Floer Homology on the Extended Moduli Space

Ciprian Manolescu and Christopher Woodward

Abstract Starting from a Heegaard splitting of a three-manifold, we use Lagrangian Floer homology to construct a three-manifold invariant in the form of a relatively $\mathbb{Z}/8\mathbb{Z}$-graded abelian group. Our motivation is to have a well-defined symplectic version of the Atiyah–Floer conjecture for arbitrary three-manifolds. The symplectic manifold used in the construction is the extended moduli space of flat $\mathrm{SU}(2)$-connections on the Heegaard surface. An open subset of this moduli space carries a symplectic form, and each of the two handlebodies in the decomposition gives rise to a Lagrangian inside the open set. In order to define their Floer homology, we compactify the open subset by symplectic cutting; the resulting manifold is only semipositive, but we show that one can still develop a version of Floer homology in this setting.

Keywords Floer homology • Three-manifold • Moduli space • Heegaard surface

1 Introduction

Floer's instanton homology [15] is an invariant of integral homology three-spheres Y that serves as target for the relative Donaldson invariants of four-manifolds with boundary; see [13]. It is defined from a complex whose generators are (suitably perturbed) irreducible flat connections in a trivial $\mathrm{SU}(2)$-bundle over Y, and whose differentials arise from counting anti-self-dual $\mathrm{SU}(2)$-connections on $Y \times \mathbb{R}$. There

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is also a version of instanton Floer homology using connections in $U(2)$-bundles with c_1 odd [9, 17], an equivariant version [4, 5], and several other variants that use both irreducible and reducible flat connections [13]. More recently, Kronheimer and Mrowka [29] have developed instanton homology for sutured manifolds; a particular case of their theory leads to a version of instanton homology that can be defined for arbitrary closed three-manifolds.

In another remarkable paper [16], Floer associated a homology theory to two Lagrangian submanifolds of a symplectic manifold, under suitable assumptions. This homology is defined from a complex whose generators are intersection points between the two Lagrangians, and whose differentials count pseudoholomorphic strips. The Atiyah–Floer conjecture [2] states that Floer's two constructions are related: for any decomposition of the homology sphere Y into two handlebodies glued along a Riemann surface Σ, instanton Floer homology should be the same as the Lagrangian Floer homology of the $SU(2)$-character varieties of the two handlebodies, viewed as subspaces of the character variety of Σ.

As stated, an obvious problem with the Atiyah–Floer conjecture is that the symplectic side is ill defined: due to the presence of reducible connections, the $SU(2)$-character variety of Σ is not smooth. One way of dealing with the singularities is to use a version of Lagrangian Floer homology defined via the symplectic vortex equations on the infinite-dimensional space of all connections. This approach was pursued by Salamon and Wehrheim, who obtained partial results toward the conjecture in this setup; see [49, 50, 56]. Another approach is to avoid reducibles altogether by using nontrivial $PU(2)$-bundles instead. This road was taken by Dostoglou and Salamon [14], who proved a variant of the conjecture for mapping tori.

The goal of this paper is to construct another candidate that could sit on the symplectic side of the (suitably modified) Atiyah–Floer conjecture.

Here is a short sketch of the construction. Let Σ be a Riemann surface of genus $h \geq 1$, and $z \in \Sigma$ a base point. The moduli space $\mathscr{M}(\Sigma)$ of flat connections in a trivial $SU(2)$-bundle over Σ can be identified with the character variety $\{\rho : \pi_1(\Sigma) \to SU(2)\}/PU(2)$. The moduli space $\mathscr{M}(\Sigma)$ is typically singular. However, Jeffrey [26] and independently Huebschmann [23], showed that $\mathscr{M}(\Sigma)$ is the symplectic quotient of a different space, called the extended moduli space, by a Hamiltonian $PU(2)$-action. The extended moduli space is naturally associated not to Σ, but to Σ', a surface with boundary obtained from Σ by deleting a small disk around z. The extended moduli space has an open smooth stratum, which Jeffrey and Huebschmann equip with a natural closed two-form. This form is nondegenerate on a certain open set $\mathscr{N}(\Sigma')$, which we take as our ambient symplectic manifold. In fact, $\mathscr{N}(\Sigma')$ can also be viewed as an open subset of the Cartesian product $SU(2)^{2h} \cong \{\rho : \pi_1(\Sigma') \to SU(2)\}$. More precisely, if we pick $2h$ generators for the free groups $\pi_1(\Sigma')$, we can describe this subset as

$$\mathscr{N}(\Sigma') = \left\{ (A_1, B_1, \ldots, A_h, B_h) \in SU(2)^{2h} \mid \prod_{i=1}^{h} [A_i, B_i] \neq -I \right\}. \quad (62)$$

Consider a Heegaard decomposition of a three-manifold Y as $Y = H_0 \cup H_1$, where the handlebodies H_0 and H_1 are glued along their common boundary Σ. There are smooth Lagrangians $L_i = \{\pi_1(H_i) \to \mathrm{SU}(2)\} \subset \mathscr{N}(\Sigma')$ for $i = 0, 1$. In order to take the Lagrangian Floer homology of L_0 and L_1, care must be taken with holomorphic strips going out to infinity; indeed, the symplectic manifold $\mathscr{N}(\Sigma')$ is not weakly convex at infinity. Our remedy is to compactify $\mathscr{N}(\Sigma')$ by (nonabelian) symplectic cutting. The resulting manifold $\mathscr{N}^c(\Sigma')$ is the union of $\mathscr{N}(\Sigma')$ and a codimension-two submanifold R. A new problem shows up here, because the natural two-form $\tilde{\omega}$ on $\mathscr{N}^c(\Sigma')$ has degeneracies on R. Nevertheless, $(\mathscr{N}^c(\Sigma'), \tilde{\omega})$ is monotone, in a suitable sense. One can deform $\tilde{\omega}$ into a symplectic form ω, at the expense of losing monotonicity. We are thus led to develop a version of Lagrangian Floer theory on $\mathscr{N}^c(\Sigma')$ by making use of the interplay between the forms $\tilde{\omega}$ and ω. Our Floer complex uses only holomorphic disks lying in the open part $\mathscr{N}(\Sigma')$ of $\mathscr{N}^c(\Sigma')$. We show that while holomorphic strips with boundary on L_0 and L_1 can go to infinity in $\mathscr{N}(\Sigma')$, they do so only in high codimension, without affecting the Floer differential. The resulting Floer homology group is denoted by

$$HSI(\Sigma; H_0, H_1) = HF(L_0, L_1 \text{ in } \mathscr{N}(\Sigma')),$$

and it admits a relative $\mathbb{Z}/8\mathbb{Z}$-grading. We call it *symplectic instanton homology*.

Using the theory of Lagrangian correspondences and pseudoholomorphic quilts developed in Wehrheim–Woodward [59] and Lekili–Lipyanskiy [30], we prove the following theorem.

Theorem 1. *The relatively $\mathbb{Z}/8\mathbb{Z}$-graded group $HSI(Y) = HSI(\Sigma; H_0, H_1)$ is an invariant of the three-manifold Y.*

Strictly speaking, if we are interested in canonical isomorphisms, then the symplectic instanton homology also depends on the base point $z \in \Sigma \subset Y$: as z varies inside Y, the corresponding groups form a local system. However, we drop z from the notation for simplicity.

Let us explain how we expect $HSI(Y)$ to be related to the traditional instanton theory on 3-manifolds. We restrict our attention to the original setup for Floer's instanton theory $I(Y)$ from [15] when Y is an integral homology sphere. It is then decidedly not the case that $HSI(Y)$ coincides with Floer's theory; for example, we have $HSI(S^3) \cong \mathbb{Z}$, but $I(S^3) = 0$. Nevertheless, in [13, Sect. 7.3.3], Donaldson introduced a different version of instanton homology, a $\mathbb{Z}/8\mathbb{Z}$-graded vector field over $\mathbb{Q}$ denoted by $\widetilde{HF}$ that satisfies $\widetilde{HF}(S^3) \cong \mathbb{Q}$. (Floer's theory I is denoted by HF in [13].) We state the following variant of the Atiyah–Floer conjecture:

Conjecture 1. For every integral homology sphere Y, the symplectic instanton homology $HSI(Y) \otimes \mathbb{Q}$ and the Donaldson–Floer homology $\widetilde{HF}(Y)$ from [13] are isomorphic as relatively $\mathbb{Z}/8\mathbb{Z}$-graded vector spaces.

Alternatively, one could hope to relate HSI to the sutured version of instanton Floer homology developed by Kronheimer and Mrowka in [29]. More open questions, and speculations along these lines, are presented in Sect. 7.3.

2 Floer Homology

2.1 *The Monotone, Nondegenerate Case*

Lagrangian Floer homology was originally constructed in [16] under some restrictive conditions, and later generalized by various authors to many different settings. We review here its definition in the monotone case, due to Oh [39, 41], together with a discussion of orientations following Fukaya–Oh–Ohta–Ono [18].

Let (M,ω) be a compact connected symplectic manifold. We denote by $\mathcal{J}(M,\omega)$ the space of compatible almost complex structures on (M,ω), and by $\mathcal{J}_t(M,\omega) = C^\infty([0,1],\mathcal{J}(M,\omega))$ the space of *time-dependent* compatible almost complex structures. Any compatible almost complex structure J defines a complex structure on the tangent bundle TM. Since $\mathcal{J}(M,\omega)$ is contractible, the first Chern class $c_1(TM) \in H^2(M,\mathbb{Z})$ depends only on ω, not on J. The minimal Chern number N_M of M is defined as the positive generator of the image of $c_1(TM) : \pi_2(M) \to \mathbb{Z}$.

Definition 1. Let (M,ω) be a symplectic manifold. Then M is called *monotone* if there exists $\kappa > 0$ such that

$$[\omega] = \kappa \cdot c_1(TM).$$

In that case, κ is called the monotonicity constant.

Definition 2. A Lagrangian submanifold $L \subset (M,\omega)$ is called *monotone* if there exists a constant $\kappa > 0$ such that

$$2[\omega]|_{\pi_2(M,L)} = \kappa \cdot \mu_L,$$

where $\mu_L : \pi_2(M,L) \to \mathbb{Z}$ is the Maslov index.

Necessarily, if L is monotone, then M is monotone with the same monotonicity constant. The minimal Maslov number N_L of a monotone Lagrangian L is defined as the positive generator of the image of μ_L in $\mathbb{Z}$.

From now on, we will assume that M is monotone with monotonicity constant κ and that we are given two closed, simply connected Lagrangians $L_0, L_1 \subset M$. These conditions imply that L_0 and L_1 are monotone with the same monotonicity constant and

$$N_{L_0} = N_{L_1} = 2N_M.$$

We assume that $N_M > 1$ and set $N = 2N_M \geq 4$. We also assume that $w_2(L_0) = w_2(L_1) = 0$.

After a small Hamiltonian perturbation, we can arrange things so that the intersection $L_0 \cap L_1$ is transverse. Let $(J_t)_{0\leq t\leq 1} \in \mathcal{J}_t(M,\omega)$. For any $x_\pm \in L_0 \cap L_1$, we denote by $\widetilde{\mathfrak{M}}(x_+,x_-)$ the space of *Floer trajectories* (or J_t*-holomorphic strips*) from x_+ to x_-, i.e., finite-energy solutions to Floer's equation

$$\begin{cases} u : \mathbb{R} \times [0,1] \to M, \\ u(s,j) \in L_j, \quad j = 0,1, \\ \partial_s u + J_t(u)\partial_t u = 0, \\ \lim_{s\to\pm\infty} u(s,\cdot) = x_\pm. \end{cases} \tag{13.1}$$

Let $\mathfrak{M}(x_+,x_-)$ denote the quotient of $\tilde{\mathfrak{M}}(x_+,x_-)$ by the translational action of $\mathbb{R}$. For $(J_t)_{0\le t\le 1}$ chosen from a comeager[1] subset $\mathcal{J}_t^{\mathrm{reg}}(L_0,L_1) \subset \mathcal{J}_t(M,\omega)$ of (L_0,L_1)-*regular*, time-dependent compatible almost complex structures, $\mathfrak{M}(x_+,x_-)$ is a smooth, finite-dimensional manifold with dimension a nonconstant $u \in \mathfrak{M}(x_+,x_-)$ given by $\dim T_u\mathfrak{M}(x_+,x_-) = I(u) - 1$. We denote by $\mathfrak{M}(x_+,x_-)_d$ the subset with $I(u) - 1 = d$ (note that $\mathfrak{M}(x_+,x_-)_{-1}$ is nonempty if $x_+ = x_-$). As explained in Oh [39], after shrinking $\mathcal{J}_t^{\mathrm{reg}}(L_0,L_1)$ further, we may assume that $\mathfrak{M}(x_+,x_-)_0$ is finite and $\mathfrak{M}(x_+,x_-)_1$ is compact up to breaking of trajectories:

$$\partial\mathfrak{M}(x_+,x_-)_1 = \bigcup_{y\in L_0\cap L_1} \mathfrak{M}(x_+,y)_0 \times \mathfrak{M}(y,x_-)_0. \tag{13.2}$$

The condition that the Lagrangians have vanishing w_2 is used in defining orientations on the moduli spaces, compatible with the identity (13.2). The Floer chain complex is then defined to be the free abelian group generated by the intersection points

$$CF(L_0,L_1) = \bigoplus_{x\in L_0\cap L_1} \mathbb{Z}\langle x\rangle .$$

The Floer differential is

$$\partial\langle x_+\rangle = \sum_{u\in\mathfrak{M}(x_+,x_-)_0} \varepsilon(u)\langle x_-\rangle ,$$

where $\varepsilon(u) \in \{\pm 1\}$ is the sign comparing the orientation of the moduli space to the canonical orientation of a point; see, for example, [58].

Our assumptions allow one to define a relative Maslov index $I(x,y) \in \mathbb{Z}/N\mathbb{Z}$ for every $x,y \in L_0\cap L_1$ such that $I(x,y) \equiv I(u) \pmod N$ for any $u \in \mathfrak{M}(x,y)$. The relative index satisfies $I(x,y) + I(y,z) = I(x,z)$, and it induces a relative $\mathbb{Z}/N\mathbb{Z}$-grading on the chain complex.

The Lagrangian Floer homology groups $HF(L_0,L_1)$ are the homology groups of $CF_*(L_0,L_1)$ with respect to the differential ∂. Equation (13.2) implies that $\partial^2 = 0$. An important property of the Floer homology groups $HF(L_0,L_1)$ is that they are

[1] A subset of a topological space is *comeager* if it is the intersection of countably many open dense subsets. Many authors use the term "Baire second category," which, however, denotes more generally subsets that are not meager, i.e., not the complement of a comeager subset. See, for example, [48, Chap. 7.8].

independent of the choice of path of almost complex structures, and invariant under Hamiltonian isotopies of both L_0 and L_1. Since $H_1(L_0) = H_1(L_1) = 0$, any isotopy of L_0 or L_1 through Lagrangians can be embedded in an ambient Hamiltonian isotopy; see, for example [44, Sect. 6.1] or the discussion in [52, Sect. 4(D)].

2.2 A Relative Version

Let $R \subset M$ denote a symplectic hypersurface disjoint from the Lagrangians L_0, L_1. Each pseudoholomorphic strip $u : \mathbb{R} \times [0,1] \to M$ meeting R in a finite number of points has a well-defined intersection number $u \cdot R$, defined by a signed count of intersection points of generic perturbations. The intersection numbers $u \cdot R$ depend only on the relative homology class of u, and are additive under concatenation of trajectories:

$$(u\#v) \cdot R = (u \cdot R) + (v \cdot R).$$

Let $\mathcal{J}(M,\omega,R)$ denote the space of compatible almost complex structures J for which R is a J-holomorphic submanifold. Let also $\mathcal{J}_t(M,\omega,R) = C^\infty([0,1], \mathcal{J}(M,\omega,R))$ be the corresponding space of time-dependent almost complex structures. If $(J_t) \in \mathcal{J}_t(M,\omega,R)$, then the intersection number of any J_t-holomorphic strip with R is a finite sum of positive local intersection numbers; see, for example, Cieliebak–Mohnke [11, Proposition 7.1]. In particular, if a J_t-holomorphic strip has trivial intersection number with R, it must be disjoint from R.

One can show that $\mathcal{J}_t^{\mathrm{reg}}(L_0,L_1,R) = \mathcal{J}_t^{\mathrm{reg}}(L_0,L_1) \cap \mathcal{J}_t(M,\omega,R)$ is comeager in $\mathcal{J}_t(M,\omega,R)$. Since L_0, L_1 are disjoint from R, Floer homology may be defined using $J_t \in \mathcal{J}_t(M,\omega,R)$. Moreover, for $J \in \mathcal{J}_t^{\mathrm{reg}}(L_0,L_1,R)$, the Floer differential decomposes as the sum

$$\partial = \sum_{m \geq 0} \partial_m,$$

where ∂_m counts the trajectories with intersection number m with R. By additivity of the intersection numbers, the square of the Floer differential satisfies the refined equality

$$\sum_{i+j=m} \partial_i \partial_j = 0.$$

In particular, $\partial_0^2 = 0$. Let $HF(L_0,L_1;R)$ denote the homology of ∂_0, counting Floer trajectories disjoint from R. We call $HF(L_0,L_1;R)$ the *Lagrangian Floer homology of* L_0, L_1 *relative to the hypersurface* R. This kind of construction has previously appeared in the literature in various guises; see, for example, Seidel's deformation of the Fukaya category [51, p. 8] or the hat version of Heegaard Floer homology [43]. Note that $HF(L_0,L_1;R)$ admits a relative $\mathbb{Z}/N'\mathbb{Z}$-grading, where $N' = 2N_{M\setminus R}$ is a positive multiple of N.

A standard continuation argument then shows that $HF(L_0, L_1; R)$ is independent of the choice of $J_t \in \mathcal{J}_t^{\mathrm{reg}}(L_0, L_1, R)$. Indeed, any two such compatible almost complex structures can be joined by a path $J_{t,\rho}, \rho \in [0,1]$, which equips the fiber bundle $\mathbb{R} \times [0,1] \times M \to \mathbb{R} \times [0,1]$ with an almost complex structure. The part of the continuation map counting pseudoholomorphic sections with zero intersection number with the almost complex submanifold $\mathbb{R} \times [0,1] \times R$ defines an isomorphism from the two Floer homology groups.

In fact, we may assume that all Floer trajectories are transverse to R by the following argument, which holds for not necessarily monotone M.

For any $k \in \mathbb{N}$, we denote by $\mathfrak{M}(x_+, x_-; k)$ the subset of $\mathfrak{M}(x_+, x_-)$ with a tangency of order exactly k to R. Given an open subset $\mathcal{W} \subset M$ containing L_0 and L_1 with closure disjoint from R and a $\tilde{J} \in \mathcal{J}(M, \omega, R)$, we denote by $\mathcal{J}_t(M, \omega, \mathcal{W}, \tilde{J})$ the space of compatible almost complex structures that agree with $\tilde{J}$ outside $\mathcal{W}$.

