^0$, $h(\gamma) = \boldsymbol{x}$, then $\psi^{-1}(\gamma)$ is close to $\boldsymbol{p}_{1,\boldsymbol{x}}$ or to $\boldsymbol{p}_{2,\boldsymbol{x}}$ according to whether $\mathrm{mt}(\gamma) = (1,0)$ or $(0,1)$.
- If $\gamma \in G^{(\mathrm{dm})} \cap \Gamma^0$, $h(\gamma) = \boldsymbol{x}$, then $\psi^{-1}(\gamma)$ consists of two points, one close to $\boldsymbol{p}_{1,\boldsymbol{x}}$ and the other close to $\boldsymbol{p}_{2,\boldsymbol{x}}$.

Next we construct a set $\mathcal{C}' \subset \mathcal{C}(\Delta, g, \overline{\boldsymbol{q}}, 1, \{\beta^\sigma\}_{\sigma\subset\partial\Delta})$ of $M(Q)$ curves with the tropicalization h_*Q. By Lemma 8, they are irreducible, nodal, of genus g, and with specified tangency conditions along $\mathrm{Tor}_{\mathbb{K}}(\partial\Delta)$.

Step 3. Similarly to Step 1, we obtain the limit curves from a collection of polynomials in $\mathbb{C}[x,y]$ associated with the edges and vertices of the parameterizing graph Γ of Q:

(i) Let $\gamma \in G \setminus \Gamma^0$ lie on the edge $e \in \overline{\Gamma}^1$. Then we associate with the edge e a polynomial $f_e(x,y)$ given by (22) with the parameters described in Step 1.

(ii) Let $\gamma \in G_0$ be a (trivalent) vertex v of Γ. Then $f_v(x,y)$ is a polynomial with Newton triangle Δ_v (see Sect. 2.2) defining in $\mathrm{Tor}(\Delta_v)$ a rational curve $C_v \in |\mathcal{L}_{\Delta_v}|$ that crosses each toric divisor of $\mathrm{Tor}(\Delta_v)$ at one point, where it is nonsingular, and that passes through the two points $\mathrm{ini}(\psi^{-1}(\gamma))$. Observe that by [15, Lemma 2.4], up to a constant factor there are precisely $|\Delta_v|$ polynomials f_v as above. (Although the assertion and the proof of [15, Lemma 2.4] are restricted to the real case, the proof works well in the same manner in the complex case regardless of the parity of the side length of Δ_v.)

(iii) Edges emanating from a vertex $v \in \Gamma^0 \cap G_0$ do not contain any other point of G due to the Δ-general position, and we define polynomials f_e for them by formula (22), where ξ,η are the (quasihomogeneous) coordinates of the intersection point of C_v with $\mathrm{Tor}(\sigma)$, with σ the side of Δ_v orthogonal to $\overline{h}(e)$.

(iv) Pick a connected component K of $\overline{\Gamma} \setminus G$ and orient it as in Lemma 3(ii). Then we inductively define polynomials for the vertices and closed edges of K: In each stage we define polynomials f_v and f_e for a vertex v and a simple closed edge e emanating from v, whereas the polynomials $f_{e'}$ for all the edges e' of K merging to v are given. Each of the latter polynomials defines a point on $\mathrm{Tor}(\partial\Delta_v)$, and these points are distinct. We denote their set by X. Then we choose a polynomial $f_v(x,y)$ with Newton triangle Δ_v defining an irreducible rational curve $C_v \subset \mathrm{Tor}(\Delta_v)$ that

   - is nonsingular along $\mathrm{Tor}(\partial\Delta_v)$
   - crosses $\mathrm{Tor}(\partial\Delta_v)$ at each point $z \in X$ with multiplicity $w(e')$, where the edge $e' \in \overline{\Gamma}^1$ merging to v is associated with a polynomial $f_{e'}$ that determines the point z
   - crosses $\mathrm{Tor}(\partial\Delta_v) \setminus X$ at precisely one point z_0.

Notice that z_0 is the unique intersection point of C_v with the toric divisor $\text{Tor}(\sigma) \subset \text{Tor}(\Delta_v)$, where σ is orthogonal to $\overline{h}(e)$, and $(C_v \cdot \text{Tor}(\sigma))_{z_0} = w(e)$. We claim that up to a constant factor there are precisely $M(Q,v)$ polynomials f_v, as required.

