Rational SFT, Linearized Legendrian Contact Homology, and Lagrangian Floer Cohomology

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This paper is dedicated to Oleg Viro on the occasion of his 60th birthday

Abstract We relate the version of rational symplectic field theory for exact Lagrangian cobordisms introduced in [6] to linearized Legendrian contact homology. More precisely, if $L \subset X$ is an exact Lagrangian submanifold of an exact symplectic manifold with convex end $\Lambda \subset Y$, where Y is a contact manifold and Λ is a Legendrian submanifold, and if L has empty concave end, then the linearized Legendrian contact cohomology of Λ, linearized with respect to the augmentation induced by L, equals the rational SFT of (X, L). Following ideas of Seidel [15], this equality in combination with a version of Lagrangian Floer cohomology of L leads us to a conjectural exact sequence that in particular implies that if $X = \mathbb{C}^n$, then the linearized Legendrian contact cohomology of $\Lambda \subset S^{2n-1}$ is isomorphic to the singular homology of L. We outline a proof of the conjecture and show how to interpret the duality exact sequence for linearized contact homology of [7] in terms of the resulting isomorphism.

Keywords Floer cohomology • Symplectic field theory • Legendrian contact homology • Cobordism • Holomorphic disk

1 Introduction

Let Y be a contact $(2n-1)$-manifold with contact 1-form λ (i.e., $\lambda \wedge (\mathrm{d}\lambda)^{n-1}$ is a volume form on Y). The *Reeb vector field* R_λ of λ is the unique vector field that satisfies $\lambda(R_\lambda) = 1$ and $\mathrm{d}\lambda(R_\lambda, \cdot) = 0$. The *symplectization* of Y is the symplectic

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manifold $Y \times \mathbb{R}$ with symplectic form $d(e^t\lambda)$, where t is a coordinate in the $\mathbb{R}$-factor. A *symplectic manifold with cylindrical ends* is a symplectic $2n$-manifold X that contains a compact subset K such that $X - K$ is symplectomorphic to a disjoint union of two half-symplectizations $Y^+ \times \mathbb{R}_+ \cup Y^- \times \mathbb{R}_-$, for some contact $(2n-1)$-manifolds $Y^\pm$, where $\mathbb{R}_+ = [0,\infty)$ and $\mathbb{R}_- = (-\infty,0]$. We call $Y^+ \times \mathbb{R}_+$ and $Y^- \times \mathbb{R}_-$ the *positive and negative ends* of X, respectively, and Y^+ and Y^-, $(+\infty)$- *and* $(-\infty)$-*boundaries* of X, respectively.

The relative counterpart of a symplectic manifold with cylindrical ends is a pair (X,L) of a symplectic $2n$-manifold X with a Lagrangian n-submanifold $L \subset X$ (i.e., the restriction of the symplectic form in X to any tangent space of L vanishes) such that outside a compact subset, (X,L) is symplectomorphic to the disjoint union of $\left(Y^+ \times \mathbb{R}_+, \Lambda^+ \times \mathbb{R}_+\right)$ and $\left(Y^- \times \mathbb{R}_-, \Lambda^- \times \mathbb{R}_-\right)$, where $\Lambda^\pm \subset Y^\pm$ are Legendrian $(n-1)$-submanifolds (i.e., $\Lambda^\pm$ are everywhere tangent to the kernels of the contact forms on $Y^\pm$). If the symplectic manifold X is exact (i.e., if the symplectic form ω on X satisfies $\omega = \mathrm{d}\beta$ for some 1-form β) and if the Lagrangian submanifold L is exact as well (i.e., if the restriction $\beta|L$ satisfies $\beta|L = \mathrm{d}f$ for some function f), then we call the pair (X,L) of exact manifolds an *exact cobordism*. We assume throughout the paper that X is simply connected, that the first Chern class of TX, viewed as a complex bundle using any almost complex structure compatible with the symplectic form on X, is trivial, and that the Maslov class of L is trivial as well. (These assumptions are made in order to have well-defined gradings in contact homology algebras over $\mathbb{Z}_2$. In more general cases, one would work with contact homology algebras with suitable Novikov coefficients in order to have appropriate gradings.)

In [6], a version of rational symplectic field theory (SFT) (see [12] for a general description of SFT) for exact cobordisms with good ends was developed; see Sect. 2. (The additional condition that ends be good allows us to disregard Reeb orbits in the ends when setting up the theory. Standard contact spheres as well as 1-jet spaces with their standard contact structures are good.) It associates to an exact cobordism (X,L), where L has k components, a $\mathbb{Z}$-graded filtered $\mathbb{Z}_2$-vector space $\mathbf{V}(X,L)$, with k filtration levels and with a filtration-preserving differential $d^f\colon \mathbf{V}(X,L) \to \mathbf{V}(X,L)$. Elements in $\mathbf{V}(X,L)$ are formal sums of admissible formal disks in which the number of summands with $(+)$-action below any given number is finite; see Sect. 2 for definitions of these notions.

The differential increases $(+)$-action, and hence if $\mathbf{V}_{[\alpha]}(X,L)$ denotes $\mathbf{V}(X,L)$ divided out by the subcomplex of all formal sums in which all disks have $(+)$-action larger than α, then the differential induces filtration-preserving differentials $d^f_\alpha\colon \mathbf{V}_{[\alpha]}(X,L) \to \mathbf{V}_{[\alpha]}(X,L)$ with associated spectral sequences $\left\{E^{p,q}_{r;[\alpha]}(X,L)\right\}_{r=1}^k$. The projection maps $\pi^\alpha_\beta\colon \mathbf{V}_{[\alpha]}(X,L) \to \mathbf{V}_{[\beta]}(X,L)$, $\alpha > \beta$, give an inverse system of chain maps. The limit $E^*_r(X,L) = \varprojlim_\alpha E^*_{r;[\alpha]}(X,L)$ is invariant under deformations of (X,L) through exact cobordisms with good ends and in particular under deformations of L through exact Lagrangian submanifolds with cylindrical ends; see Theorem 2.1.

In this paper we will use only the simplest version of the theory just described, which is as follows. Let (X,L) be an exact cobordism such that L is connected and without negative end, i.e., $\Lambda^- = \emptyset$. In this case, admissible formal disks have only one positive puncture, and we identify (a quotient of) $\mathbf{V}(X,L)$ with the $\mathbb{Z}_2$-vector space of formal sums of Reeb chords of $\Lambda = \Lambda^+$. Furthermore, our assumptions on $\pi_1(X)$ and vanishing of $c_1(TX)$ and of the Maslov class of L imply that the grading of a formal disks depends only on the Reeb chord at its positive puncture. We let $|c|$ denote the grading of a chord $c \in \mathbf{V}(X,L)$. Rational SFT then provides a differential

$$d^f : \mathbf{V}(X,L) \to \mathbf{V}(X,L)$$

with $|d^f(c)| = |c|+1$ that increases action in the sense that if $\mathfrak{a}(c)$ denotes the action of the Reeb chord c and if the Reeb chord b appears with nonzero coefficient in ∂c, then $\mathfrak{a}(b) > \mathfrak{a}(c)$. Furthermore, since L is connected, the spectral sequences have only one level, and

$$E_1^*(X,L) = \varprojlim_\alpha \left(\ker d_\alpha^f / \operatorname{im} d_\alpha^f\right).$$

Our first result relates $E_1^*(X,L)$ to linearized Legendrian contact cohomology; see Sect. 3. Legendrian contact homology was introduced in [5, 12]. It was worked out in detail in special cases including 1-jet spaces in [9–11]. From the point of view of Legendrian contact homology, an exact cobordism (X,L) with good ends induces a chain map from the contact homology algebra of (Y^+,Λ^+) to that of (Y^-,Λ^-). In particular, if $\Lambda^- = \emptyset$, then the latter equals the ground field $\mathbb{Z}_2$ with the trivial differential. Such a chain map ϵ is called an *augmentation*, and it gives rise to a linearization of the contact homology algebra of Λ^+. That is, it endows the chain complex $Q(\Lambda)$ generated by Reeb chords of Λ with a differential ∂^ϵ.

The resulting homology is called *ϵ-linearized contact homology* and denoted by $\mathrm{LCH}_*(Y,\Lambda;\epsilon)$. We let $\mathrm{LCH}^*(Y;\Lambda;\epsilon)$ be the homology of the dual complex $Q'(\Lambda) = \mathrm{Hom}(Q(\Lambda);\mathbb{Z}_2)$ and call it the *ϵ-linearized contact cohomology* of Λ.

We say that (X,L) satisfies a *monotonicity condition* if there are constants C_0 and $C_1 > 0$ such that for any Reeb chord c of $\Lambda \subset Y$, $|c| > C_1\mathfrak{a}(c) + C_0$. Note that if Y is a 1-jet space or the sphere endowed with a generic small perturbation of the standard contact form and if Λ is in general position with respect to the Reeb flow, then (X,L) satisfies a monotonicity condition.

Theorem 1.1. *Let (X,L) be an exact cobordism with good ends. Let (Y,Λ) denote the positive end of (X,L) and assume that the $(-\infty)$-boundary of L is empty. Let ϵ denote the augmentation on the contact homology algebra of Λ induced by L. Then the natural map $Q'(\Lambda) \to \mathbf{V}(X,L)$, which takes an element in $Q'(\Lambda)$ thought of as a formal sum of covectors dual to Reeb chords in $Q(\Lambda)$ to the corresponding formal sum of Reeb chords in $\mathbf{V}(X,L)$, is a chain map. Furthermore, if (X,L) satisfies a monotonicity condition, then the corresponding map on homology*

$$\mathrm{LCH}^*(Y,\Lambda;\epsilon) \to E_1^*(X,L),$$

is an isomorphism.

Theorem 1.1 is proved in Sect. 3.2. We point out that when (X,L) satisfies a monotonicity condition, it follows from this result that $\mathrm{LCH}_*(Y,\Lambda;\epsilon)$ depends only on the symplectic topology of (X,L).

We next consider two exact cobordisms (X,L_0) and (X,L_1) with good ends and with the following properties: both L_0 and L_1 have empty $(-\infty)$-boundaries; if Λ_j denotes the $(+\infty)$-boundary of L_j, then $\Lambda_0 \cap \Lambda_1 = \emptyset$; L_0 and L_1 intersect transversely; and the Reeb flow of Λ_0 along a Reeb chord connecting Λ_0 to Λ_1 is transverse to Λ_1 at its endpoint. For such pairs of exact cobordisms we define Lagrangian Floer cohomology $HF^*(X;L_0,L_1)$ as an inverse limit of the cohomologies $HF^*_{[\alpha]}(X;L_0,L_1)$ of cochain complexes $C_{[\alpha]}(X;L_0,L_1)$ generated by Reeb chords between Λ_0 and Λ_1 of action at most α and by points in $L_0 \cap L_1$. This Floer cohomology has a relative $\mathbb{Z}$-grading and is invariant under exact deformations of L_1.

Consider an exact cobordism (X,L) where L has empty $(-\infty)$-boundary and $(+\infty)$-boundary Λ. Let L' be a slight push-off of L, which is an extension of a small push-off Λ' of Λ along the Reeb vector field.

Conjecture 1.2. For any $\alpha > 0$, there is a long exact sequence

$$\begin{array}{lllll} \cdots \xrightarrow{\delta_{\alpha;L,L'}} & E^*_{1;[\alpha]}(X,L) & \longrightarrow & HF^*_{[\alpha]}(X;L,L') & \longrightarrow & H_{n-*}(L) \\ \xrightarrow{\delta_{\alpha;L,L'}} & E^{*+1}_{1;[\alpha]}(X,L) & \longrightarrow & HF^{*+1}_{[\alpha]}(X;L,L') & \longrightarrow & H_{n-*-1}(L) \\ \xrightarrow{\delta_{\alpha;L,L'}} & \quad \cdots, & & & & \end{array} \tag{1}$$

where $H_*(L)$ is the ordinary homology of L with $\mathbb{Z}_2$-coefficients. It follows in particular that if $X = \mathbb{C}^n$ or $X = J^1(\mathbb{R}^{n-1}) \times \mathbb{R}$, then $HF^*(X;L,L') = 0$, and the map $\delta_{L,L'} \colon H_{n-*+1}(L) \to E^*_1(X,L) \approx \mathrm{LCH}^*(Y,\Lambda;\epsilon)$ induced by the maps $\delta_{\alpha;L,L'}$ is an isomorphism.

The author learned about the isomorphism above, between linearized contact homology of a Legendrian submanifold with a Lagrangian filling and the ordinary homology of the filling, from Seidel [15], who explained it using an exact sequence in wrapped Floer homology [1, 14] similar to (1). Borrowing Seidel's argument, we outline in Sect. 4.4 a proof of Conjecture 1.2 in which the Lagrangian Floer cohomology $HF(X;L,L')$ plays the role of wrapped Floer homology.

In [7], a duality exact sequence for linearized contact homology of a Legendrian submanifold $\Lambda \subset Y$, where $Y = P \times \mathbb{R}$ for some exact symplectic manifold P and where the projection of Λ into P is displaceable, was found. In what follows, we restrict attention to the case $Y = J^1(\mathbb{R}^{n-1})$. Then every compact Legendrian submanifold has displaceable projection, and the duality exact sequence is the following, where ϵ denotes any augmentation and where we suppress the ambient manifold $Y = J^1(\mathbb{R}^{n-1})$ from the notation:

$$\begin{array}{l} \cdots \xrightarrow{\rho} H_{k+1}(\Lambda) \xrightarrow{\sigma} \mathrm{LCH}^{(n-1)-k-1}(\Lambda;\epsilon) \xrightarrow{\theta} \mathrm{LCH}_k(\Lambda;\epsilon) \\ \xrightarrow{\rho} H_k(\Lambda) \xrightarrow{\sigma} \mathrm{LCH}^{(n-1)-k}(\Lambda;\epsilon) \xrightarrow{\theta} \mathrm{LCH}_{k-1}(\Lambda;\epsilon)\cdots. \end{array} \tag{2}$$

Here, if $\beta = \rho(\alpha) \in H_k(\Lambda)$, then the Poincaré dual $\gamma \in H_{n-k}(\Lambda)$ of β satisfies $\langle \sigma(\gamma), \alpha \rangle = 1$, where $\langle , \rangle$ is the pairing between the homology and cohomology of $Q(\Lambda)$. Furthermore, the maps ρ and σ are defined through a count of rigid configurations of holomorphic disks with boundary on Λ with a flow line emanating from its boundary, and the map θ is defined through a count of rigid holomorphic disks with boundary on Λ with two positive punctures.

