# Topology of Spaces of $S$-Immersions

**Y.M. Eliashberg and N.M. Mishachev**

*To Oleg Viro of his 60th birthday.*

**Abstract** We use the wrinkling theorem proven in [EM97] to fully describe the homotopy type of the space of $S$-immersions, i.e., equidimensional folded maps with prescribed folds.

**Keywords** S-immersion • Wrinkling • Zigzag • Folded mapping

## 1 Equidimensional Folded Maps

Let $V$ and $W$ be two manifolds of dimension $q$. The manifold $V$ will always be assumed *closed* and *connected*. A map $f : V \to W$ is called *folded* if it has only fold-type singularities. We will discuss in this paper an $h$-principle for folded maps $f : V \to W$ with a *prescribed fold* $\Sigma^{10}(f) \subset V$. Folded maps with the fold $S = \Sigma^{1,0}(f)$ are also called *$S$-immersions*; see [El70]. Throughout this paper we assume that $S \neq \emptyset$. We provide in this paper an essentially complete description of the homotopy type of the spaces of $S$-immersions. More precisely, we prove that in most cases, one has a result of $h$-principle type, while sometimes the space of $S$-immersions may have a number of additional components of a different nature. The topology of these components is also fully described.



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*Remarks.* 1. The results proven in this paper were formulated in [El70, El72], but the proof of the injectivity part of the $h$-principle claim has never been published previously.

2. The approach described in this paper to the problem of construction of mappings with prescribed singularities can be generalized to the case of maps $V^n \to W^q$ with $n > q$. However, in contrast to the case $n = q$, the results that can be proven using the current techniques are essentially equivalent to the results of [El72].

3. The subject of this paper is the geometry of singularities in the *source* manifold. The geometry of singularities in the *image* is much more subtle; see [Gr07].

## 1.1 S-Immersions

We refer the reader to Sect. 4 below for the definition and basic properties of folds and wrinkles, and for the formulation of the wrinkling theorem from [EM97].

Let $f : V \to W$ be an $S$-immersion, i.e., a map with only a fold-type singularity $S = \Sigma^{1,0}(f)$. The fold $S \subset V$ has a neighborhood $U$ that admits an involution $\alpha_{\mathrm{loc}} : U \to U$ such that $f \circ \alpha_{\mathrm{loc}} = f$. In particular, if $S$ divides $V$ into two submanifolds $V_\pm$ with common boundary $\partial V_\pm = S$, then an $S$-immersion is just a pair of immersions $f_\pm : V_\pm \to W$ such that $f_+ = f_- \circ \alpha_{\mathrm{loc}}$ near $S$. Denote by $T_S V$ an $n$-dimensional tangent bundle over $V$ that is obtained from $TV$ by regluing $TV$ along $S$ with $\mathrm{d}\alpha_{\mathrm{loc}}$. For example, if $V = S^q$ and $S$ is the equator $S^{q-1} \subset S^q$, then $T_S V = S^q \times \mathbb{R}^q$. We will call $T_S V$ the *tangent bundle of V folded along S*. The differential $\mathrm{d}f : TV \to TW$ of any $S$-immersion $f : V \to W$ has a canonical (bijective) regularization $d_S f : T_S V \to TW$, the *folded differential* of $f$.

Let us denote by $\mathfrak{M}(V,W,S)$ the space of $S$-immersions $V \to W$. We also consider the space $\mathfrak{m}(V,W,S)$, a formal analogue of $\mathfrak{M}(V,W,S)$, which consists of bijective homomorphisms $T_S V \to TW$. The folded differential induces a natural inclusion $d : \mathfrak{M}(V,W,S) \to \mathfrak{m}(V,W,S)$. Our goal is to study the homotopic properties of the map $d$. We prove that in most cases, the map $d$ is a (weak) homotopy equivalence. However, there are some exceptional cases that arise when the map $d$ is a homotopy equivalence on some of the components of $\mathfrak{M}(V,W,S)$, while the structure of the remaining components can also be completely understood.

## 1.2 Taut–Soft Dichotomy for S-Immersions

A map $f \in \mathfrak{M}(V,W,S)$ is called *taut* if there exists an involution $\alpha : V \to V$ such that $\mathrm{Fix}\,\alpha = S$ and $f \circ \alpha = f$. We denote by $\mathfrak{M}_{\mathrm{taut}}(V,W,S)$ the subspace of $\mathfrak{M}(V,W,S)$ that consists of taut maps. Nontaut maps are called *soft*, and we define $\mathfrak{M}_{\mathrm{soft}}(V,W,S) := \mathfrak{M}(V,W,S) \setminus \mathfrak{M}_{\mathrm{taut}}(V,W,S)$. A map $f \in \mathfrak{M}_{\mathrm{taut}}(V,W,S)$ uniquely determines the corresponding involution $\alpha$, and thus the topological structure of the space of taut $S$-immersions:

**1.2.1. (Topological Structure of the Space of Taut $S$-Immersions)** *The space $\mathfrak{M}_{\mathrm{taut}}(V,W,S)$ is the space of pairs $(\alpha,h)$, where $\alpha : V \to V$ is an involution with $\operatorname{Fix}\alpha = S$ and $h$ is an immersion of the quotient manifold $V/\alpha$ with boundary $S$ to $W$.*

Of course, for most pairs $(V,S)$, the space $\mathfrak{I}(V,S)$ of such involutions is empty, and hence in such cases the space $\mathfrak{M}_{\mathrm{taut}}(V,W,S)$ is empty as well.

The topology of the space $\mathfrak{M}_{\mathrm{taut}}(V,W,S)$ is especially simple if $S$ divides $V$ into two submanifolds $V_\pm$ with common boundary $\partial V_\pm = S$, e.g., when both manifolds $V$ and $S$ are orientable. Clearly, we have the following result:

**1.2.2. (Topological Structure of the Space of Taut $S$-Immersions: The Orientable Case)** *If $S$ divides $V$ into two submanifolds $V_\pm$ with common boundary $\partial V_\pm = S$ such that there exists a diffeomorphism $V_+ \to V_-$ fixed along the boundary, then the space $\mathfrak{M}_{\mathrm{taut}}(V,W,S)$ is homeomorphic to the product*

$$\operatorname{Diff}_S(V_+) \times \operatorname{Imm}(V_+,W)\,,$$

*where $\operatorname{Diff}_S(V_+)$ is the group of diffeomorphisms $V_+ \to V_+$ fixed at the boundary together with their $\infty$-jet, and $\operatorname{Imm}(V_+,W)$ is the space of immersions $V_+ \to W$.*

Note that according to Hirsch's theorem [Hi], the space $\operatorname{Imm}(V_+,W)$ is homotopy equivalent to the space $Iso(V,W)$ of fiberwise isomorphic bundle maps $TV_+ \to TW$. For instance, when $V = S^q$, $W = \mathbb{R}^q$, and $S$ is the equator $S^{q-1} \subset S^q$, then we get

$$\mathfrak{M}_{\mathrm{taut}}(S^q,\mathbb{R}^q,S^{q-1}) \overset{\text{h.e.}}{\simeq} \operatorname{Diff}_{\partial D^q} D^q \times O(q).$$

In particular, when $q = 2$, the space of taut $S^1$-immersions $S^2 \to \mathbb{R}^2$ that preserve orientation on $S^2_+$ is homotopy equivalent to $S^1$.

**1.2.3. (Subspaces of Taut and Soft $S$-Immersions are Open–Closed)** *Let $S \subset V$ be a closed $(q-1)$-dimensional submanifold of $V$. Then the subspaces $\mathfrak{M}_{\mathrm{taut}}(V,W,S) \subset \mathfrak{M}(V,W,S)$ and $\mathfrak{M}_{\mathrm{soft}}(V,W,S)$ are open and closed, i.e., they consist of whole connected components of $\mathfrak{M}(V,W,S)$.*

*Proof.* Given any $f \in \mathfrak{M}(V,W,S)$ there exists $\varepsilon = \varepsilon(f) > 0$ such that the local involution $\alpha^f_{\mathrm{loc}} : \mathcal{O}p\,S \to \mathcal{O}p\,S$ with $f \circ \alpha^f_{\mathrm{loc}} = f$ is defined on an $\varepsilon$-tubular neighborhood $U_\varepsilon \supset S$. If $f$ is taut, then $\alpha^f_{\mathrm{loc}}$ extends as a global involution $\alpha^f : V \to V$ such that $f \circ \alpha^f = f$. Hence, for any $x \in V \setminus U_{\varepsilon/2}$, the distance $d(x,\alpha^f(x))$ is greater than or equal to $\varepsilon$. This implies that there exists a $\delta(\varepsilon) > 0$ such that if $||f' - f||_{C^2} < \delta(\varepsilon)$, then $\varepsilon(f') > \frac{\varepsilon}{2}$. Therefore, the local involution $\alpha^{f'}_{\mathrm{loc}}$ is defined on $U_{\varepsilon(f)/2}$. We extend it to $V \setminus U_{\varepsilon(f)/2}$ as a global involution $\alpha^{f'}$ by defining $\alpha^{f'}(x)$ as the unique point from $(f')^{-1}(x)$ whose distance from $\alpha^f(x)$ is less than $\frac{\varepsilon(f)}{2}$. Hence, $\mathfrak{M}_{\mathrm{taut}}(V,W,S)$ is open. On the other hand, using the same argument together with the implicit

