On Symplectic Caps

David T. Gay and András I. Stipsicz

We dedicate this paper to Oleg Viro on the occasion of his 60th birthday

Abstract An important class of contact 3-manifolds comprises those that arise as links of rational surface singularities with reduced fundamental cycle. We explicitly describe symplectic caps (concave fillings) of such contact 3-manifolds. As an application, we present a new obstruction for such singularities to admit rational homology disk smoothings.

Keywords Three-manifold • Symplectic cap • Handle attachment • Link • Open book

1 Introduction

Our understanding of topological properties of (weak) symplectic fillings of certain contact 3-manifolds has improved dramatically in the recent past. These developments have rested on recent results in symplectic topology, most notably on McDuff's characterization of (closed) rational symplectic 4-manifolds [14]. In order to apply results of McDuff, however, *symplectic caps* were needed to close up the fillings at hand. General results of Eliashberg and Etnyre [5,6] showed that such caps do exist in general, but these results can be used powerfully only when a detailed description of the cap is also available. This was the case, for example, for lens spaces with their standard contact structures [12] and for certain 3-manifolds that can be given as links of isolated surface singularities [2, 3, 16].

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In the following we will show an explicit construction of symplectic caps for contact 3-manifolds that can be given as links (with their Milnor fillable structures) of rational singularities with reduced fundamental cycle. In topological terms, this means that the 3-manifold can be given as a plumbing of spheres along a negative definite tree, with the additional assumption that the absolute value of the framing at each vertex is at least the valency of the vertex. The construction of the cap in this case relies on a symplectic handle attachment along a component of the binding of a compatible open-book decomposition. In the terminology of open-book decompositions, our construction coincides with the cap-off procedure initiated and further studied by Baldwin [1].

The success of the rational blowdown procedure (initiated by Fintushel and Stern [8] and then extended by J. Park [17]) led to the search for isolated surface singularities that admit rational homology disk smoothings. Strong restrictions on the combinatorics of the resolution graph of such a singularity were found in [20], and by identifying Neumann's $\overline{\mu}$-invariant with a Heegaard Floer-theoretic invariant of the underlying 3-manifold, further obstructions to the existence of such a smoothing were given in [19]. More recently, the question was answered in [3] for all singularities with star-shaped resolution graphs (in particular, for weighted homogeneous singularities), but the general problem remained open. Motivated by our construction of a symplectic cap for special types of Milnor fillable contact 3-manifolds, we show examples of surface singularities that pass all tests provided by [19, 20] but still do not admit rational homology disk smoothings.

The paper is organized as follows. In Sect. 2, we describe the symplectic handle attachment that caps off a boundary component of a compatible open-book decomposition. Section 3 is devoted to a detailed description of the topology of the symplectic cap, and also an example is worked out. In Sect. 4, we show that certain singularities do not admit rational homology disk smoothings.

2 Symplectic Handle Attachments

Throughout this section, suppose that (Y,ξ) is a strongly convex boundary component of a symplectic 4-manifold (X,ω); that ξ is supported by an open-book decomposition with oriented page Σ, oriented binding $B=\partial\Sigma$, and monodromy h; and that L is a sublink of B. For each component K of L, let $\mathrm{pf(K)}$ denote the page-framing of K, the framing induced by the page Σ. Note that if Y_L is the result of performing surgery on Y along each component K of L with framing $\mathrm{pf(K)}$, and if $L \neq B$, then the open book on Y induces a natural open book on Y_L with page Σ_L equal to $\Sigma \cup_L (\sqcup^{|L|} D^2)$, the result of capping off each K with a disk, and with monodromy equal to h extended by the identity on the D^2 caps. (In [1] this construction has been examined from the Heegaard Floer-theoretic point of view.)

If instead, for $|L| = 1$ and $L = K$, Y_K is obtained by surgery along K with framing $\mathrm{pf(K)} \pm 1$, then the open book on Y induces a natural open book on Y_K with page $\Sigma_K = \Sigma$ and with monodromy $h_K = h \circ \tau_K^{\mp 1}$, where τ_K is a right-handed Dehn twist along a circle in the interior of Σ parallel to K. In fact, if $K \neq B$, then surgery with framing $\mathrm{pf(K)} - 1$ coincides with Legendrian surgery along a Legendrian realization of K on the page; hence the 4-dimensional cobordism resulting from the construction supports a symplectic structure. In the following two theorems we extend the existence of such a symplectic structure to the cases in which the surgery coefficients are $\mathrm{pf(K)}$ and $\mathrm{pf(K)} + 1$.

In the first case, in which the surgery coefficient is $\mathrm{pf(K)}$, we have a rather technical extra condition in terms of the existence of a closed 1-form with certain behavior near K. Later, we will state one case in which this condition is always satisfied, but for the moment we leave it technical because the theorem is most general that way. When we discuss the behavior of anything near a component K of B, we always use oriented coordinates (r, μ, λ) near K such that $\mu, \lambda \in S^1$ are the meridional and longitudinal coordinates, respectively, chosen to represent the page framing. In other words, $\mu^{-1}(\theta)$, for any $\theta \in S^1$, is the intersection of a page with this coordinate neighborhood, and the closure of $\lambda^{-1}(\theta)$ is a meridional disk. Also, we assume that ∂_λ points in the direction of the orientation of K, oriented with the boundary of the page.

