Regularity of Plurisubharmonic Upper Envelopes in Big Cohomology Classes

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Dedicated to Professor Oleg Viro for his deep contributions to mathematics

Abstract The goal of this work is to prove the regularity of certain quasi-plurisubharmonic upper envelopes. Such envelopes appear in a natural way in the construction of Hermitian metrics with minimal singularities on a big line bundle over a compact complex manifold. We prove that the complex Hessian forms of these envelopes are locally bounded outside an analytic set of singularities. It is furthermore shown that a parametrized version of this result yields a priori inequalities for the solution of the Dirichlet problem for a degenerate Monge–Ampère operator; applications to geodesics in the space of Kähler metrics are discussed. A similar technique provides a logarithmic modulus of continuity for Tsuji's "supercanonical" metrics, which generalize a well-known construction of Narasimhan and Simha.

Keywords Plurisubharmonic function • Upper envelope • Hermitian line bundle • Singular metric • Logarithmic poles • Legendre–Kiselman transform • Pseudoeffective cone • Volume • Monge–Ampère measure • Supercanonical metric • Ohsawa–Takegoshi theorem

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1 Main Regularity Theorem

Let X be a compact complex manifold and ω a Hermitian metric on X, viewed as a smooth positive $(1,1)$-form. As usual, we put $d^c = \frac{1}{4i\pi}(\partial - \overline{\partial})$, so that $dd^c = \frac{1}{2i\pi}\partial\overline{\partial}$. Consider the dd^c-cohomology class $\{\alpha\}$ of a smooth real d-closed form α of type $(1,1)$ on X. (In general, one has to consider the Bott–Chern cohomology group, for which boundaries are dd^c-exact $(1,1)$-forms $dd^c\varphi$, but in the case in which X is Kähler, this group is isomorphic to the Dolbeault cohomology group $H^{1,1}(X)$.)

Recall that a function ψ is said to be quasiplurisubharmonic (or quasi-psh) if $idd^c\psi$ is locally bounded from below, or equivalently, if it can be written locally as a sum $\psi = \varphi + u$ of a psh function φ and a smooth function u. More precisely, it is said to be α-plurisubharmonic (or α-psh) if $\alpha + dd^c\psi \geq 0$. We denote by $\mathrm{PSH}(X,\alpha)$ the set of α-psh functions on X.

Definition 1.1. The class $\{\alpha\} \in H^{1,1}(X,\mathbb{R})$ is said to be pseudoeffective if it contains a closed (semi)positive current $T = \alpha + dd^c\psi \geq 0$, and big if it contains a closed "Kähler current" $T = \alpha + dd^c\psi$ such that $T \geq \varepsilon\omega > 0$ for some $\varepsilon > 0$.

From now on in this section, we assume that $\{\alpha\}$ is *big*. We know by [Dem92] that we can then find $T_0 \in \{\alpha\}$ of the form

$$T_0 = \alpha + dd^c\psi_0 \geq \varepsilon_0\omega \tag{1.2}$$

with a possibly slightly smaller $\varepsilon_0 > 0$ than the ε in the definition, and ψ_0 a quasi-psh function with analytic singularities, i.e., locally

$$\psi_0 = c\log\sum|g_j|^2 + u, \quad \text{where } c > 0,\ u \in C^\infty,\ g_j \text{ holomorphic.} \tag{1.3}$$

By [DP04], X carries such a class $\{\alpha\}$ if and only if X is in the Fujiki class $\mathcal{C}$ of smooth varieties that are bimeromorphic to compact Kähler manifolds. Our main result is the following.

Theorem 1.4. *Let X be a compact complex manifold in the Fujiki class $\mathcal{C}$, and let α be a smooth closed form of type $(1,1)$ on X such that the cohomology class $\{\alpha\}$ is big. Pick $T_0 = \alpha + dd^c\psi_0 \in \{\alpha\}$ satisfying (1.2) and (1.3) for some Hermitian metric ω on X, and let Z_0 be the analytic set $Z_0 = \psi_0^{-1}(-\infty)$. Then the upper envelope*

$$\varphi := \sup\{\psi \leq 0,\ \psi\ \alpha\text{-psh}\}$$

is a quasiplurisubharmonic function that has locally bounded second-order derivatives $\partial^2\varphi/\partial z_j\partial\overline{z}_k$ on $X \smallsetminus Z_0$, and moreover, for suitable constants $C, B > 0$, there is a global bound

$$|dd^c\varphi|_\omega \leq C(|\psi_0| + 1)^2 \mathrm{e}^{B|\psi_0|}$$

that explains how these derivatives blow up near Z_0. In particular, φ is $C^{1,1-\delta}$ on $X \smallsetminus Z_0$ for every $\delta > 0$, and the second derivatives $D^2\varphi$ are in $L^p_{\mathrm{loc}}(X \smallsetminus Z_0)$ for every $p > 0$.

An important special case is the situation in which we have a Hermitian line bundle (L, h_L) *and* $\alpha = \Theta_{L,h_L}$*, with the assumption that* L *is big, i.e., that there exists a singular Hermitian* $h_0 = h_L \mathrm{e}^{-\psi_0}$ *that has analytic singularities and a curvature current* $\Theta_{L,h_0} = \alpha + dd^c\psi_0 \geq \varepsilon_0\omega$*. We then infer that the metric with minimal singularities* $h_{\min} = h_L e^{-\varphi}$ *has the regularity properties prescribed by Theorem 4.1 outside of the analytic set* $Z_0 = \psi_0^{-1}(-\infty)$*. In fact, [Ber07, Theorem 3.4 (a)] proves in this case the slightly stronger result that* φ *in* $C^{1,1}$ *on* $X \smallsetminus Z_0$ *(using the fact that* X *is then Moishezon and that the total space of* L^* *has many holomorphic vector fields). The present approach is by necessity different, since we can no longer rely on the existence of vector fields when* X *is not algebraic. Even then, our proof will be in fact somewhat simpler.*

Proof. Notice that in order to get a quasi-psh function φ, we should a priori replace φ by its upper semicontinuous regularization $\varphi^*(z) = \limsup_{\zeta\to z}\varphi(\zeta)$, but since $\varphi^* \leq 0$ and φ^* is α-psh as well, $\psi = \varphi^*$ contributes to the envelope, and therefore $\varphi = \varphi^*$. Without loss of generality, after subtracting a constant from ψ_0, we may assume that $\psi_0 \leq 0$. Then ψ_0 contributes to the upper envelope, and therefore $\varphi \geq \psi_0$. This already implies that φ is locally bounded on $X \smallsetminus Z_0$. Following [Dem94], for every $\delta > 0$, we consider the regularization operator

$$\psi \mapsto \rho_\delta \psi \tag{1.5}$$

defined by $\rho_\delta\psi(z) = \Psi(z,\delta)$ and

$$\Psi(z,w) = \int_{\zeta\in T_{X,z}} \psi\big(\mathrm{exph}_z(w\zeta)\big)\,\chi(|\zeta|^2)\,\mathrm{d}V_\omega(\zeta), \qquad (z,w)\in X\times\mathbb{C}, \tag{1.6}$$

where $\mathrm{exph} : T_X \to X$, $T_{X,z} \ni \zeta \mapsto \mathrm{exph}_z(\zeta)$, is the formal holomorphic part of the Taylor expansion of the exponential map of the Chern connection on T_X associated with the metric ω, and $\chi : \mathbb{R} \to \mathbb{R}_+$ is a smooth function with support in $]-\infty, 1]$ defined by

$$\chi(t) = \frac{C}{(1-t)^2}\exp\frac{1}{t-1} \quad \text{for } t<1, \qquad \chi(t) = 0 \quad \text{for } t \geq 1,$$

with $C > 0$ adjusted so that $\int_{|x|\leq 1}\chi(|x|^2)\,\mathrm{d}x = 1$ with respect to the Lebesgue measure $\mathrm{d}x$ on $\mathbb{C}^n$.

Also, $\mathrm{d}V_\omega(\zeta)$ denotes the standard Hermitian Lebesgue measure on (T_X, ω). Clearly, $\Psi(z,w)$ depends only on $|w|$. With the relevant change of notation, the estimates proved in Sects. 3 and 4 of [Dem94] (see especially Theorem 4.1 and estimates (4.3), (4.5) therein) show that if one assumes $\alpha + dd^c\psi \geq 0$, then there are constants $\delta_0, K > 0$ such that for $(z,w) \in X\times\mathbb{C}$,

$$[0,\delta_0] \ni t \mapsto \Psi(z,t) + Kt^2 \quad \text{is increasing,} \tag{1.7}$$

$$\alpha(z) + dd^c\Psi(z,w) \geq -A\lambda(z,|w|)|\mathrm{d}z|^2 - K\big(|w|^2|\mathrm{d}z|^2 + |\mathrm{d}z||\mathrm{d}w| + |\mathrm{d}w|^2\big), \tag{1.8}$$

where $A = \sup_{|\zeta|\le 1, |\xi|\le 1}\{-c_{jk\ell m}\zeta_j\overline{\zeta}_k\xi_\ell\overline{\xi}_m\}$ is a bound for the negative part of the curvature tensor $(c_{jk\ell m})$ of (T_X, ω) and

$$\lambda(z,t) = \frac{\mathrm{d}}{d\log t}(\Psi(z,t) + Kt^2) \underset{t\to 0_+}{\longrightarrow} \nu(\psi, z) \qquad \text{(Lelong number)}. \tag{1.9}$$

In fact, this is clear from [Dem94] if $\alpha = 0$, and otherwise we simply apply the above estimates (1.7)–(1.9) locally to $u+\psi$, where u is a local potential of α, and then subtract the resulting regularization $U(z,w)$ of u, which is such that

$$dd^c(U(z,w) - u(z)) = O(|w|^2|\mathrm{d}z|^2 + |w||\mathrm{d}z||\mathrm{d}w| + |\mathrm{d}w|^2), \tag{1.10}$$

because the left-hand side is smooth and $U(z,w) - u(z) = O(|w|^2)$.

As a consequence, the regularization operator ρ_δ transforms quasi-psh functions into quasi-psh functions, while providing very good control on the complex Hessian. We exploit this, again quite similarly as in [Dem94], by introducing the Kiselman–Legendre transform (cf. [Kis78, Kis94])

$$\psi_{c,\delta}(z) = \inf_{t\in]0,\delta]} \rho_t\psi(z) + Kt^2 - K\delta^2 - c\log\frac{t}{\delta}, \qquad c > 0,\ \delta \in]0, \delta_0]. \tag{1.11}$$

We need the following basic lower bound on the Hessian form.

