Khovanov Homology Theories and Their Applications

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To my teacher and advisor, Oleg Yanovich Viro, on the occasion of his 60th birthday

Abstract This is an expository paper discussing various versions of Khovanov homology theories, interrelations between them, their properties, and their applications to other areas of knot theory and low-dimensional topology.

Keywords Khovanov homology • Categorification • Writhe number • Kauffman state • Link • Homological width

1 Introduction

Khovanov homology is a special case of *categorification*, a novel approach to construction of knot (or link) invariants that has been under active development over the last decade following a seminal paper [Kh1] by Mikhail Khovanov. The idea of categorification is to replace a known polynomial knot (or link) invariant with a family of chain complexes such that the coefficients of the original polynomial are the Euler characteristics of these complexes. Although the chain complexes themselves depend heavily on a diagram that represents the link, their homology depends on the isotopy class of the link only. Khovanov homology categorifies the Jones polynomial [J].

More specifically, let L be an oriented link in $\mathcal{R}^3$ represented by a planar diagram D and let $J_L(q)$ be a version of the Jones polynomial of L that satisfies the following identities (called the *Jones skein relation* and *normalization*):

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I. Itenberg et al. (eds.), *Perspectives in Analysis, Geometry, and Topology: On the Occasion of the 60th Birthday of Oleg Viro*, Progress in Mathematics 296,
DOI 10.1007/978-0-8176-8277-4_17,

$$-q^{-2}J_{\times_+}(q)+q^{2}J_{\times_-}(q)=(q-1/q)J_{)(_0}(q); \qquad J_{\bigcirc}(q)=q+1/q. \tag{1.1}$$

The skein relation should be understood as relating the Jones polynomials of three links whose planar diagrams are identical everywhere except in a small disk, where they are different, as depicted in (1.1). The normalization fixes the value of the Jones polynomial on the trivial knot; $J_L(q)$ is a Laurent polynomial in q for every link L and is completely determined by its skein relation and normalization.

In [Kh1], Mikhail Khovanov assigned to D a family of abelian groups $\mathcal{H}^{i,j}(L)$ whose isomorphism classes depend on the isotopy class of L only. These groups are defined as homology groups of an appropriate (graded) chain complex $\mathcal{C}^{i,j}(D)$ with integer coefficients. Groups $\mathcal{H}^{i,j}(L)$ are nontrivial for finitely many values of the pair (i,j) only. The gist of the categorification is that the graded Euler characteristic of the Khovanov chain complex equals $J_L(q)$:

$$J_L(q)=\sum_{i,j}(-1)^i q^j h^{i,j}(L), \tag{1.2}$$

where $h^{i,j}(L)=\operatorname{rk}(\mathcal{H}^{i,j}(L))$, the Betti numbers of $\mathcal{H}$. The reader is referred to Sect. 2 for detailed treatment (see also [BN1, Kh1]).

In our paper we also make use of another version of the Jones polynomial, denoted by $\widetilde{J}_L(q)$, that satisfies the same skein relation (1.1) but is normalized to equal 1 on the trivial knot. For the sake of completeness, we also list the skein relation for the original Jones polynomial $V_L(t)$ from [J]:

$$t^{-1}V_{\times_+}(t)-tV_{\times_-}(t)=(t^{1/2}-t^{-1/2})V_{)(_0}(t); \qquad V_{\bigcirc}(t)=1. \tag{1.3}$$

We note that $J_L(q)\in\mathbb{Z}[q,q^{-1}]$, while $V_L(t)\in\mathbb{Z}[t^{1/2},t^{-1/2}]$. In fact, the terms of $V_L(t)$ have half-integer exponents if L has an even number of components. This is one of the main motivations for our convention (1.1) to be different from (1.3). We also want to ensure that the Jones polynomial of the trivial link has only positive coefficients. The different versions of the Jones polynomial are related as follows:

$$J_L(q)=(q+1/q)\widetilde{J}_L(q), \qquad \widetilde{J}_L(-t^{1/2})=V_L(t), \qquad V_L(q^2)=\widetilde{J}_L(q). \tag{1.4}$$

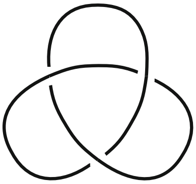
Another way to look at Khovanov's identity (1.2) is via the *Poincaré polynomial* of the Khovanov homology:

$$Kh_L(t,q)=\sum_{i,j}t^i q^j h^{i,j}(L). \tag{1.5}$$

With this notation, we get

$$J_L(q)=Kh_L(-1,q). \tag{1.6}$$

Fig. 1 Right trefoil and its Khovanov homology



	0	1	2	3
9				**1**
7				$\mathbf{1_2}$
5			**1**	
3	**1**			
1	**1**			

Example 1.A. Consider the right trefoil K. Its nonzero homology groups are tabulated in Fig. 1, where the i-grading is represented horizontally and the j-grading vertically. The homology is nontrivial for odd j-grading only, and thus even q-degrees are not shown in the table. A table entry $\mathbf{1}$ or $\mathbf{1_2}$ means that the corresponding group is $\mathbb{Z}$ or $\mathbb{Z}_2$, respectively (one can find a more interesting example in Fig. 9). In general, an entry of the form $\mathbf{a}, \mathbf{b_2}$ corresponds to the group $\mathbb{Z}^a \oplus \mathbb{Z}_2^b$. For the trefoil K, we have that $\mathcal{H}^{0,1}(K) \simeq \mathcal{H}^{0,3}(K) \simeq \mathcal{H}^{2,5}(K) \simeq \mathcal{H}^{3,9}(K) \simeq \mathbb{Z}$ and $\mathcal{H}^{3,7}(K) \simeq \mathbb{Z}_2$. Therefore, $Kh_K(t,q) = q + q^3 + t^2 q^5 + t^3 q^9$. On the other hand, the Jones polynomial of K equals $V_{\mathrm{K}}(t) = t + t^3 - t^4$. Relation (1.4) implies that $J_{\mathrm{K}}(q) = (q + 1/q)(q^2 + q^6 - q^8) = q + q^3 + q^5 - q^9 = Kh_{\mathrm{K}}(-1,q)$.

Without going into details, we note that the initial categorification of the Jones polynomial by Khovanov was followed by a flurry of activity. Categorifications of the colored Jones polynomial [Kh3, BW] and skein $\mathfrak{sl}(3)$ polynomial [Kh4] were based on Khovanov's original construction. A Matrix factorization technique was used to categorify the $\mathfrak{sl}(n)$ skein polynomials [KhR1], the HOMFLY-PT polynomial [KhR2], the Kauffman polynomial [KhR3], and more recently, colored $\mathfrak{sl}(n)$ polynomials [Wu, Y]. Ozsváth, Szabó, and independently Rasmussen used a completely different method of Floer homology to categorify the Alexander polynomial [OS2, Ra1]. Ideas of categorification were successfully applied to tangles, virtual links, skein modules, and polynomial invariants of graphs.

One of the most important recent developments in the Khovanov homology theory was the introduction in 2007 of its odd version by Ozsváth, Rasmussen, and Szabó [ORS]. The odd Khovanov homology equals the original (even) one modulo 2, and in particular, categorifies the same Jones polynomial. On the other hand, the odd and even homology theories often have drastically different properties (see Sects. 2.4 and 3 for details). The odd Khovanov homology appears to be one of the connecting links between the Khovanov and Heegaard–Floer homology theories [OS3].

The importance of the Khovanov homology became apparent after a seminal result by Jacob Rasmussen [Ra2], who used the Khovanov chain complex to give the first purely combinatorial proof of the Milnor conjecture. This conjecture states that the four-dimensional (slice) genus (and hence the genus) of a (p,q)-torus knot equals $\frac{(p-1)(q-1)}{2}$. It was originally proved by Kronheimer and Mrowka [KM] using gauge theory in 1993.

There are numerous other applications of Khovanov homology theories. They can be used to provide combinatorial proofs of the slice–Bennequin inequality and to give upper bounds on the Thurston–Bennequin number of Legendrian links, detect quasialternating links, and find topologically locally flatly slice knots that are not smoothly slice. We refer the reader to Sect. 4 for the details.

The goal of this paper is to give an overview of the current state of research in Khovanov homology. The exposition is mostly self-contained, and no advanced knowledge of the subject is required from the reader. We intentionally limit the scope of our paper to the categorifications of the Jones polynomial, so as to keep its size under control. The reader is referred to other expository papers on the subject [AKh, Kh6, Ra3] to learn more about the interrelations between different types of categorifications.

We also pay significant attention to experimental aspects of the Khovanov homology. As is often the case with new theories, the initial discovery was the result of experimentation. This is especially true for Khovanov homology, since it can be computed by hand for a very limited family of knots only. At the moment, there are two programs [BNG, Sh1] that compute Khovanov homology. The first one was written by Dror Bar-Natan and his student Jeremy Green in 2005 and implements the methods from [BN2]. It works significantly faster for knots with sufficiently many crossings (say more than 15) than the older program `KhoHo` by the author. On the other hand, `KhoHo` can compute all the versions of the Khovanov homology that are mentioned in this paper. It is currently the only program that can deal with the odd Khovanov homology. Most of the experimental results that are referred to in this paper were obtained with `KhoHo`.

This paper is organized as follows. In Sect. 2 we give a quick overview of constructions involved in the definition of various Khovanov homology theories. We compare these theories with each other and list their basic properties in Sect. 3. Section 4 is devoted to some of the more important applications of the Khovanov homology to other areas of low-dimensional topology.

2 Definition of the Khovanov Homology

In this section we give a brief outline of various Khovanov homology theories starting with Khovanov's original construction. Our setting is slightly more general than the one in the introduction, for we allow different coefficient rings, not only $\mathbb{Z}$.

2.1 *Algebraic Preliminaries*

Let R be a commutative ring with unity. In this paper, we are mainly interested in the cases $R = \mathbb{Z}$, $\mathbb{Q}$, or $\mathbb{Z}_2$.

Definition 2.1.A. A $\mathbb{Z}$-*graded* (or simply *graded*) R-module M is an R-module decomposed into a direct sum $M = \bigoplus_{j\in\mathbb{Z}} M_j$, where each M_j is an R-module itself. The summands M_j are called the *homogeneous components* of M, and elements of M_j are called the *homogeneous elements of degree* j.

Definition 2.1.B. Let $M = \bigoplus_{j\in\mathbb{Z}} M_j$ be a graded free R-module. The *graded dimension* of M is the power series $\dim_q(M) = \sum_{j\in\mathbb{Z}} q^j \dim(M_j)$ in the variable q. If $k \in \mathbb{Z}$, the *shifted module* $M\{k\}$ is defined as having homogeneous components $M\{k\}_j = M_{j-k}$.

