Remarks on Khovanov Homology and the Potts Model

Louis H. Kauffman

Dedicated to Oleg Viro on his 60th birthday

Abstract In the paper we explore how the Potts model in statistical mechanics is related to Khovanov homology. This exploration is made possible because the underlying combinatorics for the bracket state sum for the Jones polynomial are shared by the Potts model for planar graphs. We show that Euler characteristics of Khovanov homology figure in the computation of the Potts model at certain imaginary temperatures and that these aspects of the Potts model can be reformulated as physical quantum amplitudes via Wick rotation. The paper concludes with a new conceptually transparent quantum algorithm for the Jones polynomial and with many further questions about Khovanov homology.

Keywords Khovanov homology • Potts model • Jones polynomial • Bracket state sum • Dichromatic polynomial

1 Introduction

This paper is about Khovanov homology and its relationships with finite combinatorial statistical mechanics models such as the Ising model and the Potts model.

Partition functions in statistical mechanics take the form

$$Z_G = \sum_{\sigma} \mathrm{e}^{(-1/kT)E(\sigma)},$$

L.H. Kauffman (✉)
Math UIC, 851 South Morgan Street, Chicago, Illinois 60607-7045
e-mail: kauffman@uic.edu

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where σ runs over the different physical states of a system G, and $E(\sigma)$ is the energy of the state σ. The probability of the system being in the state σ is taken to be

$$\mathrm{prob}(\sigma) = \mathrm{e}^{(-1/kT)E(\sigma)}/Z_G.$$

Since Onsager's work showing that the partition function of the Ising model has a phase transition, it has been a significant subject in mathematical physics to study the properties of partition functions for simply defined models based on graphs G. The underlying physical system is modeled by a graph G. and the states σ are certain discrete labelings of G. The reader can consult Baxter's book [4] for many beautiful examples. The Potts model, discussed below, is a generalization of the Ising model, and it is an example of a statistical-mechanical model that is intimately related to knot theory and to the Jones polynomial [8]. Since there are many connections between statistical mechanics and the Jones polynomial, it is remarkable that there have not been many connections between statistical mechanics and categorifications of the Jones polynomial. The reader will be find other points of view in [5, 6].

The partition function for the Potts model is given by the formula

$$P_G(Q,T) = \sum_{\sigma} \mathrm{e}^{(J/kT)E(\sigma)} = \sum_{\sigma} \mathrm{e}^{KE(\sigma)},$$

where σ is an assignment of one element of the set $\{1,2,\ldots,Q\}$ to each node of the graph G, and $E(\sigma)$ denotes the number of edges of a graph G whose end nodes receive the same assignment from σ. In this model, σ is regarded as a *physical state* of the Potts system, and $E(\sigma)$ is the *energy* of this state. Here $K = J\frac{1}{kT}$, where J is ± 1 (ferromagnetic and antiferromagnetic cases), k is Boltzmann's constant, and T is the temperature. The Potts partition function can be expressed in terms of the dichromatic polynomial of the graph G. Letting

$$v = e^K - 1,$$

it is shown in [12, 13] that the dichromatic polynomial $Z[G](v,Q)$ for a plane graph can be expressed in terms of a bracket state summation of the form

$$\{\times\} = \{\asymp\} + Q^{-\frac{1}{2}} v \{)(\},$$

with

$$\{\bigcirc\} = Q^{\frac{1}{2}}.$$

Then the Potts partition function is given by the formula

$$P_G = Q^{N/2}\{K(G)\},$$

where $K(G)$ is an alternating link diagram associated with the plane graph G such that the projection of $K(G)$ to the plane is a medial diagram for the graph. This translation of the Potts model in terms of a bracket expansion makes it possible to examine how the Khovanov homology of the states of the bracket is related to the evaluation of the partition function.

There are five sections in this paper beyond the introduction. In Sect. 2, we review the definition of Khovanov homology and observe, in parallel with [27], that it is very natural to begin by defining the Khovanov chain complex via enhanced states of the bracket polynomial model for the Jones polynomial. In fact, we begin with Khovanov's rewrite of the bracket state sum in the form

$$\langle \times \rangle = \langle \asymp \rangle - q \langle \rangle\langle \rangle,$$

with $\langle \bigcirc \rangle = (q + q^{-1})$. We rewrite the state sum formula for this version of the bracket in terms of enhanced states (each loop is labeled $+1$ or -1 corresponding to q^{+1} and q^{-1} respectively) and show how, by collecting terms, the formula for the state sum has the form of a graded sum of Euler characteristics

$$\langle K \rangle = \sum_j q^j \sum_i (-1)^i \dim(\mathcal{C}^{ij}) = \sum_j q^j \chi(\mathcal{C}^{\bullet j}) = \sum_j q^j \chi(\mathcal{H}^{\bullet j}),$$

where $\mathcal{C}^{ij}$ is a module with basis the set of enhanced states s with i smoothings of type B (see Sect. 2 for definitions), $\mathcal{H}^{\bullet j}$ is its homology, and $j = j(s)$, where

$$j(s) = n_B(s) + \lambda(s),$$

where $n_B(s)$ denotes the number of B-smoothings in s and $\lambda(s)$ denotes the number of loops with positive label minus the number of loops with negative label in s. This formula suggests that there should be differentials $\partial : \mathcal{C}^{i,j} \longrightarrow \mathcal{C}^{i+1,j}$ such that j is preserved under the differential. We show that the restriction $j(s) = j(\partial(s))$ uniquely determines the differential in the complex, and how this leads to the Frobenius algebra structure that Khovanov used to define the differential. This part of our remarks is well known, but I believe that the method by which we arrive at the graded Euler characteristic is particularly useful for our subsequent discussion. We see clearly here that one should look for subcomplexes on which Euler characteristics can be defined, and one should attempt to shape the state summation so that these characteristics appear in the state sum. This happens miraculously for the bracket state sum and makes the combinatorics of that model dovetail with the Khovanov homology of its states, so that the bracket polynomial is seen as a graded Euler characteristic of the homology theory. The section ends with a discussion of how Grassmann algebra can be used, in analogy with de Rahm cohomology, to define the integral Khovanov chain complex. We further note that this analogy with de Rahm cohomology leads to other possibilities for chain complexes associated with the bracket states. These complexes will be the subject of a separate paper.

In Sect. 3 we recall the definition of the dichromatic polynomial and the fact [12] that the dichromatic polynomial for a planar graph G can be expressed as a special bracket state summation on an associated alternating knot $K(G)$. We recall that for special values of its two parameters, the Potts model can be expressed as a dichromatic polynomial. This sets the stage for examining the role of Khovanov homology in the evaluation of the Potts model and the dichromatic polynomial. In both cases (dichromatic polynomial in general and the Potts model in particular), one finds that the state summation does not so easily rearrange itself as a sum over Euler characteristics, as we have explained in Sect. 2. The states in the bracket summation for the dichromatic polynomial of $K(G)$ are the same as the states for the bracket polynomial of $K(G)$, so the Khovanov chain complex is present at the level of the states.