Theorem 2.1. *Suppose that L is a sublink of B, not equal to B, and that $X_L \supset X$ is the result of attaching a 2-handle to X along each component K of L with framing* $\mathrm{pf(K)}$. *Suppose furthermore that there exists a closed 1-form α_0 defined on $Y \setminus L$ that near each component K of L, has the form $m_K d\mu + l_K d\lambda$ for some constants m_K and l_K, with $l_K > 0$. (The coordinates (r, μ, λ) near K are as described in the preceding paragraph.) Then ω extends to a symplectic form ω_L on X_L, and the new boundary Y_L is ω_L-convex. The new contact structure ξ_L is supported by the natural open book on Y_L described above.*

Proof. Let $\pi\colon Y \setminus B \to S^1$ be the fibration associated with our given open book on Y, and let $\pi_L\colon Y_L \setminus (B \setminus L) \to S^1$ be the fibration for the induced open book on Y_L. Let Z be $[-1, 0] \times Y$ together with the 2-handles attached along $\{0\} \times L \subset \{0\} \times Y$, and identify Y with $\{0\} \times Y$. Thus Z is a cobordism from $\{-1\} \times Y$ to Y_L, and $Y \cap Y_L$ is nonempty and is in fact the complement of a neighborhood of L in Y. We will show that there is a symplectic structure η on Z that on $[-1, 0] \times Y$, is equal to the symplectization of a certain contact form α on Y supported by (B, π) and such that Y_L is η-convex, with induced contact structure ξ_L supported by the natural open book $(B \setminus L, \pi_L)$ on Y_L described above. This proves the theorem.

As mentioned above, for each component K of L we use coordinates (r, μ, λ) on a neighborhood $\nu \cong D^2 \times S^1$ of K, with (r, μ) being polar coordinates on the D^2-factor and λ being the S^1-coordinate, in such a way that $\mu = \pi|_\nu$. Thus the pages are the level sets for μ. We will also add now the convention that r is always parameterized so as to take values in $[0, 1 + \epsilon]$ for some small positive ϵ.

Let ν' be the corresponding neighborhood in Y_L of the belt-sphere for the 2-handle H_K that is attached along K, with corresponding coordinates (r', μ', λ'),

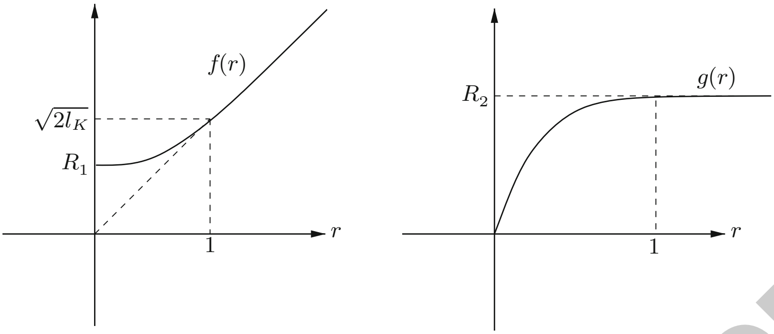


Fig. 1 Graphs of the functions f and g

with the natural diffeomorphism $\nu \setminus \{r=0\} \to \nu' \setminus \{r'=0\}$ given by $r'=r$, $\mu' = -\lambda$, and $\lambda' = \mu$. Note that $\pi_L|_{\nu'} = \lambda'$, which is defined on all of ν'.

There are, of course, many different contact structures supported by the given open book on Y, but they are all isotopic, and up to isotopy, we can always assume that ξ has the following behavior in each neighborhood ν of each component K of L:

1. ξ is (μ,λ)-invariant. That is, there exist functions $F(r)$ and $G(r)$ such that ξ is spanned by ∂_r and $F(r)\partial_\mu + G(r)\partial_\lambda$. We necessarily have $G(0)=0$, and we will adopt the convention that $F(0)>0$, so that $G'(0)<0$ and thus $G(r)<0$ for r close to 0.
2. As r ranges from 0 to 1, ξ makes a full quarter-turn in the (μ,λ)-plane. In other words, the vector $(F(r),G(r)) \in \mathbb{R}^2$ goes from $F(0)>0, G(0)=0$ to $F(1)=0$, $G(1)<0$, with $F(r)>0$ and $G(r)<0$ for all $r \in (0,1)$. (We can make this assumption precisely because $L \neq B$. One way to see this is to think of the construction of a contact structure supported by a given open book as beginning with a Weinstein structure on the page. This Weinstein structure comes from a handle decomposition of the page, and if we choose a handle decomposition starting with collar neighborhoods of the components of L and then adding 1-handles, we will get the desired behavior.)

So now we assume that ξ has the form above.

Next we claim that we can find a contact form α for this ξ satisfying certain special properties. To understand the local properties of α near each K, consider Fig. 1.

