Exotic Structures on Smooth Four-Manifolds

Selman Akbulut

Dedicated to Oleg Viro on the occasion of his 60th birthday.

Abstract A short survey of exotic smooth structures on 4-manifolds is given with a special emphasis on the corresponding cork structures. Along the way we discuss some of the more recent results in this direction, obtained jointly with R. Matveyev, B. Ozbagci, C. Karakurt, and K. Yasui.

Keywords Four-manifold • Exotic structure • Lefschetz fibration • Cork

1 Corks

Let M be a smooth closed simply connected four-manifold, and M' an exotic copy of M (a smooth manifold homeomorphic but not diffeomorphic to M). Then we can find a compact contractible codimension-zero submanifold $W \subset M$ with complement N, and an involution $f : \partial W \to \partial W$ giving decompositions (identifications are by diffeomorphisms)

$$M = N \cup_{\mathrm{id}} W, \quad M' = N \cup_{\mathrm{f}} W \tag{1}$$

The existence of this structure was first observed in an example in [A1]. Then in [M] and [CFHS], this was generalized to the general form discussed above. Also, the 5-dimensional h-cobordisms induced by corks were studied in [K]. Since then, the contractible pieces W appearing in this decomposition have come to be known as *corks*.

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Author's Proof

AQ1

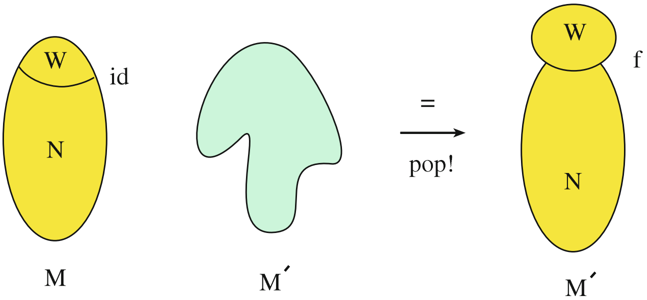

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Fig. 1

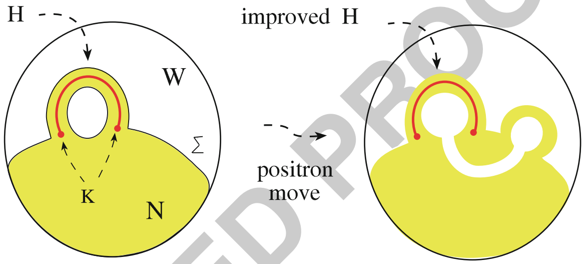

Fig. 2

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Recall that a properly embedded complex submanifold of an affine space $X \subset \mathbf{C}^N$ is called a *Stein manifold*. Also, in topology, a smooth submanifold $M \subset X$ is called a *compact Stein* manifold if it is cut out from X by $f \leq c$, where $f : X \to \mathbf{R}$ is a strictly plurisubharmonic (proper) Morse function and c is a regular value. In particular, M is a symplectic manifold with convex boundary and the symplectic form $\omega = \frac{1}{2}\partial\bar{\partial} f$. The form ω induces a contact structure ξ on the boundary ∂M. We call (M, ω) a *Stein filling* of the boundary contact manifold $(\partial M, \xi)$. Stein manifolds have been a useful tool for studying smooth four-manifolds. In this paper, we will sometimes for the sake of brevity abuse conventions and call compact Stein manifolds just Stein manifolds (Figs. 1 and 2).

AQ2

By [AM], in the cork decomposition (1), each W and each N piece can be made Stein. This is achieved by a useful technique (called "creating positrons" in [AM]) that amounts to moving the common boundary $\Sigma = \partial W = \partial N$ in M by a convenient homotopy: First, by handle exchanges we can assume that each W and N side has only 1- and 2-handles. Eliashberg's criterion (cf. [G]) says that manifolds with 1- and 2-handles are Stein if the attaching framings of the 2-handles are sufficiently negative (let us call these *admissible*). This means that any 2-handle H has to be attached along a knot K with framing less than the Thurston–Bennequin framing $\mathrm{tb}(K)$ of any Legendrian representative of K (i.e., K is tangent to ξ, and $\mathrm{tb}(K)$ is the framing induced by ξ). The idea is that when the attaching framing is bigger than $\mathrm{tb}(K) - 1$, by local handle exchanges near H (but away from H) to alter $\Sigma \rightsquigarrow \Sigma'$, which results in an increase in the Thurston–Bennequin numbers $\mathrm{tb}(K) \rightsquigarrow \mathrm{tb}(K') = \mathrm{tb}(K) + 3$.