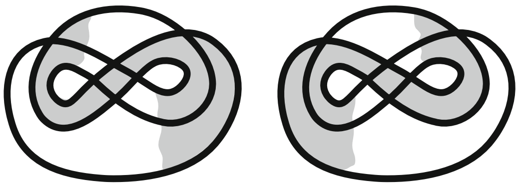


**Fig. 1** $f \in \mathfrak{M}_{\text{soft}}(S^2, \mathbb{R}^2, S^1)$ as a pair $f_1, f_2 : D^2 \to \mathbb{R}^2$

function theorem, we conclude that given a sequence $f_n \in \mathfrak{M}_{\text{taut}}(V,W,S)$ such that $f_n \xrightarrow{C^2} f$, then $\alpha^{f_n} \xrightarrow{C^1} \alpha^f$, and hence $f \in \mathfrak{M}_{\text{taut}}(V,W,S)$ and $\mathfrak{M}_{\text{taut}}(V,W,S)$ is closed. □

The following theorem completes the description of the homotopy type of spaces of $S$-immersions.

**1.2.4. (Homotopic Structure of the Space of Soft $S$-Immersions)** *Let $S \subset V$ be a closed nonempty $(q-1)$-dimensional submanifold of $V$. Suppose that $q \geq 3$, or $q = 2$ but $W$ is open. Then the inclusion*

$$d : \mathfrak{M}_{\text{soft}}(V,W,S) \to \mathfrak{m}(V,W,S)$$

*is a (weak) homotopy equivalence. In particular, when the space $\mathfrak{I}(V,S)$ is empty (and hence $\mathfrak{M}_{\text{taut}}(V,W,S)$ is empty as well), then $d : \mathfrak{M}(V,W,S) \to \mathfrak{m}(V,W,S)$ is a (weak) homotopy equivalence.*

For instance, when $V = S^q, W = \mathbb{R}^q$ and $S$ is the equator $S^{q-1} \subset S^q$, then we get

$$\mathfrak{M}_{\text{soft}}(S^q, \mathbb{R}^q, S^{q-1}) \overset{\text{h.e.}}{\simeq} O(q) \times \Omega_q(O(q)),$$

where $\Omega_q(O(q))$ is the based $q$-loops space of the orthogonal group $O(q)$. In
AQ2 particular, using also 1.2.2, we conclude that the space of *all* $S^1$-immersions $S^2 \to \mathbb{R}^2$ that preserve orientation on $S^2_+$ consists of two components homotopy equivalent to $S^1$. The standard projection represents the taut component, while an example of a soft $S^1$-immersion $f : S^2 \to \mathbb{R}^2$ (as a pair of maps $f_1, f_2 : D^2 \to \mathbb{R}^2$, $f_1|_{S^1} = f_2|_{S^1}$) is presented in Fig. 1.

*Remarks.* 1. Theorem 1.2.4 was formulated in [El72], but only the epimorphism part of the statement was proven there.
2. Theorem 1.2.4 trivially holds for $q = 1$ and any $W$.
3. In the case $q = 2$ and $W$ is a closed surface, we can prove only that the map $d$ induces an epimorphism on homotopy groups.

# 2 Zigzags

## 2.1 Zigzags and Soft S-Immersions

A *model zigzag* is any smooth function $\mathcal{Z} : [a,b] \to \mathbb{R}$ such that

- $\mathcal{Z}$ is increasing near $a$ and $b$;
- $\mathcal{Z}$ has exactly two interior nondegenerate critical points $m, M \in (a,b)$, $m < M$;
- $\mathcal{Z}(b) > \mathcal{Z}(a)$.

If, in addition, $\mathcal{Z}(b) > \mathcal{Z}(M)$ and $\mathcal{Z}(a) < \mathcal{Z}(m)$, then the model zigzag is called *long*.

*Example.* The function $\mathcal{Z}(z) = z^3 - 3z$ is a model zigzag on any interval $[a,b]$ such that $a < -1$ and $b > 1$ and $\mathcal{Z}(b) > \mathcal{Z}(a)$. If $a < -2$ and $b > 2$, then $\mathcal{Z}$ is long.

An embedding $h : [a,b] \to V$ is called a *zigzag* (respectively *long zigzag*) of a map $f \in \mathfrak{M}(V,W,S)$ if

- It is transversal to $S$, and
- The composition $f \circ h$ can be presented as $g \circ \mathcal{Z}$, where $\mathcal{Z} : [a,b] \to \mathbb{R}$ is a model zigzag (respectively long model zigzag) and $g : [A,B] \to W$ is an immersion defined on an interval $[A,B]$ such that $(A,B) \supset \mathcal{Z}([a,b])$ (Fig. 2).



The image $h([a,b])$ of the embedding $h$ will also be called a (long) zigzag. The points $h(a), h(b) \in V$ are called the *endpoints* of the zigzag $h$. Note the following extensions of long zigzags:

**2.1.1. (Extension of Long Zigzags)** *Let $h : [a,b] \to V$ be a* long *zigzag. Then any extension $h' : [a',b'] \to V$ of the embedding $h$ that does not have additional intersection points with $S$ is a long zigzag.*

The implicit function theorem implies the following:

**2.1.2. (Local Flexibility of Zigzags)** *Let $f_t \in \mathfrak{M}(V,W,S)$ defined for $t \in \mathcal{O}p\,0 \subset \mathbb{R}$. Suppose that $f_0$ admits a zigzag $h_0 : [a,b] \to V$ such that the composition $f_0 \circ h_0$ can be factored as $g_0 \circ \mathcal{Z}$ for an immersion $g_0 : [A,B] \to V$. Suppose that $g_0$ is included in a family of immersions $g_t : [A,B] \to W$, $t \in \mathcal{O}p\,0$. Then there exists a family of zigzags $h_t : [a,b] \to V$ defined for $t \in \mathcal{O}p\,0$ such that $f_t \circ h_t = g_t \circ \mathcal{Z}$.*

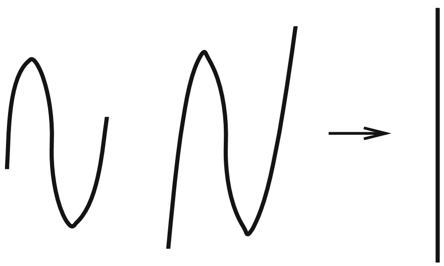


**Fig. 2** Zigzag and long zigzag

**2.1.3. (Softness Criterion)** *A map $f \in \mathfrak{M}(V,W,S)$ is soft if and only if it admits a zigzag.*

*Proof.* Existence of any zigzag is incompatible with the existence of an involution, and hence a map admitting a zigzag is soft. On the other hand, if $f \in \mathfrak{M}(V,W,S)$ does not admit a zigzag, we can define an involution $\alpha = \alpha^f : V \to V$, $f \circ \alpha = f$, as follows. Given $v \in V$, take any arc $C \subset V$ connecting $v$ in $V \setminus S$ with a point $s \in S$. Then the absence of zigzags guarantees that $f^{-1}(C)$ contains a unique candidate $v'$ for $\alpha(v)$. Similarly, if for another path we had another candidate $v''$, this would create a zigzag. Hence, the involution $\alpha : V \to V$ with $f \circ \alpha = f$ is correctly defined, and therefore $f$ is taut. □

## 2.2 Zigzags Adjacent to a Chamber

The components of $V \setminus S$ are called *chambers*. We say that a zigzag $h$ is *adjacent to a chamber* $C$ if this chamber contains one of the endpoints of the zigzag.

A family of maps $f_s \in \mathfrak{M}_{\text{soft}}(V,W,S)$, $s \in K$, parameterized by a connected compact set $K$ can be viewed as a fibered map $\widetilde{f} : K \times V \to K \times W$. We will call such maps *fibered (over $K$) $S$-immersions*. A fibered $S$-immersion is called *soft* if it is fiberwise soft. According to 1.2.3, $\widetilde{f}$ is soft if it is soft over a point $s \in K$. We denote the space of soft $S$-immersions fibered over $K$ by $\mathfrak{M}^K_{\text{soft}}(V,W,S)$. A family $Z_s$ of zigzags for $f_s$, $s \in K$, will be referred to as a *fibered* over $K$ zigzag $Z$. A fibered zigzag is called *special* if the projections $f_s(Z^s)$ are independent of $s \in K$. We say that a fibered soft map $\widetilde{f} : K \times V \to K \times W$ admits a *set of fibered (respectively special fibered) zigzags subordinated to a covering* $K = \bigcup_1^N U_j$ if there exist fibered (respectively special fibered) *disjoint* zigzags $\widetilde{Z}_1, \dots, \widetilde{Z}_N$ for the fibered maps $\widetilde{f}_j = \widetilde{f}|_{U_j \times V} : U_j \times V \to U_j \times W$, $j = 1, \dots, N$.