Lemma 1. *There exists a comeager subset $\mathcal{J}_t^{\mathrm{reg}}(L_0, L_1, \mathcal{W}, \tilde{J})$ of $\mathcal{J}_t(M, \omega, \mathcal{W}, \tilde{J})$ contained in $\mathcal{J}_t^{\mathrm{reg}}(L_0, L_1, R)$ such that for any $(J_t) \in \mathcal{J}_t^{\mathrm{reg}}(L_0, L_1, \mathcal{W}, \tilde{J})$, the corresponding moduli space $\mathfrak{M}(x_+, x_-)$ is a smooth manifold, and for every $k \in \mathbb{N}$ and $x_\pm \in L_0 \cap L_1$, $\mathfrak{M}(x_+, x_-; k)$ is a smooth submanifold of $\mathfrak{M}(x_+, x_-)$ of codimension $2k$.*

Proof. For the closed case, see Cieliebak–Mohnke [11, Proposition 6.9]. The proofs for Floer trajectories and compatible almost complex structures are the same, since the Lagrangians are disjoint from R. Note that [11] uses tamed almost complex structures; however, the arguments apply equally well to compatible almost complex structures; see [33, p. 47].

Corollary 1. *If $(J_t) \in \mathcal{J}_t^{\mathrm{reg}}(L_0, L_1, \mathcal{W}, \tilde{J})$, then for every element of $\mathfrak{M}(x_+, x_-)_0$ and $\mathfrak{M}(x_+, x_-)_1$, the intersection with R is transversal, and the number of intersection points equals the intersection pairing with R.*

2.3 Floer Homology on Semipositive Manifolds

In this section, we extend the definition of Floer homology to a semipositive setting. More precisely, we assume the following:

Assumption 2.1. (i) *(M, ω) is a compact symplectic manifold.*

(ii) *$\tilde{\omega}$ is a closed two-form on M.*

(iii) *The degeneracy locus $R \subset M$ of $\tilde{\omega}$ is a symplectic hypersurface with respect to ω.*

(iv) *$\tilde{\omega}$ is monotone, i.e., $[\tilde{\omega}] = \kappa \cdot c_1(TM)$ for some $\kappa > 0$.*

(v) *The restrictions of $\tilde{\omega}$ and ω to $M \setminus R$ have the same cohomology class in $H^2(M \setminus R)$.*

(vi) *The forms $\tilde{\omega}$ and ω themselves coincide on an open subset $\mathcal{W} \subset M \setminus R$.*

(vii) *We are given two closed submanifolds $L_0, L_1 \subset \mathcal{W}$ that are Lagrangian with respect to ω (hence Lagrangians with respect to $\tilde{\omega}$ as well).*

(viii) *L_0 and L_1 intersect transversely.*
(ix) $\pi_1(L_0) = \pi_1(L_1) = 1$ *and* $w_2(L_0) = w_2(L_1) = 0$.
(x) *The minimal Chern number $N_{M\setminus R}$ (with respect to ω) is at least 2, so that* $N = 2N_{M\setminus R} \geq 4$.
(xi) *There exists an almost complex structure that is compatible with respect to ω on M and compatible with respect to $\tilde{\omega}$ on $M \setminus R$, and for which R is an almost complex submanifold. We fix such a $\tilde{J}$, which we call the* base almost complex structure.
(xii) *Any $\tilde{J}$-holomorphic sphere in M of index zero (necessarily contained in R) has intersection number with R equal to a negative multiple of* 2.

Let us remark that because $\tilde{J}$ is compatible with respect to $\tilde{\omega}$ on $M \setminus R$, by continuity it follows that $\tilde{J}$ is semipositive with respect to $\tilde{\omega}$ on all of M; i.e., $\tilde{\omega}(v, \tilde{J}v) \geq 0$ for any $m \in M$ and $v \in T_m M$.

Our goal is to define a relatively $\mathbb{Z}/N\mathbb{Z}$-graded Floer homology group $HF(L_0, L_1, \tilde{J}; R)$ using Floer trajectories away from R and a path of almost complex structures that are small perturbations of $\tilde{J}$ supported in a neighborhood of $L_0 \cup L_1$. The construction is similar to the one in Sect. 2.2, but a priori it depends on $\tilde{J}$.

Definition 3. (a) We say that $J \in \mathcal{J}(M, \omega)$ is *spherically semipositive* if every J-holomorphic sphere has nonnegative Chern number $c_1(TM)[u] \geq 0$.
(b) We say that $J \in \mathcal{J}(M, \omega)$ is *hemispherically semipositive* if J is spherically semipositive and every J-holomorphic map $(D^2, \partial D^2) \to (M, L_i)$, $i \in 0, 1$, has nonnegative Maslov index $I(u)$, and further, if $I(u) = 0$, then u is constant.

Given a continuous map $u : (D^2, \partial D^2) \to (M, L_i)$, $i = 0, 1$, we define the *canonical area* of u by

$$\tilde{A}(u) := \frac{[\tilde{\omega}](u)}{\kappa}.$$

Lemma 2. *We have $I(u) = \tilde{A}(u)$ for any $u : (D^2, \partial D^2) \to (M, L_i)$.*

Proof. Since L_i is simply connected, we can find a disk v contained in L_i with boundary equal to that of u, but with reversed orientation. Let $u\#v : S^2 \to M$ be the map formed by gluing. By additivity of the Maslov index,

$$I(u) = I(u) + I(v) = I(u\#v) = 2\frac{[\tilde{\omega}](u\#v)}{\kappa} = 2\frac{[\tilde{\omega}](u)}{\kappa},$$

since both the index and the area of v are trivial.

We define a *strip with decay near the ends* to be a continuous map

$$u : (\mathbb{R} \times [0,1], \mathbb{R} \times \{0\}, \mathbb{R} \times \{1\}) \to (M, L_0, L_1) \qquad (13.3)$$

such that $\lim_{s\to\infty} u(s,t)$, $\lim_{s\to-\infty} u(s,t) \in L_0 \cap L_1$ exist. Every strip with decay near the ends admits a relative homology class in $H_2(M, L_0 \cup L_1)$, and therefore has a well-defined canonical area

$$\tilde{A}(u) := \frac{[\tilde{\omega}](u)}{\kappa}$$

and a Maslov index $I(u)$.

The following lemma is [39, Proposition 2.7].

Lemma 3. *Strips* (13.3) *satisfy an index-action relation*

$$I(u) = \tilde{A}(u) + C$$

for some constant C depending only on the endpoints of u.

Proof (Proof (sketch)). Pick u_0 a reference strip with the same endpoints as u. Using the fact that $\pi_1(L_0) = 1$, we can find a map $v : D^2 \to L_0$ such that half of its boundary is taken to the image of $u_0(\mathbb{R} \times \{0\})$ and the other half to the image of $u(\mathbb{R} \times \{0\})$. By adjoining v to u and u_0 (the latter taken with reversed orientation), we obtain a disk $(-u_0)\#v\#u$ with boundary in L_1. Applying Lemma 2 to this disk and using the additivity of the index and canonical action under gluing, we obtain

$$I(u) - I(u_0) = \tilde{A}(u) - \tilde{A}(u_0).$$

We then take $C = I(u_0) - \tilde{A}(u_0)$.

As in Sect. 2.2, $\mathcal{J}(M, \omega, \mathcal{W}, \tilde{J})$ denotes the space of compatible almost complex structures agreeing with $\tilde{J}$ outside $\mathcal{W}$. We let $\mathcal{J}_t(M, \omega, \mathcal{W}, \tilde{J}) = C^\infty([0,1], \mathcal{J}(M, \omega, \mathcal{W}, \tilde{J})$.

Lemma 4. *Every J in $\mathcal{J}(M, \omega, \mathcal{W}, \tilde{J})$ is hemispherically semipositive.*

Proof. Since $\tilde{\omega}$ agrees with ω on $\mathcal{W}$, we have $\tilde{\omega}(v, Jv) \geq 0$ for every $v \in T_m M$, where $m \in \mathcal{W}$. Since J agrees with $\tilde{J}$ outside $\mathcal{W}$, we in fact have $\tilde{\omega}(v, Jv) \geq 0$ everywhere. Nonnegativity of I then follows from the monotonicity of $\tilde{\omega}$ (for spheres) and Lemma 2 for disks. If a J-holomorphic disk u has $I(u) = 0$, its canonical area must be zero. Since J is compatible with respect to $\tilde{\omega}$ on $M \setminus R$, the disk should be contained in R. However, this is impossible, because the disk has boundary on a Lagrangian L_i with $L_i \cap R = \emptyset$. (By contrast, we could have $I(u) = 0$ for nonconstant $\tilde{J}$-holomorphic spheres contained in R.)

Let $\mathcal{J}_t^{\mathrm{reg}}(L_0, L_1, \mathcal{W}, \tilde{J}) \subset \mathcal{J}_t(M, \omega, \mathcal{W}, \tilde{J}) \cap \mathcal{J}_t^{\mathrm{reg}}(L_0, L_1, R)$ be as in Lemma 1.

Proposition 1. *Let $M, L_0, L_1, \tilde{\omega}, \omega, \tilde{J}$ satisfy Assumption 2.1. If we choose $(J_t) \in \mathcal{J}_t^{\mathrm{reg}}(L_0, L_1, \mathcal{W}, \tilde{J})$, then the relative Floer differential counting trajectories disjoint from R is finite and satisfies $\partial_0^2 = 0$. The resulting (relatively $\mathbb{Z}/N\mathbb{Z}$-graded) Floer homology groups $HF_*(L_0, L_1, \tilde{J}; R)$ are independent of the choice of path (J_t), and are preserved under Hamiltonian isotopies of either Lagrangian, as long as Assumption 2.1 is still satisfied.*

Proof. Using parts (v) and (vi) of Assumption 2.1, we see that on the complement of R we have $\omega - \tilde{\omega} = \mathrm{d}a$, for some $a \in \Omega^1(M \setminus R)$ satisfying $da = 0$ in the neighborhood $\mathcal{W}$ of $L_0 \cup L_1$. Let u be a pseudoholomorphic strip whose image is contained in $M \setminus R$. Then

$$E(u) - \kappa\tilde{A}(u) = \int_{\mathbb{R}\times[0,1]} u^*(\omega - \tilde{\omega}) = \int_{\mathbb{R}\times[0,1]} \mathrm{d}(u^*a) = \int_{\gamma_0} u^*a - \int_{\gamma_1} u^*a,$$

where γ_i is a path in the Lagrangian L_i joining the endpoints of u. Since $\mathrm{d}a = 0$ on L_i, Stokes's theorem implies that $\int_\gamma u^*a$ is independent of γ; it depends only on the endpoints. Therefore, $E(u) - \kappa\tilde{A}(u)$ depends only on the endpoints of u. Together with Lemma 3 this gives an energy index relation as follows: for any u in $M \setminus R$, we have

$$I(u) = E(u)/\kappa + C',$$

where C' is a constant depending on the endpoints of u. Since there are only finitely many possibilities for these endpoints, it follows that there exists a constant $K > 0$ such that the energy of any such trajectory u is bounded above by K.

Let $(J_t) \in \mathcal{J}_t^{\mathrm{reg}}(L_0, L_1, \mathcal{W}, \tilde{J})$. By Proposition 4 each J_t is hemispherically semipositive. We define the Floer differential by counting J_t-holomorphic strips in $M \setminus R$. By Lemma 1, a sequence of such strips cannot converge to a strip that intersects R, unless further bubbling occurs.

We seek to rule out sphere bubbles and disk bubbles in the boundary of the zero- and one-dimensional moduli spaces of such strips (i.e., those of index 1 or 2). Assume that we have a sequence (u_ν) of pseudoholomorphic strips of index 1 or 2. Because of the energy bound, a subsequence Gromov converges to a limiting configuration consisting of a broken trajectory and a collection of disk and sphere bubbles. Since the J_t's are hemispherically semipositive, it follows that the indices of the bubbles are nonnegative. Further, by part (x) of Assumption 2.1, the index of each bubble is a multiple of 4. Since we started with a configuration of index at most 2, all bubbles have index zero. By the definition of hemispherical semipositivity, the index-zero disks are constant.

By item (xii) of Assumption 2.1, each index sphere bubble contributes a multiple of two to the intersection number with R. By Lemma 1, the intersection number of the limiting trajectory u_∞ (with sphere bubbles removed) is given by the number of intersection points, and each of these is transverse. Hence at most half of the intersection points with R have sphere bubbles attached. In particular, there exists a point $z \in \mathbb{R} \times [0,1]$ such that $u_\infty(z) \in R$ is a transverse intersection point but z is not in the bubbling set. Since the intersection points are stable under perturbation, it follows that u_∞ cannot be a limit of Floer trajectories disjoint from R. Indeed, by definition this convergence is uniform in all derivatives on the complement of the bubbling set, and in particular, on an open subset containing z. Since there are no sphere bubbles, there cannot be any disk bubbles either, since at least one disk bubble would have to be nonconstant. Hence the limit is a (possibly broken) Floer trajectory.

The rest of the argument is then as in the monotone case. In particular, the statement about the invariance of $HF(L_0,L_1,\tilde{J};R)$ follows from the usual continuation arguments in Floer theory.

Remark 1. If $M,L_0,L_1,\tilde{\omega},\omega,\tilde{J}$ satisfy Assumption 2.1, we can define $HF_*(L_0,L_1,\tilde{J};R)$ even if L_0 and L_1 do not intersect transversely: one can simply isotope one of the Lagrangians to achieve transversality, and take the resulting Floer homology.

Remark 2. A priori, the construction of the Floer homologies $HF(L_0,L_1,\tilde{J};R)$ depends on the open set $\mathcal{W}$, because (J_t) is chosen from the corresponding set $\mathcal{J}_t^{\text{reg}}(L_0,L_1,\mathcal{W},\tilde{J})$. However, suppose we have another open set $\mathcal{W}' \subset M\setminus R$ satisfying $L_0\cap L_1 \subset \mathcal{W}'$ and $\omega=\tilde{\omega}$ on $\mathcal{W}'$. Note that

$$\mathcal{J}_t^{\text{reg}}(L_0,L_1,\mathcal{W},\tilde{J})\cap\mathcal{J}_t^{\text{reg}}(L_0,L_1,\mathcal{W}',\tilde{J})=\mathcal{J}_t^{\text{reg}}(L_0,L_1,\mathcal{W}\cap\mathcal{W}',\tilde{J}), \tag{13.4}$$

because the regularity condition in Lemma 1 is intrinsic for (J_t) (it boils down to the surjectivity of certain linear operators). It follows that by choosing (J_t) in the (necessarily nonempty) intersection (13.4), the Floer homologies $HF(L_0,L_1,\tilde{J};R)$ defined from $\mathcal{W}$ and $\mathcal{W}'$ are isomorphic. Thus, we can safely drop $\mathcal{W}$ from the notation.

Remark 3. A smooth variation of the base almost complex structure $\tilde{J}$ induces an isomorphism between the respective Floer homologies $HF(L_0,L_1,\tilde{J};R)$. However, if we are given only $\tilde{\omega}$ and ω, it is not clear whether the space of possible $\tilde{J}$'s is contractible. This justifies keeping $\tilde{J}$ in the notation $HF(L_0,L_1,\tilde{J};R)$.

3 Moduli Spaces

3.1 Notation

Throughout the rest of the paper, G will denote the Lie group $\mathrm{SU}(2)$, and $G^{\text{ad}}=\mathrm{PU}(2)=\mathrm{SO}(3)$ the corresponding group of adjoint type. We identify the Lie algebra $\mathfrak{g}=\mathfrak{su}(2)$ with its dual $\mathfrak{g}^*$ using the basic invariant bilinear form

$$\langle\cdot,\cdot\rangle:\mathfrak{g}\times\mathfrak{g}\to\mathbb{R},\ \ \langle A,B\rangle=-\mathrm{Tr}\,(AB).$$

The maximal torus $T\cong S^1\subset G$ consists of the diagonal matrices diag $(\mathrm{e}^{2\pi ti},\mathrm{e}^{-2\pi ti}), t\in\mathbb{R}$. We let $T^{\text{ad}}=T/(\mathbb{Z}/2\mathbb{Z})\subset G^{\text{ad}}$ and identify their Lie algebra $\mathfrak{t}$ with $\mathbb{R}$ by sending $\mathrm{diag}(\mathrm{i},-\mathrm{i})$ to 1. Under this identification, the restriction of the inner product $\langle\cdot,\cdot\rangle$ to $\mathfrak{t}$ is twice the Euclidean metric. We use this inner product to identify $\mathfrak{t}$ with $\mathfrak{t}^*$ as well. Finally, we let $\mathfrak{t}^{\perp}$ denote the orthocomplement of $\mathfrak{t}$ in $\mathfrak{g}$.

Conjugacy classes in $\mathfrak{g}$ (under the adjoint action of G) are parameterized by the positive Weyl chamber $\mathfrak{t}_+ = [0,\infty)$. Indeed, the adjoint quotient map

$$Q : \mathfrak{g} \to [0,\infty)$$

takes $\theta \in \mathfrak{g}$ to t such that θ is conjugate to $\mathrm{diag}(ti,-ti)$.

On the other hand, conjugacy classes in G are parameterized by the fundamental alcove $\mathfrak{A} = [0,1/2]$. Indeed, for any $g \in G$, there is a unique $t \in [0,1/2]$ such that g is conjugate to the diagonal matrix $\mathrm{diag}(\mathrm{e}^{2\pi ti},\mathrm{e}^{-2\pi ti})$.

3.2 *The Extended Moduli Space*

We review here the construction of the *extended moduli space* [23, 26], mostly following Jeffrey's gauge-theoretic approach from [26].

Let Σ be a compact connected Riemann surface of genus $h \geq 1$. Fix some $z \in \Sigma$ and let Σ' denote the complement in Σ of a small disk around z, so that $S = \partial\Sigma'$ is a circle. Identify a neighborhood of S in Σ' with $[0,\varepsilon) \times S$, and let $s \in \mathbb{R}/2\pi\mathbb{Z}$ be the coordinate on the circle S.

Consider the space $\mathscr{A}(\Sigma') \cong \Omega^1(\Sigma') \otimes \mathfrak{g}$ of smooth connections on the trivial G-bundle over Σ', and set

$$\mathscr{A}^{\mathfrak{g}}(\Sigma') = \{A \in \mathscr{A}(\Sigma') \mid F_A = 0,\ A = \theta \mathrm{d}s \text{ on some neighborhood of } S \text{ for some } \theta \in \mathfrak{g}\}.$$

The space $\mathscr{A}^{\mathfrak{g}}(\Sigma')$ is acted on by the gauge group

$$\mathscr{G}^c(\Sigma') = \{f : \Sigma' \to G) \mid f = I \text{ on some neighborhood of } S\}.$$

The extended moduli space is then defined as

$$\mathscr{M}^{\mathfrak{g}}(\Sigma') = \mathscr{A}^{\mathfrak{g}}(\Sigma')/\mathscr{G}^c(\Sigma').$$

A more explicit description of the extended moduli space is obtained by fixing a collection of simple closed curves α_i, β_i $(i = 1,\dots,h)$ on Σ', based at a point in S, such that $\pi_1(\Sigma')$ is generated by their equivalence classes and the class of a curve γ around S, with the relation $\prod_{i=1}^h [\alpha_i, \beta_j] = \gamma$.