Author's Proof

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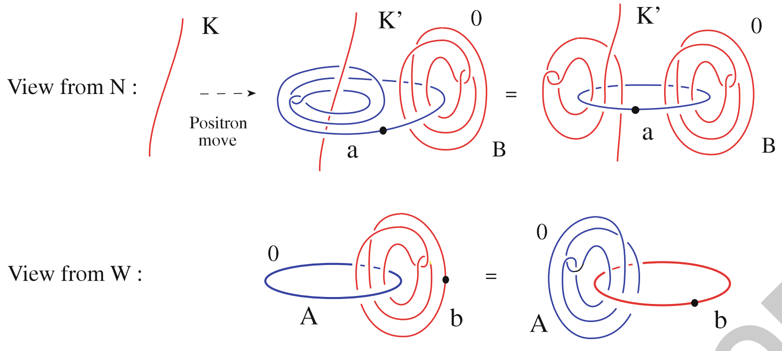

Fig. 3 Making W and N Stein

For example, as indicated in Fig. 3, carving out a tubular neighborhood of a properly embedded 2-disk **a** from the interior of N increases $\mathrm{tb}(K)$ by 3. Carving in the N side corresponds to attaching a 2-handle **A** from the W side, which itself might be attached with a "bad" framing. To prevent this, we also attach a 2-handle **B** to N near **a**, which corresponds to carving out a 2-disk from the W side. This makes the framing of the 2-handle **A** in the W side admissible. Furthermore, **B** itself is admissible. So by carving a 2-disk **a** and attaching a 2-handle **B**, we have improved the attaching framing of the 2-handle H without changing other handles (we changed Σ by a homotopy). This technique gives the following result.

Theorem 1. *([AM]) Given any decomposition of a closed smooth four-manifold $M = N_1 \cup_\partial N_2$ by codimension-zero submanifolds, with each piece consisting of 1- and 2-handles, after altering pieces by a homotopy, we can obtain a similar decomposition $M = N_1' \cup N_2'$, where both pieces N_1' and N_2' are Stein manifolds.*

Using [Gi], one can also assume that the two open books on the common contact three-manifold boundaries match, but with the wrong orientation [B1].

Definition 1. A cork is a pair (W, f), where W is a compact Stein manifold and $f : \partial W \to \partial W$ is an involution that extends to a self-homeomorphism of W but does not extend to a self-diffeomorphism of W. We say that (W, f) is a cork of M if we have the decomposition (1) for some exotic copy M' of M.

In particular, a cork is a fake copy of itself. There are some natural families of corks W_n, $n = 1, 2, \ldots$, which are generalizations of the Mazur manifold W used in [A1], and $\overline{W}_n$, $n = 1, 2, \ldots$ (Fig. 4), which were introduced in [AM] (so-called positrons). Recently, in [AY1], all these infinite families were shown to be corks. Now it is a natural question to ask whether these small standard corks are sufficient to explain all exotic smooth structures on four-manifolds? (Fig. 5). For example, we know that W_1 is a cork of the blown-up Kummer surface $E(2)\#\overline{\mathbb{CP}}^2$ [A1], and $\overline{W}_1$ is a cork of the Dolgachev surface $E(1)_{2,3}$ [A2], where $E(n)$ is the elliptic surface of signature $-8n$.

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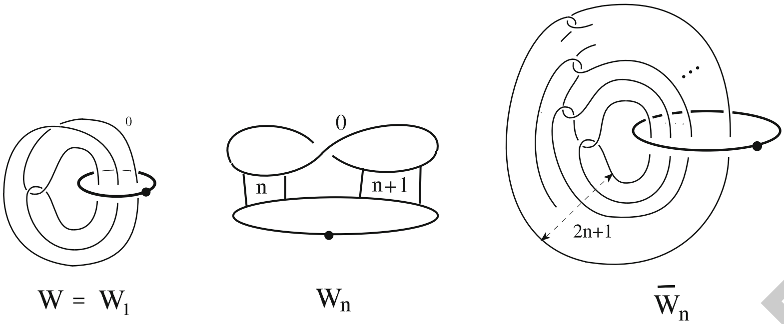

Fig. 4 Variety of corks

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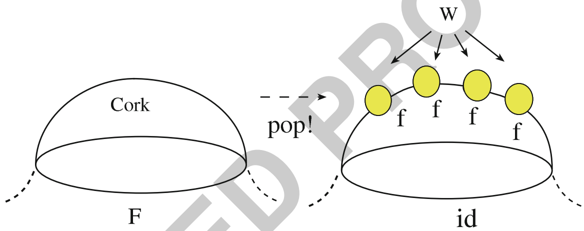

Fig. 5 Do all corks decompose into standard corks?

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We require corks to be Stein manifolds in order to rule out trivial examples, as well as to introduce rigidity in their structures. For example, a theorem of Eliashberg says that if a Stein manifold has boundary S^3 or $S^1 \times S^2$, then it has to be a B^4 or $S^1 \times B^3$, respectively.

1.1 How to Recognize a Cork

In general, it is hard to recognize when a codimension-zero contractible submanifold $W \subset M^4$ is a cork of M. In fact, all the corks obtained in the general cork decomposition theorem of [M] have the property that $W \cup -W = S^4$ and $W \cup_f -W = S^4$. So it is easy to embed W's into charts of M without being corks of M. One quick way of showing that (W, f) is a cork of a manifold M with nontrivial Seiberg–Witten invariants is to show that the change $M \rightsquigarrow M'$ in (1) gives a split manifold M', implying zero (or different) Seiberg–Witten invariants. In [AY1] and [A2], many interesting corks were located using this strategy.