Definition 2.1.C. Let M and N be two graded R-modules. A map $\varphi : M \to N$ is said to be *graded of degree* k if $\varphi(M_j) \subset N_{j+k}$ for each $j \in \mathbb{Z}$.

2.1.D. It is an easy exercise to check that $\dim_q(M\{k\}) = q^k \dim_q(M)$, $\dim_q(M \oplus N) = \dim_q(M) + \dim_q(N)$, and $\dim_q(M \otimes N) = \dim_q(M)\dim_q(N)$, where M and N are graded R-modules. Moreover, if $\varphi : M \to N$ is a graded map of degree k', then the *shifted map* $\varphi : M \to N\{k\}$ is graded of degree $k' + k$. We slightly abuse notation here by denoting the shifted map in the same way as the map itself.

Definition 2.1.E. Let $(\mathcal{C}, d) = \cdots \longrightarrow \mathcal{C}^{i-1} \xrightarrow{d^{i-1}} \mathcal{C}^i \xrightarrow{d^i} \mathcal{C}^{i+1} \longrightarrow \cdots$ be a (co)chain complex of graded free R-modules with differentials d^i having degree 0 for all $i \in \mathbb{Z}$. Then the *graded Euler characteristic* of $\mathcal{C}$ is defined as $\chi_q(\mathcal{C}) = \sum_{i\in\mathbb{Z}} (-1)^i \dim_q(\mathcal{C}^i)$.

Remark. One can think of a graded (co)chain complex of R-modules as a *bigraded R-module* in which the homogeneous components are indexed by pairs of numbers $(i, j) \in \mathbb{Z}^2$.

Let $A = R[X]/X^2$ be the algebra of truncated polynomials. As an R-module, A is freely generated by 1 and X. We put a grading on A by specifying that $\deg(1) = 1$ and $\deg(X) = -1$ (we follow the original grading convention from [Kh1] and [BN1] here; it is different by a sign from that of [AKh]). In other words, $A \simeq R\{1\} \oplus R\{-1\}$ and $\dim_q(A) = q + q^{-1}$. At the same time, A is a (graded) commutative algebra with unity 1 and multiplication $m : A \otimes A \to A$ given by

$$m(1 \otimes 1) = 1, \qquad m(1 \otimes X) = m(X \otimes 1) = X, \qquad m(X \otimes X) = 0. \tag{2.1}$$

The module A can also be equipped with a coalgebra structure with comultiplication $\Delta : A \to A \otimes A$ and counit $\varepsilon : A \to R$ defined as

$$\Delta(1) = 1 \otimes X + X \otimes 1, \qquad \Delta(X) = X \otimes X; \tag{2.2}$$

$$\varepsilon(1) = 0, \qquad \varepsilon(X) = 1. \tag{2.3}$$

The comultiplication Δ is coassociative and cocommutative and satisfies

$$(m \otimes \mathrm{id}_A) \circ (\mathrm{id}_A \otimes \Delta) = \Delta \circ m, \tag{2.4}$$

$$(\varepsilon \otimes \mathrm{id}_A) \circ \Delta = \mathrm{id}_A\,. \tag{2.5}$$

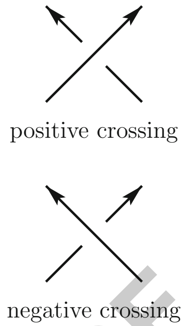


Fig. 2 Positive and negative crossings

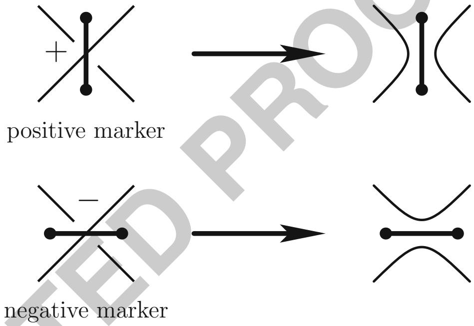


Fig. 3 Positive and negative markers and the corresponding resolutions of a diagram

Together with the unit map $\iota : R \to A$ given by $\iota(1) = 1$, this makes A into a commutative Frobenius algebra over R [Kh5].

It follows directly from the definitions that ι, ε, m, and Δ are graded maps with $\deg(\iota) = \deg(\varepsilon) = 1$ and $\deg(m) = \deg(\Delta) = -1$.

2.2 *Khovanov Chain Complex*

Let L be an oriented link and D its planar diagram. We assign a number ± 1, called the *sign*, to every crossing of D according to the rule depicted in Fig. 2. The sum of these signs over all the crossings of D is called the *writhe number* of D and is denoted by $w(D)$.

Every crossing of D can be *resolved* in two different ways according to a choice of a *marker*, which can be either *positive* or *negative*, at this crossing (see Fig. 3). A collection of markers chosen at every crossing of a diagram D is called a *(Kauffman) state* of D. For a diagram with n crossings, there are obviously 2^n different states. Denote by $\sigma(s)$ the difference between the numbers of positive and negative markers in a given state s. Define

$$i(s) = \frac{w(D) - \sigma(s)}{2}, \qquad j(s) = \frac{3w(D) - \sigma(s)}{2}. \tag{2.6}$$

Since both $w(D)$ and $\sigma(s)$ are congruent to n modulo 2, $i(s)$ and $j(s)$ are always integers. For a given state s, the result of the resolution of D at each crossing according to s is a family D_s of disjointly embedded circles. Denote the number of these circles by $|D_s|$.

For each state s of D, let $\mathcal{A}(s) = A^{\otimes |D_s|}\{j(s)\}$. One should understand this construction as assigning a copy of the algebra A to each circle from D_s, taking the tensor product of all of these copies, and shifting the grading of the result by $j(s)$. By construction, $\mathcal{A}(s)$ is a graded free R-module of graded dimension $\dim_q(\mathcal{A}(s)) = q^{j(s)}(q+q^{-1})^{|D_s|}$. Let $\mathcal{C}^i(D) = \bigoplus_{i(s)=i} \mathcal{A}(s)$ for each $i \in \mathbb{Z}$. In order to make $\mathcal{C}(D)$ into a graded complex, we need to define a (graded) differential $d^i : \mathcal{C}^i(D) \to \mathcal{C}^{i+1}(D)$ of degree 0. But even before the differential is defined, the (graded) Euler characteristic of $\mathcal{C}(D)$ makes sense.

Lemma 2.2.A. *The graded Euler characteristic of $\mathcal{C}(D)$ equals the Jones polynomial of the link L. That is, $\chi_q(\mathcal{C}(D)) = J_L(q)$.*

Proof.

$$\begin{aligned}
\chi_q(\mathcal{C}(D)) &= \sum_{i\in\mathbb{Z}} (-1)^i \dim_q(\mathcal{C}^i(D)) \\
&= \sum_{i\in\mathbb{Z}} (-1)^i \sum_{i(s)=i} \dim_q(\mathcal{A}(s)) \\
&= \sum_s (-1)^{i(s)} q^{j(s)} (q+q^{-1})^{|D_s|} \\
&= \sum_s (-1)^{\frac{w(D)-\sigma(s)}{2}} q^{\frac{3w(D)-\sigma(s)}{2}} (q+q^{-1})^{|D_s|}.
\end{aligned}$$

Let us forget for a moment that A denotes an algebra and (temporarily) use this letter for a variable. Substituting $(-A^{-2})$ instead of q and noticing that $w(D) \equiv \sigma(s) \pmod 2$, we arrive at

$$\chi_q(\mathcal{C}(D)) = (-A)^{-3w(D)} \sum_s A^{\sigma(s)} (-A^2 - A^{-2})^{|D_s|} = (-A^2 - A^{-2}) \langle L \rangle_N,$$

where $\langle L \rangle_N$ is the normalized Kauffman bracket polynomial of L (see [K] for details). The normalized bracket polynomial of a link is related to the bracket polynomial of its diagram by $\langle L \rangle_N = (-A)^{-3w(D)} \langle D \rangle$. Kauffman proved [K] that $\langle L \rangle_N$ equals the Jones polynomial $V_L(t)$ of L after substituting $t^{-1/4}$ instead of A. The relation (1.4) between $V_L(t)$ and $J_L(q)$ completes our proof. □

Let s_+ and s_- be two states of D that differ at a single crossing, where s_+ has a positive marker, while s_- has a negative one. We call two such states *adjacent*. In this case, $\sigma(s_-) = \sigma(s_+) - 2$, and consequently, $i(s_-) = i(s_+) + 1$ and $j(s_-) = j(s_+) + 1$. Consider now the resolutions of D corresponding to s_+ and s_-. One can readily see that D_{s_-} is obtained from D_{s_+} by either merging two circles into one or splitting one circle into two (see Fig. 4). All the circles that do not pass through the crossing at which s_+ and s_- differ remain unchanged. We define $d_{s_+:s_-} : \mathcal{A}(s_+) \to \mathcal{A}(s_-)$ as either $m \otimes \mathrm{id}$ or $\Delta \otimes \mathrm{id}$ depending on whether the circles merge or split.

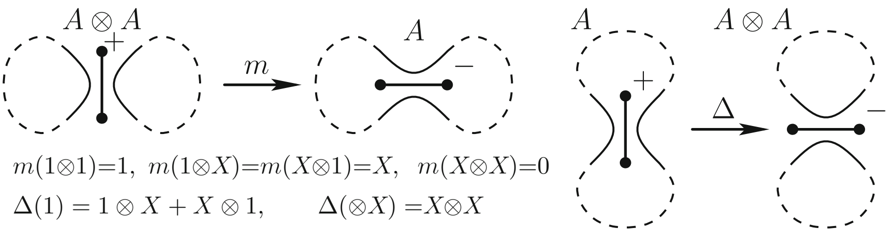


Fig. 4 Diagram resolutions corresponding to adjacent states and maps between the algebras assigned to the circles

Here, the multiplication or comultiplication is performed on the copies of A that are assigned to the affected circles, as in Fig. 4, while $d_{s_+:s_-}$ acts as the identity on all the A's corresponding to the unaffected ones. The difference in grading shift between $\mathcal{A}(s_+)$ and $\mathcal{A}(s_-)$ ensures that $\deg(d_{s_+:s_-}) = 0$ by 2.1.D.

We need one more ingredient in order to finish the definition of the differential on $\mathcal{C}(D)$, namely, an ordering of the crossings of D. For an adjacent pair of states (s_+, s_-), define $\varepsilon(s_+, s_-)$ to be the number of *negative* markers in s_+ (or s_-) that appear in the ordering of the crossings *after* the crossing at which s_+ and s_- differ. Finally, let $d^i = \sum_{(s_+,s_-)}(-1)^{\varepsilon(s_+,s_-)} d_{s_+:s_-}$, where (s_+, s_-) runs over all adjacent pairs of states with $i(s_+) = i$. It is straightforward to verify [Kh1] that $d^{i+1} \circ d^i = 0$, and hence $d : \mathcal{C}(D) \to \mathcal{C}(D)$ is indeed a differential.