We clarify this relationship using a two-variable bracket expansion, the ρ-*bracket*, that reduces to the Khovanov version of the bracket as a function of q when ρ is equal to 1:

$$[\times] = [\asymp] - q\rho[)(]$$

with

$$[\bigcirc] = q + q^{-1}.$$

We can regard this expansion as an intermediary between the Potts model (dichromatic polynomial) and the topological bracket. The ρ-bracket can be rewritten in the following form:

$$[K] = \sum_{i,j}(-\rho)^i q^j \dim(\mathcal{C}^{ij}) = \sum_j q^j \sum_i (-\rho)^i \dim(\mathcal{C}^{ij}) = \sum_j q^j \chi_\rho(\mathcal{C}^{\bullet j}),$$

where we define $\mathcal{C}^{ij}$ to be the linear span of the set of enhanced states with $n_B(s) = i$ and $j(s) = j$. Then the number of such states is the dimension $\dim(\mathcal{C}^{ij})$. We have expressed this ρ-bracket expansion in terms of generalized Euler characteristics of the complexes $\mathcal{C}^{\bullet j}$:

$$\chi_\rho(\mathcal{C}^{\bullet j}) = \sum_i (-\rho)^i \dim(\mathcal{C}^{ij}).$$

These generalized Euler characteristics become classical Euler characteristics when $\rho = 1$, and in that case, are the same as the Euler characteristic of the homology. With ρ not equal to 1, we do not have direct access to the homology. In that case, the polynomial is expressed in terms of ranks of chain modules, but not in terms of ranks of corresponding homology groups.

In Sect. 3 we analyze those cases of the Potts model in which $\rho = 1$ (so that Euler characteristics of Khovanov homology appear as coefficients in the Potts partition function), and we find that at criticality this requires $Q = 4$ and $e^K = -1$ ($K = J/kT$

as in the second paragraph of this introduction), hence an imaginary value of the temperature. When we simply require that $\rho = 1$, not necessarily at criticality, then we find that for $Q = 2$ we have $e^K = \pm \mathrm{i}$. For $Q = 3$, we have $e^K = \frac{-1 \pm \sqrt{3}i}{2}$. For $Q = 4$ we have $e^K = -1$. For $Q > 4$ it is easy to verify that e^K is real and negative. Thus in all cases of $\rho = 1$ we find that the Potts model has complex temperature values. Further work is called for to see how the evaluations of the Potts model at complex values influence its behavior for real temperature values in relation to the Khovanov homology. Such relationships between complex and real temperature evaluations are already known for the accumulation of the zeros of the partition function (the Lee–Yang zeros [20]). In Sect. 5 we return to the matter of imaginary temperature. This section is described below.

In Sect. 4 we discuss Stosic's categorification of the dichromatic polynomial, which involves using a differential motivated directly by the graphical structure and gives a homology theory distinct from Khovanov homology. We examine Stosic's categorification in relation to the Potts model and again show that it will work (in the sense that the coefficients of a partition function are Euler characteristics of the homology) when the temperature is imaginary. Specifically, we find this behavior for $K = i\pi + \ln(q + q^2 + \cdots + q^n)$, and again the challenge is to discover the influence of this graph homology on the Potts model at real temperatures. The results of this section do not require planarity of the graph G.

More generally, we see that in the case of this model, the differential in the homology is related to the combinatorial structure of the physical model. In that model, a given state has graphical regions of constant spin (calling the discrete assignments $\{1, 2, \ldots, Q\}$ the *spins*). We interpret the partial differentials in this model as taking two such regions and making them into a single region, by adding an edge that joins them, and reassigning a spin to the new region that is a combination of the spins of the two formerly separate regions. This form of partial differential is a way to think about relating the different states of the physical system. It is, however, not directly related to a classical physical process, since it invokes a global change in the spin configuration of the model. (One might consider such global transitions in quantum processes.)

The Stosic homology measures such global changes, and it is probably the nonclassical physicality of this patterning that makes it manifest at imaginary temperatures. The direct physical transitions from state to state that are relevant to the classical physics of the model may indeed involve changes of the form of the regions of constant spin, but this will happen by a local change, so that two disjoint regions with the same spin join into a single region, or a single region of constant spin bifurcates into two regions with this spin. These are the sorts of local-classical-time transitions that one works with in a statistical-mechanics model. Such transitions are part of the larger and more global transitions that are described by the differentials in our interpretation of the Stosic homology for the Potts model. Obviously, much more work needs to be done in this field. We have made some first steps in this paper.

In Sect. 5 we formulate a version of Wick rotation for the Potts model so that it is seen (for imaginary temperature) as a quantum amplitude. In this way, for those cases in which the temperature is pure imaginary, we obtain a quantum-physical interpretation of the Potts model and hence a relationship between Khovanov homology and quantum amplitudes at the special values discussed in Sect. 3.

In Sect. 6, we remark that the bracket state sum itself can be given a quantum-statistical interpretation, by choosing a Hilbert space whose basis is the set of enhanced states of a diagram K. We use the evaluation of the bracket at each enhanced state as a matrix element for a linear transformation on the Hilbert space. This transformation is unitary when the bracket variable (here denoted as above by q) is on the unit circle. In this way, using the Hadamard test, *we obtain a new quantum algorithm for the Jones polynomial* at all values of the Laurent polynomial variable that lie on the unit circle in the complex plane. This is not an efficient quantum algorithm, but it is conceptually transparent and it will allow us in subsequent work to analyze relationships between Khovanov homology and quantum computation.

2 Khovanov Homology

In this section, we describe Khovanov homology along the lines of [2, 19], and we tell the story in way that allows the gradings and the structure of the differential to emerge in a natural way. This approach to motivating the Khovanov homology uses elements of Khovanov's original approach, Viro's use of enhanced states for the bracket polynomial [27], and Bar-Natan's emphasis on tangle cobordisms [3]. We use similar considerations in our paper [17].

Two key motivating ideas are involved in finding the Khovanov invariant. First of all, one would like to *categorify* a link polynomial such as $\langle K \rangle$. There are many meanings to the term categorify, but here the quest is to find a way to express the link polynomial as a *graded Euler characteristic* $\langle K \rangle = \chi_q \langle \mathcal{H}(K) \rangle$ for some homology theory associated with $\langle K \rangle$.

The bracket polynomial [11] model for the Jones polynomial [8–10, 28] is usually described by the expansion

$$\langle \times \rangle = A \langle \asymp \rangle + A^{-1} \langle)(\rangle,$$

and we have

$$\langle K \bigcirc \rangle = (-A^2 - A^{-2}) \langle K \rangle$$

$$\langle \gamma \rangle = (-A^3) \langle \smile \rangle$$

$$\langle \gamma' \rangle = (-A^{-3}) \langle \smile \rangle.$$

Letting $c(K)$ denote the number of crossings in the diagram K, if we replace $\langle K \rangle$ by $A^{-c(K)}\langle K \rangle$ and then replace A^2 by $-q^{-1}$, the bracket will be rewritten in the following form:

$$\langle \times \rangle = \langle \asymp \rangle - q\langle \,)(\, \rangle$$

with $\langle \bigcirc \rangle = (q + q^{-1})$. It is useful to use this form of the bracket state sum for the sake of the grading in the Khovanov homology (to be described below). We shall continue to refer to the smoothings labeled q (or A^{-1} in the original bracket formulation) as *B-smoothings*. We should further note that we use the well-known convention of *enhanced states*, where an enhanced state has a label of 1 or X on each of its component loops. We then regard the value of the loop $q + q^{-1}$ as the value of a circle labeled with a 1 (the value is q) added to the value of a circle labeled with an X (the value is q^{-1}). We could have chosen the more neutral labels of $+1$ and -1 so that

$$q^{+1} \Longleftrightarrow +1 \Longleftrightarrow 1$$

and

$$q^{-1} \Longleftrightarrow -1 \Longleftrightarrow X,$$

but since an algebra involving 1 and X naturally appears later, we take this form of labeling from the beginning.

