Cauchy–Pompeiu-Type Formulas for $\bar{\partial}$ on Affine Algebraic Riemann Surfaces with Applications

Gennadi M. Henkin

Dedicated to Oleg Viro on the occasion of his 60th birthday

Abstract We present explicit solution formulas $f = \hat{R}\varphi$ and $u = R_\lambda f$ for the equations $\bar{\partial} f = \varphi$ and $(\partial + \lambda \mathrm{d}z_1)u = f - \mathcal{H}_\lambda f$ on an affine algebraic curve $V \subset \mathbb{C}^2$. Here $\mathcal{H}_\lambda f$ denotes the projection of $f \in \tilde{W}^{1,\tilde{p}}_{1,0}(V)$ to the subspace of pseudoholomorphic (1,0)-forms on V: $\bar{\partial}\mathcal{H}_\lambda f = \bar{\lambda}\mathrm{d}\bar{z}_1 \wedge \mathcal{H}_\lambda f$. These formulas can be interpreted as explicit versions and refinements of the Hodge–Riemann decomposition on Riemann surfaces. The main application consists in the construction of the Faddeev–Green function for $\bar{\partial}(\partial + \lambda \mathrm{d}z_1)$ on V as the kernel of the operator $R_\lambda \circ \hat{R}$. This Faddeev–Green function is the main tool for the solution of the inverse conductivity problem on bordered Riemann surfaces $X \subset V$, that is, for the reconstruction of the conductivity function σ in the equation $d(\sigma d^c U) = 0$ from the Dirichlet-to-Neumann mapping $U\big|_{bX} \mapsto \sigma d^c U\big|_{bX}$. The case $V = \mathbb{C}$ was treated by R. Novikov [N1]. In Sect. 4 we give a correction to the paper [HM], in which the case of a general algebraic curve V was first considered.

Keywords Riemann surface • Inverse conductivity problem • $\bar{\partial}$-method • Homotopy formulas

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Introduction

This paper is motivated by a problem from two-dimensional electrical impedance tomography, namely the question of how to reconstruct the conductivity function σ on a bordered Riemann surface X from the knowledge of the Dirichlet-to-Neumann mapping $u\big|_{bX} \to \sigma d^c U\big|_{bX}$ for solutions U of the Dirichlet problem

$$d(\sigma d^c U)\big|_X = 0,\ \ U\big|_{bX} = u, \quad \text{where } d = \partial + \bar\partial,\ \ d^c = i(\bar\partial - \partial).$$

For the case $X = \Omega \subset \mathbb{R}^2 \simeq \mathbb{C}$, $z = x_1 + ix_2$, the exact reconstruction scheme was given first by R. Novikov [N1] under a certain restriction on the conductivity function σ. This restriction was eliminated later by A. Nachman [Na].

The scheme consists in the following.

Let $\sigma(x) > 0$ for $x \in \bar\Omega$ and $\sigma \in C^{(2)}(\bar\Omega)$. Put $\sigma(x) = 1$ for $x \in \mathbb{R}^2\backslash\bar\Omega$. The substitution $\psi = \sqrt{\sigma}U$ transforms the equation $d(\sigma d^c U) = 0$ into the equation $dd^c\psi = \frac{dd^c\sqrt{\sigma}}{\sqrt{\sigma}}\psi$ on $\mathbb{R}^2$. From L. Faddeev's [F1] result (with additional arguments [BC2] and [Na]), it follows that for each $\lambda \in \mathbb{C}$, there exists a unique solution $\psi(z,\lambda)$ of the above equation, with asymptotics

$$\psi(z,\lambda)\cdot \mathrm{e}^{-\lambda z} \overset{\text{def}}{=} \mu(z,\lambda) = 1 + o(1), \quad z \to \infty.$$

Such a solution can be found from the integral equation

$$\mu(z,\lambda) = 1 + \frac{i}{2}\int\limits_{\xi\in\Omega} g(z-\xi,\lambda)\frac{\mu(\xi,\lambda)dd^c\sqrt{\sigma}}{\sqrt{\sigma}},$$

where the function

$$g(z,\lambda) = \frac{-1}{2i(2\pi)^2}\int\limits_{w\in\mathbb{C}} \frac{\mathrm{e}^{i(w\bar z + \bar w z)}\mathrm{d}w\wedge \mathrm{d}\bar w}{w(\bar w - i\lambda)}, \quad z\in\mathbb{C},\ \ \lambda\in\mathbb{C},$$

is called the Faddeev–Green function for the operator $\mu \mapsto \bar\partial(\partial + \lambda\mathrm{d}z)\mu$.

From the work of R. Novikov [N1] it follows that the function $\psi\big|_{b\Omega}$ can be found through the Dirichlet-to-Neumann mapping by the integral equation

$$\psi(z,\lambda)\big|_{b\Omega} = \mathrm{e}^{\lambda z} + \int\limits_{\xi\in b\Omega} \mathrm{e}^{\lambda(z-\xi)}g(z-\xi,\lambda)(\hat\Phi\psi(\xi,\lambda) - \hat\Phi_0\psi(\xi,\lambda)),$$

where

$$\hat\Phi\psi = \bar\partial\psi\big|_{b\Omega},\ \ \hat\Phi_0\psi = \bar\partial\psi_0\big|_{b\Omega},\ \ \psi_0\big|_{b\Omega} = \psi,\ \ \bar\partial\partial\psi_0\big|_{\Omega} = 0.$$

By results of R. Beals, R. Coifmann [BC1], P. Grinevich, S. Novikov [GN], and R. Novikov [N2], it then follows that $\psi(z,\lambda)$ satisfies a $\bar\partial$-equation of Bers–Vekua type with respect to $\lambda \in \mathbb{C}$:

$$\frac{\partial \psi}{\partial \bar\lambda} = b(\lambda)\bar\psi,$$

where $\lambda \mapsto b(\lambda) \in L^{2+\varepsilon}(\mathbb{C}) \cap L^{2-\varepsilon}(\mathbb{C})$, and $\psi(z,\lambda)\mathrm{e}^{-\lambda z} \to 1$ as $\lambda \to \infty$, for all $z \in \mathbb{C}$.

This $\bar\partial$-equation combined with R. Novikov's integral equation permits us to find, starting from the Dirichlet-to-Neumann mapping, first the boundary values $\psi\big|_{b\Omega}$, second the "$\bar\partial$-scattering data" $b(\lambda)$, and third $\psi\big|_{\Omega}$.

Summarizing, the conductivity function $\sigma\big|_{\Omega}$ is thus retrieved from the given Dirichlet-to-Neumann data by means of the scheme

$$\text{DN data} \;\to\; \psi\big|_{b\Omega} \to \bar\partial-\text{scattering data} \;\to\; \psi\big|_{\Omega} \to \frac{dd^c\sqrt{\sigma}}{\sqrt{\sigma}}\Big|_{\Omega}.$$

Main result

We suppose that instead of $\mathbb{C}$, we have a smooth algebraic Riemann surface V in $\mathbb{C}^2$, given by an equation $V = \{z \in \mathbb{C}^2; P(z) = 0\}$, where P is a holomorphic polynomial of degree $d \geq 1$. Put $z_1 = w_1/w_0$, $z_2 = w_2/w_0$, and suppose that the projective compactification $\tilde V$ of V in $\mathbb{C}P^2 \supset \mathbb{C}^2$ with coordinates $w = (w_0 : w_1 : w_2)$ intersects $CP^1_\infty = \{z \in \mathbb{C}P^2; w_0 = 0\}$ transversally in d points. In order to extend the Novikov reconstruction scheme to the case of a Riemann surface $V \subset \mathbb{C}^2$, we need first to find an appropriate Faddeev-type Green function for $\bar\partial(\partial + \lambda \mathrm{d}z_1)$ on V. One can check that for the case $V = \mathbb{C}$, the Faddeev–Green function $g(z,\lambda)$ is a composition of Cauchy–Green Pompeiu kernels for the operators $f \mapsto \varphi = \bar\partial f$ and $u \mapsto f = (\partial + \lambda \mathrm{d}z)u$, where u, f, and φ are respectively a function, a $(1,0)$-form, and a $(1,1)$-form on $\mathbb{C}$. More precisely, one has the formula

$$g(z,\lambda) = \frac{-1}{i(2\pi)^2} \int\limits_{w\in\mathbb{C}} \frac{\mathrm{e}^{\lambda w - \bar\lambda \bar w}\mathrm{d}w \wedge \mathrm{d}\bar w}{(w+z)\cdot \bar w}.$$

The main purpose of this paper is to construct an analogue of the Faddeev–Green function on the Riemann surface V. To do this, we need to find explicit formulas $f = \hat R\varphi$ and $u = R_\lambda f$ (with appropriate estimates) for solutions of the two equations $\bar\partial f = \varphi$ and $(\partial + \lambda \mathrm{d}z_1)u = f - \mathcal{H}_\lambda f$ on V. Here we consider V equipped with the Euclidean volume form $dd^c|z|^2$, and we require $\varphi \in L^1_{1,1}(V)$, $f \in \tilde W^{1,\tilde p}_{1,0}(V)$, and $u \in L^\infty(V)$, $\tilde p > 2$, with $\mathcal{H}_\lambda f$ the projection of f on the subspace of pseudoholomorphic (1,0)-forms on $\tilde V$: $\bar\partial \mathcal{H}_\lambda f = \bar\lambda \mathrm{d}\bar z_1 \wedge \mathcal{H}_\lambda f$.

The new formulas obtained in this paper for the solution of $\bar\partial f = \varphi$ and $(\partial + \lambda \mathrm{d}z_1)u = f$ on V can be interpreted as explicit and more precise versions of the classical Hodge–Riemann decomposition results on Riemann surfaces. We will define the Faddeev-type Green function for $\bar\partial(\partial+\lambda \mathrm{d}z_1)$ on V as the kernel $g_\lambda(z,\xi)$ of the integral operator $R_\lambda \circ \hat{R}$.

Further results

Let $\sigma \in C^{(2)}(V)$, with $\sigma > 0$ on V, and $\sigma \equiv$ const on a neighborhood of $\tilde{V}\backslash V$. Let $a_1,\dots,a_g$ be generic points in this neighborhood, with g the genus of $\tilde{V}$. Using the Faddeev-type Green function constructed here, we have in [HM] obtained natural analogues of all steps of the Novikov reconstruction scheme in the case of a Riemann surface V. In particular, under a smallness assumption on $d\log\sqrt{\sigma}$, the existence (and uniqueness) of the solution $\mu(z,\lambda)$ of the Faddeev-type integral equation

$$\mu(z,\lambda) = 1 + \frac{i}{2}\int\limits_{\xi\in V} g_\lambda(z,\xi)\frac{\mu(\xi,\lambda)dd^c\sqrt{\sigma}}{\sqrt{\sigma}} + i\sum_{l=1}^{g} c_l g_\lambda(z,a_l), \quad z\in V,\ \lambda\in\mathbb{C},$$

holds for any a priori fixed constants $c_1,\dots,c_g$. However (and this was overlooked in [HM]), there exists only one choice of constants $c_l = c_l(\lambda,\sigma)$ for which the integral equation above is equivalent to the differential equation

$$\bar\partial(\partial+\lambda \mathrm{d}z_1)\mu = \frac{i}{2}\left(\frac{dd^c\sqrt{\sigma}}{\sqrt{\sigma}}\mu\right) + i\sum_{l=1}^{g} c_l\delta(z,a_l),$$

where $\delta(z,a_l)$ are Dirac measures concentrated in the points a_l (see also Sect. 4 below).