This figure shows graphs of two functions f and g, specified by constants R_1, l_K, and R_2. The properties of f and g are as follows:

1. The function f is monotone increasing with $f'(0)=0$ and $f'(r)>0$ for $r>0$.
2. $f(0)=R_1$ and $f(r)=\sqrt{2l_K}r$ for $r \geq 1$. (Hence $\sqrt{2l_K} > R_1$.)
3. $g(0)=0$.
4. The function g is monotone increasing with $g'(r)>0$ on $[0,1)$.
5. $g(r)=R_2$ for $r \geq 1$.

The claim, then, is that there exists a contact form α for ξ such that the following conditions hold:

1. The 1-form $\alpha - \alpha_0$ is a positive contact form on the complement of the neighborhoods of radius $r \leq 1$ of each component K of L, and it also satisfies the support condition for the given open book outside these neighborhoods.
2. For each component K of L there are constants R_1 and R_2 and associated functions f and g, as in Fig. 1 (with the constant l_K coming from $\alpha_0 = m_K \mathrm{d}\mu + l_K \mathrm{d}\lambda$), with $\frac{1}{2}R_2^2 > m_K$ such that in the neighborhood ν of K, α has the form

$$\alpha = \frac{1}{2}g(r)^2 \mathrm{d}\mu + \left(l_K - \frac{1}{2}f(r)^2\right)\mathrm{d}\lambda.$$

   (We might need to reparameterize the coordinate r, but only via a reparameterization fixing 0 and 1.)

The condition $\frac{1}{2}R_2^2 > m_K$ is necessary to guarantee that $\alpha - \alpha_0$ is positive contact when $r \geq 1$; this condition will also be used later.

To verify this claim, first choose any contact form α' for ξ satisfying the support condition for the given open book. Now note that for any suitably large constant $k > 0$, $k\alpha' - \alpha_0$ is a positive contact form satisfying the support condition. We know that in ν, $k\alpha' = -G(r)\mathrm{d}\mu + F(r)\mathrm{d}\lambda$ for functions $F(r), G(r)$ such that the vector $(F(r), G(r))$ makes one quarter-turn through the fourth quadrant as r goes from 0 to 1. Because k is large, we may assume that $G(1) < -m_K$. We can then scale $k\alpha'$ by a positive function $\phi(r)$ supported inside $r \leq 1 + \epsilon$ so as to arrange that the pair of functions $(\tilde{F}(r) = \phi(r)F(r), \tilde{G}(r) = \phi(r)G(r))$ has the appropriate shape, and then we let $\frac{1}{2}g(r)^2 = -\tilde{G}(r)$ and $l_K - \frac{1}{2}f(r)^2 = \tilde{F}(r)$. Then we have $\alpha = \phi(r)k\alpha'$.

Now embed ν and ν' in $\mathbb{R}^4$ as follows, using polar coordinates $(r_1, \theta_1, r_2, \theta_2)$ on $\mathbb{R}^4$. The embedding of ν is given by $(r_1 = f(r), \theta_1 = -\lambda, r_2 = g(r), \theta_2 = \mu)$. The embedding of ν' is given by $(r_1 = \sqrt{2l_K}r', \theta_1 = \mu', r_2 = R_2, \theta_2 = \lambda')$. This is illustrated in Fig. 2, which also shows that the region between ν and ν' is precisely our 2-handle H attached along K with framing pf(K). The overlap $\nu \cap \nu'$ is the set $\{r_1 \geq \sqrt{2l_K}, r_2 = R_2\}$, which in ν-coordinates is $\{r \geq 1\}$ and in ν'-coordinates is $\{r' \geq 1\}$.

Consider the standard symplectic form $\omega_0 = r_1 \mathrm{d}r_1 \mathrm{d}\theta_1 + r_2 \mathrm{d}r_2 \mathrm{d}\theta_2$ on $\mathbb{R}^4$. Note that $\omega_0|_\nu = gg'\mathrm{d}r\mathrm{d}\mu - ff'\mathrm{d}r\mathrm{d}\lambda = \mathrm{d}\alpha$, so that H equipped with this symplectic form can be glued symplectically to $[-1, 0] \times Y$ with the symplectization of α. Next note that $\omega_0|_{\nu'} = 2l_K r'\mathrm{d}r'\mathrm{d}\mu' = \mathrm{d}\alpha'$, where $\alpha' = \frac{1}{2}(\sqrt{2l_K}r')^2\mathrm{d}\mu' + (\frac{1}{2}R_2^2 - m_K)\mathrm{d}\lambda'$. (Here we see that $\frac{1}{2}R_2^2 > m_K$ is necessary for α' to be a positive contact form and to be supported by the open book inside this neighborhood ν'.) On the overlap $\nu \cap \nu' \subset \mathbb{R}^4$, using the coordinates (r, μ, λ) from ν, we see that $\alpha' = \left(\frac{1}{2}R_2^2 - m_K\right)\mathrm{d}\mu + \left(-\frac{1}{2}\left(\sqrt{2l_K}r\right)^2\right)\mathrm{d}\lambda = \alpha - \alpha_0$. Thus we see that α' extends to the rest of Y_L as $\alpha - \alpha_0$, concluding the proof of the theorem. □