There are also some hard-to-calculate algebraic ways of checking whether $W \subset M$ is a cork, provided that we know the Heegard–Floer homology groups of the boundary of W [OS]. This follows from the computation of the Ozsváth–Szabó 4-manifold invariant, i.e., by first removing two B^4's from M as shown in Fig. 6, and computing a certain trace of the induced map on the Floer homology of the two S^3 boundary components (induced from the cobordism). For example, we have the following theorem.

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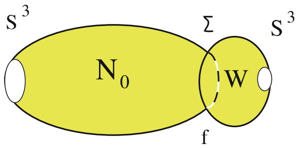
Fig. 6



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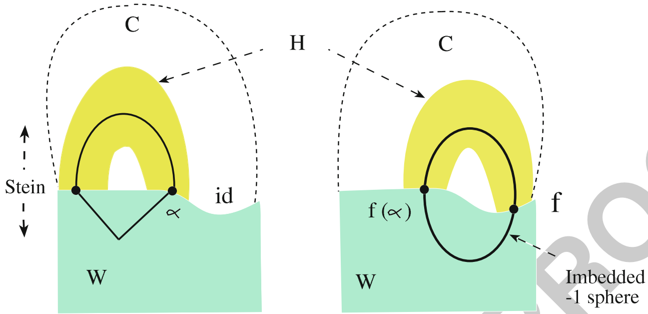

Fig. 7 Inflating a cork to exotic manifold pairs

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Theorem 2. *([AD]): Let $M = N \cup_\partial W$ be a cork decomposition of a smooth closed four-manifold, where W is the Mazur manifold and $b_2^+(M) > 1$ (the union is along the common boundary Σ). Let N_0 be the cobordism from S^3 and Σ obtained from N by removing a B^4 from its interior. Then $Q' = N \cup_{\mathrm{f}} W$ is a fake copy of Q if the image of the "mix map" $F^{\mathrm{mix}}_{(N_0,s)}$ (defined in [OS]) lies in T_0^+ for some* Spin^c *structure s.*

$$F^{\mathrm{mix}}_{(N_0,s)} : HF^-(S^3) \to HF^+(\Sigma) \cong T_0^+ \oplus \mathbf{Z}_{(0)} \oplus \mathbf{Z}_{(0)}.$$

1.2 Constructing Exotic Manifolds from Corks

By thickening a cork in two different ways one can obtain absolutely exotic manifold pairs (i.e., homeomorphic but not diffeomorphic manifolds). Here is a quick review of [A3]: Let (W, f) be the Mazur cork. Then its involution $f : \partial W \to \partial W$ has an amazing property: There is a pair of loops α, β with the following properties:

- $f(\alpha) = \beta$.
- $M := W +$ (2-handle to α with -1 framing) is a Stein manifold.
- β is slice in W; hence $M' := W +$ (2-handle to β with -1 framing) contains an embedded (-1)-sphere.

So M' is an absolutely exotic copy of M; if not, M' would be a Stein manifold also, but any Stein manifold compactifies into an irreducible symplectic manifold [LM], contradicting the existence of the smoothly embedded (-1)-sphere (Fig. 7).

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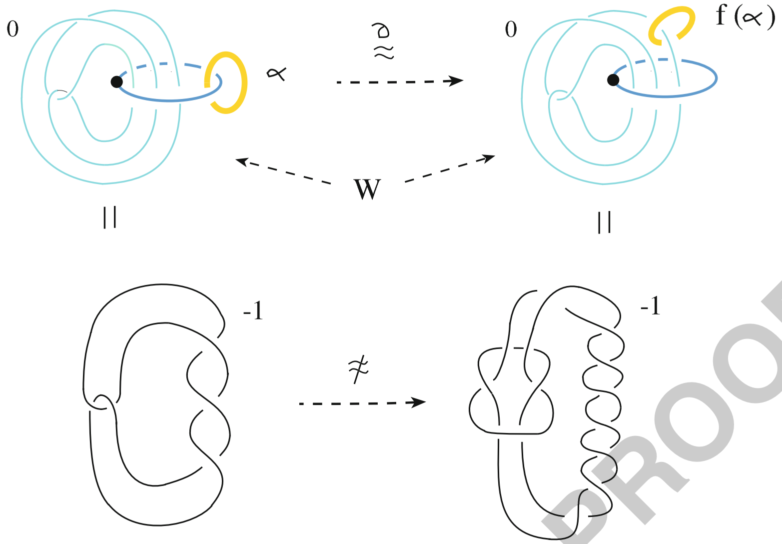

Fig. 8 Exotic manifold pairs

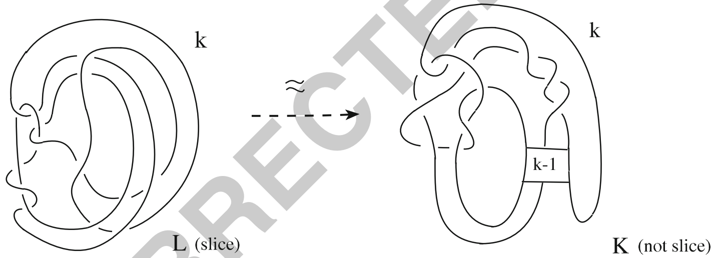

Fig. 9 Diffeomorphic manifold pairs

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Interestingly, by handle slides one can show that each of M, M' is obtained by attaching a 2-handle to B^4 along a knot, as shown in Fig. 8 [A3]. The reader should contrast this with [A5], where examples of other knot pairs $K, L \subset S^3$ are given (one is a slice; the other is not a slice) such that attaching 2-handles to B^4 along K, L gives diffeomorphic four-manifolds, as in Fig. 9.