**2.2.1. (Set of Zigzags Subordinated to an Inscribed Covering)** *Let $\widetilde{Z}_1, \dots, \widetilde{Z}_N$ be a set of (special) fibered zigzags for a fibered map $\widetilde{f}$ subordinated to a covering $K = \bigcup_1^N U_j$. Let $K = \bigcup_1^{N'} U'_j$ be another covering inscribed in the first one, i.e., for every $U'_i$, $i = 1, \dots, N'$, there is $U_j$, $j = 1, \dots, N$, such that $U'_i \subset U_j$. Then there exists a set of fibered (special) zigzags $\widetilde{Z}'_1, \dots, \widetilde{Z}'_{N'}$ subordinated to the covering $K = \bigcup_1^{N'} U'_j$ such that for each $s \in U'_j$, the zigzag $Z'^s_j$ is $C^\infty$-close to the zigzag $Z^s_i$ for some $i = 1, \dots, N$.*

*Proof.* Lemma 2.1.2 implies that a neighborhood of any (special) fibered zigzag is foliated by (special) fibered zigzags, and hence near any (special) fibered zigzag $\widetilde{Z}$ one can always find arbitrarily many disjoint copies of $\widetilde{Z}$. □

We would like to prove that a fibered soft map admits a set of fibered zigzags adjacent to any of its chambers $C$. Below, we explain two methods for proving this. However, each of the methods requires some additional assumptions. The first one

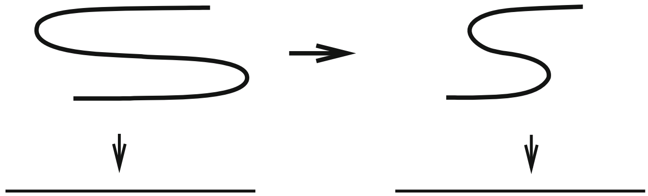


**Fig. 3** Making zigzags long

works for any $W$, but only if $q \geq 3$, while the second works for $q \geq 2$, but only if the target manifold is open. We do not know whether the statement still holds if $W$ is a closed surface.[1]

**2.2.2. (Fibered Zigzags Adjacent to a Chamber)** *Let $\widetilde{f}$ be a fibered over $K$ soft $S$-immersion. Suppose that $q \geq 3$, or $q = 2$ but $W$ is open. Then given any chamber $C \subset V \setminus S$, there is a homotopy $\widetilde{f_t} \in \mathfrak{M}^K_{\mathrm{soft}}(V,W,S)$, $t \in [0,1]$, such that $\widetilde{f_0} = \widetilde{f}$ and $\widetilde{f_1}$ admits a set of special fibered zigzags adjacent to the chamber $C$.*

Case $q \geq 3$. We begin with two lemmas.

**2.2.3. (Making Zigzags Long)** *Let $\widetilde{f} \in \mathfrak{M}^K(V,W,S)$. Suppose that $q \geq 3$. Then there exist a homotopy $\widetilde{f_t} \in \mathfrak{M}^K(V,W,S)$, $t \in [0,1]$, and a covering $K = \bigcup_1^N U_j$ such that $\widetilde{f_0} = \widetilde{f}$ and $\widetilde{f_1}$ admits a set of special fibered long zigzags subordinated to a covering inscribed in the covering $K = \bigcup_1^N U_j$.*

*Proof.* Choose a zigzag $h_s : [a,b] \to V$ for each $f_s$, $s \in K$, in the family $\widetilde{f}$. Denote by $g_s$ the corresponding immersion $[A,B] \to W$. According to Lemma 2.1.2, there is a neighborhood $U_s \ni s$ in $K$ such that for each $s' \in U_s$ there exists a zigzag $h'_{s'} : [a,b] \to V$ such that it factors through the same immersion $g_s$. Due to compactness of $K$, we can choose a finite subcovering $U_j = U_{s_j}$, $j = 1,\dots,N$. Lemma 2.1.2 further implies that if we $C^\infty$-small perturb the immersions $g_j := g_{s_j}$, then there still exist for each $j = 1,\dots,N$ and $s \in U_j$ zigzags $Z_{s,j} \subset V$ that factor through them. Hence, if $q \geq 3$, a general position argument allows us to assume that the images of the immersions $g_j$, and hence of the fibered zigzags $\widetilde{Z}_j$, do not intersect. Now for each $j = 1,\dots,N$, there exists a deformation $f_{s,t}$, $s \in U_j$, $t \in [0,1]$, of the family $f_s$ supported for each $s$ in an arbitrarily small neighborhood of $Z_s$ that makes the zigzags long; see Fig. 3. But the images of constructed zigzags do not intersect, and hence if the neighborhoods of zigzags are chosen sufficiently small, then the above deformation can be done simultaneously over all $U_j$, $j = 1,\dots,N$. □

[1]The second approach also works for $q = 1$ and any $W$. The case $W = T^2$ can also be treated by a slight modification of this method.

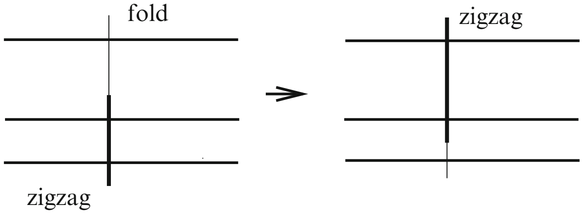



**Fig. 4** Penetration through a wall

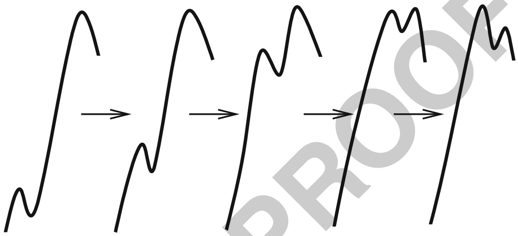


**Fig. 5** Homotopy $f_t$ on $h([a,c'])$

**2.2.4. (Penetration Through Walls)** *Let $C$ be one of the chambers for $f \in \mathfrak{M}(V,W,S)$. Let $h:[a,c] \to V$ be an embedding such that*

- *There exists $b \in (a,c)$ such that $h|_{[a,b]}$ is a long zigzag for $f$;*
- *$h$ is transversal to $S$ and $h(c) \in C$.*

*Then there exists a deformation $f_t \in \mathfrak{M}(V,W,S)$, $t \in [0,1]$, with $f_0 = f$ that is supported in a neighborhood of the image $h([a,c])$ and such that for some $b' \in (a,c)$, the embedding $h|_{[b',c]}$ is a zigzag for $f_1$ adjacent to $C$.*

*Proof.* Suppose that the embedding $h|_{[b,c]}$ intersects the wall $S$ in a sequence of points $h(p_1),\dots,h(p_k)$. We can consequently push the original zigzag through these points. Let $k=1$. By a small perturbation of $f$ near $h([a,c])$ we can make $h$ invariant with respect to the local involution $\alpha_{\mathrm{loc}}$ on $\mathcal{O}p\, p$. Then there exists $c' \in (p,c)$ such that $f \circ h|_{[a,c']}$ can be factored as $[a,c'] \to \mathbb{R} \to W$. Then the deformation shown in Figs. 4 and 5 allows us to move the zigzag from $[a,b]$ to $[b',c']$, where $c > p_1$, and thus the new long zigzag ends in the next chamber through which the embedding $h$ travels. For $k>1$ we apply inductively the same procedure. □

*Proof of Lemma 2.2.2 for $q \geq 3$.* We begin with a set of fibered long zigzags $\widetilde{Z}_j$ subordinated to a covering $K = \bigcup_1^N U_j$. Given a point $s \in U_j$, for each $j = 1,\dots,N$ we denote by $Z_j^s$ the zigzag over $s \in U_j$, and by $h_j^s:[a,b] \to V$ its parameterization.

Fix $j = 1,\dots,N$ and denote by $C_0$ one of the chambers to which the zigzag $\widetilde{Z}_j$ is adjacent, say $h_j^s(b) \in C_0$ for $s \in U_j$. Passing, if necessary, to a set of zigzags subordinated to a finer covering of $K$, we can extend the family of embeddings $h_j^s$, $s \in U_j$, to a family of embeddings $[a,c] \to V$, $c > b$, still denoted by $h_j^s$, such that

- For each $s \in U_j$ the embedding $h_j^s$ is transversal to $S$ and $H_j^s(c) \in C$;
- The image $f_j^s([a,c]) \subset W$ is independent of $s$ and disjoint from images of all other zigzags.