Lemma 1.12. *For all $c > 0$ and $\delta \in]0, \delta_0]$, we have*

$$\alpha + dd^c\psi_{c,\delta} \ge -\big(A\min\big(c, \lambda(z,\delta)\big) + K\delta^2\big)\omega.$$

Proof of lemma. In general, an infimum $\inf_{\eta\in E} u(z,\eta)$ of psh functions $z \mapsto u(z,\eta)$ is not psh, but this is the case if $u(z,\eta)$ is psh with respect to (z,η) and $u(z,\eta)$ depends only on $\mathrm{Re}\,\eta$, in which case it is actually a convex function of $\mathrm{Re}\,\eta$. This fundamental fact is known as Kiselman's infimum principle. We apply it here by putting $w = \mathrm{e}^\eta$ and $t = |w| = \mathrm{e}^{\mathrm{Re}\,\eta}$. At all points of $E_c(\psi) = \{z \in X\,;\ \nu(\psi, z) \ge c\}$, the infimum occurring in (1.11) is attained at $t = 0$. However, for $z \in X \smallsetminus E_c(\psi)$ it is attained for $t = t_{\min}$, where

$$\begin{cases} t_{\min} = \delta & \mathit{if}\, \lambda(z,\delta) \le c, \\ t_{\min} < \delta & \textit{such that } c = \lambda(z, t_{\min}) = \frac{\mathrm{d}}{\mathrm{d}t}(\Psi(z,t) + Kt^2)_{t=t_{\min}}\, \mathit{if}\, \lambda(z,\delta) > c. \end{cases}$$

In a neighborhood of such a point $z \in X \smallsetminus E_c(\psi)$, the infimum coincides with the infimum taken for t close to $t_{\min}$, and all functions involved have (modulo addition of α) a Hessian form bounded below by $-(A\lambda(z,t_{\min}) + K\delta^2)\omega$ by (1.8). Since $\lambda(z, t_{\min}) \le \min(c, \lambda(z,\delta))$, we get the desired estimate on the dense open set $X \smallsetminus E_c(\psi)$ by Kiselman's infimum principle. However, $\psi_{c,\delta}$ is quasi-psh on X, and $E_c(\psi)$ is of measure zero, so the estimate is in fact valid on all of X, in the sense of currents. $\square$

We now proceed to complete the proof or Theorem 4.1. Lemma 1 implies the more brutal estimate

$$\alpha + dd^c\psi_{c,\delta} \geq -(Ac+K\delta^2)\,\omega \qquad \text{for } \delta \in \,]0,\delta_0]. \tag{1.13}$$

Consider the convex linear combination

$$\theta = \frac{Ac+K\delta^2}{\varepsilon_0}\psi_0 + \left(1 - \frac{Ac+K\delta^2}{\varepsilon_0}\right)\varphi_{c,\delta},$$

where φ is the upper envelope of all α-psh functions $\psi \leq 0$. Since $\alpha + dd^c\varphi \geq 0$, (1.2) and (1.13) imply

$$\alpha + dd^c\theta \geq (Ac+K\delta^2)\,\omega - \left(1 - \frac{Ac+K\delta^2}{\varepsilon_0}\right)(Ac+K\delta^2)\,\omega \geq 0.$$

Also $\varphi \leq 0$, and therefore $\varphi_{c,\delta} \leq \rho_\delta\varphi \leq 0$ and $\theta \leq 0$ likewise. In particular θ contributes to the envelope, and as a consequence we get $\varphi \geq \theta$.

Returning to the definition of $\varphi_{c,\delta}$, we infer that for every point $z \in X \smallsetminus Z_0$ and every $\delta > 0$, there exists $t \in \,]0,\delta]$ such that

$$\begin{aligned}\varphi(z) &\geq \frac{Ac+K\delta^2}{\varepsilon_0}\psi_0(z) + \left(1 - \frac{Ac+K\delta^2}{\varepsilon_0}\right)(\rho_t\varphi(z) + Kt^2 - K\delta^2 - c\log t/\delta)\\ &\geq \frac{Ac+K\delta^2}{\varepsilon_0}\psi_0(z) + (\rho_t\varphi(z) + Kt^2 - K\delta^2 - c\log t/\delta)\end{aligned}$$

(using the fact that the infimum is ≤ 0 and reached for some $t \in \,]0,\delta]$, since $t \mapsto \rho_t\varphi(z)$ is bounded for $z \in X \smallsetminus Z_0$). Therefore, we get

$$\rho_t\varphi(z) + Kt^2 \leq \varphi(z) + K\delta^2 - (Ac+K\delta^2)\varepsilon_0^{-1}\psi_0(z) + c\log\frac{t}{\delta}\,. \tag{1.14}$$

Since $t \mapsto \rho_t\varphi(z) + Kt^2$ is increasing and equal to $\varphi(z)$ for $t = 0$, we infer that

$$K\delta^2 - (Ac+K\delta^2)\varepsilon_0^{-1}\psi_0(z) + c\log\frac{t}{\delta} \geq 0,$$

or equivalently, since $\psi_0 \leq 0$,

$$t \geq \delta\,\exp\bigl(-(A+K\delta^2/c)\varepsilon_0^{-1}|\psi_0(z)| - K\delta^2/c\bigr).$$

Now (1.14) implies the weaker estimate

$$\rho_t\varphi(z) \leq \varphi(z) + K\delta^2 + (Ac+K\delta^2)\varepsilon_0^{-1}|\psi_0(z)|;$$

hence, by combining the last two inequalities, we get

$$\frac{\rho_t\varphi(z)-\varphi(z)}{t^2} \le K\Big(1+\Big(\frac{Ac}{K\delta^2}+1\Big)\varepsilon_0^{-1}|\psi_0(z)|\Big)\exp\Big(2\Big(A+K\frac{\delta^2}{c}\Big)\varepsilon_0^{-1}|\psi_0(z)|+2K\frac{\delta^2}{c}\Big).$$

We exploit this by letting $0<t\le\delta$ and c tend to 0 in such a way that $Ac/K\delta^2$ converges to a positive limit ℓ (if $A=0$, just enlarge A slightly and then let $A\to 0$). In this way, we get for every $\ell>0$,

$$\liminf_{t\to 0_+}\frac{\rho_t\varphi(z)-\varphi(z)}{t^2} \le K\big(1+(\ell+1)\varepsilon_0^{-1}|\psi_0(z)|\big)\exp\Big(2A\big((1+\ell^{-1})\varepsilon_0^{-1}|\psi_0(z)|+\ell^{-1}\big)\Big).$$

The special (essentially optimal) choice $\ell=\varepsilon_0^{-1}|\psi_0(z)|+1$ yields

$$\liminf_{t\to 0_+}\frac{\rho_t\varphi(z)-\varphi(z)}{t^2}\le K(\varepsilon_0^{-1}|\psi_0(z)|+1)^2\exp\big(2A(\varepsilon_0^{-1}|\psi_0(z)|+1)\big). \quad (1.15)$$

Now, putting as usual $\nu(\varphi,z,r)=\frac{1}{\pi^{n-1}r^{2n-2}/(n-1)!}\int_{B(z,r)}\Delta\varphi(\zeta)\,\mathrm{d}\zeta$, we infer from estimate (4.5) of [Dem94] the Lelong–Jensen-like inequality

$$\begin{aligned}\rho_t\varphi(z)-\varphi(z) &= \int_0^t\frac{\mathrm{d}}{\mathrm{d}\tau}\Phi(z,\tau)\,\mathrm{d}\tau\\ &\ge \int_0^t\frac{\mathrm{d}\tau}{\tau}\Big(\int_{B(0,1)}\nu(\varphi,z,\tau|\zeta|)\,\chi(|\zeta|^2)\,\mathrm{d}\zeta-O(\tau^2)\Big)\\ &\ge c(a)\,\nu(\varphi,z,at)-C_2t^2 \quad [\text{where } a<1,\ c(a)>0 \text{ and } C_2\gg 1]\\ &= \frac{c'(a)}{t^{2n-2}}\int_{B(z,at)}\Delta\varphi(\zeta)\,\mathrm{d}\zeta-C_2t^2, \end{aligned} \quad (1.16)$$

where the third line is obtained by integrating for $\tau\in[a^{1/2}t,t]$ and for ζ in the corona $a^{1/2}<|\zeta|<a^{1/4}$ (here we assume that χ is taken to be decreasing with $\chi(t)>0$ for all $t<1$, and we compute the Laplacian Δ in normalized coordinates at z given by $\zeta\mapsto \mathrm{exph}_z(\zeta)$).

Hence by Lebesgue's theorem on the existence almost everywhere of the density of a positive measure (see, e.g., [Rud66, 7.14]), we obtain

$$\lim_{t\to 0_+}\frac{1}{t^2}\big(\rho_t\varphi(z)-\varphi(z)\big)\ge c''(\Delta_\omega\varphi)_{\mathrm{ac}}(z)-C_2 \quad \text{a.e. on } X, \quad (1.17)$$

where the subscript "ac" means the absolutely continuous part of the measure $\Delta_\omega \varphi$. By combining (1.15) and (1.17) and using the quasiplurisubharmonicity of φ, we conclude that

$$|dd^c \varphi|_\omega \le \Delta_\omega \varphi + C_3 \le C\,(|\psi_0|+1)^2\,\mathrm{e}^{2A\varepsilon_0^{-1}\psi_0(z)} \quad \text{a.e. on } X \smallsetminus Z_0$$

for some constant $C > 0$. There cannot be any singular measure part μ in $\Delta_\omega \varphi$ either, since we know that the Lebesgue density would then be equal to $+\infty$ μ-a.e. [Rud66, 7.15], in contradiction to (1.15). This gives the required estimates for the complex derivatives $\partial^2\varphi/\partial z_j \partial \bar{z}_k$. The other real derivatives $\partial^2\varphi/\partial x_i \partial x_j$ are obtained from $\Delta\varphi = \sum_k \partial^2\varphi/\partial z_k \partial \bar{z}_k$ via singular integral operators, and it is well known that these operate boundedly on L^p for all $p < \infty$. Theorem (1.4) follows. □

Remark 1.18. The proof gave us in fact the very explicit value $B = 2A\varepsilon_0^{-1}$, where A is an upper bound of the negative part of the curvature of (T_X, ω). The slightly more refined estimates obtained in [Dem94] show that we could even replace B by the possibly smaller constant $B_\eta = 2(A' + \eta)\varepsilon_0^{-1}$, where

$$A' = \sup_{|\zeta|=1,\,|\xi|=1,\,\zeta\perp\xi} -c_{jk\ell m}\zeta_j \overline{\zeta}_k \xi_\ell \overline{\xi}_m,$$

and the dependence of the other constants on η could then be made explicit.

Remark 1.19. In Theorem (1.4), one can replace the assumption that α is smooth by the assumption that α has L^∞ coefficients. In fact, we used the smoothness of α only as a cheap argument to get the validity of estimate (1.10) for the local potentials u of α. However, the results of [Dem94] easily imply the same estimates when α is L^∞, since both u and $-u$ are then quasi-psh; this follows, for instance, from (1.8) applied with respect to a smooth α_∞ and $\psi = \pm u$ if we observe that $\lambda(z, |w|) = O(|w|^2)$ when $|dd^c\psi|_\omega$ is bounded. Therefore, only the constant K will be affected in the proof.

2 Applications to Volume and Monge–Ampère Measures

Recall that the *volume* of a big class $\{\alpha\}$ is defined, in the work [Bou02] of S. Boucksom, as

$$\mathrm{Vol}(\{\alpha\}) = \sup_T \int_{X \smallsetminus \mathrm{sing}(T)} T^n, \tag{2.1}$$

with T ranging over all positive currents in the class $\{\alpha\}$ with *analytic singularities*, whose locus is denoted by $\mathrm{sing}(T)$. If the class is not big, then the volume is defined to be zero. With this definition, it is clear that $\{\alpha\}$ is big precisely when $\mathrm{Vol}(\{\alpha\}) > 0$.