To see how the Khovanov grading arises, consider the form of the expansion of this version of the bracket polynomial in enhanced states. We have the formula as a sum over enhanced states s:

$$\langle K \rangle = \sum_s (-1)^{n_B(s)} q^{j(s)},$$

where $n_B(s)$ is the number of B-type smoothings in s, $\lambda(s)$ is the number of loops in s labeled 1 minus the number of loops labeled X, and $j(s) = n_B(s) + \lambda(s)$. This can be rewritten in the following form:

$$\langle K \rangle = \sum_{i,j} (-1)^i q^j \dim(\mathcal{C}^{ij}),$$

where we define $\mathcal{C}^{ij}$ to be the linear span (over $k = Z/2Z$, since we will work with coefficients modulo 2) of the set of enhanced states with $n_B(s) = i$ and $j(s) = j$. Then the number of such states is the dimension $\dim(\mathcal{C}^{ij})$.

We would like to have a bigraded complex comprising the $\mathcal{C}^{ij}$ with a differential

$$\partial : \mathcal{C}^{ij} \longrightarrow \mathcal{C}^{i+1\,j}.$$

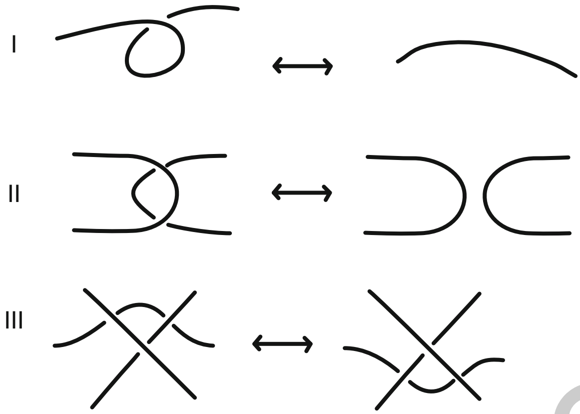

Fig. 1 Reidemeister moves

The differential should increase the *homological grading* i by 1 and preserve the *quantum grading* j. Then we could write

$$\langle K\rangle = \sum_j q^j \sum_i (-1)^i \dim(\mathcal{C}^{ij}) = \sum_j q^j \chi(\mathcal{C}^{\bullet j}),$$

where $\chi(\mathcal{C}^{\bullet j})$ is the Euler characteristic of the subcomplex $\mathcal{C}^{\bullet j}$ for a fixed value of j.

This formula would constitute a categorification of the bracket polynomial. Below, we shall see how *the original Khovanov differential* ∂ *is uniquely determined by the restriction that* $j(\partial s) = j(s)$ *for each enhanced state* s. Since j is preserved by the differential, these subcomplexes $\mathcal{C}^{\bullet j}$ have their own Euler characteristics and homology. We have

$$\chi(H(\mathcal{C}^{\bullet j})) = \chi(\mathcal{C}^{\bullet j}),$$

where $H(\mathcal{C}^{\bullet j})$ denotes the homology of the complex $\mathcal{C}^{\bullet j}$. We can write

$$\langle K\rangle = \sum_j q^j \chi(H(\mathcal{C}^{\bullet j})).$$

This last formula expresses the bracket polynomial as a *graded Euler characteristic* of a homology theory associated with the enhanced states of the bracket state summation. This is the categorification of the bracket polynomial. Khovanov proves that this homology theory is an invariant of knots and links (via the Reidemeister moves of Fig. 1), creating a new invariant that is stronger than the original Jones polynomial.

We will construct the differential in this complex first for mod-2 coefficients. The differential is based on regarding two states as *adjacent* if one differs from the other by a single smoothing at some site. Thus if (s, τ) denotes a pair consisting of an enhanced state s and site τ of that state with τ of type A, then we consider all enhanced states s' obtained from s by smoothing at τ and relabeling only those loops that are affected by the resmoothing. Call this set of enhanced states $S'[s,\tau]$.

Then we shall define the *partial differential* $\partial_\tau(s)$ as a sum over certain elements in $S'[s,\tau]$ and the differential by the formula

$$\partial(s) = \sum_\tau \partial_\tau(s)$$

with the sum over all type-A sites τ in s. It then remains to determine the possibilities for $\partial_\tau(s)$ for which $j(s)$ is preserved.

Note that if $s' \in S'[s,\tau]$, then $n_B(s') = n_B(s) + 1$. Thus

$$j(s') = n_B(s') + \lambda(s') = 1 + n_B(s) + \lambda(s').$$

From this we conclude that $j(s) = j(s')$ if and only if $\lambda(s') = \lambda(s) - 1$. Recall that

$$\lambda(s) = [s:+] - [s:-],$$

where $[s:+]$ is the number of loops in s labeled $+1$, and $[s:-]$ is the number of loops labeled -1 (same as labeled with X) and $j(s) = n_B(s) + \lambda(s)$.

Proposition. *The partial differentials $\partial_\tau(s)$ are uniquely determined by the condition $j(s') = j(s)$ for all s' involved in the action of the partial differential on the enhanced state s. This unique form of the partial differential can be described by the following structures of multiplication and comultiplication on the algebra $\mathcal{A} = k[X]/(X^2)$, where $k = Z/2Z$ for mod-2 coefficients, or $k = Z$ for integral coefficients:*

1. *The element 1 is a multiplicative unit and* $X^2 = 0$.
2. $\Delta(1) = 1 \otimes X + X \otimes 1$ *and* $\Delta(X) = X \otimes X$.

These rules describe the local relabeling process for loops in a state. Multiplication corresponds to the case that two loops merge to a single loop, while comultiplication corresponds to the case in which one loop bifurcates into two loops.

Proof. Using the above description of the differential, suppose that there are two loops at τ that merge in the smoothing. If both loops are labeled 1 in s, then the local contribution to $\lambda(s)$ is 2. Let s' denote a smoothing in $S[s,\tau]$. In order for the local λ contribution to become 1, we see that the merged loop must be labeled 1. Similarly, if the two loops are labeled 1 and X, then the merged loop must be labeled X, so that the local contribution for λ goes from 0 to -1. Finally, if the two loops are labeled X and X, then there is no label available for a single loop that will give -3, so we define ∂ to be zero in this case. We can summarize the result by saying that there is a multiplicative structure m such that $m(1,1) = 1, breakm(1,X) = m(X,1) = x, m(X,X) = 0$, and this multiplication describes the structure of the partial differential when two loops merge. Since this is the multiplicative structure of the algebra $\mathcal{A} = k[X]/(X^2)$, we take this algebra as summarizing the differential.