Using Lemma 2.2.4, we can construct a deformation of the fibered map $\widetilde{f}|_{U_j}$ supported in $\mathcal{O}p\,\widetilde{h}_j([a,c])$ that creates a fibered zigzag adjacent to the chamber $C$. Note that for different $i = 1,\dots,k$, and $j = 1,\dots,N$, all these deformations are supported in nonintersecting neighborhoods, and hence can be done simultaneously. $\square$

CASE OF AN OPEN $W$ AND $q \geq 2$. Let us consider a function $\phi : W \to \mathbb{R}$ without critical points. Let $\mathcal{F}$ be a 1-dimensional foliation by the gradient trajectories of $\phi$ for some Riemannian metric.

**2.2.5. (The Case of $K$ a Point)** *Any $f \in \mathfrak{M}(V,W,S)$ admits a zigzag adjacent to any of its chambers that projects to one of the leaves of the foliation $\mathcal{F}$.*

*Proof.* Let us call a leaf $L$ of $\mathcal{F}$ *regular* for $f$ if the map $f|_S : S \to W$ is transversal to it. Sard's theorem implies that a generic leaf of $\mathcal{F}$ is regular. Let $C$ be one of the chambers. Given a leaf $L$ that is regular for $f$, we denote by $C_L$ the union of those connected components of the closed 1-dimensional manifold $f^{-1}(L) \subset V$ that intersect $C$. We claim that for some regular leaf $L$, there exists a nonempty component $\Sigma$ of $C_L$ such that $f|_S : S \to L$ has more than two critical points. Indeed, otherwise we could reconstruct an involution $\alpha : V \to V$ such that $f \circ \alpha = f$, which would imply that $f$ is taut. The intersection of $C$ with the circle $\Sigma$ consists of one or several arcs. Let $A$ be one of these arcs. Its endpoints, $p_1$ and $p_2$, belong to the fold $S$, and are critical points of $f|_\Sigma : \Sigma \to L$. Let us assume that $p_1$ is a local minimum and $p_2$ is a local maximum. Recall that the leaf $L$ is oriented by the gradient vector field of the function $\phi$. We orient the arc $A$ from $p_1$ to $p_2$ and orient the circle $\Sigma$ accordingly. Let $p_3$ be the next critical point of $f|_\Sigma$. Choose a point $q_1 \in A$ close to $p_1$ and a point $q_2$ close to $p_3$ and *after* it in terms of the orientation. If $f(p_3) > f(p_1)$, then the arc $Z = [q_1, q_2]$ is a zigzag adjacent to $C$ beginning at $q_1$ and ending at $q_2$. If $f(p_3) < f(p_1)$, then the same arc $Z = [q_1, q_2]$ is again is a zigzag adjacent to $C$, but beginning at $q_2$ and ending at $q_1$. $\square$

*Proof of Lemma 2.2.2 for an open $W$ and $q \geq 2$.* Lemma 2.2.5 implies that over any point $s \in K$, the map $f_s$ admits a zigzag adjacent to the chamber $C$. Moreover, we can assume that all these zigzags lie over different leaves of $\mathcal{F}$. These zigzags extend to a neighborhood $\mathcal{O}p\,s \in K$ as fibered zigzags over this neighborhood that for each $s' \in \mathcal{O}p\,s$ project to the same leaves of $\mathcal{F}$. Hence, we can choose a finite covering $K = \bigcup_1^N U_j$ by neighborhoods $U_j$ over which there exist ample sets of special fibered zigzags. Clearly we can arrange that zigzags over different $U_j$ project to different leaves of $\mathcal{F}$, and thus their images do not intersect. $\square$

# 3 Proof of the Main Theorem

## *3.1 Wrinkled S-Immersions*

A map $f : V \to W$ is called a *wrinkled S-immersion* if it has $S$ as its fold singularity and in the complement of $S$ it is a wrinkled map; see Sect. 4.2. Let $C$ be one of the chambers (i.e., a connected component of $V \setminus S$). We denote by $\mathfrak{M}_w(V,W,S,C)$ the space of wrinkled $S$-immersions that have all wrinkles in the chamber $C$. We will call $C$ the *designated* chamber. The regularized differential construction (see Sect. 4.2) provides a map $d_R : \mathfrak{M}_w(V,W,S,C) \to \mathfrak{m}(V,W,S)$, and the wrinkling Theorem 4.4.2 implies the following result:

**3.1.1. (From Formal to Wrinkled $S$-Immersions)** *The map $d_R$ is a (weak) homotopy equivalence.*

*Proof.* Let $C_1,\dots,C_l$ be a sequence of all chambers in $V \setminus S$ such that $C_l = C$ and each $C_i$, $i < l$, has a common wall with $C_j$, $j > i$. We apply Hirsch's $h$-principle (see [Hi]) for equidimensional immersions of open (i.e., nonclosed!) manifolds to $\overline{C}_1$ and then make a fold on $\partial\overline{C}_1$. Next, we apply the relative version the same $h$-principle to the pair $(C_2, \partial C_1)$ and so on. At the last step, we apply the wrinkling Theorem 4.2 to the pair $(\overline{C}, \partial\overline{C})$. □

A slightly stronger statement can be formulated in the language of fibered maps. A fibered over $K$ map is called a *fibered wrinkled S-immersion* if it has $S$ as its fiberwise fold singularity, and in the complement of $K \times S \subset K \times V$ it is a fibered wrinkled map. One can also talk about fibered over $K$ formal $S$-immersions, i.e., parameterized by $K$ families $F_s \in \mathfrak{m}(V,W,S)$. Then a regularized differential of a fibered $S$-immersion is a fibered formal $S$-immersion. The following proposition is a slight improvement of the above homotopy equivalence claim, and it also follows from 4.4.2.

**3.1.2. (From Formal to Wrinkled $S$-Immersions; A Fibered Version)** *Given any fibered over $K$ formal $S$-immersion $\widetilde{F} \in \mathfrak{m}^K(V,W,S)$, there exists a fibered over $K$ wrinkled map $\widetilde{g} \in \mathfrak{M}^K_w(V,W,S,C)$ whose regularized fibered differential is homotopic to $\widetilde{F}$. Moreover, if over a closed $L \subset K$ we have $\widetilde{F} = \mathrm{d}\widetilde{f}$, where $\widetilde{f}$ is a genuine fibered $S$-immersion, then the map $\widetilde{g}$ can be chosen equal to $\widetilde{f}$ over $L$, and the homotopy can be made fixed over $L$.*

Our next goal is to supply a wrinkled $S$-immersion with zigzags adjacent to the designated chamber.

**3.1.3. (Zigzags for Fibered Wrinkled $S$-Immersions)** *Let $\widetilde{f}$ be a fibered over $K$ wrinkled map that has all wrinkles in the chamber $C$. Suppose that over a closed subset $L \subset K$, the fibered map $\widetilde{f}$ consists of genuine (i.e., nonwrinkled) $S$-immersions. Let $\bigcup_1^N U_j \supset L$ be a covering of $L$ by contractible and sets open in $K$, and let $\widetilde{Z}_1,\dots,\widetilde{Z}_N$ be a set of fibered zigzags adjacent to $C$ subordinated to the covering $\bigcup_1^N U_j$. Then there is a homotopy $\widetilde{f}_t \in \mathfrak{M}^K_w(V,W,S,C)$, $t \in [0,1]$, $\widetilde{f}_0 = \widetilde{f}$, such that*

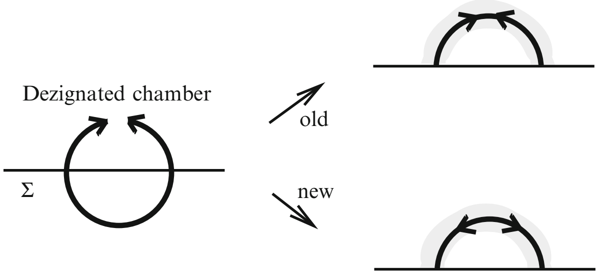



**Fig. 6** Local birth of a zigzag

- *$\widetilde{f}_t$ is fixed over $L$ in a neighborhood of the zigzags $\widetilde{Z}_1,\dots,\widetilde{Z}_N$;*
- *There exists a zigzag $\widetilde{Z}$, fibered over a domain $U$, $K\setminus\bigcup_1^N U_j \subset U \subset K\setminus \mathcal{O}p\,L$, for $\widetilde{f}_1$, adjacent to $C$ and disjoint from $\widetilde{Z}_1,\dots,\widetilde{Z}_N$.*

Let us define $\Sigma := \partial C \subset S$. The following two lemmas will be needed in the proof of 3.1.3.