Now fix a *smooth* representative α in a *pseudoeffective* class $\{\alpha\}$. We then obtain a uniquely defined α-plurisubharmonic function $\varphi = \psi_{\min} \geq 0$ with minimal singularities defined as in Theorem (1.4) by

$$\varphi := \sup\left\{\psi \leq 0,\ \psi\ \alpha\text{-psh}\right\}; \tag{2.2}$$

notice that the supremum is nonempty by our assumption that $\{\alpha\}$ is pseudoeffective. If $\{\alpha\}$ is big and ψ is α-psh and locally bounded in the complement of an analytic $Z \subset X$, one can define the Monge–Ampère measure $\mathrm{MA}_\alpha(\psi)$ by

$$\mathrm{MA}_\alpha(\psi) := \mathbf{1}_{X \smallsetminus Z}(\alpha + dd^c\psi)^n, \tag{2.3}$$

as follows from the work of Bedford and Taylor [BT76, BT82]. In particular, if $\{\alpha\}$ is big, there is a well-defined positive measure on $\mathrm{MA}_\alpha(\varphi) = \mathrm{MA}_\alpha(\psi_{\min})$ on X; its total mass coincides with $\mathrm{Vol}(\{\alpha\})$, i.e.,

$$\mathrm{Vol}(\{\alpha\}) = \int_X \mathrm{MA}_\alpha(\varphi)$$

(this follows from the comparison theorem and the fact that Monge–Ampère measures of locally bounded psh functions do not carry mass on analytic sets; see, e.g., [BEGZ08]). Next, notice that in general, the α-psh envelope $\varphi = \psi_{\min}$ corresponds *canonically* to α, so we may associate to α the following subset of X:

$$D = \{\varphi = 0\}. \tag{2.4}$$

Since φ is upper semicontinuous, the set D is compact. Moreover, a simple application of the maximum principle shows that $\alpha \geq 0$ pointwise on D (precisely as in Proposition 3.1 of [Ber07]: at any point z_0 where α is not semipositive, we can find complex coordinates and a small $\varepsilon > 0$ such that $\varphi(z) - \varepsilon|z - z_0|^2$ is subharmonic near z_0, using the fact that $dd^c\varphi \geq -\alpha$ or rather the induced inequality between traces, and so integrating over a small ball B_δ centered at z_0 gives $\varphi(z_0) - 0 \leq \int_{B_\delta} \varphi(z) - \varepsilon|z - z_0|^2 < 0$, showing that z_0 is not in D).

In particular, $\mathbf{1}_D\alpha$ is a positive $(1,1)$-form on X. From Theorem (1.4) we infer the following.

Corollary 2.5. *Assume that X is a Kähler manifold. For any smooth closed form α of type $(1,1)$ in a pseudoeffective class and $\varphi \leq 0$ the α-psh upper envelope, we have*

$$\mathrm{MA}_\alpha(\varphi) = \mathbf{1}_D\alpha^n, \qquad D = \{\varphi = 0\}, \tag{2.6}$$

as measures on X (provided the left-hand side is interpreted as a suitable weak limit) and

$$\mathrm{Vol}(\{\alpha\}) = \int_D \alpha^n \geq 0. \tag{2.7}$$

In particular, $\{\alpha\}$ is big if and only if $\int_D \alpha^n > 0$.

Proof. Let ω be a Kähler metric on X. First assume that the class $\{\alpha\}$ is big and let Z_0 be the singularity set of some strictly positive representative $\alpha + dd^c\psi_0 \geq \varepsilon\omega$ with analytic singularities. By Theorem (1.4), $\alpha + dd^c\varphi$ is in $L^\infty_{\mathrm{loc}}(X \smallsetminus Z_0)$. In particular (see [Dem89]), the Monge–Ampère measure $(\alpha + dd^c\varphi)^n$ has a locally bounded density on $X \smallsetminus Z_0$ with respect to ω^n. Since by definition, the Monge–Ampère measure puts no mass on Z_0, it is enough to prove the identity (2.6) pointwise almost everywhere on X.

To this end, one argues essentially as in [Ber07] (where the class was assumed to be integral). First, a well-known local argument based on the solution of the Dirichlet problem for $(dd^c)^n$ (see, e.g., [BT76, BT82], and also Proposition 1.10 in [BB08]) proves that the Monge–Ampère measure $(\alpha + dd^c\varphi)^n$ of the envelope φ vanishes on the open set $(X \smallsetminus Z_0) \smallsetminus D$ (this uses only the fact that α has continuous potentials and the continuity of φ on $X \smallsetminus Z_0$). Moreover, Theorem (1.4) implies that $\varphi \in C^1(X \smallsetminus Z_0)$ and

$$\frac{\partial^2\varphi}{\partial x_i \partial x_j} \in L^p_{\mathrm{loc}} \tag{2.8}$$

for any $p \in]1,\infty[$ and $i,j \in [1,2n]$. Even if this is slightly weaker than the situation in [Ber07], where it was shown that one can take $p = \infty$, the argument given in [Ber07] still goes through. Indeed, by well-known properties of measurable sets, D has Lebesgue density $\lim_{r\to 0} \lambda(D \cap B(x,r))/\lambda(B(x,r) = 1$ at almost every point $x \in D$, and since $\varphi = 0$ on D, we conclude that $\partial\varphi/\partial x_i = 0$ at those points (if the density is 1, no open cone of vertex x can be omitted and thus we can approach x from any direction by a sequence $x_\nu \to x$).

But the first derivative is Hölder continuous on $D \smallsetminus Z_0$; hence $\partial\varphi/\partial x_i = 0$ everywhere on $D \smallsetminus Z_0$. By repeating the argument for $\partial\varphi/\partial x_i$, which has a derivative in L^p (L^1 would even be enough), we conclude from Lebesgue's theorem that $\partial^2\varphi/\partial x_i \partial x_j = 0$ a.e. on $D \smallsetminus Z_0$, hence that $\alpha + dd^c\varphi = \alpha$ on $D \smallsetminus E$, where the set E has measure zero with respect to ω^n. This proves formula (2.6) in the case of a big class.

Finally, assume that $\{\alpha\}$ is pseudoeffective but not big. For any given positive number ε, we let $\alpha_\varepsilon = \alpha + \varepsilon\omega$ and denote by D_ε the corresponding set (2.4). Clearly α_ε represents a big class. Moreover, by the continuity of the volume function up to the boundary of the big cone [Bou02],

$$\mathrm{Vol}(\{\alpha_\varepsilon\}) \to \mathrm{Vol}(\{\alpha\}) \quad (= 0) \tag{2.9}$$

as ε tends to zero. Now observe that $D \subset D_\varepsilon$ (there are more $(\alpha + \varepsilon\omega)$-psh functions than α-psh functions, and so $\varphi \leq \varphi_\varepsilon \leq 0$; clearly, φ_ε increases with ε and $\varphi = \lim_{\varepsilon\to 0}\varphi_\varepsilon$; compare with Proposition 3.3 in [Ber07]). Therefore

$$\int_D \alpha^n \leq \int_{D_\varepsilon} \alpha^n \leq \int_{D_\varepsilon} \alpha_\varepsilon^n,$$

where we used that $\alpha \leq \alpha_\varepsilon$ in the second step.

Finally, since by the big case treated above, the right-hand side above is precisely $\mathrm{Vol}(\{\alpha_\varepsilon\})$, letting ε tend to zero and using (2.9) proves that $\int_D \alpha^n = 0 = \mathrm{Vol}(\{\alpha\})$ (and that $\mathrm{MA}_\alpha(\varphi) = 0$ if we interpret it as the limit of $\mathrm{MA}_{\alpha_\varepsilon}(\varphi_\varepsilon)$). This concludes the proof. □

In the case that $\{\alpha\}$ is an integer class, i.e., when it is the first Chern class $c_1(L)$ of a holomorphic line bundle L over X, the result of the corollary was obtained in [Ber07] under the additional assumption that X is a *projective* manifold; it was conjectured there that the result was also valid for integral classes over a nonprojective Kähler manifold.

Remark 2.10. In particular, the corollary shows that if $\{\alpha\}$ is big, there is always an α-plurisubharmonic function φ with minimal singularities such that $\mathrm{MA}_\alpha(\varphi)$ has an L^∞-density with respect to ω^n. This is a very useful fact when one is dealing with big classes that are not Kähler (see, for example, [BBGZ09]).

3 Application to Regularity of a Boundary Value Problem and a Variational Principle

In this section we will see how the main theorem may be interpreted as a regularity result for (1) a free boundary value problem for the Monge–Ampère operator and (2) a variational principle. For simplicity we consider only the case of a Kähler class.

3.1 *A Free Boundary Value Problem for the Monge–Ampère Operator*

Let (X,ω) be a Kähler manifold. Given a function $f \in \mathcal{C}^2(X)$, consider the following *free* boundary value problem:

$$\begin{cases} \mathrm{MA}_\omega(u) = 0 & \text{on } \Omega, \\ u = f & \text{on } \partial\Omega, \\ \mathrm{d}u = \mathrm{d}f & \end{cases}$$

for a pair (u,Ω), where u is an ω-psh function on $\overline{\Omega}$ that is in $\mathcal{C}^1(\overline{\Omega})$, and Ω is an open set in X. We have used the notation $\partial\Omega := \overline{\Omega} \setminus \Omega$, but no regularity of the boundary is assumed. The reason that the set Ω is assumed to be part of the solution is that for a fixed Ω, the equations are overdetermined. Setting $u := \varphi + f$ and $\Omega := X \setminus D$, where φ is the upper envelope with respect to $\alpha := dd^c f + \omega$, yields a solution. In fact, by Theorem (1.4), $u \in \mathcal{C}^{1,1-\delta}(\overline{\Omega})$ for any $\delta > 0$.

3.2 A Variational Principle

Fix a form α in a Kähler class $\{\alpha\}$ possessing continuous potentials. Consider the following energy functional defined on the convex space $\mathrm{PSH}(X,\alpha)\cap L^{\infty}$ of all α-psh functions that are bounded on X:

$$\mathcal{E}[\psi] := \frac{1}{n+1}\sum_{j=0}^{n}\int_X \psi(\alpha+dd^c\psi)^j\wedge\alpha^{n-j}. \qquad (3.2.1)$$

This functional seems to first have appeared, independently, in the work of Aubin and Mabuchi on Kähler–Einstein geometry (in the case that α is a Kähler form). More geometrically, up to an additive constant, $\mathcal{E}$ can be defined as a primitive of the one-form on $\mathrm{PSH}(X,\alpha)\cap L^{\infty}$ defined by the measure-valued operator $\psi\mapsto \mathrm{MA}_{\alpha}(\psi)$.

As shown in [BB08] (version 1), the following variational characterization of the envelope φ holds:

Proposition 3.2.2. *The functional*

$$\psi\mapsto\mathcal{E}[\psi]-\int_X\psi(\alpha+dd^c\psi)^n$$

achieves its minimum value on the space $\mathrm{PSH}(X,\alpha)\cap L^{\infty}$ *precisely when* ψ *is equal to the envelope* φ *(defined with respect to* α*). Moreover, the minimum is achieved only at* φ*, up to an additive constant.*

Hence, the main theorem above can be interpreted as a regularity result for the functions in $\mathrm{PSH}(X,\alpha)\cap L^{\infty}$ minimizing the functional (3.2.1) in the case that α is assumed to have $L^{\infty}_{\mathrm{loc}}$ coefficients. More generally, a similar variational characterization of φ can be given in the case of a *big* class $[\alpha]$ [BBGZ09].

4 Degenerate Monge–Ampère Equations and Geodesics in the Space of Kähler Metrics

Assume that (X,ω) is a compact Kähler manifold and that Σ is a Stein manifold with strictly pseudoconvex boundary, i.e., Σ admits a smooth strictly psh nonpositive function η_{Σ} that vanishes precisely on $\partial\Sigma$. The corresponding product manifold will be denoted by $M:=\Sigma\times X$. By taking pullbacks, we identify η_{Σ} with a function on M and ω with a semipositive form on M. In this way, we obtain a Kähler form $\omega_M:=\omega+dd^c\eta_{\Sigma}$ on M. Given a function f on M and a point s in Σ, we use the notation $f_s:=f(s,\cdot)$ for the induced function on X.