In $J^1(\mathbb{R}^{n-1})$, a generic Legendrian submanifold has finitely many Reeb chords. Furthermore, if L is an exact Lagrangian cobordism in the symplectization $J^1(\mathbb{R}^{n-1}) \times \mathbb{R}$ with empty $(-\infty)$-boundary, then L is displaceable. Hence, both Theorem 1.1 and Conjecture 1.2 give isomorphisms. Combining (1) and (2) leads to the following.

Corollary 1.3. *Let L be an exact Lagrangian cobordism in $J^1(\mathbb{R}^{n-1}) \times \mathbb{R}$ with empty $(-\infty)$-boundary and with $(+\infty)$-boundary Λ and let ϵ denote the augmentation on the contact homology algebra of Λ induced by L. Then the following diagram with exact rows commutes, and all vertical maps are isomorphisms:*

$$\begin{array}{ccccccc} H_{k+1}(\Lambda) & \longrightarrow & H_{k+1}(L) & \longrightarrow & H_{k+1}(L,\Lambda) & \longrightarrow & H_k(\Lambda) \\ \cdots \quad \downarrow \mathrm{id} & & \downarrow \delta_{L,L'} & & \downarrow H^{-1}\circ\delta'_{L,L'} & & \downarrow \mathrm{id} \quad \cdots \\ H_{k+1}(\Lambda) & \xrightarrow{\sigma} & \mathrm{LCH}^{n-k-2}(\Lambda;\epsilon) & \longrightarrow & \mathrm{LCH}_k(\Lambda;\epsilon) & \xrightarrow{\rho} & H_k(\Lambda) \end{array}$$

Here the top row is the long exact homology sequence of (L,Λ), the bottom row is the duality exact sequence, the map $\delta_{L,L'}$ is the map in Conjecture 1.2, the map $\delta'_{L,L'}$ is analogous to $\delta_{L,L'}$, and the map H counts disks in the symplectization with boundary on Λ and with two positive punctures; see Sect. 4.5 for details.

The proof of Corollary 1.3 is discussed in Sect. 4.5.

2 A Brief Sketch of Relative SFT of Lagrangian Cobordisms

Although we will use only the simplest version of relative SFT introduced in [6] in this paper, we give a brief introduction to the full theory for two reasons. First, it is reasonable to expect that this theory is related to product structures on linearized contact homology, see [4], in much the same way as the simplest version of the theory appears in Conjecture 1.2. Second, some of the moduli spaces of holomorphic disks that we will make use of are analogous to those needed for more involved versions of the theory.

2.1 Formal and Admissible Disks

In order to describe relative rational SFT, we introduce the following notation. Let (X,L) be an exact cobordism with ends $(Y^{\pm}\times\mathbb{R}_{\pm},\Lambda^{\pm}\times\mathbb{R}_{\pm})$. Write $(\bar{X},\bar{L})$ for a compact part of (X,L) obtained by cutting the infinite parts of the cylindrical ends off at some $|t|=T>0$. We will sometimes think of Reeb chords of $\Lambda^{\pm}$ in the $(\pm\infty)$-boundary as lying in $\partial\bar{X}$ with endpoints on $\partial\bar{L}$. A *formal disk* of (X,L) is a homotopy class of maps of the 2-disk D, with m marked disjoint closed subintervals in ∂D, into $\bar{X}$, where the m marked intervals are required to map in an orientation-preserving (reversing) manner to Reeb chords of $\partial\bar{L}$ in the $(+\infty)$-boundary (in the $(-\infty)$-boundary) and where remaining parts of the boundary ∂D map to $\bar{L}$.

If L^b and L^a are exact Lagrangian cobordisms in X^b and X^a, respectively, such that a component (Y,Λ) of the $(-\infty)$-boundary of (X^a,L^a) agrees with a component of the $(+\infty)$-boundary of (X^b,L^b), then these cobordisms can be joined to an exact cobordism L^{ba} in X^{ba}, where (X^{ba},L^{ba}) is obtained by gluing the positive end (Y,Λ) of (X^b,L^b) to the corresponding negative end of (X^a,L^a). Furthermore, if v^b and v^a are collections of formal disks of (X^b,L^b) and (X^a,L^a), respectively, then we can construct formal disks in L^{ba} in the following way: start with a disk v^a_1 from v^a, and let $c_1,\dots,c_{r_1}$ denote the Reeb chords at its negative punctures. Attach positive punctures of disks $v^b_{1;1},\dots,v^b_{1;r_1}$ in v^b mapping the Reeb chords $c_1,\dots,c_{r_1}$ to the corresponding negative punctures of the disk v^a_1. This gives a disk v^{ba}_1 with some positive punctures mapping to chords $c_1,\dots,c_{r_2}$ of Λ. Attach negative punctures of the disk $v^a_{2;1},\dots,v^a_{2;r_2}$ to v^{ba}_1 at $c_1,\dots,c_{r_2}$. This gives a disk v^{ba}_2 with some negative punctures mapping to Reeb chords in Λ. Continue this process until there are no punctures mapping to Λ. We call the resulting disk a formal disk in L^{ba} *with factors from v^a and v^b*.

Assume that the set of connected components of L has been subdivided into subsets L_j so that L is a disjoint union $L=L_1\cup\dots\cup L_k$, where each L_j is a collection of connected components of L. We call $L_1,\dots,L_k$ the *pieces* of L. With respect to such a subdivision, Reeb chords fall into two classes: *pure*, with both endpoints on the same piece, and *mixed*, with endpoints on distinct pieces.

A formal disk represented by a map $u\colon D\to\bar{X}$ is *admissible* if for any arc α in D that connects two unmarked segments in ∂D that are mapped to the same piece by u, all marked segments on the boundary of one of the components of $D-\alpha$ map to pure Reeb chords in the $(-\infty)$-boundary.

2.2 Holomorphic Disks

Let (X,L) be an exact cobordism. Fix an almost complex structure J on X that is adjusted to its symplectic form. Let S be a punctured Riemann surface with complex structure j and with boundary ∂S. A J-holomorphic curve with boundary on L is a map $u\colon S\to X$ such that

$$du+J\circ du\circ j=0$$

and such that $u(\partial S) \subset L$. For details on holomorphic curves in this setting we refer to [6, Appendix B] and references therein. Here we summarize the main properties we will use.

By definition, an adjusted almost complex structure J is invariant under $\mathbb{R}$-translations in the ends of X and pairs the Reeb vector field in the $(\pm\infty)$-boundary with the symplectization direction. Consequently, strips that are cylinders over Reeb chords as well as cylinders over Reeb orbits are J-holomorphic. Furthermore, any J-holomorphic disk of finite energy is asymptotic to such Reeb chord strips at its boundary punctures and to Reeb orbit cylinders at interior punctures; see [6, Sect. B.1]. We say that a puncture of a J-holomorphic disk is positive (negative) if the disk is asymptotic to a Reeb chord strip (Reeb orbit cylinder) in the positive (negative) end of (X,L). Note that exactness of (X,L) and the fact that the symplectic form is positive on J-complex tangent lines imply that any J-holomorphic curve has at least one positive puncture.

These results on asymptotics imply that any J-holomorphic disk in X with boundary on L determines a formal disk. Let $\mathcal{M}(v)$ denote the moduli space of J-holomorphic disks with associated formal disk equal to v. The formal dimension of $\mathcal{M}(v)$ is determined by the Fredholm index of the linearized $\bar{\partial}_J$-operator along a representative of v; see [6, Sect. 3.1].

A sequence of J-holomorphic disks with boundary on L may converge to a broken disk of two components that intersects at a boundary point. We will refer to this phenomenon as boundary bubbling. However, if all elements in the sequence have only one positive puncture, then boundary bubbling is impossible by exactness: each component in the limit curve must have at least one positive puncture. The reason for using admissible disks to set up relative SFT is the following: In a sequence of holomorphic disks with corresponding formal disks admissible, boundary bubbling is impossible for topological reasons. As a consequence, if v is a formal disk, then the boundary of $\mathcal{M}(v)$ consists of several level J-holomorphic disks and spheres joined at Reeb chords or at Reeb orbits; see [2].

Recall from Sect. 1 that we require the ends of our exact cobordisms (X,L) to be good. The precise formulation of this condition is as follows. If γ^+ (γ^-) is a Reeb orbit in the $(+\infty)$-boundary Y^+ (in the $(-\infty)$-boundary Y^-) of X, then the formal dimension of any moduli space of holomorphic spheres in X (in $Y^- \times \mathbb{R}$) with positive puncture at γ^+ (at γ^-) is ≥ 2. Together with transversality arguments these conditions guarantee that broken curves in the boundary of $\mathcal{M}(v)$, where v is an admissible formal disk, cannot contain any spheres if $\dim(\mathcal{M}(v)) \leq 1$, or if $\dim(\mathcal{M}(v)) = 2$ when (X,L) is a trivial cobordism; see [6, Lemma B.6]. In particular, in the boundary of $\mathcal{M}(v)$, where $\dim(\mathcal{M}(v))$ satisfies these dimensional constraints and where v is admissible, there can be only two level curves, all pieces of which are admissible disks; see [6, Lemma 2.5].

Under our additional assumptions ($\pi_1(X)$ trivial, first Chern class of X and Maslov class of L vanish), the grading of a formal disk depends only on the Reeb chords at its punctures. For later reference, we describe this more precisely in the case that L is connected and its $(-\infty)$-boundary is empty. Let (Y,Λ) denote the $(+\infty)$-boundary of (X,L). If c is a Reeb chord of $\Lambda \subset Y$, then let γ be any path in L

joining its endpoints. Since X is simply connected, $\gamma \cup c$ bounds a disk $\Gamma : D \to X$. Fix a trivialization of TX along Γ such that the linearized Reeb flow along c is represented by the identity transformation with respect to this trivialization. Then the tangent space $T_s(\Lambda \times \mathbb{R})$ at the initial point s of c is transported to a subspace $V_s \times \mathbb{R}$ in the tangent space T_eX at the final point e of c where V_s is transverse to $T_e\Lambda$ in the contact hyperplane ξ_e at e. Let R denote a negative rotation along the complex angle taking V_s to $T_e\Lambda$ in ξ_e; see [6, Sect. 3.1]. Then the Lagrangian tangent planes of L along γ capped off with R form a loop Δ_Γ of Lagrangian subspaces in $\mathbb{C}^n$ with respect to the trivialization, and if $\mathcal{M}(c)$ denotes the moduli space of holomorphic disks in X with one positive boundary puncture at which they are asymptotic to the Reeb chord strip of the Reeb chord c, then

$$\dim(\mathcal{M}(c)) = n - 3 + \mu(\Delta_\Gamma) + 1,$$

where μ denotes the Maslov index; see [6, p. 655]. To see that this is independent of Γ, note that the difference of two trivializations along the disks is measured by $c_1(TX)$. To see that it is independent of the path γ, note that the difference in the dimension formula corresponding to two different paths γ and γ' is measured by the Maslov class of L evaluated on the loop $\gamma \cup -\gamma'$. Define

$$|c| = \dim(\mathcal{M}(c)). \tag{3}$$

If a and $b_1, \dots, b_m$ are Reeb chords of Λ and if $\mathcal{M}(a; b_1, \dots, b_m)$ denotes the moduli space of holomorphic disks in $Y \times \mathbb{R}$ with boundary on $\Lambda \times \mathbb{R}$ with positive puncture at the Reeb chord a and negative punctures at the Reeb chords $b_1, \dots, b_k$ in the order given by following the boundary orientation of the disk starting at the positive puncture, then additivity of the index gives

$$\dim(\mathcal{M}(a; b_1, \dots, b_k)) = |a| - \sum_j |b_j|. \tag{4}$$

2.3 Hamiltonian and Potential Vectors and Differentials

Let (X, L) be an exact cobordism and let v be a formal disk of (X, L). Define the $(+)$-action of v as the sum of the actions

$$\mathfrak{a}(c) = \int_c \lambda^+$$

over the Reeb chords c at their positive punctures. Here λ^+ is the contact form in the $(+\infty)$-boundary Y^+ of X. Note that for generic Legendrian $(+\infty)$-boundary, $\Lambda^+ \subset Y^+$, the set of actions of Reeb chords, is a discrete subset of $\mathbb{R}$. Let $\mathbf{V}(X, L)$ denote the $\mathbb{Z}$-graded vector space over $\mathbb{Z}_2$ with elements that are formal sums of admissible formal disks that contain only a finite number of summands below any given $(+)$-action. The grading on $\mathbf{V}(X,L)$ is the following: the degree of a formal disk v is the formal dimension of the moduli space $\mathcal{M}(v)$ of J-holomorphic disks homotopic to the formal disk. We use the natural filtration

$$0 \subset F^k\mathbf{V}(X,L) \subset \cdots \subset F^2\mathbf{V}(X,L) \subset F^1\mathbf{V}(X,L) = \mathbf{V}(X,L)$$

of $\mathbf{V}(X,L)$, where k is the number of pieces of L and where the filtration level is determined by the number of positive punctures. (It is straightforward to check that an admissible formal disk has at most k Reeb chords at the positive end.)