1 A Cauchy–Pompeiu-Type Formula on an Affine Algebraic Riemann Surface

By $L_{p,q}(V)$ we denote the space of (p,q)-forms on V whose coefficients are distributions of measure type on V. By $L^s_{p,q}(V)$ we denote the space of (p,q)-forms on V with coefficients absolutely integrable in degree $s \geq 1$ with respect to the Euclidean volume form on V. If $V=\mathbb{C}$ and f is a function from $L^1(\mathbb{C})$ such that $\bar\partial f \in L_{0,1}(\mathbb{C})$, then the generalized Cauchy formula has the following form:

$$f(z) = -\frac{1}{2\pi i}\int\limits_{\xi\in\mathbb{C}} \frac{\bar\partial f(\xi)\wedge \mathrm{d}\xi}{\xi - z}, \quad z\in\mathbb{C}.$$

This formula becomes the classical Cauchy formula when $f=0$ on $\mathbb{C}\backslash\Omega$ and $f\in\mathcal{O}(\Omega)$, where Ω is some bounded domain with rectifiable boundary in $\mathbb{C}$.

The generalized Cauchy formula was discovered by Pompeiu [P1] in connection with his solution of the Painlevé problem. Pompeiu proved the existence of a nonzero function $f \in \mathcal{O}(\mathbb{C}\backslash E) \cap C(\mathbb{C}) \cap L^1(\mathbb{C})$ for any totally disconnected compact set E of positive Lebesgue measure. The Cauchy–Pompeiu formula has a large number of fundamental applications: in the theory of distributions (L. Schwartz), in approximation problems (E. Bishop, S. Mergelyan, A. Vitushkin), in the solution of the corona problem (L. Carleson), in the theory of pseudoanalytic functions (L. Bers, I. Vekua), and in inverse scattering and integrable equations (R. Beals, R. Coifman, M. Ablowitz, D. Bar Yaacov, A. Fokas).

Motivated by applications to electrical impedance tomography, we develop in this paper the Cauchy–Pompeiu-type formulas on affine algebraic Riemann surfaces $V \subset \mathbb{C}^2$ and give some applications.

Let $\tilde{V}$ be a smooth algebraic curve in $\mathbb{C}P^2$ given by the equation

$$\tilde{V} = \{w \in \mathbb{C}P^2;\ \tilde{P}(w) = 0\},$$

with $\tilde{P}$ a homogeneous holomorphic polynomial in the homogeneous coordinates $w = (w_0 : w_1 : w_2) \in \mathbb{C}P^2$. Without loss of generality, we may suppose the following:

(i) $\tilde{V}$ intersects $\mathbb{C}P^1_\infty = \{w \in \mathbb{C}P^2;\ w_0 = 0\}$ transversally, $\tilde{V} \cap \mathbb{C}P^1_\infty = \{a_1, \dots, a_d\}$, $d = \deg \tilde{P}$;

(ii) $V = \tilde{V} \backslash \mathbb{C}P^1_\infty$ is a connected curve in $\mathbb{C}^2$ with an equation of the form $V = \{z \in \mathbb{C}^2;\ P(z) = 0\}$, where $P(z) = \tilde{P}(1, z_1, z_2)$ such that

$$\left|\frac{\partial P/\partial z_1}{\partial P/\partial z_2}\right| \le \text{const}(V), \quad \text{if } |z_1| \ge r_0 = \text{const}(V);$$

(iii) For any $z^* \in V$ such that $\frac{\partial P}{\partial z_2}(z^*) = 0$, we have $\frac{\partial^2 P}{\partial z_2^2}(z^*) \neq 0$.

By the Hurwitz–Riemann theorem, the number of such ramification points is equal to $d(d-1)$. Let us equip V with the Euclidean volume form $dd^c|z|^2$.

Notation

Let $\tilde{W}^{1,\tilde{p}}(V) = \{F \in L^\infty(V); \bar{\partial} F \in L^{\tilde{p}}_{0,1}(V)\}$, $\tilde{p} > 2$. Let us denote by $H^p_{0,1}(V)$ the subspace in $L^p_{0,1}(V)$, $1 < p < 2$, consisting of antiholomorphic forms. For all $p \in (1,2)$, the space $H^p_{0,1}(V)$ coincides with the space of antiholomorphic forms on V admitting an antiholomorphic extension to the compactification $\tilde{V} \subset \mathbb{C}P^2$. Hence, by the Riemann–Clebsch theorem, one has $\dim_{\mathbb{C}} H^p_{0,1}(V) = (d-1)(d-2)/2$ for all $p \in (1,2)$.

Proposition 1. *Let $\{V_j\}$ be the connected components of $\{z \in V;\ |z_1| > r_0\}$. Then for all $j \in \{1, \dots, d\}$ there exist operators $R_1\colon\ L^p_{0,1}(V) \to L^{\tilde{p}}(V)$, $R_0\colon\ L^p_{0,1}(V) \to \tilde{W}^{1,\tilde{p}}(V)$, and $\mathcal{H}\colon\ L^p_{0,1}(V) \to H^p_{0,1}(V)$, $1 < p < 2$, $1/\tilde{p} = 1/p - 1/2$, such that for* *all* $\Phi \in L^p_{0,1}(V)$*, one has the decomposition*

$$\Phi = \bar{\partial} R\Phi + \mathcal{H}\Phi, \quad \text{where } R = R_1 + R_0, \tag{1.1}$$

$$R_1\Phi = \frac{1}{2\pi i}\int_{\xi\in V} \Phi(\xi)\frac{d\xi_1}{\frac{\partial P}{\partial \xi_2}} \det\left[\frac{\partial P}{\partial \xi}(\xi), \frac{\bar{\xi}-\bar{z}}{|\xi - z|^2}\right], \tag{1.2}$$

$$\mathcal{H}\Phi = \sum_{j=1}^{g}\left(\int_V \Phi\wedge\omega_j\right)\bar{\omega}_j,$$

with $\{\omega_j\}$ *an orthonormal basis for the holomorphic (1,0)-forms on* $\tilde{V}$*, i.e.,*

$$\int_V \omega_j \wedge \bar{\omega}_k = \delta_{jk}, \quad j,k = 1,2,\ldots,g,$$

and

$$\lim_{\substack{z\in V_j\\ z\to\infty}} R\Phi(z) = 0.$$

Remark 1. If $p \in [1,2)$ and $q \in (2,\infty]$, the condition $\Phi \in L^p_{0,1}(V)\cap L^q_{0,1}(V)$ implies that $R\Phi \in C(\tilde{V})$.

Remark 2. For the case $V = \mathbb{C} = \{z \in \mathbb{C}^2;\ z_2 = 0\}$, Proposition 1 and Remark 1 reduce to the classical results of Pompeiu [P1], [P2] and of Vekua [V].

Remark 3. Based on the technique of [HP], one can construct an explicit formula not only for the main part R_1 of the R-operator, but for the whole operator R.

Proof of Proposition 1: Let $Q(\xi,z) = \{Q_1(\xi,z), Q_2(\xi,z)\}$ be a pair of holomorphic polynomials in the variables $\xi = (\xi_1,\xi_2)$ and $z = (z_1,z_2)$ such that

$$Q(\xi,\xi) = \frac{\partial P}{\partial \xi}(\xi) \text{ and}$$

$$P(\xi) - P(z) = Q_1(\xi,z)(\xi_1 - z_1) + Q_2(\xi,z)(\xi_2 - z_2) \stackrel{\text{def}}{=} \langle Q(\xi,z), \xi - z\rangle. \tag{1.3}$$

The conditions (i) and (ii) imply that for ε_0 small enough, there exists a holomorphic retraction $z \to r(z)$ of the domain $\mathcal{U}_{\varepsilon_0} = \{z \in \mathbb{C}^2 : |P(z)| < \varepsilon_0\}$ onto the curve V.

Put $\mathcal{U}_{\varepsilon,r} = \{z \in \mathbb{C}^2;\ |P(z)| < \varepsilon,\ |z_1| < r\}$, where $0 < \varepsilon \le \varepsilon_0$ and $r \ge r_0$. Put also $V^c = \{z \in \mathbb{C}^2;\ P(z) = c\}$, where $c \in \mathbb{C}$, $|c| \le \varepsilon_0$, and $\tilde{\Phi}(z) = \Phi(r(z))$, $z \in \mathcal{U}_{\varepsilon_0}$. The condition $\Phi \in L^p_{0,1}(V)$ and properties of the retraction $z \to r(z)$ together imply that $\bar{\partial}\tilde{\Phi} = 0$ on $\mathcal{U}_{\varepsilon_0}$ and

$$\|\tilde{\Phi}\|_{L^p(V^c)} \le \text{const}(V)\cdot\|\Phi\|_{L^p(V)}, \tag{1.4}$$

uniformly with respect to c, for $|c| \leq \varepsilon_0$. By results from [H] and [Po] we can choose the following explicit solution $\tilde{F}_{\varepsilon,r}$ on $\mathfrak{U}_{\varepsilon,r}$ of the $\bar{\partial}$-equation $\bar{\partial}\tilde{F}_{\varepsilon,r} = \tilde{\Phi}|_{\mathfrak{U}_{\varepsilon,r}}$:

$$\begin{aligned}\tilde{F}_{\varepsilon,r}(z) = \left(\frac{1}{2\pi i}\right)\Bigg\{ & \int\limits_{\xi\in\mathfrak{U}_{\varepsilon,r}} \tilde{\Phi}\wedge\det\left[\frac{\bar{\xi}-\bar{z}}{|\xi-z|^2},\bar{\partial}_{\xi}\frac{\bar{\xi}-\bar{z}}{|\xi-z|^2}\right]\wedge d\xi_1\wedge d\xi_2 \\ & + \int\limits_{\xi\in b\mathfrak{U}_{\varepsilon,r}:\ |\xi_1|=r} \tilde{\Phi}\wedge\left[-\frac{(\bar{\xi}_2-\bar{z}_2)}{(\xi_1-z_1)|\xi-z|^2}\right]\wedge d\xi_1\wedge d\xi_2 \\ & + \int\limits_{\xi\in b\mathfrak{U}_{\varepsilon,r}:\ |P(\xi)|=\varepsilon} \tilde{\Phi}\wedge\det\left[\frac{\bar{\xi}-\bar{z}}{|\xi-z|^2},\frac{Q}{P(\xi)-P(z)}\right]\wedge d\xi_1\wedge d\xi_2\Bigg\},\ z\in\mathfrak{U}_{\varepsilon,R}.\end{aligned} \tag{1.5}$$