Further, given a closed $(1,1)$-form α on M with bounded coefficients and a continuous function f on ∂M, we define the upper envelope

$$\varphi_{\alpha,f} := \sup\left\{\psi : \psi \in \mathrm{PSH}(M,\alpha)\cap C^0(M),\ \psi_{\partial M} \le f\right\}. \tag{4.1}$$

Note that when Σ is a point and $f = 0$, this definition coincides with the one introduced in Sect. 1. Also, when F is a smooth function on the whole of M, the obvious translation $\psi \mapsto \psi' = \psi - F$ yields the relation

$$\varphi_{\beta,f-F} = \varphi_{\alpha,f} - F, \quad \text{where } \beta = \alpha + dd^cF. \tag{4.2}$$

The proof of the following lemma is a straightforward adaptation of the proof of Bedford–Taylor [BT76] in the case that M is a strictly pseudoconvex domain in $\mathbb{C}^n$.

Lemma 4.3. *Let α be a closed real $(1,1)$-form on M with bounded coefficients, such that $\alpha_{|\{s\}\times X} \ge \varepsilon_0\omega$ is positive definite for all $s \in \Sigma$. Then the corresponding envelope $\varphi = \varphi_{\alpha,0}$ vanishes on the boundary of M and is continuous on M. Moreover, $MA_\alpha(\varphi)$ vanishes in the interior of M.*

Proof. By (4.2), we have $\varphi_{\alpha,0} = \varphi_{\beta,0} + C\eta_\Sigma$, where $\beta = \alpha + Cdd^c\eta_\Sigma$ can be taken to be positive definite on M for $C \gg 1$, as is easily seen from the Cauchy–Schwarz inequality and the hypotheses on α. Therefore, we can assume without loss of generality that α is positive definite on M. Since 0 is a candidate for the supremum defining φ, it follows immediately that $0 \le \varphi$ and hence $\varphi_{\partial M} = 0$. To see that φ is continuous on ∂M (from the inside), take an arbitrary candidate ψ for the sup and observe that

$$\psi \le -C\eta_\Sigma$$

for $C \gg 1$, independent of ψ.

Indeed, since $dd^c\psi \ge -\alpha$, there is a large positive constant C such that the function $\psi + C\eta_\Sigma$ is strictly plurisubharmonic on $\Sigma \times \{x\}$ for all x. Thus the inequality above follows from the maximum principle applied to all slices $\Sigma \times \{x\}$. All in all, taking the sup over all such ψ gives

$$0 \le \varphi \le -C\eta_\Sigma.$$

But since $\eta_{\Sigma|\partial M} = 0$ and η_Σ is continuous, it follows that $\varphi(x_i) \to 0 = \varphi(x)$ when $x_i \to x \in \partial M$.

Next, fix a compact subset K in the interior of M and $\varepsilon > 0$. Let $M_\delta := \{\eta_\Sigma < -\delta\}$, where δ is sufficiently small to ensure that K is contained in $M_{4\delta}$. By the regularization results in [Dem92] or [Dem94], there is a sequence φ_j in $\mathrm{PSH}(M, \alpha - 2^{-j}\alpha) \cap C^0(M_{\delta/2})$ decreasing to the upper semicontinuous regularization φ^*. By replacing φ_j with $(1-2^{-j})^{-1}\varphi_j$, we can even assume $\varphi_j \in \mathrm{PSH}(M,\alpha)\cap C^0(M_{\delta/2})$. Put

$$\varphi_j' := \max\{\varphi_j - \varepsilon, C\eta_\Sigma\} \text{ on } M_\delta, \quad \text{and} \quad \varphi_j' := C\eta_\Sigma \text{ on } M \smallsetminus M_\delta.$$

On ∂M_δ we have $C\eta_\Sigma = -C\delta$, and we can take j so large that

$$\varphi_j < -C\eta_\Sigma + \varepsilon/2 = C\delta + \varepsilon/2,$$

so we will have $\varphi_j - \varepsilon < C\eta_\Sigma$ as soon as $2C\delta \leq \varepsilon/2$. We simply take $\varepsilon = 4C\delta$. Then φ'_j is a well-defined continuous α-psh function on M, and φ'_j is equal to $\varphi_j - \varepsilon$ on $K \subset M_{4\delta}$, since $C\eta_\Sigma \leq -4C\delta \leq -\varepsilon \leq \varphi_j - \varepsilon$ there. In particular, φ'_j is a candidate for the sup defining φ; hence $\varphi'_j \leq \varphi \leq \varphi^*$, and so

$$\varphi^* \leq \varphi_j \leq \varphi'_j + \varepsilon \leq \varphi^* + \varepsilon$$

on K. This means that φ_j converges to φ uniformly on K, and therefore φ is continuous on K.

All in all this shows that $\varphi \in C^0(M)$. The last statement of the proposition follows from standard local considerations for envelopes due to Bedford–Taylor [BT76] (see also the exposition in [Dem89]). □

Theorem 4.4. *Let α be a closed real $(1,1)$-form on M with bounded coefficients such that $\alpha_{|\{s\}\times X} \geq \varepsilon_0\omega$ is positive definite for all $s \in \Sigma$. Consider a continuous function f on ∂M such that $f_s \in \mathrm{PSH}(X, \alpha_s)$ for all $s \in \partial\Sigma$. Then the upper envelope $\varphi = \varphi_{\alpha,f}$ is the unique α-psh continuous solution of the Dirichlet problem*

$$\varphi = f \ \text{ on } \partial M, \qquad (dd^c u + \alpha)^{\dim M} = 0 \text{ on the interior } M^\circ. \tag{4.5}$$

Moreover, if f is $C^{1,1}$ on ∂M, then for any s in Σ, the restriction φ_s of φ on $\{s\}\times X$ has a dd^c in L^∞_{loc}. More precisely, we have a uniform bound $|dd^c\varphi_s|_\omega \leq C$ a.e. on X, where C is a constant independent of s.

AQ2 *Proof.* Without loss of generality, we may assume as in Lemma (4.4) that α is positive definite on M. Also, after adding a positive constant to f, which has only the effect of adding the same constant to $\varphi = \varphi_{\alpha,f}$, we may suppose that $\sup_{\partial M} f > 0$ (this will simplify a little bit the arguments below).

Continuity. Let us first prove the continuity statement in the theorem. In the case that f extends to a smooth function F in $\mathrm{PSH}(M, (1-\varepsilon)\alpha)$, the statement follows immediately from (4.2) and Lemma (4.3), since

$$f - F = 0 \ \text{ on } \partial M \text{ and } \ \beta = \alpha + dd^c F \geq \varepsilon\alpha \geq \varepsilon\varepsilon_0\omega.$$

Next, assume that f is *smooth* on ∂M and that $f_s \in \mathrm{PSH}(X, (1-\varepsilon)\alpha_s)$ for all $s \in \partial\Sigma$. If we take a smooth extension $\widetilde{f}$ of f to M and $C \gg 1$, we will get

$$\alpha + dd^c(\widetilde{f}(x,s) + C\eta_\Sigma(s)) \geq (\varepsilon/2)\alpha$$

on a sufficiently small neighborhood V of ∂M (again using Cauchy–Schwarz). Therefore, after enlarging C if necessary, we can define

$$F(x,s) = \max_\varepsilon(\widetilde{f}(x,s) + C\eta_\Sigma(s), 0)$$

with a regularized max function $\max_\varepsilon$ in such a way that the maximum is equal to 0 on a neighborhood of $M \setminus V$ ($C \gg 1$ being used to ensure that $\widetilde{f} + C\eta_\Sigma < 0$ on $M \setminus V$). Then F equals f on ∂M and satisfies

$$\alpha + dd^c F \geq (\varepsilon/2)\alpha \geq (\varepsilon\varepsilon_0/2)\omega$$

on M, and we can argue as previously. Finally, to handle the general case in which f is continuous with $f_s \in \mathrm{PSH}(X, \alpha_s)$ for every $s \in \Sigma$, we may, by a parametrized version of Richberg's regularization theorem applied to $(1-2^{-\nu})f + C2^{-\nu}$ (see, e.g., [Dem91]), write f as a decreasing uniform limit of smooth functions f_ν on ∂M satisfying $f_{\nu,s} \in \mathrm{PSH}(X, (1-2^{-\nu-1})\alpha_s)$ for every $s \in \partial\Sigma$. Then $\varphi_{\omega,f}$ is a decreasing uniform limit on M of the continuous functions φ_{ω,f_ν}, (as follows easily from the definition of $\varphi_{\omega,f}$ as an upper envelope).

Observe also that the uniqueness of a continuous solution of the Dirichlet problem (4.5) results from a standard application of the maximum principle for the Monge–Ampère operator. This proves the general case of the continuity statement.

Smoothness. Next, we turn to the proof of the smoothness statement. Since the proof is a straightforward adaptation of the proof of the main regularity result above, we will just briefly indicate the relevant modification. Quite similarly to what we did in Sect. 1, we consider an α-psh function ψ with $\psi \leq f$ on ∂M, and introduce the fiberwise transform Ψ_s of ψ_s on each $\{s\} \times X$, which is defined in terms of the exponential map $\mathrm{exph} : T_X \to X$, and we put

$$\Psi(z,s,t) = \Psi_s(z,t).$$

Then essentially the same calculations as in the previous case show that all properties of Ψ are still valid with the constant K depending on the $C^{1,1}$-norm of the local potentials $u(z,s)$ of α, the constant A depending only on ω and with

$$\partial\Psi(z,s,t)/\partial(\log t) := \lambda(z,s,t) \to \nu(\psi_s),$$

as $t \to 0^+$, where $\nu(\psi_s)$ is the Lelong number of the function ψ_s on X at z.

Moreover, the local vector-valued differential $\mathrm{d}z$ should be replaced by the differential $\mathrm{d}(z,s) = \mathrm{d}z + \mathrm{d}s$ in the previous formulas. Next, performing a Kiselman–Legendre transform fiberwise, we let

$$\psi_{c,\delta}(z,s) := (\psi_s)_{c,\delta}(z).$$

Then, using a parametrized version of the estimates of [Dem94] and the properties of $\Psi(z,s,w)$ as in Sect. 1, arguments derived from Kiselman's infimum principle show that

$$\alpha + dd^c\psi_{c,\delta} \geq (-A\min(c,\lambda(z,s,\delta)) - K\delta^2)\omega_M \geq -(Ac+K\delta^2)\omega_M, \tag{4.6}$$

where ω_M is the Kähler form on M.

In addition to this, we have $|\psi_{c,\delta} - f| \leq K'\delta^2$ on ∂M by the hypothesis that f is $C^{1,1}$. For a sufficiently large constant C_1, we infer from this that $\theta = (1 - C_1(Ac + K\delta^2))\psi_{c,\delta}$ satisfies $\theta \leq f$ on ∂M (here we use the fact that $f > 0$ and hence that $\psi_0 \equiv 0$ is a candidate for the upper envelope). Moreover, $\alpha + dd^c\theta \geq 0$ on M thanks to (4.6) and the positivity of α. Therefore, θ is a candidate for the upper envelope, and so $\theta \leq \varphi = \varphi_{f,\alpha}$.

Repeating the arguments of Sect. 1 almost word for word, we obtain for $(\rho_t\varphi)(z,s) := \Phi(z,s,t)$ the analogue of estimate (1.15), which reduces simply to

$$\liminf_{t\to 0_+} \frac{\rho_t\varphi(z,s) - \varphi(z,s)}{t^2} \leq C_2,$$

since $\psi_0 \equiv 0$ in the present situation. The final conclusion follows from (1.16) and the related arguments already explained. □

In connection to the study of Wess–Zumino–Witten-type equations [Don99], [Don02] and geodesics in the space of Kähler metrics [Don99], [Don02], [Che00], it is useful to formulate the result of the previous theorem as an extension problem from $\partial\Sigma$, in the case that $\alpha(z,s) = \omega(z)$ does not depend on s.