2 PALFs

It turns out that Stein manifolds admit finer structures as primitives. They are "positive allowable Lefschetz fibrations" over the 2-disk, where the regular fibers are surfaces F with boundaries; in [AO1], we called them PALFs. Here "allowable"

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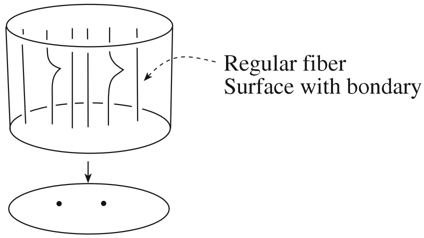
Fig. 10 PALF



means that the monodromies of Lefschetz singularities over the singular points are products of positive Dehn twists along nonseparating loops (this last condition is not a restriction; it comes for free from the proofs).

So PALFs are certain topological structures underlying a Stein manifold: Stein $=$ $|\text{PALF}|$. Existence of these structure on Stein manifolds was first proven in [LP]; later, in [AO1], a constructive topological proof along with its converse was given, thereby establishing the following result.

Theorem 3. *There is a surjection*

$$\{\text{PALFs}\} \Longrightarrow \{\text{Stein manifolds}\}.$$

Here, by Eliashberg's characterization [G], a *Stein manifold* means a handlebody consisting of 1- and 2-handles, where the 2-handles are attached along a Legendrian framed link, with each of its components K framed with $\text{tb}(K) - 1$ framing.

The proof that a PALF $\mathcal{F}$ gives a Stein manifold $|\mathcal{F}|$ goes as follows: By [Ka], $\mathcal{F}$ is obtained by starting with the trivial fibration $X_0 = F \times B^2 \to B^2$ and attaching a sequence of 2-handles to the curves $k_i \subset F$, $i = 1, 2, \dots$, on the fibers, with framing one less than the page framing: $X_0 \rightsquigarrow X_1 \rightsquigarrow \cdots \rightsquigarrow X_n = \mathcal{F}$. On an $F \times B^2$, we start with the standard Stein structure and assume that to the contact boundary F there is a convex surface with the "dividing set" ∂F [T]. Then by applying the "Legendrian realization principle" of [H], after an isotopy we make the surface framings of $k_i \subset F$ into the Thurston–Bennequin framings, and then the result follows from Eliashberg's theorem (Fig. 10).

Conversely, to show that a Stein manifold W admits a PALF (here we indicate the proof only for the case in which there are no 1-handles), we isotope the Legendrian framed link to square bridge position (by turning each component counterclockwise $45°$), and put the framed link on a fiber F of the (p,q) torus knot L as indicated in Fig. 11. This gives a PALF structure on $B^4 = |\mathcal{F}|$. Attaching handles to this framed link has the effect of enhancing the monodromy of the (p,q) torus knot by the Dehn twist along them, resulting in a bigger PALF. An improved version of this theorem is given in [Ar].

A PALF structure $\mathcal{F}$, like a triangulation or handlebody structure on a smooth manifold, should be viewed as an auxiliary topological structure on a Stein manifold $X = |\mathcal{F}|$. On the boundary, a PALF gives an open book compatible with the induced contact manifold $\partial X = |\partial \mathcal{F}|$. Usually, geometric structures come as primitives of

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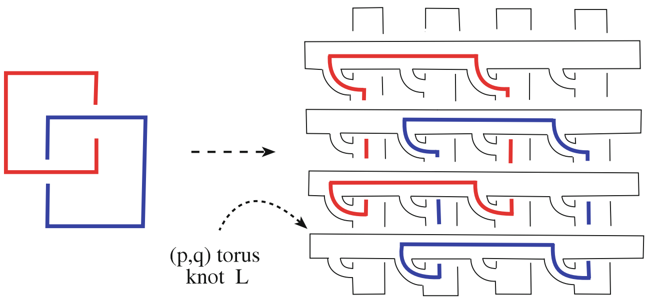

Fig. 11 Surgery on a framed link induces Dehn twists on the page

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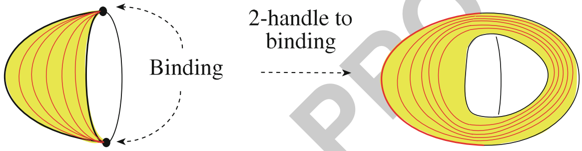
Fig. 12 Adding a 2-handle to a binding



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topological structures: topology $= |\text{geometry}|$, such as the real algebraic structures or complex structures on a smooth manifold; but surprisingly, in this case the roles are reversed: geometry $= |\text{topology}|$. For example, B^4 has a unique Stein structure, whereas it has infinitely many PALF structures corresponding to fibered links.