We will define a differential $d^f : \mathbf{V}(X,L) \to \mathbf{V}(X,L)$ that respects this filtration using 1-dimensional moduli spaces of holomorphic disks. To this end, fix an almost complex structure J on X that is compatible with the symplectic form and adjusted to $d(e^t\lambda^\pm)$ in the ends, where $\lambda^\pm$ is the contact form in the $(\pm\infty)$-boundary. Assume that J is generic with respect to 0- and 1-dimensional moduli spaces of holomorphic disks; see [6, Lemma B.8]. Since J is invariant under translations in the ends, $\mathbb{R}$ acts on moduli spaces $\mathcal{M}(u)$, where u is a formal disk of $(Y^\pm \times \mathbb{R}, \Lambda^\pm \times \mathbb{R})$. In this case we define the reduced moduli spaces as $\widehat{\mathcal{M}}(u) = \mathcal{M}(u)/\mathbb{R}$. Let $h^\pm \in \mathbf{V}(Y^\pm \times \mathbb{R}, \Lambda^\pm \times \mathbb{R})$ denote the vector of admissible formal disks in $Y^\pm \times \mathbb{R}$ with boundary on $\Lambda^\pm \times \mathbb{R}$ represented by J-holomorphic disks:

$$h^\pm = \sum_{\dim(\widehat{\mathcal{M}}(v))=0} |\widehat{\mathcal{M}}(v)|\, v \in \mathbf{V}(\Lambda^\pm \times \mathbb{R}), \tag{5}$$

where the sum ranges over all formal disks of $\Lambda^\pm \times \mathbb{R}$ and where $|\widehat{\mathcal{M}}|$ denotes the mod-2 number of points in the compact 0-manifold $\widehat{\mathcal{M}}$. We call h^+ and h^- the *Hamiltonian vectors* of the positive and negative ends, respectively. Similarly, let f denote the generating function of rigid disks in the cobordism:

$$f = \sum_{\dim(\mathcal{M}(v))=0} |\mathcal{M}(v)|\, v \in \mathbf{V}(X,L), \tag{6}$$

where the sum ranges over all formal disks of (X,L). We call f the *potential vector of* (X,L).

We view elements w in $\mathbf{V}(Y^\pm \times \mathbb{R}, \Lambda^\pm \times \mathbb{R})$ and $\mathbf{V}(X,L)$ as sets of admissible formal disks, where the set consists of those formal disks that appear with nonzero coefficient in w. Define the differential $d^f : \mathbf{V}(X,L) \to \mathbf{V}(X,L)$ as the linear map such that if v is an admissible formal disk (a generator of $\mathbf{V}(X,L)$), then $d^f(v)$ is the sum of all admissible formal disks obtained in the following way:

(i) Attach a positive puncture of v to a negative puncture of an h^+-disk.
(ii) Then attach f-disks at remaining negative punctures of the h^+-disk, or
(iii) Attach a negative puncture of v to a positive puncture of an h^--disk.
(iv) Then attach f-disks at remaining positive punctures of the h^--disk.

The fact that this is a differential is a consequence of the product structure of the boundary of the moduli space mentioned above in the case of 1-dimensional moduli spaces; see [6, Lemma 3.7]. Furthermore, the differential increases the grading by 1 and respects the filtration, since any disk in $h^{\pm}$ or in f has at least one positive puncture.

2.4 The Rational Admissible SFT Spectral Sequence

Fix $\alpha > 0$. If $\mathbf{V}_{[\alpha+]}(X,L) \subset \mathbf{V}(X,L)$ denotes the subspace of formal sums of formal disks with $(+)$-action at least α, then since holomorphic disks have positive symplectic area, it follows that $d^f\left(\mathbf{V}_{[\alpha+]}(X,L)\right) \subset \mathbf{V}_{[\alpha+]}(X,L)$. If $\mathbf{V}_{[\alpha]}(X,L) = \mathbf{V}(X,L)/\mathbf{V}_{[\alpha+]}(X,L)$, then $\mathbf{V}_{[\alpha]}(X,L)$ is isomorphic to the vector space generated by formal disks of $(+)$-action less than α, and there is a short exact sequence of chain complexes

$$0 \longrightarrow \mathbf{V}_{[\alpha+]}(X,L) \longrightarrow \mathbf{V}(X,L) \longrightarrow \mathbf{V}_{[\alpha]}(X,L) \longrightarrow 0.$$

The quotients $\mathbf{V}_{[\alpha]}(X,L)$ form an inverse system

$$\pi^{\alpha}_{\beta} \colon \mathbf{V}_{[\alpha]}(X,L) \to \mathbf{V}_{[\beta]}(X,L), \quad \alpha > \beta,$$

of graded chain complexes, where π^{β}_{α} are the natural projections. Consequently, the k-level spectral sequences corresponding to the filtrations

$$0 \subset F^k\mathbf{V}_{[\alpha]}(X,L) \subset \cdots \subset F^2\mathbf{V}_{[\alpha]}(X,L) \subset F^1\mathbf{V}_{[\alpha]}(X,L) = \mathbf{V}_{[\alpha]}(X,L),$$

which we denote by

$$\left\{E^{p,q}_{r;[\alpha]}(X,L)\right\}_{r=1}^{k},$$

form an inverse system as well, and we define the rational admissible SFT invariant as

$$\{E^{p,q}_r(X,L)\}_{r=1}^k = \varprojlim_{\alpha} \left\{E^{p,q}_{r;[\alpha]}(X,L)\right\}_{r=1}^{k}.$$

This is in general not a spectral sequence, but it is so under some finiteness conditions. The following result is a consequence of [6, Theorems 1.1 and 1.2].

Theorem 2.1. *Let (X,L) be an exact cobordism with a subdivision $L = L_1 \cup \cdots \cup L_k$ into pieces. Then $\left\{E^{p,q}_r(X,L)\right\}$ does not depend on the choice of adjusted almost complex structure J, and is invariant under deformations of (X,L) through exact cobordisms with good ends.*

Proof. Any such deformation can be subdivided into a compactly supported deformation and a Legendrian isotopy at infinity. Deformations of the former type are shown in [6, Theorem 1.1.] to induce isomorphisms of $\left\{E_r^{p,q}(X,L)\right\}$.

To show that a deformation of the latter type induces an isomorphism, we note that it gives rise to an invertible exact cobordism, see [6, Appendix A], and use the same argument as in the proof of [6, Theorem 1.2.] as follows. Let C_{01} be the exact cobordism of the Legendrian isotopy at infinity and let C_{10} be its inverse cobordism. We use the symbol $A\#B$ to denote the result of joining two cobordisms along a common end. Consider first the cobordism

$$L\#C_{01}\#C_{10}.$$

Since this cobordism can be deformed by a compact deformation to L, we find that the composition of the maps $\Phi\colon \mathbf{V}(X,L\#C_{01}) \to \mathbf{V}(X,L\#C_{01}\#C_{10})$ and $\Psi\colon \mathbf{V}(X,L) \to \mathbf{V}(X,L\#C_{01})$ is chain homotopic to the identity. Hence Ψ is injective on homology. Consider second the cobordism

$$L\#C_{01}\#C_{10}\#C_{01}.$$

Since this cobordism can be deformed to $L\#C_{01}$, we find similarly that there is a map Θ such that $\Psi\circ\Theta$ is chain homotopic to the identity on $\mathbf{V}(X,L\#C_{01})$; hence Ψ is surjective on homology as well.

2.5 A Simple Version of Rational Admissible SFT

As mentioned in Sect. 1, in the present paper, we will use the rational admissible spectral sequence in the simplest case: for (X,L), where L has only one component. Since there is only one piece, the spectral sequence has only one level, and

$$E_1^{1,q}(X,L) = \varprojlim_\alpha E_{1;[\alpha]}^{1,q}(X,L) = \varprojlim_\alpha \ker(d_\alpha^f)/\operatorname{im}(d_\alpha^f)$$

is the invariant that we will compute. To simplify things further, we will work not with the chain complex $\mathbf{V}(X,L)$ as described above but with the quotient of it obtained by forgetting the homotopy classes of formal disks. We view this quotient, using our assumption that $\pi_1(X)$ is trivial, as the space of formal sums of Reeb chords of the $(+\infty)$-boundary Λ of L. Further, our assumptions $c_1(TX)=0$ and vanishing Maslov class of L imply that the grading descends to the quotient; see (3). For simplicity, we keep the notation $\mathbf{V}(X,L)$ and $\mathbf{V}_{[\alpha]}(X,L)$ for the corresponding quotients.

3 Legendrian Contact Homology, Augmentation, and Linearization

In this section we will define Legendrian contact homology and its linearization. We work in the following setting: (X,L) is an exact cobordism with good ends, the $(-\infty)$-boundary of L is empty, and the $(+\infty)$-boundary of (X,L) will be denoted by (Y,Λ).

Recall that the assumption on good ends allows us to disregard Reeb orbits. Furthermore, our additional assumptions on (X,L), i.e., $\pi_1(X)$ trivial and first Chern class and Maslov class trivial, allows us to work with coefficients in $\mathbb{Z}_2$ and still retain the grading.

3.1 Legendrian Contact Homology

Assume that $\Lambda \subset Y$ is generic with respect to the Reeb flow on Y. If c is a Reeb chord of Λ, let $|c| \in \mathbb{Z}$ be as in (3).

Definition 3.1. The DGA of (Y,Λ) is the unital noncommutative algebra $\mathcal{A}(Y,\Lambda)$ over $\mathbb{Z}_2$ generated by the Reeb chords of Λ. The grading of a Reeb chord c is $|c|$.

Definition 3.2. The contact homology differential is the map $\partial\colon \mathcal{A}(Y,\Lambda) \to \mathcal{A}(Y,\Lambda)$ that is linear over $\mathbb{Z}_2$ satisfies the Leibniz rule, and is defined as follows on generators:

$$\partial c = \sum_{\dim(\mathcal{M}(c;\overline{b}))=1} |\widehat{\mathcal{M}}(c;\overline{b})|\,\overline{b},$$

where c is a Reeb chord and $\overline{b} = b_1,\dots,b_k$ is a word of Reeb chords. (For notation, see (4).)

We give a brief explanation of why ∂ in Definition 3.2 is a differential, i.e., why $\partial^2 = 0$. Consider the boundary of the 2-dimensional moduli space $\mathcal{M}$ (which becomes 1-dimensional after the $\mathbb{R}$-action has been divided out) of holomorphic disks with one positive puncture at a. As explained in Sect. 2.2, the boundary of such a moduli space consists of two level curves such that all components except two are Reeb chord strips. Since these configurations are exactly what is counted by $\partial^2 c$ and since they correspond to the boundary points of the compact 1-manifold $\mathcal{M}/\mathbb{R}$, we conclude that $\partial^2 c = 0$.

Definition 3.3. An *augmentation* of $\mathcal{A}(Y,\Lambda)$ is a chain map $\epsilon\colon \mathcal{A}(Y,\Lambda) \to \mathbb{Z}_2$, where $\mathbb{Z}_2$ is equipped with the trivial differential.

Given an augmentation ϵ, define the algebra isomorphism $E_\epsilon\colon \mathcal{A}(Y,\Lambda) \to \mathcal{A}(Y,\Lambda)$ by letting

$$E_\epsilon(c) = c + \epsilon(c),$$

for each generator c. Consider the word-length filtration of $\mathcal{A}(Y,\Lambda)$,

$$\mathcal{A}(Y,\Lambda) = \mathcal{A}_0(Y,\Lambda) \supset \mathcal{A}_1(Y,\Lambda) \supset \mathcal{A}_2(Y,\Lambda) \supset \cdots.$$

The differential $\partial^\epsilon = E_\epsilon \circ \partial \circ E_\epsilon^{-1}\colon \mathcal{A}(Y,\Lambda) \to \mathcal{A}(Y,\Lambda)$ respects this filtration: $\partial^\epsilon(\mathcal{A}_j(Y,\Lambda)) \subset \mathcal{A}_j(Y,\Lambda)$. In particular, we obtain the ϵ-*linearized differential*

$$\partial_1^\epsilon\colon \mathcal{A}_1(Y,\Lambda)/\mathcal{A}_2(Y,\Lambda) \to \mathcal{A}_1(Y,\Lambda)/\mathcal{A}_2(Y,\Lambda). \tag{7}$$

Definition 3.4. The ϵ-*linearized contact homology* is the $\mathbb{Z}_2$-vector space

$$\mathrm{LCH}_*(Y,\Lambda;\epsilon) = \ker(\partial_1^\epsilon)/\operatorname{im}(\partial_1^\epsilon). \tag{8}$$

For simpler notation below, we write

$$Q(Y,\Lambda) = \mathcal{A}_1(Y,\Lambda)/\mathcal{A}_2(Y,\Lambda)$$

and think of $Q(Y,\Lambda)$ as the graded vector space generated by the Reeb chords of Λ. Furthermore, the augmentation will often be clear from the context, and we will drop it from the notation and write the differential as

$$\partial_1\colon Q(Y,\Lambda) \to Q(Y,\Lambda).$$

Consider an exact cobordism (X,L) with $(+\infty)$-boundary (Y^+,Λ^+) and $(-\infty)$-boundary (Y^-,Λ^-). Define the algebra map $\Phi\colon \mathcal{A}(Y^+,\Lambda^+) \to \mathcal{A}(Y^-,\Lambda^-)$ by mapping generators c of $\mathcal{A}(Y^+,\Lambda^+)$ as follows:

$$\Phi(c) = \sum_{\dim(\mathcal{M}(c;\overline{b}))=0} |\mathcal{M}(c;\overline{b})|\,\overline{b},$$

where $\overline{b} = b_1,\dots,b_k$ is a word of Reeb chords of Λ^-, where $\mathcal{M}(c;\overline{b})$ denotes the moduli space of holomorphic disks in (X,L) with boundary on L, with positive puncture at a and negative punctures at $b_1,\dots,b_k$. An argument completely analogous to the argument above showing that $\partial^2 = 0$, looking at the boundary of 1-dimensional moduli spaces, shows that $\Phi \circ \partial^+ = \partial^- \circ \Phi$, where $\partial^\pm$ is the differential on $\mathcal{A}(Y^\pm,\Lambda^\pm)$, i.e., that Φ is a chain map. Consequently, if $\epsilon^-\colon \mathcal{A}(Y^-,\Lambda^-) \to \mathbb{Z}_2$ is an augmentation, then so is $\epsilon^+ = \epsilon^- \circ \Phi$. In particular if $\Lambda^- = \emptyset$ and $(Y,\Lambda) = (Y^+,\Lambda^+)$, then $\mathcal{A}(Y^-,\Lambda^-) = \mathbb{Z}_2$ with the trivial differential, and $\epsilon = \epsilon^+ = \Phi$ is an augmentation of $\mathcal{A}(Y,\Lambda)$.