To each connection on Σ' one can then associate the holonomies $A_i, B_i \in G$ around the loops α_i and β_i, respectively, $i = 1,\dots,h$. This allows us to view the extended moduli space as

$$\mathscr{M}^{\mathfrak{g}}(\Sigma') = \left\{ (A_1,B_1,\dots,A_h,B_h) \in G^{2h}, \theta \in \mathfrak{g} \mid \prod_{i=1}^{h}[A_i,B_i] = \exp(2\pi\theta) \right\}. \quad (13.1)$$

There is a proper map

$$\Phi : \mathscr{M}^{\mathfrak{g}}(\Sigma') \to \mathfrak{g}$$

that takes the class $[A]$ of a connection A to the value $\theta = \Phi(A)$ such that $A|_S = \theta \mathrm{d}s$. (This corresponds to the variable θ appearing in (13.1).) There is also a natural G-action on $\mathscr{M}^{\mathfrak{g}}(\Sigma')$ given by constant gauge transformations. With respect to the identification (13.1), it is

$$g \in G : (A_i, B_i, \theta) \to (gA_i g^{-1}, gB_i g^{-1}, \mathrm{Ad}(g)\theta). \tag{13.2}$$

Observe that this action factors through G^{ad}. The map Φ is equivariant with respect to this action on its domain, and the adjoint action on its target. Set

$$\tilde{\Phi} : \mathscr{M}^{\mathfrak{g}}(\Sigma') \to [0,\infty), \quad \tilde{\Phi} = Q \circ \Phi.$$

Now consider the subspace

$$\mathscr{M}^{\mathfrak{g}}_s(\Sigma') = \{x \in \mathscr{M}^{\mathfrak{g}}(\Sigma') \mid \tilde{\Phi}(x) \notin \mathbb{Z} \setminus \{0\}\}.$$

Proposition 2. (a) *The space $\mathscr{M}^{\mathfrak{g}}_s(\Sigma')$ is a smooth manifold of real dimension $6h$.*
(b) *Every nonzero element $\theta \in \mathfrak{g}$ is a regular value for the restriction of Φ to $\mathscr{M}^{\mathfrak{g}}_s(\Sigma')$.*

Proof. Part (a) is proved in [26, Theorem 2.7]. We copy the proof here, and explain how the same arguments can be used to deduce part (b) as well.

Consider the commutator map $c : G^{2h} \to G, c(A_1, B_1, \ldots, A_h, B_h) = \prod_{i=1}^{h} [A_i, B_i]$. For $\rho = (A_1, B_1, \ldots, A_h, B_h) \in G^{2h}$, we denote by $Z(\rho) \subset G$ its stabilizer (under the diagonal action by conjugation). Let $z(\rho) \subset \mathfrak{g}$ be the Lie algebra of $Z(\rho)$. Note that $Z(\rho) = \{\pm I\}$ unless $c(\rho) = I$.

The image of $\mathrm{d}c_\rho \cdot c(\rho)^{-1}$ is $z(\rho)^\perp$; see, for example, [19, proof of Proposition 3.7]. In particular, the differential $\mathrm{d}c_\rho$ is surjective whenever $c(\rho) \neq I$.

Define the maps

$$f_1 : G^{2h} \times \mathfrak{g} \to G, \quad f_1(\rho, \theta) = c(\rho) \cdot \exp(-2\pi\theta)$$

and

$$f_2 : G^{2h} \times \mathfrak{g} \to G \times \mathfrak{g}, \quad f_2(\rho, \theta) = (f_1(\rho, \theta), \theta).$$

On the extended moduli space $\mathscr{M}^{\mathfrak{g}}(\Sigma') = f_1^{-1}(I)$, we have

$$(\mathrm{d}f_1)_{(\rho,\theta)} = (\mathrm{d}c)_\rho \exp(-2\pi\theta) + 2\pi \exp(2\pi\theta)(\mathrm{d}\exp)_{-2\pi\theta}.$$

When $c(\rho) = \exp(2\pi\theta) \neq I$, we have that $(dc)_\rho$ is surjective, hence so is $(\mathrm{d}f_1)_{(\rho,\theta)}$. Also, when $\theta = 0$, $(\mathrm{d}\exp)_{-2\pi\theta}$ is just the identity, so again $(\mathrm{d}f_1)_{(\rho,\theta)}$ is surjective. Claim (a) follows.

Next, observe that

$$(\mathrm{d}f_2)_{(\rho,\theta)}(\alpha,\lambda) = ((\mathrm{d}f_1)_{(\rho,\theta)}(\alpha,\lambda),\lambda) = (\mathrm{d}c_\rho(\alpha)\cdot\exp(-2\pi\theta)+l(\lambda),\lambda),$$

where $l(\lambda)$ does not depend on α. Hence, when $c(\rho) = \exp(2\pi\theta) \neq I$, the differential $(\mathrm{d}f_2)_{(\rho,\theta)}$ is surjective. This implies that any $\theta \in \mathfrak{g}$ with $Q(\theta) \notin \mathbb{Z}$ is a regular value for $\Phi|_{\mathscr{M}_s^{\mathfrak{g}}(\Sigma')}$. Since the values $\theta \in \mathfrak{g}$ with $Q(\theta) \in \mathbb{Z}\setminus\{0\}$ are not in the image of $\Phi|_{\mathscr{M}_s^{\mathfrak{g}}(\Sigma')}$, they are automatically regular values, and claim (b) follows.

Consider also the subspace

$$\mathscr{N}(\Sigma') = \tilde{\Phi}^{-1}\big([0,1/2)\big) \subset \mathscr{M}_s^{\mathfrak{g}}(\Sigma').$$

Note that the restriction of the exponential map $\theta \to \exp(2\pi\theta)$ to $Q^{-1}\big[0,1/2\big)$ is a diffeomorphism onto its image $G\setminus\{-I\}$. Therefore, using the identification (13.1), we can describe $\mathscr{N}(\Sigma')$ as

$$\mathscr{N}(\Sigma') = \left\{ (A_1,B_1,\dots,A_h,B_h) \in G^{2h} \;\middle|\; \prod_{i=1}^{h}[A_i,B_i] \neq -I \right\}. \qquad (13.3)$$

3.3 Hamiltonian Actions

Let K be a compact connected Lie group with Lie algebra $\mathfrak{k}$. We let K act on the dual Lie algebra $\mathfrak{k}^*$ by the coadjoint action.

A *presymplectic manifold* is a smooth manifold M together with a closed form $\omega \in \Omega^2(M)$, possibly degenerate. A *Hamiltonian presymplectic K-manifold* (M,ω,Φ) is a presymplectic manifold (M,ω) together with a K-equivariant smooth map $\Phi : M \to \mathfrak{k}^*$ such that for any $\xi \in \mathfrak{g}$, if X_ξ denotes the vector field on M generated by the one-parameter subgroup $\{\exp(-t\xi) \mid t \in \mathbb{R}\} \subset K$, we have

$$d\big(\langle \Phi,\xi\rangle\big) = -\iota(X_\xi)\omega.$$

Under these hypotheses, the K-action on M is called Hamiltonian, and Φ is called the *moment map*. The quotient

$$M/\!/K \;:=\; \Phi^{-1}(0)/K$$

is called the *presymplectic quotient* of M by K. The following result is known as the reduction theorem [32],[36], [20, Theorem 5.1].

Theorem 2. *Let (M,ω,Φ) be a Hamiltonian presymplectic K-manifold. Suppose that the level set $\Phi^{-1}(0)$ is a smooth manifold on which K acts freely. Let $i : \Phi^{-1}(0) \hookrightarrow M$ be the inclusion and $\pi : \Phi^{-1}(0) \to M/\!/K$ the projection. Then there* *exists a unique closed form* ω_{red} *on the smooth manifold* $M//K$ *with the property that* $i^*\omega = \pi^*\omega_{\text{red}}$*. The reduced form* ω_{red} *is nondegenerate on* $M//K$ *if and only if* ω *is nondegenerate on* M *at the points of* $\Phi^{-1}(0)$.

Furthermore, if M *admits another Hamiltonian* K'*-action (for some compact Lie group* K'*) that commutes with the* K*-action, then* $(M//K, \omega_{\text{red}})$ *has an induced Hamiltonian* K'*-action.*

When the form ω is symplectic, (M, ω, Φ) is called simply a *Hamiltonian K-manifold*. In this case we can drop the condition that $\Phi^{-1}(0)$ be smooth from the hypotheses of Theorem 2; indeed, this condition is automatically implied by the assumption that K acts freely on $\Phi^{-1}(0)$.

3.4 A Closed Two-Form on the Extended Moduli Space

According to [26, (2.7)], the tangent space to the smooth stratum $\mathscr{M}_s^{\mathfrak{g}}(\Sigma') \subset \mathscr{M}^{\mathfrak{g}}(\Sigma')$ at some class $[A]$ can be naturally identified with

$$T_{[A]}\mathscr{M}_s^{\mathfrak{g}}(\Sigma') = \frac{\operatorname{Ker}(\mathrm{d}_A : \Omega^{1,\mathfrak{g}}(\Sigma') \to \Omega_c^2(\Sigma') \otimes \mathfrak{g})}{\operatorname{Im}(\mathrm{d}_A : \Omega_c^0(\Sigma') \otimes \mathfrak{g} \to \Omega^{1,\mathfrak{g}}(\Sigma'))}, \tag{13.4}$$

where $\Omega_c^p(\Sigma')$ denotes the space of p-forms compactly supported in the interior of Σ', and $\Omega^{1,\mathfrak{g}}(\Sigma')$ denotes the space of 1-forms A such that $A = \theta \mathrm{d}s$ near $S = \partial\Sigma'$ for some $\theta \in \mathfrak{g}$.

Define a bilinear form ω on $\Omega^{1,\mathfrak{g}}(\Sigma')$ by

$$\omega(a,b) = \int_{\Sigma'} \operatorname{Tr}\,(a \wedge b),$$

where the wedge operation on $\mathfrak{g}$-valued forms combines the usual exterior product with the inner product on $\mathfrak{g}$. Stokes's theorem implies that ω descends to a bilinear form on the tangent space to $\mathscr{M}_s^{\mathfrak{g}}(\Sigma')$ described in (13.4) above. Thus we can think of ω as a two-form on $\mathscr{M}_s^{\mathfrak{g}}(\Sigma')$.

Theorem 3 (Huebschmann–Jeffrey). *The two-form* $\omega \in \Omega^2(\mathscr{M}_s^{\mathfrak{g}}(\Sigma'))$ *is closed. It is nondegenerate when restricted to* $\mathscr{N}(\Sigma') \subset \mathscr{M}_s^{\mathfrak{g}}(\Sigma')$*. Moreover, the restriction of the* G^{ad}*-action (13.2) to* $\mathscr{M}_s^{\mathfrak{g}}(\Sigma')$ *is Hamiltonian with respect to* ω*. Its moment map is the restriction of* Φ *to* $\mathscr{N}(\Sigma')$*, which we henceforth also denote by* Φ.

For the proof, we refer to Jeffrey [26]; see also [34].

Theorem 3 says that $(\mathscr{M}_s^{\mathfrak{g}}(\Sigma'), \omega, \Phi)$ is a Hamiltonian presymplectic G^{ad}-manifold in the sense of Sect. 3.3, and that its subset $(\mathscr{N}(\Sigma'), \omega, \Phi)$ is a (symplectic) Hamiltonian G^{ad}-manifold. The symplectic quotient

$$\mathscr{N}(\Sigma')//G^{\text{ad}} = \Phi^{-1}(0)/G^{\text{ad}} = \mathscr{M}(\Sigma)$$

is the usual moduli space of flat G-connections on Σ, with the symplectic form (on its smooth stratum) being the one constructed by Atiyah and Bott [3]. If Σ is given a complex structure, $\mathscr{M}(\Sigma)$ can also be viewed as the moduli space of semistable bundles of rank two on Σ with trivial determinant, cf. [38].

For an alternative (group-theoretic) description of the form ω on $\mathscr{N}(\Sigma')$, see [27], [23], or [24].

Let us mention two results about the two-form ω. The first is proved in [35].

Theorem 4 (Meinrenken–Woodward). *$(\mathscr{N}(\Sigma'),\omega)$ is a monotone symplectic manifold, with monotonicity constant $1/4$.*

The second result is given in the following lemma.

Lemma 5. *The cohomology class of the symplectic form $\omega \in \Omega^2(\mathscr{N}(\Sigma'))$ is integral.*

Proof. The extended moduli space $\mathscr{M}^{\mathfrak{g}}(\Sigma')$ embeds in the moduli space $\mathscr{M}(\Sigma')$ of all flat connections on Σ'. The latter is an infinite-dimensional Banach manifold with a natural symplectic form that restricts to ω on $\mathscr{M}^{\mathfrak{g}}_s(\Sigma')$. Moreover, Donaldson [12] showed that $\mathscr{M}(\Sigma')$ has the structure of a Hamiltonian LG-manifold, where $LG = \mathrm{Map}(S^1, G)$ is the loop group of G.

Recall that a prequantum line bundle E for a symplectic manifold (M,ω) is a Hermitian line bundle equipped with an invariant connection ∇ whose curvature is $-2\pi i$ times the symplectic form. If M is finite-dimensional, this implies that $[\omega] = c_1(E) \in H^2(M;\mathbb{Z})$. In our situation, a prequantum line bundle on $M = \mathscr{N}(\Sigma')$ can be obtained by restricting the well-known LG-equivariant prequantum line bundle on the infinite-dimensional symplectic manifold $\mathscr{M}(\Sigma')$. We refer the reader to [37], [46], and [60] for the construction of the latter; see also [34].

Corollary 2. *The minimal Chern number of the symplectic manifold $\mathscr{N}(\Sigma')$ is a positive multiple of* 4.

Proof. Use Theorem 4 and Lemma 5.

3.5 Other Versions

Although our main interest lies in the extended moduli space $\mathscr{M}^{\mathfrak{g}}(\Sigma')$ and its open subset $\mathscr{N}(\Sigma')$, in order to understand them better, we need to introduce two other moduli spaces. Both of them appeared in [26], where their main properties are spelled out. An alternative viewpoint on them is given in [34, Sect. 3.4.2], where they are interpreted as cross-sections of the full moduli space $\mathscr{M}(\Sigma')$.

The first auxiliary space that we consider is the *toroidal extended moduli space*

$$\mathscr{M}^{\mathfrak{t}}(\Sigma') = \Phi^{-1}(\mathfrak{t}) \subset \mathscr{M}^{\mathfrak{g}}(\Sigma').$$

It has a smooth stratum

$$\mathscr{M}_s^{\mathfrak{t}}(\Sigma') = \{x \in \mathscr{M}^{\mathfrak{t}}(\Sigma') \mid \tilde{\Phi}(x) \notin \mathbb{Z}\}.$$

The restrictions of ω and Φ to $\mathscr{M}_s^{\mathfrak{t}}(\Sigma')$ turn it into a Hamiltonian presymplectic T^{ad}-manifold. On the open subset $\mathscr{M}^{\mathfrak{t}}(\Sigma') \cap \tilde{\Phi}^{-1}(0,1/2)$, the two-form is nondegenerate.

The second space is the *twisted extended moduli space* from [26, Sect. 5.3]. In terms of coordinates, it is

$$\mathscr{M}_{\mathrm{tw}}^{\mathfrak{g}}(\Sigma') = \left\{(A_1,B_1,\ldots,A_h,B_h) \in G^{2h}, \theta \in \mathfrak{g} \,\middle|\, \prod_{i=1}^{h}[A_i,B_i] = -\exp(2\pi\theta)\right\}.$$

This space admits a G^{ad}-action just like that on $\mathscr{M}^{\mathfrak{g}}(\Sigma')$ and also a natural projection $\Phi_{\mathrm{tw}} : \mathscr{M}_{\mathrm{tw}}^{\mathfrak{g}} \to \mathfrak{g}$. Set $\tilde{\Phi}_{\mathrm{tw}} = Q \circ \Phi_{\mathrm{tw}}$. The smooth stratum of $\mathscr{M}_{\mathrm{tw}}^{\mathfrak{g}}(\Sigma')$ is

$$\mathscr{M}_{\mathrm{tw},s}^{\mathfrak{g}}(\Sigma') = \left\{x \in \mathscr{M}_{\mathrm{tw}}^{\mathfrak{g}}(\Sigma') \,\middle|\, \tilde{\Phi}_{\mathrm{tw}}(x) \notin \mathbb{Z} + \frac{1}{2}\right\}.$$

Furthermore, $\mathscr{M}_{\mathrm{tw},s}^{\mathfrak{g}}(\Sigma')$ admits a natural two-form ω_{tw}, which turns it into a Hamiltonian presymplectic G^{ad}-manifold, with moment map Φ_{tw}. The restriction of ω_{tw} to the subspace

$$\mathscr{N}_{\mathrm{tw}}(\Sigma') = \tilde{\Phi}_{\mathrm{tw}}^{-1}\big([0,1/2)\big)$$

is nondegenerate.

Observe that the subspace $\Phi_{\mathrm{tw}}^{-1}(\mathfrak{t}) \subset \mathscr{M}_{\mathrm{tw}}^{\mathfrak{g}}(\Sigma')$ can be identified with the toroidal extended moduli space $\mathscr{M}^{\mathfrak{t}}(\Sigma')$, via the map

$$(A_1,B_1,\ldots,A_h,B_h,t) \to (A_1,B_1,\ldots,A_h,B_h,1/2-t).$$

This map is a diffeomorphism of the smooth strata and is compatible with the restrictions of the presymplectic forms ω and ω_{tw}.

3.6 The Structure of Degeneracies of $\mathscr{M}_s^{\mathfrak{g}}(\Sigma')$

Recall from Theorem 3 that the degeneracy locus of the presymplectic manifold $\mathscr{M}_s^{\mathfrak{g}}(\Sigma')$ is contained in the preimage $\tilde{\Phi}^{-1}(1/2)$. We seek to understand the structure of the degeneracies.

Let $\mu = \mathrm{diag}(\mathrm{i}/2, -\mathrm{i}/2)$. Note that the stabilizer G^{ad} of $\exp(2\pi\mu) = -I$ is bigger than the stabilizer $T^{\mathrm{ad}} = S^1$ of μ. Thus, we have an obvious diffeomorphism

$$\tilde{\Phi}^{-1}(1/2) \cong \mathcal{O}_{\mu} \times \Phi^{-1}(\mu),$$

where $\mathscr{O}_\mu$ denotes the coadjoint orbit of μ. The first factor $\mathscr{O}_\mu$ is diffeomorphic to the flag variety $G^{\mathrm{ad}}/T^{\mathrm{ad}} \cong \mathbb{P}^1$. The second factor $\Phi^{-1}(\mu)$ is smooth by Proposition 2(b).