The case of a trivalent vertex v was considered in Step 1. In general, observe that the set of the required curves is finite, since we impose

$$(C_v \cdot \text{Tor}(\partial\Delta_v)) - 1 = -C_v K_{\text{Tor}(\Delta_v)} - 1$$

conditions on the rational curves $C_v \in |\mathcal{L}_{\Delta_v}|$, and the conditions are independent by Riemann–Roch. The cardinality of this set depends neither on the choice of a generic configuration of fixed points on $\text{Tor}(\partial\Delta_v)$ nor on the choice of an algebraically closed ground field of characteristic zero. Thus, we consider the field $\mathbb{K}$ and pick the fixed points on $\text{Tor}_{\mathbb{K}}(\partial\Delta_v)$ so that the valuation takes them injectively to a Δ_v-generic configuration in $\partial\mathbb{R}^2_{\Delta_v}$. Then the rule (M5) and the construction in Step 1 provide $M(Q,v)$ curves as required. The fact that there are no other curves under consideration follows from a slightly modified Mikhalkin's correspondence theorem (for details, see, for instance, [18]).

We then define $f_e(x,y)$ via (22) with ξ,η the (quasihomogeneous) coordinates of z_0 in $\text{Tor}(\sigma)$.

Summarizing, we deduce that the number of choices of the curves C_v, $v \in \Gamma^0$, and C_e, $e \in \overline{\Gamma}^1$, is

$$\prod_{v\in\Gamma^0} M(Q,v).$$

Step 4. Now we define the limit curves, the deformation patterns, and the refined conditions to pass through fixed points.

For each polygon Δ_k of the subdivision S of Δ, the limit curve $C_k \subset \text{Tor}(\Delta_k)$ is defined by the product of the polynomials f_v, f_e constructed above corresponding to the summands in the decomposition (6) of Δ_k.

The deformation pattern for each edge $e \in \Gamma^1$ such that $w(e) > 1$ is defined in the way described in Step 1.

Finally, the condition to pass through a given point $\boldsymbol{q} \in \overline{\boldsymbol{q}}$ such that $\gamma = \psi(\boldsymbol{q}) \in G$ lies in the interior of an edge $e \in \overline{\Gamma}^1$ with $w(e) > 1$ admits a refinement (see [13, Sects. 5.4] and [7, 2.5.9]) that in turn is defined up to the choice of a $w(e)$th root of unity, where $e \in \overline{\Gamma}^1$ contains γ.

So, the total number of choices we made up to now is

$$\prod_{v\in\Gamma^0} M(Q,v) \cdot \prod_{e\in\Gamma^1} M(Q,e) \cdot \prod_{\gamma\in G} M(Q,\gamma) = M(Q).$$

Step 5. Let us verify the hypotheses of the patchworking theorem from [13, 16].

First, the requirement on the limit curves (see [13], conditions (A), (B), (C) in Sect. 5.1, or [16], conditions (C1), (C2) in Sect. 2.1) is ensured by the generic choice of $\mathrm{ini}(\boldsymbol{q})$, $\boldsymbol{q} \in \overline{\boldsymbol{q}}$. Namely, the limit curves do not contain multiple nonbinomial components (i.e., defined by polynomials with nondegenerate Newton polygons), any two distinct components of any limit curve $C_k \subset \mathrm{Tor}(\Delta_k)$ intersect transversally at nonsingular points all of which lie in the big torus $(\mathbb{C}^*)^2 \subset \mathrm{Tor}(\Delta_k)$, and finally, the intersection points of any component of a limit curve C_k with $\mathrm{Tor}(\partial\Delta_k)$ are nonsingular.

The main requirement is the transversality condition for the limit curves and deformation patterns (see [13, Sect. 5.2] and [16, Sect. 2.2]), which is relative to the choice of an orientation of the edges of the underlying tropical curve. In [13, 16], one considers an orientation of edges of the embedded plane tropical curve (cf. Sect. 2.2), which in our setting is just $h_*(Q) \subset \mathbb{R}^2$. Here we consider the orientation of the edges of the connected components of $\overline{\Gamma} \setminus G$ as defined in Lemma 3(ii). Since this orientation does not define oriented cycles and since the intersection points of distinct components of any limit curve with toric divisors are distinct, the proof of [13, Theorem 5] and [16, Theorem 2.4] with the orientation of Γ is a word-for-word copy of the proof with the orientation of $h(\Gamma)$. Moreover, comparing with [13, 16], here we impose extra conditions to pass through the points $\mathrm{ini}(\boldsymbol{q})$, $\boldsymbol{q} \in \overline{\boldsymbol{q}}$.

The deformation patterns are transversal in the sense of [13, Definition 5.2], due to [13, Lemma 5.5(ii)], where both inequalities hold, since the deformation patterns are nodal ([13, Lemma 3.9]) and thus do not contribute to the left-hand side of the inequalities, whereas their right-hand sides are positive.