Definition 2.2.B (Khovanov, [Kh1]). The resulting (co)chain complex $\mathcal{C}(D) = \cdots \longrightarrow \mathcal{C}^{i-1}(D) \xrightarrow{d^{i-1}} \mathcal{C}^i(D) \xrightarrow{d^i} \mathcal{C}^{i+1}(D) \longrightarrow \cdots$ is called the *Khovanov chain complex* of the diagram D. The homology of $\mathcal{C}(D)$ with respect to d is called the *Khovanov homology* of L and is denoted by $\mathcal{H}(L)$. We write $\mathcal{C}(D;R)$ and $\mathcal{H}(L;R)$ if we want to emphasize the ring of coefficients with which we work. If R is omitted from the notation, integer coefficients are assumed.

Theorem 2.2.C (Khovanov, [Kh1]; see also [BN1]). *The isomorphism class of $\mathcal{H}(L;R)$ depends on the isotopy class of L only and hence is a link invariant. In particular, it does not depend on the ordering chosen for the crossings of D. The Khovanov homology $\mathcal{H}(L;R)$ categorifies $J_L(q)$, a version of the Jones polynomial defined by* (1.1).

2.2.D. Let $\#L$ be the number of components of a link L. One can check that $j(s) + |D_s|$ is congruent modulo 2 to $\#L$ for every state s. It follows that $\mathcal{C}(D;R)$ has nontrivial homogeneous components only in the degrees that have the same parity as $\#L$. Consequently, $\mathcal{H}(L;R)$ is nontrivial only if the q-grading has this parity (see Example 1.A).

Remark. One can think of $\mathcal{C}(D;R)$ as a bigraded (co)chain complex $\mathcal{C}^{i,j}(D;R)$ with a differential of bidegree $(1,0)$. In this case, i is the homological grading of this complex, and j is its q-grading, also called the *Jones grading*. Correspondingly, $\mathcal{H}(L;R)$ can be considered to be a bigraded R-module as well.

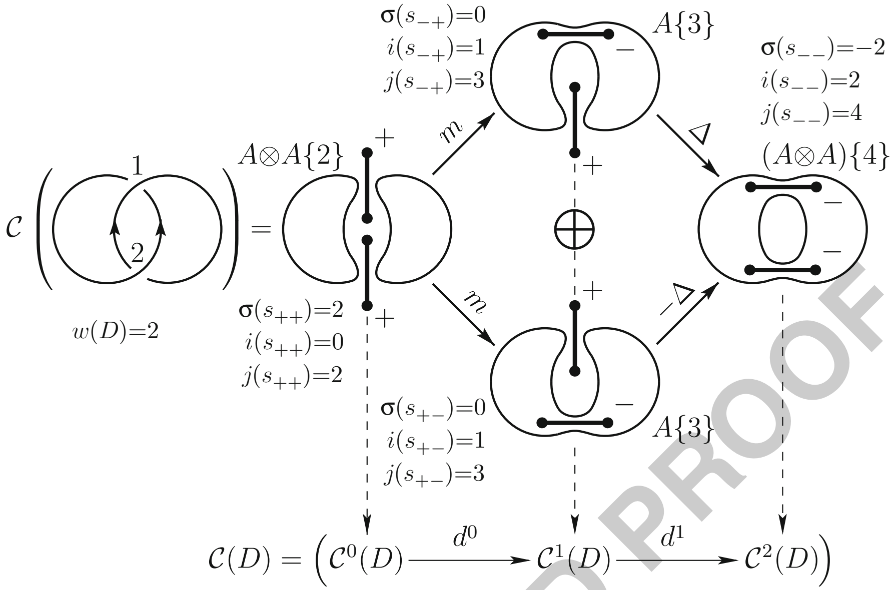


Fig. 5 Khovanov chain complex for the Hopf link

Example 2.2.E. Figure 5 shows the Khovanov chain complex for the Hopf link with the indicated orientation. The diagram has two positive crossings, so its writhe number is 2. Let $s_{\pm\pm}$ be the four possible resolutions of this diagram, where each "+" or "−" describes the sign of the marker at the corresponding crossings. The chosen ordering of crossings is depicted by numbers placed next to them. By looking at Fig. 5, one easily computes that $\mathcal{A}(s_{++}) = A^{\otimes 2}\{2\}$, $\mathcal{A}(s_{+-}) = \mathcal{A}(s_{-+}) = A\{3\}$, and $\mathcal{A}(s_{--}) = A^{\otimes 2}\{4\}$. Correspondingly, $\mathcal{C}^0(D) = \mathcal{A}(s_{++}) = A^{\otimes 2}\{2\}$, $\mathcal{C}^1(D) = \mathcal{A}(s_{+-}) \oplus \mathcal{A}(s_{-+}) = (A \oplus A)\{3\}$, and $\mathcal{C}^2(D) = \mathcal{A}(s_{--}) = A^{\otimes 2}\{4\}$. It is convenient to arrange the four resolutions in the corners of a square placed in the plane in such a way that its diagonal from s_{++} to s_{--} is horizontal. Then the edges of this square correspond to the maps between the adjacent states (see Fig. 5). We notice that only one of the maps, namely the one corresponding to the edge from s_{+-} to s_{--}, comes with the negative sign.

In general, 2^n resolutions of a diagram D with n crossings can be arranged into an n-dimensional *cube of resolutions*, where vertices correspond to the 2^n states of D. The edges of this cube connect adjacent pairs of states and can be oriented from s_+ to s_-. Every edge is assigned either m or Δ with the sign $(-1)^{\varepsilon(s_+,s_-)}$, as described above. It is easy to check that this makes each square (that is, a two-dimensional face) of the cube anticommutative (all squares are commutative without the signs). Finally, the differential d^i restricted to each summand $\mathcal{A}(s)$ with $i(s) = i$ equals the sum of all the maps assigned to the edges that originate at s.

2.3 Reduced Khovanov Homology

Let, as before, D be a diagram of an oriented link L. Fix a base point on D that is different from all the crossings. For each state s, we define $\widetilde{\mathcal{A}}(s)$ in almost the same way as $\mathcal{A}(s)$, except that we assign XA instead of A to the circle from the resolution D_s of D that contains the base point. That is, $\widetilde{\mathcal{A}}(s) = \big((XA) \otimes A^{\otimes(|D_s|-1)}\big)\{j(s)\}$. We can now build the *reduced Khovanov chain complex* $\widetilde{\mathcal{C}}(D;R)$ in exactly the same way as $\mathcal{C}(D;R)$ by replacing $\mathcal{A}$ with $\widetilde{\mathcal{A}}$ everywhere. The grading shifts and differentials remain the same. It is easy to see that $\widetilde{\mathcal{C}}(D;R)$ is a subcomplex of $\mathcal{C}(D;R)$ of index 2. In fact, it is the image of the chain map $X : \mathcal{C}(D;R) \to \mathcal{C}(D;R)$ that acts by multiplying elements assigned to the circle containing the base point by X.

Definition 2.3.A (Khovanov [Kh2]). The homology of $\widetilde{\mathcal{C}}(D;R)$ is called the *reduced Khovanov homology of* L and is denoted by $\widetilde{\mathcal{H}}(L;R)$. It is clear from the construction of $\widetilde{\mathcal{C}}(D;R)$ that its graded Euler characteristic equals $\widetilde{J}_L(q)$.

Theorem 2.3.B (Khovanov [Kh2]). *The isomorphism class of $\widetilde{\mathcal{H}}(L;R)$ is a link invariant that categorifies $\widetilde{J}_L(q)$, a version of the Jones polynomial defined by* (1.1) *and* (1.4)*. Moreover, if two base points are chosen on the same component of L, then the corresponding reduced Khovanov homologies are isomorphic. On the other hand, $\widetilde{\mathcal{H}}(L;R)$ might depend on the component of L on which the base point is chosen.*

Although $\widetilde{\mathcal{C}}(D;R)$ can be determined from $\mathcal{C}(D;R)$, it is in general not clear how $\mathcal{H}(L;R)$ and $\widetilde{\mathcal{H}}(L;R)$ are related. There are several examples of pairs of knots (the smallest ones having 14 crossings) that have the same rational Khovanov homology, but different rational reduced Khovanov homologies. No such examples are known for homologies over $\mathbb{Z}$ among all prime knots with at most 15 crossings. On the other hand, it is proved that $\mathcal{H}(L;\mathbb{Z}_2)$ and $\widetilde{\mathcal{H}}(L;\mathbb{Z}_2)$ determine each other completely.

Theorem 2.3.C ([Sh2]). $\mathcal{H}(L;\mathbb{Z}_2) \simeq \widetilde{\mathcal{H}}(L;\mathbb{Z}_2) \otimes_{\mathbb{Z}_2} A_{\mathbb{Z}_2}$. *In particular, $\widetilde{\mathcal{H}}(L;\mathbb{Z}_2)$ does not depend on the component on which the base point is chosen.*

Remark. $XA \simeq R\{0\}$ as a graded R-module. It follows that $\widetilde{\mathcal{C}}$ and $\widetilde{\mathcal{H}}$ are nontrivial only in the q-gradings with parity different from that of $\#L$, the number of components of L (cf. 2.2.D).

2.4 Odd Khovanov Homology

In 2007, Ozsváth, Rasmussen, and Szabó introduced [ORS] an odd version of the Khovanov homology. In their theory, the nilpotent variables X assigned to each circle in the resolutions of the link diagram (see Sect. 2.2) anticommute rather than commute. The odd Khovanov homology equals the original (*even*) one modulo 2, and in particular, categorifies the same Jones polynomial. In fact, the corresponding chain complexes are isomorphic as free bigraded R-modules, and their differentials

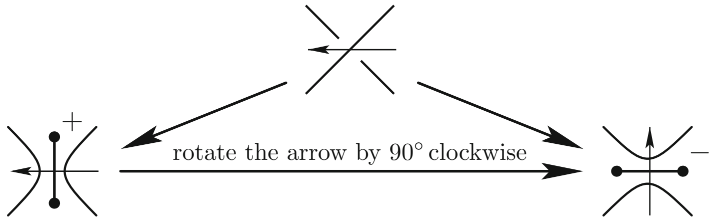


Fig. 6 Choice of arrows at the diagram crossings

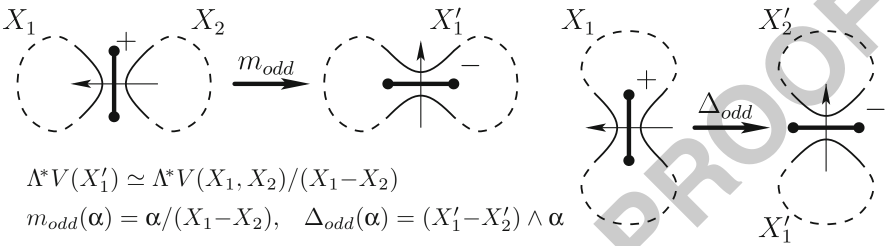


Fig. 7 Adjacent states and differentials in the odd Khovanov chain complex

are different only by sign. On the other hand, the resulting homology theories often have drastically different properties. We define the odd Khovanov homology below.