Now consider the case that s has a single loop at the site τ. Smoothing produces two loops. If the single loop is labeled X, then we must label each of the two loops by X in order to make λ decrease by 1. If the single loop is labeled 1, then we can label the two loops by X and 1 in either order. In this second case we take the partial differential of s to be the sum of these two labeled states. This structure can be described by taking a coproduct structure with $\Delta(X) = X \otimes X$ and $\Delta(1) = 1 \otimes X + X \otimes 1$. We now have the algebra $\mathcal{A} = k[X]/(X^2)$ with product $m : \mathcal{A} \otimes \mathcal{A} \longrightarrow \mathcal{A}$ and coproduct $\Delta : \mathcal{A} \longrightarrow \mathcal{A} \otimes \mathcal{A}$, describing the differential completely. This completes the proof. □

Partial differentials are defined on each enhanced state s and a site τ of type A in that state. We consider states obtained from the given state by smoothing the given site τ. The result of smoothing τ is to produce a new state s' with one more site of type B than s. Forming s' from s, we either amalgamate two loops to a single loop at τ or divide a loop at τ into two distinct loops. In the case of amalgamation, the new state s acquires the label on the amalgamated circle that is the product of the labels on the two circles that are its ancestors in s. This case of the partial differential is described by the multiplication in the algebra. If one circle becomes two circles, then we apply the coproduct. Thus if the circle is labeled X, then the resultant two circles are each labeled X corresponding to $\Delta(X) = X \otimes X$. If the original circle is labeled 1, then we take the partial boundary to be a sum of two enhanced states with labels 1 and X in one case, and labels X and 1 in the other case, on the respective circles. This corresponds to $\Delta(1) = 1 \otimes X + X \otimes 1$. Modulo two, the boundary of an enhanced state is the sum, over all sites of type A in the state, of the partial boundaries at these sites. It is not hard to verify directly that the square of the boundary mapping is zero (this is the identity of mixed partials!) and that it behaves as advertised, keeping $j(s)$ constant. There is more to say about the nature of this construction with respect to Frobenius algebras and tangle cobordisms. In Figs. 2 and 3 we illustrate how the partial boundaries can be conceptualized in terms of surface cobordisms. The equality of mixed partials corresponds to topological equivalence of the corresponding surface cobordisms, and to the relationships between Frobenius algebras and the surface cobordism category. The proof of invariance of Khovanov homology with respect to the Reidemeister moves (respecting grading changes) will not be given here. See [2, 3, 19]. It is remarkable that this version of Khovanov homology is uniquely specified by natural ideas about adjacency of states in the bracket polynomial.

Remark on integral differentials. Choose an ordering for the crossings in the link diagram K and denote them by $1, 2, \ldots, n$. Let s be any enhanced state of K and let $\partial_i(s)$ denote the chain obtained from s by applying a partial boundary at the ith site of s. If the ith site is a smoothing of type A^{-1}, then $\partial_i(s) = 0$. If the ith site is a smoothing of type A, then $\partial_i(s)$ is given by the rules discussed above (with the same signs). The compatibility conditions that we have discussed show that partials commute in the sense that $\partial_i(\partial_j(s)) = \partial_j(\partial_i(s))$ for all i and j. One then defines signed boundary formulas in the usual way of algebraic topology. One way to think of this regards the complex as the analogue of a complex in de Rahm

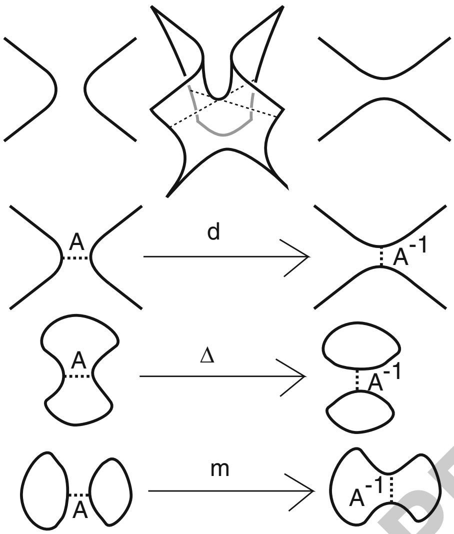

Fig. 2 Saddle points and state smoothings

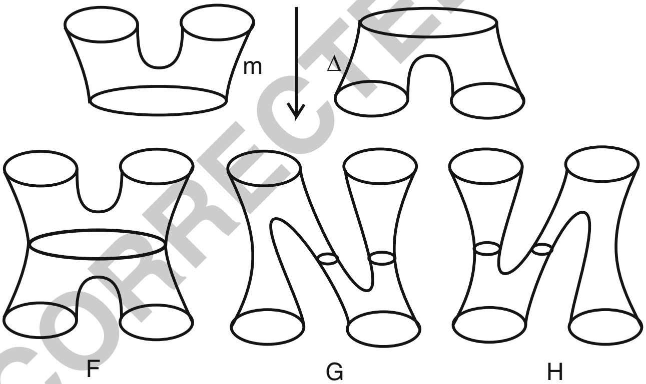

Fig. 3 Surface cobordisms

cohomology. Let $\{\mathrm{d}x_1, \mathrm{d}x_2, \ldots, \mathrm{d}x_n\}$ be a formal basis for a Grassmann algebra such that $\mathrm{d}x_i \wedge \mathrm{d}x_j = -\mathrm{d}x_i \wedge \mathrm{d}x_j$. Starting with enhanced states s in $C^0(K)$ (that is, states with all A-type smoothings), define formally $d_i(s) = \partial_i(s)\mathrm{d}x_i$ and regard $d_i(s)$ as identical with $\partial_i(s)$, as we have previously regarded it in $C^1(K)$. In general, given an enhanced state s in $C^k(K)$ with B-smoothings at locations $i_1 < i_2 < \cdots < i_k$, we represent this chain as $s\,\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k}$ and define

$$\partial(s\,\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k}) = \sum_{j=1}^{n} \partial_j(s)\,\mathrm{d}x_j \wedge \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k},$$

just as in a de Rahm complex. The Grassmann algebra automatically computes the correct signs in the chain complex, and this boundary formula gives the original boundary formula when we take coefficients modulo two. Note that in this formalism, partial differentials ∂_i of enhanced states with a B-smoothing at the site i are zero due to the fact that $\mathrm{d}x_i \wedge \mathrm{d}x_i = 0$ in the Grassmann algebra. There is more to discuss about the use of Grassmann algebras in this context. For example, this approach clarifies parts of the construction in [18].

It of interest to examine this analogy between the Khovanov (co)homology and de Rahm cohomology. In that analogy the enhanced states correspond to the differentiable functions on a manifold. The Khovanov complex $C^k(K)$ is generated by elements of the form $s\,\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k}$, where the enhanced state s has B-smoothings at exactly the sites $i_1, \ldots, i_k$. If we were to follow the analogy with de Rahm cohomology literally, we would define a new complex $DR(K)$, where $DR^k(K)$ is generated by elements $s\,\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k}$, where s is *any* enhanced state of the link K. The partial boundaries are defined in the same way as before, and the global boundary formula is just as we have written it above. This gives a *new* chain complex associated with the link K. Whether its homology contains new topological information about the link K will be the subject of a subsequent paper.