3.2 *Proof of Theorem 1.1*

If $Q_{[\alpha]}(Y,\Lambda)$ denotes the subspace of $Q(Y,\Lambda)$ generated by Reeb chords c of action $\mathfrak{a}(c) < \alpha$, then $Q_{[\alpha]}(Y,\Lambda)$ is a subcomplex of $Q(Y,\Lambda)$. By definition, the map that takes a Reeb chord c viewed as a generator of $\mathbf{V}_{[\alpha]}(X,L)$ to the dual c^* of c in the cochain complex $Q'_{[\alpha]}(Y,\Lambda)$ of $Q_{[\alpha]}(Y,\Lambda)$ is an isomorphism intertwining the respective differentials. To prove the theorem, since the complex is finite-dimensional below the action α, it remains only to show that the monotonicity condition implies $H^r(Q'_{[\alpha]}(Y,\Lambda)) = H^r(Q'(Y,\Lambda))$ for $\alpha > 0$ large enough. This is straightforward: if $|c| = C_1\mathfrak{a}(c) + C_0$, then

$$H^r(Q'_{[\alpha]}(Y,\Lambda)) = H^r(Q'(Y,\Lambda)),$$

for $\alpha > \frac{r+1-C_0}{C_1}$. □

4 Lagrangian Floer Cohomology of Exact Cobordisms

In this section we introduce a Lagrangian Floer cohomology of exact cobordisms. It is a generalization of the two-copy version of the relative SFT of an exact cobordism (X,L), $L = L_0 \cup L_1$, to the case that each piece of L is embedded but $L_0 \cap L_1 \neq \emptyset$. To prove that this theory has the desired properties, we will use a mixture of results from Floer homology of compact Lagrangian submanifolds and the SFT framework explained in Sect. 2. After setting up the theory, we state a conjectural lemma about how moduli spaces of holomorphic disks with boundary on $L \cup L'$, where L' is a small perturbation of L, can be described in terms of holomorphic disks with boundary on L and a version of Morse theory on L. We then show how Conjecture 1.2 and Corollary 1.3 follow from this conjectural description.

Remark 4.1. The Lagrangian Floer cohomology considered here is closely related to the wrapped Floer cohomology introduced in [1, 14]. Indeed, in analogy with results relating the symplectic homology of a Liouville domain to the linearized contact homology of its boundary, see [2], one expects that the Lagrangian Floer cohomology considered here is isomorphic to the wrapped Floer cohomology.

4.1 *The Chain Complex*

Let X be a simply connected exact symplectic cobordism with $c_1(TX) = 0$ and with good ends. Let L_0 and L_1 be exact Lagrangian cobordisms in X with empty negative ends and with trivial Maslov classes. In other words, (X,L_0) and (X,L_1)

are exact cobordisms with empty $(-\infty)$-boundaries. Let the $(+\infty)$-boundaries of (X,L_0) and (X,L_1) be (Y,Λ_0) and (Y,Λ_1), respectively.

Define

$$C(X;L_0,L_1) = C_\infty(X;L_0,L_1) \oplus C_0(X;L_0,L_1)$$

as follows. The summand $C_\infty(X;L_0,L_1)$ is the $\mathbb{Z}_2$-vector space of formal sums of Reeb chords that start on Λ_0 and end on Λ_1. The summand $C_0(X;L_0,L_1)$ is the $\mathbb{Z}_2$-vector space generated by the transverse intersection points in $L_0 \cap L_1$.

In order to define the grading and a differential on $C(X;L_0,L_1)$, we will consider the following three types of moduli spaces. The first type was considered already in (4); if a is a Reeb chord and $\overline{b} = b_1,\ldots,b_k$ is a word of Reeb chords, we write

$$\mathcal{M}(a;\overline{b})$$

for the moduli space of holomorphic curves in the symplectization $Y \times \mathbb{R}$ with boundary on $\Lambda_0 \times \mathbb{R} \cup \Lambda_1 \times \mathbb{R}$, with positive puncture at a and negative punctures at $b_1,\ldots,b_k$.

The second kind is the standard moduli spaces for Lagrangian Floer homology; if x and y are intersection points of L_0 and L_1, we write

$$\mathcal{M}(x;y)$$

for the moduli space of holomorphic disks in X with two boundary punctures at which the disks are asymptotic to x and y, with boundary on $L_0 \cup L_1$, and that are such that in the orientation on the boundary induced by the complex orientation, the incoming boundary component at x maps to L_0.

Finally, the third kind is a mixture of these; if c is a Reeb chord connecting Λ_0 to Λ_1 and if y is an intersection point of L_0 and L_1, we write $\mathcal{M}(c;y)$ for the moduli space of holomorphic disks in X with two boundary punctures; at one, the disk has a positive puncture at c, and at the other, the disk is asymptotic to y.

If a and c are both Reeb chord generators, then we define the grading difference between a and c to equal the formal dimension $\dim(\mathcal{M}(a;c))$. If g is a Reeb chord or an intersection point generator and if x is an intersection point generator, then we take the grading difference between g and x to equal $\dim(\mathcal{M}(g;x))+1$. Our assumptions on the exact cobordism guarantee that this is well defined.

Write $C = C(X;L_0,L_1)$, $C_0 = C_0(X;L_0,L_1)$, and $C_\infty = C_\infty(X;L_0,L_1)$. Define the differential $d\colon C \to C$, using the decomposition $C = C_\infty \oplus C_0$ given by the matrix

$$d = \begin{pmatrix} d_\infty & \rho \\ 0 & d_0 \end{pmatrix},$$

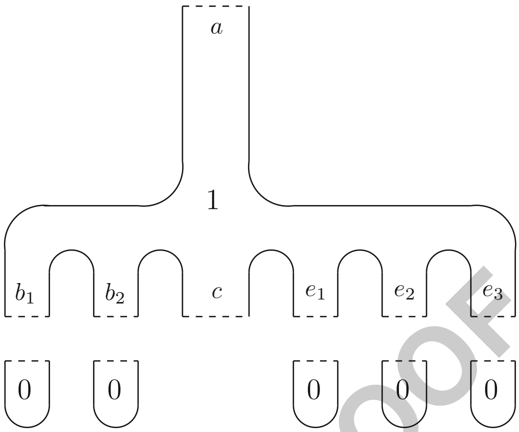

Fig. 1 A disk configuration contributing to the differential of c. Reeb chords are *dashed*. Numbers inside disk components indicate their dimension

where $d_\infty\colon C_\infty \to C_\infty$, $\rho\colon C_0 \to C_\infty$, and $d_0\colon C_0 \to C_0$ are defined as follows. Let c be a Reeb chord from Λ_0 to Λ_1 and let $\theta\colon \mathcal{A}(Y,\Lambda_0) \to \mathbb{Z}_2$ and $\epsilon\colon \mathcal{A}(Y,\Lambda_1) \to \mathbb{Z}_2$ denote the augmentations induced by L_0 and L_1, respectively. Define

$$d_\infty(c) = \sum_{\dim(\mathcal{M}(a;\overline{b}c\overline{e}))=1} |\widehat{\mathcal{M}}(a;\overline{b}c\overline{e})|\epsilon(\overline{b})\theta(\overline{e})\,a, \tag{9}$$

where $\overline{b}$ and $\overline{e}$ are words of Reeb chords from Λ_0 to Λ_0 and from Λ_1 to Λ_1, respectively; see Fig. 1. Let x be an intersection point of L_0 and L_1. Define

$$\rho(x) = \sum_{\dim(\mathcal{M}(c;x))=0} |\mathcal{M}(c;x)|\,c, \tag{10}$$

where c is a Reeb chord from Λ_0 to Λ_1, and

$$d_0(x) = \sum_{\dim(\mathcal{M}(y;x))=0} |\mathcal{M}(y;x)|\,y, \tag{11}$$

where y is an intersection point of L_0 to L_1. See Fig. 2. Then d increases the grading by 1.

Lemma 4.2. *The map d is a differential, i.e., $d^2 = 0$.*

Proof. We first check that $d_\infty^2 = 0$. Let c and a be Reeb chords of index difference 2. Consider two holomorphic disks in $\mathcal{M}(a';\overline{b}_- c\overline{e}_-)$ and $\mathcal{M}(a;\overline{b}_+ a'\overline{e}_+)$ contributing to the coefficient of a in $d^2(c)$. Gluing these two 1-dimensional families at a' and completing with Reeb chord strips at chords in $\overline{b}_+$ and $\overline{e}_+$, we find that the broken disk corresponds to one endpoint of a reduced moduli space $\widehat{\mathcal{M}}(a;\overline{b}c\overline{e})$, where $\overline{b} = \overline{b}_+\overline{b}_-$ and $\overline{e} = \overline{e}_-\overline{e}_+$. Note that there are three possible types of breaking at the boundary of $\widehat{\mathcal{M}}(a;\overline{b}c\overline{e})$:

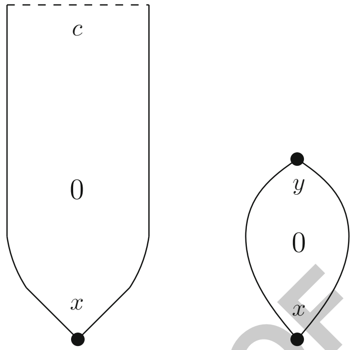

Fig. 2 Disks contributing to the differential of *x*. Reeb chords are *dashed*, and double points appear as *dots*. Numbers inside disk components indicate their dimension

(a) Breaking at a Reeb chord from Λ_0 to Λ_1,
(b) Breaking at a Reeb chord from Λ_0 to Λ_0,
(c) Breaking at a Reeb chord from Λ_1 to Λ_1.

Summing the formal disks of all these boundary configurations gives 0, since the ends of a compact 1-manifold cancel in pairs. To interpret this algebraically, we define

$$\tilde{d}_\infty\colon \mathcal{A}(Y,\Lambda_0)\otimes C_\infty\otimes\mathcal{A}(Y,\Lambda_1)\to\mathcal{A}(Y,\Lambda_0)\otimes C_\infty\otimes\mathcal{A}(Y,\Lambda_1)$$

as follows on generators:

$$\tilde{d}_\infty(w_0\otimes c\otimes w_1)=\sum_{\dim(\mathcal{M}(a;\overline{b}c\overline{e}))=1}|\widehat{\mathcal{M}}(a;\overline{b}c\overline{e})|\,w_0\overline{b}\otimes a\otimes\overline{e}w_1.$$

Then the cancellation mentioned above implies with $\hat{c}=1\otimes c\otimes 1$ that

$$\tilde{d}_\infty^2(\hat{c})+\big(\partial_0\otimes 1\otimes 1\big)(\tilde{d}_\infty(\hat{c}))+\big(1\otimes 1\otimes\partial_1\big)(\tilde{d}_\infty(\hat{c}))=0,$$

where $\partial_j\colon\mathcal{A}(Y,\Lambda_j)\to\mathcal{A}(Y,\Lambda_j)$, $j=0,1$, denotes the contact homology differential. Here the first term corresponds to breaking of type (a), the second to type (c), and the third to type (b). By definition,

$$d_\infty(c)=\big(\epsilon\otimes 1\otimes\theta\big)(\tilde{d}_\infty(\hat{c})).$$

Consequently,

$$\begin{aligned} d_\infty^2(c)&=\big(\epsilon\otimes 1\otimes\theta\big)(\tilde{d}_\infty^2(\hat{c}))\\ &=\Big(\epsilon\otimes 1\otimes\theta\Big)\Big(\big(\partial_0\otimes 1\otimes 1\big)(\tilde{d}_\infty(\hat{c}))+\big(1\otimes 1\otimes\partial_1\big)(\tilde{d}_\infty(\hat{c}))\Big)=0, \end{aligned}$$

since $\epsilon\circ\partial_0=0$ and $\theta\circ\partial_1=0$.

The equation $d_0^2 = 0$ follows as in usual Lagrangian Floer homology from the fact that the terms contributing to d_0^2 are in 1-to-1 correspondence with the ends of the 1-dimensional moduli spaces of the form $\mathcal{M}(x;z)$, where x and z are intersection points of grading difference 2.

Finally, to see that $d_\infty \circ \rho + \rho \circ d_0 = 0$, we consider 1-dimensional moduli spaces of the form $\mathcal{M}(c,x)$, where c is a Reeb chord from Λ_0 to Λ_1 and where x is an intersection point. The analysis of breaking, see [2], in combination with standard arguments from Lagrangian Floer theory, shows that there are two possible breakings in the boundary of $\mathcal{M}(c,x)$:

(a) Breaking at an intersection point,
(b) Breaking at Reeb chords.