Property (1.4) implies that for any $z \in V$, we have

$$\int\limits_{\xi\in\mathfrak{U}_{\varepsilon,r}} \tilde{\Phi}\wedge\det\left[\frac{\bar{\xi}-\bar{z}}{|\xi-z|^2},\bar{\partial}_{\xi}\frac{\bar{\xi}-\bar{z}}{|\xi-z|^2}\right]\wedge d\xi_1\wedge d\xi_2 \to 0,\ \varepsilon\to 0 \ \text{ and}$$

$$\int\limits_{\xi\in b\mathfrak{U}_{\varepsilon,r}:\ |\xi_1|=r} \tilde{\Phi}\wedge\left[-\frac{(\bar{\xi}_2-\bar{z}_2)}{(\xi_1-z_1)|\xi-z|^2}\right]\wedge d\xi_1\wedge d\xi_2 \to 0,\ \varepsilon\to 0,\ r\to\infty.$$

Hence for all $z \in V$ there exists $\lim\limits_{\substack{\varepsilon\to 0\\ r\to\infty}} \tilde{F}_{\varepsilon,r} = \tilde{F}(z)$, where

$$\tilde{F}(z) = -\frac{1}{2\pi i}\int\limits_{\xi\in V}\frac{\Phi d\xi_1}{\frac{\partial P}{\partial\xi_2}(\xi)}\wedge\det\left[\frac{\bar{\xi}-\bar{z}}{|\xi-z|^2},Q(\xi,z)\right]. \tag{1.6}$$

From (1.5) and (1.6) it follows that

$$\bar{\partial}_z\tilde{F}\big|_V = \Phi(z). \tag{1.7}$$

Now put $F_1 = R_1\Phi$. Using (1.2), (1.3), (1.6), and (1.7), we obtain

$$\begin{aligned}\bar{\partial}_z F_1(z)\big|_V &= \frac{1}{2\pi i}\int\limits_{\xi\in V}\Phi(\xi)\wedge\frac{d\xi_1}{\frac{\partial P}{\partial\xi_2}}\wedge\det\left[\frac{\partial P}{\partial\xi}(\xi),\bar{\partial}_z\frac{\bar{\xi}-\bar{z}}{|\xi-z|^2}\right] \\ &= \Phi + \frac{1}{2\pi i}\int\limits_{\xi\in V}\Phi(\xi)\wedge\frac{d\xi_1}{\frac{\partial P}{\partial\xi_2}}\wedge\frac{1}{|\xi-z|^4}\det\begin{vmatrix}\frac{\partial P}{\partial\xi_1}(\xi) & \xi_2-z_2\\ \frac{\partial P}{\partial\xi_2}(\xi) & -(\xi_1-z_1)\end{vmatrix}\det\begin{vmatrix}\bar{\xi}_1-\bar{z}_1 & d\bar{z}_1\\ \bar{\xi}_2-\bar{z}_2 & d\bar{z}_2\end{vmatrix} \\ &= \Phi + K\Phi,\end{aligned}$$

where

$$K\Phi = \frac{1}{2\pi i}\int\limits_{\xi\in V} \frac{\Phi(\xi)\wedge \mathrm{d}\xi_1}{|\xi - z|^4} \wedge \frac{\langle \frac{\partial P}{\partial \xi}(\xi), \xi - z\rangle \cdot \langle \frac{\partial \bar{P}}{\partial \bar{\xi}}(z), \bar{\xi} - \bar{z}\rangle}{\frac{\partial P}{\partial \xi_2}(\xi)\cdot \frac{\partial \bar{P}}{\partial \bar{\xi}_2}(z)} \mathrm{d}\bar{z}_1. \tag{1.8}$$

The estimate $R_1\Phi = F_1 \in L^{\tilde{p}}(V)$ follows from the property $\Phi \in L^p(V)$ and the following estimate of the kernel for the operator R_1:

$$\left|\left(\frac{\partial P}{\partial \xi_2}(\xi)\right)^{-1} \det\left[\frac{\partial P}{\partial \xi}(\xi), \frac{\bar{\xi} - \bar{z}}{|\xi - z|^2}\right]\mathrm{d}\xi_1\right| = O\left(\frac{1}{|\xi - z|}\right)(|\mathrm{d}\xi_1| + |\mathrm{d}\xi_2|),$$

where $\xi, z \in V$.

For the kernels of the operators $\Phi \mapsto K\Phi$ and $\Phi \mapsto \partial_z K\Phi$ we have the corresponding estimates

$$\left|\frac{\langle \frac{\partial P}{\partial \xi}(\xi), \xi - z\rangle \cdot \langle \frac{\partial \bar{P}}{\partial \bar{\xi}}(z), \bar{\xi} - \bar{z}\rangle\, \mathrm{d}\xi_1 \wedge \mathrm{d}\bar{z}_1}{\frac{\partial P}{\partial \xi_2}(\xi)\cdot |\xi - z|^4 \cdot \frac{\partial \bar{P}}{\partial \bar{\xi}_2}(z)}\right|$$

$$= \begin{cases} O\left(\frac{1}{1+|z|^2}\right)|(\mathrm{d}\xi_1 + \mathrm{d}\xi_2)\wedge(\mathrm{d}\bar{z}_1 + \mathrm{d}\bar{z}_2)| & \text{if } |\xi - z| \le 1, \\ O\left(\frac{1}{|\xi - z|^2}\right)|(\mathrm{d}\xi_1 + \mathrm{d}\xi_2)\wedge(\mathrm{d}\bar{z}_1 + \mathrm{d}\bar{z}_2)| & \text{if } |\xi - z| \ge 1. \end{cases} \tag{1.9}$$

$$\left|\partial_z \frac{\langle \frac{\partial P}{\partial \xi}(\xi), \xi - z\rangle \cdot \langle \frac{\partial \bar{P}}{\partial \bar{\xi}}(z), \bar{\xi} - \bar{z}\rangle\, \mathrm{d}\xi_1 \wedge \mathrm{d}\bar{z}_1}{\frac{\partial P}{\partial \xi_2}(\xi)\cdot |\xi - z|^4 \cdot \frac{\partial \bar{P}}{\partial \bar{\xi}_2}(z)}\right|$$

$$= \begin{cases} O\left(\frac{1}{(1+|z|^2)|\xi - z|}\right)|(\mathrm{d}\xi_1 + \mathrm{d}\xi_2)\wedge(\mathrm{d}\bar{z}_1 + \mathrm{d}\bar{z}_2)\wedge(\mathrm{d}z_1 + \mathrm{d}z_2)| & \text{if } |\xi - z| < 1, \\ O\left(\frac{1}{|\xi - z|^3}\right)|(\mathrm{d}\xi_1 + \mathrm{d}\xi_2)\wedge(\mathrm{d}z_1 + \mathrm{d}z_2)\wedge(\mathrm{d}\bar{z}_1 + \mathrm{d}\bar{z}_2)| & \text{if } |\xi - z| \ge 1, \\ & \xi, z \in V \end{cases} \tag{1.10}$$

These estimates imply that for all $\tilde{p} > 2$ and $p > 1$, one has

$$\Phi_0 \stackrel{\text{def}}{=} K\Phi \in W^{1,\tilde{p}}_{0,1}(V) \cap L^p_{0,1}(V). \tag{1.11}$$

From estimates (1.9)–(1.11), it follows that the (0,1)-form $\Phi_0 = K\Phi$ on V can be considered also as a (0,1)-form on the compactification $\tilde{V}$ of V in $\mathbb{C}P^2$ belonging to the spaces $W^{1,p}_{0,1}(\tilde{V})$ for all $p < 2$, where $\tilde{V}$ is equipped with the projective volume form $dd^c \ln(1 + |z|^2)$.

From the Hodge–Riemann decomposition theorem [Ho], [W], we have

$$\Phi_0 = \bar{\partial}(\bar{\partial}^* G\Phi_0) + \mathcal{H}\Phi_0, \tag{1.12}$$

where $\mathcal{H}\Phi_0 \in H_{0,1}(\tilde{V})$, and G is the Hodge–Green operator for the Laplacian $\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$ on $\tilde{V}$ with the properties

$$G(H_{0,1}(\tilde{V})) = 0, \ \ \bar{\partial}G = G\bar{\partial}, \ \ \bar{\partial}^*G = G\bar{\partial}^*.$$

The decomposition (1.12) implies that

$$\bar{\partial}^* G\Phi_0 \in W^{2,p}(\tilde{V}), \ p \in (1,2) \text{ and } \mathcal{H}\Phi_0 \in H_{0,1}(\tilde{V}),$$

and this in turn implies that $\bar{\partial}^* G\Phi_0 \in C(\tilde{V})$. Returning to the affine curve V with the Euclidean volume form, we obtain that

$$\tilde{R}_0\Phi \stackrel{\text{def}}{=} \bar{\partial}^* GK\Phi\big|_V \in \tilde{W}^{1,\tilde{p}}(V), \ \forall \tilde{p} > 2, \ \text{ where}$$

$$\tilde{W}^{1,\tilde{p}}(V) \stackrel{\text{def}}{=} \{F \in L^\infty(V); \ \bar{\partial}F \in L^{\tilde{p}}_{0,1}(V)\},$$

$$\text{and } \ \mathcal{H}\Phi \stackrel{\text{def}}{=} \mathcal{H}K\Phi\big|_V \in H^p_{0,1}(V), \ p > 1. \tag{1.13}$$

Now put $\tilde{R} = R_1 + \tilde{R}_0$. Then for all $\Phi \in L^p_{0,1}(V)$, we have $\tilde{R}_0\Phi \in \tilde{W}^{1,\tilde{p}}(V)$ and $\tilde{R}\Phi \in L^\infty(V) \cup L^{\tilde{p}}(V)$.

By Corollary 1 below, which is based only on (1.13), it follows that for any form $\Phi \in L^p_{0,1}(V)$, one has a limit

$$\lim_{\substack{z\to\infty \\ z\in V_j}} \tilde{R}\Phi(z) \stackrel{\text{def}}{=} \tilde{R}\Phi(\infty_j).$$

Put $R_0\Phi = \tilde{R}_0\Phi - \tilde{R}\Phi(\infty_j)$ and $R\Phi = \tilde{R}\Phi - \tilde{R}\Phi(\infty_j)$. We then have property (1.1) for $R = R_1 + R_0$. This concludes the proof of Proposition 1.

Corollary 1. *Let $F \in L^\infty(V)$ and $\bar{\partial}F \in L^p_{0,1}(V)$, $1 < p < 2$. Then for all $j \in \{1,\dots,d\}$, there exists a limit $\lim_{\substack{z\to\infty \\ z\in V_j}} F(z) \stackrel{\text{def}}{=} F(\infty_j)$ such that $(F - F(\infty_j))\big|_{V_j} \in L^{\tilde{p}}$.*

Proof. Put $\bar{\partial}F = \Phi$. Then by (1.13) we have $\tilde{R}\Phi \in L^\infty(V) \cup L^{\tilde{p}}(V)$ and $\bar{\partial}(F - \tilde{R}\Phi) = \mathcal{H}\Phi$. Then the function $h = F - \tilde{R}\Phi$ is harmonic on V. The estimates $F \in L^\infty(V)$ and $\tilde{R}\Phi \in L^{\tilde{p}}(V) \cup L^\infty(V)$ imply by the Riemann extension theorem that h can be extended to a harmonic function $\tilde{h}$ on $\tilde{V}$. Hence, one has $h = F - \tilde{R}\Phi \equiv \text{const} = c$. This implies that there exists $\lim_{\substack{z\to\infty \\ z\in V_j}} F(z) = c_j \stackrel{\text{def}}{=} F(\infty_j)$. Corollary 1 is proved.