A further remark on de Rham cohomology. There is another deep relationship with the de Rham complex: In [21], it was observed that Khovanov homology is related to Hochschild homology, and Hochschild homology is thought to be an algebraic version of de Rham chain complex (cyclic cohomology corresponds to de Rham cohomology); compare [22].

3 The Dichromatic Polynomial and the Potts Model

We define the *dichromatic polynomial* as follows:

$$Z[G](v,Q) = Z[G'](v,Q) + vZ[G''](v,Q),$$
$$Z[\bullet \sqcup G] = QZ[G],$$

where G' is the result of deleting an edge from G, while G'' is the result of contracting that same edge so that its end nodes have been collapsed to a single node. In the second equation, $\bullet$ represents a graph with one node and no edges, and $\bullet \sqcup G$ represents the disjoint union of the single-node graph with the graph G.

In [12, 13] it is shown that the dichromatic polynomial $Z[G](v,Q)$ for a plane graph can be expressed in terms of a bracket state summation of the form

$$\{\times\} = \{\asymp\} + Q^{-\frac{1}{2}} v \{)\,(\}$$

with

$$\{\bigcirc\} = Q^{\frac{1}{2}}.$$

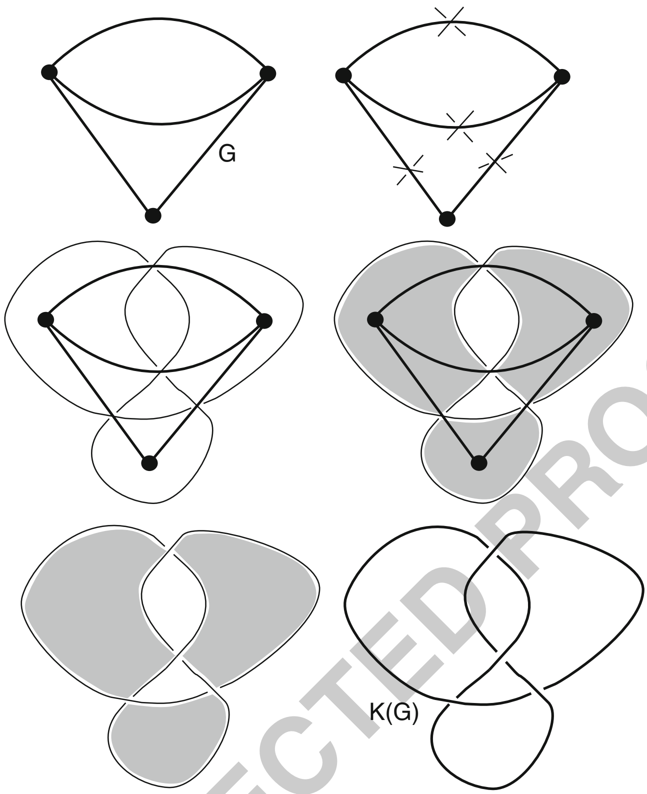

Fig. 4 Medial graph, Tait checkerboard graph, and $K(G)$

Here 355

$$Z[G](v,Q) = Q^{N/2}\{K(G)\},$$

where $K(G)$ is an alternating link diagram associated with the plane graph G in such a way that the projection of $K(G)$ to the plane is a medial diagram for the graph. Here we use the opposite convention from [13] in associating crossings to edges in the graph. We set $K(G)$ so that smoothing $K(G)$ along edges of the graph gives rise to B-smoothings of $K(G)$. See Fig. 4. The formula above, in bracket expansion form, is derived from the graphical contraction–deletion formula by translating first to the medial graph as indicated in the formulas below: 356 357 358 359 360 361 362

$$Z[\times] = Z[\asymp] + vZ[)(],$$

$$Z[R \sqcup K] = QZ[K].$$

Here the shaded medial graph is indicated by the shaded glyphs in these formulas. The medial graph is obtained by placing a crossing at each edge of G and then connecting all these crossings around each face of G as shown in Fig. 4. The medial can be checkerboard-shaded in relation to the original graph G (this is usually called the Tait checkerboard graph after Peter Guthrie Tait, who introduced these ideas into graph theory), and encoded with a crossing structure so that it represents a link diagram. Here R denotes a connected shaded region in the shaded medial graph. Such a region corresponds to a collection of nodes in the original graph, all labeled with the same color. The proof of the formula $Z[G] = Q^{N/2}\{K(G)\}$ then involves recounting boundaries of regions in correspondence with the loops in the link diagram. The advantage of the bracket expansion of the dichromatic polynomial is that it shows that this graph invariant is part of a family of polynomials that includes the Jones polynomial, and it shows how the dichromatic polynomial for a graph whose medial is a braid closure can be expressed in terms of the Temperley–Lieb algebra. This, in turn, reflects on the structure of the Potts model for planar graphs, as we remark below.

It is well known that the partition function $P_G(Q,T)$ for the Q-state Potts model in statistical mechanics on a graph G is equal to the dichromatic polynomial when

$$v = \mathrm{e}^{J\frac{1}{kT}} - 1,$$

where T is the temperature for the model and k is Boltzmann's constant. Here $J = \pm 1$ according to whether we work with the ferromagnetic or antiferromagnetic model (see [4, Chap. 12]). For simplicity, we define

$$K = J\frac{1}{kT},$$

so that

$$v = e^K - 1.$$

We have the identity

$$P_G(Q,T) = Z[G](e^K - 1, Q).$$

The partition function is given by the formula

$$P_G(Q,T) = \sum_{\sigma} e^{KE(\sigma)},$$

where σ is an assignment of one element of the set $\{1,2,\ldots,Q\}$ to each node of the graph G, and $E(\sigma)$ denotes the number of edges of the graph G whose end nodes receive the same assignment from σ. In this model, σ is regarded as a *physical* *state* of the Potts system, and $E(\sigma)$ is the *energy* of that state. Thus we have a link diagrammatic formulation for the Potts partition function for planar graphs G:

$$P_G(Q,T) = Q^{N/2}\{K(G)\}(Q, v = e^K - 1),$$

where N is the number of nodes in the graph G.

This bracket expansion for the Potts model is very useful in thinking about the physical structure of the model. For example, since the bracket expansion can be expressed in terms of the Temperley–Lieb algebra, one can use this formalism to express the expansion of the Potts model in terms of the Temperley–Lieb algebra. This method clarifies the fundamental relationship between the Potts model and the algebra of Temperley and Lieb. Furthermore, the conjectured critical temperature for the Potts model occurs for T when $Q^{-\frac{1}{2}}v = 1$. We see clearly in the bracket expansion that this value of T corresponds to a point of symmetry of the model where the value of the partition function does not depend on the designation of over- and undercrossings in the associated knot or link. This corresponds to a symmetry between the plane graph G and its dual.