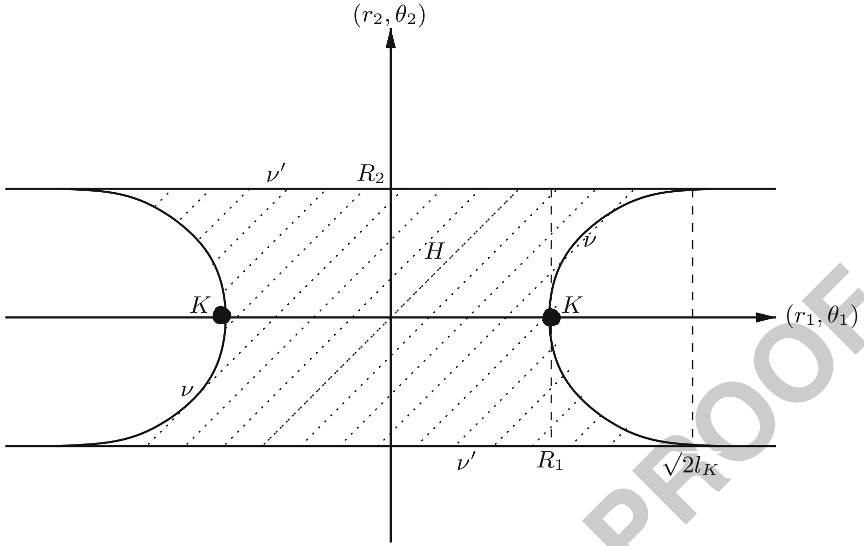


Fig. 2 Embeddings of ν, ν', and H into $\mathbb{R}^4$

In fact, 2-handles can be attached with framing $\mathrm{pf(K)}+1$ to boundary components of a compatible open book, and the symplectic structure will still extend. In this case, however, the convex boundary will become concave. More precisely, we have the following theorem.

Theorem 2.2. *Suppose that $K = B$ and that $X_K \supset X$ is the result of attaching a 2-handle H to X along K with framing* $\mathrm{pf(K)}+1$. *Then ω extends to a symplectic form ω_K on X_K, and the new boundary Y_K is ω_K-concave. The new (negative) contact structure ξ_K is supported by the natural open book on Y_K described above.*

Proof. This is [9, Theorem 1.2]. However in that paper, which predates Giroux's work on open-book decompositions, the terminology is slightly different. [9, Definition 2.4] defines what it means for a transverse link L in a contact 3-manifold (M,ξ) to be "nicely fibered." It is easy to see that if L is the binding of an open book supporting ξ, then L is nicely fibered. (The notion of "nicely fibered" is more general because, in open-book language, it allows for "pages" whose boundaries multiply cover the binding.) Theorem 1.2 in [9] then says that if we attach 2-handles to all the components of a nicely fibered link in the strongly convex boundary of a symplectic 4-manifold, with framings that are more positive than the framings coming from the fibration, then the symplectic form extends across the 2-handles to make the new boundary strongly concave. In our case, we have a single component, and we are attaching with framing exactly one more than the framing coming from the fibration. Finally, [9, Addendum 5.1] characterizes the negative contact structure induced on the new boundary as follows: There exists a constant k such that $\alpha_K = k\mathrm{d}\pi - \alpha$ on the complement of the surgery knots.

(Here we are identifying $Y \setminus K$ with the complement in Y_K of the belt sphere for H in the obvious way.) The constant k is simply the appropriate constant such that α_K extends to all of Y_K. Then $\mathrm{d}\pi \wedge \mathrm{d}\alpha_K = -\mathrm{d}\pi \wedge \mathrm{d}\alpha$, which is positive on $-Y_K$. Since $\mathrm{d}\alpha_K = -\mathrm{d}\alpha$, and the Reeb vector field for α is tangent to the level sets for the radial function r on a neighborhood of K (see [9, Definition 2.4]), the Reeb vector field for α_K is necessarily tangent to the new binding of Y_K, and it is not hard to check that it points in the correct direction, so that α_K is supported by the natural open book on Y_K. □

We have the following application. (For a similar result, see [22, Theorem 4'].)

Corollary 2.3. *If the open book on Y is planar (i.e.,* $\mathrm{genus}(\Sigma) = 0$*), then (X, ω) embeds in a closed symplectic 4-manifold (Z, η) that contains a symplectic $(+1)$-sphere disjoint from X.*

As preparation we need the following lemma:

Lemma 2.4. *Let B be the (disconnected) binding of a planar open book on Y, and let $L \subset B$ be the complement of a single component of B. Then there exists a 1-form α_0 on $Y \setminus L$ such that near each component K of L, α_0 has the form $\alpha_0 = m_K \mathrm{d}\mu + l_K \mathrm{d}\lambda$, for $l_K > 0$. (The coordinates near K are as in Theorem 2.1 and are determined by the open book.)*

Proof. Let Y_L be the result of page-framed surgery on L, with the corresponding oriented link $L' \subset Y_L$ (the cores of the surgeries). Note that $Y_L \cong S^3$, because the induced open book on Y_L has disk pages. Thus L' is an oriented link in S^3, and there exists a map $\sigma\colon S^3 \setminus L' \to S^1$ with the closure of each $\sigma^{-1}(\theta)$, for each regular value θ, an oriented Seifert surface for L'. Pull σ back to $Y \setminus L = Y_L \setminus L'$ and let $\alpha_0 = d\sigma$. □