Let L be an oriented link and D its planar diagram. To each resolution s of D we assign a free graded R-module $\Lambda(s)$ as follows. Label all circles from the resolution D_s by some independent variables, say $X_1, X_2, \ldots, X_{|D_s|}$, and let $V_s = V(X_1, X_2, \ldots, X_{|D_s|})$ be a free R-module generated by them. We define $\Lambda(s) = \Lambda^*(V_s)$, the exterior algebra of V_s. Then $\Lambda(s) = \Lambda^0(V_s) \oplus \Lambda^1(V_s) \oplus \cdots \oplus \Lambda^{|D_s|}(V_s)$, and we grade $\Lambda(s)$ by specifying $\Lambda(s)_{|D_s|-2k} = \Lambda^k(V_s)$ for each $0 \leq k \leq |D_s|$, where $\Lambda(s)_{|D_s|-2k}$ is the homogeneous component of $\Lambda(s)$ of degree $|D_s| - 2k$. It is an easy exercise for the reader to check that $\dim_q(\Lambda(s)) = \dim_q(A^{\otimes |D_s|})$.

Just as in the case of the even Khovanov homology, these R-modules $\Lambda(s)$ can be arranged into an n-dimensional cube of resolutions. Let $\mathcal{C}^i_{\text{odd}}(D) = \bigoplus_{i(s)=i} \Lambda(s)\{j(s)\}$. Then, similarly to Lemma 2.2.A, we have that $\chi_q(\mathcal{C}_{\text{odd}}(D)) = J_L(q)$. In fact, $\mathcal{C}_{\text{odd}}(D) \simeq \mathcal{C}(D)$ as bigraded R-modules. In order to define the differential on $\mathcal{C}_{\text{odd}}$, we need to introduce an additional structure, a choice of an arrow at each crossing of D that is parallel to the *negative* marker at that crossing (see Fig. 6). There are obviously 2^n such choices. For every state s on D, we place arrows that connect two branches of D_s near each (former) crossing according to the rule from Fig. 6.

We now assign (graded) maps m_{odd} and Δ_{odd} to each edge of the cube of resolutions that connects adjacent states s_+ and s_-. If s_- is obtained from s_+ by merging two circles, then $\Lambda(s_-) \simeq \Lambda(s_+)/(X_1 - X_2)$, where X_1 and X_2 are the generators of V_{s_+} corresponding to the two merging circles, as depicted in Fig. 7. We define $m_{\text{odd}} : \Lambda(s_+) \to \Lambda(s_-)$ to be this isomorphism composed with the projection $\Lambda(s_+) \to \Lambda(s_+)/(X_1 - X_2)$. The case in which one circle splits into two is more interesting. Let X_1' and X_2' be the generators of V_{s_-} corresponding to these two circles such that the arrow points from X_1' to X_2' (see Fig. 7). Now for each generator X_k of V_{s_+}, we define $\Delta_{\mathrm{odd}}(X_k) = (X_1' - X_2') \wedge X_{\eta(k)}'$, where η is the correspondence between circles in D_{s_+} and D_{s_-}. While $\eta(1)$ can equal either 1 or 2, this choice does not matter, since $(X_1' - X_2') \wedge X_2' = X_1' \wedge X_2' = -X_2' \wedge X_1' = (X_1' - X_2') \wedge X_1'$.

This definition makes each square in the cube of resolutions commutative, anticommutative, or both. The latter case means that both double-composites corresponding to the square are trivial. This is a major departure from the situation that we had in the even case, in which each square was commutative. In particular, it makes the choice of signs on the edges of the cube much more involved.

Theorem 2.4.A (Ozsváth–Rasmussen–Szabó [ORS]). *It is possible to assign a sign to each edge in this cube of resolutions in such a way that every square becomes anticommutative. This results in a graded (co)chain complex $\mathcal{C}_{\mathrm{odd}}(D;R)$. The homology $\mathcal{H}_{\mathrm{odd}}(L;R)$ of $\mathcal{C}_{\mathrm{odd}}(D;R)$ does not depend on the choice of arrows at the crossings, the choice of edge signs, and some other choices needed in the construction. Moreover, the isomorphism class of $\mathcal{H}_{\mathrm{odd}}(L;R)$ is a link invariant, called* odd Khovanov homology*, that categorifies $J_L(q)$.*

Remark. There is no explicit construction for an assignment of signs to the edges of the cube of resolutions in the case of the odd Khovanov chain complex. The theorem above ensures only that signs exist.

2.4.B. By comparing the definitions of $\mathcal{C}_{\mathrm{odd}}(D;\mathbb{Z}_2)$ and $\mathcal{C}(D;\mathbb{Z}_2)$, it is easy to see that they are isomorphic as graded chain complexes (since the signs do not matter modulo 2). It follows that $\mathcal{H}_{\mathrm{odd}}(D;\mathbb{Z}_2) \simeq \mathcal{H}(D;\mathbb{Z}_2)$ as well.

2.4.C. One can construct a *reduced* odd Khovanov chain complex $\widetilde{\mathcal{C}}_{\mathrm{odd}}(D;R)$ and *reduced* odd Khovanov homology $\widetilde{\mathcal{H}}_{\mathrm{odd}}(L;R)$ using methods similar to those from Sect. 2.3. In this case, in contrast to the even situation, reduced and nonreduced odd Khovanov homologies determine each other completely (see [ORS]). Namely, $\mathcal{H}_{\mathrm{odd}}(L;R) \simeq \widetilde{\mathcal{H}}_{\mathrm{odd}}(L;R)\{1\} \oplus \widetilde{\mathcal{H}}_{\mathrm{odd}}(L;R)\{-1\}$ (cf. Theorem 2.3.C). It is therefore enough to consider the reduced version of the odd Khovanov homology only.

Example 2.4.D. The odd Khovanov chain complex for the Hopf link is depicted in Fig. 8. All the grading shifts in this case are the same as in Example 2.2.E and in Fig. 5, so we do not list them again. We observe that the resulting square of resolutions is anticommutative, so no adjustment of signs is needed.

The odd Khovanov homology should provide an insight into interrelations between Khovanov and Heegaard–Floer [OS1] homology theories. Its definition was motivated by the following result.

Theorem 2.4.E (Ozsváth–Szabó [OS3]). *For each link L with a diagram D, there exists a spectral sequence with $E^1 = \widetilde{\mathcal{C}}(D;\mathbb{Z}_2)$ and $E^2 = \widetilde{\mathcal{H}}(L;\mathbb{Z}_2)$ that converges to the $\mathbb{Z}_2$-Heegaard–Floer homology $\widehat{HF}(\Sigma(L);\mathbb{Z}_2)$ of the double branched cover $\Sigma(L)$ of S^3 along L.*

Conjecture 2.4.F. There exists a spectral sequence that starts with $\widetilde{\mathcal{C}}_{\mathrm{odd}}(D;\mathbb{Z})$ and $\widetilde{\mathcal{H}}_{\mathrm{odd}}(L;\mathbb{Z})$ and converges to $\widehat{HF}(\Sigma(L);\mathbb{Z})$.

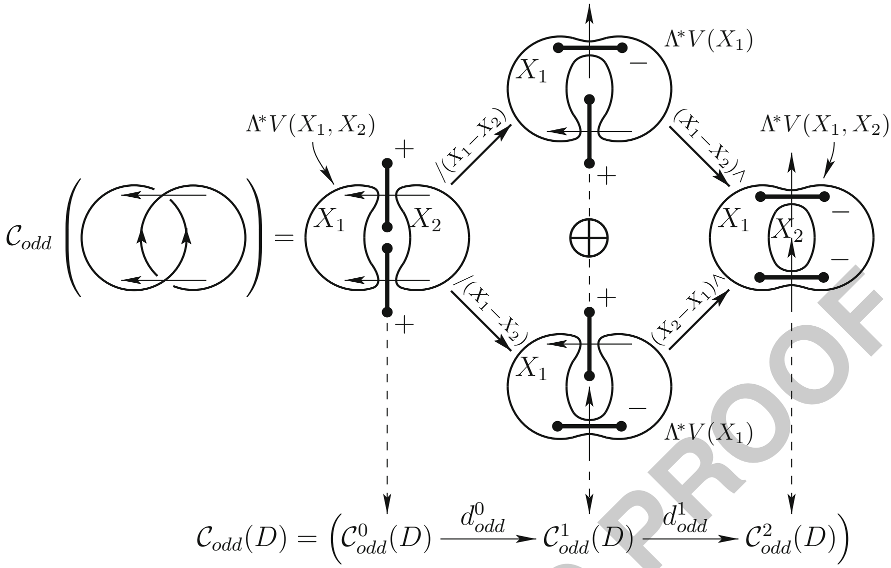


Fig. 8 Odd Khovanov chain complex for the Hopf link

3 Properties of the Khovanov Homology

In this section we summarize the main properties of the Khovanov homology and list related constructions. We emphasize similarities and differences in properties exhibited by different versions of the Khovanov homology. Some of them were already mentioned in the previous sections.