Choosing an underlying PALF is often useful for solving problems in Stein manifolds. A striking application of this principle was the approach in [AO2] to the compactification problem of Stein manifolds, which was later strengthened by [E] and [Et]. The problem of compactifying a Stein manifold W into a closed symplectic manifold was first solved in [LM]. Then in [AO2], an algorithmic solution was given using PALFs. An analogous case involves compactifying the interior of a compact smooth manifold to a closed manifold by first choosing a handlebody on W, then canonically closing it up by attaching dual handles (doubling).

In the symplectic case, we first choose a PALF on $W = |\mathcal{F}|$, then attach a 2-handle to the binding of the open book on the boundary (Fig. 12), thereby obtaining a closed surface F^*-bundle over the 2-disk with monodromy a product of positive Dehn twists $\alpha_1 \cdot \alpha_2 \cdots \alpha_k$. We then extend this fibration $\mathcal{F}$ by doubling monodromies $\alpha_1 \cdot \alpha_2 \cdots \alpha_k \cdots \alpha_k^{-1} \cdots \alpha_1^{-1}$ (i.e., attaching corresponding 2-handles) and capping off with $F^* \times B^2$ on the other side. We do this after converting each negative Dehn twist α_i^{-1} in this expression into products of positive Dehn twists using the relation $(a_1 b_1 \cdots a_g b_g)^{4g+2} = 1$ among the standard Dehn twist generators of the surface F^* of genus g (cf. [AO2]).

Choosing a PALF on W makes this an algorithmic canonical process. Even in the case of the Stein ball B^4, using different PALF structures on B^4 we get a variety

Author's Proof

of different symplectic compactifications of B^4. For example, $B^4 = |\mathcal{U}| \rightsquigarrow S^2 \times S^2$, where $\mathcal{U}$ is the trivial PALF $D^2 \times D^2 \to D^2$, whereas $B^4 = |\mathcal{T}| \rightsquigarrow K3$ Surface, where $\mathcal{T}$ is the PALF induced by the trefoil knot.

3 BLFs

Lack of a uniqueness result in the "cork decompositions" and the differing orientations of the two Stein pieces obtained from Theorem 1 make it hard to define four-manifold invariants. In [ADK], a more general version of the Lefschetz fibration (or pencil) structure on four-manifolds is introduced, namely the "broken Lefschetz fibration" BLF (or broken Lefschetz pencil BLP), where Lefschetz fibrations or pencils $\pi : X^4 \to S^2$ (with closed surfaces as regular fibers) are allowed to have circle singularities; that is, on a neighborhood of some circles, π can look like a map $S^1 \times B^3 \to \mathbb{R}^2$ given by $(t, x_1, x_2, x_3) \mapsto (t, x_1^2 + x_2^2 - x_3^2)$ (otherwise, it is a Lefschetz fibration). In [ADK], using analytic techniques, it was shown that every four-manifold X with $b_+^2 > 0$ is a BLP. Also, after a useful partial result in [GK], in [L] and [AK] two independent proofs that all four-manifolds are BLFs were given; the first proof uses singularity theory, and the second uses handlebody theory (in [B2], another singularity approach is employed, resulting in a weaker version of [L]). In [P] there is an approach using BLFs to construct four-manifold invariants (Fig. 13).

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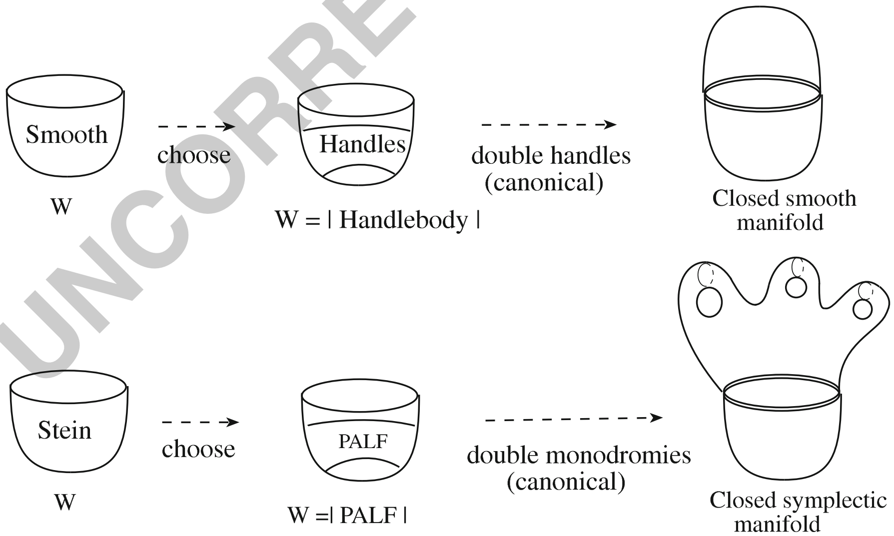