To this end, let $F : \partial\Sigma \to \mathrm{PSH}(X,\omega)$ be the map defined by $F(s) = f_s$. Then the previous theorem gives a continuous "maximal plurisubharmonic" extension U of F to Σ, where $U(s) := u_s$, so that $U : \partial\Sigma \to \mathrm{PSH}(X,\omega)$.

Let us next specialize to the case in which $\Sigma := A$ is an annulus $R_1 < |s| < R_2$ in $\mathbb{C}$ and the boundary datum $f(x,s)$ is invariant under rotations $s \mapsto s\,\mathrm{e}^{i\theta}$. Denote by f^0 and f^1 the elements in $\mathrm{PSH}(X,\omega)$ corresponding to the two boundary circles of A. Then the previous theorem furnishes a continuous path f^t in $\mathrm{PSH}(X,\omega)$ if we put $t = \log|s|$, or rather $t = \log(|s|/R_1)/\log(R_2/R_1)$, to be precise. Following [PS08], the corresponding path of semipositive forms $\omega^t := \omega + dd^c f^t$ will be called a *(generalized) geodesic* in $\mathrm{PSH}(X,\omega)$ (compare also with Remark 4.8).

Corollary 4.7. *Assume that the semipositive closed* $(1,1)$*-forms* ω^0 *and* ω^1 *belong to the same Kähler class* $\{\omega\}$ *and have bounded coefficients. Then the geodesic* ω^t *connecting* ω^0 *and* ω^1 *is continuous on* $[0,1]\times X$*, and there is a constant* C *such that* $\omega^t \leq C\omega$ *on* X*, i.e.,* ω^t *has uniformly bounded coefficients.*

In particular, the previous corollary shows that the space of all semipositive forms with bounded coefficients in a given Kähler class is "geodesically convex."

Remark 4.8. As shown in the work of Semmes, Mabuchi, and Donaldson, the space of Kähler metrics $\mathcal{H}_\omega$ in a given Kähler class $\{\omega\}$ admits a natural Riemannian structure defined in the following way (see [Che00] and references therein). First note that the map $u \mapsto \omega + dd^c u$ identifies $\mathcal{H}_\omega$ with the space of all smooth and strictly ω-psh functions, modulo constants. Now with the tangent space of $\mathcal{H}_\omega$ identified at the point $\omega + dd^c u \in \mathcal{H}_\omega$ with $C^\infty(X)/\mathbb{R}$, the squared norm of a tangent vector v at the point u is defined as

$$\int_X v^2 (\omega + dd^c u)^n / n!.$$

Then the potentials f^t of any given geodesic ω^t in $\mathcal{H}_\omega$ are in fact solutions of the Dirichlet problem (4.5) above, with Σ an annulus and $t := \log|s|$; see [Che00].

However, the *existence* of a geodesic u_t in $\mathcal{H}_\omega$ connecting any given points u_0 and u_1 is an open and even dubious problem. In the case that Σ is a Riemann surface and the boundary datum f is smooth with $\alpha_s + dd^c f_s > 0$ on X for $s \in \partial\Sigma$, it was shown in [Che00] that the solution φ of the Dirichlet problem (4.5) has a total Laplacian that is bounded on M. See also [Blo08] for a detailed analysis of the proof in [Che00] and some refinements.

On the other hand, it is not known whether $\alpha_s + dd^c \varphi_s > 0$ for all $s \in \Sigma$, even under the assumption of rotational invariance, which appears in the case of geodesics as above. See [CT08], however, for results in this direction. A case similar to the degenerate setting in the previous corollary was also considered very recently in [PS08], building on [Blo08].

Remark 4.9. Note that the assumption $f \in C^2(\partial M)$ is not sufficient to obtain uniform estimates on the total Laplacian on M with respect to ω_M of the envelope u up to the boundary. To see this, let Σ be the unit ball in $\mathbb{C}^2$ and write $s = (s_1, s_2) \in \mathbb{C}^2$. Then $f(s) := (1 + \mathrm{Re}\, s_1)^{2-\varepsilon}$ is in $C^{4-2\varepsilon}(\partial M)$, and $u(x,s) := f(s)$ is the continuous solution of the Dirichlet problem (4.5). However, u is not in $C^{1,1}(M)$ at $(x; -1, 0) \in \partial M$ for any $x \in X$. Note that this example is the trivial extension of the example in [CNS86] for the real Monge–Ampère equation on the disk.

5 Regularity of "Supercanonical" Metrics

Let X be a compact complex manifold and $(L, h_{L,\gamma})$ a holomorphic line bundle over X equipped with a singular Hermitian metric $h_{L,\gamma} = \mathrm{e}^{-\gamma} h_L$ that satisfies $\int \mathrm{e}^{-\gamma} < +\infty$ locally on X, where h_L is a smooth metric on L. In fact, we can more generally consider the case in which $(L, h_{L,\gamma})$ is a "Hermitian $\mathbb{R}$-line bundle"; by this we mean that we have chosen a smooth real d-closed $(1,1)$-form α_L on X (whose dd^c cohomology class is equal to $c_1(L)$), and a specific current $T_{L,\gamma}$ representing it, namely $T_{L,\gamma} = \alpha_L + dd^c \gamma$, such that γ is a locally integrable function satisfying $\int \mathrm{e}^{-\gamma} < +\infty$.

An important special case is obtained by considering a klt (Kawamata log terminal) effective divisor Δ. In this situation, $\Delta = \sum c_j \Delta_j$ with $c_j \in \mathbb{R}$, and if g_j is a local generator of the ideal sheaf $\mathcal{O}(-\Delta_j)$ identifying it with the trivial invertible sheaf $g_j\mathcal{O}$, we take $\gamma = \sum c_j \log|g_j|^2$, $T_{L,\gamma} = \sum c_j[\Delta_j]$ (current of integration on Δ) and α_L given by any smooth representative of the same dd^c-cohomology class; the klt condition means precisely that

$$\int_V \mathrm{e}^{-\gamma} = \int_V \prod |g_j|^{-2c_j} < +\infty \tag{5.1}$$

on a small neighborhood V of any point in the support $|\Delta| = \bigcup \Delta_j$. (Condition (5.1) implies $c_j < 1$ for every j, and this in turn is sufficient to imply Δ klt if Δ is a normal crossing divisor; the line bundle L is then the real line bundle $\mathcal{O}(\Delta)$, which makes sense as a genuine line bundle only if $c_j \in \mathbb{Z}$.)

For each klt pair (X,Δ) such that $K_X + \Delta$ is pseudoeffective, Tsuji [Ts07a, Ts07b] has introduced a "supercanonical metric" that generalizes the metric introduced by Narasimhan and Simha [NS68] for projective algebraic varieties with ample canonical divisor. We take the opportunity to present here a simpler, more direct, and more general approach.

We assume from now on that $K_X + L$ is *pseudoeffective*, i.e., that the class $c_1(K_X) + \{\alpha_L\}$ is pseudoeffective, and under this condition, we are going to define a "supercanonical metric" on $K_X + L$. Select an arbitrary smooth Hermitian metric ω on X. We then find induced Hermitian metrics h_{K_X} on K_X and $h_{K_X+L} = h_{K_X} h_L$ on $K_X + L$ whose curvature is the smooth real $(1,1)$-form

$$\alpha = \Theta_{K_X+L,h_{K_X+L}} = \Theta_{K_X,\omega} + \alpha_L.$$

A singular Hermitian metric on $K_X + L$ is a metric of the form $h_{K_X+L,\varphi} = \mathrm{e}^{-\varphi} h_{K_X+L}$, where φ is locally integrable, and by the pseudoeffectivity assumption, we can find quasi-psh functions φ such that $\alpha + dd^c\varphi \geq 0$.

The metrics on L and $K_X + L$ can now be "subtracted" to give rise to a metric

$$h_{L,\gamma} h^{-1}_{K_X+L,\varphi} = \mathrm{e}^{\varphi-\gamma} h_L h^{-1}_{K_X+L} = \mathrm{e}^{\varphi-\gamma} h^{-1}_{K_X} = \mathrm{e}^{\varphi-\gamma} \mathrm{d}V_\omega$$

on $K_X^{-1} = \Lambda^n T_X$, since $h^{-1}_{K_X} = \mathrm{d}V_\omega$ is just the Hermitian (n,n) volume form on X. Therefore the integral $\int_X h_{L,\gamma} h^{-1}_{K_X+L,\varphi}$ has an intrinsic meaning, and it makes sense to require that

$$\int_X h_{L,\gamma} h^{-1}_{K_X+L,\varphi} = \int_X \mathrm{e}^{\varphi-\gamma} \mathrm{d}V_\omega \leq 1, \tag{5.2}$$

in view of the fact that φ is locally bounded from above and because of the assumption $\int \mathrm{e}^{-\gamma} < +\infty$. Observe that condition (5.2) can always be achieved by subtracting a constant from φ. We can now generalize Tsuji's supercanonical metrics on klt pairs (cf. [Ts07b]) as follows.

Definition 5.3. Let X be a compact complex manifold and let (L, h_L) be a Hermitian $\mathbb{R}$-line bundle on X associated with a smooth, real, closed $(1,1)$-form α_L. Assume that $K_X + L$ is pseudoeffective and that L is equipped with a singular Hermitian metric $h_{L,\gamma} = \mathrm{e}^{-\gamma} h_L$ such that $\int \mathrm{e}^{-\gamma} < +\infty$ locally on X. Take a Hermitian metric ω on X and define $\alpha = \Theta_{K_X+L,h_{K_X+L}} = \Theta_{K_X,\omega} + \alpha_L$. Then we define the supercanonical metric h_{can} of $K_X + L$ to be

$$h_{K_X+L,\mathrm{can}} = \inf_{\varphi} h_{K_X+L,\varphi} \quad \text{i.e.} \quad h_{K_X+L,\mathrm{can}} = \mathrm{e}^{-\varphi_{\mathrm{can}}} h_{K_X+L}, \ \text{ where}$$
$$\varphi_{\mathrm{can}}(x) = \sup_{\varphi} \varphi(x) \ \text{ for all } \varphi \text{ with } \ \alpha + dd^c\varphi \geq 0, \quad \int_X \mathrm{e}^{\varphi-\gamma} \mathrm{d}V_\omega \leq 1.$$

In particular, this gives a definition of the supercanonical metric on $K_X + \Delta$ for every klt pair (X, Δ) such that $K_X + \Delta$ is pseudoeffective, and as an even more special case, a supercanonical metric on K_X when K_X is pseudoeffective.

In the sequel, we assume that γ has analytic singularities, for otherwise, not much can be said. The mean value inequality then immediately shows that the quasi-psh functions φ involved in Definition (5.3) are globally uniformly bounded outside of the poles of γ, and therefore everywhere on X. Hence the envelopes $\varphi_{\mathrm{can}} = \sup_\varphi \varphi$ are indeed well defined and bounded above. As a consequence, we get a "supercanonical" current $T_{\mathrm{can}} = \alpha + dd^c\varphi_{\mathrm{can}} \geq 0$, and $h_{K_X+L,\mathrm{can}}$ satisfies

$$\int_X h_{L,\gamma} h^{-1}_{K_X+L,\mathrm{can}} = \int_X \mathrm{e}^{\varphi_{\mathrm{can}}-\gamma} \mathrm{d}V_\omega < +\infty. \tag{5.4}$$

It is easy to see that in Definition (5.3) the supremum is a maximum and that $\varphi_{\mathrm{can}} = (\varphi_{\mathrm{can}})^*$ everywhere, so that taking the upper semicontinuous regularization is not needed.