The transversality of the limit curve $C_k \subset \Sigma_k := \mathrm{Tor}(\Delta_k)$ in the sense of [13, Definition 5.1] means the triviality (i.e., zero-dimensionality) of the Zariski tangent space at C_k to the stratum in $|\mathcal{L}_{\Delta_k}|$ formed by the curves that split into the same number of rational components as C_k (i.e., the components of C_k do not glue up when one is deforming along such a stratum), each of them having the same number of intersection points with $\mathrm{Tor}(\partial\Delta_k)$ as the respective component of C_k and with the same intersection number, and such that all but one of these intersection points are fixed. In other words, the conditions imposed on each of the components of C_k determine a stratum with the one-point Zariski tangent space. Indeed, the above fixation of intersection numbers of a component C' of C_k and all but one intersection point of C' with $\mathrm{Tor}(\partial\Delta_k)$ has imposed $-C'K_{\Sigma_k} - 1$ conditions, all of which are independent due to Riemann–Roch on C'.

Thus, [13, Theorem 5] applies, and each of the $M(Q)$ refined patchworking data constructed above produces a curve $C \subset \mathrm{Tor}_{\mathbb{K}}(\Delta)$ as asserted in Theorem 2.

Step 6. Now we specialize the configuration $\overline{\boldsymbol{q}}$ to $\overline{\boldsymbol{p}}$ and prove that each of the curves $C \in \mathcal{C}'$ constructed above tends (in an appropriate topology) to some curve $\hat{C} \in |\mathcal{L}_\Delta|$.

To obtain the required limits, we introduce a suitable topology. Since the variation of $\overline{\boldsymbol{q}}$ does not affect its valuation image, the same holds for the (variable) curves $C' \in \mathcal{C}'$, and hence one can fix once and for all the function $\nu : \Delta \to \mathbb{R}$. Then, writing each coordinate of any point $\boldsymbol{q} \in \overline{\boldsymbol{q}}$ as $X = t^{-\mathrm{Val}(X)}\Psi_{\boldsymbol{q},X}(t)$ and each coefficient of the defining polynomials of C as $A_\omega = t^{\nu(\omega)}\Psi_\omega(t)$, and assuming (without loss of generality) that all the exponents of t in the above coordinates and coefficients are integral, we deal with the following topology in the space of the functions $\psi_\omega(t)$ holomorphic in a neighborhood of zero: Take the C^0 topology in each subspace consisting of the functions convergent in $|t| \le \varepsilon$ and then define the inductive limit topology in the whole space.

So we assume that the variation of $\overline{\boldsymbol{q}}$ reduces to only variation of $\mathrm{ini}(\boldsymbol{q})$, $\boldsymbol{q} \in \overline{\boldsymbol{q}} \cap (\mathbb{K}^*)^2$, whereas the reminders of the corresponding series in t are unchanged.

To show that the families of the curves $C \in \mathcal{C}'$ have limits, we recall that their coefficients appear as solutions to a system of analytic equations that is soluble by the implicit function theorem due to the transversality of the initial (refined) patchworking data (cf. [13, 16]). Thus, to confirm the existence of the limits of the curves $C \in \mathcal{C}'$, it is sufficient to show that the system of equations and the (refined) patchworking data have limits and that the latter limit is transverse. In particular, we shall obtain that in each coefficient $A_\omega = t^{\nu(\omega)}\Psi_\omega(t)$, $\omega \in \Delta \cap \mathbb{Z}^2$, the factor Ψ_ω converges uniformly in the family.

We start with analyzing the specialization of limit curves. Since the given tropical curve Q stays the same, we go through the curves C_v, $v \in \Gamma^0$, and C_e, $e \in \overline{\Gamma}^1$. Clearly, the curves C_e, $e \in \overline{\Gamma}^1$, keep their form (22) with the parameters ξ, η possibly changing as $\mathrm{ini}(\boldsymbol{q})$ tends to $\mathrm{ini}(\boldsymbol{p})$, $\boldsymbol{p} \in \overline{\boldsymbol{p}}$. Similarly, the curves C_v corresponding to the vertices $v \in \Gamma^0$ of valency 3 remain as described in Step 1, i.e., nodal nonsingular along $\mathrm{Tor}(\partial\Delta_v)$, and crossing each toric divisor at one point. Furthermore, the curves C_v corresponding to the nonspecial vertices $v \in \Gamma^0$ of valency greater than 3, remain as described in Step 3, paragraph (iv), since the intersection points of C_v with the toric divisors that correspond to the edges of $\overline{\Gamma}$, merging to v, do not collate and remain generic in the specialization, since they are not affected by possible collisions of the points $\mathrm{ini}(\boldsymbol{q})$, $\boldsymbol{q} \in \overline{\boldsymbol{q}}$. So let us consider the case of a special vertex $v \in \Gamma^0$. By (T4), C_v cannot split into proper components, and hence it specializes to an irreducible rational curve. Furthermore, the intersection points of C_v with toric divisors that correspond to the special edges may collate, forming singular points, centers of several smooth branches.