Corollary 1 admits the following useful reformulation.

Corollary 2. *In the notation of Proposition 1, for any bounded function ψ on V such that $\bar\partial\psi \in L^p(V)$, $1<p<2$, the following formula is valid:*

$$\psi(z)=\psi(\infty_j)+R_0\bar\partial\psi+\frac{1}{2\pi i}\int\limits_{\xi\in V}\bar\partial_\xi\left(\frac{\det\left[\frac{\partial P}{\partial\xi}(\xi),\bar\xi-\bar z\right]d\xi_1}{\frac{\partial P}{\partial\xi_2}(\xi)\cdot|\xi-z|^2}\right)\wedge\psi,$$

where $R_0\bar\partial\psi\in\tilde W^{1,\tilde p}(V)$, $1/\tilde p=1/2-1/p$ and $R_0\bar\partial\psi(z)\to 0$, for $V_j\ni z\to\infty$.

2 Kernels and Estimates for $\bar\partial f=\varphi$ with $\varphi\in L^1_{1,1}(V)$

Let φ be a $(1,1)$-form of class $L^\infty_{1,1}(V)$ with support in $V_0=\{z\in V;\ |z_1|\le r_0\}$, where r_0 satisfies condition (ii) of Sect. 1.

If $V=\mathbb{C}$, then by classical results from [P1] and [V], the Cauchy–Pompeiu operator

$$\varphi\mapsto\frac{dz}{2\pi i}\int\limits_{\xi\in V_0}\frac{\varphi(\xi)}{\xi-z}\stackrel{\text{def}}{=}\hat R\varphi$$

determines a solution $f=\hat R\varphi$ for the equation $\bar\partial f=\varphi$ on $\mathbb{C}$ with the property

$$f\in W^{1,\tilde p}_{1,0}(\mathbb{C})\cap\mathcal{O}_{1,0}(\mathbb{C}\setminus V_0)\quad\text{for all}\quad\tilde p>2.$$

In this section we derive an analogous result for the case of an affine algebraic Riemann surface $V\subset\mathbb{C}^2$.

Let $V\setminus V_0=\cup_{j=1}^d V_j$, where $\{V_j\}$ are the connected components of $V\setminus V_0$.

Lemma 1. *Let $\Phi=\mathrm{d}z_1\rfloor\varphi$ and $f=F\mathrm{d}z_1=(R\Phi)\mathrm{d}z_1$, where R is the operator from Proposition 1. Then*

(i) $\Phi\in L^p_{0,1}(V_0)$, $p\in[1,2)$, $\Phi=0$ on $V\setminus V_0$;

(ii) $F\big|_{V_0}\in W^{1,p}(V_0)$ for all $p\in(1,2)$, $f\in\tilde W^{1,\tilde p}_{1,0}(V)$ for all $\tilde p\in(2,\infty)$, $\bar\partial F=\Phi-\mathcal{H}\Phi$, where $\mathcal{H}$ is the operator from Proposition 1, $\bar\partial f=\varphi-(\mathcal{H}\Phi)\wedge\mathrm{d}z_1$, and $\|F\|_{L^\infty(V\setminus V_0)}+\|F\|_{L^{\tilde p}(V_0)}\le\mathrm{const}(V,p)\|\Phi\|_{L^p(V)}$, $1/\tilde p=1/p-1/2$;

(iii) If, in addition, $\varphi\in W^{1,\infty}(V)$, then $f\in\tilde W^{2,\tilde p}(V)$.

Proof. (i) The property $\Phi\big|_{V\setminus V_0}=0$ follows from $\varphi\big|_{V\setminus V_0}=0$.

Put $V_0^\pm=\{z\in V_0;\ \pm\left|\frac{\partial P}{\partial z_2}\right|\ge\pm\left|\frac{\partial P}{\partial z_1}\right|\}$. The definition $\Phi=dz\rfloor\varphi$ implies that

$$\Phi\big|_{V_j^+}=\Phi^+\mathrm{d}\bar z_1,\ \text{where}\ \Phi^+\in L^\infty(V_0^+)\ \text{and}$$

$$\Phi\big|_{V_j^-}=\Phi^-\mathrm{d}\bar z_2/(\partial z_1/\partial z_2),\ \text{where}\ \Phi^-\in L^\infty(V_0^-).\qquad(2.1)$$

The properties (2.1) imply that $\Phi \in L^p_{0,1}(V_0)$ for all $p \in (1,2)$.

(ii) The equalities $\bar\partial F = \Phi - \mathcal{H}\Phi$ and $\bar\partial f = \varphi - (\mathcal{H}\Phi) \wedge \mathrm{d}z_1$ follow from Proposition 1 together with the definitions $\Phi = \mathrm{d}z_1 \rfloor \varphi$ and $f = F\mathrm{d}z_1$. The inclusions $F \in L^\infty(V\backslash V_0)$ and $F\big|_{V_0} \in W^{1,p}(V_0)$ follow from the formula $F = R\Phi$ and Proposition 1. The inclusion $f \in \tilde W^{1,\tilde p}_{1,0}(V)$ follows from the equalities $\bar\partial f = \varphi - (\mathcal{H}\Phi) \wedge \mathrm{d}z_1$, $f = F\mathrm{d}z_1$, and Proposition 1.

(iii) If, in addition, $\varphi \in W^{1,\infty}_{1,1}(V)$, then $f \in \tilde W^{2,\tilde p}_{1,0}(V)$. This follows from the equalities above with $\varphi \in W^{1,\infty}_{1,1}(V)$ and supp, $\varphi \subset V_0$.

Lemma 2. *For each (0,1)-form $g \in H^p_{0,1}(V)$ there exists a (1,0)-form $h \in L^p_{1,0}(\tilde V)$ ($1 \le p < 2$), unique up to adding holomorphic (1,0)-forms on $\tilde V$, such that*

$$\bar\partial h\big|_{\tilde V} = g\mathrm{d}z_1. \tag{2.2}$$

Proof. For any $g \in H^p_{0,1}(V)$ the $(1,1)$-form $g \wedge \mathrm{d}z_1$ determines a current G on $\tilde Y$ by the equality

$$\langle G, \chi\rangle \stackrel{\mathrm{def}}{=} \lim_{R\to\infty} \sum_{j=1}^{d} \left[\int\limits_{V_j} (\chi - \chi_j(\infty)) g\mathrm{d}z_1 + \chi_j(\infty) \int\limits_{\{z\in V_j;\ |z_1|<r\}} g \wedge \mathrm{d}z_1 \right],$$

where $\chi \in C^{(\varepsilon)}(\tilde V)$, $\varepsilon > 0$, and $\chi_j(\infty) = \lim\limits_{\substack{z\in V_j\\ z\to\infty}} \chi(z)$.

By Serre duality [S], the current G is $\bar\partial$-exact on $\tilde V$ if and only if

$$\langle G, 1\rangle = \lim_{R\to\infty} \int\limits_{\{z\in V;\ |z_1|\le r\}} g \wedge \mathrm{d}z_1 = 0. \tag{2.3}$$

Let us check (2.3). We have

$$\int\limits_{\{z\in V:\ |z_1|\le r\}} g \wedge \mathrm{d}z_1 \;=\; -\int\limits_{\{z\in V:\ |z_1|=r\}} z_1 \wedge g.$$

Putting $w_1 = 1/z_1$ into the right-hand side of this equality, we obtain

$$\int\limits_{\{z\in V:\ |z_1|\le r\}} g \wedge \mathrm{d}z_1 \;=\; -\sum_{j=1}^{d} \int\limits_{|w_1|=1/r} g_j(\bar w_1) \frac{\mathrm{d}\bar w_1}{w_1} = 0.$$

Here the last equality follows from the properties

$$g_j(\bar w_1)\mathrm{d}\bar w_1 = g\big|_{V_j\cap\{|w_1|\le 1/r\}} \text{ and } \bar g_j \in \mathcal{O}(D(0,1/r)).$$

Hence by (2.3), there exists $h \in L^1_{1,0}(\tilde V)$ such that equality (2.2) is valid in the sense of currents. Moreover, any solution of (2.2) automatically belongs to $L^p_{1,0}(\tilde V)$, $1 < p < 2$. Such a solution h of (2.2) is unique up to holomorphic (1,0)-forms on $\tilde V$ because the conditions $h \in L^p_{1,0}(\tilde V)$ and $\bar\partial h = 0$ on V imply that h extends as a holomorphic (1,0)-form on $\tilde V$.

Notation: Let $\mathcal{H}^\perp : H^p_{0,1}(V) \to L^p_{1,0}(\tilde V)$ $(1 < p < 2)$ be the operator defined by the formula $g \mapsto \mathcal{H}^\perp g$, where $\mathcal{H}^\perp g$ is the unique solution h of (2.2) in $L^p_{1,0}(\tilde V)$ with the property

$$\int_V h \wedge \tilde g = 0 \quad \text{for all} \quad \tilde g \in H^p_{0,1}(V).$$

Lemma 2 guarantees the existence and uniqueness of $H^\perp g \in L^p_{1,0}(\tilde V)$ for any $g \in H^p_{0,1}(V)$.

Proposition 2. *Let R be the operator defined by formula (1.1), and $\mathcal{H}$ the operator defined by formula (1.13). For any $(1,1)$-form $\varphi \in L^1_{1,1}(V) \cap L^\infty_{1,1}(V)$ with support in V_0, put*

$$\hat R\varphi = R^1\varphi + R^0\varphi, \tag{2.4}$$

where

$$R^1\varphi = (R(\mathrm{d}z_1 \rfloor \varphi))\mathrm{d}z_1, \quad R^0\varphi = \mathcal{H}^\perp \circ \mathcal{H}(\mathrm{d}z_1 \rfloor \varphi).$$

Then

$$\bar\partial \hat R\varphi = \varphi, \tag{2.5}$$

$$f = F\mathrm{d}z_1 = \hat R\varphi \in \tilde W^{1,\tilde p}_{1,0}(V) \text{ for all } \tilde p \in (2,\infty),$$

$$F\big|_{V_0} \in W^{1,p}(V_0) \text{ for all } p \in (1,2) \tag{2.6}$$

$$\text{and } f\big|_{V_l} = \sum_{k=1}^{\infty} \frac{c_k^{(l)}}{z_1^k}\mathrm{d}z_1 + b_l \mathrm{d}z_1, \text{ if } |z_1| \geq r_0.$$

Here $l = 1,\dots,d$, and $b_l = 0$ for $l = j$.

Proof. The properties (2.5) and $f = \hat R\varphi \in \tilde W^{1,\tilde p}_{1,0}(V)$ follow from Proposition 1 and Lemmas 1 and 2. The properties (2.5) and $\varphi\big|_{V\setminus V_0} = 0$ imply analyticity of f on $V\setminus V_0$. The series expansion (2.6) follows from the analyticity of $f\big|_{V\setminus V_0}$ and the inclusion $f\big|_{V\setminus V_0} \in L^\infty_{1,0}(V\setminus V_0)$.