3.A. Let L be an oriented link and D its planar diagram. Then

- $\widetilde{\mathcal{C}}(D;R)$ is a subcomplex of $\mathcal{C}(D;R)$ of index 2.
- $\mathcal{H}_{\text{odd}}(L;\mathbb{Z}_2) \simeq \mathcal{H}(L;\mathbb{Z}_2)$ and $\widetilde{\mathcal{H}}_{\text{odd}}(L;\mathbb{Z}_2) \simeq \widetilde{\mathcal{H}}(L;\mathbb{Z}_2)$.
- $\chi_q(\mathcal{H}(L;R)) = \chi_q(\mathcal{H}_{\text{odd}}(L;R)) = J_L(q)$ and $\chi_q(\widetilde{\mathcal{H}}(L;R)) = \chi_q(\widetilde{\mathcal{H}}_{\text{odd}}(L;R)) = \widetilde{J}_L(q)$.
- $\mathcal{H}(L;\mathbb{Z}_2) \simeq \widetilde{\mathcal{H}}(L;\mathbb{Z}_2) \otimes_{\mathbb{Z}_2} A_{\mathbb{Z}_2}$ [Sh2] and $\mathcal{H}_{\text{odd}}(L;R) \simeq \widetilde{\mathcal{H}}_{\text{odd}}(L;R)\{1\} \oplus \widetilde{\mathcal{H}}_{\text{odd}}(L;R)\{-1\}$ [ORS]. On the other hand, $\mathcal{H}(L;\mathbb{Z})$ and $\mathcal{H}(L;\mathbb{Q})$ do not split in general.
- For links, $\widetilde{\mathcal{H}}(L;\mathbb{Z}_2)$ and $\widetilde{\mathcal{H}}_{odd}(L;R)$ do not depend on the choice of a component with the base point. This is, in general, not the case for $\mathcal{H}(L;\mathbb{Z})$ and $\mathcal{H}(L;\mathbb{Q})$.
- If L is a nonsplit alternating link, then $\mathcal{H}(L;\mathbb{Q})$, $\widetilde{\mathcal{H}}(L;R)$, and $\widetilde{\mathcal{H}}_{\text{odd}}(L;R)$ are completely determined by the Jones polynomial and signature of L [Kh2,L,ORS].
- $\mathcal{H}(L;\mathbb{Z}_2)$ and $\mathcal{H}_{\text{odd}}(L;\mathbb{Z})$ are invariant under the component-preserving link mutations [B,W2]. It is unclear whether the same holds for $\mathcal{H}(L;\mathbb{Z})$. On the other hand, $\mathcal{H}(L;\mathbb{Z})$ is not preserved under a mutation that exchanges components of a link [W1] and under a cabled mutation [DGShT].

- $\mathcal{H}(L;\mathbb{Z})$ almost always has torsion (except for several special cases), but mostly of order 2. The first knot with 4-torsion is the (4,5)-torus knot, which has 15 crossings. The first known knot with 3-torsion is the (5,6)-torus knot with 24 crossings. On the other hand, $\mathcal{H}_{\mathrm{odd}}(L;\mathbb{Z})$ has plenty of torsion of all orders (mostly 2 and 3).
- $\widetilde{\mathcal{H}}(L;\mathbb{Z})$ has very little torsion. The first knot with torsion has 13 crossings. On the other hand, $\widetilde{\mathcal{H}}_{\mathrm{odd}}(L;\mathbb{Z})$ has as much torsion as $\mathcal{H}_{\mathrm{odd}}(L;\mathbb{Z})$.

Remark. The properties above show that $\widetilde{\mathcal{H}}_{\mathrm{odd}}(L;\mathbb{Z})$ behaves similarly to $\widetilde{\mathcal{H}}(L;\mathbb{Z}_2)$ but not to $\widetilde{\mathcal{H}}(L;\mathbb{Z})$. This is by design (see 2.4.E and 2.4.F).

3.1 Homological Thickness

Definition 3.1.A. Let L be a link. The *homological width* of L over a ring R is the minimal number of adjacent diagonals $j-2i=const$ that support $\mathcal{H}(L;R)$. It is denoted by $\mathrm{hw}_R(L)$. The *reduced homological width* $\widetilde{\mathrm{hw}_R(L)}$ of L, *odd homological width* $\mathrm{ohw}_R(L)$ of L, and *reduced odd homological width* $\widetilde{\mathrm{ohw}}_R(L)$ of L are defined similarly.

3.1.B. It follows from 2.4.C that $\widetilde{\mathrm{ohw}}_R(L)=\mathrm{ohw}_R(L)-1$. The same holds in the case of the even Khovanov homology over $\mathbb{Q}$: $\widetilde{\mathrm{hw}}_{\mathbb{Q}}(L)=\mathrm{hw}_{\mathbb{Q}}(L)-1$ (see [Kh2]).

Definition 3.1.C. A link L is said to be *homologically thin* over a ring R, or simply H-thin, if $\mathrm{hw}_R(L)=2$; L is *homologically thick*, or H-thick, otherwise. We define *odd-homologically* thin and thick, or simply OH-thin and OH-thick, links similarly.

Theorem 3.1.D (Lee, Ozsváth–Rasmussen–Szabó, Manolescu–Ozsváth [L, ORS, MO]). *Quasialternating links (see Sect. 4.3 for the definition) are H-thin and OH-thin over every ring R. In particular, this is true for nonsplit alternating links.*

Theorem 3.1.E (Khovanov [Kh2]). *Adequate links are H-thick over every R.*

3.1.F. Homological thickness of a link L often does not depend on the base ring. The first prime knot with $\mathrm{hw}_{\mathbb{Q}}(L)<\mathrm{hw}_{\mathbb{Z}_2}(L)$ and $\mathrm{hw}_{\mathbb{Q}}(L)<\mathrm{hw}_{\mathbb{Z}}(L)$ has 15 crossings. The first prime knot that is $\mathbb{Q}$H-thin but $\mathbb{Z}$H-thick, 16^n_{197566}, has 16 crossings; see Fig. 9. Please observe that $\mathcal{H}^{9,25}(16^n_{197566};\mathbb{Z})$ and $\mathcal{H}^{-8,-25}(\overline{16}^n_{197566};\mathbb{Z})$ have 4-torsion, indicated by a small box in the tables.

Remark. Throughout this paper we use the following notation for knots: knots with 10 crossings or fewer are numbered according to Rolfsen's knot table [Ro], and knots with 11 crossings or more are numbered according to the knot table from Knotscape [HTh]. Mirror images of knots from the table are denoted with a bar on top. For example, the knot $\overline{9}_{46}$ is the mirror image of knot number 46 with 9 crossings from Rolfsen's table, and the knot 16^n_{197566} is a nonalternating knot, number 197566, with 16 crossings from Knotscape's table.

	-2	-1	0	1	2	3	4	5	6	7	8	9	10
29													$\mathbf{1}$
27												$\mathbf{4}$	$\mathbf{1_2}$
25											$\mathbf{7}$	$\mathbf{1, 3_2}\boxed{\mathbf{1_4}}$	
23										$\mathbf{12}$	$\mathbf{4, 8_2}$		
21									$\mathbf{15}$	$\mathbf{7, 12_2}$	$\mathbf{1_2}$		
19								$\mathbf{17}$	$\mathbf{12, 16_2}$				
17							$\mathbf{16}$	$\mathbf{15, 18_2}$	$\mathbf{1_2}$				
15						$\mathbf{15}$	$\mathbf{17, 16_2}$	$\mathbf{1_2}$					
13					$\mathbf{10}$	$\mathbf{16, 15_2}$							
11				$\mathbf{6}$	$\mathbf{15, 10_2}$								
9			$\mathbf{3}$	$\mathbf{10, 6_2}$									
7		$\mathbf{1}$	$\mathbf{7, 2_2}$										
5		$\mathbf{2, 1_2}$											
3	$\mathbf{1}$												

The free part of $\mathcal{H}(16^n_{197566};\mathbb{Z})$ is supported on diagonals $j-2i=7$ and $j-2i=9$. On the other hand, there is 2-torsion on the diagonal $j-2i=5$. Therefore, 16^n_{197566} is $\mathbb{Q}$H-thin, but $\mathbb{Z}$H-thick and $\mathbb{Z}_2$H-thick.

	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2
-3													$\mathbf{1}$
-5												$\mathbf{2}$	$\mathbf{1_2}$
-7											$\mathbf{7}$	$\mathbf{1, 2_2}$	
-9										$\mathbf{10}$	$\mathbf{3, 6_2}$		
-11									$\mathbf{15}$	$\mathbf{6, 10_2}$			
-13								$\mathbf{16}$	$\mathbf{10, 15_2}$				
-15							$\mathbf{17, 1_2}$	$\mathbf{15, 16_2}$					
-17						$\mathbf{15, 1_2}$	$\mathbf{16, 18_2}$						
-19					$\mathbf{12}$	$\mathbf{17, 16_2}$							
-21				$\mathbf{7, 1_2}$	$\mathbf{15, 12_2}$								
-23			$\mathbf{4}$	$\mathbf{12, 8_2}$									
-25		$\mathbf{1}$	$\mathbf{7, 3_2}\boxed{\mathbf{1_4}}$										
-27		$\mathbf{4, 1_2}$											
-29	$\mathbf{1}$												

$\mathcal{H}(\overline{16}^n_{197566};\mathbb{Z})$ is supported on diagonals $j-2i=-7$ and $j-2i=-9$. But there is 2-torsion on the diagonal $j-2i=-7$. Therefore, $\overline{16}^n_{197566}$ is $\mathbb{Q}$H-thin and $\mathbb{Z}$H-thin, but $\mathbb{Z}_2$H-thick.

Fig. 9 Integral Khovanov homology of the knots 16^n_{197566} and $\overline{16}^n_{197566}$

	-7	-6	-5	-4	-3	-2	-1	0	1	2	3
8											**1**
6										**1**	
4									**1**		
2							**1**	**2**			
0						**1**		$\mathbf{1_2}$			
-2					**1**	**1**					
-4				**2**	**1**						
-6			**1**								
-8		**1**									
-10	**1**										

$\mathrm{hw}_{\mathbb{Q}} = 3$, $\widetilde{\mathrm{hw}}_{\mathbb{Q}} = 2$, $\mathrm{hw}_{\mathbb{Z}} = 4$, $\widetilde{\mathrm{hw}}_{\mathbb{Z}} = 3$

	-7	-6	-5	-4	-3	-2	-1	0	1	2	3
8											**1**
6										**1**	
4									**1**		
2								**1**, $\mathbf{1_2}$			
0							$\boxed{\mathbf{1_6}}$	**1**			
-2						$\boxed{\mathbf{1_6}}$					
-4				**1**	$\mathbf{1_2}$						
-6			**1**								
-8		**1**									
-10	**1**										

$\mathrm{ohw}_{\mathbb{Q}} = 4$, $\widetilde{\mathrm{ohw}}_{\mathbb{Q}} = 3$, $\mathrm{ohw}_{\mathbb{Z}} = 4$, $\widetilde{\mathrm{ohw}}_{\mathbb{Z}} = 3$

Fig. 10 Integral reduced even Khovanov homology (above) and odd Khovanov Homology (below) of the knot 15^n_{41127}

3.1.G. Odd Khovanov homology is often thicker than the even homology over $\mathbb{Z}$. This is crucial for applications (see Sect. 4). On the other hand, $\widetilde{\mathrm{ohw}}_{\mathbb{Q}}(L) \leq \widetilde{\mathrm{hw}}_{\mathbb{Q}}(L)$ for all but one prime knot with at most 15 crossings. The homology for this knot, 15^n_{41127}, is shown in Fig. 10. Please note that $\widetilde{\mathcal{H}}_{\mathrm{odd}}(15^n_{41127})$ has 3-torsion (in gradings $(-2,-2)$ and $(-1,0)$), while $\widetilde{\mathcal{H}}(15^n_{41127})$ has none.