Proof (of Corollary 2.3). Let the components of B be $K_1, \ldots, K_n$. Attach 2-handles to $K_1, \ldots, K_{n-1}$ with framings $\mathrm{pf(K_i)}$, as in Theorem 2.1. This gives $(X', \omega') \supset (X, \omega)$ with ω'-convex boundary (Y', ξ'). Now attach a 2-handle to K_n with framing $\mathrm{pf(K_n)} + 1$ as in Theorem 2.2; the resulting concave end is S^3 with its negative contact structure supported by the standard disk open book, i.e., the contact structure is the standard negative tight contact structure. Thus we can fill in the concave end with the standard symplectic structure on B^4. Alternatively, we can note that on Y', the positive contact structure ξ' is supported by an open book with page diffeomorphic to a disk. In other words, Y' is diffeomorphic to S^3, and ξ' is the standard positive tight contact structure on S^3. Thus we can remove a standard (B^4, ω_0) from $\mathbb{CP}^2$ with its standard Kähler form, and replace (B^4, ω_0) with (X', ω') to get (Z, η). Since there is a symplectic $(+1)$-sphere in $\mathbb{CP}^2$ disjoint from B^4, we end up with a symplectic $(+1)$-sphere in (Z, η) disjoint from X', and hence disjoint from X. □

By [14], the symplectic 4-manifold Z found in the proof of Corollary 2.3 is diffeomorphic to a blowup of $\mathbb{CP}^2$. Let Z' be the result of anti–blowing down the symplectic $(+1)$-sphere in Z (i.e., Z' is the union of the 4-manifold X' in the proof of the corollary above with B^4). Then Z' (still containing X) is diffeomorphic to the connected sum of a number of copies of $\overline{\mathbb{CP}^2}$. Let D be the closure of $Z' \setminus X$ in Z'; we will call this the *dual configuration* (or *compactification*) for X. Thus we get embeddings of the intersection forms $H_2(X;\mathbb{Z})$ and $H_2(D;\mathbb{Z})$ into a negative definite diagonal lattice, and therefore both $H_2(X;\mathbb{Z})$ and $H_2(D;\mathbb{Z})$ are negative definite.

Remark 2.5. A very similar compactification has been found by Némethi and Popescu-Pampu in [15], using rather different methods.

3 Examples: Rational Surface Singularities with Reduced Fundamental Cycle

Suppose that Γ is a plumbing tree of spheres that is negative definite, and at each vertex the absolute value of the framing is at least the number of edges emanating from the vertex. Every negative definite plumbing graph Γ gives rise to a (not necessarily unique) surface singularity, and the further assumptions on Γ ensure that the singularity has reduced fundamental cycle. According to Laufer's algorithm, for example, this property implies that the singularity is rational; cf. [19, Sect. 3]. The Milnor fillable contact structure on such a 3-manifold is known to be compatible with a planar open-book decomposition [7, 18]. A fairly explicit description of such an open-book decomposition can be given by a construction resting on results of [10]. By [10, Proposition 5.3], the Milnor fillable contact structure is compatible with an open-book decomposition resting on a toric construction (cf. [10, Sect. 4]), and therefore by [10, Proposition 4.2], a compatible planar open book can be explicitly given as follows.

View the tree Γ as a planar graph in $\mathbb{R}^2$ and consider the boundary sphere of an ϵ neighborhood of it in $\mathbb{R}^3$. Suppose that v is a vertex of Γ with framing e_v and valency d_v. Then near v, drill $-e_v - d_v \geq 0$ holes on the sphere. The resulting planar surface will be the page of the open-book decomposition. Consider a parallel circle to each boundary component and further curves near each edge, as shown by the example of Fig. 3. The monodromy of the open-book decomposition is simply the product of the right-handed Dehn twists defined by all these curves on the planar surface.

Consider now the Kirby diagram for Y based on the open-book decomposition as follows: Regard the planar page as a multipunctured disk. (This step involves

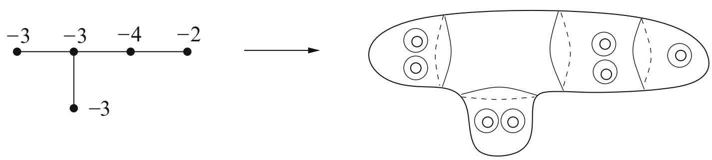


Fig. 3 Light circles on the punctured sphere define the monodromy of the open book

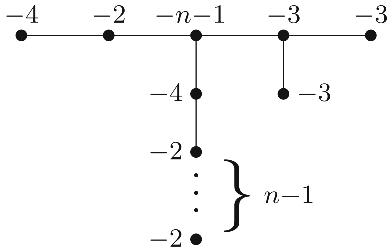
a choice of an "outer circle.") Every hole on the disk defines a 0-framed unknot linking the boundary of the hole, while the light circles defining the monodromy through right-handed Dehn twists give rise to (-1)-framed unknots. In fact, the 0-framed unknots can be turned into dotted circles and then viewed as 4-dimensional 1-handles (for these notions of Kirby calculus, see [11]). These will build up a Lefschetz fibration with fiber diffeomorphic to the page of the open book, and the addition of the (-1)-framed circles corresponds to the vanishing cycles of the Lefschetz fibration, giving the right monodromy.