In fact, if $x_0 \in X$ is given and we write

$$(\varphi_{\mathrm{can}})^*(x_0) = \limsup_{x\to x_0} \varphi_{\mathrm{can}}(x) = \lim_{\nu\to+\infty} \varphi_{\mathrm{can}}(x_\nu) = \lim_{\nu\to+\infty} \varphi_\nu(x_\nu)$$

with suitable sequences $x_\nu \to x_0$ and (φ_ν) such that $\int_X \mathrm{e}^{\varphi_\nu-\gamma}\mathrm{d}V_\omega \leq 1$, the well-known weak compactness properties of quasi-psh functions in the L^1 topology imply the existence of a subsequence of (φ_ν) converging in L^1 and almost everywhere to a quasi-psh limit φ. Since $\int_X \mathrm{e}^{\varphi_\nu-\gamma}\mathrm{d}V_\omega \leq 1$ holds for every ν, Fatou's lemma implies that we have $\int_X e^{\varphi-\gamma}\mathrm{d}V_\omega \leq 1$ in the limit. By taking a subsequence, we can assume that $\varphi_\nu \to \varphi$ in $L^1(X)$. Then for every $\varepsilon > 0$, the mean value $⨍_{B(x_\nu,\varepsilon)} \varphi_\nu$ satisfies

$$⨍_{B(x_0,\varepsilon)} \varphi = \lim_{\nu\to+\infty} ⨍_{B(x_\nu,\varepsilon)} \varphi_\nu \geq \lim_{\nu\to+\infty} \varphi_\nu(x_\nu) = (\varphi_{\mathrm{can}})^*(x_0),$$

and hence we get $\varphi(x_0) = \lim_{\varepsilon\to 0} ⨍_{B(x_0,\varepsilon)} \varphi \geq (\varphi_{\mathrm{can}})^*(x_0) \geq \varphi_{\mathrm{can}}(x_0)$, and therefore the sup is a maximum and $\varphi_{\mathrm{can}} = \varphi^*_{\mathrm{can}}$.

By elaborating on this argument, we can infer certain regularity properties of the envelope. However, there is no reason why the integral occurring in (5.4) should be equal to 1 when we take the upper envelope. As a consequence, neither the upper envelope nor its regularizations participate in the family of admissible metrics. This is why the estimates that we will be able to obtain are much weaker than in the case of envelopes normalized by a condition $\varphi \le 0$.

Theorem 5.5. *Let X be a compact complex manifold and (L, h_L) a holomorphic $\mathbb{R}$-line bundle such that $K_X + L$ is big. Assume that L is equipped with a singular Hermitian metric $h_{L,\gamma} = \mathrm{e}^{-\gamma} h_L$ with analytic singularities such that $\int \mathrm{e}^{-\gamma} < +\infty$ (klt condition). Denote by Z_0 the set of poles of a singular metric $h_0 = \mathrm{e}^{-\psi_0} h_{K_X+L}$ with analytic singularities on $K_X + L$ and by Z_γ the poles of γ (assumed analytic). Then the associated supercanonical metric h_{can} is continuous on $X \smallsetminus (Z_0 \cup Z_\gamma)$ and possesses some computable logarithmic modulus of continuity.*

Proof. With the notation already introduced, let $h_{K_X+L,\varphi} = \mathrm{e}^{-\varphi} h_{K_X+L}$ be a singular Hermitian metric such that its curvature satisfies $\alpha + dd^c\varphi \ge 0$ and $\int_X e^{\varphi-\gamma} \mathrm{d}V_\omega \le 1$. We apply to φ the regularization procedure defined in (1.6). Jensen's inequality implies

$$\mathrm{e}^{\Phi(z,w)} \le \int_{\zeta \in T_{X,z}} \mathrm{e}^{\varphi(\mathrm{exph}_z(w\zeta))} \chi(|\zeta|^2)\, \mathrm{d}V_\omega(\zeta).$$

If we change variables by putting $u = \mathrm{exph}_z(w\zeta)$, then in a neighborhood of the diagonal of $X \times X$ we have an inverse map $\mathrm{logh} : X \times X \to T_X$ such that $\mathrm{exph}_z(\mathrm{logh}(z,u)) = u$, and we obtain for w small enough,

$$\begin{aligned}
&\int_X \mathrm{e}^{\Phi(z,w)-\gamma(z)} \mathrm{d}V_\omega(z) \\
&\quad \le \int_{z\in X} \left(\int_{u \in X} \mathrm{e}^{\varphi(u)-\gamma(z)} \chi\left(\frac{|\mathrm{logh}(z,u)|^2}{|w|^2} \right) \frac{1}{|w|^{2n}} \mathrm{d}V_\omega(\mathrm{logh}(z,u)) \right) \mathrm{d}V_\omega(z) \\
&\quad = \int_{u \in X} P(u,w)\, \mathrm{e}^{\varphi(u)-\gamma(u)} \mathrm{d}V_\omega(u),
\end{aligned}$$

where P is a kernel on $X \times D(0, \delta_0)$ such that

$$P(u,w) = \int_{z \in X} \frac{1}{|w|^{2n}} \chi\left(\frac{|\mathrm{logh}(z,u)|^2}{|w|^2} \right) \frac{\mathrm{e}^{\gamma(u)-\gamma(z)} \mathrm{d}V_\omega(\mathrm{logh}(z,u))}{\mathrm{d}V_\omega(u)}\, \mathrm{d}V_\omega(z).$$

Let us first assume that γ is smooth (the case in which γ has logarithmic poles will be considered later). Then a change of variable $\zeta = \frac{1}{w} \mathrm{logh}(z,u)$ shows that P is smooth, and we have $P(u,0) = 1$. Since $P(u,w)$ depends only on $|w|$, we infer

$$P(u,w) \le 1 + C_0 |w|^2$$

for w small. This shows that the integral of $z \mapsto \mathrm{e}^{\Phi(z,w) - C_0|w|^2}$ will be at most equal to 1, and therefore if we define

$$\varphi_{c,\delta}(z) = \inf_{t \in]0,\delta]} \Phi(z,t) + Kt^2 - K\delta^2 - c \log \frac{t}{\delta} \tag{5.6}$$

as in (1.10), the function $\varphi_{c,\delta}(z) \le \Phi(z,\delta)$ will also satisfy

$$\int_X \mathrm{e}^{\varphi_{c,\delta}(z) - C_0\delta^2 - \gamma(z)} \mathrm{d}V_\omega \le 1. \tag{5.7}$$

Now, thanks to the assumption that $K_X + L$ is big, there exists a quasi-psh function ψ_0 with analytic singularities such that $\alpha + dd^c\psi_0 \ge \varepsilon_0\omega$. We can assume $\int_X \mathrm{e}^{\psi_0-\gamma}\mathrm{d}V_\omega = 1$ after adjusting ψ_0 with a suitable constant. Consider a pair of points $x, y \in X$. We take φ such that $\varphi(x) = \varphi_{\mathrm{can}}(x)$ (this is possible by the above discussion). We define

$$\varphi_\lambda = \log\left(\lambda \mathrm{e}^{\psi_0} + (1-\lambda)e^{\varphi}\right) \tag{5.8}$$

with a suitable constant $\lambda \in [0, 1/2]$, which will be fixed later, and obtain in this way regularized functions $\Phi_\lambda(z,w)$ and $\varphi_{\lambda,c,\delta}(z)$. This is obviously a compact family, and therefore the associated constants K needed in (5.6) are uniform in λ. Also, as in Sect. 1, we have

$$\alpha + dd^c\varphi_{\lambda,c,\delta} \ge -(Ac + K\delta^2)\,\omega \qquad \text{for all } \delta \in \,]0,\delta_0]. \tag{5.9}$$

Finally, we consider the linear combination

$$\theta = \frac{Ac + K\delta^2}{\varepsilon_0}\psi_0 + \left(1 - \frac{Ac+K\delta^2}{\varepsilon_0}\right)(\varphi_{\lambda,c,\delta} - C_0\delta^2). \tag{5.10}$$

Clearly, $\int_X \mathrm{e}^{\varphi_\lambda - \gamma}\mathrm{d}V_\omega \le 1$, and therefore θ also satisfies $\int_X e^{\theta-\gamma}\mathrm{d}V_\omega \le 1$ by Hölder's inequality. Our linear combination is precisely taken so that $\alpha + dd^c\theta \ge 0$. Therefore, by definition of φ_{can}, we find that

$$\varphi_{\mathrm{can}} \ge \theta = \frac{Ac + K\delta^2}{\varepsilon_0}\psi_0 + \left(1 - \frac{Ac+K\delta^2}{\varepsilon_0}\right)(\varphi_{\lambda,c,\delta} - C_0\delta^2). \tag{5.11}$$

Assume $x \in X \smallsetminus Z_0$, so that $\varphi_\lambda(x) > -\infty$ and $\nu(\varphi_\lambda, x) = 0$. In (5.6), the infimum is reached either for $t = \delta$ or for t such that $c = t\frac{\mathrm{d}}{\mathrm{d}t}(\Phi_\lambda(z,t) + Kt^2)$. The function $t \mapsto \Phi_\lambda(z,t) + Kt^2$ is convex increasing in $\log t$ and tends to $\varphi_\lambda(z)$ as $t \to 0$. By convexity, this implies

$$\begin{aligned} c = t\frac{\mathrm{d}}{\mathrm{d}t}(\Phi_\lambda(z,t) + Kt^2) &\le \frac{(\Phi_\lambda(x,\delta_0) + K\delta_0^2) - (\Phi_\lambda(z,t) + Kt^2)}{\log(\delta_0/t)} \\ &\le \frac{C_1 - \varphi_\lambda(x)}{\log(\delta_0/t)} \le \frac{C_1 + |\psi_0(z)| + \log(1/\lambda)}{\log(\delta_0/t)}, \end{aligned}$$

and hence

$$\frac{1}{t} \le \max\left(\frac{1}{\delta}, \frac{1}{\delta_0}\exp\left(\frac{C_1 + |\psi_0(z)| + \log(1/\lambda)}{c}\right)\right). \tag{5.12}$$

This shows that t cannot be too small when the infimum is reached.

When t is taken equal to the value that achieves the infimum for $z=y$, we find that

$$\varphi_{\lambda,c,\delta}(y)=\Phi_\lambda(y,t)+Kt^2-K\delta^2-c\log\frac{t}{\delta}\geq\Phi_\lambda(y,t)+Kt^2-K\delta^2. \tag{5.13}$$

Since $z\mapsto\Phi_\lambda(z,t)$ is a convolution of φ_λ, we get a bound of the first-order derivative

$$|D_z\Phi_\lambda(z,t)|\leq\|\varphi_\lambda\|_{L^1(X)}\frac{C_2}{t}\leq\frac{C_3}{t},$$

and with respect to the geodesic distance $d(x,y)$ we infer from this that

$$\Phi_\lambda(y,t)\geq\Phi_\lambda(x,t)-\frac{C_3}{t}d(x,y). \tag{5.14}$$

A combination of (5.11), (5.13), and (5.14) yields

$$\begin{aligned}\varphi_{\mathrm{can}}(y)&\geq\frac{Ac+K\delta^2}{\varepsilon_0}\psi_0(y)+\left(1-\frac{Ac+K\delta^2}{\varepsilon_0}\right)\left(\Phi_\lambda(x,t)+Kt^2-K\delta^2-\frac{C_3}{t}d(x,y)\right)\\&\geq\frac{Ac+K\delta^2}{\varepsilon_0}\psi_0(y)+\left(1-\frac{Ac+K\delta^2}{\varepsilon_0}\right)\left(\varphi_\lambda(x)-K\delta^2-\frac{C_3}{t}d(x,y)\right)\\&\geq\log\left(\lambda e^{\psi_0(x)}+(1-\lambda)e^{\varphi(x)}\right)-C_4\left((c+\delta^2)(|\psi_0(y)|+1)+\frac{1}{t}d(x,y)\right)\\&\geq\varphi_{\mathrm{can}}(x)-C_5\left(\lambda+(c+\delta^2)(|\psi_0(y)|+1)+\frac{1}{t}d(x,y)\right),\end{aligned}$$

if we use the fact that $\varphi_\lambda(x)\leq C_6$, $\varphi(x)=\varphi_{\mathrm{can}}(x)$, and $\log(1-\lambda)\geq-(2\log 2)\lambda$ for all $\lambda\in[0,1/2]$.