In case (a), the broken configuration contributes to $\rho \circ d_0$. In case (b), the disk has two levels: the top level is a curve in a moduli space $\mathcal{M}(a;\overline{b}c\overline{e})$, and the second level is a collection of rigid disks in X with boundary on L_0 and L_1 and with positive punctures at Reeb chords in $\overline{b}$ and $\overline{e}$, respectively, and a rigid disk in $\mathcal{M}(c;x)$. Such a configuration contributes to $d_\infty \circ \rho$, and the desired equation follows.

As above, we let $\mathfrak{a}(c)$ denote the action of a Reeb chord c. Define the action $\mathfrak{a}(x) = 0$ for intersection points $x \in L_0 \cap L_1$. Then the differential on $C = C(X;L_0,L_1)$ increases the action. Define $C_{[\alpha+]} \subset C$ as the subcomplex of formal sums in which all summands have action at least α and let $C_{[\alpha]} = C/C_{[\alpha+]}$ denote the corresponding quotient complex. Let $FH^*_{[\alpha]}(X;L_0,L_1)$ denote the cohomology of $C_{[\alpha]}$ and note that the natural projections give an inverse system of cochain maps

$$\pi^\alpha_\beta \colon C_{[\alpha]} \to C_{[\beta]}, \quad \alpha > \beta.$$

Define the Lagrangian Floer cohomology $FH^*(X;L_0,L_1)$ as the inverse limit of the corresponding inverse system of cohomologies

$$FH^*(X;L_0,L_1) = \varprojlim_\alpha FH^*_{[\alpha]}(X;L_0,L_1).$$

4.2 Chain Maps and Invariance

Our proof of the invariance of the Lagrangian Floer cohomology $FH^*(X;L_0,L_1)$ under isotopies of L_1 uses three ingredients: homology isomorphisms induced by compactly supported isotopies, chain maps induced by joining cobordisms, and chain homotopies induced by compactly supported deformations of adjoined cobordisms. Before proving invariance, we consider these three separately.

4.2.1 Compactly Supported Deformations

Let (X, L_0) and (X, L_1) be exact cobordisms as above. We first consider deformations of (X, L_1) that are fixed in the positive end. More precisely, let L_1^t, $0 \le t \le 1$, be a 1-parameter family of exact Lagrangian cobordisms such that the positive end Λ_1^t is fixed at Λ_1 for $0 \le t \le 1$. Our proof of invariance is a generalization of a standard argument in Floer theory.

We first consider changes of the chain complex. Assuming that L_1^t is generic, there is a finite number of birth/death instances $0 < t_1 < \cdots < t_m < 1$ when two double points cancel or are born at a standard Lagrangian tangency moment. (At such a moment t_j, there is exactly one nontransverse intersection point $x \in L_0 \cap L_1^{t_j}$, $\dim(T_x L_0 \cap T_x L_1^{t_j}) = 1$, and if v denotes the deformation vector field of L_1 at x, then v is not symplectically orthogonal to $T_x L_0 \cap T_x L_1^{t_j}$.)

For $0 \le t \le 1$, let $\mathcal{M}^t$ denote a moduli space of the form $\mathcal{M}^t(g;x)$ or $\mathcal{M}^t(c)$, where x is an intersection point, c a Reeb chord, and g either a Reeb chord or an intersection point of holomorphic disks as considered in the definition of the differential, with boundary on L_0 and L_1^t. If L_1^t is chosen generically, then such a moduli space $\mathcal{M}^t$ is empty for all t, provided $\dim(\mathcal{M}^t) < -1$ and there is a finite number of instances $0 < \tau_1 < \cdots < \tau_k < 1$ where there is exactly one disk of formal dimension -1 that is transversely cut out as a 0-dimensional parameterized moduli space; see [6, Lemma B.8]. We call these instances (-1)-*disk instances*. Furthermore, for generic L_1^t, birth/death instances and (-1)-disk instances are distinct.

For $I \subset [0,1]$, consider the parameterized moduli space of disks

$$\mathcal{M}_I = \cup_{t \in I} \mathcal{M}^t,$$

where $\dim(\mathcal{M}^t) = 0$. If I contains neither (-1)-disk instances nor birth/death instances, then $\mathcal{M}_I$ is a 1-manifold with boundary that consists of rigid disks in $\mathcal{M}^t$ and $\mathcal{M}^{t'}$, where $\partial I = \{t, t'\}$, and it follows by the definition of the differential that the chain complexes $C(X, L_0, L_1^t)$ and $C(X; L_0, L_1^{t'})$ are canonically isomorphic. If, on the other hand, I does contain (-1)-disk instances or birth/death instances, then $\mathcal{M}_I$ has additional boundary points corresponding to broken disks at these instances. It is clear that in order to show invariance, it is enough to show that the homology is unchanged over intervals containing only one (-1)-disk instance or birth/death. For simpler notation, we take $I = [-1, 1]$ and assume that there is a (-1)-disk instance or a birth/death at $t = 0$.

We start with the case of a (-1)-disk. There are two cases to consider: either the (-1)-disk is mixed (i.e., has boundary components mapping both to L_0 and L_1^0), or it is pure (i.e., all of its boundary maps to L_1^0). Consider first the case of a mixed (-1)-disk. Since L_1^t is fixed at infinity, the (-1)-disk must lie in a moduli space $\mathcal{M}(g;x)$, where g is a Reeb chord or an intersection point and where x is an intersection point. Write $C(-) = C(X; L_0, L_1^{-1})$ and $C(+) = C(X; L_0, L_1^1)$ and let d^- and d^+ denote the corresponding differentials. Note that there is a canonical identification between generators of $C(-)$ and $C(+)$.

Lemma 4.3. *Let $\phi\colon C(-)\to C(+)$ be the linear map defined on generators h as follows:*

$$\phi(h)=\begin{cases} x+g & \text{if } h=x,\\ h & \text{otherwise.}\end{cases}$$

If $\Phi(h)=\phi(h+d^-h)$, then $\Phi\colon C(-)\to C(+)$ is a chain isomorphism.

Proof. We first note that the differentials d^+ and d^- agree on the canonically isomorphic subspaces $C_\infty(-)$ and $C_\infty(+)$. In order to study the remaining part of the differential, we consider parameterized 1-dimensional moduli spaces of the form $\mathcal{M}_I(h;y)$, where y is an intersection point and where h is either an intersection point or a Reeb chord. Note that disks at a fixed generic instance that lie in a parameterized moduli of dimension 1 are exactly those that contribute to the differential. Write $\mathcal{M}_I(g;x)$ for the transversely cut-out 0-manifold that is the parameterized moduli space containing the (-1)-disk.

If all punctures at intersection points are considered mixed, then any disk that contribute to the differential is admissible, see Sect. 2.1, and since the (-1)-disk is mixed, it follows from [6, Lemma B.9] (or from a standard result in Floer theory in case both generators are double points; see [13, Lemma 3.5 and Proposition 4.2]) that the boundary of the compactified 1-manifold $\mathcal{M}_I(h;y)$ satisfies the following:

- If $h\neq g$, $y\neq x$, and $[s,t]\subset I$, then

$$\partial\mathcal{M}_{[s,t]}(h;y)=\mathcal{M}^s(h;y)\cup\mathcal{M}^t(h;y);$$

- If $h=g$, then $y\neq x$ and

$$\partial\mathcal{M}_I(g;y)=\mathcal{M}^{-1}(g;y)\cup\mathcal{M}^1(g;y)\cup(\mathcal{M}_I(g;x)\times\mathcal{M}^0(x;y));$$

- If $y=x$, then $h\neq g$ and

$$\partial\mathcal{M}_I(h;x)=\mathcal{M}^{-1}(h;x)\cup\mathcal{M}^1(h;x)\cup(\widehat{\mathcal{M}}^0(h;g)\times\mathcal{M}_I(g;x)).$$

Here $\widehat{\mathcal{M}}^0(h,g)$ denotes the moduli space divided by the $\mathbb{R}$-action if both h and g are Reeb chords, and the moduli space itself otherwise.

Translating this into algebra, we find that for any generator h the following holds:

$$d^+h=\begin{cases} d^-h+\phi(d^-h) & \text{if } |h|=|x|-1,\\ d^-h+x^*(h)\,d^-g & \text{if } |h|=|x|,\\ d^-h & \text{otherwise,}\end{cases}$$

where $x^*\colon C(+)\to\mathbb{Z}_2$ is the map given by $x^*(h)=0$ if $h\neq x$ and $x^*(h)=1$ if $h=x$. The lemma follows.

Consider next the case of a pure (-1)-disk. Since L_1^t is fixed at infinity, such a disk must lie in a moduli space $\mathcal{M}(c)$, where c is a Reeb chord of Λ_1. This case is less straightforward than the case of a mixed (-1)-disk considered above. In order to get control of the resulting change in differential, we need to introduce abstract perturbations of the $\bar{\partial}_J$-operator near the moduli space of J-holomorphic disks. We give a short description here and refer to [6, Sect. B.6] for details. Note first that the change in differential is caused by the change that the augmentation induced by L_1^t undergoes as t passes 0. To describe this change, we study parameterized moduli spaces of the form $\mathcal{M}_I(b)$, $I = [-1,1]$ of dimension 1, where b is a Reeb chord of Λ_1. A priori, broken disks in the boundary of such a moduli space consist of a several level disks with one positive puncture at b and several negative punctures at $c_1,\dots,c_m$, where each c_j is capped off with a disk in $\mathcal{M}(c_j)$ and the only requirement is that the sum of dimensions of the components equal 0. In particular, if $c_j = c$ for several indices j, then since the cap at c has dimension -1, the sum of dimensions over the disks in the symplectization must be larger than 1.

This situation is impossible to control algebraically. In order to gain algebraic control, a perturbation that time orders the complex structures at the negative pure Λ_1-punctures of any disk in the symplectization $(Y\times\mathbb{R},\Lambda_1\times\mathbb{R})$ is introduced. This perturbation needs to be extended over the entire moduli space of 1-punctured holomorphic disks (below a fixed $(+)$-action) in the symplectization $(Y\times\mathbb{R},\Lambda_1\times\mathbb{R})$. Such a perturbation is defined energy level by energy level starting from the lowest one. The time ordering of the negative punctures implies that only one (-1)-disk at a time can be attached to any disk in the symplectization. It is important to note that the time ordering itself may introduce new (-1)-disks, but the positive punctures of such introduced disks all lie close to $t = 0$. More precisely, the count of (-1)-disks with positive puncture at a Reeb chord b depends on the perturbation used on energy levels below $\mathfrak{a}(b)$, and the positive puncture of any such (-1)-disk lies close to $t = 0$ compared to the size of the time-ordering perturbation of negative punctures mapping to b.

Let ϵ^- and ϵ^+ denote the augmentations on $\mathcal{A}(Y,\Lambda)$ induced by (X,L_1^{-1}) and (X,L_1^1), respectively. It is a consequence of [6, Lemma B.15] (which uses the perturbation scheme above) that there is a map K from the set of generators of $\mathcal{A}(Y,\Lambda_1)$ into $\mathbb{Z}_2$ such that if c is a Reeb chord, then $K(c)$ counts (-1)-disks with positive puncture at c, and such that

$$\epsilon^-(c)+\epsilon^+(c) = \Omega_K(\partial c). \tag{12}$$

Here $\partial\colon \mathcal{A}(Y,\Lambda_1)\to\mathcal{A}(Y,\Lambda_1)$ is the contact homology differential, and if $w = b_1\dots b_m$ is a word of Reeb chords, then

$$\Omega_K(w) = \sum_j \epsilon^-(b_1)\dots\epsilon^-(b_{j-1})K(b_j)\epsilon^+(b_{j+1})\dots\epsilon^+(b_m).$$

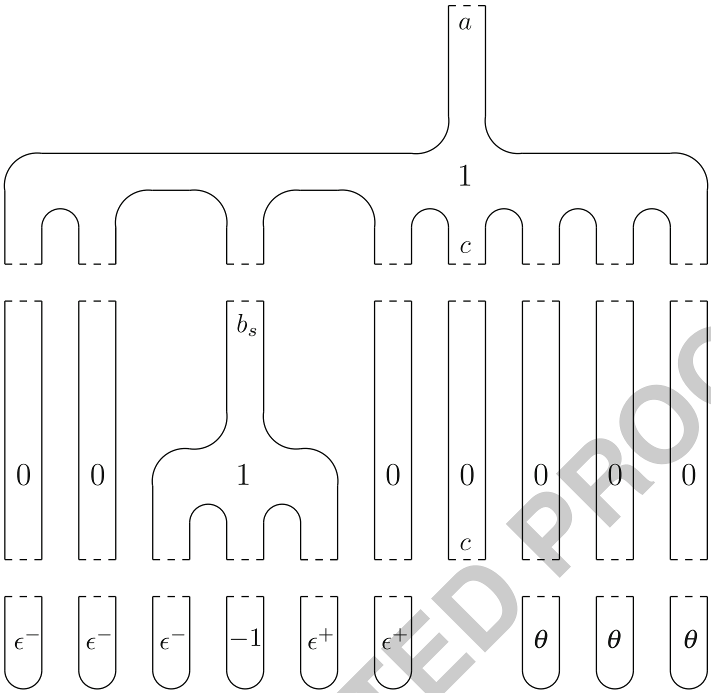

Fig. 3 A disk contributing to $d^+(c)+d^-(c)$: the disk breaks at the pure Reeb chord b_s, and the (-1)-disk is attached to one of the negative ends of the disk with positive puncture at b_s

Define the linear map $\phi\colon C(-)\to C(+)$ as

$$\phi(c)=\sum_{\dim(\mathcal{M}(a;\overline{b}\,c\,\overline{e}))=1}|\widehat{\mathcal{M}}(a;\overline{b}\,c\,\overline{e})|\,\Omega_K(\overline{b})\,\theta(\overline{e})\,a$$

for generators $c\in C_\infty(-)$ and $\phi(x)=0$ for generators $x\in C_0(-)$.