We first analyze how our heuristics leading to the Khovanov homology look when generalized to the context of the dichromatic polynomial. (This approach to the question is different from the methods of Stosic [26] and [7, 24], but see the next section for a discussion of Stosic's approach to categorifying the dichromatic polynomial.) We then ask questions about the relationship between Khovanov homology and the Potts model. It is natural to ask such questions, since the adjacency of states in the Khovanov homology corresponds to an adjacency for energetic states of the physical system described by the Potts model, as we shall describe below.

For this purpose we now adopt yet another bracket expansion as indicated below. We call this two-variable bracket expansion the ρ*-bracket*. It reduces to the Khovanov version of the bracket as a function of q when ρ is equal to one:

$$[\times] = [\asymp] - q\rho\,[\,)(\,]$$

with

$$[\bigcirc] = q + q^{-1}.$$

We can regard this expansion as an intermediary between the Potts model (dichromatic polynomial) and the topological bracket. When $\rho = 1$, we have the topological bracket expansion in Khovanov form. When

$$-q\rho = Q^{-\frac{1}{2}}v$$

and

$$q + q^{-1} = Q^{\frac{1}{2}},$$

we have the Potts model. We shall return to these parameterizations shortly.

Just as in the last section, we have

$$[K] = \sum_s (-\rho)^{n_B(s)} q^{j(s)},$$

where $n_B(s)$ is the number of B-type smoothings in s, $\lambda(s)$ is the number of loops in s labeled 1 minus the number of loops labeled X, and $j(s) = n_B(s) + \lambda(s)$. This can be rewritten in the following form:

$$[K] = \sum_{i,j} (-\rho)^i q^j \dim(\mathcal{C}^{ij}) = \sum_j q^j \sum_i (-\rho)^i \dim(\mathcal{C}^{ij}) = \sum_j q^j \chi_\rho(\mathcal{C}^{\bullet j}),$$

where we define $\mathcal{C}^{ij}$ to be the linear span of the set of enhanced states with $n_B(s) = i$ and $j(s) = j$. Then the number of such states is the dimension $\dim(\mathcal{C}^{ij})$. Now we have expressed this general bracket expansion in terms of generalized Euler characteristics of the complexes:

$$\chi_\rho(\mathcal{C}^{\bullet j}) = \sum_i (-\rho)^i \dim(\mathcal{C}^{ij}).$$

These generalized Euler characteristics become classical Euler characteristics when $\rho = 1$, and in that case are the same as the Euler characteristic of the homology. With ρ not equal to 1, we do not have direct access to the homology.

Nevertheless, I believe that this raises a significant question about the relationship between $[K](q,\rho)$ and Khovanov homology. We get the Khovanov version of the bracket polynomial for $\rho = 1$ such that for $\rho = 1$, we have

$$[K](q,\rho = 1) = \sum_j q^j \chi_\rho(\mathcal{C}^{\bullet j}) = \sum_j q^j \chi(H(\mathcal{C}^{\bullet j})).$$

Away from $\rho = 1$ one can inquire into the influence of the homology groups on the coefficients of the expansion of $[K](q,\rho)$ and the corresponding questions about the Potts model. This is a way to generalize questions about the relationship between the Jones polynomial and the Potts model. In the case of the Khovanov formalism, we have the same structure of the states and the same homology theory for the states in both cases, but in the case of the Jones polynomial (ρ-bracket expansion with $\rho = 1$), we have expressions for the coefficients of the Jones polynomial in terms of ranks of the Khovanov homology groups. Only the ranks of the chain complexes figure in the Potts model itself. Thus we are suggesting here that it is worth asking about the relationship between the Khovanov homology and the dichromatic polynomial, the ρ-bracket and the Potts model, without changing the definition of the homology groups or chain spaces. This also raises the question of the relationship of the Khovanov homology to those constructions that have been made (e.g., [26]) where the homology has been adjusted to fit directly with the dichromatic polynomial. We will take up this comparison in the next section.

We now look more closely at the Potts model by writing a translation between the variables q, ρ and Q, v. We have

$$-q\rho = Q^{-\frac{1}{2}}v$$

and

$$q + q^{-1} = Q^{\frac{1}{2}},$$

and from this we conclude that

$$q^2 - \sqrt{Q}q + 1 = 0,$$

whence

$$q = \frac{\sqrt{Q} \pm \sqrt{Q-4}}{2}$$

and

$$\frac{1}{q} = \frac{\sqrt{Q} \mp \sqrt{Q-4}}{2}.$$

Thus

$$\rho = -\frac{v}{\sqrt{Q}q} = v\left(\frac{-1 \pm \sqrt{1-4/Q}}{2}\right).$$

For physical applications, Q is a positive integer greater than or equal to 2. Let us begin by analyzing the Potts model at criticality (see discussion above), where $-\rho q = 1$. Then

$$\rho = -\frac{1}{q} = \frac{-\sqrt{Q} \pm \sqrt{Q-4}}{2}.$$

For the Khovanov homology (its Euler characteristics) to appear directly in the partition function, we want

$$\rho = 1.$$

Thus we want

$$2 = -\sqrt{Q} \pm \sqrt{Q-4}.$$

Squaring both sides and collecting terms, we find that $4 - Q = \mp\sqrt{Q}\sqrt{Q-4}$. Squaring once more and collecting terms, we find that the only possibility for $\rho = 1$ is $Q = 4$. Returning to the equation for ρ, we see that this will be satisfied when we take $\sqrt{4} = -2$. This can be done in the parameterization, and then the partition function will have Khovanov topological terms. However, note that with this choice, $q = -1$, and so $v/\sqrt{Q} = -\rho q = 1$ implies that $v = -2$. Thus $e^K - 1 = -2$, and so

$$e^K = -1.$$

From this we see that in order to have $\rho = 1$ at criticality, we need a four-state Potts model with imaginary temperature variable $K = (2n+1)i\pi$. It is worthwhile considering the Potts models at imaginary temperature values. For example, the Lee–Yang theorem [20] shows that under certain circumstances, the zeros on the partition function are on the unit circle in the complex plane. We take the present calculation as an indication of the need for further investigation of the Potts model with real and complex values for its parameters.

Now we go back and consider $\rho = 1$ without insisting on criticality. Then we have $1 = -v/(q\sqrt{Q})$, so that

$$v = -q\sqrt{Q} = \frac{-Q \mp \sqrt{Q}\sqrt{Q-4}}{2}.$$

From this we see that

$$e^K = 1 + v = \frac{2 - Q \mp \sqrt{Q}\sqrt{Q-4}}{2}.$$

From this we get the following formulas for e^K: For $Q = 2$, we have $e^K = \pm\mathrm{i}$. For $Q = 3$, we have $e^K = \frac{-1 \pm \sqrt{3}i}{2}$. For $Q = 4$, we have $e^K = -1$. For $Q > 4$, it is easy to verify that e^K is real and negative. Thus in all cases of $\rho = 1$, we find that the Potts model has complex temperature values. In a subsequent paper, we shall attempt to analyze the influence of the Khovanov homology at these complex values on the behavior of the model for real temperatures.

4 The Potts Model and Stosic's Categorification of the Dichromatic Polynomial

In [26], Stosic gives a categorification for certain specializations of the dichromatic polynomial. In this section we describe this categorification, and discuss its relationship to the Potts model. The reader should also note that the relationships between dichromatic (and chromatic) homology and Khovanov homology are observed in [7, Theorem 24]. In this paper we use the Stosic formulation for our analysis.