Supplement. Let $\tilde V_0 = \{z \in V : |z_1| \leq \tilde r_0\}$, where $\tilde r_0 > r_0$. If $\operatorname{supp}\varphi \subseteq V_0$ and

$$\left(\varphi - \sum_{l=1}^{g} c_l \delta(z, a_l)\right) \in L^\infty_{1,1}(V), \text{ where } a_l \in V_{j(l)} \cap \tilde V_0,$$

then

$$\left(\hat{R}\varphi - \sum_{l=1}^{g} c_l \hat{R}(\delta(z,a_l))\right) \in W_{1,0}^{1,\tilde{p}}(V).$$

3 Kernels and Estimates for $(\partial + \lambda \mathrm{d}z_1)u = f$, with $f \in \tilde{W}_{1,0}^{1,\tilde{p}}(V)$

If $V = \mathbb{C}$, then the equation $\partial u + \lambda u \mathrm{d}z_1 = f$ was also introduced by Pompeiu [P2]. One can check that this equation can be solved by the explicit formula

$$\mathrm{e}^{\lambda z - \bar{\lambda}\bar{z}} u(z) = \frac{1}{2\pi i} \int\limits_{\xi \in \mathbb{C}} \frac{\mathrm{e}^{\lambda\xi - \bar{\lambda}\bar{\xi}} f(\xi) \mathrm{d}\bar{\xi}}{\bar{\xi} - \bar{z}} \overset{\mathrm{def}}{=} \lim_{r\to\infty} \frac{1}{2\pi i} \int\limits_{\{\xi \in \mathbb{C}:\ |\xi| < r\}} s \frac{\mathrm{e}^{\lambda\xi - \bar{\lambda}\bar{\xi}} f(\xi) \mathrm{d}\bar{\xi}}{\bar{\xi} - \bar{z}}.$$

For a Riemann surface $V = \{z \in \mathbb{C}^2 :\ P(z) = 0\}$ we will obtain the following generalization of this formula.

Proposition 3. *Let $f = F\mathrm{d}z_1$ be a $(1,0)$-form as in Proposition 2, i.e., $F\big|_{V_0} \in W^{1,p}(V_0)$ for all $p \in (1,2)$, $f \in \tilde{W}_{1,0}^{1,\tilde{p}}(V)$ for all $\tilde{p} \in (2,\infty)$, and $\operatorname{supp} \bar{\partial} f \subset V_0$. Let $e_\lambda(\xi) = \mathrm{e}^{\lambda\xi_1 - \bar{\lambda}\bar{\xi}_1}$. Put*

$$\overline{R_1(\bar{e}_\lambda \bar{f})} \overset{\mathrm{def}}{=} -\frac{1}{2\pi i} \lim_{r\to\infty} \int\limits_{\{\xi \in V:\ |\xi| < r\}} e_\lambda(\xi) f(\xi) \frac{\mathrm{d}\bar{\xi}_1}{\partial\bar{P}/\partial\bar{\xi}_2} \det\left[\frac{\partial\bar{P}}{\partial\bar{\xi}}(\xi), \frac{\xi - z}{|\xi - z|^2}\right].$$

Put also $\mathcal{H} f \overset{\mathrm{def}}{=} \overline{\mathcal{H}\bar{f}}$, where $\mathcal{H}$ is the operator from Proposition 1. Finally, let

$$u = R_\lambda f = R_\lambda^1 f + R_\lambda^0 f, \tag{3.1}$$

where $R_\lambda^1 f + R_\lambda^0 f = e_{-\lambda}(z) \cdot \overline{R_1(\bar{e}_\lambda \bar{f})} + e_{-\lambda}(z) \cdot \overline{R_0(\bar{e}_\lambda \bar{f})}$, with R_1 and R_0 being the operators from Proposition 1.

Then for all $\lambda \neq 0$ one has the following:

(i) $(\partial + \lambda \mathrm{d}z_1) R_\lambda f = f - \mathcal{H}_\lambda(f)$, *where* $\mathcal{H}_\lambda(f) = e_{-\lambda}(z)\mathcal{H}(e_\lambda f)$.

(ii)

$$\|u - u(\infty_l)\|_{L^\infty(V)} \leq$$

$$\mathrm{const}(V,p) \cdot \min\left(\frac{1}{\sqrt{|\lambda|}}, \frac{1}{|\lambda|}\right) \left(\|F\|_{L^{\tilde{p}}(V_0)} + \|F\|_{L^\infty(V \setminus V_0)} + \|\partial F\|_{L_{1,0}^{p}(V)}\right),$$

$$\|\partial u\|_{L_{1,0}^{\tilde{p}}(V)} \leq \mathrm{const}(V,p) \cdot \|\partial F\|_{L_{1,0}^{p}(V)}, \text{ where } 1/\tilde{p} = 1/p - 1/2,\ l = 1, \ldots, d.$$

(iii) In addition, if $|\lambda| \le 1$*, then*

$$\|(1+|z_1|)(u-u(\infty_l))\|_{L^\infty(V_l)} \le \frac{\mathrm{const}(V,\tilde{p})}{\sqrt{|\lambda|}}(\|F\|_{L^{\tilde{p}}(V_0)} + \|F\|_{L^\infty(V\setminus V_0)}),$$

$$\|(1+|z_1|)\partial u\|_{L^\infty_{1,0}(V)} \le \mathrm{const}(V,\tilde{p})(\sqrt{|\lambda|}+1)(\|F\|_{L^{\tilde{p}}(V_0)} + \|F\|_{L^\infty(V\setminus V_0)}), \forall \tilde{p} > 2.$$

Supplement. Put

$$L^{2\pm\varepsilon}(V) = \{u; u|_{\tilde{V}_0} \in L^{2-\varepsilon}(\tilde{V}_0),\ \ u|_{V\setminus\tilde{V}_0} \in L^{2+\varepsilon}(V\setminus\tilde{V}_0)\}.$$

If $f = f_0 - f_1$, where $f_0 \in \tilde{W}^{1,\tilde{p}}_{1,0}(V)$, $\mathrm{supp}\,\bar{\partial} f_0 \subset V_0$, and $f_1 = \sum_{l=1}^{g} c_l \hat{R}(\delta(z,a_l))$, $a_l \in V_{j(l)} \cap \tilde{V}_0$, then instead of (i)–(iii) we have (i) and the following conclusion:

(ii′) We have

$$\|R_\lambda f - R_\lambda f(\infty_l)\|_{L^{2+\varepsilon}(V_l)} \le \frac{\mathrm{const}(V,\tilde{p})}{\varepsilon}\min\left(|\lambda|^{-1/2},|\lambda|^{-1}\right)\left(\|f_0\|_{\tilde{W}^{1,\tilde{p}}_{1,0}} + \sum_{l=1}^{g}|c_l|\right),$$

$$\|\partial R_\lambda f\|_{L^{2\pm\varepsilon}_{1,0}(V)} \le \frac{\mathrm{const}(V,\tilde{p})}{\varepsilon}\left(\|f_0\|_{\tilde{W}^{1,\tilde{p}}_{1,0}} + \sum_{l=1}^{g}|c_l|\right),$$

$$\|\mathcal{H}_\lambda(f)\|_{L^\infty_{1,0}(V)} \le \frac{\mathrm{const}(V,\tilde{p})}{(1+|\lambda|)}\left(\|f_0\|_{\tilde{W}^{1,\tilde{p}}_{1,0}} + \sum_{l=1}^{g}|c_l|\right), \text{ where } \tilde{p} > 2, 0 < \varepsilon < 1/2.$$

Lemma 3. *Put*

$$J(z) = \int\limits_{\{\xi\in\mathbb{C}:\ |\xi|<\rho\}} \frac{\psi(\xi)\mathrm{d}\xi \wedge \mathrm{d}\bar{\xi}}{|\xi|\cdot|\xi - z|},\ \ z \in \mathbb{C},$$

where $\psi \in L^p(V_0)$, $p > 1$*. Then for any* $\varepsilon > 0$ *and any* $\tilde{p} > 2$*, one has the estimate*

$$\|J(z)\|_{L^{\tilde{p}}(\mathbb{C})} \le \frac{1}{\varepsilon} O(\rho^{(2-2\varepsilon\tilde{p})/\tilde{p}}) \cdot \|\psi\|_{L^{(1+\varepsilon)/\varepsilon}(V_0)}.$$

Proof. Using the notation $\|\psi\|_\varepsilon = \|\psi\|_{L^{(1+\varepsilon)/\varepsilon}(V_0)}$, we obtain from the expression for $J(z)$ the following estimates:

$$|J(z)| \le \left(\int\limits_{|\xi|\le\rho} \frac{|\mathrm{d}\xi\wedge\mathrm{d}\bar{\xi}|}{|\xi|^{1+\varepsilon}|\xi - z|^{1+\varepsilon}}\right)^{1/(1+\varepsilon)} \cdot \|\psi\|_\varepsilon$$

$$\leq O\left(\int\limits_{r=0}^{\rho}\frac{\mathrm{d}r}{r^{\varepsilon}}\int\limits_{0}^{1}\frac{\mathrm{d}\varphi}{(|r-|z||+|z|\varphi)^{1+\varepsilon}}\right)^{1/(1+\varepsilon)}\cdot\|\psi\|_{\varepsilon}$$

$$\leq O\left(\int\limits_{r=0}^{\rho}\frac{\mathrm{d}r}{r^{\varepsilon}}\frac{1}{|z|^{1+\varepsilon}}\int\limits_{0}^{1}\frac{\mathrm{d}\varphi}{(|\frac{r}{|z|}-1|+\varphi)^{1+\varepsilon}}\right)^{1/(1+\varepsilon)}\cdot\|\psi\|_{\varepsilon}$$

AQ1

$$\leq\frac{1}{\varepsilon}O\left(\int\limits_{0}^{\rho}\frac{\mathrm{d}r}{r^{\varepsilon}}\,\frac{1}{|z|}\left(\frac{1}{|r-|z||^{\varepsilon}}-\frac{1}{(|r-|z||+|z|)^{\varepsilon}}\right)\right)^{1/(1+\varepsilon)}\cdot\|\psi\|_{\varepsilon}.$$

From the last estimate we deduce

AQ2

$$|J(z)|\leq\frac{1}{\varepsilon}O\Bigg(\frac{1}{|z|}\Bigg(\int\limits_{0}^{|z|}\frac{\mathrm{d}r}{r^{\varepsilon}|z|^{\varepsilon}}+\int\limits_{|z|}^{\rho}\frac{\varepsilon|z|}{r^{\varepsilon}r^{\varepsilon}r}\Bigg)\Bigg)^{1/(1+\varepsilon)}\|\psi\|_{\varepsilon},\ \text{if } |z|\leq\rho,$$

$$\text{and } |J(z)|\leq\frac{1}{\varepsilon}O\Bigg(\frac{1}{|z|}\int\limits_{0}^{\rho}\frac{\mathrm{d}r}{r^{\varepsilon}|z|^{\varepsilon}}\Bigg)\Bigg)^{1/(1+\varepsilon)}\|\psi\|_{\varepsilon},\ \text{if } |z|\geq\rho.$$

These equalities imply

AQ3

$$|J(z)|\leq\frac{1}{\varepsilon}O\left(\left(\frac{1}{|z|}\right)^{2\varepsilon/(1+\varepsilon)}\right)\|\psi\|_{\varepsilon},\ \text{ if }\ |z|\leq\rho,$$

$$|J(z)|\leq\frac{1}{\varepsilon}O\left(\left(\frac{1}{|z|}\,\rho^{(1-\varepsilon)/(1+\varepsilon)}\right)\|\psi\|_{\varepsilon},\ \text{ if }\ |z|\geq\rho.$$

Putting $|z|=t$, we obtain finally that

$$\|J\|_{L^{\tilde{p}}(\mathbb{C})}\leq\frac{1}{\varepsilon}O\left(\int\limits_{0}^{\rho}\frac{\mathrm{d}t}{t^{2\varepsilon\tilde{p}/(1+\varepsilon)-1}}+\rho^{\frac{1-\varepsilon}{1+\varepsilon}\tilde{p}}\int\limits_{\rho}^{\infty}\frac{\mathrm{d}t}{t^{\tilde{p}-1}}\right)^{1/\tilde{p}}\|\psi\|_{\varepsilon}\leq\frac{1}{\varepsilon}O\left(\rho^{\frac{2-\varepsilon\tilde{p}}{\tilde{p}}}\right)\|\psi\|_{\varepsilon}.$$

Lemma 3 is proved.