There is a residual T^{ad}-action on the space $\Phi^{-1}(\mu)$. Thus $\Phi^{-1}(\mu)$ is an S^1-bundle over

$$\mathscr{M}_\mu(\Sigma') = \Phi^{-1}(\mu)/T^{\mathrm{ad}}.$$

Finally, $\mathscr{M}_\mu(\Sigma')$ is a $\mathbb{P}^1$-bundle over

$$\mathscr{M}_{-I}(\Sigma') = \left\{(A_1, B_1, \dots, A_h, B_h) \in G^{2h} \,\middle|\, \prod_{i=1}^{h} [A_i, B_i] = -I\right\}/G^{\mathrm{ad}}.$$

This last space $\mathscr{M}_{-I}(\Sigma')$ can be identified with the moduli space $\mathscr{M}_{\mathrm{tw}}(\Sigma)$ of projectively flat connections on $\mathfrak{E}$ with fixed central curvature, where $\mathfrak{E}$ is a $\mathrm{U}(2)$-bundle of odd degree over the closed surface $\Sigma = \Sigma' \cup D^2$. Alternatively, it is the moduli space of rank-two stable bundles on Σ having fixed determinant of odd degree; cf. [3, 38]. It can also be viewed as the symplectic quotient of the twisted extended moduli space from Sect. 3.5:

$$\mathscr{M}_{\mathrm{tw}}(\Sigma) = \mathscr{N}_{\mathrm{tw}}(\Sigma')/\!/G^{\mathrm{ad}} = \Phi_{\mathrm{tw}}^{-1}(0)/G^{\mathrm{ad}}.$$

We have described a string of fibrations that gives a clue to the structure of the space $\tilde{\Phi}^{-1}(1/2)$. Let us now reshuffle these fibrations and view $\tilde{\Phi}^{-1}(1/2)$ as a G^{ad}-bundle over the space $\mathscr{O}_\mu \times \mathscr{M}_{-I}(\Sigma')$. Its fiberwise tangent space (at any point) is $\mathfrak{g}$, which can be decomposed as $\mathfrak{t} \oplus \mathfrak{t}^\perp$, with $\mathfrak{t}^\perp \cong \mathbb{C}$.

Proposition 3. *Let $x \in \tilde{\Phi}^{-1}(1/2) \subset \mathscr{M}_s^{\mathfrak{g}}(\Sigma')$. The null space of the form ω at x consists of the fiber directions corresponding to $\mathfrak{t}^\perp \subset \mathfrak{g}$.*

Proof. Our strategy for proving Proposition 3 is to reduce it to a similar statement for the toroidal extended moduli space $\mathscr{M}^{\mathfrak{t}}(\Sigma')$, and then study the latter via its embedding into the twisted extended moduli space $\mathscr{M}_{\mathrm{tw}}^{\mathfrak{g}}(\Sigma')$.

First, note that by G^{ad}-invariance, we can assume without loss of generality that $\Phi(x) = \mu$. The symplectic cross-section theorem [21] says that near $\Phi^{-1}(\mu)$, the two-form on $\mathscr{M}^{\mathfrak{g}}(\Sigma')$ is obtained from the one on $\mathscr{M}^{\mathfrak{t}}(\Sigma') = \Phi^{-1}(\mathfrak{t})$ by a procedure called symplectic induction. (Strictly speaking, symplectic induction is described in [21] for nondegenerate forms; however, it applies to the Hamiltonian presymplectic case as well.) More concretely, we have a (noncanonical) decomposition

$$T_x\mathscr{M}^{\mathfrak{g}}(\Sigma') = T_x\mathscr{M}^{\mathfrak{t}}(\Sigma') \oplus T_\mu(\mathscr{O}_\mu) \tag{13.5}$$

such that $\omega|_x$ is the direct sum of its restriction to the first summand in (13.5) with the canonical symplectic form on the second summand.

Recall that we are viewing $\tilde{\Phi}^{-1}(1/2)$ as a G-bundle over $\mathscr{O}_\mu \times \mathscr{M}_{-I}(\Sigma')$. Its intersection with $\mathscr{M}^{\mathfrak{t}}(\Sigma')$ is $\Phi^{-1}(\mu)$, which is the part of the G^{ad}-bundle that lies over $\{\mu\} \times \mathscr{M}_{-I}(\Sigma')$. The decomposition (13.5) implies that in order to prove the final claim about the null space of $\omega|_x$, it suffices to show that the null space of $\omega|_{\mathscr{M}^{\mathfrak{t}}(\Sigma')}$ at x consists of the fiber directions corresponding to $\mathfrak{t}^{\perp} \subset \mathfrak{g}$.

Let us use the observation in the last paragraph of Sect. 3.5, and view $\mathscr{M}^{\mathfrak{t}}(\Sigma')$ as $\Phi_{\text{tw}}^{-1}(\mathfrak{t}) \subset \mathscr{M}_{\text{tw}}^{\mathfrak{g}}(\Sigma')$. The point x now lies in $\Phi_{\text{tw}}^{-1}(0)$.

Recall from Sect. 3.5 that the two-form $\mathscr{M}_{\text{tw}}^{\mathfrak{g}}(\Sigma')$ is nondegenerate near $\Phi_{\text{tw}}^{-1}(0)$. Further, it is easy to check that the action of G^{ad} on $\Phi_{\text{tw}}^{-1}(0)$ is free. This action is Hamiltonian; hence, the quotient $\Phi_{\text{tw}}^{-1}(0)/G^{\text{ad}} = \mathscr{M}_{-I}(\Sigma')$ is smooth, and the reduced two-form on it is nondegenerate. Further, there is a (noncanonical) decomposition

$$T_x\mathscr{M}_{\text{tw}}^{\mathfrak{g}}(\Sigma') \cong \pi^* T_{\pi(x)}\mathscr{M}_{-I}(\Sigma') \oplus \mathfrak{g} \oplus \mathfrak{g}^*, \tag{13.6}$$

where $\pi : \Phi_{\text{tw}}^{-1}(0) \to \mathscr{M}_{-I}(\Sigma')$ is the quotient map. (See, for example, [20, (5.6)].) The two-form ω_{tw} at x is the direct summand of the reduced form at $\pi(x)$ and the natural pairing of the two last factors in (13.6).

With respect to the decomposition (13.6), the subspace $T_x\mathscr{M}^{\mathfrak{t}}(\Sigma') \subset T_x\mathscr{M}_{\text{tw}}^{\mathfrak{g}}(\Sigma')$ corresponds to

$$T_x\mathscr{M}^{\mathfrak{t}}(\Sigma') \cong \pi^* T_{\pi(x)}\mathscr{M}_{-I}(\Sigma') \oplus \mathfrak{g} \oplus \mathfrak{t}^*.$$

Therefore, the null space of ω_{tw} on $T_x\mathscr{M}^{\mathfrak{t}}(\Sigma')$ is the null space of the restriction of the natural pairing on $\mathfrak{g} \oplus \mathfrak{g}^*$ to $\mathfrak{g} \oplus \mathfrak{t}^*$. This is $\mathfrak{g}/\mathfrak{t} \cong \mathfrak{t}^{\perp}$, as claimed.

4 Symplectic Cutting

4.1 Abelian Symplectic Cutting

We review here Lerman's definition of (abelian) symplectic cutting, following [31]. Consider a symplectic manifold (M, ω) with a Hamiltonian S^1-action and moment map $\Phi : M \to \mathbb{R}$. Pick some $\lambda \in \mathbb{R}$. The diagonal S^1-action on the space $M \times \mathbb{C}^-$ (endowed with the standard product symplectic structure, where $\mathbb{C}^-$ is $\mathbb{C}$ with negative the usual area form) is Hamiltonian with respect to the moment map

$$\Psi : M \times \mathbb{C}^- \to \mathbb{R}, \quad \Psi(m,z) = \Phi(m) + \frac{1}{2}|z|^2 - \lambda.$$

The symplectic quotient

$$M_{\leq\lambda} := \Psi^{-1}(0)/S^1 \cong \Phi^{-1}(\lambda)/S^1 \cup \Phi^{-1}(-\infty, \lambda)$$

is called the *symplectic cut* of M at λ. If the action of S^1 on $\Phi^{-1}(\lambda)$ is free, then $M_{\leq\lambda}$ is a symplectic manifold, and it contains $\Phi^{-1}(\lambda)/S^1$ (with its reduced form) as a symplectic hypersurface, i.e., a symplectic submanifold of real codimension two.

Remark 4. The normal bundle to $\Phi^{-1}(\lambda)/S^1$ in $M_{\leq\lambda}$ is the complex line bundle whose associated circle bundle is $\Phi^{-1}(\lambda) \to \Phi^{-1}(\lambda)/S^1$.

Remark 5. Symplectic cutting is a local construction. In particular, if (M,ω) is symplectic and $\Phi : M \to \mathbb{R}$ is a continuous map that induces a smooth Hamiltonian S^1-action on an open set $\mathcal{U} \subset M$ containing $\Phi^{-1}(\lambda)$, then we can still define $M_{\leq\lambda}$ as the union $(M \setminus \mathcal{U}) \cup \mathcal{U}_{\leq\lambda}$.

Remark 6. If M has an additional Hamiltonian K-action (for some other compact group K) commuting with that of S^1, then $M_{\leq\lambda}$ has an induced Hamiltonian K-action. This follows from a similar statement for symplectic reduction; cf. Theorem 2.

4.2 Nonabelian Symplectic Cutting

An analogue of symplectic cutting for nonabelian Hamiltonian actions was defined in [61]. We explain here the case of Hamiltonian PU(2)-actions, since this is all we need for our purposes.

We keep the notation from Sect. 3.1, with $G = \mathrm{SU}(2)$ and $G^{\mathrm{ad}} = \mathrm{PU}(2)$. Let (M,ω,Φ) be a Hamiltonian G^{ad}-manifold. Since $\mathfrak{g}$ and $\mathfrak{g}^*$ are identified using the bilinear form, from now on we will view the moment map Φ as taking values in $\mathfrak{g}$. Recall that

$$Q : \mathfrak{g} \to \mathfrak{g}/G^{\mathrm{ad}} \cong [0,\infty)$$

denotes the adjoint quotient map. The map Q is continuous and is smooth outside $Q^{-1}(0)$. Set

$$\tilde{\Phi} = Q \circ \Phi.$$

On the complement $\mathcal{U} = \mathcal{U}$ of $\Phi^{-1}(0)$ in M, the map $\tilde{\Phi}$ induces a Hamiltonian S^1-action. Explicitly, $u \in S^1 = \mathbb{R}/2\pi\mathbb{Z}$ acts on $m \in \mathcal{U}$ by

$$m \to \exp\left(u \cdot \frac{\Phi(m)}{2\tilde{\Phi}(m)}\right) \cdot m. \tag{13.1}$$

This action is well defined because $\exp(\pi H) = I$ in G^{ad}. We can describe it alternatively as follows: on $\Phi^{-1}(\mathfrak{t}) \subset M$, it coincides with the action of $T^{\mathrm{ad}} \subset G^{\mathrm{ad}}$; then it is extended to all of M in a G^{ad}-equivariant manner.

Fix $\lambda > 0$. Using the local version (from Remark 5) of abelian symplectic cutting for the action (13.1), we define the *nonabelian symplectic cut of M at λ* to be

$$M_{\leq\lambda} = \Phi^{-1}(0) \cup \mathcal{U}_{\leq\lambda} = M_{<\lambda} \cup R,$$

where

$$M_{<\lambda} = \Phi_1^{-1}([0,\lambda)), \;\; R_\lambda = \tilde{\Phi}^{-1}(\lambda)/S^1.$$

If S^1 acts freely on $\tilde{\Phi}^{-1}(\lambda)$, then $M_{\leq\lambda}$ is a smooth manifold. It can be naturally equipped with a symplectic form $\omega_{\leq\lambda}$, coming from the symplectic form ω on M. In fact, $M_{\leq\lambda}$ is a Hamiltonian G^{ad}-manifold; cf. Remark 6. With respect to the form $\omega_{\leq\lambda}$, R is a symplectic hypersurface in $M_{\leq\lambda}$.

4.3 Monotonicity

We aim to find a condition that guarantees that a nonabelian symplectic cut is monotone. As a toy model for our future results, we start with a general fact about symplectic reduction:

Lemma 6. *Let K be a Lie group with $H^2(K;\mathbb{R}) = 0$, and let (M,ω,Φ) be a Hamiltonian K-manifold that is monotone, with monotonicity constant κ. Assume that the moment map Φ is proper, and the K-action on $\Phi^{-1}(0)$ is free. Then the symplectic quotient $M//K = \Phi^{-1}(0)/K$ (with the reduced symplectic form ω^{red}) is also monotone, with the same monotonicity constant κ.*

Proof. Consider the Kirwan map from [28]:

$$H^2_K(M;\mathbb{R}) \to H^2(M//K;\mathbb{R}),$$

which is obtained by composing the map $H^2_K(M;\mathbb{R}) \to H^2_K(\Phi^{-1}(0);\mathbb{R})$ (induced by the inclusion) with the Cartan isomorphism $H^2_K(\Phi^{-1}(0);\mathbb{R}) \cong H^2(M//K;\mathbb{R})$. The Kirwan map takes the first equivariant Chern class $c_1^K(TM)$ to $c_1(T(M//K))$, and the equivariant two-form $\tilde{\omega} = \omega - \Phi$ to ω^{red}. Since $H^2_K(M;\mathbb{R}) \cong H^2(M;\mathbb{R})$, with c_1^K corresponding to c_1 and $[\tilde{\omega}]$ to $[\omega]$), the conclusion follows.

Let us now specialize to the case $K = G^{\mathrm{ad}} = \mathrm{PU}(2)$. For $\lambda \in (0,\infty)$, let $\mathcal{O}_\lambda \cong \mathbb{P}^1$ be the coadjoint orbit of $\mathrm{diag}(\mathrm{i}\lambda, -\mathrm{i}\lambda)$, endowed with the Kostant–Kirillov–Souriau form $\omega_{KKS}(\lambda)$. It has a Hamiltonian G^{ad}-action with moment map the inclusion $\iota : \mathcal{O}_\lambda \to \mathfrak{g}$. Let $\gamma = \mathrm{P.D.}(pt)$ denote the generator of $H^2(\mathcal{O}_\lambda;\mathbb{Z}) \subset H^2(\mathcal{O}_\lambda;\mathbb{R})$, so that $c_1(\mathcal{O}_\lambda) = 2\gamma$. Then $c_1(\mathcal{O}_\lambda) = [\omega_{KKS}(1)]$ [6, Sects. 7.5 and 7.6], and so $[\omega_{KKS}(\lambda)] = 2\lambda\gamma$.

If (M,ω,Φ) is a Hamiltonian G^{ad}-manifold, let $M \times \mathcal{O}_\lambda{}^-$ denote the Hamiltonian manifold $(M \times \mathcal{O}_\lambda, \omega \times -\omega_{KKS}(\lambda), \Phi - \iota)$. The *reduction of M with respect to $\mathcal{O}_\lambda$* is defined as

$$M_\lambda = (M \times \mathcal{O}_\lambda{}^-)//G^{\mathrm{ad}} = \Phi^{-1}(\mathcal{O}_\lambda)/G^{\mathrm{ad}}.$$

If the G^{ad}-action on $\Phi^{-1}(\mathcal{O}_\lambda)$ is free, the quotient M_λ is smooth and admits a natural symplectic form ω_λ. It can be viewed as $\Phi^{-1}(\mathrm{diag}(\mathrm{i}\lambda, -\mathrm{i}\lambda))/T^{\mathrm{ad}}$; we let E_λ denote the complex line bundle on M_λ associated with the respective T^{ad}-fibration.

Lemma 7. *Let (M,ω,Φ) be a Hamiltonian G^{ad}-manifold such that the moment map Φ is proper and the action of G^{ad} is free outside $\Phi^{-1}(0)$. Assume that M*

is monotone, with monotonicity constant κ*. Then the cohomology class of the reduced form* ω_λ *is given by the formula*

$$[\omega_\lambda] = \kappa \cdot c_1(TM_\lambda) + (\lambda - \kappa) \cdot c_1(E_\lambda).$$

Proof. First, note that for any Hamiltonian G^{ad}-manifold M, we have $H^2_{G^{\mathrm{ad}}}(M;\mathbb{R}) \cong H^2(M;\mathbb{R})$, because $H^i(BG^{\mathrm{ad}};\mathbb{R}) = 0$ for $i = 1,2$. Thus the Kirwan map can viewed as going from $H^2(M;\mathbb{R})$ into $H^2(M/\!/G^{\mathrm{ad}};\mathbb{R})$.

Let us consider the Kirwan map for the manifold $M \times \mathcal{O}_\lambda{}^-$, whose symplectic reduction is M_λ. By abuse of notation, we denote classes in $H^2(M)$ or $H^2(\mathcal{O}_\lambda{}^-)$ in the same way as their pullbacks to $H^2(M \times \mathcal{O}_\lambda{}^-)$.

Just as in the proof of Lemma 6, we get that the Kirwan map takes $[\omega] - [\omega_{KKS}(\lambda)] = \kappa c_1(TM) - 2\lambda\gamma$ to the reduced form $[\omega_\lambda]$, and $c_1(TM) - c_1(T\mathcal{O}_\lambda) = c_1(TM) - 2\gamma$ to the reduced Chern class $c_1(TM_\lambda)$. Note also that the image of $c_1(T\mathcal{O}_\lambda{}^-) = -2\gamma$ under the Kirwan map is $c_1(E_\lambda)$. Hence

$$[\omega_\lambda] - \kappa \cdot c_1(TM_\lambda) = (\lambda - \kappa) \cdot c_1(E_\lambda),$$

as desired.

We are now ready to study monotonicity for nonabelian cuts.

Proposition 4. *Let* $G^{\mathrm{ad}} = \mathrm{PU}(2)$*, and let* (M, ω, Φ) *be a Hamiltonian* G^{ad}*-manifold that is monotone with monotonicity constant* $\kappa > 0$*. Assume that the moment map* Φ *is proper, and that* G^{ad} *acts freely outside* $\Phi^{-1}(0)$*. Then the symplectic cut* $M_{\leq\lambda}$ *at the value* $\lambda = 2\kappa \in (0,\infty)$ *is also monotone, with the same monotonicity constant* κ*.*

Proof. Recall that the symplectic cut $M_{\leq\lambda}$ is the union of the open piece $M_{<\lambda}$ and the hypersurface $R_\lambda = \Phi^{-1}(\mathcal{O}_\lambda)/S^1$. Note that there is a natural symplectomorphism

$$R_\lambda \xrightarrow{\cong} \mathcal{O}_\lambda \times M_\lambda, \quad m \to \big(\Phi(m), [m]\big). \tag{13.2}$$

The inverse to this symplectomorphism is given by the map $([g],[m]) \to [gm]$.

By Remark 4, the normal bundle to R_λ is the line bundle associated with the defining T^{ad}-bundle on R_λ. We denote this T^{ad}-bundle by N_λ; it is the product of $G^{\mathrm{ad}} \to G^{\mathrm{ad}}/T^{\mathrm{ad}} \cong \mathcal{O}_\lambda$ on the $\mathcal{O}_\lambda$ factor and the circle bundle of E_λ on the M_λ factor.

Let $\nu(R_\lambda)$ be a regular neighborhood of R_λ, so that the intersection $M_{<\lambda} \cap \nu(R_\lambda)$ admits a deformation retract into a copy of N_λ.

We have a Mayer–Vietoris sequence

$$\ldots \to H^1(M_{<\lambda}) \oplus H^1(\nu(R_\lambda)) \to H^1(N_\lambda) \to H^2(M_{\leq\lambda}) \to H^2(M_{<\lambda}) \oplus H^2(\nu(R_\lambda)) \to \ldots.$$

Note that the first Chern class of the bundle $N_\lambda \to R_\lambda$ is nontorsion in $H^2(R_\lambda)$, because it is so on the $\mathcal{O}_\lambda$ factor. Hence, the map $H^1(\nu(R_\lambda);\mathbb{R}) \to H^1(N_\lambda;\mathbb{R})$ is onto. The Mayer–Vietoris sequence then tells us that the map

$$H^2(M_{\leq\lambda};\mathbb{R}) \to H^2(M_{<\lambda};\mathbb{R}) \oplus H^2(\nu(R_\lambda);\mathbb{R})$$

is injective. Therefore, in order to check the monotonicity of $M_{\leq\lambda}$, it suffices to check it on $M_{<\lambda}$ and $\nu(R_\lambda)$.