For this purpose, we define, as in the previous section, the dichromatic polynomial through the formulas

$$Z[G](v,Q) = Z[G'](v,Q) + vZ[G''](v,Q)$$

and

$$Z[\bullet \sqcup G] = QZ[G],$$

where G' is the result of deleting an edge from G, while G'' is the result of contracting that same edge so that its end nodes have been collapsed to a single node. In the second equation, $\bullet$ represents a graph with one node and no edges, and $\bullet \sqcup G$ represents the disjoint union of the single-node graph with the graph G. The graph G is an arbitrary finite (multi)graph. This formulation of the dichromatic polynomial reveals its origins as a generalization of the chromatic polynomial for a graph G. The case $v = -1$ is that of the chromatic polynomial. In that case, the first equation asserts that the number of proper colorings of the nodes of G using Q colors is equal to the number of colorings of the deleted graph G' minus the number of colorings of the contracted graph G''. This statement is a tautology, since a proper coloring demands that nodes connected by an edge be colored with distinct colors, whence the deleted graph allows all colors, while the contracted graph allows only colorings where the nodes at the original edge receive the same color. The difference is then equal to the number of colorings that are proper at the given edge.

We reformulate this recursion for the dichromatic polynomial as follows: Instead of contracting an edge of the graph to a point in the second term of the formula, simply label that edge (say with the letter x) so that we know that it has been used in the recursion. For thinking of colorings from the set $\{1,2,\ldots,Q\}$ when Q is a positive integer, regard an edge marked with x as indicating that the colors on its two nodes are the same. This rule coincides with our interpretation of the coloring polynomial in the last paragraph. Then G'' in the deletion–contraction formula above denotes the labeling of the edge by the letter x. We then see that we can write the following formula for the dichromatic polynomial:

$$Z[G] = \sum_{H \subset G} Q^{|H|} v^{e(H)},$$

where H is a subgraph of G, $|H|$ is the number of components of H, and $e(H)$ is the number of edges of H. The subgraphs H correspond to the graphs generated by the new interpretation of the deletion–contraction formula, where contraction is replaced by edge labeling.

Moving now in the direction of Euler characteristics, we let $w = -v$, so that

$$Z[G] = Z[G'] - wZ[G''],$$

$$Z[\bullet \sqcup G] = QZ[G],$$

and

$$Z[G] = \sum_{H \subset G} (-1)^{e(H)} Q^{|H|} w^{e(H)}.$$

This suggests that differentials should increase the number of edges on the subgraphs, and that the terms $Q^{|H|}w^{e(H)}$ should not change under the application of the (partial) differentials. Stosic's solution to this requirement is to take

$$Q = q^n$$

and

$$w = 1 + q + q^2 + \cdots + q^n$$

so that

$$Z[G] = \sum_{H \subset G} (-1)^{e(H)} q^{n|H|} (1 + q + a^2 + \cdots + q^n)^{e(H)}.$$

To see how this works, we first rewrite this state sum (over states H that are subgraphs of G) as a sum over *enhanced states* h, where we define enhanced states h for a graph G to be *labeled subgraphs* h, where a labeling of h consists in an assignment of one of the elements of the set $S = \{1, X, X^2, \ldots, X^n\}$ to each component of h. Regard the elements of S as generators of the ring $R = Z[X]/(X^{n+1})$. Define the *degree* of X^i by the formula $\deg(X^i) = n - i$ and let

$$j(h) = n|h| + \sum_{\gamma \in C(h)} \deg(label(\gamma)),$$

where the sum goes over all γ in $C(h)$, the set of components of h (each component is labeled from S and $|h|$ denotes the number of components in h). Then it is easy to see that

$$Z[G] = \sum_{h \in S(G)} (-1)^{e(h)} q^{j(h)},$$

where $S(G)$ denotes the set of enhanced states of G.

We now define a chain complex for a corresponding homology theory. Let $C_i(G)$ be the module generated by the enhanced states of G with i edges. Partial boundaries applied to an enhanced state h simply add new edges between nodes, or from a node to itself. If A and B are components of an enhanced state that are joined by a partial boundary to form a new component C, then C is assigned the label X^{i+j} when A and B have respective labels X^i and X^j. This partial boundary does not change the labels on other components of h. It may happen that a component A is transformed by adding an edge to itself to form a new component A'. In this case, if A has label 1, we assign label X^n to A' and otherwise take the partial boundary to be zero if the label of A is not equal to 1. It is then easy to check that the partial boundaries defined in this way preserve $j(h)$ as defined in the last paragraph, and are compatible so that the composition of the boundary with itself is zero. We have described Stosic's homology theory for a specialization of the dichromatic polynomial. We have, as in the first section of this paper,

$$Z[G] = \sum_j q^j \chi(C^{\bullet j}(G)) = \sum_j q^j \chi(H^{\bullet j}(G)), vspace * -3pt$$

where $\chi(C^{\bullet j}(G))$ denotes the complex defined above generated by enhanced states h with $j = j(h)$, and correspondingly for the homology.

Now let us turn to a discussion of the Stosic homology in relation to the Potts model. In the Potts model we have that $Q = q^n$ is the number of spins in the model. Thus we can take any q such that q^n is a natural number greater than or equal to 2. For example, we could take q to be an nth root of 2, and then this would be a two-state Potts model. On the other hand, we have $w = -v = 1 - e^K$ as in our previous analysis for the Potts model. Thus we have

$$-e^K = q + q^2 + \cdots + q^n.$$

With q real and positive, we can take

$$K = \mathrm{i}\pi + \ln(q + q^2 + \cdots + q^n),$$

arriving at an imaginary temperature for the values of the Potts model where the partition function is expressed in terms of the homology. If we take $q = (1+n)^{1/n}$, then the model will have $Q = n+1$ states, and so in this case, we can identify the enhanced states of this model as corresponding to the spin assignments of $\{1, X, X^2, \ldots, X^n\}$ to the subgraphs, interpreted as regions of constant spin. The partial boundaries for this homology theory describe particular (global) ways to transit between spin-labeled regions where the regions themselves change locally. Usually in thinking about the dynamics of a model in statistical physics, one looks for evolutions that are strictly local. In the case of the partial differentials, we change the configuration of regions at a single bond (edge in the graph), but we make a global change in the spin-labeling (for example, from X^i and X^j on two separate regions to X^{i+j} on the joined region). It is likely that the reason we see the results of this cohomology in the partition function only at imaginary temperature is related to this nonlocal structure. Nevertheless, the categorified homology is seen in direct relation to the Potts partition, function and this connection deserves further examination.