So finally, the transversality conditions for such a curve reduce to the fact that the Zariski tangent space at C_v to the stratum in $|\mathcal{L}_{\Delta_v}|$ consisting of rational curves with given intersection points along the two toric divisors that are related to the oriented edges of $\overline{\Gamma}$ merging to v is zero-dimensional. These are precisely the same stratum conditions as in Step 5, and the argument of Step 5 (Riemann–Roch on the rational curve C_v) shows that all the $-C_vK_{\mathrm{Tor}(\Delta_v)} - 1$ conditions defining the stratum in the Severi variety parameterizing the rational curves in $|\mathcal{L}_{\Delta_v}|$ are independent.

Next, we notice that by assumption (T4), the possible collision of intersection points of C_v with $\mathrm{Tor}(\partial\Delta_v)$ concerns only transverse intersection points (i.e., those that correspond to edges of weight 1), and hence affects neither the deformation patterns nor the refined conditions to pass through $\overline{\boldsymbol{p}}$. Thus, each of the curves $C \in \mathcal{C}'$ degenerates into some curve $\hat{C} \in |\mathcal{L}_\Delta|$ that is given by a polynomial with coefficients $A_\omega = t^{\nu(\omega)}\Psi_\omega(t)$, $\omega \in \Delta$, containing factors Ψ_ω convergent uniformly in some neighborhood of 0 in $\mathbb{C}$.

Remark 11. (1) Observe that the genus of $\hat{C}$ does not exceed the genus of C.

(2) Notice also that there is no need to study refinements of possible singular points appearing in the above collisions of the intersection points of C_v with $\mathrm{Tor}(\partial\Delta_v)$. Indeed, the number of transverse conditions we found equals the number of parameters; hence no extra ramification is possible.

Step 7. Next we show that at each point $\boldsymbol{p} \in \overline{\boldsymbol{p}}$ with $\mu(\boldsymbol{p}) > 1$, the obtained curve $\hat{C}$ has $\mu(\boldsymbol{p})$ local branches.

Considering the point $\boldsymbol{p} \in (\mathbb{K}^*)^2$ as a family of points $\boldsymbol{p}^{(t)} \in (\mathbb{C}^*)^2$, $t \neq 0$, we claim that the curves $\hat{C}^{(t)} \subset \mathrm{Tor}(\Delta)$ have $\mu(\boldsymbol{p})$ branches at $\boldsymbol{p}^{(t)}$, $t \neq 0$. Indeed, we will describe how to glue up the limit curves forming $\hat{C}^{(0)}$ when $\hat{C}^{(0)}$ deforms into $\hat{C}^{(t)}$, $t \neq 0$. Our approach is to compare the above gluing with the gluing of the limit curves in the deformation of $C^{(0)}$ into $C^{(t)}$, $t \neq 0$, where $C \in \mathcal{C}'$ passes through the configuration $\overline{\boldsymbol{q}}$, and this comparison heavily relies on the one-to-one correspondence between the limit curves of $\hat{C}$ and $C \in \mathcal{C}'$ established in Step 6.

Let $\boldsymbol{q}_1, \ldots, \boldsymbol{q}_s$ be all the points of the configuration $\overline{\boldsymbol{q}}$ that appear in the dissipation of the point $\boldsymbol{p} \in \overline{\boldsymbol{p}}$ (cf. Step 2), and let $\gamma_i = \psi(\boldsymbol{q}_i)$, $i = 1, \ldots, s$, be the corresponding marked points on Γ, so that $\gamma_i \in e_i \in \Gamma^1$, $i = 1, \ldots, s$. If the edges $h(e_i), h(e_j)$ intersect transversally at $V = h(\gamma_i) = h(\gamma_j)$, then V is a vertex of the plane tropical curve $h_*(Q)$ dual to a polygon Δ_V of the corresponding subdivision of Δ. The components $C_i, C_j \subset \mathrm{Tor}(\Delta_V)$ of the curve $C^{(0)}$ passing through $\mathrm{ini}(\boldsymbol{q}_i), \mathrm{ini}(\boldsymbol{q}_j) \in (\mathbb{C}^*)^2 \subset \mathrm{Tor}(\Delta_V)$, respectively, intersect transversally in $(\mathbb{C}^*)^2$, and their intersection points in $(\mathbb{C}^*)^2$ do not smooth up in the deformation $C^{(t)}$, $t \neq 0$, and the same holds for the corresponding components $\hat{C}_i, \hat{C}_j$ of $\hat{C}^{(0)}$ meeting at $\mathrm{ini}(\boldsymbol{p}) \in (\mathbb{C}^*)^2 \subset \mathrm{Tor}(\Delta_V)$, since the smoothing out of an intersection point $\mathrm{ini}(\boldsymbol{p})$ of $\hat{C}_i$ and $\hat{C}_j$ would raise the genus of $\hat{C}$ above the genus of C, contrary to Remark 11.