Having this Kirby diagram for Y, a relative handlebody diagram for the dual configuration D (built on $-Y$) can be easily deduced by performing 0-surgery along all the boundary circles except the outer one. This operation corresponds to capping off all but the last boundary component of the open book defining the Milnor fillable structure on Y. Since after all the capping off we get an open book with a disk as a page, the 4-manifold D is a cobordism from $-Y$ to S^3.

It is usually more convenient to have an absolute handlebody than a relative one, and since the other boundary component of D is S^3, by turning D upside down we can easily derive a handlebody description first for $-D$ and then, after the reversal of the orientation, for D. After appropriate handleslides, in fact, the diagram for D can be given by a simple algorithm. Since we dualize only 2-handles, D can be given by attaching 2-handles to D^4. The framed link can be given by a braid, which is derived from the plumbing tree by the following inductive procedure. To start, we choose a vertex v where the strict inequality $-e_v - d_v > 0$ holds. (Such a vertex always exists; for example, we can take a leaf.) We will choose the outer circle to be the boundary of one of the holes near v. Now associate to every inner boundary component a string and to every light circle a box symbolizing a full negative twist of the strings passing through the box, which in our case consists of those strings that correspond to the boundary components encircled by the light circle. The framing on a string is given by the negative of the "distance" of the boundary component from the outer circle: this distance is simply the number of light circles we have to cross when traveling from the boundary component to the outer circle. Another (obviously equivalent) way of describing the same braid purely in terms of the graph Γ goes as follows: Choose again a vertex v with $-e_v - d_v > 0$, and consider $-e_u - d_u$ strings for each vertex u, except for v, for which we take only $-e_v - d_v - 1$ strings. Introduce a full negative twist on the resulting trivial braid (corresponding to the light circle parallel to the outer circle), and then introduce a further full negative twist for every edge e in the graph, where the strings affected by the negative twist can be characterized by the property that they correspond to vertices that are in a component of $\Gamma - \{e\}$ not containing the distinguished vertex v. Finally, equip every string corresponding to a vertex u with $r_{uv} - 2$, where r_{uv} is the negative of the minimal number of edges we traverse when passing from u to v.

We will demonstrate this procedure through an explicit family of examples. (For a similar result see [21, Theorem 3].) To this end, suppose that the graph Γ_n is given by Fig. 4. It is easy to see that the graphs in the family for $n \geq 1$ are all negative definite, and for $n \geq 2$, they define a rational singularity with reduced fundamental cycle. Assume that $n \geq 3$ and choose a boundary circle near the $(-n-1)$-framed

Fig. 4 An interesting family of plumbing graphs.



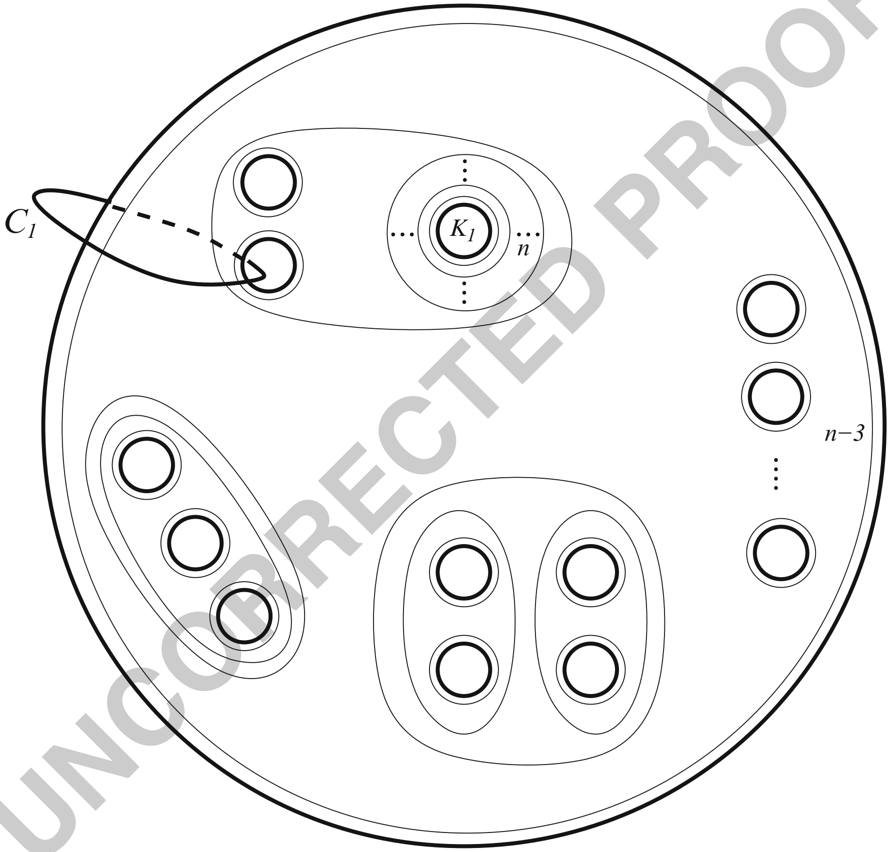


Fig. 5 The light circles on the disk define the monodromy of the open book. There are n concentric light circles around the boundary component labeled by K, and there are $n-3$ boundary circles on the right-hand side of the disk. For each of the interior boundary components there should be a corresponding unknot C_i linking it and the exterior boundary component; here we have drawn only C_1

vertex to be the outer circle. The page of the planar open book, together with the light circles (giving rise to the monodromy through right-handed Dehn twists), is pictured by Fig. 5 (with the circle C_1 disregarded for a moment).