Since $M_{<\lambda}$ is symplectomorphic to a subset of M, monotonicity is satisfied there by assumption. Let us check it on $\nu(R_\lambda)$, or equivalently, on its deformation retract R_λ. We will use the symplectomorphism (13.2), and by abuse of notation, we will denote the objects on $\mathcal{O}_\lambda$ or M_λ in the same way as we denote their pullbacks to R_λ. Let γ be the generator of $H^2(\mathcal{O}_\lambda;\mathbb{Z})$ as in the proof of Lemma 7. By the result of that lemma, we have

$$[\omega_{\leq\lambda}|_{R_\lambda}] = 2\lambda\gamma + \kappa c_1(TM_\lambda) + (\lambda-\kappa)c_1(E_\lambda). \tag{13.3}$$

On the other hand, the tangent space to $M_{\leq\lambda}$ at a point of R_λ decomposes into the tangent and normal bundles to R_λ. Therefore,

$$c_1(TM_{\leq\lambda}|_{R_\lambda}) = c_1(TR_\lambda) + 2\gamma + c_1(E_\lambda) = 4\gamma + c_1(TM_\lambda) + c_1(E_\lambda).$$

Taking into account (13.3), for $\lambda = 2\kappa$ we conclude that $[\omega_{\leq\lambda}|_{R_\lambda}] = \kappa \cdot c_1(TM_{\leq\lambda}|_{R_\lambda})$.

4.4 Extensions to Presymplectic Manifolds

Abelian cutting and nonabelian cutting are simply particular instances of symplectic reduction. Since the latter can be extended to the presymplectic setting, one can also define abelian and nonabelian cutting for Hamiltonian presymplectic manifolds.

In general, one cannot define $c_1(TM)$ (and the notion of monotonicity) for presymplectic manifolds, because there is no good notion of compatible almost complex structure. In order to fix that, we introduce the following definition.

Definition 4. An *ϵ-symplectic manifold* $(M,\{\omega_t\})$ is a smooth manifold M together with a smooth family of closed two-forms $\omega_t \in \Omega^2(M)$, $t \in [0,\epsilon]$, for some $\epsilon > 0$, such that ω_t is symplectic for all $t \in (0,\epsilon]$.

One should think of an ϵ-symplectic manifold $(M,\{\omega_t\})$ as the presymplectic manifold (M,ω_0) together with some additional data given by the other ω_t's. In particular, by the *degeneracy locus of* $(M,\{\omega_t\})$ we mean the degeneracy locus of ω_0, i.e.,

$$R(\omega_0) = \{m \in M \mid \omega_0 \text{ is degenerate on } T_mM\}.$$

If $(M,\{\omega_t\})$ is any ϵ-symplectic manifold, we can define its first Chern class $c_1(TM) \in H^2(M;\mathbb{Z})$ by giving TM an almost complex structure compatible with some ω_t for $t > 0$. (Note that the resulting $c_1(TM)$ does not depend on t.) Thus, we can define the minimal Chern number of an ϵ-symplectic manifold just as we did for symplectic manifolds. Moreover, we can talk about monotonicity:

Definition 5. The ϵ-symplectic manifold $(M, \{\omega_t\})$ is called *monotone* (with monotonicity constant $\kappa > 0$) if

$$[\omega_0] = \kappa \cdot c_1(TM).$$

One source of ϵ-symplectic manifolds is symplectic reduction. Indeed, suppose we have a Hamiltonian presymplectic S^1-manifold (M, ω, Φ) with the moment map $\Phi : M \to \mathbb{R}$ proper. The form ω may have some degeneracies on $\Phi^{-1}(0)$; however, we assume that it is nondegenerate on $\Phi^{-1}\big((0,\epsilon]\big)$ for some $\epsilon > 0$. Assume also that S^1 acts freely on $\Phi^{-1}\big([0,\epsilon]\big)$ (hence any $t \in (0,\epsilon]$ is a regular value for Φ), and further, 0 is a regular value for Φ as well. Then the presymplectic quotients $M_t = \Phi^{-1}(t)/S^1$ for $t \in [0,\epsilon]$ form a smooth fibration over the interval $[0,\epsilon]$. By choosing a connection for this fiber bundle, we can find a smooth family of diffeomorphisms $\phi_t : M_0 \to M_t$, $t \in [0,\epsilon]$, with $\phi_0 = \mathrm{id}_{M_0}$. We can then put a structure of an ϵ-symplectic manifold on M_0 by using the forms $\phi_t^* \omega_t$, $t \in [0,\epsilon]$, where ω_t is the reduced form on M_t. Note that the space of choices involved in this construction (i.e., connections) is contractible. Therefore, whether $(M_0, \phi_t^* \omega_t)$ is monotone is independent of these choices.

Since abelian cutting and nonabelian cutting are instances of (pre)symplectic reduction, one can also turn presymplectic cuts into ϵ-symplectic manifolds in an essentially canonical way, provided that the form is nondegenerate on the nearby cuts. (By "nearby" we implicitly assume that we have chosen a preferred side for approximating the cut value: either from above or from below.) In this context, we have the following analogue of Proposition 4:

Proposition 5. *Let $G^{\mathrm{ad}} = \mathrm{PU}(2)$, and let (M, ω, Φ) be a Hamiltonian presymplectic G^{ad}-manifold. Set $\tilde{\Phi} = Q \circ \Phi : M \to [0,\infty)$ as usual. Assume that the following hold:*

- *The moment map Φ is proper.*
- *The form ω is nondegenerate on the open subset $M_{<\lambda} = \tilde{\Phi}^{-1}\big([0,\lambda)\big)$, for some value $\lambda \in (0,\infty)$.*
- *G^{ad} acts freely on $\tilde{\Phi}^{-1}\big((0,\lambda]\big)$ (hence any $t \in (0,\lambda)$ is a regular value for $\tilde{\Phi}$).*
- *λ is also a regular value for $\tilde{\Phi}$.*
- *As a symplectic manifold, $M_{<\lambda}$ is monotone, with monotonicity constant $\kappa = \lambda/2$.*

Fix some $\epsilon \in (0,\lambda)$ and view the presymplectic cut $M_{\leq\lambda}$ as an ϵ-symplectic manifold with respect to forms $\phi_t^* \omega_{\leq\lambda-t}$ for a smooth family of diffeomorphisms $\phi_t : M_{\leq\lambda} \to M_{\leq\lambda-t}$, $t \in [0,\epsilon]$, $\phi_0 = id$.

Then, $M_{\leq\lambda}$ is monotone, with the same monotonicity constant $\kappa = \lambda/2$.

Proof. We can run the same arguments as in the proof of Proposition 4, as long as we apply them to the Hamiltonian manifold $M_{<\lambda}$, where ω is nondegenerate. This gives us the corresponding formulas for the cohomology classes $[\omega_{\leq\lambda-t}]$ and $c_1(TM_{\leq\lambda-t})$, for $t \in (0,\epsilon)$. In the limit $t \to 0$, we get monotonicity.

4.5 Cutting the Extended Moduli Space

Recall from Sect. 3.4 that the smooth part $\mathscr{M}_s^{\mathfrak{g}}(\Sigma')$ of the extended moduli space is a Hamiltonian presymplectic G^{ad}-manifold. Let us consider its nonabelian cut at the value $\lambda = 1/2$:

$$\mathscr{N}^c(\Sigma') = \mathscr{M}_s^{\mathfrak{g}}(\Sigma')_{\leq 1/2}.$$

The notation $\mathscr{N}^c(\Sigma')$ indicates that this space is a compactification of $\mathscr{N}(\Sigma') = \mathscr{M}_s^{\mathfrak{g}}(\Sigma')_{<1/2}$. Indeed, we have

$$\mathscr{N}^c(\Sigma') = \mathscr{N}(\Sigma') \cup R,$$

where

$$R \cong \left\{ (A_1, B_1, \ldots, A_h, B_h, \theta) \in G^{2h} \times \mathfrak{g} \,\middle|\, \prod_{i=1}^{h} [A_i, B_i] = \exp(2\pi\theta) = -1 \right\} / S^1. \tag{13.4}$$

Here $u \in S^1 = \mathbb{R}/2\pi\mathbb{Z}$ acts by conjugating each A_i and B_i by $\exp(u\theta)$ and preserving θ.

The G^{ad}-action on $\tilde{\Phi}^{-1}\big((0,1/2]\big) \subset \mathscr{M}_s^{\mathfrak{g}}(\Sigma')$ is free. Since ω is nondegenerate on $\tilde{\Phi}^{-1}\big((0,1/2]\big)$ by Theorem 3, this implies that any $\theta \in \mathfrak{g}$ with $Q(\theta) \in (0,1/2]$ is a regular value for Φ. The last statement also follows from Proposition 2(b), which further says that the values $\theta \in \mathfrak{g}$ with $Q(\theta) = 1/2$ are also regular. Hence, any $t \in (0,1/2]$ is a regular value for $\tilde{\Phi}$. Lastly, note that Theorem 4 says that $\tilde{\Phi}^{-1}\big([0,1/2)\big)$ is monotone, with monotonicity constant $\kappa = 1/4 = \lambda/2$. We conclude that the hypotheses of Proposition 5 are satisfied.

Proposition 6. *Fix $\epsilon \in (0,1/2)$. Endow $\mathscr{N}^c(\Sigma')$ with the structure of an ϵ-symplectic manifold, using the forms $\phi_t^*\omega_{\leq 1/2-t}$, coming from a smooth family of diffeomorphisms*

$$\phi_t : \mathscr{N}^c(\Sigma') = M_{\leq 1/2} \to M_{\leq 1/2-t}, \quad t \in [0,\epsilon],\ \phi_0 = id.$$

Then $\mathscr{N}^c(\Sigma')$ is monotone with monotonicity constant $1/4$.

Thus, we have succeeded in compactifying the symplectic manifold $\mathscr{N}(\Sigma')$ while preserving monotonicity. The downside is that $\mathscr{N}^c(\Sigma')$ is only presymplectic. The resulting two-form has degeneracies on R.

Lemma 8. *Let us view $R = \tilde{\Phi}^{-1}(1/2)/S^1$ as a $\mathbb{P}^1$-bundle over the space $\mathcal{O}_\mu \times \mathscr{M}_{-I}(\Sigma')$; cf. Sect. 3.6. Then the null space of the form $\omega_{\leq 1/2}$ at $x \in R$ consists of the fiber directions. Furthermore, the intersection number (inside $\mathscr{N}^c(\Sigma)$) of R with any $\mathbb{P}^1$ fiber of R is -2.*

Proof. The first claim follows from Proposition 3. The second holds because the normal bundle is the associated bundle to $\mathfrak{t}^\perp$, which is a weight space with weight -2.

In a family of forms that make $\mathscr{N}^c(\Sigma')$ into an ϵ-symplectic manifold (as in Proposition 6), the degenerate form $\omega_{\le 1/2}$ always corresponds to $t = 0$. Hence from now on, we will denote it by ω_0.

Proposition 7. *In addition to the degenerate form ω_0 coming from the cut, the space $\mathscr{N}^c(\Sigma') = \mathscr{N}(\Sigma') \cup R$ also admits a symplectic form ω_ϵ with the following properties:*

(i) R is a symplectic hypersurface with respect to ω_ϵ.
(ii) The restrictions of ω_0 and ω_ϵ to $\mathscr{N}(\Sigma')$ have the same cohomology class in $H^2(\mathscr{N}(\Sigma');\mathbb{R})$.
(iii) The forms ω_0 and ω_ϵ themselves coincide on the open subset $\mathcal{W} = \tilde{\Phi}^{-1}\big([0,1/4)\big) \subset \mathscr{N}(\Sigma')$.
(iv) There exists an almost complex structure $\tilde{J}$ on $\mathscr{N}^c(\Sigma')$ that preserves R, is compatible with respect to ω_ϵ on $\mathscr{N}^c(\Sigma')$, and compatible with respect to ω_0 on $\mathscr{N}(\Sigma')$, and for which any $\tilde{J}$-holomorphic sphere of index zero has intersection number with R a negative multiple of two.

Proof. As the name suggests, the form ω_ϵ will be part of a family $(\omega_t), t \in [0,\epsilon]$ of the type used to turn $\mathscr{N}^c(\Sigma')$ into an ϵ-symplectic manifold. In fact, it is easy to find such a form that satisfies conditions (i)–(iii) above. One needs to choose $\epsilon < 1/4$ and a smooth family of diffeomorphisms $\phi_t : \mathscr{N}^c(\Sigma') = M_{\le 1/2} \to M_{\le 1/2-t}$, $t \in [0,\epsilon]$, $\phi_0 = id$, such that $\phi_t = id$ on $\mathcal{W}$ and ϕ_t takes R to $R_{1/2-t} = \tilde{\Phi}^{-1}(1/2-t)/S^1$. Then set $\omega_\epsilon = \phi_\epsilon^* \omega_0$. Note that condition (ii) is automatic from (iii), because $\mathcal{W}$ is a deformation retract of $\mathscr{N}(\Sigma')$.

However, in order to ensure that condition (iv) is satisfied, more care is needed in choosing the diffeomorphisms above. We will construct only $\phi = \phi_\epsilon$, since this is all we need for our purposes; however, it will be easy to see that one could interpolate between ϕ and the identity.

The strategy for constructing ϕ and $\tilde{J}$ is the same as in the proofs of Proposition 3 and Lemma 8: we construct a diffeomorphism and an almost complex structure on the toroidal extended moduli space $\mathscr{M}^{\mathfrak{t}}(\Sigma')$, by looking at it as a subset of the twisted extended moduli space $\mathscr{M}^{\mathfrak{g}}_{\mathrm{tw}}(\Sigma')$; then we lift them to $\mathscr{M}^{\mathfrak{g}}(\Sigma')$; finally, we show how they descend to the cut.

Let $\mu = \mathrm{diag}(\mathrm{i}/2, -\mathrm{i}/2)$ as in Sect. 3.6. We start by carefully examining the restriction of the form ω to $\mathscr{M}^{\mathfrak{t}}(\Sigma')$, in a neighborhood of $\Phi^{-1}(\mu)$. By the remark at the end of Sect. 3.5, this is the same as looking at the restriction of ω_{tw} to $\Phi_{\mathrm{tw}}^{-1}(\mathfrak{t}^*)$ in a neighborhood of $\Phi_{\mathrm{tw}}^{-1}(0)$.

The zero set Z of the moment map Φ_{tw} on (the smooth, symplectic part of) $\mathscr{M}^{\mathfrak{g}}_{\mathrm{tw}}(\Sigma')$ is a coisotropic submanifold. Let $\omega_{\mathrm{tw},0}$ be the reduced form on $Z/G^{\mathrm{ad}} = \mathscr{M}_{-I}(\Sigma')$. Pick a connection form $\alpha \in \Omega^1(Z) \otimes \mathfrak{g}$ for the G^{ad}-action on Z. By the equivariant coisotropic embedding theorem [21, Proposition 39.2], we can find a G^{ad}-equivariant diffeomorphism between a neighborhood of $Z = \Phi_{\mathrm{tw}}^{-1}(0)$ in $\mathscr{M}^{\mathfrak{g}}_{\mathrm{tw}}(\Sigma')$ and a neighborhood of $Z \times \{0\}$ in $Z \times \mathfrak{g}^*$ such that the form ω_{tw} looks like

$$\omega_{\mathrm{tw}} = \pi_1^* \omega_{\mathrm{tw},0} + \mathrm{d}(\alpha, \pi_2),$$

where $\pi_1 : Z \times \mathfrak{g} \to Z \to Z/G^{\mathrm{ad}}$ and $\pi_2 : Z \times \mathfrak{g}^* \to \mathfrak{g}^*$ are projections. We can assume that π_2 corresponds to the moment map.

Restricting this diffeomorphism to $\Phi_{\mathrm{tw}}^{-1}(\mathfrak{t}^*)$, we obtain a local model $Z \times \mathfrak{t}^*$ for that space. This implies that locally near Z, we get a decomposition of its tangent spaces into several (nontrivial) bundles

$$T(\Phi_{\mathrm{tw}}^{-1}(\mathfrak{t}^*)) \cong T(Z/S^1) \oplus \mathfrak{g} \oplus \mathfrak{t}^* \cong T(\mathscr{M}_{-I}(\Sigma')) \oplus \mathfrak{t}^{\perp} \oplus (\mathfrak{t} \oplus \mathfrak{t}^*). \tag{13.5}$$

(We omitted the pullback symbols from the notation for simplicity.)

The restriction of ω_{tw} to $\Phi_{\mathrm{tw}}^{-1}(\mathfrak{t}^*)$ is nondegenerate in the horizontal directions $T\mathscr{M}_{-I}(\Sigma')$ as well as on $\mathfrak{t} \oplus \mathfrak{t}^*$. Let us compute it on the subbundle $\mathfrak{t}^{\perp} \subset \mathfrak{g}$. For a point x with $\Phi_{\mathrm{tw}}(x) = t\mu \in \mathfrak{t}^*$, and for $\xi_1, \xi_2 \in \mathfrak{t}^{\perp} \subset T_x\Phi_{\mathrm{tw}}^{-1}(\mathfrak{t}^*)$, we have

$$\omega_{\mathrm{tw}}(\xi_1, \xi_2) = (\mathrm{d}\alpha(\xi_1, \xi_2), t\mu) = -\frac{t}{2}\langle [\xi_1, \xi_2], \mu\rangle. \tag{13.6}$$

Thus the restriction of the form to $\mathfrak{t}^{\perp}$ is nondegenerate as long as $t \neq 0$. (For $t = 0$, we already knew that it was degenerate from the proof of Proposition 3.)

We construct a G^{ad}-equivariant almost complex structure J in a neighborhood of Z in $\Phi_{\mathrm{tw}}^{-1}(\mathfrak{t}^*)$ such that J is split with respect to the decomposition (13.5) and is compatible with ω_{tw} "as much as possible." More precisely, we choose G^{ad}-equivariant complex structures J_1, J_3 on each of the subbundles $T(\mathscr{M}_{-I}(\Sigma'))$ and $\mathfrak{t} \oplus \mathfrak{t}^*$ that are compatible with respect to the restriction of ω_{tw} on the respective subbundle. We also choose a G^{ad}-equivariant complex structure J_2 on $\mathfrak{t}^{\perp}$ that is compatible with respect to the form σ given by

$$\sigma(\xi_1, \xi_2) = -\langle [\xi_1, \xi_2], \mu\rangle.$$

By 13.6, we have $\omega_{\mathrm{tw}} = t\sigma/2$; hence J_2 is compatible with respect to ω_{tw} away from $t = 0$. We then let $J = J_1 \oplus J_2 \oplus J_3$ be the almost complex structure on $\Phi_{\mathrm{tw}}^{-1}(\mathfrak{t}^*)$ near Z.