Suppose that in the above notation, $h(e_i)$ and $h(e_j)$ lie on the same straight line, but e_i, e_j have no vertex in common (see Fig. 3a). We consider the case of finite-length edges e_i, e_j; the case of ends can be treated similarly. Let v_i, v_i' be the vertices of e_i, and let v_j, v_j' be the vertices of e_j. Their dual polygons $\Delta_{v_i}, \Delta_{v_i'}, \Delta_{v_j}, \Delta_{v_j'}$ (see Sect. 2.2) have sides E_i, E_i', E_j, E_j' orthogonal to $h(e_i)$. In the deformation $C^{(0)} \to C^{(t)}$, $t \neq 0$, the limit curves $C_i \subset \mathrm{Tor}(\Delta_{v_i})$ and $C_i' \subset \mathrm{Tor}(\Delta_{v_i'})$ passing through $\mathrm{ini}(\boldsymbol{q}_i) \in \mathrm{Tor}(E_i) = \mathrm{Tor}(E_i')$ glue up to form a branch centered at $\boldsymbol{q}_i^{(t)}$, and similarly the limit curves $C_j \subset \mathrm{Tor}(\Delta_{v_j})$ and $C_j' \subset \mathrm{Tor}(\Delta_{v_j'})$ passing through $\mathrm{ini}(\boldsymbol{q}_j) \in \mathrm{Tor}(E_j) = \mathrm{Tor}(E_j')$ glue up to form a branch centered at $\boldsymbol{q}_j^{(t)}$. The same happens when C specializes to $\hat{C}$ and $\boldsymbol{q}_i, \boldsymbol{q}_j$ specialize to $\boldsymbol{p}$, since again the aforementioned restriction $g(\hat{C}) \leq g(C)$ does not allow the limit curves $\hat{C}_i, \hat{C}_i'$ to glue up with the limit curves $\hat{C}_j, \hat{C}_j'$.

The remaining case to study is given by the tropical data described in condition (T7), Sect. 3.1. Without loss of generality we can assume that all the edges $e_1, \ldots, e_s$ have a common vertex v and that their h-images lie on the same line (see an example in Fig. 3b). Applying an appropriate invertible integral–affine transformation, we

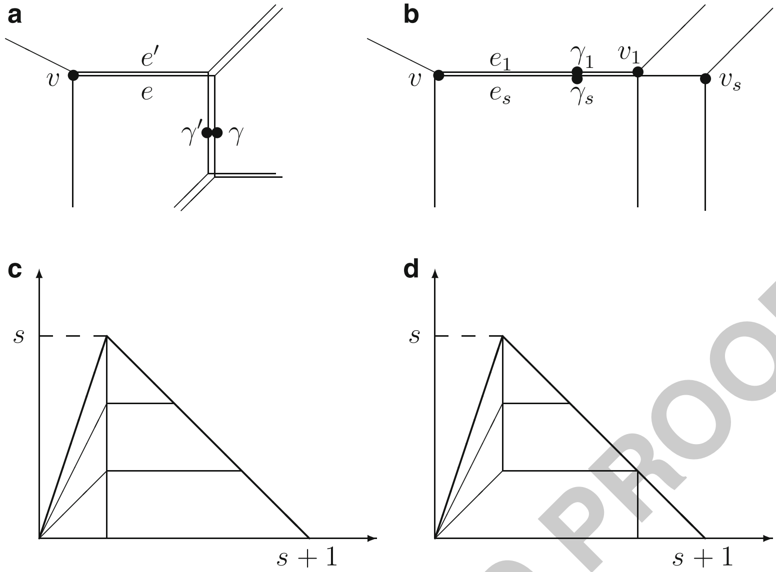

Fig. 3 Illustration to Step 7 of the proof of Theorem 2

can make the edges $e_1,\dots,e_s$ horizontal and the point $\boldsymbol{x} = \mathrm{Val}(\boldsymbol{p}) \in \mathbb{R}^2$ to be the origin. Correspondingly, $v = (-\alpha, 0)$, $v_i = (\alpha_i, 0)$, $i = 1,\dots,s$, with $0 < \alpha_1 \le \dots \le \alpha_s \le \infty$ and

$$\alpha > \sum_{1 \le i < s-1} \alpha_i + 2\alpha_{s-1}. \qquad (23)$$

In what follows we suppose that $\alpha_s < \infty$. The case $\alpha_s = \infty$ admits the same treatment as the case of finite $\alpha_s \gg \alpha$.