Proof of Proposition 3:

(i) We have

$$(\partial+\lambda\mathrm{d}z_1)R_{\lambda}f=(\partial+\lambda\mathrm{d}z_1)e_{-\lambda}(z)\cdot\overline{R(\bar{e}_{\lambda}\bar{f})}$$
$$=\partial(e_{-\lambda}(z))\cdot\overline{R(\bar{e}_{\lambda}\bar{f})}+e_{-\lambda}(z)\partial(\overline{R(\bar{e}_{\lambda}\bar{f})})+\lambda\mathrm{d}z_1e_{-\lambda}(z)\cdot\overline{R(\bar{e}_{\lambda}\bar{f})}$$

$$= (-\lambda \mathrm{d}z_1 + \lambda \mathrm{d}z_1) e_{-\lambda}(z) \cdot \overline{R(\bar{e}_\lambda \bar{f})}$$
$$+ e_{-\lambda}(z) \cdot (e_\lambda(z) f - \overline{\mathcal{H} \bar{e}_\lambda \bar{f}}) = f - e_{-\lambda} \mathcal{H}(e_\lambda f) \stackrel{\text{def}}{=} f - \mathcal{H}_\lambda f,$$

where we have used the equality (1.1) from Proposition 1.

(iii) Let $r \geq r_0$. Let the functions $\chi_\pm \in C^{(1)}(V)$ be such that $\chi_+ + \chi_- \equiv 1$ on V, $\operatorname{supp} \chi_+ \subset \{\xi \in V : |\xi_1| < 2r\}$, $\operatorname{supp} \chi_- \subset \{\xi \in V : |\xi_1| \geq r\}$, and $|d\chi_\pm| = O(1/r)$. We then have $u = u_+ + u_-$, where

$$u_\pm(z) = R_\lambda(\chi_\pm f). \tag{3.1$_\pm$}$$

Using the properties $f \in L^\infty(V)$ and $|e_\lambda| \equiv 1$, in combination with the equality $\partial u_+ = \chi_+ F \mathrm{d}z_1 - \lambda u_+ \mathrm{d}z_1 - \mathcal{H}_\lambda(\chi_+ f)$, we obtain for u_+ and $\frac{\partial u_\pm}{\partial z_1}$ the estimates

$$\|(1+|z|)(u_+(z) - u_+(\infty_l))\|_{L^\infty(V_l)} = O(r)\|f\|_{L^\infty_{1,0}(V)}, \; l = 1, \ldots, d,$$
$$\|(1+|z|)\partial u_+(z)\|_{L^\infty_{1,0}(V)} = O(\lambda r + 1)\|f\|_{L^\infty_{1,0}(V)}. \tag{3.2}$$

In order to estimate u_-, we transform the expression $(3.1)_-$ using the series expansion (2.6) for $f\big|_{V_j}$, and we integrate by parts. We thus obtain

$$u_-(z) = R_\lambda \chi_- f = R^1_\lambda \chi_- f + R^0_\lambda \chi_- f$$
$$= -\frac{e_{-\lambda}(z)}{2\pi i} \frac{1}{\lambda} \int\limits_{\xi \in V} \frac{\mathrm{e}^{\lambda \xi_1 - \bar{\lambda} \bar{\xi}_1} (\mathrm{d}\chi_-) F \wedge \mathrm{d}\bar{\xi}_1 \det\left[\frac{\partial \bar{P}}{\partial \bar{\xi}}(\xi), \xi - z\right]}{\frac{\partial \bar{P}}{\partial \bar{\xi}_2}(\xi) \cdot |\xi - z|^2}$$
$$+ \frac{e_{-\lambda}(z)}{2\pi i} \frac{1}{\lambda} \sum_j \int\limits_{\xi \in V_j} \mathrm{e}^{\lambda \xi_1 - \bar{\lambda} \bar{\xi}_1} \chi_- \left(\sum_{k=1}^{\infty} k \frac{c_k^{(j)}}{\xi_1^{k+1}} \right) \frac{\mathrm{d}\xi_1 \wedge \mathrm{d}\bar{\xi}_1 \det\left[\frac{\partial \bar{P}}{\partial \bar{\xi}}(\xi), \xi - z\right]}{\frac{\partial \bar{P}}{\partial \bar{\xi}_2}(\xi) \cdot |\xi - z|^2}$$
$$- \frac{e_{-\lambda}(z)}{2\pi i} \frac{1}{\lambda} \int\limits_{\xi \in V} \mathrm{e}^{\lambda \xi_1 - \bar{\lambda} \bar{\xi}_1} \chi_- F \partial_\xi \left(\frac{\det\left[\frac{\partial \bar{P}}{\partial \bar{\xi}}(\xi), \xi - z\right] \mathrm{d}\bar{\xi}_1}{\frac{\partial \bar{P}}{\partial \bar{\xi}_2}(\xi) \cdot |\xi - z|^2} \right)$$
$$+ e_{-\lambda}(z) \bar{R}_0(e_\lambda \chi_- f), \tag{3.3}$$

where the operator $R_0 = \bar{\partial}^* GK$ is defined by (1.13). Using Corollary 2 we have, in addition,

$$
-\frac{e_{-\lambda}(z)}{2\pi i}\frac{1}{\lambda}\int\limits_{\xi\in V} \mathrm{e}^{\lambda\xi_1-\bar\lambda\bar\xi_1}\chi_- F\partial_\xi\left(\frac{\det\left[\frac{\partial\bar P}{\partial\bar\xi}(\xi),\xi-z\right]\mathrm{d}\bar\xi_1}{\frac{\partial\bar P}{\partial\bar\xi_2}(\xi)\cdot|\xi-z|^2}\right)
$$

$$
=\frac{e_{-\lambda}(z)}{2\pi i}\frac{1}{\lambda}e_\lambda(z)\chi_-(z)F(z)-\frac{e_{-\lambda}(z)}{2\pi i}\frac{1}{\lambda}\bar R_0(\partial(e_\lambda\chi_- F))
$$

$$
=\frac{1}{2\pi i}\frac{1}{\lambda}\chi_-(z)F(z)-\frac{e_{-\lambda}(z)}{2\pi i}\frac{1}{\lambda}\bar R_0(\partial(e_\lambda\chi_- F)).
$$

Putting the last equality into (3.3) and making use of the properties $|e_\lambda|\equiv 1$, $|d\chi_-|=O(1/r)$, $\partial u_-=\chi_- F\mathrm{d}z_1-\lambda u_-\mathrm{d}z_1-\mathcal{H}_\lambda(\chi_- f)$, and the property of R_0, we obtain from Proposition 1

$$
\|(1+|z_1|)(u_- - u_-(\infty_l))\|_{L^\infty(V_l)}
$$

$$
=O\left(\frac{1}{|\lambda|r}\right)(\|F\|_{L^{\tilde p}(V_0)}+\|F\|_{L^\infty(V\setminus V_0)})+\|(1+|z_1|)\bar R_0(e_\lambda\chi_- f)\|_{L^\infty(V)}
$$

$$
+\frac{1}{2\pi|\lambda|}\|(1+|z_1|)\bar R_0\partial(e_\lambda\chi_- F)\|_{L^\infty(V)}\le O\left(\frac{1}{|\lambda|r}\right)\|F\|_{L^{\tilde p}(V)},\ l=1,\ldots,d
$$

$$
\text{and}\quad \|(1+|z_1|)\frac{\partial u_-}{\partial z_1}\|_{L^\infty(V)}=O(1/r+1)(\|F\|_{L^{\tilde p}(V_0)}+\|F\|_{L^\infty(V\setminus V_0)}). \tag{3.4}
$$

The estimates (3.2) and (3.4) imply

$$
\|(1+|z_1|)(u-u(\infty_l))\|_{L^\infty(V_l)}
$$

$$
=O\left(r+\frac{1}{|\lambda|r}\right)(\|F\|_{L^{\tilde p}(V_0)}+\|F\|_{L^\infty(V\setminus V_0)}),
$$

$$
\text{and}\quad \|(1+|z_1|)\partial u\|_{L^\infty_{1,0}(V)}
$$

$$
=O(|\lambda|r+1/r+1)(\|F\|_{L^{\tilde p}(V_0)}+\|F\|_{L^\infty(V\setminus V_0)}),\ \forall\tilde p>2. \tag{3.5}
$$

Putting in (3.5) $r=r_0/\sqrt{|\lambda|}$, we obtain (iii).