Choose $\epsilon \in (0, 1/8)$ sufficiently small that $Z \times (-3\epsilon, 3\epsilon)$ is part of the local model for $\Phi_{\mathrm{tw}}^{-1}(\mathfrak{t}^*)$ described above. Pick a smooth function $f : \mathbb{R} \to \mathbb{R}$ with the following properties: $f(t) = t + \epsilon$ for t in a neighborhood of 0; $f(t) = t$ for $|t| \geq 2\epsilon$; and $f'(t) > 0$ everywhere. This induces a G^{ad}-equivariant self-diffeomorphism of the open subset $Z \times (-3\epsilon, 3\epsilon) \subset \Phi_{\mathrm{tw}}^{-1}(\mathfrak{t}^*)$, given by $(z, t) \to (z, f(t))$. Note that this diffeomorphism preserves J, it is the identity near the boundary, and it takes $Z \times [0, 2\epsilon)$ to $Z \times [\epsilon, 2\epsilon)$.

Now let us look at the constructions we have made in light of the identification between $\Phi_{\mathrm{tw}}^{-1}(\mathfrak{t}^*)$ and $\mathscr{M}^{\mathfrak{t}}(\Sigma') = \Phi^{-1}(\mathfrak{t}) \subset \mathscr{M}^{\mathfrak{g}}(\Sigma')$. We have obtained a local model $Z \times (-3\epsilon, 3\epsilon)$ for the neighborhood $N = \Phi^{-1}(-3\epsilon\mu, 3\epsilon\mu)$ of $\Phi^{-1}(\mu)$ in $\mathscr{M}^{\mathfrak{t}}(\Sigma')$, an almost complex structure on N, and a self-diffeomorphism of N.

The symplectic cross-section theorem [21] says that locally near $\tilde{\Phi}^{-1}(1/2)$, the extended moduli space $\mathscr{M}^{\mathfrak{g}}(\Sigma')$ looks like $G \times_T \mathscr{M}^{\mathfrak{t}}(\Sigma')$. Thus, we can lift the local model for $\mathscr{M}^{\mathfrak{t}}(\Sigma')$ and obtain a G^{ad}-equivariant local model $(G \times_T Z) \times (-3\epsilon, 3\epsilon)$

for $\mathscr{M}^{\mathfrak{g}}(\Sigma')$. Projection on the second factor corresponds to the map $1/2-\tilde{\Phi}$. Further, locally we can decompose the tangent bundle to $\mathscr{M}^{\mathfrak{g}}(\Sigma')$ as in (13.5). The form ω is nondegenerate when restricted to $T_\mu(\mathscr{O}_\mu)$. Let us choose a G^{ad}-equivariant complex structure on this subbundle that is compatible with the restriction of ω there. By combining it with J, we obtain an equivariant almost complex structure $\tilde{J}$ on

$$\tilde{N} = \tilde{\Phi}^{-1}(1/2-3\epsilon, 1/2+3\epsilon) \subset \mathscr{M}^{\mathfrak{g}}(\Sigma').$$

We can also lift the self-diffeomorphism of $N \subset \mathscr{M}^{\mathfrak{t}}(\Sigma')$ to $\tilde{N} = G \times_T N$ in an equivariant manner. Since this self-diffeomorphism is the identity near the boundary, we can extend it by the identity to all of $\mathscr{M}_s^{\mathfrak{g}}(\Sigma')$. The result is a G^{ad}-equivariant diffeomorphism

$$\mathscr{M}_s^{\mathfrak{g}}(\Sigma') \to \mathscr{M}_s^{\mathfrak{g}}(\Sigma')$$

that preserves $\tilde{J}$ on $\tilde{N}$, takes $\tilde{\Phi}^{-1}(1/2)$ to $\tilde{\Phi}^{-1}(1/2-\epsilon)$, and is the identity on $\tilde{\Phi}^{-1}\big([0,1/2-2\epsilon)\big)$. This diffeomorphism descends to one between the corresponding cut spaces:

$$\phi : \mathscr{N}^c(\Sigma') = \mathscr{M}_s^{\mathfrak{g}}(\Sigma')_{\leq 1/2} \to \mathscr{M}_s^{\mathfrak{g}}(\Sigma')_{\leq 1/2-\epsilon}.$$

We set $\omega_\epsilon = \phi^*\omega_0$. Note that ω_0 and ω_ϵ coincide on the subset $\tilde{\Phi}^{-1}\big([0,1/2-2\epsilon)\big)$. Since we chose $2\epsilon < 1/4$, the latter subset contains $\mathcal{W} = \tilde{\Phi}^{-1}\big([0,1/4)\big)$.

The almost complex structure $\tilde{J}$ on $\tilde{N}$ descends to the cut $\tilde{N}_{\leq 1/2}$ as well. Indeed, if $\mathfrak{t} \subset T\tilde{N}$ denotes the line bundle in the direction of the T^{ad}-action used for cutting, by construction we have $\tilde{J}\mathfrak{t} \cap T\big(\tilde{\Phi}^{-1}(1/2)\big) = 0$. Since $\tilde{J}$ equivariance, it is easy to see that it induces an almost complex structure on the cut, which we still denote by $\tilde{J}$. We extend $\tilde{J}$ to $\tilde{\Phi}^{-1}\big([0,1/2-2\epsilon)\big)$ by choosing it to be compatible with $\omega_0 = \omega_\epsilon$ there.

It is easy to see that the resulting $\tilde{J}$ and ω_ϵ satisfy the required conditions (i)–(iv). With respect to the last claim in (iv), note that any $\tilde{J}$-holomorphic sphere of index zero is necessarily a multiple cover of one of the fibers of the $\mathbb{P}^1$-bundle $R \to \mathscr{M}_{-I}(\Sigma')$. Hence it has intersection number with R a positive multiple of the intersection number of the fiber, which by Lemma 8 is -2.

Remark 7. There were several choices made in the construction of ω_ϵ and $\tilde{J}$ in Proposition 7, for example, the connection α, the structures J_1, J_2, J_3, and the function f. The space of all these choices is contractible.

5 Symplectic Instanton Homology

5.1 Lagrangians from Handlebodies

Let H be a handlebody of genus $h \geq 1$ whose boundary is the compact Riemann surface Σ. We view Σ' and Σ as subsets of H, with $\Sigma' = \Sigma \setminus D^2$.

Let $\mathscr{A}^{\mathfrak{g}}(\Sigma'|H) \subset \mathscr{A}^{\mathfrak{g}}(\Sigma')$ be the subspace of connections that extend to flat connections on the trivial G-bundle over H. Consider also $\mathscr{A}(H)$, the space of flat connections on H, which is acted on by the based gauge group $\mathscr{G}_0(H) = \{f : H \to G \mid f(z) = I\}$. Since $\pi_1(G) = 1$ and Σ' has the homotopy type of a wedge of spheres, every map $\Sigma' \to G$ must be null homotopic. This implies that $\mathscr{G}^c(\Sigma')$ preserves $\mathscr{A}^{\mathfrak{g}}(\Sigma'|H)$, and furthermore, the natural map

$$\mathscr{A}(H)/\mathscr{G}_0(H) \longrightarrow \mathscr{A}^{\mathfrak{g}}(\Sigma'|H)/\mathscr{G}^c(\Sigma') \tag{13.1}$$

is a diffeomorphism.

Set

$$L(H) = \mathscr{A}(H)/\mathscr{G}_0(H) \cong \mathscr{A}^{\mathfrak{g}}(\Sigma'|H)/\mathscr{G}^c(\Sigma') \subset \mathscr{M}^{\mathfrak{g}}(\Sigma') = \mathscr{A}^{\mathfrak{g}}(\Sigma')/\mathscr{G}^c(\Sigma').$$

The left-hand side of (13.1) is the moduli space of flat connections on H. After a set of h simple closed curves $\alpha_1, \ldots, \alpha_h$ on H whose classes generate $\pi_1(H)$ has been chosen, the space $\mathscr{A}(H)/\mathscr{G}(H)$ can be identified with the space of homomorphisms $\pi_1(H) \to G$ or, alternatively, with the Cartesian product G^h.

In fact, if the curves $\alpha_1, \ldots, \alpha_h$ are the same as those chosen on Σ' for the identification (13.1), so that the remaining curves β_i are null homotopic in H, then with respect to the identification (13.3), we have

$$L(H) \cong \{(A_1, B_1, \ldots, A_h, B_h) \in G^{2h} \mid B_i = I, \quad i = 1, \ldots, h\} \subset \mathscr{N}(\Sigma'). \tag{13.2}$$

Let us now view $L(H)$ as $\mathscr{A}^{\mathfrak{g}}(\Sigma'|H)/\mathscr{G}^c(\Sigma')$ via (13.1). Note that connections A that extend to H extend in particular to Σ, which means that the value $\theta \in \mathfrak{g}$ such that $A|_S = \theta \mathrm{d}s$ is zero. In other words, $L(H)$ lies in $\Phi^{-1}(0) \subset \mathscr{N}(\Sigma')$.

Lemma 9. *With respect to the Huebschmann–Jeffrey symplectic form ω from Sect. 3.4, $L(H)$ is a Lagrangian submanifold of $\mathscr{N}(\Sigma')$.*

Proof. Let $\tilde{A}$ be a flat connection on H and A its restriction to Σ'. With respect to the description (13.4) of $T_{[A]}\mathscr{N}(\Sigma')$, the tangent space to $L(H)$ at A consists of equivalence classes of d_A-closed forms $a \in \Omega^{1,\mathfrak{g}}(\Sigma')$ that extend to $\mathrm{d}_{\tilde{A}}$-closed forms $\tilde{a} \in \Omega^1(H) \otimes \mathfrak{g}$. Let a, b be two such forms and $\tilde{a}, \tilde{b}$ their extensions to H. We have $a|_S = b|_S = 0$. Furthermore, by the Poincaré lemma for connections, on the disk D^2 that is the complement of Σ' in Σ there exists $\lambda \in \Omega^0(D^2; \mathfrak{g})$ such that $\mathrm{d}_{\tilde{A}}\lambda = \tilde{a}|_{D^2}$. By Stokes's theorem,

$$\int_{D^2} \langle a \wedge b \rangle = \int_S \langle \lambda \wedge b \rangle = 0.$$

Another application of Stokes's theorem gives

$$\int_{\Sigma'} \langle a \wedge b \rangle = \int_{\Sigma} \langle \tilde{a} \wedge \tilde{b} \rangle = \int_H \langle \mathrm{d}_{\tilde{A}}(\tilde{a} \wedge \tilde{b}) \rangle = 0.$$

This shows that ω vanishes on the tangent space to $L(H) \cong G^h$, which is half-dimensional.

5.2 Symplectic Instanton Homology

Let $Y = H_0 \cup H_1$ be a Heegaard decomposition of a three-manifold Y, where H_0 and H_1 are handlebodies of genus h, with $\partial H_0 = -\partial H_1 = \Sigma$. Let $L_0 = L(H_0)$ and $L_1 = L(H_1) \subset \mathscr{N}(\Sigma')$ be the Lagrangians associated respectively with H_0 and H_1, as in 5.1. View $\mathscr{N}(\Sigma')$ as an open subset of the compactified space $\mathscr{N}^c(\Sigma')$, as in Sect. 4.5, with R being its complement.

In Sect. 4.5 we gave $\mathscr{N}^c(\Sigma')$ the structure of an ϵ-symplectic manifold. By Lemma 8, its degeneracy locus is exactly R. Using the variant of Floer homology described in Sect. 2.3 and letting $\tilde{\omega} = \omega_0$, $\omega = \omega_\epsilon$, and $\tilde{J}$ be as in Proposition 7, we define

$$HSI(\Sigma'; H_0, H_1) = HF(L_0, L_1, \tilde{J}; R).$$

In order to ensure that the Floer homology is well defined, we should check that the hypotheses (i)–(ix) listed at the beginning of Sect. 2.3 are satisfied. Indeed, (i), (ii), (iii), (v), and (x) are subsumed in Proposition 7, while (iv), (v), and (ix) follow respectively from Proposition 6, Lemma 9, and Corollary 2. For (viii), the Lagrangians are simply connected and spin because they are diffeomorphic to G^h. By Theorem 4 and Lemma 5, the minimal Chern number of the open subset $\mathscr{N}(\Sigma)$ is a multiple of 4; therefore, the Floer groups admit a relative $\mathbb{Z}/8\mathbb{Z}$-grading.

A priori, the Floer homology depends on $\tilde{J}$. However, the set of choices used in the construction of $\tilde{J}$ is contractible; cf. Remark 7. By the usual continuation arguments in Floer theory, if we change $\tilde{J}$, the corresponding Floer homology groups are canonically isomorphic.

5.3 Dependence on the Base Point

Recall that the surface Σ' is obtained from a closed surface Σ by deleting a disk around some base point $z \in \Sigma$. Let $z_0, z_1 \in \Sigma$ be two choices of base point. Any choice of path $\gamma : [0,1] \to \Sigma$, $j \mapsto z_j$, $j = 0,1$, induces an identification of fundamental groups $\Sigma'_0 \to \Sigma'_1$ and equivariant presymplectomorphisms $T_\gamma : \mathscr{N}^c(\Sigma'_0) \to \mathscr{N}^c(\Sigma'_1)$ preserving the cut locus R. The pullbacks of the form ω and the almost complex structure $\tilde{J}$ from Proposition 7 (applied to $\mathscr{N}^c(\Sigma'_1)$) can act as the corresponding form and almost complex structure in Proposition 7 applied to $\mathscr{N}^c(\Sigma'_0)$. Moreover, if H_0, H_1 are handlebodies, the symplectomorphism T_γ preserves the corresponding Lagrangians L_0, L_1, since the vanishing holonomy condition is invariant under conjugation by paths. Therefore, the continuation arguments in Floer theory show that T_γ induces an isomorphism

$$HSI(\Sigma'_0; H_0, H_1) \to HSI(\Sigma'_1; H_0, H_1).$$

This isomorphism depends only on the homotopy class of γ relative to its endpoints. We conclude that the symplectic instanton homology groups naturally form a flat bundle over Σ. In particular, there is a natural action of $\pi_1(\Sigma, z_0)$ on $HSI(\Sigma_0'; H_0, H_1)$.

When we care about the Floer homology group only up to isomorphism (not canonical isomorphism), we drop the base point from the notation and write $HSI(\Sigma'; H_0, H_1) = HSI(\Sigma; H_0, H_1)$, as in the introduction.

6 Invariance

We prove here that the groups $HSI(\Sigma; H_0, H_1)$ are invariants of the 3-manifold $Y = H_0 \cup H_1$. The proof is based on the theory of Lagrangian correspondences in Floer theory; cf. [59]. We start by reviewing this theory.

6.1 *Quilted Floer Homology*

Let M_0, M_1 be compact symplectic manifolds. A *Lagrangian correspondence* from M_0 to M_1 is a Lagrangian submanifold $L_{01} \subset M_0^- \times M_1$. (The minus superscript means that we are considering the same manifold equipped with the negative of the given symplectic form.) Given Lagrangian correspondences $L_{01} \subset M_0^- \times M_1$, $L_{12} \subset M_1^- \times M_2$, their *composition* is the subset of $M_0^- \times M_2$ defined by

$$L_{01} \circ L_{12} = \pi_{02}(L_{01} \times_{M_1} L_{12}),$$

where $\pi_{02} : M_0^- \times M_1 \times M_1^- \times M_2 \to M_0^- \times M_2$ is the projection. If the intersection

$$L_{01} \times_{M_1} L_{12} = (L_{01} \times L_{12}) \cap (M_0^- \times \Delta_{M_1} \times M_2)$$

is transverse (hence smooth) in $M_0^- \times M_1 \times M_1^- \times M_2$, and the projection $\pi_{02} : L_{01} \times_{M_1} L_{12} \to L_{01} \circ L_{12}$ is embedded, we say that the composition $L_{02} = L_{01} \circ L_{12}$ is *embedded*. An embedded composition L_{02} is a smooth Lagrangian correspondence from M_0 to M_2.

Suppose now that M_0, M_1, M_2 are compact symplectic manifolds, monotone with the same monotonicity constant, and with minimal Chern number at least 2. Suppose that $L_0 \subset M_0, L_{01} \subset M_0^- \times M_1, L_{12} \subset M_1^- \times M_2, L_2 \subset M_2$ are simply connected Lagrangian submanifolds. (This implies that their minimal Maslov numbers are at least 4.)

Define

$$HF(L_0, L_{12}, L_{12}, L_2) := HF(L_0 \times L_{12}, L_{01} \times L_2)$$

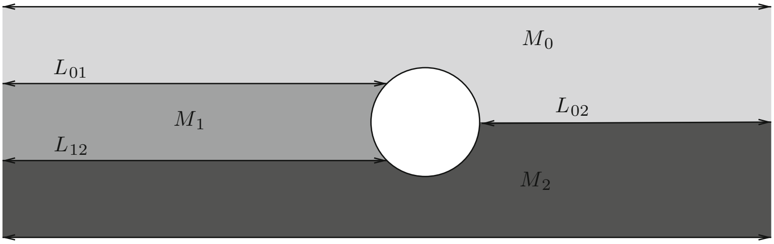

Fig. 1 Geometric composition via a quilt count of Y-maps

and

$$HF(L_0, L_{02}, L_2) := HF(L_0 \times L_2, L_{01} \circ L_{12}).$$

The main theorem of [59] implies the following.

Theorem 5. *With $M_0, M_1, M_2, L_0, L_{01}, L_{12}, L_2$ monotone as above, if $L_{02} := L_{01} \circ L_{12}$ is embedded, then there exists a canonical isomorphism of Lagrangian Floer homology groups*

$$HF(L_0, L_{01}, L_{12}, L_2) \to HF(L_0, L_{02}, L_2). \tag{13.1}$$

In Wehrheim–Woodward [59], an isomorphism is defined using pseudoholomorphic quilts, i.e., in this case, triples of strips in M_0, M_1, M_2 with boundary conditions in L_0, L_{01}, L_{12}, and L_2. The count of such quilts is used in the left-hand side of (13.1). In the limit whereby the width δ of the middle strip goes to 0, the same count produces the right-hand side. An alternative proof was given in Lekili–Lipyanskiy [30] using a count of *Y-maps*. This approach is better suited for the semipositive case in which we will need it, so we review the construction. Given $x_- \in (L_0 \times L_{12}) \cap (L_{01} \times L_2), x_+ \in (L_0 \times L_2) \cap L_{02}$, let $\mathfrak{M}(x_-, x_+)$ denote the set of holomorphic quilts with two striplike ends and one cylindrical end as shown in Fig. 1, with finite energy and limits $x_\pm$. The authors show that for a comeager subset of the space of point-dependent compatible almost complex structures, the moduli space $\mathfrak{M}(x_-, x_+)$ of Y-maps has the structure of a finite-dimensional manifold, and counting the zero-dimensional component $\mathfrak{M}(x_-, x_+)_0$ defines a cochain map

$$\Phi : CF(L_0, L_{01}, L_{12}, L_2) \to CF(L_0, L_{02}, L_2), \quad \langle x_- \rangle \mapsto \sum_{u \in \mathfrak{M}(x_-, x_+)_0} \varepsilon(u) \langle x_+ \rangle .$$

Here, in the case of integer coefficients, the map

$$\varepsilon : \mathfrak{M}(x_-, x_+)_0 \to \{\pm 1\}$$

is defined by comparing the orientations constructed in [58] with the canonical orientation of a point.