Let $\boldsymbol{q}_i = \psi^{-1}(\gamma_i)$, $1 \le i \le s$, be the points of the configuration $\overline{\boldsymbol{q}}$ that appear in the deformation of the point $\boldsymbol{p}$ described in Step 2. Our assumptions yield that

$$\boldsymbol{p} = (\xi + O(t^{>0}), \eta + O(t^{>0})), \quad \boldsymbol{q}_i = (\xi_i + O(t^{>0}), \eta_i + O(t^{>0})),\ i = 1,\dots,s,$$

with some $\xi, \eta \in \mathbb{C}^*$, ξ_i close to ξ, η_i close to η, $i = 1,\dots,s$. Furthermore, the triangles Δ_v and Δ_{v_i} dual to the vertices v and v_i, $1 \le i \le s$, respectively, have vertical edges $\sigma \subset \partial\Delta_v$ and $\sigma_i \subset \partial\Delta_{v_i}$, $1 \le i \le s$, along which the function ν (see Sect. 2.2) is constant. By assumptions (T4)–(T6), the limit curve $C_v \subset \mathrm{Tor}(\Delta_v)$ crosses the toric divisor $\mathrm{Tor}(\sigma)$ at the points $\eta_1,\dots,\eta_s$ with total intersection multiplicity s, and each of the limit curves $C_{v_i} \subset \mathrm{Tor}(\Delta_{v_i})$, $1 \le i \le s$, crosses the toric divisor $\mathrm{Tor}(\sigma_i)$ at the unique point η_i transversally, and the corresponding limit curve $\hat{C}_{v_i}$ crosses $\mathrm{Tor}(\sigma_i)$ at the point η transversally, too.

Now we move the points $\boldsymbol{q}_1,\dots,\boldsymbol{q}_s$, keeping their x-coordinates and making $(\boldsymbol{q}_1)_y=\dots=(\boldsymbol{q}_s)_y=(\boldsymbol{p})_y$. As shown in Step 6, the curve C (depending on $\boldsymbol{q}_1,\dots,\boldsymbol{q}_s$) converges to a curve C' with the same Newton polygon, genus, and tropicalization, and the limit curves of C componentwise converge to limit curves of C'. Consider now the polynomial $\widetilde{F}(x,y):=F'(x,y+(\boldsymbol{p})_y)$, where the polynomial $F'(x,y)$ defines the curve C'. As in the refinement procedure described in [13, Sect. 3.4] or [7, Sect. 2.5.8], the subdivision of the Newton polygon $\widetilde{\Delta}$ of $\widetilde{F}$ contains the fragment bounded by the triangle $\delta=\operatorname{conv}\{(0,0),(1,s),(s+1,0)\}$ (see Fig. 3d), which matches the points $\boldsymbol{q}_1,\dots,\boldsymbol{q}_s$. The corresponding function $\widetilde{\nu}:\widetilde{\Delta}\to\mathbb{R}$ takes the values

$$\widetilde{\nu}(0,0)=\alpha,\quad \widetilde{\nu}(1,s)=0,\quad \widetilde{\nu}(k,s+1-k)=\sum_{1\le i<k}\alpha_i,\ k=2,\dots,s+1,$$

along the inclined part of $\partial\delta$. The tropical limit of $\widetilde{F}$ restricted to the above fragment consists of a subdivision of δ determined by some extension of the function $\widetilde{\nu}$ inside δ and of limit curves that must meet the following conditions:

- These limit curves glue up into a rational curve (with Newton triangle δ), since in the original tropical curve, the aforementioned fragment corresponds to a tree (see Fig. 3b).
- The intersection points $\boldsymbol{q}$ of the curve $C_\delta:=\{\widetilde{F}_\delta=0\}$ with the line $x=(\boldsymbol{p})_x$ such that $\operatorname{Val}(\boldsymbol{q})_y\le 0$ converge to $\boldsymbol{p}$ as $\boldsymbol{q}_1,\dots,\boldsymbol{q}_s$ tend to $\boldsymbol{p}$, where $\widetilde{F}_\delta$ is the sum of the monomials of $\widetilde{F}$ matching the set $\Delta\cap\mathbb{Z}^2$, and the convergence is understood in the topology of Step 6.
- The subdivision of δ contains a segment $\widetilde{\sigma}$ of length s lying inside the edge $[(0,0),(s+1,0)]$, along which the function $\widetilde{\nu}$ is constant and such that the corresponding toric divisor $\operatorname{Tor}(\widetilde{\sigma})$ intersects the limit curves at the points $\xi_1,\dots,\xi_s$.