Lemma 4.4. *The map* $\Phi\colon C(-)\to C(+)$*,*

$$\Phi(c)=c+\phi(c),$$

is a chain isomorphism.

Proof. The map is an isomorphism, since the action of any chord in $\phi(c)$ is larger than that of c. We thus need only show that it is a chain map, or in other words, that

$$d^+ + d^- = d^+\circ\phi+\phi\circ d^-.$$

Consider first the operator on the left-hand side acting on a Reeb chord c. According to (12), broken disk configurations that contribute to $d^+(c)+d^-(c)$ are of the following form; see Fig. 3:

1. The top level is a 1-dimensional disk in $Y \times \mathbb{R}$ with positive puncture at a Reeb chord a connecting Λ_0 to Λ_1, followed by k negative punctures at Reeb chords $b_1, \dots, b_k$ connecting Λ_1 to itself, followed by a negative puncture at c connecting Λ_0 to Λ_1, in turn followed by r negative punctures at Reeb chords $e_1, \dots e_r$ connecting Λ_0 to itself.
2. The middle level consists of Reeb chord strips at all negative punctures except b_s for some $1 \le s \le k$. At b_s, a 1-dimensional disk in $Y \times \mathbb{R}$ with boundary on $\Lambda_1 \times \mathbb{R}$ and with negative punctures at $q_1, \dots, q_m$ is attached.
3. The bottom level consists of rigid disks with boundary on L_0 and positive puncture at e_j attached at all punctures e_j, $j = 1, \dots, r$, rigid disks with boundary on L_1^{-1} attached at punctures b_j, $1 \le j \le s-1$, and at punctures q_j, $1 \le j < v$, a (-1)-disk attached at q_v, and rigid disks with boundary on L_1^1 at punctures b_j, $s+1 \le j \le k$, and q_j, $v < j \le m$.

Consider gluing the top and middle levels above in the symplectization. This gives one boundary component of a reduced 1-dimensional moduli space. The other boundary component corresponds to one of three breakings: at a pure Λ_1-chord, at a chord connecting Λ_0 to Λ_1, or at a pure Λ_0 chord. The first type of breaking contributes to $d_-(c) + d_+(c)$ as well; see Fig. 3. The second type contributes to either $d^+ \circ \phi(c)$ or $\phi \circ d^-(c)$ depending on the factor to which the (-1)-end goes – see Figs. 4 and 5, respectively – and finally, the total contribution of the third type of breaking is 0, since $\theta \circ \partial = 0$, where $\partial\colon \mathcal{A}(Y, \Lambda_0) \to \mathcal{A}(Y, \Lambda_0)$ is the contact homology differential and where $\theta\colon \mathcal{A}(Y, \Lambda_0) \to \mathbb{Z}_2$ is the augmentation; see Fig. 6.

Next, consider the operators acting on a double point x. The chain map property in this case follows from an argument similar to the one just given. Additional boundary components of the parameterized moduli space $\mathcal{M}^t(c; x)$ correspond to two level disks with top level a disk as in (1) above and with bottom level as in (3) above with the addition that there is a rigid disk with positive puncture at c and a puncture at x. We conclude that Φ is a chain map.

Finally, consider a birth/death moment involving intersection points x and y. Assume that $x, y \in C(+)$ (birth moment). Then $d^+ x = y + v$, where v does not contain any y-term; see [13, Lemma 3.7 and Proposition 5.1]. Define the map $\Phi\colon C(+) \to C(-)$ by

$$\Phi(x) = 0, \quad \Phi(y) = v, \quad \text{and} \quad \Phi(w) = w \text{ for } w \neq x, y.$$

Define the map $\Psi\colon C(-) \to C(+)$ by

$$\Psi(c) = c + y^*(d^+ c)x.$$

Lemma 4.5. *The maps Φ and Ψ are chain maps that induce isomorphisms on homology.*

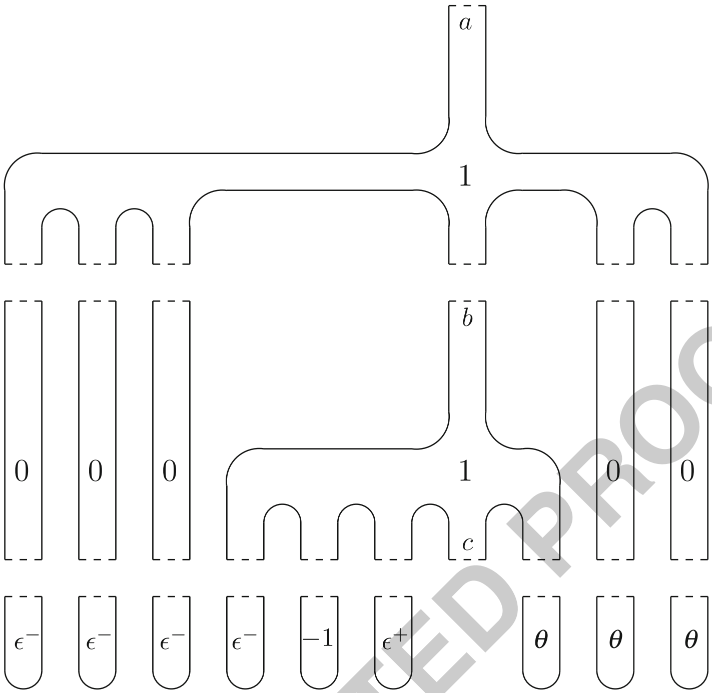

Fig. 4 A disk contributing to $d^+ \circ \phi(c)$: the disk breaks at the mixed chord b, and the (-1)-disk is attached to the disk with positive puncture at b

Proof. A straightforward generalization of the gluing theorem [9, Proposition 2.16] (in the case that only one disk is glued at the degenerate intersection) shows that if g is a generator of $C(-)$, then

$$d^-(c) = d^+(c) + y^*(d^+(c))v.$$

The lemma then follows from a straightforward calculation.

4.2.2 Joining Cobordisms

Let (X, L_0) and (X, L_1) be cobordisms as above. Consider the trivial cobordism $(Y \times \mathbb{R}, \Lambda_0 \times \mathbb{R})$ and some cobordism $(Y \times \mathbb{R}, L_1^a)$, where the $(-\infty)$-boundary of L_1^a equals Λ_1 and its $(+\infty)$-boundary equals Λ_1^a. Then we can join these cobordisms to (X, L_0) and (X, L_1), respectively. This results in a new pair of cobordisms (X, L_0) and $(X, \tilde{L}_1)$. Consider the subdivision

$$C(X; L_0, \tilde{L}_1) = C_\infty(X; L_0, \tilde{L}_1) \oplus C_0(X; L_0, \tilde{L}_1).$$

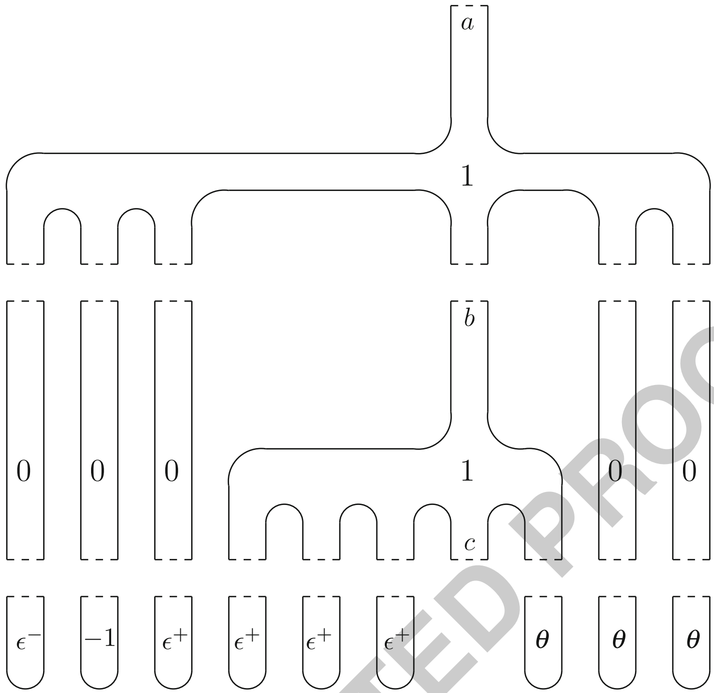

Fig. 5 A disk contributing to $\phi \circ d^{-}(c)$: the disk breaks at the mixed chord b, and the (-1)-disk is attached to the disk with positive puncture at a

If $z \in L_0 \cap \tilde{L}_1$, then either $z \in L_0 \cap L_1$ or $z \in (\Lambda_0 \times \mathbb{R}) \cap L_1^a$, and we have the further subdivision

$$C_0(X; L_0, \tilde{L}_1) = C_0(X; L_0, L_1) \oplus C_0(Y \times \mathbb{R}; \Lambda_0 \times \mathbb{R}, L_1^a).$$

We define a map

$$\begin{aligned} \Phi \colon & C_\infty(X; L_0, L_1) \oplus C_0(X; L_0, L_1) \to \\ & C_\infty(X; L_0, \tilde{L}_1) \oplus C_0(Y \times \mathbb{R}; \Lambda_0 \times \mathbb{R}, L_1^a) \oplus C_0(X; L_0, L_1), \end{aligned}$$

with matrix

$$\begin{pmatrix} \phi_\infty & 0 \\ \phi_0 & 0 \\ 0 & \mathrm{id} \end{pmatrix},$$

as follows, using moduli spaces of holomorphic disks in $Y \times \mathbb{R}$ with boundary on $(\Lambda_0 \times \mathbb{R}) \cup L_1^a$. If c is a generator of $C_\infty(X; L_0, L_1)$, then c is a Reeb chord from Λ_0 to Λ_1. Thinking of c as lying in the negative end of $(Y \times \mathbb{R}, \Lambda_0 \times \mathbb{R} \cup L_1^a)$, we define

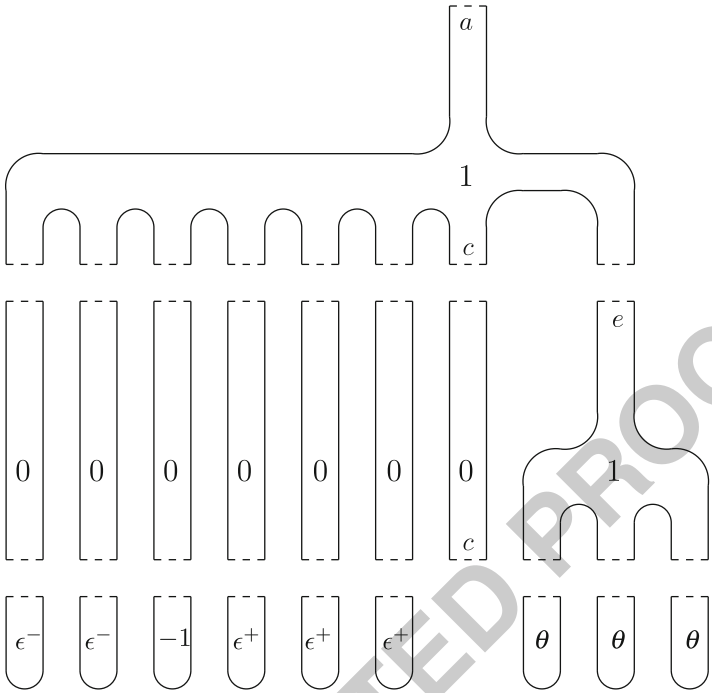

Fig. 6 Disks with total contribution 0: the disk breaks at the pure Λ_0 chord e, and the total contribution vanishes, since $\theta \circ \partial = 0$

$$\phi_0(c) = \sum_{\dim \mathcal{M}(z;\overline{b}c\overline{e})=0} |\mathcal{M}(x;\overline{b}c\overline{e})|\epsilon(\overline{b})\theta(\overline{e})z,$$

where the sum ranges over intersection points $z \in C_0(Y \times \mathbb{R}; \Lambda \times \mathbb{R}, L_1^a)$, and

$$\phi_\infty(c) = \sum_{\dim \mathcal{M}(a;\overline{b}c\overline{e})=0} |\mathcal{M}(a;\overline{b}c\overline{e})|\epsilon(\overline{b})\theta(\overline{e})a,$$

where the sum ranges over Reeb chords a connecting Λ_0 to Λ_1^a, i.e., over generators of $C_\infty(X; L_0, \tilde{L}_1)$.

Lemma 4.6. *The map Φ is a chain map. That is, if d and $\tilde{d}$ denote the differentials on $C(X, L_0, L_1)$ and $C(X, L_0, \tilde{L}_1)$, respectively, then*

$$\tilde{d} \circ \Phi = \Phi \circ d.$$

Proof. As above, we write the differentials d and $\tilde{d}$ in matrix form with respect to the splittings $C = C_\infty \oplus C_0$:

$$d = \begin{pmatrix} d_\infty & \rho \\ 0 & d_0 \end{pmatrix} \quad \text{and} \quad \tilde{d} = \begin{pmatrix} \tilde{d}_\infty & \tilde{\rho} \\ 0 & \tilde{d}_0 \end{pmatrix}.$$

Furthermore, we decompose $\tilde{d}_0$ as $\tilde{d}_0 = \tilde{d}_0' \oplus \tilde{d}_0''$ with respect to the decomposition

$$C_0(X; L_0, \tilde{L}_1) = C_0(Y \times \mathbb{R}; \Lambda_0 \times \mathbb{R}, L_1^a) \oplus C_0(X; L_0, L_1),$$

i.e., d_0' maps into the first summand and d_0'' into the second.