Counting Y-maps in the opposite direction defines a chain map

$$\Psi : CF(L_0, L_{02}, L_2) \to CF(L_0, L_{01}, L_{12}, L_2).$$

Lekili and Lipyanskiy [30] prove that the monotonicity constant for these Y-maps is the same as the monotonicity constant for Floer trajectories. They then show that Φ and Ψ induce isomorphisms on homology.

6.2 *Relative Quilted Floer Homology in Semipositive Manifolds*

We wish to have a version of the quilted Floer homology and composition theorem, Theorem 5, that holds for Floer homology *relative to hypersurfaces* in *semipositive* manifolds, as in Sect. 2.3. Suppose that R_0, R_1 are symplectic hypersurfaces in M_0, M_1. From them we obtain two hypersurfaces $\tilde{R}_0 = R_0^- \times M_1$, $\tilde{R}_1 = M_0^- \times R_1$ in $M_0^- \times M_1$. Let $\tilde{N}_{R_0}$, $\tilde{N}_{R_1}$ denote their normal bundles N_{R_0}, N_{R_1}, that is, the pullbacks of N_{R_0}, N_{R_1} to $\tilde{R}_0$, $\tilde{R}_1$. Because R_0, R_1 are symplectic, N_{R_0}, N_{R_1} are *oriented* rank-2 bundles, or equivalently up to homotopy, rank-1 complex line bundles. As we will see below, the following definition gives sufficient conditions for a sort of combined intersection number with R_0, R_1 to be well defined and given by the usual geometric formulas.

Definition 6. A simply connected Lagrangian correspondence $L_{01} \subset M_0^- \times M_1$ is called *compatible* with the pair (R_0, R_1) if

$$(R_0 \times M_1) \cap L_{01} = (M_0 \times R_1) \cap L_{01} = (R_0 \times R_1) \cap L_{01}$$

and there exist an isomorphism

$$\varphi : (\tilde{N}_{R_0})|_{(R_0 \times R_1) \cap L_{01}} \cong (\tilde{N}_{R_1})|_{(R_0 \times R_1) \cap L_{01}}$$

and tubular neighborhoods

$$\tau_0 : N_{R_0} \to M_0, \quad \tau_1 : N_{R_1} \to M_1$$

of R_0 and R_1 respectively such that $(\tau_0 \times \tau_1)^{-1}(L_{01}) \subset N_{R_0} \times N_{R_1} = \tilde{N}_{R_0} \times_{M_0 \times M_1} \tilde{N}_{R_1}$ is equal to the graph of φ.

To explain the conditions in the definition, note that the existence of φ implies that any map of a compact oriented surface with boundary to M with boundary conditions in L_{01} has a well-defined intersection number with $\tilde{R}_0 \cup \tilde{R}_1$. For example, suppose that $u : (D, \partial D) \to (M_0 \times M_1, L_{01})$ is a disk with Lagrangian boundary conditions. The sum of dual classes $[\tilde{R}_0]^\vee + [\tilde{R}_1]^\vee$ has trivial restriction to $H^2(L_{01}; \mathbb{Z})$. If L_{01} is simply connected, then $H^2(M, L_{01})$ is the kernel of $H^2(M) \to H^2(L_{01})$, and so we may consider $[\tilde{R}_0]^\vee + [\tilde{R}_1]^\vee$ as a class in $H^2(M, L_{01})$. Then the intersection number of a map $u : (D, \partial D) \to (M_0 \times M_1, L_{01})$ with $[\tilde{R}_0] + [\tilde{R}_1]$ is well defined and denoted by $u \cdot R$.

The existence of the tubular neighborhoods τ_0, τ_1 implies that $u \cdot R$ is given by a geometric count of intersection points. Indeed, we may identify a neighborhood of R_j with the normal bundle $\pi_j : N_j \to R_j$ via the tubular neighborhood τ_j. Then $\pi_j^* N_j$ is trivial on the complement of R_j, since the map π_j gives a nonvanishing section, and extends to a bundle L_{R_j} on M_j trivial on the complement of R_j. Then the dual class $[R_j]^\vee$ is given by a Thom class in the tubular neighborhood of R_j, and hence equals the Euler class of L_{R_j}. The bundles L_{R_0} and L_{R_1} are isomorphic on ∂D via φ, and so glue together to a bundle denoted by $u^* L_R$ over $S^2 = D \cup_{\partial D} D$. The intersection number is then the Euler number of $u^* L_R$, that is,

$$u \cdot R = ([S^2], \mathrm{Eul}(u^* L_R)).$$

The compatibility condition on the maps τ_j implies that the maps u_0, u_1 considered as sections of L_{R_j} near R_j glue together to a section of $u^* L_R$, which by abuse of notation we denote by u. If each u_j meets R_j in a finite number of points, then $u \cdot R$ is a sum of local intersection numbers $(u \cdot R)_z$, given by the image of a small loop around each intersection point z in $H_1(u^* L_R|_V - 0, \mathbb{Z}) \cong \mathbb{Z}$ in a small neighborhood V of z. Note that since we have constructed $u^* L_R$ only as a *topological* (or rather, piecewise smooth) bundle, such a loop will be piecewise smooth only if $z \in \partial D$.

Our examples will arise as follows:

Example 1. Suppose $\iota : C \to M_1$ is a fibered coisotropic submanifold of M_1 with structure group C, the fibration being $\pi : C \to M_0$. Then $(\pi \times \iota) : C \to M_0^- \times M_1$ defines a Lagrangian correspondence; cf. [59, Example 2.0.3(b)]. Suppose further that M_1 is a Hamiltonian U(1)-manifold with moment map Φ_1 and C is U(1)-invariant and meets $\Phi_0^{-1}(\lambda)$ transversely. Then the symplectic cut $M_{1,\le\lambda}$ contains the closure $C_{\le\lambda}$ of the image C as a fibered coisotropic submanifold, whose graph is a Lagrangian correspondence in $M_{0,\le\lambda} \times M_{1,\le\lambda}$. Furthermore, the submanifolds $R_0 := M_{0,\lambda}, R_1 := M_{1,\lambda}$ are symplectic submanifolds with the properties described in Definition 6. Indeed, any tubular neighborhood $N_{R_1} \to M_{1,\le\lambda}$ of R_1 that is U(1)-invariant, maps $N_{R_1}|_{C_{\le\lambda}}$ to $C_{\le\lambda}$, and maps fibers to fibers induces a tubular neighborhood $N_{R_0} \to M_{0,\le\lambda}$ with the required properties.

The intersection numbers described above are well defined more generally for quilted strips, as we now explain. Given symplectic manifolds $M_1, \ldots, M_k$, Lagrangian submanifolds $L_1 \subset M_1, L_k \subset M_k$ disjoint respectively from R_1 and R_k, and Lagrangian correspondences

$$L_{12} \subset M_1 \times M_2, \ldots, L_{(k-1)k} \subset M_{k-1}^- \times M_k$$

compatible with hypersurfaces $\underline{R} = (R_j \subset M_j)_{j=1,\ldots,k}$, the intersection number $\underline{u} \cdot \underline{R}$ of a quilted Floer trajectory

$$\underline{u} = (u_j : \mathbb{R} \times [0,1] \to M_j)_{j=1}^k$$

is the pairing of $\underline{u}$ with the sum of the dual classes $[R_j]^\vee$ to R_j.

If the intersection of $\underline{u}$ with $\underline{R}$ is finite, then the intersection number is the sum of local intersection numbers defined as follows. By assumption, there exists an isomorphism

$$N_{j-1}|_{L_{(j-1)j} \cap (R_{j-1} \times R_j)} \xrightarrow{\cong} N_j|_{L_{(j-1)j} \cap (R_{j-1} \times R_j)},$$

and this extends to an isomorphism of $\tilde{N}_{R_j}|_{L_{(j-1)j}}$ and $\tilde{N}_{R_{j-1}}|_{L_{(j-1)j}}$ by the assumption about the tubular neighborhoods. Thus the pullback bundles $u_j^* \tilde{N}_{R_j}$ patch together to a bundle on the quilted surface $\underline{S} = \cup_j S_j$, which we denote by $\underline{u}^* \tilde{N}_{\underline{R}}$. The intersection number is then the relative Euler number of $\underline{u}^* \underline{N}_{\underline{R}} \to \underline{S}$, that is, the pairing of the relative Euler class with the generator of $H^2(cl(\underline{S}), \partial cl(\underline{S}))$, where $cl(\underline{S})$ is the closed disk obtained by adding points at $\pm\infty$. The map $\underline{u}$ then provides a section of $\underline{u}^* \underline{N}_{\underline{R}}$, by the compatibility conditions in Definition 6. If the intersection is finite, then

$$\underline{u} \cdot \underline{R} = \sum_{\{z \in \underline{S} | \underline{u}(z) \in \underline{R}\}} (\underline{u} \cdot \underline{R})_z, \qquad (13.2)$$

where $(\underline{u} \cdot \underline{R})_z \in \mathbb{Z}$ is, as in the case of disks discussed before, the image of a small loop around z in the complement $\underline{u}^* \underline{N}_{\underline{R}} - 0$ of the zero section, as a multiple of the generator of the first homology of the fiber, and the condition $\underline{u}(z) \in \underline{R}$ means that if z lies in the component S_j, then $u_j(z_j) \in R_j$. Note in particular that these local intersection numbers are topologically continuous, that is, given any loop in the domain of the quilt, the sum of the local intersection numbers is constant in any continuous family as long as none of the intersection points cross the loop.

If the intersection is not only finite but transverse, and the hypersurfaces $\underline{R}$ are almost complex, then the intersection number is the usual one counted with weight $1/2$ for the seam points:

Lemma 10. *Suppose that L_0 respectively L_k is disjoint from R_0 respectively R_k and each $L_{(j-1)j}$ is compatible with (R_{j-1}, R_j). Suppose that the almost complex structure on $M_0 \times \cdots \times M_k$ is of product form $J_0 \times \cdots \times J_k$ near each $\tilde{R}_j$, so that each R_j is an almost complex submanifold of M_j with respect to J_j. Let $\underline{u} : \underline{S} \to \underline{M}$ be a quilted Floer trajectory with Lagrangian boundary and seam conditions in $\underline{L}$ meeting each $\tilde{R}_j$ transversally. Then*

$$\underline{u} \cdot \underline{R} = \sum_{j=0}^k \#\{z_j \in \mathrm{int}(S_j) | u_j(z_j) \in R_j\} + \frac{1}{2} \#\{z_j \in \partial S_j | u_j(z_j) \in R_j\}.$$

Proof. The local intersection number in (13.2) at a transversal point of intersection $z \in \underline{S}$ is the homology class of the image of a small loop around z, considered as an element of $H_1(\underline{N}_z) \cong \mathbb{Z}$. We consider only the case of an intersection point z on the seam; the loop is divided into two loops, one coming from each component of the quilt, and is only piecewise smooth. The case of an interior intersection is easier and left to the reader.

Suppose z is on the seam $L_{(j-1)j}$ where the components u_{j-1} and u_j of the quilt meet. For $l = j-1$ or j, let us view u_l as a section of a piecewise smooth line bundle. Using a local trivialization of the bundle and a coordinate chart for $\underline{S}$ centered at z, we have that u_l near z (now viewed as a map to $\mathbb{C}$) is given approximately by its linearization at z:

$$|u_l(r\exp(it)) - (Du_l(z))r\exp(it)| < Cr^2.$$

We use here that since R_l is almost complex, the linearization Du_l is complex linear. Fix $\epsilon > 0$. For r sufficiently small, we have

$$|\arg(u_l(r\exp(it))) - \arg(Du_l(z)r\exp(it))| < \epsilon.$$

This implies that

$$\left|\int_0^1 u_l^* \mathrm{d}\theta - \pi\right| < \epsilon, \quad l = j-1, j,$$

and so

$$\int_0^1 u_{j-1}^* \mathrm{d}\theta + \int_0^1 u_j^* \mathrm{d}\theta \in (2\pi - 2\epsilon, 2\pi + 2\epsilon).$$

Since the integral must be an integer multiple of 2π (and ϵ can be chosen arbitrarily small), the integral must in fact equal 2π. It follows that the two paths patch together to a positive generator of $H_1(\mathbb{C}^*, \mathbb{Z})$, as claimed.

We can now define relative quilted Floer homology in semipositive manifolds.

Theorem 6. *Suppose that $\underline{M} = (M_i)_{i=0}^k$ are semipositive manifolds as in the first six items of Assumption 2.1, with a collection of open sets $\underline{\mathcal{W}} = (\mathcal{W}_i)_{i=0}^k$ on which the respective forms ω_i and $\tilde{\omega}_i$ coincide. Suppose the manifolds M_i come equipped with almost complex structures $\tilde{J}_i$, so that the degeneracy loci R_i of the forms $\tilde{\omega}_i$ are almost complex hypersurfaces in M_i, disjoint from $\mathcal{W}_i$. We denote by $\mathcal{J}_t(\underline{M}, \underline{\mathcal{W}}, \tilde{J})$ the space of time-dependent almost complex structures on $M_0 \times \cdots \times M_k$ that agree with $\tilde{J} = \tilde{J}_0 \times \cdots \times \tilde{J}_k$ on $\mathcal{W} := \prod_{i=0}^k \mathcal{W}_i$.*

We are also given simply connected Lagrangians $L_0 \subset M_0, L_{01} \subset M_0^- \times M_1, \ldots, L_{(k-1)k} \subset M_{k-1}^- \times M_k, L_k \subset M_k$ such that the seam conditions $L_{(i-1)i}$ are compatible with (R_{i-1}, R_i), and L_0 and L_k are contained in $\mathcal{W}_0$ respectively $\mathcal{W}_k$. Also, we assume that

$$(L_0 \times L_{12} \times \cdots) \cap (L_{01} \times L_{23} \times \cdots) \subset \mathcal{W}_0 \times \cdots \times \mathcal{W}_k. \tag{13.3}$$

Suppose further that any holomorphic disk with boundary in $L_{(i-1)i}$, $i = 1, \ldots, k$, or holomorphic sphere with zero canonical area has intersection number with $\underline{R}$ given by a negative multiple of 2.

Then there exists a comeager subset $\mathcal{J}_t^{\mathrm{reg}}(\underline{L},\underline{\mathcal{W}},\tilde{J})$ *of* $\mathcal{J}_t(\underline{M},\underline{\mathcal{W}},\tilde{J})$ *such that if the almost complex structure* (J_t) *is chosen from* $\mathcal{J}_t^{\mathrm{reg}}(\underline{L},\underline{\mathcal{W}},\tilde{J})$, *then the part of the Floer differential of* $CF(\underline{L}) = CF(L_0 \times L_{12} \times \cdots, L_{01} \times L_{23} \times \cdots)$ *counting trajectories disjoint from* R_i, $i = 1,\ldots,m$, *is finite and squares to zero. We denote by*

$$HF(\underline{L};\underline{R}) := HF(\underline{L},\tilde{J};\underline{R})$$

the resulting Floer homology group; it is independent up to isomorphism of all choices except possibly the base almost complex structures $\tilde{J}_i$.

Proof. First, note that the condition (13.3) implies that the endpoints of any holomorphic quilt are contained in $\mathcal{W} = \mathcal{W}_0 \times \cdots \times \mathcal{W}_k$. Hence, every quilt component u_i contains a point in the respective open set $\mathcal{W}_i$. This implies that the usual transversality arguments for holomorphic quilts apply, even when we restrict to almost complex structures J_t that are required to agree with $\tilde{J}$ on $\mathcal{W}$.

Next, we discuss compactness. We must rule out sphere and disk bubbling in the zero- and one-dimensional moduli spaces. For a suitable comeager subset of almost complex structures agreeing with the given $\tilde{J}_i$, the trajectories are transverse to the R_j in the zero- and one-dimensional moduli spaces, by the same argument we gave previously for the unquilted case (Corollary 1).

Suppose that $\underline{u}_\infty$ is the limit of a sequence of trajectories of index 1 or 2 disjoint from $\underline{R}$. By the assumption on the intersection number, any sphere bubble or disk bubble with boundary in some $L_{(j-1)j}$ contributes at least -2 to the intersection number with $\underline{R}$. It follows that at least one intersection point does not have a bubble attached. But then, since the intersection point is transverse, $\underline{u}_\infty$ cannot be the limit of a sequence of trajectories disjoint from $\underline{R}$, since transverse intersection points persist under deformation. Hence there is no such bubbling, and the limit is a (possibly broken) trajectory, as desired. Independence of the choice of almost complex structures is proved by the usual continuation argument, ruling out disk bubbles of index one and sphere bubbles by the same reasoning.

Remark 8. If the Lagrangian correspondences above are associated to fibered coisotropics, then the almost complex structures may be taken of split form, that is, products of the almost complex structures on $M_0,\ldots,M_k$. This will be the case in our application.

Theorem 7. *Suppose that* $\underline{M} = (M_0,M_1,M_2)$ *and* $\underline{L} = (L_0,L_{01},L_{12},L_2)$ *satisfy the assumptions of Theorem 6. Suppose further that* $L_{01} \circ L_{12}$ *is an embedded composition, is simply connected, and is compatible with* (R_0,R_2), *and that all holomorphic quilted cylinders with seams in* $L_{01},L_{12},L_{01}\circ L_{12}$ *with zero canonical area have intersection number equal to a negative multiple of 2. Then the relative Lagrangian Floer homology groups* $HF(L_0,L_{01},L_{12},L_2;R_0,R_1,R_2)$, $HF(L_0,L_{01}\circ L_{12},L_2;R_0,R_2)$ *are isomorphic. Similar statements hold for the composition of any two adjacent pairs, as long as the compositions are smooth and embedded.*

Proof. If the Lagrangian correspondences had been monotone, the result would have been a slight extension of Theorem 5 in [59], by counting only those trajectories disjoint from R_i; indeed, since the intersection numbers are homotopy invariants, they do not change on taking the limit $\delta \to 0$.

In the semipositive case at hand, one can rule out disk and sphere bubbling as in the proof of Proposition 1, but not the figure-eight bubbles mentioned in [59, Sect. 5.3]. Indeed, removal of singularities, transversality, and Fredholm theory for figure-eight bubbles have not yet been developed. For this reason, we use instead the approach of Lekili–Lipyanskiy [30].

First, one checks that for a comeager subset of compatible almost complex structures, the ends of the cylinders of Y-maps will not map to R in the zero- and one-dimensional components of the moduli space, since this is a codimension-2 condition. Indeed, an examination of the weighted Sobolev space construction of the moduli space of Y-maps in [30] shows that the evaluation map at the end of the cylinder is smooth; indeed, it projects onto the factor of asymptotically constant maps in the Banach manifolds in which the moduli space of Y-maps is locally embedded: $W^{1,p,\varepsilon}(S;\underline{u}^*T\underline{M},\underline{u}^*T\underline{L}) \oplus T_{(\underline{u})_{02}(\infty)}L_{02}$, where the former is the space of from S with Lagrangian boundary conditions with finite ε-weighted Sobolev norm of class $(1,p)$, and the latter is the intersection of the linearized Lagrangian boundary conditions at infinity on the cylindrical end.