These restrictions and inequality (23) leave only one possibility: the subdivision of δ shown in Fig. 3c,d (the subdivision (c) for the case $\alpha>\alpha_1+\dots+\alpha_s$, and the subdivision (d) for the case $\alpha<\alpha_1+\dots+\alpha_s$). The limit curve $C_{\delta'}\subset\operatorname{Tor}(\delta')$ for a triangle $\delta'\subset\delta$ having a horizontal base splits into $H(\delta)$ distinct straight lines (any of them crossing each toric divisor at one point), where $H(\delta')$ is the height. The limit curve $\mathbb{C}_{\delta'}\subset\operatorname{Tor}(\delta')$ for a trapeze $\delta'\subset\delta$ splits into $H(\delta')$ straight lines as above and a suitable number of straight lines $x=\mathrm{const}$ (which reflect the splitting of the trapeze into the Minkowski sum of a triangle with a horizontal segment). All the limit curves are uniquely defined by the intersections with the toric divisors $\operatorname{Tor}(\sigma')$ for inclined segments σ' (in our construction, these data are determined by the points $\overline{\boldsymbol{q}}\setminus\{\boldsymbol{q}_1,\dots,\boldsymbol{q}_s\}$ and by the condition $(\boldsymbol{q}_1)_y=\dots=(\boldsymbol{q}_s)_y=0)$ and by intersections with $\operatorname{Tor}(\widetilde{\sigma})$ introduced above. When $\boldsymbol{q}_1,\dots,\boldsymbol{q}_s$ tend to $\boldsymbol{p}$, the subdivision of δ remains unchanged, whereas the limit curves naturally converge componentwise. Then we immediately derive that components of the limit curves passing through $\operatorname{ini}(\boldsymbol{p})$ do not glue up together in the deformation $\hat{C}^{(t)}$, $t\in(\mathbb{C},0)$, since otherwise, the (geometric) genus of $\hat{C}^{(t)}$, $t\ne 0$, would jump above the genus of C, which is impossible (see Remark 11).

Step 8. By assumption (A5), Sect. 3.1, the curves $\hat{C} \subset \mathrm{Tor}_{\mathbb{K}}(\Delta)$ are immersed, irreducible, of genus g, have multiplicity $\mu(\boldsymbol{p})$ at each point $\boldsymbol{p} \in \overline{\boldsymbol{p}} \cap (\mathbb{K}^*)^2$, and satisfy the tangency conditions with $\mathrm{Tor}_{\mathbb{K}}(\partial\Delta)$ as specified in the assertion of Theorem 2. It remains to show that we have constructed precisely $M(Q)$ curves $\hat{C}$.

Indeed, condition (16) implies that for any dissipation of each point $\boldsymbol{p} \in \overline{\boldsymbol{p}} \cap (\mathbb{K}^*)^2$ into $\mu(\boldsymbol{p})$ distinct points there exists a unique deformation of $\hat{C}$ into a curve $C \in \mathcal{C}$ such that a priori prescribed branches of $\hat{C}$ at $\boldsymbol{p}$ will pass through prescribed points of the dissipation.

Finally, we notice that the sets $\mathcal{C}(\hat{Q}_1)$ and $\mathcal{C}(\hat{Q}_2)$ are disjoint for distinct (nonisomorphic) PPT-curves $\hat{Q}_1, \hat{Q}_2$. Indeed, the collections of limit curves as constructed in Steps 1 and 3 appear to be distinct for distinct curves $\hat{Q}_1$ and $\hat{Q}_2$ and the given configuration $\overline{\boldsymbol{p}}$. □

3.5 *Proof of Theorem 3*

The curves $C \in \Re\mathcal{C}(\hat{Q})$ constructed in the proof of Theorem 2 are immersed, and hence the formula (1) for the Welschinger weight applies. Thus the left-hand side of (21) is well defined.

Next we go through the proof of Theorem 2, counting the contribution to the right-hand side of (21).

First, we deform the configuration $\overline{\boldsymbol{p}}$ as described in Step 2, assuming that the deformed configuration $\overline{\boldsymbol{q}}$ is Conj-invariant and that the map $\psi : \overline{\boldsymbol{q}} \to G$ sends $\Re\overline{\boldsymbol{q}} = \overline{\boldsymbol{q}} \cap \mathrm{Fix}(\mathrm{Conj})$ to $\Re G$ and sends $\Im\overline{\boldsymbol{q}} = \overline{\boldsymbol{q}} \setminus \Re\overline{\boldsymbol{q}}$ to $\Im G$, respectively. In particular, if $\boldsymbol{p} \in \Re\overline{\boldsymbol{p}}$, and the points $\mathrm{Val}(\boldsymbol{p})$ is an image of r points of $\Re G \cap \Re\overline{\Gamma}$ and s pairs of points of $\Re G \cap \Im\overline{\Gamma}$, then $\boldsymbol{p}$ deforms into r real points and s pairs of imaginary conjugate points.