**3.1.4.** *Let $U_1,\dots,U_N$ be as in 3.1.3 and $\sigma_j : U_j \to U_j\times\Sigma$, $j = 1,\dots,N$, disjoint sections. Then there exists a section $\sigma : K \to K\times\Sigma$ disjoint from the sections $\sigma_1,\dots,\sigma_N$ and homotopic to a constant section.*

*Proof.* Arguing by induction over $j = 1,\dots,N$, we construct a fiberwise isotopy $\widetilde{g}_t : K\times V \to V$, $t\in[0,1]$, that makes sections $\sigma_j : U_j \to U_j\times\Sigma$ constant, i.e., $\widetilde{g}_1\circ\sigma_j(s) = (s,c_j)$, $c_j\in\Sigma$, $j = 1,\dots,N$. Take a point $c\in\Sigma$ different from $c_1,\dots,c_N$ and define $\sigma'(s) := (s,c)$ for any $s\in K$. Then the section $\sigma = \widetilde{g}_1^{-1}\circ\sigma' : K\to K\to V$ has the required properties. □

**3.1.5. (Local Birth of a Zigzag Near $\sigma$)** *Let $M\subset K$ be a closed subset and $\sigma : M\to M\times\Sigma\subset K\times V$ a section. Then there exists a deformation $\widetilde{f}_t \in \mathfrak{M}^K_w(V,W,S,C)$ of $\widetilde{f}_0 = \widetilde{f}$ that is supported in $\mathcal{O}p\,\sigma(M)\subset K\times V$ such that $\widetilde{f}_1$ admits a fibered over $\mathcal{O}p\,M$ zigzag adjacent to $C$ and supported in $\mathcal{O}p\,\sigma(M)\subset K\times V$.*

*Proof.* First we give a sketch of the construction. Take a fibered embedding $\widetilde{h} : M\times I \to M\times\mathcal{O}p\,\Sigma$ such that for all $s\in M$, the image $h(s\times I)$ lies on a small $\alpha_{\mathrm{loc}}$-invariant circle near $\sigma(s)\in\Sigma$; see Fig. 6. There exists a homotopy $\widetilde{g}_t$ of the map $\widetilde{g}_0 = \widetilde{f}|_{\mathcal{O}p\,h(M\times I)}$ such that $\widetilde{g}_1$ has a fibered zigzag over $M$; see Fig. 6. This local homotopy can be extended as a homotopy $\widetilde{f}_t\in\mathfrak{M}^K_w(V,W,S,C)$ of the whole fibered map $\widetilde{f}$.

Let us give now a more detailed description. Let $S^1\subset\mathbb{C}$ be the unit circle, and $\exp : \mathbb{R}\to S^1$ a covering map $u\mapsto e^{iu}$, $u\in\mathbb{R}$. Choose a neighborhood $\Omega\supset\sigma(M)$. Let $\widetilde{h} : M\times S^1 \to \Omega\subset M\times V$ be a fibered embedding such that for all $s\in M$, the image $h^s(S^1)$ is a small $\alpha_{\mathrm{loc}}$-invariant circle near $\sigma(s)\in\Sigma$; see Fig. 6. We can assume

$h^s(\exp u) \in \overline{C}$ for $u \in [\pi/2, 3\pi/2]$ and $h^s(\exp u) \notin \overline{C}$ for $u \in (-\pi/2, \pi/2)$. Consider also a fibered embedding

$$\widetilde{\varphi} : \mathcal{O}p\, M \times B \to \mathcal{O}p\, M \times C \subset \Omega,$$

where $B$ is an open $n$-ball, such that $\varphi^s(B)$ is a small ball centered at the point $h^s(-1) \in C$ and such that $\varphi^s(D) \cap h^s(S^1) = h^s(\exp((\pi-\varepsilon, \pi+\varepsilon)), s \in M$. There exists a compactly supported fibered regular homotopy (see Fig. 6)

$$\widetilde{\psi}_t : \Omega \setminus \widetilde{\varphi}(\mathcal{O}p\, M \times B) \to \Omega, \;\; t \in [0,1],$$

such that

$$\widetilde{\psi}_t(\widetilde{h}(s, \exp u)) = \widetilde{h}(s, \exp(1+t/2)u), \; s \in M, \; u \in [-\pi+\varepsilon, \pi-\varepsilon].$$

Notice that $\psi_1^s$ maps an embedded arc $h^s([-\pi+\varepsilon, \pi-\varepsilon])$ onto an overlapping arc $h^s([-\frac{3}{2}(-\pi+\varepsilon), \frac{3}{2}(\pi-\varepsilon))$. Using Theorem 4.4.2, we can extend the regular homotopy $\widetilde{\psi}_t$ to a compactly supported fibered wrinkled homotopy $\Omega \to \Omega$, and then further extend it to the rest of $K \times V$ as the identity map. We will use the same notation $\widetilde{\psi}_t$ for this extension. Finally, the wrinkled homotopy $\widetilde{f}_t = \widetilde{f}_0 \circ \widetilde{\psi}_t : K \times V \to K \times V$ connects $\widetilde{f}_0$ with a wrinkled map $\widetilde{f}_1$ that has the embedding $\widetilde{\psi}_1 \circ \widetilde{h}|_{M \times [-\pi-\varepsilon, \pi+\varepsilon])}$ as its fibered over $M$ zigzag. □

*Proof of Proposition 3.1.3.* Note that the zigzags $\widetilde{Z}_j$ over $U_j \subset K$, $j = 1, \dots, N$, intersect the closure $\overline{C}$ of the chamber $C$ along intervals with one end on $\Sigma$ and the second inside $C$. Taking the endpoints in $\Sigma$, we get sections $\sigma_j : U_j \to U_j \times \Sigma$, $j = 1, \dots, N$. Let us apply Lemma 3.1.4 and construct a section $\sigma : K \to K \times \Sigma$, disjoint from the sections $\sigma_1, \dots, \sigma_N$. Let $M \subset K \setminus L$ be a closed set such that $K \setminus M \subset \bigcup_1^N U_j$. There exists a neighborhood $\Omega \supset \sigma(M) \subset K \times V$ that does not intersect the fibered zigzags $\widetilde{Z}_j$, $j = 1, \dots, N$. Let $U \supset M$ be an open neighborhood whose closure is contained in $K \setminus L$ and such that $\sigma(U) \subset \Omega$. We conclude the proof of 3.1.3 by applying Lemma 3.1.5. □



## 3.2 *Engulfing Wrinkles by Zigzags*

**3.2.1. (Getting rid of Wrinkles)** *Suppose that $\widetilde{f} \in \mathfrak{M}_w^K(V, W, S, C)$ admits a set of fibered zigzags adjacent to the chamber $C$. Let $\widetilde{f}$ be a genuine fibered $S$-immersion over a closed $L \subset K$. Then $\widetilde{f}$ is homotopic in $\mathfrak{M}_w^K(V, W, S, C)$ to a genuine fibered $S$-immersion via a homotopy fixed over $L$.*

*Proof.* Let $\widetilde{Z}_1, \dots, \widetilde{Z}_N$ be fibered zigzags adjacent to $C$ and subordinated to a covering $\bigcup_1^N U_j = K$. First of all, we can apply the enhanced wrinkling theorem (see the remark after Theorem 4.4.2) to $\widetilde{f}|_{K \times C}$ and get a modified $\widetilde{f}$ such that each fibered wrinkle is supported over one of the elements of the covering. Using Lemma 2.2.1, we can assume that wrinkles and zigzags are in 1-to-1 correspondence with the elements of the covering. We will eliminate inductively all the wrinkles by a procedure that we call *engulfing of wrinkles by zigzags*. We will discuss this construction only in the nonparametric case, i.e., when $K$ is a point. The case of a general $K$ differs only in the notation.

Let $w = w(q)$ be the standard wrinkle with membrane $D^q$ (see 4.2). Let us recall that $w$ is a fibered over $\mathbb{R}^{q-1}$ map $\mathbb{R}^q \to \mathbb{R}^q$ defined by the formula

$$(y,z) \mapsto \left(y, z^3 + 3(|y|^2 - 1)z\right),$$

where $y \in \mathbb{R}^{q-1}$, $z \in \mathbb{R}^1$, and $|y|^2 = \sum_1^{q-1} y_i^2$. Note that for a sufficiently small $\varepsilon > 0$ we have $w(D^q_{1+\varepsilon}) \subset \{|y| \le 1+\varepsilon, |z| \in [-3,3]\} = D^{q-1}_{1+\varepsilon} \times [-3,3]$.

We will need the following lemma.