As a result, the intersection number $\underline{u} \cdot \underline{R}$ of any Y-map $\underline{u}$ of index zero and one with the collection $\underline{R}$ is well defined and given by the formula (10). (More generally, one could make the intersection number with *any* Y-map well defined by imposing the compatibility condition $\varphi_{01} \circ \varphi_{12} = \varphi_{02}$, so that the bundle $\underline{u}^* L_{\underline{R}}$ is well defined. But we will not need this.) In the zero- and one-dimensional moduli spaces, all intersections with the manifolds R_j are transverse for $\underline{J}$ chosen from a comeager subset of the space of compatible almost complex structures making R_j almost complex, by standard arguments [11, Sect. 6].

A Gromov compactness argument shows that finite-energy Y-maps have as limits configurations consisting of a (possibly broken) Y-map together with some sphere bubbles, disk bubbles, and cylinder bubbles. The cylinder bubbles may form when there is an accumulation of energy at the Y-end.

In the case at hand, sphere and disk bubbles are ruled out as in the proof of Theorem 6: any sphere or disk bubble appearing in the limit configuration $\underline{u}_\infty$ must have index zero, and therefore intersection number at most -2 with $\underline{R}$. By (10), any intersection point contributes at most 1 to the intersection number, and therefore at some intersection point with $\underline{R}$ it is not attached to a bubble. But then $\underline{u}_\infty$ cannot be the limit of a sequence of trajectories disjoint from $\underline{R}$, since the local intersection number of $\underline{u}_\infty$ is nonzero.

It remains to rule out cylinder bubbles. Since no trajectory of index zero or one maps the end of the cylinder to $\underline{R}$, any quilted cylinder bubble must capture positive canonical area. But then, for index reasons explained in Lekili–Lipyanskiy [30], the cylinder bubble must capture at least index two, so the index of the remaining Y-map is at most -1. (Here working with Y-maps, rather than strip-shrinking, provides an advantage: by exponential decay for holomorphic strips with boundary values in Lagrangians intersecting cleanly, one knows that these cylinder bubbles connect to a point outside of $\underline{R}$, whereas for figure-eight bubbles, such exponential decay estimates are missing.) But such a trajectory does not exist, since transversality is achieved for the chosen $\underline{J}$.

It follows that the moduli spaces of Y-maps of dimension zero and one that are disjoint from $\underline{R}$ are compact up to breaking off trajectories disjoint from $\underline{R}$. Furthermore, for these trajectories and Y-maps we have the same relationship as in [30], since the complements of $\underline{R}$ are monotone. The rest of the argument now goes as in [30, Sect. 3.1].

6.3 *Proof of Invariance*

Returning to topology, let Σ_0, Σ_1 be Riemann surfaces of genus h and $h+1$ respectively. Let H_{01} be a compression body with boundary $\Sigma_0^- \times \Sigma_1$, that is, a cobordism consisting of attaching a single handle of index one. Associated to H_{01} we have a Lagrangian correspondence

$$L_{01} \subset \mathscr{N}(\Sigma_0')^- \times \mathscr{N}(\Sigma_1')$$

defined as follows. Suppose that γ is a path from the base points z_0 to z_1, equipped with a framing of the normal bundle. Let H_{01}' denote the noncompact surface obtained from H_{01} by removing a regular neighborhood of γ. The boundary of H_{01}' then consists of Σ_0', Σ_1', and a cylinder $S \times [0,1]$. Let $\mathscr{N}(H_{01}')$ denote the moduli space of flat connections on H_{01} of the form $\theta \mathrm{d}s$ near $S \times [0,1]$ (where s is the coordinate on the circle S), for some $\theta \in \mathfrak{g}$, modulo gauge transformations equal to the identity in a neighborhood of $S \times [0,1]$. The same arguments as in the proof of Lemma 9 show that L_{01} is a Lagrangian correspondence.

The Lagrangian correspondence L_{01} has the following explicit description in terms of holonomies, similar to (13.3) and (13.2). Suppose that H_{01} consists in attaching a one-handle whose meridian is the generator B_{h+1} of $\pi_1(\Sigma_1)$. We have the following lemma.

Lemma 11. *The Lagrangian correspondence L_{01} is given by*

$$L_{01} = \{((A_1,\ldots,B_h) \in \mathscr{N}(\Sigma_0'), (A_1,\ldots,B_h,A_{h+1},B_{h+1}) \in \mathscr{N}(\Sigma_1')) \mid B_{h+1} = I\}.$$

Proof. H_{01}' has the homotopy type of the wedge product of Σ_0' with a circle, corresponding to a single additional generator a_{h+1}. Thus $\pi_1(H_{01}')$ is freely generated by $(a_1,\ldots,b_h,a_{h+1})$, and the lemma follows.

Recall from Sect. 4.5 that $\mathscr{N}(\Sigma_0')$ admits a compactification $\mathscr{N}^c(\Sigma_0') = \mathscr{N}(\Sigma_0') \cup R_0$. We equip $\mathscr{N}^c(\Sigma')$ with the (nonmonotone) symplectic form constructed in Proposition 7, which we denote by $\omega_{\epsilon,0}$. Then R_0 is a symplectic hypersurface. Similarly, we have a symplectic form $\omega_{\epsilon,1}$ on $\mathscr{N}^c(\Sigma_1') = \mathscr{N}(\Sigma_1') \cup R_1$. Let L_{01}^c denote the closure of L_{01} in the compactification $\mathscr{N}^c(\Sigma_0')^- \times \mathscr{N}^c(\Sigma_1')$.

Lemma 12. *The Lagrangian correspondence L_{01}^c is compatible with the pair (R_0, R_1). Furthermore, any disk bubble with boundary in L_{01}^c with index zero has intersection number with (R_0, R_1) a negative multiple of 2.*

Proof. View L_{01}^c as a coisotropic submanifold of $\mathscr{N}^c(\Sigma_1')$, fibered over $\mathscr{N}^c(\Sigma_0')$ with fiber G. We are then exactly in the setting of Example 1. To prove the claim on the intersection number, note that any fiber of R_1 that intersects L_{01} is mapped symplectomorphically onto the corresponding fiber of R_0 via the projection of the fibered coisotropic $B_{h+1} = I$. Hence the patches of any such disk bubble, after projection to $\mathscr{N}^c(\Sigma_0')$, glue together to a sphere bubble in the $\mathbb{P}^1$-fiber of R_0. Furthermore, the projection induces an isomorphism of normal bundles by assumption, so the intersection number is equal to the intersection number of the sphere with R_0, which is a negative multiple of 2 as claimed.

Lemma 13. *Let $L_0 \subset \mathscr{N}^c(\Sigma_0')$, respectively $L_1 \subset \mathscr{N}^c(\Sigma_1')$, be the Lagrangian for the handlebody given by contracting the cycle $b_1, \ldots, b_h$, respectively $b_1, \ldots, b_{h+1}$. Then the composition $L_0 \circ L_{01}^c$ is embedded, and equals L_1.*

Proof. Immediate from Lemma 11 and the fact that L_0 does not meet the hypersurface R_0.

Lemma 14. *Let $L_{01}^c \subset \mathscr{N}^c(\Sigma_0')^- \times \mathscr{N}^c(\Sigma_1')$ be the Lagrangian correspondence for attaching a handle corresponding to adding the cycle a_{h+1}, and let $L_{10}^c \subset \mathscr{N}^c(\Sigma_1')^- \times \mathscr{N}^c(\Sigma_0')$ be the Lagrangian correspondence corresponding to contracting the cycle b_{h+1}. Then the composition $L_{01}^c \circ L_{10}^c$ is embedded, and it equals the diagonal $\Delta_0 \subset \mathscr{N}^c(\Sigma_0')^- \times \mathscr{N}^c(\Sigma_0')$. Furthermore, any quilted cylinder with seams in $L_{10}^c, L_{01}^c, \Delta_0$ with index zero has intersection number with (R_0, R_1, R_0) a negative multiple of 2.*

Proof. The first claim is immediate from Lemma 11. To see the assertion on the quilted cylinders, note that any quilted cylinder of index zero has zero canonical area, and so each component is contained in the corresponding R_j and maps onto a single fiber of the degeneracy locus. As in the proof of Lemma 12, the three holomorphic strips patch together to an orientation-preserving map of a sphere to a fiber of R_0, which must have intersection number a positive multiple of the intersection number of the fiber, which is -2.

Proof (Proof of Theorem 1). We seek to show that the Floer homology groups

$$HSI(\Sigma'; H_0, H_1) = HF(L_0, L_1; R)$$

are independent of the choice of Heegaard splitting of the 3-manifold Y.

By the Reidemeister–Singer theorem [47, 53], any two Heegaard splittings $Y = H_0 \cup_{\Sigma_0} H_1$, $Y = H_0' \cup_{\Sigma_1} H_1'$, are related by a sequence of stabilizations and destabilizations. Therefore, it suffices to consider the case that H_0', H_1' are obtained from H_0, H_1 by stabilization. That is,

$$H_0' = H_0 \cup_{\Sigma_0} H_{01}, \quad H_1' = H_1 \cup_{\Sigma_0} (-H_{10}),$$

where H_{01}, H_{10} are the compression bodies corresponding to adding the cycle a_{h+1}, respectively contracting b_{h+1}. Then, after three applications of Theorem 7, and taking into account Lemmas 13, 14, we have

$$\begin{aligned} HF(L_0, L_1; R_0) &\cong HF(L_0, \Delta_0, L_1; R_0, R_0) \\ &\cong HF(L_0, L_{01}^c, L_{10}^c, L_1; R_0, R_1, R_0) \\ &\cong HF(L_0 \circ L_{01}^c, L_{10}^c \circ L_1; R_1, R_1) \\ &= HF(L_0', L_1'; R_1). \end{aligned}$$

Remark 9. The symplectic instanton homology groups $HSI(Y, z)$ depend on the choice of base point $z \in \Sigma \subset Y$; cf. Sect. 5.3. As z varies, the groups naturally form a flat bundle over Y. Still, we usually drop z from the notation and denote them by $HSI(Y)$.

7 Properties and Examples

7.1 *The Euler Characteristic*

In general, the Euler characteristic of Lagrangian Floer homology is the intersection number of the two Lagrangians. In our situation, the corresponding intersection number is computed (up to a sign) in [1, Proposition 1.1(a),(b)]:

$$\chi\big(HSI(Y)\big) = [L_0] \cdot [L_1] = \begin{cases} \pm |H_1(Y; \mathbb{Z})| & \text{if } b_1(Y) = 0; \\ 0 & \text{otherwise.} \end{cases} \tag{13.1}$$

7.2 *Examples*

Proposition 8. *We have an isomorphism*

$$HSI(S^3) \cong \mathbb{Z}.$$

Proof. Let $\mathcal{H}_h$ denote the Heegaard decomposition $S^3 = H_0 \cup_\Sigma H_1$ of genus $h \geq 1$ such that there is a system of $2h$ curves α_i, β_i on Σ' as in Sect. 3.2 with the property that the β_i are null homotopic in H_0 and the α_i's are null homotopic in H_1.

With respect to the identification (13.3), the Lagrangians corresponding to H_0 and H_1 are given by

$$L_0 = \{(A_1, B_1, \ldots, A_h, B_h) \in G^{2h} \mid B_i = I,\ i = 1, \ldots, h\},$$

$$L_1 = \{(A_1, B_1, \ldots, A_h, B_h) \in G^{2h} \mid A_i = I,\ i = 1, \ldots, h\}.$$

These have exactly one intersection point, the reducible $A_i = B_i = I$. Clearly L_0 and L_1 intersect transversely in $\mathcal{N}(\Sigma') \subset G^{2h}$ at that point. It is somewhat counterintuitive that L_0 and L_1 can intersect transversely at I, because they both live in the subspace $\Phi^{-1}(0)$ of codimension three in $\mathcal{N}(\Sigma')$. However, that subspace is not smooth, so there is no contradiction. We conclude that the Floer chain group has one generator; hence so does the homology.

Proposition 9. *For $h \geq 1$, we have an isomorphism*

$$HSI(\#^h(S^1 \times S^2)) \cong \left(H_*(S^3; \mathbb{Z}/2\mathbb{Z})\right)^{\otimes h},$$

where the grading of the latter vector space is collapsed modulo 8.

Proof. Let $\mathcal{H}'_h$ be the Heegaard splitting of genus $h \geq 1$ for $\#^h(S^1 \times S^2)$. Since $L_0 = L_1 \cong G^h \cong (S^3)^h$, the cohomology ring of L_0 is generated by its degree-d ($d = 3$) part. Under the monotonicity assumptions that are satisfied in our setting, Oh [40] constructed a spectral sequence whose E^1 term is $H_*(L_0; \mathbb{Z}/2\mathbb{Z})$ and that converges to $HF_*(L_0, L_0; \mathbb{Z}/2\mathbb{Z})$. This sequence is multiplicative by the results of Buhovski [10] and Biran–Cornea [7, 8]. A consequence of multiplicativity is that the spectral sequence collapses at the E_1 stage, provided that $N_L > d+1$; see, for example, [8, Theorem 1.2.2]. This is satisfied in our case because $N_{L_0} = N \geq 8$. Hence $HF_*(L_0, L_0; \mathbb{Z}/2\mathbb{Z}) \cong H_*(G^h; \mathbb{Z}/2\mathbb{Z})$.

Note that the results of Oh, Buhovski, and Biran–Cornea were originally formulated for monotone symplectic manifolds, i.e., in the setting of Section 2.1. However, they also apply to the Floer homology groups defined in Sect. 2.3. Indeed, the arguments in the proof of Proposition 1 about the finiteness of the Floer differential and the fact that $\partial^2 = 0$ apply equally well to the "string of pearls" complex used in [7, 8].

Proposition 10. *For a lens space $L(p,q)$, with $\gcd(p,q) = 1$, the symplectic instanton homology $HSI(L(p,q))$ is a free abelian group of rank p.*

Proof. Denote by $\mathcal{H}(p,q)$ the genus-one Heegaard splitting of $L(p,q)$. In terms of the coordinates $A = A_1$ and $B = B_1$, the two Lagrangians are given by $L_0 = B = 1$ and $L_1 = A^p B^{-q} = 1$. Their intersection consists of the space of representations $\pi_1(L(p,q)) \cong \mathbb{Z}/p \to \mathrm{SU}(2)$, which has several components: when p is odd, there are the reducible point ($A = B = I$) and $(p-1)/2$ copies of S^2; when p is even, there are two reducibles ($A = B = I$ and $A = -I, B = I$) and $(p-2)/2$ copies of S^2. It is straightforward to check that each component is a clean intersection in the sense of Poźniak [45]. Therefore, there exists a spectral sequence that starts at $H_*(L_0 \cap L_1) \cong \mathbb{Z}^p$ and converges to $HF(L_0, L_1)$; cf. [45]. Since the Euler characteristic of $HF(L_0, L_1)$ is p by (13.1), the sequence must collapse at the first stage.

Remark 10. More generally, whenever we have a Heegaard decomposition $\mathcal{H}$ of a three-manifold Y with $H^1(Y) = 0$, the two Lagrangians L_0 and L_1 will intersect transversely at the reducible I; cf. [1, Proposition 1.1(c)]. We could then fix an absolute $\mathbb{Z}/8\mathbb{Z}$-grading on $HSI(\mathcal{H})$ by requiring that the $\mathbb{Z}$ summand corresponding to I lie in grading zero.

7.3 Comparison with Other Approaches

Let $Y = H_0 \cup_\Sigma H_1$ be a Heegaard splitting of a 3-manifold, with Σ of genus h. Recall that the Lagrangians $L_0 = L(H_0)$ and $L_1 = L(H_1)$ live inside the subspace

$$\Phi^{-1}(0) = \left\{ (A_1, B_1, \dots, A_h, B_h) \in G^{2h} \,\middle|\, \prod_{i=1}^{h} [A_i, B_i] = I \right\} \subset \mathscr{N}(\Sigma').$$

There is an alternative way of embedding $\Phi^{-1}(0)$ inside a symplectic manifold of dimension $6h$. Namely, let Σ_+ be the closed surface (of genus $h+1$) obtained by gluing a copy of $T^2 \setminus D^2$ onto the boundary of $\Sigma' = \Sigma \setminus D^2$. Consider the moduli space $\mathscr{M}_{\mathrm{tw}}(\Sigma_+)$ of projectively flat connections (with fixed central curvature) in an odd-degree $\mathrm{U}(2)$-bundle over Σ_+, as in Sect. 3.6:

$$\mathscr{M}_{\mathrm{tw}}(\Sigma_+) = \left\{ (A_1, B_1, \dots, A_{h+1}, B_{h+1}) \in G^{2h+2} \,\middle|\, \prod_{i=1}^{h+1} [A_i, B_i] = -I \right\} / G.$$

Pick two particular matrices $X, Y \in G$ with the property that $[X, Y] = -I$. Then we can embed $\Phi^{-1}(0)$ into $\mathscr{M}_{\mathrm{tw}}(\Sigma_+)$ by the map

$$(A_1, B_1, \dots, A_h, B_h) \to [(A_1, B_1, \dots, A_h, B_h, X, Y)].$$

With respect to the natural symplectic form on $\mathscr{M}_{\mathrm{tw}}(\Sigma_+)$, the spaces $L_0, L_1 \subset \Phi^{-1}(0)$ are still Lagrangians. One can take their Floer homology and obtain a $\mathbb{Z}/4\mathbb{Z}$-graded abelian group. This was studied in [57, Sect. 4.1], where it is shown that it is a 3-manifold invariant. It is not obvious how this invariant relates to HSI.

The advantage of using $\mathscr{M}_{\mathrm{tw}}(\Sigma_+)$ instead of $\mathscr{N}(\Sigma')$ is that the former is already compact (and monotone); therefore, the definition of Floer homology is less technical, and this allows one to prove invariance. Nevertheless, the construction presented in this paper (using $\mathscr{N}(\Sigma')$) has certain advantages as well: first, the resulting groups are $\mathbb{Z}/8\mathbb{Z}$-graded rather than $\mathbb{Z}/4\mathbb{Z}$-graded. Second, it is better suited to defining an equivariant version of symplectic instanton homology. Indeed, unlike $\mathscr{M}_{\text{tw}}(\Sigma_+)$, the space $\mathscr{N}(\Sigma')$ comes with a natural action of G that preserves the symplectic form and the Lagrangians. Following the ideas of Viterbo from [54, 55], we expect that one should be able to use this action to define equivariant Floer groups $HSI_*^G(Y)$ in the form of $H^*(BG)$-modules. For integral homology spheres, a suitable Atiyah–Floer conjecture would relate these to the equivariant instanton homology of Austin and Braam [5].

In a different direction, it would be interesting to study the connection between our construction and the Heegaard Floer homology groups $\widehat{HF}, HF^+$ of Ozsváth and Szabó [42, 43]. In particular, we ask the following question:

Question 7.1. For an arbitrary 3-manifold Y, are the total ranks of $HSI(Y) \otimes \mathbb{Q}$ and $\widehat{HF}(Y) \otimes \mathbb{Q}$ equal?

Finally, we remark that Jacobsson and Rubinsztein [25] have recently described a construction similar to the one in this paper, but for the case of knots in S^3 rather than 3-manifolds. Given a representation of a knot as a braid closure, they define two Lagrangians inside a certain symplectic manifold; this manifold was first constructed in [22] and is a version of the extended moduli space. Conjecturally, one should be able to take the Floer homology of the two Lagrangians and obtain a knot invariant.

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AUTHOR QUERIES

AQ1. First author has been consider as the corresponding author. Please check.
AQ2. Please update references [7,8,10,25,29,30,57,58]