Consider first an intersection point $x \in L_0 \cap L_1$. In this case, we must show that

$$\phi_\infty(\rho x) + \phi_0(\rho x) + d_0 x = \tilde{\rho} x + \tilde{d}_0' x + \tilde{d}_0'' x.$$

Note first that holomorphic disks that contribute to $d_0 x$ also contribute to $\tilde{d}_0'' x$, and hence the last terms on the left- and right-hand sides cancel.

A moduli space contributing to $\tilde{\rho} x$ or $\tilde{d}_0' x$ is of the form $\mathcal{M}(g; x)$, where g is respectively a Reeb chord connecting Λ_0 to Λ_1^a or an intersection point in $(\Lambda_0 \times \mathbb{R}) \cap L_1^a$. Consider stretching along the hypersurface $(Y, \Lambda_0 \cup \Lambda_1)$ where the cobordisms are joined. It is a consequence of [2] that families of disks in $\mathcal{M}(g; x)$ converge to broken disks with one part in $(X, L_0 \cup L_1)$, one in $(Y \times \mathbb{R}; \Lambda_0 \times \mathbb{R} \cup L_1^a)$, and possibly other levels in the symplectizations, in the limit. Since $\dim(\mathcal{M}(g; x)) = 0$, every level in the limit must have dimension 0 by transversality. By admissibility of the disks in $\mathcal{M}(g; x)$ there is exactly one Reeb chord c' connecting Λ_0 to Λ_1 in the hypersurface Y in the broken disk that arises in the limit. By definition, the sum of the two first terms on the left-hand side counts broken disks of this type, and the chain map equation follows in this case.

Consider second a Reeb chord c connecting Λ_0 to Λ_1. In this case, we must show that

$$\phi_\infty(d_\infty c) + \phi_0(d_\infty c) = \tilde{d}_\infty(\phi_\infty c) + \tilde{\rho}(\phi_0 c) + \tilde{d}_0''(\phi_0 c).$$

To show that the chain map equation holds in this case, we first consider 1-dimensional moduli spaces $\mathcal{M}(z; \overline{b} c \overline{e})$ of disks in $Y \times \mathbb{R}$ with boundary on $(\Lambda_0 \times \mathbb{R}) \cup L_1^a$. Here z is an intersection point in $(\Lambda_0 \times \mathbb{R}) \cap L_1^a$, $\overline{b} = b_1, \ldots, b_k$, and $\overline{e} = e_1, \ldots, e_r$ are words of Reeb chords connecting Λ_1 (i.e., the negative end of L_1^a) to itself and connecting Λ_0 to itself, respectively, and such that $|b_j| = 0$ for all j, and $|e_l| = 0$ for all l. By transversality and admissibility, the boundary points of such a moduli space consists of a broken disks with two components that are either both 0-dimensional and joined at an intersection point, or a 1-dimensional disk in the negative end joined at a Reeb chord to a 0-dimensional disk. The former broken disks contribute to $d_0''(\phi_0 c)$ and the latter to $\phi_0(d_\infty c)$. Thus these two terms cancel.

Second, we consider 1-dimensional moduli spaces $\mathcal{M}(a; \overline{b} c \overline{e})$ of disks in $Y \times \mathbb{R}$ with boundary on $(\Lambda_0 \times \mathbb{R}) \cup L_1^a$. Here a is a Reeb chord connecting Λ_0 to Λ_1^a at the positive end of $Y \times \mathbb{R}$, with $\overline{b} = b_1, \ldots, b_k$ and $\overline{e} = e_1, \ldots, e_r$ as above. By transversality and admissibility, the boundary of such a moduli space consists of two level broken disks of the following form.

- Broken disks with two 0-dimensional components joined at an intersection point. Such disks contribute to $\tilde{\rho}(\phi_0 c)$.
- Broken disks with one 1-dimensional component in the positive end joined at Reeb chords to 0-dimensional disks. Such disks contribute to $\tilde{d}_\infty(\phi_\infty c)$.
- Broken disks with one 1-dimensional component in the negative end joined at a mixed Reeb chord to a 0-dimensional disk. Such disks contribute to $\phi_\infty(d_\infty c)$.
- Broken disks with one 1-dimensional component in the negative end joined at a pure Reeb chord to a 0-dimensional disk. Contributions from such disks cancel, since $\theta \circ \partial = 0$ and $\epsilon \circ \partial = 0$.

It follows that the first term on the left-hand side cancels with the sum of the two first terms on the right-hand side. The lemma follows.

4.2.3 Joining Maps and Deformations

We next consider generic 1-parameter families of cobordisms as considered in Sect. 4.2.2. More precisely, let (X, L_0) and (X, L_1) be exact cobordisms as usual. Let $(Y \times \mathbb{R}, L_1^a(t))$, $t \in [a,b]$, be a 1-parameter family of exact cobordisms that is constant outside a compact set and such that the $(-\infty)$-boundary of $L_1^a(t)$ equals Λ_1 and its $(+\infty)$-boundary equals Λ_1^a. Adjoining $(Y \times \mathbb{R}, L_1^a(t))$ to (X, L_1), we obtain a 1-parameter family of cobordisms $(X, \tilde{L}_1(t))$, and for generic t in $[a,b]$, where moduli spaces are transversely cut out, corresponding chain maps

$$\Phi_t \colon C(X; L_0, L_1) \to C(X; L_0, \tilde{L}_1(t)).$$

Furthermore, since $(X, \tilde{L}_1(t))$ and $(X, \tilde{L}_1(t'))$ are related by a compact deformation, Lemmas 4.3–4.5 provide chain maps

$$\Psi_{tt'} \colon C(X; L_0, L_1(t)) \to C(X; L_0, \tilde{L}_1(t')),$$

where t and t' are generic, that induce isomorphisms on homology. In fact, unless the interval between t and t' contain (-1)-disk instances or birth/death instances, the map $\Psi_{tt'}$ is the canonical identification map on generators. Here the births/deaths take place in the added cobordism, and the (-1)-disk instances correspond to (-1)-disk instances in the added cobordism. As we shall see below, if the interval between t and t' contains a birth/death or a (-1)-disk instance, then there is a chain homotopy connecting the chain maps $\Psi_{tt'} \circ \Phi_t$ and $\Phi_{t'}$. The proofs of these results are similar to the proofs of results in Sects. 4.2.1 and 4.2.2, and many details from there will not be repeated. For convenient notation below we take $t = -1$, $t' = 1$ and assume that the critical instance is at $t = 0$. Furthermore, we write $C = C(X, L_0, L_1)$ with differential d, $\tilde{C}_\pm = C(X, L_0, \tilde{L}_1(\pm 1))$ with differential $\tilde{d}_\pm$, $\Phi_\pm = \Phi_{\pm 1}$, and $\Psi = \Psi_{-1 1}$.

Consider first the case of a pure (-1)-disk in $(Y \times \mathbb{R}, L_1^a(t))$. Applying our perturbation scheme that time orders the negative punctures of disks in the positive end $(Y \times \mathbb{R}, \Lambda_1^a \times \mathbb{R})$, we see that such a disk gives rise to several (-1)-disks in

$(Y \times \mathbb{R}, L_1^a(t))$. The (-1)-disks in the total cobordism $(X, \tilde{L}_1(t))$ are then these (-1)-disks in $(Y \times \mathbb{R}, L_1^a(t))$, capped off with 0-dimensional rigid disks in (X, L_1); see [6, Lemmas 4.3 and 4.4].

Lemma 4.7. *If there is a pure (-1)-disk at $t = 0$, then the following diagram commutes:*

$$\begin{array}{ccc} C & \xrightarrow{\Phi_-} & \tilde{C}_- \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \Psi} \\ C & \xrightarrow[\Phi_+]{} & \tilde{C}_+ \end{array}.$$

Proof. Consider the parameterized 1-dimensional moduli space corresponding to a 0-dimensional moduli space contributing to Φ_+. A boundary component at $t = -1$ contributes to $\Psi \circ \Phi_-$. A boundary component in the interior of $[-1, 1]$ is a broken disk consisting of a 1-dimensional disk in an end and a (-1)-disk and 0-disks in the cobordisms. If the 1-dimensional disk lies in the upper end, then the broken disk contributes to $\Psi \circ \Phi_-$, and as usual, the total contribution of disks with a 1-dimensional disk in the lower end is 0, since augmentations are chain maps. The result follows.

Second, consider the case of a mixed (-1)-disk. By admissibility, any disk contributing to the chain maps then contains at most one such (-1)-disk; see [6, Lemma 2.8]. Define the map $K\colon C \to \tilde{C}_+$ as follows:

$$K(x) = 0$$

if x is an intersection point generator, and

$$K(c) = \sum_{\dim(\mathcal{M}_I(g;\overline{b}c\overline{e}))=0} |\mathcal{M}_I(g;\overline{b}c\overline{e})| \epsilon(\overline{b}) \theta(\overline{e})\, g$$

if c is a Reeb chord generator, where $\mathcal{M}_I(g;\overline{b}c\overline{e})$ is the parameterized moduli space of disks in $Y \times \mathbb{R}$ with boundary on $\Lambda_0 \times \mathbb{R}$ and $L_1^a(t)$, and where $\overline{b}$ and $\overline{e}$ are Reeb chords of Λ_0 and Λ_1, respectively.

Lemma 4.8. *If there is a mixed (-1)-disk at $t = 0$, then the chain maps in the diagram*

$$\begin{array}{ccc} C & \xrightarrow{\Phi_-} & \tilde{C}_- \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \Psi} \\ C & \xrightarrow[\Phi_+]{} & \tilde{C}_+ \end{array}$$

satisfy $\Psi \circ \Phi_- + \Phi_+ = K \circ d + \tilde{d}_+ \circ K$.

Proof. Consider first the left-hand side acting on an intersection point $x \in L_0 \cap L_1$. Both maps Φ_- and Φ_+ are then inclusions, and by definition, $\Psi(x)$ counts (-1)-disks in $\mathcal{M}(g;x)$, where g is a generator. Since there are no (-1)-disks in the lower cobordism, we find that in a moduli space that contributes to $\Psi(x)$, the generator g is either an intersection point or a Reeb chord in the upper cobordism. Consider now the splittings of a (-1)-disk as we stretch over the joining hypersurface: it splits into a (-1)-disk in the upper cobordism and 0-dimensional disks in the lower. By definition, the count of such split disks is $K(d(x))$. Since $K(x) = 0$, the chain map equation follows.

Consider next the left-hand side acting on a Reeb chord generator c. Here $\Phi_\pm(c)$ are given by counts of 0-dimensional disks in the upper cobordism, and Ψ is a count of (-1)-disks emanating at double points. In particular, $\Psi(a) = a$ for any Reeb chord. Consider now a moduli space that contributes to Φ_+. The boundary of the corresponding parameterized moduli space consists of rigid disks over endpoints as well as broken disks with a 1-dimensional disk in either symplectization end and a (-1)-disk in the cobordism. The total count of such disks gives the desired equation after one observes that for the usual reason, splittings at pure chords do not contribute.

Third, consider a birth/death instance.

Lemma 4.9. *If there is a birth/death instance at $t = 0$, then the following diagram commutes:*

$$\begin{array}{ccc} C & \xrightarrow{\Phi_-} & \tilde{C}_- \\ \uparrow{\scriptstyle \mathrm{id}} & & \uparrow{\scriptstyle \Psi} \\ C & \xrightarrow[\Phi_+]{} & \tilde{C}_+ \end{array}.$$

Proof. Assume that the canceling pair of double points is (x,y) with $\tilde{d}_+ x = y + v$ as in Lemma 4.5. As was the case there, disks from x to v can on the one hand be glued to rigid disks ending at y, resulting in rigid disks, and on the other, can be glued to rigid disks ending at x, resulting in nonrigid disks. Commutativity then follows from a straightforward calculation.

4.2.4 Invariance

Let (X, L_0) and (X, L_1) be exact cobordisms as above.

Theorem 4.10. *The Lagrangian Floer homology $FH^*(X; L_0, L_1)$ is invariant under exact deformations of L_1.*

Proof. The proof is similar to the proof of Theorem 2.1: Any deformation considered can be subdivided into a compactly supported deformation and a Legendrian isotopy at infinity. The former type induces isomorphisms on homology by

Lemmas 4.3–4.5. The latter type of deformation gives rise to an invertible exact cobordism, which in turn gives a chain map on homology by Lemma 4.6. Lemmas 4.7–4.9 show that on the homology level these maps are independent of compact deformations of the cobordism added. A word-for-word repetition of the proof of Theorem 2.1 then finishes the proof.

4.3 Holomorphic Disks for $FH^*(X;L,L')$

Let (X,L) be an exact cobordism as considered above. Let L' denote a copy of L. In order to make L and L' transverse, we identify a neighborhood of $L \subset X$ with the cotangent bundle T^*L. Pick a Morse function $\tilde{F}\colon L \to \mathbb{R}$ such that

$$\tilde{F}(y,t) = C + t,$$

where C is a constant, for $(y,t) \in \Lambda \times [T,\infty)$ for some $T > 0$. Cut the function $\tilde{F}$ off outside a small neighborhood of L and let L'' be the image of L under the time 1-flow of the Hamiltonian vector field of $\epsilon\tilde{F}$ for small ϵ. Then L intersects L'' transversely. However, in the end where $\tilde{F} = C + t$, the Hamiltonian just shifts Λ along the Reeb flow, so that there is a Reeb chord from Λ to Λ'' at every point of Λ. In order to perturb our way out from this Morse–Bott situation, we identify a neighborhood of $\Lambda \subset Y$ with $J^1(\Lambda)$, fix a Morse function $f\colon \Lambda \to \mathbb{R}$, and let Λ' denote the graph of the 1-jet extension of f. Then Λ and Λ' are contact isotopic via the contact isotopy generated by the time-dependent contact Hamiltonian $H_t(q,p,z) = \psi(t)f(q)$, where $(q,p,z) \in T^*\Lambda \times \mathbb{R}$ and $\psi(t)$ is a cut-off function. Take f and $\frac{\mathrm{d}\psi}{\mathrm{d}t}$ very small and adjoin the cobordism $Y \times \mathbb{R}$ with the symplectic form $de^t(\lambda - H_t)$ to (X,L). This gives the desired L' with Λ' as $(+\infty)$-boundary.