**3.2.2. (Standard Model for Engulfing)** *Let us consider a fibered over $D^{q-1}_{1+\varepsilon}$ map $\Gamma : D^{q-1}_{1+\varepsilon} \times [a,c] \to D^{q-1}_{1+\varepsilon} \times [a,c]$ such that*

- *for some $b \in (a,c)$, the restriction*

$$\Gamma|_{D^{q-1}_{1+\varepsilon} \times [a,b]} : D^{q-1}_{1+\varepsilon} \times [a,b] \to D^{q-1}_{1+\varepsilon} \times [a,c]$$

  *is a fibered over $D^{q-1}_{1+\varepsilon}$ zigzag;*
- *the restriction*

$$\Gamma|_{D^{q-1}_{1+\varepsilon} \times [b,c]} : D^{q-1}_{1+\varepsilon} \times [b,c] \to D^{q-1}_{1+\varepsilon} \times [a,c]$$

  *is a fibered over $D^{q-1}_{1+\varepsilon}$ wrinkled map with a unique wrinkle whose cusp locus projects to the sphere $\partial D^{q-1}$. In other words, there exist fibered over $D^{q-1}_{1+\varepsilon}$ embeddings $\alpha : D^q_{1+\varepsilon} \to D^{q-1}_{1+\varepsilon} \times [b,c]$ and $\beta : D^{q-1}_{1+\varepsilon} \times [-3,3] \to D^{q-1}_{1+\varepsilon} \times [a,c]$ such that $\Gamma \circ \alpha = \beta \circ w$.*

*Then there exists a fibered over $D^{q-1}_{1+\varepsilon}$ homotopy*

$$\Gamma_t : D^{q-1}_{1+\varepsilon} \times [a,c] \to D^{q-1}_{1+\varepsilon} \times [a,c], t \in [0,1],$$

*that begins with $\Gamma_0 = \Gamma$, is fixed near $D^{q-1}_{1+\varepsilon} \times [a,c]$, and satisfies the following conditions:*

- *$\Gamma_t|_{D^{q-1}_{1+\varepsilon} \times [a,b]}$ is the homotopy in the space of fibered folded maps;*
- *$\Gamma_t|_{D^{q-1}_{1+\varepsilon} \times [b,c]}$ is a fibered wrinkled homotopy that eliminates the wrinkle of the map $\Gamma_0 = \Gamma$, i.e., the map $\Gamma_1|_{D^{q-1}_{1+\varepsilon} \times [b,c]}$ is nonsingular.*

We will call the homotopy $\Gamma_t$ *engulfing of the wrinkle $w$ by a zigzag*.

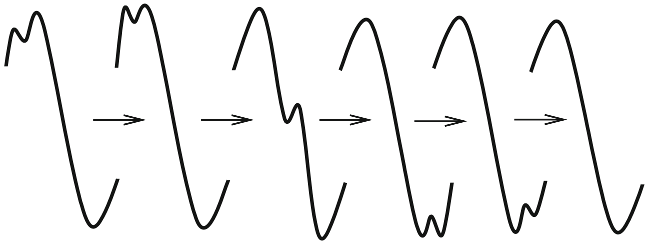
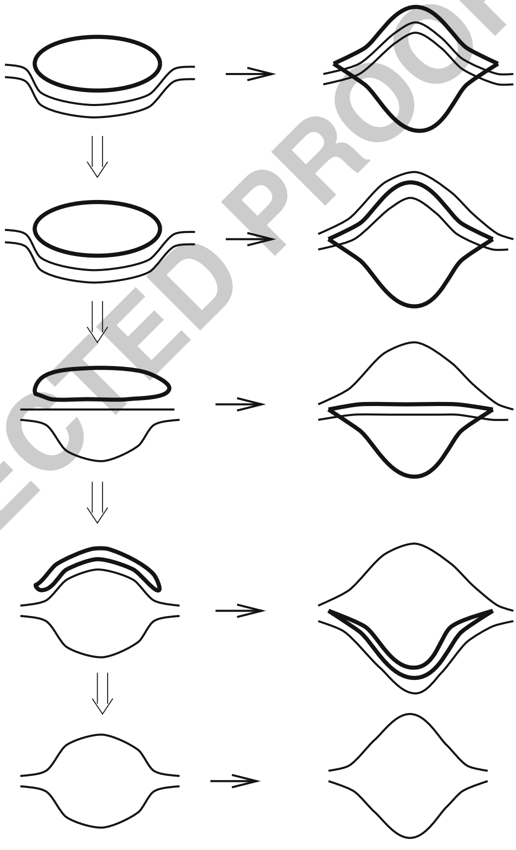


**Fig. 7** Homotopy $\Gamma_{\mathrm{t}}$ on $y \times [a,c]$, $y \in \mathrm{Int}\, D^q$

**Fig. 8** Homotopy $\Gamma_{\mathrm{t}}$, $q = 2$

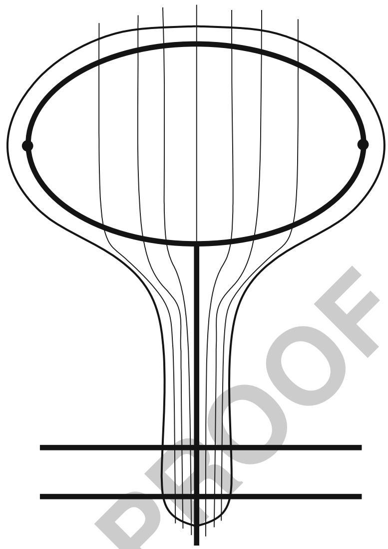




*Proof.* For $y \in \mathrm{Int}\, D^{q-1}$, the homotopy $\Gamma_t$ is shown in Fig. 7. This deformation can be done smoothly depending on the parameter $y \in D^{q-1}_{1+\varepsilon}$ and can be made to die out on $[1, 1+\varepsilon]$. See also Fig. 8, where the deformation $\Gamma_t$ is shown for $q = 2$. □

291 We continue now the proof of 3.2.1 by reducing it to the standard engulfing
292 model 3.2.2. Let us recall that there is 1-to-1 correspondence between the wrinkles
293 and zigzags. Let $h : [a,b] \to V$ be a zigzag of a map $f$ adjacent to the designated

**Fig. 9** Embedding $\widehat{H}$



chamber $C$. The embedding $h$ extends to an embedding $H : D^{q-1}_{1+\varepsilon} \times [a,b] \to V$ onto a small neighborhood of the zigzag $Z = h([a,b])$ such that $H|_{y\times[a,b]}$ is a zigzag for each $y \in D^{q-1}$. Take the corresponding wrinkle with membrane $D \subset C$. By definition, this means that for a sufficiently small $\varepsilon > 0$, there exists an embedding $\alpha : D^q_{1+\varepsilon} \to V$ such that $\alpha(D^q) = D$ and $f \circ \alpha = g \circ w(q)$ for an embedding $g : D^{q-1}_{1+\varepsilon} \times [-3,3] \to W$.

As is clear from Fig. 9, there exists an extension of the embedding $H$ to an embedding $\widehat{H} : D^{q-1}_{1+\varepsilon} \times [a,c] \to V$, $c > b$, such that

- $\widehat{H}(D^{q-1}_{1+\varepsilon} \times [b,c]) \supset D$;
- The map $f \circ \widehat{H} : D^{q-1}_{1+\varepsilon} \times [a,c]$ can be written as $g \circ \Gamma$, where $\Gamma$ is a fibered over $D^{q-1}_{1+\varepsilon}$ map $D^{q-1}_{1+\varepsilon} \times [a,c] \to D^{q-1}_{1+\varepsilon} \times [a,c]$ and $g : D^{q-1}_{1+\varepsilon} \times [a,c] \to W$ is an immersion;
- The restriction of $\Gamma$ to $D^{q-1}_{1+\varepsilon} \times [b,c]$ is a fibered over $D^{q-1}_{1+\varepsilon}$ wrinkled map with the unique wrinkle whose cusp locus projects to the sphere $\partial D^{q-1} \subset D^{q-1}_{1+\varepsilon}$.