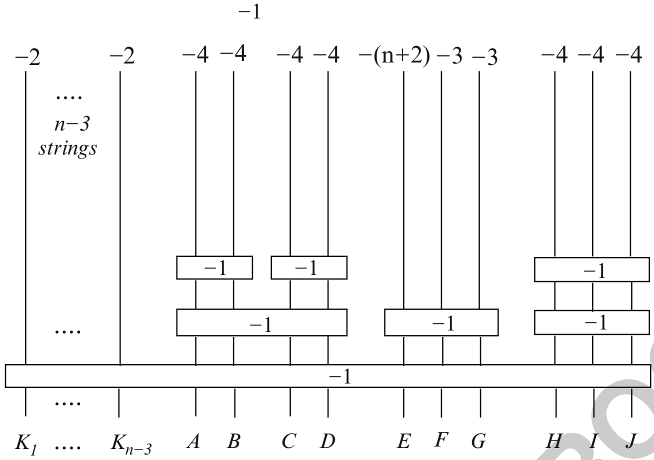


Fig. 6 Boxes in the diagram mean full negative twists

The 0-framed unknots originating from the 1-handles of the Lefschetz fibration become unknots each of which links one of the interior boundary components of the punctured disk once and the exterior boundary once. In the diagram, the unknot labeled C_1 is one of these unknots; we have not drawn the rest because they would only complicate the picture needlessly, but it is important to keep in mind that there is one such unknot for each interior boundary. Putting (-1)-framings to all light circles we get a convenient description of Y. Now add framing 0 to all boundary components except the outer one. The result is a cobordism D from $-Y$ to S^3. Mark all these circles (for example, use the convention of [11] by replacing all framing a with $\langle a \rangle$) and turn D upside down: add 0-framed meridians to the circles corresponding to the boundary components of the open book (these are the curves along which we "capped off" the open book). Now sliding and blowing down marked curves only, we end up with the diagram of $-D$, and by reversing all crossings and multiplying all framings by (-1), eventually we get a Kirby diagram for D, as shown in Fig. 6. (Every box in the diagram means a full negative twist.)

4 The Nonexistence of Rational Homology Disk Smoothings

Next we will demonstrate how the explicit topological description of the dual D can be applied to study smoothings of surface singularities. We start with a simple observation providing an obstruction for a 3-manifold to bound a rational homology disk, i.e., a 4-manifold V with $H_*(V;\mathbb{Q}) = H_*(D^4;\mathbb{Q})$.

Theorem 4.1. *Suppose that the rational homology 3-sphere $-Y$ is the boundary of a compact 4-manifold D with the property that* $\mathrm{rk} H_2(D;\mathbb{Z}) = n$ *and that the* *intersection form $(H_2(D;\mathbb{Z}), Q_D)$ does not embed into the negative definite diagonal lattice $n\langle -1\rangle$ of the same rank. Then Y cannot bound a rational homology disk.*

Proof. Suppose that such a rational homology disk V exists; then $Z = V \cup_Y D$ is a closed, negative definite 4-manifold. By Donaldson's theorem [4], the intersection form of Z is diagonalizable over $\mathbb{Z}$, and by our assumption on V, we get that $\mathrm{rk} H_2(Z;\mathbb{Z}) = \mathrm{rk} H_2(D;\mathbb{Z}) = n$. Since $H_2(D;\mathbb{Z}) \subset H_2(Z;\mathbb{Z})$ does not embed into $n\langle -1\rangle$, we get a contradiction, implying the result. □

Consider now the plumbing graph Γ_n of Fig. 4, and denote the corresponding 3-manifold by Y_n.

Proposition 4.2. *The 3-manifold Y_n does not bound a rational homology disk 4-manifold for $n \geq 7$.*

Remark 4.3. Notice that elements of this family pass all the tests provided by [20], since these graphs are elements of the family $\mathcal{A}$ of [20]: change the framing of the single (-4)-framed vertex with valency two to (-1) and blow down the graph until it becomes the defining graph of the family $\mathcal{A}$. Also, using the algorithm described, e.g., in [19], it is easy to see that $\det \Gamma_n \equiv n \bmod 2$; hence for odd n, the 3-manifold Y_n admits a unique spin structure. The corresponding Wu class can be given by the (-3)-framed vertex of valency three, the unique (-4)-framed vertex on the long chain, and then every second (-2)-framed vertex. A simple count then shows that for n odd, we have $\overline{\mu}(Y_n) = 0$, and hence the result of [19] provides no obstruction to a rational homology disk smoothing. (For the terminology used in the above argument, see [19].)