3.2 Lee Spectral Sequence and the Knight-Move Conjecture

In [L], Eun Soo Lee introduced a structure of a spectral sequence on the rational Khovanov chain complex $\mathcal{C}(D;\mathbb{Q})$ of a link diagram D. Namely, Lee defined a differential $d' : \mathcal{C}(D;\mathbb{Q}) \to \mathcal{C}(D;\mathbb{Q})$ of bidegree $(1,4)$ by setting

$$\begin{aligned} &m' : A\otimes A \to A: \quad m'(1\otimes 1) = m'(1\otimes X) = m'(X\otimes 1) = 0, \quad m'(X\otimes X) = 1, \\ &\Delta' : A \to A\otimes A: \quad \Delta'(1) = 0, \qquad \Delta'(X) = 1\otimes 1. \end{aligned} \tag{3.1}$$

It is straightforward to verify that d' is indeed a differential and that it anticommutes with d, that is, $d\circ d' + d'\circ d = 0$. This makes $(\mathcal{C}(D;\mathbb{Q}), d, d')$ into a double complex. Let d'_* be the differential induced by d' on $\mathcal{H}(L;\mathbb{Q})$. Lee proved that d'_* is functorial, that is, it commutes with the isomorphisms induced on $\mathcal{H}(L;\mathbb{Q})$ by isotopies of L. It follows that there exists a spectral sequence with $(E_1,d_1) = (\mathcal{C}(D;\mathbb{Q}),d)$ and $(E_2,d_2) = (\mathcal{H}(L;\mathbb{Q}),d'_*)$ that converges to the homology of the total (filtered) complex of $\mathcal{C}(D;\mathbb{Q})$ with respect to the differential $d+d'$. It is called the *Lee spectral sequence*. The differentials d_n in this spectral sequence have bidegree $(1,4(n-1))$.

Theorem 3.2.A (Lee [L]). *If L is an oriented link with $\#L$ components, then $H(Total(\mathcal{C}(D;\mathbb{Q})), d+d')$, the limit of the Lee spectral sequence, consists of 2^{n-1} copies of $\mathbb{Q}\oplus\mathbb{Q}$, each located in a specific homological grading that is explicitly defined by linking numbers of the components of L. In particular, if L is a knot, then the Lee spectral sequence converges to $\mathbb{Q}\oplus\mathbb{Q}$ localed in homological grading* 0.

The following theorem is the cornerstone in the definition of the Rasmussen invariant, one of the main applications of the Khovanov homology (see Sect. 4.1).

Theorem 3.2.B (Rasmussen [Ra2]). *If L is a knot, then the two copies of $\mathbb{Q}$ in the limiting term of the Lee spectral sequence for L are "neighbors," that is, their q-gradings differ by* 2.

Corollary 3.2.C. *If the Lee spectral sequence for a link L collapses after the second page (that is, $d_n = 0$ for $n \geq 3$), then $\mathcal{H}(L;\mathbb{Q})$ consists of one "pawn-move" pair in homological grading 0 and multiple "knight-move" pairs, shown below, with appropriate grading shifts.*

Pawn-move pair:

$\mathbb{Q}$
$\mathbb{Q}$

Knight-move pair:

	$\mathbb{Q}$
$\mathbb{Q}$	

Corollary 3.2.D. *Since d_3 has bidegree (1, 8), the Lee spectral sequence collapses after the second page for all knots with homological width 2 or 3, in particular, for all alternating and quasialternating knots. Hence, Corollary 3.2.C can be applied to such knots.*

Knight-Move Conjecture 3.2.E (Garoufalidis–Khovanov–Bar-Natan [BN1, Kh1]). The conclusion of Corollary 3.2.C is true for every knot.

Remark. There are currently no known counterexamples to the knight-move conjecture. In fact, the Lee spectral sequence can be proved (in one way or another) to collapse after the second page for every known example of Khovanov homology.

Remark. The Lee spectral sequence has no analogue in the odd and reduced Khovanov homology theories. The knight-move conjecture has no analogue in these theories either.

3.3 *Long Exact Sequence of the Khovanov Homology*

One of the most useful tools in studying Khovanov homology is the long exact sequence that categorifies Kauffman's *unoriented* skein relation for the Jones polynomial [Kh1]. If we forget about the grading, then it is clear from the construction from Sect. 2.2 that $\mathcal{C}(\asymp)$ is a subcomplex of $\mathcal{C}(\times)$ and $\mathcal{C}()\,() \simeq \mathcal{C}(\times)/\mathcal{C}(\asymp)$ (see also Fig. 5). Here, $\asymp$ and $)\,($ depict link diagrams where a single crossing $\times$ is resolved in a negative or respectively positive direction. This results in a short exact sequence of nongraded chain complexes:

$$0 \longrightarrow \mathcal{C}(\asymp) \xrightarrow{in} \mathcal{C}(\times) \xrightarrow{p} \mathcal{C}()\,() \longrightarrow 0, \tag{3.2}$$

where *in* is the inclusion and p is the projection.

In order to introduce grading into (3.2), we need to consider the cases in which the crossing to be resolved is either positive or negative. We get (see [Ra3])

$$\begin{gathered} 0 \longrightarrow \mathcal{C}(\asymp)\{2+3\delta\}[1+\delta] \xrightarrow{in} \mathcal{C}\left(\times_+\right) \xrightarrow{p} \mathcal{C}()\,()\{1\} \longrightarrow 0, \\ 0 \longrightarrow \mathcal{C}()\,()\{-1\} \xrightarrow{in} \mathcal{C}\left(\times_-\right) \xrightarrow{p} \mathcal{C}(\asymp)\{1+3\delta\}[\delta] \longrightarrow 0, \end{gathered} \tag{3.3}$$

where δ is the difference between the numbers of negative crossings in the unoriented resolution $\asymp$ (it has to be oriented somehow in order to define its Khovanov chain complex) and in the original diagram. The notation $\mathcal{C}[k]$ is used to represent a shift in the homological grading of a complex $\mathcal{C}$ by k. The graded versions of *in* and p are both homogeneous, that is, have bidegree $(0,0)$.

By passing to homology in (3.3), we get the following result.

Theorem 3.3.A (Khovanov, Viro, Rasmussen [Kh1, V, Ra3]). *The Khovanov homology is subject to the following long exact sequences:*

$$\begin{gathered} \cdots \longrightarrow \mathcal{H}()\,()\{1\} \xrightarrow{\partial} \mathcal{H}(\asymp)\{2+3\delta\}[1+\delta] \xrightarrow{in_*} \mathcal{H}\left(\times_+\right) \xrightarrow{p_*} \mathcal{H}()\,()\{1\} \longrightarrow \cdots \\ \cdots \longrightarrow \mathcal{H}()\,()\{-1\} \xrightarrow{in_*} \mathcal{H}\left(\times_-\right) \xrightarrow{p_*} \mathcal{H}(\asymp)\{1+3\delta\}[\delta] \xrightarrow{\partial} \mathcal{H}()\,()\{-1\} \longrightarrow \cdots \end{gathered} \tag{3.4}$$

where in_* and p_* are homogeneous and ∂ is the connecting differential with bidegree $(1,0)$.

Remark. The long exact sequences (3.4) work equally well over any ring R and for every version of the Khovanov homology, including the odd one (see [ORS]). This is both a blessing and a curse. On the one hand, this means that all of the properties of the even Khovanov homology that are proved using these long exact sequences (and most of them are) hold automatically for the odd Khovanov homology as well. On the other hand, this makes it very hard to find explanations for many differences among these homology theories.

4 Applications of the Khovanov Homology

In this section we collect some of the more prominent applications of the Khovanov homology theories. This list is by no means complete; it is chosen to provide the reader with a broader view of the type of problems that can be solved with the help of the Khovanov homology. We make a special effort to compare the performance of different versions of the homology, where applicable.

4.1 Rasmussen Invariant and Bounds on the Slice Genus

One of the most important known applications of Khovanov's construction was obtained by Jacob Rasmussen in 2004. In [Ra2], he used the structure of the Lee spectral sequence to define a new invariant of knots that gives a lower bound on the slice genus. More specifically, for a knot L, its Rasmussen invariant $s(L)$ is defined as the mean q-grading of the two copies of $\mathbb{Q}$ that remain in the homological grading 0 of the limiting term of the Lee spectral sequence; see Theorem 3.2.B. Since the q-gradings of these $\mathbb{Q}$'s are odd and differ by 2, the Rasmussen invariant is an even integer.

Theorem 4.1.A (Rasmussen [Ra2]). *Let L be a knot and $s(L)$ its Rasmussen invariant. Then*

- $|s(L)| \leq 2g_s(L)$*, where $g_s(L)$ is the slice genus of L, that is, the smallest possible genus of a smoothly embedded surface in the 4-ball D^4 that has $L \subset S^3 = \partial D^4$ as its boundary.*
- $s(L) = \sigma(L)$ *for alternating L, where $\sigma(L)$ is the signature of L.*
- $s(L) = 2g_s(L) = 2g(L)$ *for a knot L that possesses a planar diagram with positive crossings only, where $g(L)$ is the genus of L.*
- *If knots L_- and L_+ have diagrams that are different at a single crossing in such a way that this crossing is negative in L_- and positive in L_+, then $s(L_-) \leq s(L_+) \leq s(L_-) + 2$.*

Corollary 4.1.B. $s(T_{p,q}) = (p-1)(q-1)$ *for $p, q > 0$, where $T_{p,q}$ is the (p,q)-torus knot. This implies the Milnor conjecture, first proved by Kronheimer and Mrowka in 1993 using gauge theory [KM]. This conjecture states that the slice genus (and* *hence the genus) of* $T_{p,q}$ *is* $\frac{1}{2}((p-1)(q-1))$. *The upper bound on the slice genus is straightforward, so the lower bound provided by the Rasmussen invariant is sharp.*

Remark. Although the Rasmussen invariant was originally defined for knots only, its definition was later extended to the case of links by Anna Beliakova and Stephan Wehrli [BW].

The Rasmussen invariant can be used to search for knots that are topologically locally flatly slice but are not smoothly slice (see [Sh3]). A knot is slice if its slice genus is 0. Theorem 4.1.A implies that knots with nontrivial Rasmussen invariant are not smoothly slice. On the other hand, it was proved by Freedman [F] that knots with Alexander polynomial 1 are topologically locally flatly slice. There are 82 knots with up to 16 crossings that possess these two properties [Sh3]. Each such knot gives rise to a family of exotic $\mathcal{R}^4$ [GS, Exercise 9.4.23]. It is worth noticing that most of these 82 examples were not previously known.