Thus we are in a position to apply Lemma 3.2.2. Let

$$\Gamma_t : [a,c] \times D^{q-1}_{1+\varepsilon} \to [a,c] \times D^{q-1}_{1+\varepsilon},\ t \in [0,1],$$

be the engulfing homotopy of $\Gamma = \Gamma_0$ constructed in 3.2.2, which eliminates the wrinkle. Note that the homotopy $\Gamma_t$ is fixed near $\partial\left([a,c] \times D^{q-1}_{1+\varepsilon}\right)$. This enables us to define the required wrinkled homotopy $f_t : V \to W$, $t \in [0,1]$, which eliminates the wrinkle $w$ by setting it equal to $f$ on $V \setminus \widehat{H}([a,c] \times D^{q-1}_{1+\varepsilon})$, and to $g \circ \Gamma_t \circ \widehat{H}^{-1}$ on $\widehat{H}([a,c] \times D^{q-1}_{1+\varepsilon})$. □

### 3.3 Proof of Theorem 1.2.4

Let us prove that the map $d$ induces an injective homomorphism on $\pi_{k-1}$, i.e., that $\pi_k(\mathfrak{m}(V,W,S),\mathfrak{M}_{\text{soft}}(V,W,S)) = 0$. Define $K := D^k$, $L := \partial D^k = S^{k-1}$. We need to show that if $\widetilde{F} \in \mathfrak{m}^K(V,W,S)$ is a fibered formal $S$-immersion that is equal to $\mathrm{d}\widetilde{f}$ over $L$ for $\widetilde{f} \in \mathfrak{M}^L_{\text{soft}}(V,W,S)$, then there exists a homotopy $\widetilde{F}_t$ of $\widetilde{F}$ in $(\mathfrak{m}^K(V,W,S),\mathfrak{M}^L_{\text{soft}}(V,W,S)$ such that $\widetilde{F}_1 = \mathrm{d}\widetilde{f}_1$ for $\widetilde{f}_1 \in \mathfrak{M}^K_{\text{soft}}(V,W,S)$.

The construction of the required homotopy can be done in four steps:

- Using Lemma 2.2.2, we first deform $\widetilde{F}$ in $(\mathfrak{m}^K(V,W,S),\mathfrak{M}^L(V,W,S))$ so that the new $\widetilde{F}$ admits over $L$ a set of special fibered zigzags adjacent to a designated chamber $C$.
- Using 3.1.1, we further deform $\widetilde{F}$ into a differential of a fibered wrinkled $S$-immersion $\widetilde{f}$ such that all the wrinkles belong to the chamber $C$.
- Using 3.1.3, we can further deform $\widetilde{f}$ in such a way that the new $\widetilde{f}$ admits a set of zigzags adjacent to the chamber $C$. Furthermore, we can choose this set of zigzags in such a way that it includes the set of zigzags over $L$.
- Using 3.2.1, one can engulf all the fibered wrinkles.

The proof of the subjectivity claim is similar but does not use the first step.

## 4 Appendix: Folds, Cusps, and Wrinkles

We recall here, for the reader's convenience, some definitions and results from [EM97]. We consider here only the equidimensional case $n = q$, and this allows us to simplify definitions, notation, and the like from what appears in [EM97].

### 4.1 Folds and Cusps

Let $V$ and $W$ be smooth manifolds of the same dimension $q$. For a smooth map $f : V \to W$ we will denote by $\Sigma(f)$ the set of its singular points, i.e.,

$$\Sigma(f) = \{p \in V,\ \text{rank}\ d_p f < q\}.$$

A point $p \in \Sigma(f)$ is called a *fold*-type singularity or a *fold* of index $s$ if near the point $p$, the map $f$ is equivalent to the map $\mathbb{R}^{q-1} \times \mathbb{R}^1 \to \mathbb{R}^{q-1} \times \mathbb{R}^1$ given by the formula $(y,x) \to (y,\ x^2)$, where $y = (y_1,\dots,y_{q-1}) \in \mathbb{R}^{q-1}$.

Let $q > 1$. A point $p \in \Sigma(f)$ is called a *cusp* of index $s + \frac{1}{2}$ if near the point $p$, the map $f$ is equivalent to the map $\mathbb{R}^{q-1} \times \mathbb{R}^1 \to \mathbb{R}^{q-1} \times \mathbb{R}^1$ given by the formula $(y,z) \to (y, z^3 + 3y_1 z)$, where $z \in \mathbb{R}^1$, $y = (y_1,\dots,y_{q-1}) \in \mathbb{R}^{q-1}$.

For $q \geq 1$, a point $p \in \Sigma(f)$ is called an *embryo* of index $s + \frac{1}{2}$ if $f$ is equivalent near $p$ to the map $\mathbb{R}^{q-1} \times \mathbb{R}^1 \to \mathbb{R}^{q-1} \times \mathbb{R}^1$ given by the formula $(y, z) \to (y, z^3 + 3|y|^2 z)$, where $y \in \mathbb{R}^{q-1}$, $z \in \mathbb{R}^1$, $|y|^2 = \sum_1^{q-1} y_i^2$. The set of all folds of $f$ is denoted by $\Sigma^{10}(f)$, the set of cusps by $\Sigma^{11}(f)$, and the closure $\overline{\Sigma^{10}(f)}$ by $\Sigma^1(f)$.

Observe that folds and cusps are stable singularities for individual maps, while embryos are stable singularities only for 1-parametric families of mappings. For a generic perturbation of an individual map, embryos either disappear or give birth to wrinkles, which we consider in the next section.

## 4.2 *Wrinkles and Wrinkled Maps*

Consider the map $w(q) : \mathbb{R}^{q-1} \times \mathbb{R}^1 \to \mathbb{R}^{q-1} \times \mathbb{R}^1$ given by the formula

$$(y, z) \mapsto \left(y, z^3 + 3(|y|^2 - 1)z\right),$$

where $y \in \mathbb{R}^{q-1}$, $z \in \mathbb{R}^1$ and $|y|^2 = \sum_1^{q-1} y_i^2$.

The singularity $\Sigma^1(w(q))$ is the $(q-1)$-dimensional sphere $S^{q-1} \subset \mathbb{R}^q$. Its equator $\{z = 0, |y| = 1\} \subset \Sigma^1(w(q))$ consists of cusp points, while the upper and lower hemispheres consist of folds points (see Fig. 10). We will call the $q$-dimensional bounded by $\Sigma^1(w)$ disk $D^q = \{z^2 + |y|^2 \leq 1\}$ the *membrane* of the wrinkle.

A map $f : U \to W$ defined on an open subset $U \subset V$ is called a *wrinkle* if it is equivalent to the restriction of $w(q)$ to an open neighborhood $U$ of the disk $D^q \subset \mathbb{R}^q$. Sometimes, the term "wrinkle" can be also used also for the singularity $\Sigma(f)$ of the wrinkle $f$.

Observe that for $q = 1$, the wrinkle is a function with two nondegenerate critical points of indices 0 and 1 given in a neighborhood of a gradient trajectory that connects the two points.

Restrictions of the map $w(q)$ to subspaces $y_1 = t$, viewed as maps $\mathbb{R}^{q-1} \to \mathbb{R}^{q-1}$ are nonsingular maps for $|t| > 1$, equivalent to $w(q-1)$ for $|t| < 1$ and to embryos for $t = \pm 1$.

Although the differential $\mathrm{d}w(q) : T(\mathbb{R}^q) \to T(\mathbb{R}^q)$ degenerates at points of $\Sigma(w)$, it can be canonically *regularized*. Namely, we can change the element $3(z^2 + |y|^2 - 1)$ in the Jacobi matrix of $w(q)$ to a function $\gamma$ that coincides with $3(z^2 + |y|^2 - 1)$ outside an arbitrarily small neighborhood $U$ of the disk $D^q$ and does not vanish on $U$. The new bundle map $\mathcal{R}(\mathrm{d}w) : T(\mathbb{R}^q) \to T(\mathbb{R}^q)$ provides a homotopically canonical extension of the map $\mathrm{d}w : T(\mathbb{R}^q \setminus U) \to T(\mathbb{R}^q)$ to an epimorphism (fiberwise surjective bundle map) $T(\mathbb{R}^q) \to T(\mathbb{R}^q)$. We call $\mathcal{R}(\mathrm{d}w)$ the *regularized differential* of the map $w(q)$.

A map $f : V \to W$ is called *wrinkled* if there exist disjoint open subsets $U_1, \ldots, U_l \subset V$ such that $f|_{V \setminus U}$, $U = \bigcup_1^l U_i$, is an immersion (i.e., has rank $q$) and for each $i = 1, \ldots, l$, the restriction $f|_{U_i}$ is a wrinkle. Observe that the sets $U_i$, $i = 1, \ldots, l$, are included in the structure of a wrinkled map.