We will state a conjectural lemma that gives a description of holomorphic disks with boundary on $L \cup L'$. To this end, we first describe a version of Morse theory on L and then discuss intersection points in $L \cap L'$ and Reeb chords of $\Lambda \cup \Lambda'$.

Consider the gradient equation $\dot{x} = \nabla F(x)$ of the function $F\colon L \to \mathbb{R}$, where

$$F(x) = \tilde{F}(x) + \psi(t)f(y),$$

where we write $x = (y,t) \in \Lambda \times [T,\infty)$. It is easy to see that if a solution of $\dot{x} = \nabla F(x)$ leaves every compact, then it is exponentially asymptotic to a *critical-point solution* of the form $s \mapsto (y_0,s)$, where $\nabla f(y_0) = 0$ in $\Lambda \times [0,\infty)$. Furthermore, every sequence of solutions of $\dot{x} = \nabla F(x)$ has a subsequence that converges to a several-level solution with one level in L and levels in $\Lambda \times \mathbb{R}$ that are solutions to the gradient equation of $f + t$, asymptotic to critical-point solutions. We call solutions that have formal dimension 0 (after dividing out reparameterization) and that are transversely cut out *rigid flow lines*. Consider a moduli space $\mathcal{M}(c)$ of holomorphic disks in X

with boundary on L or a moduli space $\mathcal{M}(c;\overline{b})$ in $Y\times\mathbb{R}$ with boundary on $\Lambda\times\mathbb{R}$. Marking a point on the boundary, there is an evaluation map $\mathrm{ev}\colon \mathcal{M}^*(c)\to L$ or $\mathrm{ev}\colon \mathcal{M}^*(c;\overline{b})\to \Lambda\times\mathbb{R}$; see [7, Sect. 6]. Consider now a flow line of F or of $f+t$ that hits the image of ev. We call such a configuration a *generalized disk*, and we say that it is rigid if it has formal dimension 0 (after dividing out the $\mathbb{R}$-translation in $Y\times\mathbb{R}$) and if it is transversely cut out.

Note that points in $L\cap L'$ are in 1-to-1 correspondence with critical points of F. We use the terms "intersection point" and "critical point" of F interchangeably. Note also that the Reeb chords connecting Λ to Λ' are of two kinds: *short chords*, which correspond to critical points of f, and *long chords* close to each Reeb chord of Λ. As with intersection points, we will sometimes identify critical points of f with their corresponding short Reeb chords. Also note that to each Reeb chord of Λ there is a unique Reeb chord of Λ'.

The following conjectural lemma is an analogue of [7, Theorem 3.6].

Lemma 4.11 (Conjectural). *Let $\overline{b}'$ denote a word of Reeb chords of Λ' and let $\overline{b}$ denote the corresponding word of Reeb chords of Λ. Let also $\overline{e}$ denote a word of Reeb chords of Λ. If c is a long Reeb chord connecting Λ to Λ', then let $\hat{c}$ denote the corresponding Reeb chord of Λ.*

For sufficiently small shift L' of L, there are the following 1-to-1 correspondences:

1. *If a and c are long Reeb chords, then rigid disks in $\mathcal{M}(a;\overline{b}'c\overline{e})$ correspond to rigid disks in $\mathcal{M}(\hat{a};\overline{b}\,\hat{c}\,\overline{e})$.*
2. *If a is a long Reeb chord and c is a short Reeb chord, then rigid disks in $\mathcal{M}(a;\overline{b}'c\overline{e})$ correspond to rigid generalized disks with disk component in $\mathcal{M}(\hat{a};\overline{b}\,\overline{e})$ and with flow line asymptotic to the critical-point solution of c at $-\infty$ and ending at a boundary point between the last $\overline{b}$-chord and the first $\overline{e}$-chord.*
3. *If a and c are short Reeb chords, then rigid disks in $\mathcal{M}(a;c)$ correspond to rigid flow lines asymptotic to the critical-point solutions of c and of a at $-\infty$ and $+\infty$, respectively.*
4. *If x is an intersection point and a is a long Reeb chord, then rigid disks in $\mathcal{M}(a;x)$ correspond to rigid generalized disks with disk component in $\mathcal{M}(\hat{a})$ and with gradient line starting at x and ending at the boundary of the disk.*
5. *If x is an intersection point and c is a short Reeb chord, then rigid disks in $\mathcal{M}(c;x)$ correspond to rigid flow lines starting at x and asymptotic to the critical-point solution of c at $+\infty$.*

Here items (1) to (3) follow from [7, Theorem 3.6] in combination with [8, Sect. 2.7] in the special case $Y=P\times\mathbb{R}$, where P is an exact symplectic manifold. Proofs of (4) and (5) would require an analysis analogous to that of [7, Sect. 6] carried out for a symplectization, taking into account the interpolation region used in the construction of L'.

4.4 Outline of Proof of Conjecture 1.2

Choose a grading on $C(X;L,L')$ such that if c is a long Reeb chord connecting Λ to Λ', then the degree $|c|$ satisfies $|c| = |\hat{c}|$, where $\hat{c}$ is the corresponding Reeb chord of Λ; see (3). Let $C = C_{[\alpha]}(X;L,L')$ and consider the decomposition

$$C = C_+ \oplus C_0,$$

where C_+ is generated by long Reeb chords and where C_0 is generated by short Reeb chords and double points. Then C_+ is a subcomplex, and we have the exact sequence

$$0 \longrightarrow C_+ \longrightarrow C \longrightarrow \widehat{C} \longrightarrow 0,$$

where $\widehat{C} = C/C_+$. If L and L' are sufficiently close, then the augmentations ϵ and θ agree, and Lemma 4.11(1) implies that the differential on C_+ is identical to that on $\mathbf{V}_{[\alpha]}(X,L)$. Furthermore, Lemma 4.11(3),(4) implies that the differential on $\widehat{C}$ is that of the Morse complex of L, and the existence of the exact sequence follows.

Consider next the isomorphism statement. Since $\mathbb{C}^n$ and $J^1(\mathbb{R}^{n-1}) \times \mathbb{R}$ satisfy monotonicity conditions, the homology of C in a fixed degree can be computed using a fixed sufficiently large energy level. Furthermore, in $\mathbb{C}^n$ or $J^1(\mathbb{R}^{n-1})$, L is displaceable, i.e., L' can be moved by Hamiltonian isotopy in such a way that $L \cap L' = \emptyset$ and so that there are no Reeb chords connecting Λ and Λ'. Hence by the invariance of Lagrangian Floer cohomology proved in Theorem 4.10, the total complex C is acyclic. The theorem follows. □

4.5 Outline of Proof of Corollary 1.3

In order to discuss Corollary 1.3, we first describe the duality exact sequence (2) in more detail.

4.5.1 Properties of the Duality Exact Sequence

We recall how the exact sequence (2) was constructed. Let Λ'' be a copy of Λ shifted a large distance (compared to the length of any Reeb chord of Λ) away from Λ in the Reeb direction and then perturbed slightly by a Morse function f. The part of the linearized contact homology complex of $\Lambda \cup \Lambda''$ generated by mixed Reeb chords was split as a direct sum $Q \oplus C \oplus P$. Here Q is generated by the mixed Reeb chords near Reeb chords of Λ that connect the lower sheet of Λ to the upper sheet of Λ'', C is generated by the Reeb chords that correspond to critical points of f, and P is generated by the Reeb chords near Reeb chords of Λ that connect the upper sheet of Λ to the lower sheet of Λ''. The linearized contact homology differential ∂ on $Q \oplus C \oplus P$ has the form

$$\partial = \begin{pmatrix} \partial_q & 0 & 0 \\ \rho & \partial_c & 0 \\ \eta & \sigma & \partial_p \end{pmatrix}.$$

Using the analogue of Lemma 4.11, the subcomplex (P, ∂_p) can be shown to be isomorphic to the dual complex of the complex (Q, ∂_q) using the natural pairing that pairs Reeb chords in Q and P that are close to the same Reeb chord of Λ. Furthermore, the complex (Q, ∂_q) is canonically isomorphic to the linearized contact homology complex $(Q(\Lambda), \partial_1)$.

The next step is the observation that since Λ is displaceable (in the sense above), the complex $C \oplus Q \oplus P$ is acyclic. Using the coarser decomposition $(Q \oplus C) \oplus P$ and writing

$$\partial = \begin{pmatrix} \partial_{qc} & 0 \\ H & \partial_p \end{pmatrix},$$

the chain map induced by

$$H = \begin{pmatrix} \eta & \sigma \end{pmatrix}$$

induces an isomorphism on homology between $Q \oplus C$ and P. Here the map $\sigma : C \to P$ counts generalized trees, whereas the map from $\eta : Q$ to P counts disks with boundary on Λ and with two positive punctures disks. (To make sense of the latter count and have transversely cut-out moduli spaces, actual disks counted have boundary on Λ and on a nearby copy Λ'.) The exact sequence (2) is then constructed from the long exact sequence of the short exact sequence for the complex $C \oplus Q$.

Below we will also make use of the other splitting of the acyclic complex $Q \oplus C \oplus P$ as $Q \oplus (C \oplus P)$ with differential

$$\partial = \begin{pmatrix} \partial_q & 0 \\ H' & \partial_{cp} \end{pmatrix},$$

where

$$H' = \begin{pmatrix} \rho \\ \eta \end{pmatrix}$$

induces an isomorphism on homology: our proof of Corollary 1.3 relates the maps H and H' to isomorphisms coming from Lagrangian Floer homology.

Let C_* denote the Morse complex for $f\colon \Lambda \to \mathbb{R}$ and let $\bar{C}^*$ denote the Morse complex for $-f$. Then $C_* = \bar{C}^{n-1-*}$, and we have the following diagram of chain maps:

$$\begin{array}{ccccccc}
\longrightarrow & \bar{C}^{n-k-2} & \longrightarrow & P^{n-k-1} & \longrightarrow & P^{n-k-1} \oplus \bar{C}^{n-k-1} \\
 & \| & & H \uparrow & & \uparrow H' \\
\longrightarrow & C_{k+1} & \longrightarrow & Q_{k+1} \oplus C_{k+1} & \longrightarrow & Q_{k+1} \\
\longrightarrow & \bar{C}^{n-k-1} & & & & \\
 & \| & & \dots & & \\
\longrightarrow & C_k & & & &
\end{array} \tag{13}$$

Lemma 4.12. *The diagram* (13) *commutes after passing to homology.*

Proof. The first and last squares commute already on the chain level by definition of the differential. Commutativity of the middle square can be seen as follows. Starting at the lower left corner with an element from Q_{k+1}, it is clear that the results of going up then right, and right then up have common component in the P^{n-k-1}-summand. Using the commutativity already established, we see that starting with an element in C_{k+1} and going up, then right is the same thing as first pulling that element back to the left and then going up and two steps to the right. This vanishes in homology by exactness and hence gives the same result as going right then up. Finally, going right then up and projecting to the $\bar{C}^{n-k-1}$-component is the same as going right twice and then up. This vanishes in homology by exactness and gives the same result as going the other way.

4.5.2 Proof of Corollary 1.3

We write down the diagram on the chain level. Let (C_*, ∂_C) denote the Morse complex of the function $-f\colon \Lambda \to \mathbb{R}$ and let (I_*, ∂_I) denote the Morse complex of the function $-F\colon L \to \mathbb{R}$ that computes the relative homology of $(\bar{L}, \partial\bar{L})$. As above, we write $\overline{C}^*$ and $\overline{I}^*$ for the corresponding cochain complexes of f and F, respectively.

Then the complex that computes the homology for L is $(I \oplus C, \partial_{IC})$, where

$$\partial_{IC} = \begin{pmatrix} \partial_I & 0 \\ \mu & \partial_C \end{pmatrix},$$

where μ counts flow lines of F that start at a critical point of F and are asymptotic to a critical-point solution at $+\infty$. We then have the following diagram:

$$\begin{array}{ccccccc}
C_{k+1} & \longrightarrow & I_{k+1}\oplus C_{k+1} & \longrightarrow & I_{k+1} & \longrightarrow & C_k \\
\| & & \| & & \| & & \| \\
\bar{C}^{n-k-1} & \longrightarrow & \bar{I}^{n-k-1}\oplus\bar{C}^{n-k-1} & \longrightarrow & \bar{I}_{n-k-1} & \longrightarrow & \bar{C}_{n-k} \\
\| & & \delta_{L,L'}\downarrow & & \downarrow\delta'_{L,L'} & & \| \\
\bar{C}^{n-k-1} & \longrightarrow & P^{n-k} & \longrightarrow & P^{n-k}\oplus\bar{C}^{n-k} & \longrightarrow & \bar{C}^{n-k},
\end{array} \tag{14}$$

where $\delta'_{L,L'}$ is the chain map inducing an isomorphism on homology from the splitting $C(X;L,L') = (P\oplus C)\oplus I$ instead of the splitting $P\oplus(C\oplus I)$ used in the proof sketch of Conjecture 1.2. Note that the bottom row of (14) is the same as the top row of (13). Joining the diagrams along this row, we see that Corollary 1.3 follows once we show that (14) commutes on the homology level. For the left and right squares, this is true already on the chain level by definition of the differential. The fact that the middle square commutes for element in $\bar{I}^{n-k-1}$ after projection to P^{n-k-1} is immediate from the definition. The same argument, using commutativity of exterior squares, as in the proof of Lemma 4.12 then gives commutativity on the homology level. □

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