Fig. 13 Natural compactification after choosing an auxiliary structure

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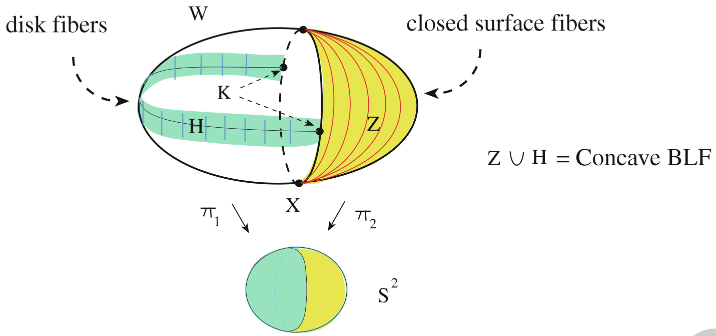

Fig. 14

The proof in [AK] proceeds along the lines of Theorems 1 and 3, discussed earlier. Roughly, it goes as follows: First define ALFs, which are weaker versions of PALFs. They are "achiral Lefschetz fibrations" over the 2-disk with bounded fibers, where we allow Lefschetz singularities to have monodromies that are negative Dehn twists. From the proof in [AO1], we get the surjection

$$\{\text{ALFs}\} \Longrightarrow \{\text{almost Stein manifolds}\}\,.$$

Here an *almost Stein manifold* means a handlebody consisting of 1- and 2-handles, where the 2-handles are attached to a Legendrian framed link, with each component K framed with $\text{tb}(K) \pm 1$ framing (it turns out that every 4-dimensional handlebody consisting of 1- and 2-handles has this nice structure).

First, we make a tubular neighborhood X_2 of any embedded surface in X a "concave BLF" (e.g., [GK]). A *concave BLF* means a BLF with 2-handles attached to circles transversal to the pages on the boundary (open book), as indicated in Fig. 14 (so a concave BLF fibers over the whole S^2, with closed-surface regular fibers on one hemisphere, and 2-disk regular fibers on the other hemisphere). Also we make sure that the complement $X_1 = X - X_2$ has only 1- and 2-handles; hence it is an ALF. Applying [Gi], we ensure that the boundary open books induced from each side X_i, $i = 1, 2$, match. So we have a (matching) union $X = X_1 \cup X_2$ consisting of an ALF X_1 and concave BLF X_2. This would have made X a BLF had X_1 been a PALF.

Now comes the crucial point. In analogy to obtaining a butterfly by drilling into its cocoon, we will turn the ALF X_1 into a PALF by removing a disk from it, i.e., we will apply the positron move of Theorem 1: For each framed knot representing a 2-handle of X_1, we pick an unknot (K in Fig. 15) with the properties that (1) it lies on a page and (2) it links that framed knot twice, as in Fig. 15. These conditions allow us to isotope K to the boundary of a properly embedded 2-disk in X_1 meeting each fiber of the ALF once (here K is isotoped to the meridian of the binding curve). Therefore, carving out the tubular neighborhood of this disk from X_1 preserves the ALF structure (this is indicated by putting a dot in K in the figure, a notation

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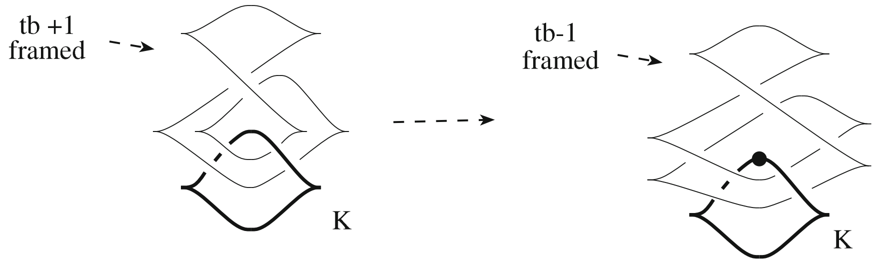

Fig. 15 Turning an ALF into a PALF by carving

from [A5]), where the new fibers are obtained by puncturing the old ones. But this magically changes the framings of each framed knot by $\mathrm{tb}(K)+1 \rightsquigarrow \mathrm{tb}(K)-1$; hence after carving out X_1, we end up with a PALF. On the other, X_2, side, this carving corresponds to attaching 2-handles to X_2 to circles transverse to fibers, so the enlarged X_2 is still a concave BLF, and so the open books of each side still match.

4 Plugs

To understand exotic structures of four-manifolds better, recently in [AY1], Yasui and I started to search for corks in the known examples of exotic four-manifolds, since we knew theoretically that they exist [AY1, Sect. 0]. From this endeavor we learned two important lessons: First, as in the case of corks, there are differently behaving codimension-zero submanifolds that are also responsible for the exoticness of four-manifolds (we named them *plugs*). Second, the position of corks and plugs in four-manifolds plays an important role. For example, it helps us to construct exotic Stein manifolds in Theorem 4, whose existence had eluded us for a long time. In some sense, plugs generalize the *Gluck twisting* operation, just as corks generalize the Mazur manifold. The rest of this section is a brief summary of [AY1, AY2].

Definition 2. A plug is a pair (W, f), where W is a compact Stein manifold and $f : \partial W \to \partial W$ is an involution that does not extend to a self-homeomorphism of W and such that there is the decomposition (1) for some exotic copy M' of M.