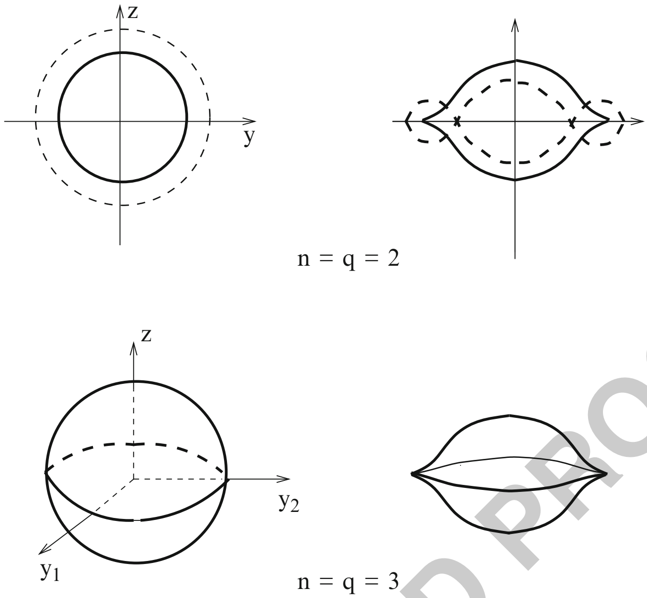



**Fig. 10** Wrinkles. Pictures in the source and in the image

The singular locus $\Sigma(f)$ of a wrinkled map $f$ is a union of $(q-1)$-dimensional wrinkles $S_i = \Sigma^1(f|_{U_i}) \subset U_i$. Each $S_i$ has a $(q-2)$-dimensional equator $T_i \subset S_i$ of cusps that divides $S_i$ into two hemispheres of folds of two neighboring indices. The differential $\mathrm{d}f : T(V) \to T(W)$ can be regularized to obtain an epimorphism $\mathcal{R}(\mathrm{d}f) : T(V) \to T(W)$. To get $\mathcal{R}(\mathrm{d}f)$, we regularize $\mathrm{d}f|_{U_i}$ for each wrinkle $f|_{U_i}$.

## 4.3 Fibered Wrinkles

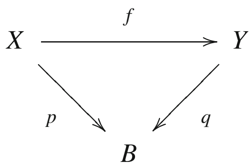
All the notions from 4.2 can be extended to the parametric case.

A *fibered* (over $B$) *map* is a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & Y \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & B & \end{array}$$

where $p$ and $q$ are submersions. A fibered map can be also denoted simply by $f : X \to Y$ if $B$, $p$, and $q$ are implied from the context.

For a fibered map




we denote by $T_B X$ and $T_B Y$ the subbundles $\operatorname{Ker} p \subset TX$ and $\operatorname{Ker} q \subset TY$. They are tangent to foliations of $X$ and $Y$ formed by preimages

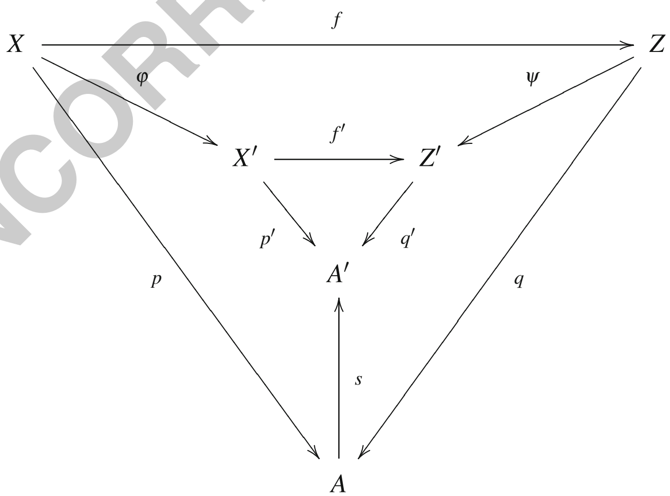
$$p^{-1}(b) \subset X\,,\; q^{-1}(b) \subset Y\,,\; b \in B\,.$$

The fibered homotopies, fibered differentials, fibered submersions, and so on are naturally defined in the category of fibered maps (see [EM97]). For example, the *fibered differential* of $f : X \to Y$ is the restriction

$$d_B f = \mathrm{d}f|_{T_B X} : T_B X \to T_B Y\,.$$

Notice that $d_B f$ itself is a map fibered over $B$.

Two fibered maps $f : X \to Y$ over $B$ and $f : X' \to Y'$ over $B'$ are called *equivalent* if there exist open subsets $A \subset B$, $A' \subset B'$, $Z \subset Y$, $Z' \subset Y'$ with $f(X) \subset Z$, $p(X) \subset A$, $f'(X') \subset Z'$, $p'(X') \subset A'$ and diffeomorphisms $\varphi : X \to X'$, $\psi : Z \to Z'$, $s : A \to A'$ such that they form the following commutative diagram:




For any integer $k > 0$, the map $w(k+q)$ can be considered a fibered map $w_k(k+q)$ over $\mathbb{R}^k \times \mathbf{0} \subset \mathbb{R}^{k+q}$. A fibered map equivalent to the restriction of $w_k(k+q)$ to an open neighborhood $U^{k+q} \supset D^{k+q}$ is called a *fibered wrinkle*. The regularized differential $\mathcal{R}(\mathrm{d}w_k(k+q))$ is a fibered (over $\mathbb{R}^k$) epimorphism

$$\mathbb{R}^k \times T(\mathbb{R}^{q-1} \times \mathbb{R}^1) \xrightarrow{\mathcal{R}(\mathrm{d}w_k(q))} \mathbb{R}^k \times T(\mathbb{R}^{q-1} \times \mathbb{R}^1)$$

A fibered map $f: V \to W$ is called a *fibered wrinkled map* if there exist disjoint open sets $U_1, U_2, \dots, U_l \subset V$ such that $f|_{V \setminus U}$, $U = \bigcup_1^l U_i$, is a fibered submersion and for each $i = 1, \dots, l$, the restriction $f|_{U_i}$ is a fibered wrinkle. The restrictions of a fibered wrinkled map to a fiber may have, in addition to wrinkles, embryo singularities.

Similarly to the nonparametric case, one can define the regularized differential of a fibered over $B$ wrinkled map $F: V \to W$, which is a fibered epimorphism $\mathcal{R}(d_B f): T_B V \to T_B W$.

## 4.4 The Wrinkling Theorem

The following theorem 4.4.1 and its parametric version 4.4.2 are adaptations of the results of our paper [EM97] to the simplest case $n = q$. In fact, in this case the results below can also be deduced from a theorem of V. Poénaru; see [Po].

**4.4.1. (Wrinkled Mappings)** *Let $F: T(V) \to T(W)$ be an epimorphism that covers a map $f: V \to W$. Suppose that $f$ is an immersion on a neighborhood of a closed subset $K \subset V$, and $F$ coincides with $\mathrm{d}f$ over that neighborhood. Then there exists a wrinkled map $g: V \to W$ that coincides with $f$ near $K$ and such that $\mathcal{R}(\mathrm{d}g)$ and $F$ are homotopic relations. $T(V)|_K$. Moreover, the map $g$ can be chosen arbitrarily $C^0$-close to $f$ and with arbitrarily small wrinkles.*

**4.4.2. (Fibered Wrinkled Mappings)** *Let $f: V \to W$ be a fibered over $B$ map covered by a fibered epimorphism $F: T_B(V) \to T_B(W)$. Suppose that $f$ is a fibered immersion on a neighborhood of a closed subset $K \subset V$, and $F$ coincides with $\mathrm{d}f$ near a closed subset $K \subset V$. Then there exists a fibered wrinkled map $g: V \to W$ that extends $f$ from a neighborhood of $K$ and such that the fibered epimorphisms $\mathcal{R}(\mathrm{d}g)$ and $F$ are homotopic rel. $T_B(V)|_K$. Moreover, the map $g$ can be chosen arbitrarily $C^0$-close to $f$ and with arbitrarily small fibered wrinkles.*

*Remark.* The proof (see [EM97]) also gives the following useful enhancement for 4.4.2: given an open covering $\{U_i\}_{i=1,\dots,N}$ of $B$, one can always choose $g$ such that for each fibered wrinkle there exists $U_i$ such that $p(D^{k+q}) \subset U_i$, where $D^{k+q}$ is the membrane of the wrinkle.

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## References


[El70] Y. Eliashberg, *On singularities of folding type*, Izv. Akad. Nauk SSSR, Ser. mat., **34**(1970), 1111–1127.

[El72] Y. Eliashberg, *Surgery of singularities of smooth maps*, Izv. Akad. Nauk SSSR Ser. Mat., **36**(1972), 1321–1347.

[EM97] Y. Eliashberg and N. Mishachev, *Wrinkling of smooth mappings and its applications I*, Invent. Math., **130**(1997), 345–369.

[Gr07] M. Gromov, *Singularities, Expanders and Topology of Maps. Part 1: Homology versus Volume in the Spaces of Cycles.* Preprint, Institut des Hautes Etudes Scientifiques, 2007.

[Hi] M.W. Hirsch, *On imbedding differentiable manifolds in Euclidean space*, Ann. of Math. **73**(1961), 566–571.

[Po] V. Poénaru, *Homotopy theory and differentiable singularities*, Manifolds-Amsterdam 1970 (Proc. Nuffic Summer School), Springer Lecture Notes in Math., **197**(1971), 106–132.

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AQ3. Kindly check whether the citation of Fig. 2 is OK as cited.