By exchanging the roles of x,y and using (5.12), we see that for all $c>0$, $\delta\in\,]0,\delta_0]$, and $\lambda\in\,]0,1/2]$, there is an inequality

$$\left|\varphi_{\mathrm{can}}(y)-\varphi_{\mathrm{can}}(x)\right|\leq C_5\left(\lambda+(c+\delta^2)\big(\max(|\psi_0(x)|,|\psi_0(y)|)+1\big)+\frac{1}{t}d(x,y)\right), \tag{5.15}$$

where

$$\frac{1}{t}\leq\max\left(\frac{1}{\delta},\,\frac{1}{\delta_0}\exp\left(\frac{C_1+\max(|\psi_0(x)|,|\psi_0(y)|)+\log(1/\lambda)}{c}\right)\right). \tag{5.16}$$

By taking c, δ, and λ small, one easily sees that this implies the continuity of φ_{can} on $X\smallsetminus Z_0$. More precisely, if we choose

$$\begin{aligned}&\delta=d(x,y)^{1/2},\qquad\lambda=\frac{1}{|\log d(x,y)|},\\&c=\frac{C_1+\max(|\psi_0(x)|,|\psi_0(y)|)+\big|\log|\log d(x,y)|\big|}{\log\delta_0/d(x,y)^{1/2}}\end{aligned}$$

with $d(x,y) < \delta_0^2 < 1$, we get $\frac{1}{t} \le d(x,y)^{-1/2}$, whence an explicit (but certainly not optimal) modulus of continuity of the form

$$\left|\varphi_{\mathrm{can}}(y) - \varphi_{\mathrm{can}}(x)\right| \le C_7\big(\max(|\psi_0(x)|,|\psi_0(y)|)+1\big)^2 \frac{\big|\log|\log d(x,y)|\big|+1}{|\log d(x,y)|+1}.$$

When the weight γ has analytic singularities, the kernel $P(u,w)$ is no longer smooth and the volume estimate (5.7). In this case, we use a modification $\mu : \widehat{X} \to X$ in such a way that the singularities of $\gamma\circ\mu$ are divisorial, given by a divisor with normal crossings. If we put

$$\widehat{L} = \mu^*L - K_{\widehat{X}/X} = \mu^*L - E$$

(E the exceptional divisor), then we get an induced singular metric on $\widehat{L}$ that still satisfies the klt condition, and the corresponding supercanonical metric on $K_{\widehat{X}} + \widehat{L}$ is just the pullback by μ of the supercanonical metric on $K_X + L$. This shows that we may assume from the start that the singularities of γ are divisorial and given by a klt divisor Δ. In this case, a solution to the problem is to introduce a complete Hermitian metric $\hat\omega$ of uniformly bounded curvature on $X \smallsetminus |\Delta|$ using the Poincaré metric on the punctured disk as a local model transversal to the components of Δ. The Poincaré metric on the punctured unit disk is given by

$$\frac{|dz|^2}{|z|^2(\log|z|)^2},$$

and the singularity of $\hat\omega$ along the component $\Delta_j = \{g_j(z)=0\}$ of Δ is given by

$$\hat\omega = \sum -dd^c \log|\log|g_j|| \qquad \mathrm{mod}\ C^\infty.$$

Since such a metric has bounded geometry and this is all that we need for the calculations of [Dem94] to work, the estimates that we have made here are still valid, especially the crucial lower bound $\alpha + dd^c\varphi_{\lambda,c,\delta} \ge -(Ac+K\delta^2)\,\hat\omega$. In order to compensate this loss of positivity, we need a quasi-psh function $\hat\psi_0$ such that $\alpha + dd^c\hat\psi_0 \ge \varepsilon_0\hat\omega$, but such a lower bound is possible by adding terms of the form $-\varepsilon_1\log|\log|g_j||$ to our previous quasi-psh function ψ_0.

With respect to the Poincaré metric, a δ-ball of center z_0 in the punctured disk is contained in the corona

$$|z_0|^{e^{-\delta}} < |z| < |z_0|^{e^{\delta}},$$

and it is easy to see from this that the mean value of $|z|^{-2a}$ on a δ-ball of center z_0 is multiplied by at most $|z_0|^{-2a\delta}$. This implies that a function of the form $\hat\varphi_{c,\delta} = \varphi_{c,\delta} + C_9\delta\sum\log|g_j|$ will actually give rise to an integral $\int_X e^{\hat\varphi_{c,\delta}-\gamma}dV_\omega \le 1$. We see that the term δ^2 in (5.15) has to be replaced by a term of the form

$$\delta\sum\max\big(|\log|g_j(x)||,|\log|g_j(x)||\big).$$

This is enough to obtain the continuity of φ_{can} on $X \smallsetminus (Z_0\cup|\Delta|)$, as well as an explicit logarithmic modulus of continuity. □

Algebraic version 5.17. Since the klt condition is open and K_X+L is assumed to be big, we can always perturb L a little bit, and after blowing up X, assume that X is projective and that $(L,h_{L,\gamma})$ is obtained as a sum of $\mathbb{Q}$-divisors

$$L=G+\Delta,$$

where Δ is klt and G is equipped with a smooth metric h_G (from which $h_{L,\gamma}$ is inferred, with Δ as its poles, so that $\Theta_{L,h_{L,\gamma}}=\Theta_{G,L_G}+[\Delta]$). Clearly this situation is "dense" in what we have been considering before, just as $\mathbb{Q}$ is dense in $\mathbb{R}$. In this case, it is possible to give a more algebraic definition of the supercanonical metric φ_{can}, following the original idea of Narasimhan–Simha [NS68] (see also Tsuji [Ts07a]) – the case considered by these authors is the special situation in which $G=0$, $h_G=1$ (and moreover, $\Delta=0$ and K_X ample, for [NS68]).

In fact, if m is a large integer that is a multiple of the denominators involved in G and Δ, we can consider sections

$$\sigma\in H^0(X,m(K_X+G+\Delta)).$$

We view them rather as sections of $m(K_X+G)$ with poles along the support $|\Delta|$ of our divisor. Then $(\sigma\wedge\overline{\sigma})^{1/m}h_G$ is a volume form with integrable poles along $|\Delta|$ (this is the klt condition for Δ). Therefore one can normalize σ by requiring that

$$\int_X(\sigma\wedge\overline{\sigma})^{1/m}h_G=1.$$

Each of these sections defines a singular Hermitian metric on $K_X+L=K_X+G+\Delta$, and we can take the regularized upper envelope

$$\varphi^{\mathrm{alg}}_{\mathrm{can}}=\left(\sup_{m,\sigma}\frac{1}{m}\log|\sigma|^2_{h^m_{K_X+L}}\right)^* \tag{5.18}$$

of the weights associated with a smooth metric h_{K_X+L}. It is clear that $\varphi^{\mathrm{alg}}_{\mathrm{can}}\le\varphi_{\mathrm{can}}$, since the supremum is taken on the smaller set of weights $\varphi=\frac{1}{m}\log|\sigma|^2_{h^m_{K_X+L}}$, and the equalities

$$\begin{aligned}\mathrm{e}^{\varphi-\gamma}\mathrm{d}V_\omega&=|\sigma|^{2/m}_{h^m_{K_X+L}}\mathrm{e}^{-\gamma}\mathrm{d}V_\omega\\&=(\sigma\wedge\overline{\sigma})^{1/m}\mathrm{e}^{-\gamma}h_L=(\sigma\wedge\overline{\sigma})^{1/m}h_{L,\gamma}=(\sigma\wedge\overline{\sigma})^{1/m}h_G\end{aligned}$$

imply $\int_X\mathrm{e}^{\varphi-\gamma}\mathrm{d}V_\omega\le 1$.

We claim that the inequality $\varphi^{\mathrm{alg}}_{\mathrm{can}}\le\varphi_{\mathrm{can}}$ is an equality. The proof is an immediate consequence of the following statement, based in turn on the Ohsawa–Takegoshi theorem and the approximation technique of [Dem92].

Proposition 5.19. *With $L = G + \Delta$, ω, $\alpha = \Theta_{K_X+L,h_{K_X+L}}$, γ as above, and $K_X + L$ assumed to be big, fix a singular Hermitian metric $\mathrm{e}^{-\varphi}h_{K_X+L}$ of curvature $\alpha + dd^c\varphi \geq 0$ such that $\int_X \mathrm{e}^{\varphi-\gamma}\mathrm{d}V_\omega \leq 1$. Then φ is equal to a regularized limit*

$$\varphi = \left(\limsup_{m\to+\infty} \frac{1}{m}\log|\sigma_m|^2_{h^m_{K_X+L}}\right)^*$$

for a suitable sequence of sections $\sigma_m \in H^0(X, m(K_X + G + \Delta))$ with $\int_X(\sigma_m \wedge \overline{\sigma}_m)^{1/m}h_G \leq 1$.

Proof. By our assumption, there exists a quasi-psh function ψ_0 with analytic singularity set Z_0 such that

$$\alpha + dd^c\psi_0 \geq \varepsilon_0\omega > 0,$$

and we can assume $\int_C \mathrm{e}^{\psi_0-\gamma}\mathrm{d}V_\omega < 1$ (the strict inequality will be useful later). For $m \geq p \geq 1$, this defines a singular metric $\exp(-(m-p)\varphi - p\psi_0)h^m_{K_X+L}$ on $m(K_X + L)$ with curvature greater than or equal to $p\varepsilon_0\omega$, and therefore a singular metric

$$h_{L'} = \exp(-(m-p)\varphi - p\psi_0)h^m_{K_X+L}h^{-1}_{K_X}$$

on $L' = (m-1)K_X + mL$ whose curvature $\Theta_{L',h_{L'}} \geq (p\varepsilon_0 - C_0)\omega$ is arbitrarily large if p is large enough.

Let us fix a finite covering of X by coordinate balls. Pick a point x_0 and one of the coordinate balls B containing x_0. By the Ohsawa–Takegoshi extension theorem applied to the ball B, we can find a section σ_B of $K_X + L' = m(K_X + L)$ that has norm 1 at x_0 with respect to the metric $h_{K_X+L'}$ and $\int_B |\sigma_B|^2_{h_{K_X+L'}}\mathrm{d}V_\omega \leq C_1$ for some uniform constant C_1 depending on the finite covering, but independent of m, p, x_0.