The Rasmussen invariant was also used [P, Sh3] to deduce the combinatorial proof of the slice–Bennequin inequality. This inequality states that

$$g_s(\widehat{\beta}) \leq \frac{1}{2}(w(\beta) - k + 1), \tag{4.1}$$

where β is a braid on k strands with the closure $\widehat{\beta}$ and $w(\beta)$ its writhe number. The slice–Bennequin inequality provides one of the upper bounds for the Thurston–Bennequin number of Legendrian links (see below). It was originally proved by Lee Rudolph [Ru] using gauge theory. The approach via the Rasmussen invariant and Khovanov homology avoids gauge theory and symplectic Floer theory and results in a purely combinatorial proof.

Remark. Since Rasmussen's construction relies on the existence and convergence of the Lee spectral sequence, the Rasmussen invariant can be defined only for the even nonreduced Khovanov homology. In fact, a knot might not have any rational homology in the homological grading 0 of the odd Khovanov homology at all; see Fig. 12.

4.2 Bounds on the Thurston–Bennequin Number

Another useful applications of the Khovanov homology is in finding upper bounds on the Thurston–Bennequin number of Legendrian links. Consider $\mathcal{R}^3$ equipped with the standard contact structure $\mathrm{d}z - y\,\mathrm{d}x$. A link $K \subset \mathcal{R}^3$ is said to be *Legendrian* if it is everywhere tangent to the two-dimensional plane distribution defined as the kernel of this 1-form. Given a Legendrian link K, one defines its *Thurston–Bennequin number* $tb(K)$ as the linking number of K with its push-off K' obtained using a vector field that is tangent to the contact planes but orthogonal to the tangent vector field of K. Roughly speaking, $tb(K)$ measures the framing of the contact plane field around K. It is well known that the TB-number can be made arbitrarily small within the same class of topological links via stabilization, but that it is bounded from above.

Definition 4.2.A. For a given *topological* link L, let $\overline{tb}(L)$, the *TB-bound* of L, be the maximal possible TB-number among all the Legendrian representatives of L. In other words, $\overline{tb}(L) = \max_K\{tb(K)\}$, where K runs over all the Legendrian links in $\mathcal{R}^3$ that are topologically isotopic to L.

Finding TB-bounds for links has attracted considerable interest lately, since such bounds can be used to demonstrate that certain contact structures on $\mathcal{R}^3$ are not isomorphic to the standard one. Such bounds can be obtained from the Bennequin and slice–Bennequin inequalities, degrees of HOMFLY-PT and Kauffman polynomials, knot Floer homology, and so on (see [Ng] for more details). The TB-bound coming from the Kauffman polynomial is usually one of the strongest, since most of the others incorporate another invariant of Legendrian links, the rotation number, into the inequality. In [Ng], Lenhard Ng used Khovanov homology to define a new bound on the TB-number.

Theorem 4.2.B (Ng [Ng]). *Let L be an oriented link. Then*

$$\overline{tb}(L) \leq \min\Big\{k \Big| \bigoplus_{j-i=k} \mathcal{H}^{i,j}(L;R) \neq 0\Big\}. \tag{4.2}$$

Moreover, this bound is sharp for alternating links.

This Khovanov bound on the TB-number is often better than those that were known before. There are only two prime knots with up to 13 crossings for which the Khovanov bound is worse than the one coming from the Kauffman polynomial [Ng]. There are 45 such knots with at most 15 crossings.

Example 4.2.C. Figure 11 shows computations of the Khovanov TB-bound for the $(4,-5)$-torus knot. The Khovanov homology groups in (4.2) can be used over any ring R, and this example shows that the bound coming from the integral homology is sometimes better than that from the rational homology, due to a strategically placed torsion. It is interesting to note that the integral Khovanov bound of -20 is computed incorrectly in [Ng]. In particular, this was one of the cases in which Ng thought that the Kauffman polynomial provided a better bound. In fact, the TB-bound of -20 is sharp for this knot.

The proof of Theorem 4.2.B is based on the long exact sequences (3.4) and hence can be applied verbatim to the reduced as well as the odd Khovanov homology. By making appropriate adjustments to the grading, we immediately get [Sh4] that

$$\overline{tb}(L) \leq -1 + \min\Big\{k \Big| \bigoplus_{j-i=k} \widetilde{\mathcal{H}}^{i,j}(L;R) \neq 0\Big\}, \tag{4.3}$$

$$\overline{tb}(L) \leq -1 + \min\Big\{k \Big| \bigoplus_{j-i=k} \widetilde{\mathcal{H}}^{i,j}_{\mathrm{odd}}(L;R) \neq 0\Big\}. \tag{4.4}$$

	-9	-8	-7	-6	-5	-4	-3	-2	-1	0
-11										**1**
-13										**1**
-15								**1**		
-17						**1**		$\mathbf{1_2}$		
-19				**1**		**1**	**1**			
-21				**1**, $\mathbf{1_2}$	**1**					
-23		**1**	**1**	$\mathbf{1_2}$	**1**					
-25		$\boxed{\mathbf{1_4}}$	**1**							
-27	**1**, $\mathbf{1_2}$									
-29	$\mathbf{1_2}$									

j-i=-18

j-i=-20

TB-bound equals -18 over $\mathbb{Q}$ and -20 over $\mathbb{Z}$

Fig. 11 Khovanov TB-bound for the $(4, -5)$-torus knot

	0	1	2	3	4	5	6	7
13								**1**
11								$\mathbf{1_2}$
9						**1**	**1**	
7					**1**	$\mathbf{1_2}$		
5					**1**, $\mathbf{1_2}$			
3			**1**	**1**				
1	**1**		$\mathbf{1_2}$					
-1	**1**	**1**						

(even) Khovanov homology
TB-bound equals -2

	0	1	2	3	4	5	6	7
12								**1**
10							**1**	
8						**1**		
6					**2**			
4				**1**				
2			**1**, $\boxed{\mathbf{1_3}}$					
0		$\boxed{\mathbf{1_8}}$						
-2	$\boxed{\mathbf{1_3}}$							

odd reduced Khovanov homology
TB-bound equals -3

Fig. 12 Khovanov TB-bounds for the knot 12^n_{475}

As it turns out, the odd Khovanov TB-bound is often better than the even one. In fact, computations performed in [Sh4] show that the odd Khovanov homology provide the best upper bound on the TB-number among all currently known bounds for all prime knots with at most 15 crossings. In particular, the odd Khovanov TB-bound equals the Kauffman bound on all the 45 knots with at most 15 crossings where the latter is better than the even Khovanov TB-bound.

Example 4.2.D. The odd Khovanov TB-bound is better than the even bound and equals the Kauffman bound for the knot 12^n_{475}, as shown in Fig. 12.

4.3 Finding Quasialternating Knots

Quasialternating links were introduced by Ozsváth and Szabó in [OS3] as a way to generalize the class of alternating links while retaining most of their important properties.

Definition 4.3.A. The class $\mathcal{Q}$ of quasialternating links is the smallest set of links such that

- The unknot belongs to $\mathcal{Q}$.
- If a link L has a planar diagram D such that the two resolutions of this diagram at one crossing represent two links, L_0 and L_1, with the properties that $L_0, L_1 \in \mathcal{Q}$ and $\det(L) = \det(L_0) + \det(L_1)$, then $L \in \mathcal{Q}$ as well.

Remark. It is well known that all nonsplit alternating links are quasialternating.

The main motivation for studying quasialternating links is the fact that the double branched covers of S^3 along such links are so-called L-spaces. A 3-manifold M is called an *L-space* if the order of its first homology group H_1 is finite and equals the rank of the Heegaard–Floer homology of M (see [OS3]). Unfortunately, due to the recursive style of Definition 4.3.A, it is often highly nontrivial to prove that a given link is quasialternating. It is equally challenging to show that it is not.

To determine that a link is not quasialternating, one usually employs the fact that such links have homologically thin Khovanov homology over $\mathbb{Z}$ and knot Floer homology over $\mathbb{Z}_2$ (see [MO]). Thus, $\mathbb{Z}$H-thick knots are not quasialternating. There are 12 such knots with up to 10 crossings. Most of the others can be shown to be quasialternating by various constructions. After the work of Champanerkar and Kofman [ChK], there were only two knots left, 9_{46} and 10_{140}, for which it was not known whether they were quasialternating. Both of them have homologically thin Khovanov and knot Floer homologies.

As it turns out, odd Khovanov homology is much better at detecting quasialternating knots. The proof of the fact that such knots are $\mathbb{Z}$H-thin is based on the long exact sequences (3.4), and therefore can be applied verbatim to the odd homology as well [ORS]. Computations show [Sh4] that the knots 9_{46} and 10_{140} have homologically thick odd Khovanov homology and hence are not quasialternating; see Figs. 13 and 14.

It is worth mentioning that the knots 9_{46} and 10_{140} are $(3,3,-3)$- and $(3,4,-3)$-pretzel knots, respectively (see Fig. 15 for the definition). Computations show that $(n,n,-n)$- and $(n,n+1,-n)$-pretzel links for $n \leq 6$ all have torsion of order n outside of the main diagonal that supports the free part of the homology. This suggests a certain n-fold symmetry on the odd Khovanov chain complexes for these pretzel links that cannot be explained by the construction.

Remark. Joshua Greene has recently determined [Gr] all quasialternating pretzel links by considering 4-manifolds that are bounded by the branched double covers of the links. In particular, he found several knots that are quasialternating, yet both H-thin and OH-thin. The smallest such knot is 11^n_{50}.