Notice that the replacement of $\overline{\boldsymbol{p}}$ by $\overline{\boldsymbol{q}}$ causes a change of sign in the left-hand side of (21) and a change in the quantity of the real curves in the count on the right-hand side of (21). We now explain the change of sign: The dissipation of a real point $\boldsymbol{p}$ as in the preceding paragraph means that for each curve $C \in \mathcal{C}'$, we count an additional r real solitary nodes in a neighborhood of $\boldsymbol{p}$, since in the nondeformed situation, the point $\boldsymbol{p}$ should be blown up for the computation of the Welschinger sign. This change is reflected in the sign $(-1)^{\ell_1}$, $\ell_1 = |\Re G \cap \Im\overline{\Gamma}|/2$, in the right-hand side of formula (10).

Next we follow the procedure in Steps 3 and 4 of the proof of Theorem 2 and construct Conj-invariant collections of limit curves, deformation patterns, and refined conditions to pass through fixed points:

- By [13, Proposition 8.1(i)], the existence of an even-weight edge $e \in \Gamma^1$, $e \subset \Re\overline{\Gamma}$, annihilates the contribution to the Welschinger number, and hence by (R7)(v), we can assume that all the edges $e \subset \Re\overline{\Gamma}$ have odd weight. In particular, with the finite-length edges $e \subset \Re\overline{\Gamma}$ one can associate a unique real deformation pattern with an even number of solitary nodes (cf. [15, Lemma 2.3]).

- The limit curves associated with the vertices of $\Re\overline{\Gamma}$ contribute as designated in rules (W2)–(W4) in Sect. 2.6 (cf. [15, Lemmas 2.3, 2.4, and 2.5]).
- The construction of limit curves and deformation patterns associated with the vertices and edges of $\Im\overline{\Gamma}'$ (a half of $\Im\overline{\Gamma}$) contributes as designated in rules (W1)–(W3) (cf. the complex formulas in the proof of Theorem 2 and [15, Sect. 2.5]). Accordingly, the data associated with $\Im\overline{\Gamma}''$ are obtained by conjugation.
- The refinement of the condition to pass through fixed points contributes as designated in rule (W2), since we have a unique refinement for $\gamma \in \Re\overline{\Gamma}$ and $w(e)$ refinements for $\gamma \in e \in \Im\overline{\Gamma}^1$.

The remaining step is to explain the factor 2^{ℓ_2}, $\ell_2 = (|\Im G \cap \Im\overline{\Gamma}| - b_0(\Im\overline{\Gamma}))/2$, in formula (10). Indeed, when constructing the limit curves associated with the vertices of $\Im\overline{\Gamma}'$, we start with the respective fixed points, which are all are imaginary in the configuration $\overline{\boldsymbol{q}}$, and thus we choose a point in each of the $|G \cap \Im\overline{\Gamma}'| = |G \cap \Im\overline{\Gamma}|/2$ pairs of the corresponding points in $\overline{\boldsymbol{q}}$.

Observe that in the degeneration $\overline{\boldsymbol{q}} \to \overline{\boldsymbol{p}}$, $|\Re G \cap \Im\overline{\Gamma}'|$ pairs ofs imaginary points of $\overline{\boldsymbol{q}}$ merge to real points in $\overline{\boldsymbol{p}}$, which leaves only $|\Im G \cap \Im\overline{\Gamma}|/2$ choices in the original configuration $\overline{\boldsymbol{p}}$. After all, we factorize by the interchange of the components of $\Im\overline{\Gamma}$, arriving at the required factor 2^{ℓ_2}. □

Acknowledgments The author was supported by the grant 465/04 from the Israel Science Foundation, a grant from the Higher Council for Scientific Cooperation between France and Israel, and a grant from Tel Aviv University. This work was completed during the author's stay at the Centre Interfacultaire Bernoulli, École Polytechnique Fédérale da Lausanne and at Laboratoire Émile Picard, Université Paul Sabatier, Toulouse. The author thanks the CIB-EPFL and UPS for the hospitality and excellent working conditions. Special thanks are due to I. Itenberg, who pointed out a mistake in the preliminary version of Theorem 3. Finally, I express my gratitude to the unknown referee for numerous remarks, corrections, and suggestions.

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