Proposition 4.4. *The lattice determined by the intersection form of the dual D_n given by Fig. 6, for $n \geq 7$, does not embed into the same rank negative definite diagonal lattice.*

Proof. The labels on the components of the braid in Fig. 6 will be used to represent the corresponding basis elements for the lattice determined by the intersection form of D_n. The rank is $n+7$. Let $E = \{e_1, \ldots, e_{n+7}\}$ be the standard basis for the negative definite diagonal lattice of rank $n+7$, so $e_i \cdot e_j = -\delta_{ij}$. Suppose that the lattice for D_n does embed into the definite diagonal lattice. Then without loss of generality, since $K_i \cdot K_i = -2$ and $K_i \cdot K_j = -1$ otherwise, we may assume that $K_i = e_1 + e_{10+i}$. Furthermore, without loss of generality we may assume that every other one of the basis elements $A, B, \ldots, J$ is of the form $e_1 + x$ where x is an expression in $e_2, \ldots, e_{10}$. Thus each basis element whose square is -3 (i.e., F and G) must be of the form $e_1 \pm u \pm v$, where u and v are distinct elements of the set $\{e_2, \ldots, e_{10}\}$. Each element whose square is -4 (i.e., A, B, C, D, H, I, and J) must be of the form $e_1 \pm q \pm r \pm s$, where q, r, and s are distinct elements of the set $\{e_2, \ldots, e_{10}\}$.

Now we can assume that $F = e_1 + e_2 + e_3$ and $G = e_1 + e_2 + e_4$ (noting that $F \cdot G = -2$). Then we note that none of the expressions for A, B, C, D, H, I, J can contain e_2, e_3, or e_4 for the following reason: For each of $X = A, B, C, D, H, I, J$ there is another basis element Y from this set such that $X \cdot Y = -3$, while $X \cdot X = Y \cdot Y = -4$. Thus if we write $X = e_1 + \alpha a + \beta b + \gamma c$ with $a, b, c \in E$ and $\alpha, \beta, \gamma \in \{-1, 1\}$, then Y must be $Y = e_1 + \alpha a + \beta b + \delta d$, with $d \in E$ and $\delta \in \{-1, 1\}$, where a, b, c, and d are distinct elements from the set $\{e_2, \dots, e_{10}\}$. Now noting that $X \cdot F = X \cdot G = Y \cdot F = Y \cdot G = -1$, we see that if $a = e_2$, then b, c, d must be in $\{e_3, e_4\}$, which cannot happen, because b, c, and d must be distinct. Similarly, b cannot be e_2. If $a = e_3$, then b or c must be e_2, but we have just seen that it cannot be b, so $c = e_2$. But the same argument also shows that $d = e_2$, but $c \neq d$. Similarly, we can rule out $a = e_4$ and also $b = e_3$ and $b = e_4$. But if one of c and d is in the set $\{e_2, e_3, e_4\}$, then one of a and b must also be, so finally we see that none of them can be.

Thus we can now take $H = e_1 + e_5 + e_6 + e_7$. There are then two possibilities for I and J (up to relabeling the members of the sets $\{e_8, e_9, e_{10}\}$ and $\{e_5, e_6, e_7\}$).

Case I: $I = e_1 + e_5 + e_6 + e_8$ and $J = e_1 + e_5 + e_6 + e_9$. In this case, we can see that A, B, C, D cannot contain e_7, e_8, or e_9. So then the only remaining possibilities are all equivalent (after changing signs of basis elements in E) to $A = e_1 + e_5 - e_6 + e_{10}$, but then we cannot find any candidates for B that give $A \cdot B = -3$. This rules out Case I.

Case II: $I = e_1 + e_5 + e_7 + e_8$ and $J = e_1 + e_5 + e_6 + e_8$. To rule out this case, write $A = e_1 + \alpha a + \beta b + \gamma c$, $a, b, c \in \{e_5, e_6, e_7, e_8, e_9, e_{10}\}$, and $\alpha, \beta, \gamma \in \{-1, 1\}$. In order to have $A \cdot H = -1$, either none or two of a, b, c must be in the set $\{e_5, e_6, e_7\}$, but not one or three of them. Similarly, using $A \cdot I = -1$, either none or two must be in $\{e_5, e_7, e_8\}$, and using $A \cdot J = -1$, either none or two must be in $\{e_5, e_6, e_8\}$. If it is none in one of these cases, it must be none for all three, but that leaves only e_9 and e_{10} for a, b, and c, an impossibility. Thus it is two in each case. We cannot have one of them e_5, because then we could not have exactly two from all three sets. So we must have $a = e_6$, $b = e_7$, $c = e_8$. But exactly the same argument holds for B, and we can never get $A \cdot B = -3$. Thus Case II is ruled out, concluding the proof of the proposition. □

Proof (of Proposition 4.2). Combine Theorem 4.1 and Proposition 4.4. □

Corollary 4.5. *Suppose that* $(S_\Gamma, 0)$ *is an isolated surface singularity with resolution graph given by Fig. 4. If* $n \geq 7$, *then* $(S_\Gamma, 0)$ *admits no rational homology disk smoothing, i.e., it has no smoothing* V *with* $H_*(V; \mathbb{Z}) = H_*(D^4; \mathbb{Z})$. □

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