	-6	-5	-4	-3	-2	-1	0
0							**2**
-2						**1**	
-4					**1**		
-6				**2**			
-8			**1**				
-10		**1**					
-12	**1**						

(even) reduced Khovanov homology

	-6	-5	-4	-3	-2	-1	0
0							**2**
-2						**1**	$\boxed{\mathbf{1_3}}$
-4					**1**		
-6				**2**			
-8			**1**				
-10		**1**					
-12	**1**						

odd reduced Khovanov homology

Fig. 13 Khovanov homology of 9_{46}, the $(3, 3, -3)$-pretzel knot

	-7	-6	-5	-4	-3	-2	-1	0
0								**1**
-2							**1**	
-4						**1**		
-6					**1**			
-8				**2**				
-10			**1**					
-12		**1**						
-14	**1**							

(even) reduced Khovanov homology

	-7	-6	-5	-4	-3	-2	-1	0
0								**1**
-2							**1**	
-4						**1**	$\boxed{\mathbf{1_3}}$	
-6					**1**			
-8				**2**				
-10			**1**					
-12		**1**						
-14	**1**							

odd reduced Khovanov homology

Fig. 14 Khovanov homology of 10_{140}, the $(3, 4, -3)$-pretzel knot

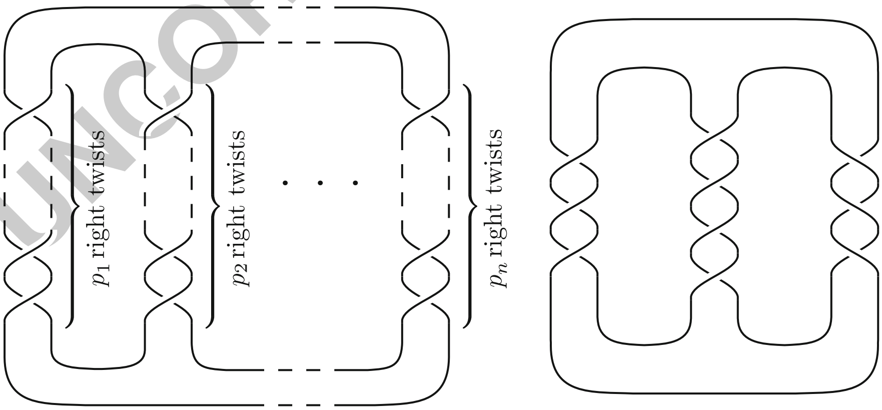


Fig. 15 $(p_1, p_2, \ldots, p_n)$-pretzel link and $(3, 4, -3)$-pretzel knot 10_{140}

4.4 Detection of the Unknot

It was conjectured since the introduction of Khovanov homology that it detects the unknot. Although this conjecture is still open as of this writing, a recent result by Matthew Hedden [H] shows that the unknot can be detected by the Khovanov homology of the 2-cable. More specifically, Hedden proved that a link L is a trivial knot if and only if $\mathrm{rk}(\mathcal{H}(L^2;\mathbb{Z}_2)) = 4$, where L^2 is the 2-cable of L. This development is a major step toward proving a longstanding conjecture that the Jones polynomial itself detects the unknot.

Acknowledgements This paper was originally presented at the Marcus Wallenberg Symposium on Perspectives in Analysis, Geometry, and Topology at Stockholm University in May 2008. The author would like to thank all the organizers of the symposium for a very successful and productive meeting. He extends his special thanks to Ilia Itenberg, Burglind Jöricke, and Mikael Passare, the editors of these proceedings, for their patience with the author. The author is indebted to Mikhail Khovanov for much advice and enlightening discussions during the work on this paper. Finally, the author would like to express his deepest gratitude to Oleg Yanovich Viro for introducing him to the wonderful world of topology twenty years ago and for continuing to be his guide in this world ever since.

References

[AKh] M. Asaeda and M. Khovanov, *Notes on link homology*, in *Low dimensional topology*, 139–195, IAS/Park City Math. Ser., 15, Amer. Math. Soc., Providence, RI, 2009; arXiv:0804.1279.

[BN1] D. Bar-Natan, *On Khovanov's categorification of the Jones polynomial*, Alg. Geom. Top., **2** (2002) 337–370; arXiv:math.QA/0201043.

[BN2] D. Bar-Natan, *Fast Khovanov Homology Computations*, J. Knot Th. and Ramif. **16** (2007), no. 3, 243–255; arXiv:math.GT/0606318.

[BNG] D. Bar-Natan and J. Green, `JavaKh`–a fast program for computing Khovanov homology, part of the `KnotTheory` Mathematica Package, `http://katlas.math.utoronto.ca/wiki/Khovanov_Homology`.

[BW] A. Beliakova and S. Wehrli, *Categorification of the colored Jones polynomial and Rasmussen invariant of links*, Canad. J. Math. **60** (2008), no. 6, 1240–1266; arXiv:math.GT/0510382.

[B] J. Bloom, *Odd Khovanov homology is mutation invariant*, arXiv:0903.3746.

[ChK] A. Champanerkar and I. Kofman, *Twisting quasi-alternating links*, Proc. Amer. Math. Soc. **137** (2009), 2451–2458; arXiv:0712.2590.

[DGShT] N. Dunfield, S. Garoufalidis, A. Shumakovitch, and M. Thistlethwaite, *Behavior of knot invariants under genus 2 mutation*, arXiv:math.GT/0607258.

[F] M. Freedman, *A surgery sequence in dimension four; the relations with knot concordance*, Invent. Math. **68** (1982), no. 2, 195–226.

[GS] R. Gompf and A. Stipsicz, 4-*manifolds and Kirby calculus*, Graduate Studies in Mathematics **20**, American Mathematical Society, Providence, RI, 1999.

[Gr] J. Greene, *Homologically thin, non-quasi-alternating links*, arXiv:0906.2222.

[H] M. Hedden, *Khovanov homology of the 2-cable detects the unknot*, arXiv:0805.4418.

[HTh] J. Hoste and M. Thistlethwaite, `Knotscape`—a program for studying knot theory and providing convenient access to tables of knots, `http://www.math.utk.edu/~morwen/knotscape.html`

[J] V. Jones, *A polynomial invariant for knots via von Neumann algebras*, Bull. Amer. Math. Soc. **12** (1985), 103–111.

[K] L. Kauffman, *State models and the Jones polynomial*, Topology **26** (1987), no. 3, 395–407.

[Kh1] M. Khovanov, *A categorification of the Jones polynomial*, Duke Math. J. **101** (2000), no. 3, 359–426; arXiv:math.QA/9908171.

[Kh2] M. Khovanov, *Patterns in knot cohomology I*, Exp. Math. **12** (2003), no. 3, 365–374; arXiv:math.QA/0201306.

[Kh3] M. Khovanov, *Categorifications of the colored Jones polynomial*, J. Knot Theory Ramif. **14** (2005), no. 1, 111–130; arXiv:math.QA/0302060.

[Kh4] M. Khovanov, $\mathfrak{sl}(3)$ *link homology I*, Algebr. Geom. Topol. **4** (2004), 1045–1081; arXiv:math.QA/0304375.

[Kh5] M. Khovanov, *Link homology and Frobenius extensions*, Fundamenta Mathematicae, **190** (2006), 179–190; arXiv:math.QA/0411447.

[Kh6] M. Khovanov, *Link homology and categorification*, ICM–2006, Madrid, Vol. II, 989–999, Eur. Math. Soc., Zürich, 2006; arXiv:math/0605339.

[KhR1] M. Khovanov and L. Rozansky, *Matrix factorizations and link homology*, Fund. Math. **199** (2008), no. 1, 1–91; arXiv:math.QA/0401268.

[KhR2] M. Khovanov and L. Rozansky, *Matrix factorizations and link homology II*, Geom. Topol. **12** (2008), no. 3, 1387–1425; arXiv:math.QA/0505056.

[KhR3] M. Khovanov and L. Rozansky, *Virtual crossings, convolutions and a categorification of the* $SO(2N)$ *Kauffman polynomial*, J. Gökova Geom. Topol. GGT **1** (2007), 116–214; arXiv:math/0701333.

[KM] P. Kronheimer and T. Mrowka, *Gauge theory for embedded surfaces I*, Topology **32** (1993), 773–826.

[L] E. S. Lee, *An endomorphism of the Khovanov invariant*, Adv. Math. **197** (2005), no. 2, 554–586; arXiv:math.GT/0210213.

[MO] C. Manolescu, P. Ozsváth, *On the Khovanov and knot Floer homologies of quasi-alternating links* in Proceedings of Gökova Geometry-Topology Conference 2007, 60–81, Gökova Geometry/Topology Conference (GGT), Gökova, 2008; arXiv: 0708.3249.

[Ng] L. Ng, *A Legendrian Thurston-Bennequin bound from Khovanov homology*, Algebr. Geom. Topol. **5** (2005) 1637–1653; arXiv:math.GT/0508649.

[ORS] P. Ozsváth, J. Rasmussen, and Z. Szabó, *Odd Khovanov homology*, arXiv:0710.4300.

[OS1] P. Ozsváth and Z. Szabó, *Holomorphic disks and topological invariants for closed three-manifolds*, Ann. Math. **159** (2004) 1027–1158; arXiv:math.SG/0101206.

[OS2] P. Ozsváth and Z. Szabó, *Holomorphic disks and knot invariants*, Adv. Math. **186** (2004), no. 1, 58–116; arXiv:math.GT/0209056.

[OS3] P. Ozsváth and Z. Szabó, *On the Heegaard Floer homology of branched double-covers*, Adv. Math. **194** (2005), no. 1, 1–33; arXiv:math.GT/0309170.

[P] O. Plamenevskaya, *Transverse knots and Khovanov homology*, Math. Res. Lett. **13** (2006), no. 4, 571–586; arXiv:math.GT/0412184.

[Ra1] J. Rasmussen, *Floer homology and knot complements*, Ph.D. Thesis, Harvard U.; arXiv:math.GT/0306378.

[Ra2] J. Rasmussen, *Khovanov homology and the slice genus*, to appear in Inventiones Math.; arXiv:math.GT/0402131.

[Ra3] J. Rasmussen, *Knot polynomials and knot homologies*, Geometry and Topology of Manifolds (Boden et al., eds.), Fields Institute Commun **47** (2005) 261–280, AMS; arXiv:math.GT/0504045.

[Ro] D. Rolfsen, *Knots and Links*, Publish or Perish, Mathematics Lecture Series 7, Wilmington 1976.

[Ru] L. Rudolph, *Quasipositivity as an obstruction to sliceness*, Bull Amer. Math. Soc. (N.S.) **29** (1993), no. 1, 51–59; arXiv:math.GT/9307233.

AQ2

[Sh1] A. Shumakovitch, `KhoHo`—a program for computing and studying Khovanov homology, `http://www.geometrie.ch/KhoHo`.

[Sh2] A. Shumakovitch, *Torsion of the Khovanov homology*, arXiv:math.GT/0405474; to appear in Fund. Math.

[Sh3] A. Shumakovitch, *Rasmussen invariant, slice–Bennequin inequality, and sliceness of knots*, J. Knot Theory Ramif., **16** (2007), no. 10, 1403–1412; arXiv:math.GT/0411643.

[Sh4] A. Shumakovitch, *Patterns in odd Khovanov homology*, Preprint, 2008.

[V] O. Viro, *Khovanov homology, its definitions and ramifications*, Fund. Math. **184** (2004), 317–342; arXiv:math.GT/0202199.

[W1] S. Wehrli, *Khovanov Homology and Conway Mutation*, arXiv:math/0301312.

[W2] S. Wehrli, *Mutation invariance of Khovanov homology over* $\mathbb{F}_2$, arXiv:0904.3401.

[Wu] H. Wu, *A colored* $\mathfrak{sl}(N)$*-homology for links in* S^3, arXiv:0907.0695.

[Y] Y. Yonezawa, *Quantum* $(\mathfrak{sl}_n, \wedge V_n)$ *link invariant and matrix factorizations*, arXiv:0906.0220.

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AQ1. Please update the reference (Rasmussen, to appear).
AQ2. Please update (Shumakovitch, to appear).