Plugs might be deformations of corks (to deform corks into each other, we might have to go through plugs). We can think of corks and plugs as freely moving particles in four-manifolds (Fig. 16), like fermions and bosons in physics, functioning like little knobs on a wall to turn on and off the ambient exotic lights.

An example of a plug that frequently appears in four-manifolds is $W_{m,n}$, where $m \geq 1, n \geq 2$ (Fig. 17). By canceling the 1-handle and the $-m$-framed 2-handle, we see that $W_{m,n}$ is obtained from B^4 by attaching a 2-handle to a knot with $-2n - n^2m^2$ framing. The involution f is induced from the symmetric link.

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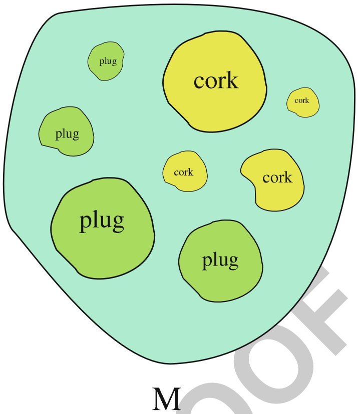

Fig. 16 Zoo of corks and plugs in a four-manifold

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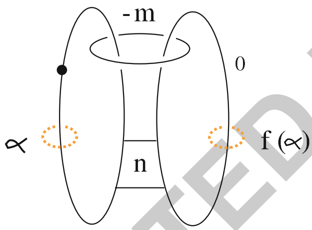

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Fig. 17 $W_{m,n}$

Here the degenerate case is also interesting. Removing $W_{1,0}$ and gluing with f corresponds to the Gluck twisting operation. Previously, we knew only one example of an exotic manifold that is obtained from the standard one by the Gluck operation, and that manifold is nonorientable [A4].

Observe that if this obvious involution $f : \partial W_{m,n} \to \partial W_{m,n}$ extended to a homeomorphism, we would get homeomorphic manifolds $W^1_{m,n}$ and $W^2_{m,n}$, obtained by attaching 2-handles to α and $f(\alpha)$ with -1 framings, respectively. But $W^2_{m,n}$ and $W^1_{m,n}$ have the following nonisomorphic intersection forms, which is a contradiction:

$$\begin{pmatrix} -2n - mn^2 & 1 \\ 1 & -1 \end{pmatrix}, \quad \begin{pmatrix} -2n - mn^2 & -1 - mn \\ -1 - mn & -1 - m \end{pmatrix}.$$

The following theorem implies that $(W_{m,n}, f)$ is a plug, and also it says that this plug can be inflated to exotic Stein manifold pairs.

Theorem 4. *([AY2]) The simply connected Stein manifolds shown in Fig. 18 are exotic copies of each other.*

Notice that the transformation $Q_1 \rightsquigarrow Q_2$ is obtained by twisting along the plug $(W_{1,3}, f)$ inside. It is easy to check that both are Stein manifolds, and clearly the

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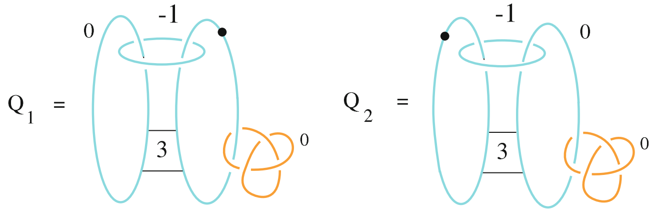

Fig. 18 Inflating a plug to an exotic Stein manifold pair

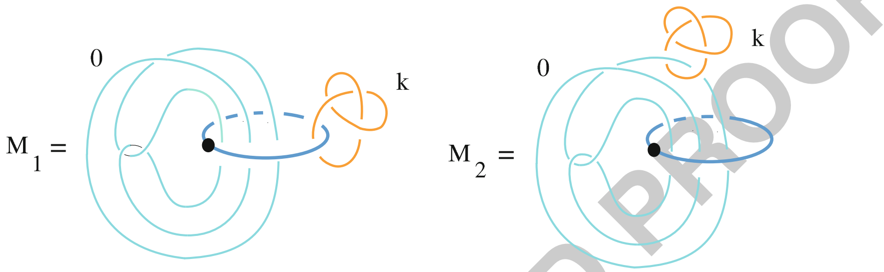

Fig. 19 Inflating a cork to an exotic Stein manifold pair ($k \leq 0$)

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boundaries of Q_1 and Q_2 are diffeomorphic, and the boundary diffeomorphism extends to a homotopy equivalence inside, since they have isomorphic intersection forms $(-1) \oplus (1)$; cf. [Bo]. Hence by Freedman's theorem they are homeomorphic. The fact that they are not diffeomorphic follows from an interesting embedding $Q_1 \subset E(2)\#2\overline{\mathbb{CP}}^2$ (so the positions of plugs are important!), where the two homology generators $\langle x_1, x_2 \rangle$ of $H_2(Q_1)$ intersect the basic class $K = \pm e_1 \pm e_2$ of $E(2)\#2\overline{\mathbb{CP}}^2$ with $x_i \cdot e_j = \delta_{ij}$, (here e_j are the two $\overline{\mathbb{CP}}^1$ factors). Now by applying the adjunction inequality we see that there is no embedded torus of self-intersection zero in Q_1, whereas there is one in Q_2.