5 Imaginary Temperature, Real Time, and Quantum Statistics

The purpose of this section is to discuss the nature of imaginary temperature in the Potts model from the point of view of quantum mechanics. We have seen that for certain values of imaginary temperature, the Potts model can be expressed in terms of Euler characteristics of Khovanov homology. The suggests looking at the analytic continuation of the partition function to relate these complex values to real values of the temperature. However, it is also useful to consider reformulating the models so that *imaginary temperature is replaced with real time*, and the context of the models is shifted to quantum mechanics. To see how this works, let us recall again the general form of a partition function in statistical mechanics. The partition function is given by the formula

$$Z_G(Q,T) = \sum_{\sigma} \mathrm{e}^{(-1/kT)E(\sigma)},$$

where the sum runs over all states σ of the physical system, T is the temperature, k is Boltzmann's constant, and $E(\sigma)$ is the energy of the state σ. In the Potts model, the underlying structure of the physical system is modeled by a graph G, and the energy has the combinatorial form that we have discussed in previous sections.

A quantum amplitude analogous to the partition function takes the form

$$A_G(Q,t) = \sum_{\sigma} \mathrm{e}^{(it/\hbar)E(\sigma)},$$

where t denotes the *time* parameter in the quantum model. We shall make precise the Hilbert space for this model below. But note that the correspondence of form between the amplitude $A_G(Q,t)$ and the partition function $Z_G(Q,T)$ suggests that we make the substitution

$$-1/kT = it/\hbar$$

or equivalently that

$$t = (\hbar/k)(1/iT).$$

Time is, up to a factor of proportionality, inverse imaginary temperature. With this substitution, we see that when one evaluates the Potts model at imaginary temperature, it can be interpreted as an evaluation of a quantum amplitude at the corresponding time given by the formula above. Thus we obtain a quantum-statistical interpretation of those places where the Potts model can be expressed directly in terms of Khovanov homology. In the process, we have given a quantum-statistical interpretation of the Khovanov homology.

To complete this section, we define the associated states and Hilbert space for the quantum amplitude $A_G(Q,t)$. Let $\mathcal{H}$ denote the vector space over the complex numbers with orthonormal basis $\{|\sigma\rangle\}$, where σ runs over the states of the Potts model for the graph G. Define a unitary operator $U(t) = \mathrm{e}^{(it/\hbar)H}$ by the formula on the basis elements

$$U(t)|\sigma\rangle = \mathrm{e}^{(it/\hbar)E(\sigma)}|\sigma\rangle.$$

The operator $U(t)$ implicitly defines the Hamiltonian for this physical system. Let

$$|\psi\rangle = \sum_{\sigma} |\sigma\rangle$$

denote an initial state and note that

$$U(t)|\psi\rangle = \sum_{\sigma} \mathrm{e}^{(it/\hbar)E(\sigma)}|\sigma\rangle$$

and

$$\langle\psi|U(t)|\psi\rangle = \sum_{\sigma} \mathrm{e}^{(it/\hbar)E(\sigma)} = A_G(Q,t).$$

Thus the Potts amplitude is the quantum-mechanical amplitude for the state $|\psi\rangle$ to evolve to the state $U(t)|\psi\rangle$. With this we have given a quantum-mechanical interpretation of the Potts model at imaginary temperature.

Note that if in the Potts model, we write $v = e^K - 1$, then in the quantum model we would write

$$e^K = \mathrm{e}^{it/\hbar}.$$

Thus we can take $t = -\hbar K i$, and if K is pure imaginary, then the time will be real in the quantum model.

Returning now to our results in Sect. 3, we recall that in the Potts model we have the following formulas for e^K: For $Q = 2$ we have $e^K = \pm \mathrm{i}$. For $Q = 3$, we have $e^K = \frac{-1 \pm \sqrt{3}i}{2}$. For $Q = 4$ we have $e^K = -1$. It is at these values that we can interpret the Potts model in terms of a quantum model at a real time value. Thus we have these interpretations for $Q = 2$, $t = \hbar\pi/2$; $Q = 3$, $t = \hbar\pi/6$; and $Q = 4$, $t = \hbar\pi$. At these values the amplitude for the quantum model is $A_G(Q,t) = \sum_{\sigma} \mathrm{e}^{(it/\hbar)E(\sigma)}$, and it is given by the formula

$$A_G(Q,t) = Q^{N/2}\{K(G)\},$$

where

$$\{K(G)\} = \sum_{j} q^j \chi(H^{\bullet j}(K(G))),$$

where $H^{\bullet j}(K(G))$ denotes the Khovanov homology of the link $K(G)$ associated with the planar graph G and $q = (1 - \mathrm{e}^{it/\hbar})/\sqrt{Q}$. At these special values, the Potts partition function in its quantum form is expressed directly in terms of the Khovanov homology and is, up to normalization, an isotopy invariant of the link $K(G)$.

6 Quantum Statistics and the Jones Polynomial

In this section we apply the point of view of the last section directly to the bracket polynomial. In keeping with the formalism of this paper, we will use the bracket in the form

$$\langle \times \rangle = \langle \asymp \rangle - q \langle)(\rangle$$

with $\langle \bigcirc \rangle = (q + q^{-1})$. We have the formula for the bracket as a sum over enhanced states s:

$$\langle K \rangle = \sum_{s} (-1)^{n_B(s)} q^{j(s)},$$

where $n_B(s)$ is the number of B-type smoothings in s, $\lambda(s)$ is the number of loops in s labeled 1 minus the number of loops labeled -1, and $j(s) = n_B(s) + \lambda(s)$. In analogy to the last section, we define a Hilbert space $\mathcal{H}(\mathcal{K})$ with orthonormal basis $\{|s\rangle\}$ in 1-to-1 correspondence with the set of enhanced states of K. Then for $q = \mathrm{e}^{\mathrm{i}\theta}$, define the unitary transformation $U : \mathcal{H}(\mathcal{K}) \longrightarrow \mathcal{H}(\mathcal{K})$ by its action on the basis elements:

$$U|s\rangle = (-1)^{n_B(s)} q^{j(s)} |s\rangle.$$

Setting $|\psi\rangle = \sum_s |s\rangle$, we conclude that

$$\langle K \rangle = \langle \psi | U | \psi \rangle.$$

Thus we can express the value of the bracket polynomial (and by normalization, the Jones polynomial) as a quantum amplitude when the polynomial variable is on the unit circle in the complex plane.

There are several conclusions that we can draw from this formula. First of all, this formulation constitutes a quantum algorithm for the computation of the bracket polynomial (and hence the Jones polynomial) at any specialization where the variable is on the unit circle. We have defined a unitary transformation U and then shown that the bracket is an evaluation of the form $\langle \psi | U | \psi \rangle$. This evaluation can be computed via the Hadamard test [23], and this gives the desired quantum algorithm. Once the unitary transformation is given as a physical construction, the algorithm will be as efficient as any application of the Hadamard test. This algorithm requires an exponentially increasing complexity of construction for the associated unitary transformation, since the dimension of the Hilbert space is equal to, $2^{c(K)}$, where $c(K)$ is the number of crossings in the diagram K.

Nevertheless, it is significant that the Jones polynomial can be formulated in such a direct way in terms of a quantum algorithm. By the same token, we can take the basic result of Khovanov homology that says that the bracket is a graded Euler characteristic of the Khovanov homology as telling us that we are taking a step in the direction of a quantum algorithm for the Khovanov homology itself. This will be the subject of a separate paper. For more information about quantum algorithms for the Jones polynomial, see [1, 14, 15, 25]. The form of this knot amplitude is also related to our research on quantum knots. See [16].

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