Now we use a cutoff function $\theta(x)$ with $\theta(x) = 1$ near x_0 to truncate σ_B and solve a $\overline{\partial}$-equation for $(n,1)$-forms with values in L to get a global section σ on X with $|\sigma(x_0)|_{h_{K_X+L'}} = 1$. For this we need to multiply our metric by a truncated factor $\exp(-2n\theta(x)\log|x - x_0|)$ so as to get solutions of $\overline{\partial}$ vanishing at x_0. However, this perturbs the curvature by bounded terms, and we can absorb them again by taking p larger. In this way, we obtain

$$\int_X |\sigma|^2_{h_{K_X+L'}}\mathrm{d}V_\omega = \int_X |\sigma|^2_{h^m_{K_X+L}}\mathrm{e}^{-(m-p)\varphi-p\psi_0}\mathrm{d}V_\omega \leq C_2. \tag{5.20}$$

Taking $p > 1$, the Hölder inequality for conjugate exponents m, $\frac{m}{m-1}$ implies

$$\begin{aligned}\int_X(\sigma\wedge\overline{\sigma})^{\frac{1}{m}}h_G &= \int_X |\sigma|^{2/m}_{h^m_{K_X+L}}\mathrm{e}^{-\gamma}\mathrm{d}V_\omega \\ &= \int_X \left(|\sigma|^2_{h^m_{K_X+L}}\mathrm{e}^{-(m-p)\varphi-p\psi_0}\right)^{\frac{1}{m}}\left(\mathrm{e}^{(1-\frac{p}{m})\varphi+\frac{p}{m}\psi_0-\gamma}\right)\mathrm{d}V_\omega\end{aligned}$$

$$
\begin{aligned}
&\le C_2^{\frac{1}{m}} \left(\int_X \left(\mathrm{e}^{(1-\frac{p}{m})\varphi + \frac{p}{m}\psi_0 - \gamma} \right)^{\frac{m}{m-1}} \mathrm{d}V_\omega \right)^{\frac{m-1}{m}} \\
&\le C_2^{\frac{1}{m}} \left(\int_X \left(\mathrm{e}^{\varphi-\gamma} \right)^{\frac{m-p}{m-1}} \left(\mathrm{e}^{\frac{p}{p-1}(\psi_0-\gamma)} \right)^{\frac{p-1}{m-1}} \mathrm{d}V_\omega \right)^{\frac{m-1}{m}} \\
&\le C_2^{\frac{1}{m}} \left(\int_X \mathrm{e}^{\frac{p}{p-1}(\psi_0-\gamma)} \mathrm{d}V_\omega \right)^{\frac{p-1}{m}}
\end{aligned}
$$

using the hypothesis $\int_X \mathrm{e}^{\varphi-\gamma}\mathrm{d}V_\omega \le 1$ and another application of Hölder's inequality. Since klt is an open condition and $\lim_{p\to+\infty} \int_X \mathrm{e}^{\frac{p}{p-1}(\psi_0-\gamma)}\mathrm{d}V_\omega = \int_X \mathrm{e}^{\psi_0-\gamma}\mathrm{d}V_\omega < 1$, we can take p large enough to ensure that

$$\int_X \mathrm{e}^{\frac{p}{p-1}(\psi_0-\gamma)}\mathrm{d}V_\omega \le C_3 < 1.$$

Therefore, we see that

$$\int_X (\sigma \wedge \overline{\sigma})^{\frac{1}{m}} h_G \le C_2^{\frac{1}{m}} C_3^{\frac{p-1}{m}} \le 1$$

for p large enough. On the other hand,

$$|\sigma(x_0)|^2_{h_{K_X+L'}} = |\sigma(x_0)|^2_{h^m_{K_X+L}} \mathrm{e}^{-(m-p)\varphi(x_0) - p\psi_0(x_0)} = 1,$$

and thus

$$\frac{1}{m} \log |\sigma(x_0)|^2_{h^m_{K_X+L}} = \left(1 - \frac{p}{m}\right)\varphi(x_0) + \frac{p}{m}\psi_0(x_0), \tag{5.21}$$

and as a consequence,

$$\frac{1}{m} \log |\sigma(x_0)|^2_{h^m_{K_X+L}} \longrightarrow \varphi(x_0)$$

whenever $m \to +\infty$, $\frac{p}{m} \to 0$, as long as $\psi_0(x_0) > -\infty$.

In the above argument, we can in fact interpolate in finitely many points $x_1, x_2, \ldots, x_q$, provided that $p \ge C_4 q$. Therefore, if we take a suitable dense subset $\{x_q\}$ and a "diagonal" sequence associated with sections $\sigma_m \in H^0(X, m(K_X+L))$ with $m \gg p = p_m \gg q = q_m \to +\infty$, we infer that

$$\left(\limsup_{m\to+\infty} \frac{1}{m} \log |\sigma_m(x)|^2_{h^m_{K_X+L}} \right)^* \ge \limsup_{x_q \to x} \varphi(x_q) = \varphi(x) \tag{5.22}$$

(the latter equality occurring if $\{x_q\}$ is suitably chosen with respect to φ). In the other direction, (5.20) implies a mean value estimate

$$\frac{1}{\pi^n r^{2n}/n!} \int_{B(x,r)} |\sigma(z)|^2_{h^m_{K_X+L}} \,\mathrm{d}z \le \frac{C_5}{r^{2n}} \sup_{B(x,r)} \mathrm{e}^{(m-p)\varphi + p\psi_0}$$

on every coordinate ball $B(x,r) \subset X$. The function $|\sigma_m|^2_{h^m_{K_X+L}}$ is plurisubharmonic after we correct the not necessarily positively curved smooth metric h_{K_X+L} by a factor of the form $\exp(C_6|z-x|^2)$. Hence the mean value inequality shows that

$$\frac{1}{m}\log|\sigma_m(x)|^2_{h^m_{K_X+L}} \le \frac{1}{m}\log\frac{C_5}{r^{2n}} + C_6 r^2 + \sup_{B(x,r)}\left(1-\frac{p_m}{m}\right)\varphi + \frac{p_m}{m}\psi_0.$$

By taking in particular $r = 1/m$ and letting $m \to +\infty$, $p_m/m \to 0$, we see that the opposite of inequality (5.22) also holds. □

Remark 5.23. We can rephrase our results in slightly different terms. In fact, let us put

$$\varphi^{\mathrm{alg}}_m = \sup_\sigma \frac{1}{m}\log|\sigma|^2_{h^m_{K_X+L}}, \qquad \sigma \in H^0(X, m(K_X+G+\Delta)),$$

with normalized sections σ such that $\int_X (\sigma\wedge\overline{\sigma})^{1/m} h_G = 1$. Then φ^{alg}_m is quasi-psh (the supremum is taken over a compact set in a finite-dimensional vector space), and by passing to the regularized supremum over all σ and all φ in (5.21), we get

$$\varphi_{\mathrm{can}} \ge \varphi^{\mathrm{alg}}_m \ge \left(1-\frac{p}{m}\right)\varphi_{\mathrm{can}}(x) + \frac{p}{m}\psi_0(x).$$

Since φ_{can} is bounded from above, we find in particular that

$$0 \le \varphi_{\mathrm{can}} - \varphi^{\mathrm{alg}}_m \le \frac{C}{m}(|\psi_0(x)|+1).$$

This implies that $(\varphi^{\mathrm{alg}}_m)$ converges uniformly to φ_{can} on every compact subset of $X \subset Z_0$, and in this way we infer again (in a purely qualitative manner) that φ_{can} is continuous on $X \smallsetminus Z_0$. Moreover, we also see that in (5.18), the upper semicontinuous regularization is not needed on $X \smallsetminus Z_0$; in case $K_X + L$ is ample, it is not needed at all, and we have uniform convergence of $(\varphi^{\mathrm{alg}}_m)$ to φ_{can} on the whole of X. Obtaining such a uniform convergence when $K_X + L$ is just big looks like a more delicate question, related, for instance, to abundance of $K_X + L$ on those subvarieties Y where the restriction $(K_X + L)_{|Y}$ would be, for example, nef but not big.

Generalization 5.24. In the general case that L is a $\mathbb{R}$-line bundle and $K_X + L$ is merely pseudo-effective, a similar algebraic approximation can be obtained. We take instead sections

$$\sigma \in H^0(X, mK_X + \lfloor mG \rfloor + \lfloor m\Delta \rfloor + p_m A)$$

where (A, h_A) is a positive line bundle, $\Theta_{A,h_A} \ge \varepsilon_0\omega$, and replace the definition of $\varphi^{\mathrm{alg}}_{\mathrm{can}}$ by

$$\varphi^{\mathrm{alg}}_{\mathrm{can}} = \left(\limsup_{m\to+\infty}\ \sup_\sigma \frac{1}{m}\log|\sigma|^2_{h_{mK_X+\lfloor mG\rfloor+p_m A}}\right)^*, \tag{5.25}$$

$$\int_X (\sigma \wedge \overline{\sigma})^{\frac{2}{m}} h^{\frac{1}{m}}_{\lfloor mG \rfloor + p_m A} \leq 1, \tag{5.26}$$

where $m \gg p_m \gg 1$ and $h^{1/m}_{\lfloor mG \rfloor}$ is chosen to converge uniformly to h_G.

We then find again $\varphi_{\mathrm{can}} = \varphi^{\mathrm{alg}}_{\mathrm{can}}$, with an almost identical proof, though we no longer have a sup in the envelope, but just a lim sup. The analogue of Proposition (5.19) also holds in this context, with an appropriate sequence of sections $\sigma_m \in H^0(X, mK_X + \lfloor mG \rfloor + \lfloor m\Delta \rfloor + p_m A)$.

Remark 5.27. The envelopes considered in Sect. 1 are envelopes constrained by an L^∞ condition, while the present ones are constrained by an L^1 condition. It is possible to interpolate and to consider envelopes constrained by an L^p condition. More precisely, assuming that $\frac{1}{p}K_X + L$ is pseudoeffective, we look at metrics $\mathrm{e}^{-\varphi} h_{\frac{1}{p}K_X + L}$ and normalize them with the L^p condition

$$\int_X \mathrm{e}^{p\varphi - \gamma} \mathrm{d}V_\omega \leq 1.$$

This is actually an L^1 condition for the induced metric on pL, and therefore we can just apply the above after replacing L by pL. If we assume, moreover, that L is pseudoeffective, it is clear that the L^p condition converges to the L^∞ condition $\varphi \leq 0$ if we normalize γ by requiring $\int_X \mathrm{e}^{-\gamma} \mathrm{d}V_\omega = 1$.

Remark 5.28. It would be nice to have a better understanding of the supercanonical metrics. In case X is a curve, this should be easier. In fact, X then has a Hermitian metric ω with constant curvature, which we normalize by requiring that $\int_X \omega = 1$, and we can also suppose $\int_X \mathrm{e}^{-\gamma}\omega = 1$. The class $\lambda = c_1(K_X + L) \geq 0$ is a number, and we take $\alpha = \lambda\omega$. Our envelope is $\varphi_{\mathrm{can}} = \sup \varphi$, where $\lambda\omega + dd^c\varphi \geq 0$ and $\int_X \mathrm{e}^{\varphi - \gamma}\omega \leq 1$.

If $\lambda = 0$, then φ must be constant, and clearly $\varphi_{\mathrm{can}} = 0$. Otherwise, if $G(z,a)$ denotes the Green function such that $\int_X G(z,a)\,\omega(z) = 0$ and $dd^c G(z,a) = \delta_a - \omega(z)$, we obtain

$$\varphi_{\mathrm{can}}(z) \geq \sup_{a \in X} \left(\lambda G(z,a) - \log \int_{z \in X} \mathrm{e}^{\lambda G(z,a) - \gamma(z)} \omega(z) \right)$$

by taking the envelope already over $\varphi(z) = \lambda G(z,a) - \mathrm{const}$. It is natural to ask whether this is always an equality, i.e., whether the extremal functions are always given by one of the Green functions, especially when $\gamma = 0$.

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AQ1. Kindly check the corresponding author.

AQ2. Kindly check the label of Lemma in the sentence "Without loss of generality, ..."

AQ3. Kindly update Ref. [Ber07], [BB08], [BBGZ09], [BEGZ08], [Dem89], [PS08], [Ts07a], [Ts07b].

AQ4. References [Blo09, Dem93, OhT87] are not cited in text. Kindly cite the same or delete them from the reference list.