We can also inflate corks to exotic Stein manifold pairs (compare these examples to the construction in Sect. 1.2).

Theorem 5. *([AY2]) The simply connected Stein manifolds shown in Fig. 19 are exotic copies of each other.*

The proof of this is similar to the previous theorem. The crucial point is finding an embedding of M_1 into a useful closed four-manifold with nontrivial Seiberg–Witten invariant. Note that by a result of Eliashberg, the only Stein filling of S^3 is B^4. The existence of simply connected exotic Stein manifold pairs was established recently in [AEMS] using the technique of knot surgery. Now a natural question is this: Are all exotic structures on four-manifolds induced from the corks W_n, $\bar{W}_n$ and plugs $W_{m,n}$? Here is a result of some recent searches.

Author's Proof

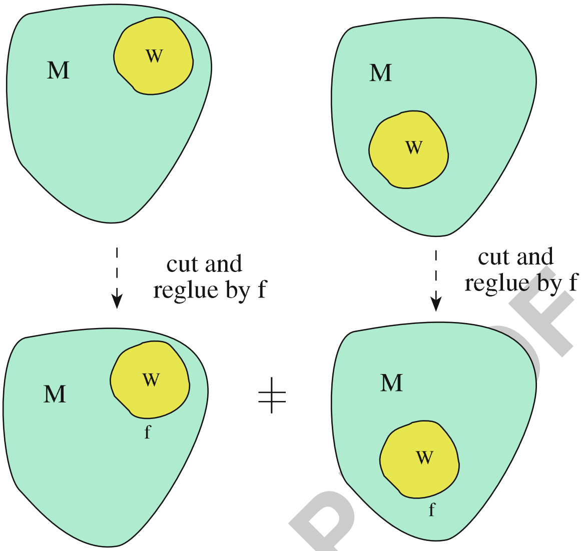

Fig. 20 Nonisotopic corks

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Theorem 6. *([AY1]) For $k,r \geq 1$, $n,p,q \geq 2$, and $\gcd(p,q) = 1$, we have the following:*

- *$E(2k)\#\bar{\mathbf{CP}}^2$ has corks (W_{2k-1}, f_{2k-1}) and (W_{2k}, f_{2k});*
- *$E(2k)\#r\bar{\mathbf{CP}}^2$ has plugs $(W_{r,2k}, f_{r,2k})$ and $(W_{r,2k+1}, f_{r,2k+1})$;*
- *$E(n)_{p,q}\#\bar{\mathbf{CP}}^2$ has cork W_1, and plug $W_{1,3}$;*
- *$E(n)_K\#\bar{\mathbf{CP}}^2$ has cork W_1, and plug $W_{1,3}$;*
- *Yasui's exotic $E(1)\#\bar{\mathbf{CP}}^2$ in [Y] has cork W_1.*

An interesting question is whether any two cork embeddings $(W,f) \subset M$ are isotopic to each other. Put another way, can you knot corks inside of four-manifolds? It turns out that there are indeed such knotted corks [AY1]. For example, there are two nonisotopic cork embeddings $(W_4, f) \subset M = \mathbb{CP}^2\#14\bar{\mathbb{CP}}^2$. It is also possible to knot some corks in infinitely many different ways [AY3]. This is proved by calculating the change in the Seiberg–Witten invariants of the two manifolds obtained by twisting M along the two embedded corks and getting different values. This calculation uses the techniques of [Y] (Fig. 20).

Another natural question arises: Since every exotic copy of a closed 4-manifold can be explained by a cork twisting, and there are many ways of constructing exotic copies of four-manifolds, e.g., logarithmic transform, rational blowing down, knot surgery operations [FS1, FS2, GS], are there ways of linking all these constructions to corks? In some cases this can be done for the rational blowing-down operation $X \rightsquigarrow X_{(p)}$, by showing that $X_{(p)}\#(p-1)\bar{\mathbf{CP}}^2$ is obtained from X by a cork twisting along some $W_n \subset X$ [AY1]. It is already known that there is a similar relation between logarithmic transforms and the rational blowings down. The difficult remaining case seems to be the problem of relating a general knot surgery operation to cork twisting.

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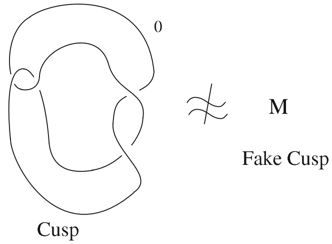
Fig. 21



Remark 1. We don't know whether an exotic copy of a manifold with boundary differs from its standard copy by a cork. It is likely that there is a relative version of the cork theorem. Perhaps the most interesting example to check is the exotic cusp of [A6], which is the smallest example of a simply connected exotic smooth manifold we know that requires 1- or 3-handles in any handlebody decomposition, whereas its standard copy has only 2-handles [AY1] (Fig. 21).

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