(ii) For proving (ii), let us put $r=\tilde r_0$ and transform $(3.1)_+$ for u_+ in the following way:

$$
u_+(z)=R_\lambda\chi_+ f=-\frac{e_{-\lambda}(z)}{2\pi i}\frac{1}{\lambda}\int\limits_{|\xi_1|\le r}\frac{\mathrm{e}^{\lambda\xi_1-\bar\lambda\bar\xi_1}\mathrm{d}\chi_+ F\wedge\mathrm{d}\bar\xi_1\det\left[\frac{\partial\bar P}{\partial\bar\xi}(\xi),\xi-z\right]}{\frac{\partial\bar P}{\partial\bar\xi_2}(\xi)\cdot|\xi-z|^2}
$$

$$- \frac{e_{-\lambda}(z)}{2\pi i}\frac{1}{\lambda}\int\limits_{|\xi_1|\le r}\frac{\mathrm{e}^{\lambda\xi_1-\bar{\lambda}\bar{\xi}_1}\chi_+\partial F\wedge \mathrm{d}\bar{\xi}_1 \det\left[\frac{\partial\bar{P}}{\partial\bar{\xi}}(\xi),\xi-z\right]}{\frac{\partial\bar{P}}{\partial\bar{\xi}_2}(\xi)\cdot|\xi-z|^2}$$

$$- \frac{e_{-\lambda}(z)}{2\pi i}\frac{1}{\lambda}\int\limits_{|\xi_1|\le r}\mathrm{e}^{\lambda\xi_1-\bar{\lambda}\bar{\xi}_1}\chi_+F\partial\left(\frac{\det\left[\frac{\partial\bar{P}}{\partial\bar{\xi}}(\xi),\xi-z\right]\mathrm{d}\bar{\xi}_1}{\frac{\partial\bar{P}}{\partial\bar{\xi}_2}(\xi)\cdot|\xi-z|^2}\right)$$

$$+e_{-\lambda}(z)\bar{R}_0(e_\lambda\chi_+f), \tag{3.6}$$

where R_0 is the operator from Proposition 1. Using the last expression for $u_+(z)$, together with the property $F\big|_{V_0}\in W^{1,p}(V_0)$ and Corollary 2, we obtain

$$\|u_+\|_{L^\infty(V)}=O(1/\lambda)\|F\|_{\tilde{W}^{1,p}(V_0)}. \tag{3.7}$$

This inequality together with (3.4) and statement (iii) proves the first part of statement (ii). Formula $u=R_\lambda f$ implies $\partial_z u=f-\lambda\mathrm{d}z_1 u-\mathcal{H}_\lambda f$. From this and from the already obtained estimates for u, we deduce the second part of statement (ii):

$$\|\partial u\|_{L^{\tilde{p}}_{1,0}(V)}\le \mathrm{const}(V,p)\|\partial F\|_{L^{p}_{1,0}(V)}.$$

(ii)′ In order to prove in this case the estimate for $u=R_\lambda f$ with $|\lambda|\le 1$, we combine the arguments above with Lemma 3, and obtain instead of (3.5) the following:

$$\|u-u(\infty_l)\|_{L^{2+\varepsilon}(V_l)}\le\frac{1}{\varepsilon}O\left(r+\frac{1}{|\lambda|r}\right)\left(\|f_0\|_{\tilde{W}^{1,\tilde{p}}_{1,0}(V)}+\sum_{l=1}^{g}|c_l|\right)$$

$$\|\partial u\|_{L^{2\pm\varepsilon}_{1,0}(V)}\le\frac{\lambda}{\varepsilon}O\left(r+\frac{1+r}{|\lambda|r}\right)\left(\|f_0\|_{\tilde{W}^{1,\tilde{p}}_{1,0}(V)}+\sum_{l=1}^{g}|c_l|\right) \tag{3.5$'$}$$

Putting in (3.5)′ $r=r_0/\sqrt{|\lambda|}$, we obtain the required estimate for $R_\lambda f$ with $|\lambda|\le 1$. To prove the estimate for $u=R_\lambda f$ with $|\lambda|\ge 1$, we use (3.6) and the Calderon–Zygmund $L^{2-\varepsilon}$-estimate for the singular integral on the right-hand side of (3.6).

In order to prove the statement concerning $H_\lambda f$, we just perform an integration by parts in the expression

$$\mathcal{H}_\lambda f=e_{-\lambda}\mathcal{H}(e_\lambda f)=\sum_{l=1}^{g}e_{-\lambda}(z)\left(\int\limits_{\tilde{V}}e_\lambda(\xi)f(\xi)\wedge\overline{\omega_l(\xi)}\right)\omega_l(z),$$

where $f=f_0+\sum\limits_{l=1}^{g}c_l\hat{R}(\delta(z,a_l))$, and where $\{\omega_l, l=1,\dots,g\}$ is an orthonormal basis of holomorphic $(1,0)$-forms on $\tilde{V}$.

4 Faddeev-Type Green Function for $\bar\partial(\partial+\lambda dz_1)u=\varphi$ and Further Results

Let $\hat R$ be the operator defined by formula (2.4) and let R_λ be the operator defined by formula (3.1).

Proposition 4. *Let $\varphi\in L^\infty_{1,1}(V)$ with support in $V_0=\{z\in V:\ |z_1|\le r_0\}$, where r_0 satisfies the condition of Sect. 1. Then for $u=G_\lambda\varphi\stackrel{\text{def}}{=}R_\lambda\circ\hat R\varphi$, where $\lambda\neq 0$, one has*

(i) $\bar\partial(\partial+\lambda dz_1)u=\varphi+\bar\lambda d\bar z_1\wedge\mathcal{H}_\lambda(\hat R\varphi)$ *on V.*

(ii) We have

$$\|u\|_{L^\infty(V)}\le \mathrm{const}(V_0,\tilde p)\cdot\min\left(1/\sqrt{|\lambda|},1/|\lambda|\right)\|\varphi\|_{L^\infty_{1,1}(V_0)},\quad \tilde p>2,$$
$$\|\partial u\|_{L^{\tilde p}_{1,0}(V)}\le \mathrm{const}(V_0,\tilde p)\,\|\varphi\|_{L^\infty_{1,1}(V_0)},\quad \tilde p>2.$$

Supplement. If we can write $\varphi=\varphi_0+\varphi_1$, where $\varphi_0\in L^\infty_{1,1}(V)$, $\operatorname{supp}\varphi_0\subset V_0$, and $\varphi_1=\sum\limits_{l=1}^{g}ic_l\delta(z,a_l)$, with $a_l\in V_{j(l)}\cap\tilde V_0$, then instead of (i)–(ii) we have (i) and the following conclusion:

(ii)′ We have

$$\|u-u(\infty_l)\|_{L^{2+\varepsilon}(V_l)}\le \mathrm{const}(V,\varepsilon)\cdot\min\left(|\lambda|^{-1/2},|\lambda|^{-1}\right)\left(\|\varphi_0\|_{L^\infty_{1,1}(V_0)}+\sum_{j=1}^{g}|c_j|\right),$$
$$\|\partial u\|_{L^{2\pm\varepsilon}_{1,0}(V)}\le \mathrm{const}(V,\varepsilon)\left(\|\varphi_0\|_{L^\infty_{1,1}(V_0)}+\sum_{l=1}^{g}|c_l|\right),$$

where $0<\varepsilon<1/2$.

Proof. By Proposition 2 we have

$$f=F dz_1=\hat R\varphi\in\tilde W^{1,\tilde p}_{1,0}(V)\ \ \forall\tilde p\in(2,\infty),\quad F\big|_{V_0}\in W^{1,p}(V_0)\ \forall p\in(1,2).$$

Propositions 2 and 3 imply that $u=R_\lambda\circ\hat R\varphi\in\tilde W^{1,\tilde p}(V)$. Let us now verify statement (i) of Proposition 4. From Proposition 3(i), we obtain

$$(\partial+\lambda dz_1)u=(\partial+\lambda dz_1)R_\lambda\circ\hat R\varphi=\hat R\varphi+\mathcal{H}_\lambda(\hat R\varphi),\quad\text{where}$$
$$\mathcal{H}_\lambda(\hat R\varphi)=e_{-\lambda}\mathcal{H}(e_\lambda\hat R\varphi).\tag{4.1}$$

From (4.1) and Proposition 2 we obtain

$$\bar\partial(\partial+\lambda \mathrm{d}z_1)u=\varphi+\bar\partial(\mathcal{H}_\lambda(\hat R\varphi))=\varphi+\bar\lambda \mathrm{d}\bar z_1\wedge\mathcal{H}_\lambda(\hat R\varphi),$$

where we have used that $\mathcal{H}(\hat R\varphi)\in H_{1,0}(\tilde V)$.

Property 4(ii) follows from Proposition 3(ii), (iii). The supplement to Proposition 4 follows from the supplement to Proposition 3.

Definition 1. We define the Faddeev-type Green function for $\bar\partial(\partial+\lambda \mathrm{d}z_1)$ on V as the kernel $g_\lambda(z,\xi)$ of the integral operator $R_\lambda\circ\hat R$.

Definition 2. Let $q\in C_{1,1}(\tilde V)$ be a form with $\operatorname{supp} q$ contained in V_0, and let g denote the genus of $\tilde V$. The function $\psi(z,\lambda)$, $z\in V$, $\lambda\in\mathbb{C}$, will be called the Faddeev-type function associated with the potential q and the points $a_1,\dots,a_g\in V\backslash\bar V_0$ if $\forall\lambda\in\mathbb{C}\backslash E$, where E is compact in $\mathbb{C}$, the function $\mu=\psi(z,\lambda)\mathrm{e}^{-\lambda z_1}$ has the following properties:

$$\bar\partial(\partial+\lambda \mathrm{d}z_1)\mu=\frac{i}{2}q\mu+i\sum_{l=1}^{g}c_l\delta(z,a_l)\ \text{and}\ \lim_{\substack{z\to\infty\\ z\in V_1}}\mu(z,\lambda)=1,$$

$$(\mu-\mu(\infty_j))\big|_{V_j}\in L^{\tilde p}(V_j),\quad \tilde p>2,\ j=1,\dots,d,$$

where $\delta(z,a_l)$ is the Dirac measure concentrated at the point a_l.

Based on the Faddeev-type Green function $g_\lambda(z,\xi)$, and on Proposition 4, we have in [HM] extended the Novikov reconstruction scheme from the case $X\subset\mathbb{C}$ to the case of a bordered Riemann surface $X\subset V$.

Definition 3. Let $\{\omega_j\}$ be an orthonormal basis for the holomorphic forms on $\tilde V$. An effective divisor $\{a_1,\dots,a_g\}$ on V will be called generic if

$$\det\left[\frac{\omega_j}{\mathrm{d}z_1}(a_k)\Big|_{j,k=1,2,\dots,g}\right]\neq 0.$$

Lemma 4. *Let $\{a_j\}$ be a generic divisor on V. Put*

$$\Delta(\lambda)=\det\left[\int_{\xi\in V}\hat R(\delta(\xi,a_j))\wedge\bar\omega_l(\xi)\mathrm{e}^{\lambda\xi_1-\bar\lambda\bar\xi_1}\right]\Big|_{j,l=1,2,\dots,g},$$

$$|\Delta(\lambda)|_\varepsilon=\sup_{|\lambda'-\lambda|<\varepsilon}|\Delta(\lambda')|,$$

where $\hat R$ is the operator from Proposition 2. Then for all $\varepsilon>0$, one has $\overline{\lim}_{\lambda\to\infty}|\lambda^g\cdot\Delta(\lambda)|<\infty$, $\underline{\lim}_{\lambda\to\infty}|\lambda^g\cdot\Delta(\lambda)|_\varepsilon>0$, and the set

$$E=\{\lambda\in\mathbb{C}:\ \Delta(\lambda)=0\}\ \text{is a closed nowhere dense subset of}\ \mathbb{C}.\qquad (*)$$

The following is a corrected version of the main results from [HM]:

1. Let X be a domain with smooth boundary on V such that $X \supset \bar{V}_0$, $\bar{X} \subset Y \subset V$. Let $\sigma \in C^{(2)}(V)$, $\sigma > 0$ on V, and $\sigma = 1$ on $V \backslash X$. Let $a_1, \dots, a_g$ be a generic divisor on $Y \backslash \bar{X}$, satisfying condition $(*)$. Then for all $\lambda \in \mathbb{C} \backslash E$, there exists a unique Faddeev-type function $\psi(z,\lambda) = \mu(z,\lambda) \mathrm{e}^{\lambda z_1}$ associated with the potential $q = \frac{dd^c \sqrt{\sigma}}{\sqrt{\sigma}}$ and the divisor $\{a_j\}$. Such a function can be found (together with constants $\{c_l\}$) from the integral equation

$$\mu(z,\lambda) = 1 + \frac{i}{2} \int\limits_{\xi \in X} g_\lambda(z,\xi) \mu(\xi,\lambda) q(\xi) + i \sum_{l=1}^{g} c_l(\lambda) g_\lambda(z, a_l), \tag{4.2}$$

where

$$\frac{1}{2} \mathcal{H}_\lambda(\hat{R}(q\mu)) = \sum_{l=1}^{g} c_l \mathcal{H}_\lambda(\hat{R}(z,a_l)),$$
$$\mu(z,\lambda) \to 1, \;\; z \in V_1, \;\; z \to \infty, \;\; \lambda \in \mathbb{C} \backslash E. \tag{4.3}$$

The relation (4.3) is equivalent to the system of equations

$$2 \sum_{l=1}^{g} c_l(\lambda) \mathrm{e}^{\lambda a_{j,1} - \bar{\lambda} \bar{a}_{j,1}} \frac{\bar{\omega}_k}{\mathrm{d}\bar{z}_1}(a_j) =$$
$$- \int\limits_{z \in X} \mathrm{e}^{\lambda z_1 - \bar{\lambda} \bar{z}_1} \left(\frac{dd^c \sqrt{\sigma}}{\sqrt{\sigma}} - 2i \partial \ln \sqrt{\sigma} \wedge \bar{\partial} \ln \sqrt{\sigma} \right) \mu(z,\lambda) \frac{\bar{\omega}_k}{\mathrm{d}\bar{z}_1}(z),$$

where $k = 1, \dots, g$ and $\{\omega_j\}$ is an orthonormal basis of holomorphic forms on $\tilde{V}$.

2. For all $\lambda \in \mathbb{C} \backslash E$, the restriction of $\mu = \mathrm{e}^{-\lambda z_1} \psi(z,\lambda)$ to bX can be found through Dirichlet-to-Neumann data for μ on bX by the Fredholm integral equation

$$\mu(z,\lambda)\big|_{bX} + \int\limits_{\xi \in bX} g_\lambda(z,\xi)(\bar{\partial}\mu(\xi,\lambda) - \bar{\partial}\mu_0(\xi,\lambda)) = 1 + i \sum_{j=1}^{g} c_j g_\lambda(z,a_j), \tag{4.4}$$

where

$$-i \sum_{j=1}^{g} (a_{j,1})^{-k} c_j = \int\limits_{z \in bX} z_1^{-k} (\partial + \lambda \mathrm{d}z_1) \mu = 0, \quad k = 2, \dots, g+1, \tag{4.5}$$

and μ_0 is the solution of the Dirichlet problem

$$\bar{\partial}(\partial + \lambda \mathrm{d}z_1)\mu_0\big|_X = 0, \;\; \mu_0\big|_{bX} = \mu\big|_{bX}.$$

The parameters $\{a_{j,1}\}$ (the first coordinates of $\{a_j\}$) are supposed to be distinct.

Equations (4.4) and (4.5) are solvable simultaneously with (4.2), (4.3).

The relations (4.5) are equivalent to the equality

$$\bar{\partial}(\partial+\lambda \mathrm{d}z_1)\mu\big|_{V\setminus X} = i\sum_{j=1}^{g} c_j\delta(z,a_j).$$

3. The Faddeev-type function $\mu = \psi(z,\lambda)\mathrm{e}^{-\lambda z_1}$ satisfies the Bers–Vekua-type $\bar{\partial}$-equation with respect to $\lambda \in \mathbb{C}\setminus E$:

$$\frac{\partial\mu(z,\lambda)}{\partial\bar{\lambda}} = b(\lambda)\bar{\mu}(z,\lambda)\mathrm{e}^{\bar{\lambda}\bar{z}_1-\lambda z_1}, \tag{4.6}$$

where

$$b(\lambda) \stackrel{\mathrm{def}}{=} \lim_{\substack{z\to\infty\\ z\in V_l}} \frac{\bar{z}_1}{\bar{\lambda}}\mathrm{e}^{\lambda z_1-\bar{\lambda}\bar{z}_1}\frac{\partial\mu}{\partial\bar{z}_1}(z,\lambda)\Big/\lim_{\substack{z\to\infty\\ z\in V_l}}\overline{\mu(z,\lambda)},$$

with $l=1,\dots,d$. The function $b(\lambda)$, referred to as nonphysical scattering data, can be found by (4.6) through $\mu\big|_{bX}$.

In addition, the following important formulas for the data $b(\lambda)$ are valid:

$$d\cdot\bar{\lambda}\cdot b(\lambda) = -\frac{1}{2\pi i}\int\limits_{z\in bY}\mathrm{e}^{\lambda z_1-\bar{\lambda}\bar{z}_1}\bar{\partial}\mu = \frac{1}{2\pi i}\int\limits_{z\in X}\frac{i}{2}\mathrm{e}^{\lambda z_1-\bar{\lambda}\bar{z}_1}q\mu + i\sum_{j=1}^{g}c_j\mathrm{e}^{\lambda a_{j,1}-\bar{\lambda}\bar{a}_{j,1}}, \tag{4.7}$$

where $\lambda\in\mathbb{C}\setminus E$.

On the basis of (4.3), (4.7), and Proposition 3, one can derive the estimate

$$|\lambda\cdot b(\lambda)| \le \mathrm{const}(V,\sigma)(1+|\lambda|)^{-g}|\Delta(\lambda)|^{-1},\ \ \lambda\in\mathbb{C}\setminus E. \tag{4.8}$$

4. Let us suppose now that the divisor $\{a_1,\dots,a_g\}$ on $Y\setminus X$ is such that the exceptional set E in $\mathbb{C}$ consists of isolated points $\lambda_1,\dots,\lambda_N$, $N\le\infty$, and

$$|\Delta(\lambda)| \ge \mathrm{const}(V)\mathrm{dist}(\lambda,E)\ \ \text{if}\ \ \mathrm{dist}(\lambda,E)\le\mathrm{const}. \tag{4.9}$$

Then the reconstruction procedure for $\mu\big|_{X\times\mathbb{C}}$ and $\sigma\big|_X$ through scattering data $b\big|_{\mathbb{C}}$ can be done in the following way.

The relations (4.2), (4.3), combined with the inequalities (4.8), (4.9), imply that the $\bar{\partial}$-equation (4.6) can be replaced by the singular integral equation

$$(\mu-1)+\frac{1}{2\pi i}\lim_{\delta\to 0}\int\limits_{\mathbb{C}\setminus\cup\{|\xi-\lambda_l|\le\delta\}} b(\xi)\mathrm{e}^{\bar{\xi}\bar{z}_1-\xi z_1}\overline{(\mu-1)}\frac{\mathrm{d}\bar{\xi}\wedge\mathrm{d}\xi}{\xi-\lambda}+\frac{1}{2\pi i}\sum_{l=1}^{N}\frac{\mu_l}{\lambda_l-\lambda}$$

$$= -\frac{1}{2\pi i}\int\limits_{\mathbb{C}} b(\xi)\mathrm{e}^{\bar{\xi}\bar{z}_1-\xi z_1}\frac{\mathrm{d}\bar{\xi}\wedge\mathrm{d}\xi}{\xi-\lambda}, \text{ where}$$

$$\mu_l = \lim_{\delta\to 0}\int\limits_{|\xi-\lambda_l|\le\delta} b\bar{\mu}\mathrm{e}^{\bar{\xi}\bar{z}_1-\xi z_1}\mathrm{d}\bar{\xi}\wedge\mathrm{d}\xi = \lim_{\delta\to 0}\int\limits_{|\xi-\lambda_l|=\delta}\mu d\xi = O_z(1),$$

$$l = 1,2,\dots,N,\ \ \lambda\in\mathbb{C}\backslash E. \tag{4.10}$$

This equation is of Fredholm–Noether type in the space of functions

$$\lambda\mapsto(\mu(\cdot,\lambda)-1):\ |\mu-1|\cdot|\Delta(\lambda)|(1+|\lambda|^g)\in L^{\tilde{p}}(\mathbb{C}),\ \ \tilde{p}>2.$$

In contrast to the planar case, when $d=1$, $g=0$, (4.10) does not necessarily have a unique solution. This makes it possible to find a basis of independent solutions of (4.10) for almost all $z_1\in\mathbb{C}$:

$$\lambda\mapsto\mu_k(z_1,\lambda),\ \ k=1,2,\dots,\tilde{d},\ \ \lambda\in\mathbb{C},\ \ \tilde{d}\ge d.$$

Put

$$\mu(z_1,z_2,\lambda)=\mu(z_1,z_{2,j}(z_1),\lambda)=\sum_{k=1}^{\tilde{d}}\gamma_{j,k}(z_1)\mu_k(z_1,\lambda),$$

where $(z_1,z_2)=(z_1,z_{2,j}(z_1))\in V$, $j=1,2,\dots,\tilde{d}$. The condition for the form $\mu^{-1}\bar{\partial}(\partial+\lambda\mathrm{d}z_1)\mu$ to be independent of λ allows us to find (maybe not uniquely) the coefficients $\gamma_{j,k}(z)$ in the expression for $\mu(z_1,z_2,\lambda)$. The equalities

$$\frac{i}{2}\frac{dd^c\sqrt{\sigma}}{\sqrt{\sigma}}\Big|_X = q\big|_X = \mu^{-1}\bar{\partial}(\partial+\lambda\mathrm{d}z_1)\mu\big|_X$$

finally permit us to find all q and σ with given scattering data $b\big|_{\mathbb{C}}$.

The uniqueness of the reconstruction of $\mu\big|_{X\times\mathbb{C}}$ and $\sigma\big|_X$ from the data b on $\mathbb{C}\setminus E$ is plausible but still unknown. Nevertheless, the uniqueness of the reconstruction of $\sigma\big|_X$ from Dirichlet-to-Neumann data of the equation $d(\sigma d^c U)\big|_X=0$ can be proved using Dirichlet-to-Neumann data not just for a single function, but for a family of Faddeev-type functions depending on a parameter θ:

$$\psi_\theta(z,\lambda)=\mathrm{e}^{\lambda(z_1+\theta z_2)}\mu_\theta(z_1,z_2,\lambda),\ \ \text{where}$$

$$\bar{\partial}(\partial+\lambda(\mathrm{d}z_1+\theta\mathrm{d}z_2))\mu_\theta=\frac{i}{2}q\mu_\theta+i\sum_{l=1}^{g}c_l\delta(z,a_l)\ \text{ and }\ \lim_{\substack{z\to\infty\\ z\in V_1}}\mu_\theta(z,\lambda)=1,$$

$$(\mu_\theta-\mu_\theta(\infty_j))\big|_{V_j}\in L^{\tilde{p}}(V_j),\ \ \tilde{p}>2,\ \ \lambda\in\mathbb{C}\backslash E_\theta,\ \ j=1,\dots,d;$$

see [HN].

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