FANO MANIFOLDS WITH BIG AND NEF TANGENT BUNDLES

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Contents

1. Introduction

An interesting open problem in uniformization theory, first formulated and studied in [CP91], asks for the structure of complex projective (or compact Kähler) manifolds X whose tangent bundles T_X are nef. This is to say - in case X is projective - that given any curve irreducible compact curve C and any quotient

$$T_X|_C \to Q \to 0,$$

the determinant of Q is non-negative: $c_1(Q) \geq 0$.
The notion of a nef tangent bundle includes the case that X carries a metric with non-negative holomorphic bisectional curvature, but is more general. Two very prominent cases have been treated in 1979 and 1988:

- Mori [Mo79] proved that the only compact Kähler manifold with *ample* tangent bundle is projective space;
- Mok [Mo88] showed that a compact Kähler manifold admitting a *Kähler* metric with non-negative holomorphic bisectional curvature, is hermitian-symmetric.

In [DPS94] the study of Kähler manifolds with nef tangent bundles was reduced to the case of Fano manifolds X. Namely, if X is a compact Kähler manifold with T_X nef, then - possibly after a finite étale cover - the Albanese map $\alpha : X \to A$ is a surjective submersion (which is flat in a certain sense), whose fibers are Fano manifolds with nef tangent bundles.

The main conjecture in [CP91] predicts that a Fano manifold X with nef tangent bundle is a rational homogeneous manifold, i.e. $X = G/P$ with G a semi-simple complex Lie group and P a parabolic subgroup. Notice that in order to prove that a Fano manifold X is rational homogeneous, it suffices to show that X is homogeneous, i.e., T_X is spanned by global sections.

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Only a few special results on the general problem were known so far, see [CP91], [DPS94], [Mo02], [SW04], [MOSWW14]. In this paper we confirm the above conjecture in case that the tangent bundle is "big".

1.1. Theorem. *Let X be a Fano manifold of dimension n with nef tangent bundle. Suppose that the top Segre class does not vanish: $s_n(X) \neq 0$. Then X is rational homogeneous.*

The key to the proof lies in considering the projectivized tangent bundle $\mathbb{P}(T_X)$. The "tautological bundle" $\mathcal{O}(1)$ on $\mathbb{P}(T_X)$ is nef (essentially by definition) and the anticanonical bundle of $\mathbb{P}(T_X)$ is a multiple of $\mathcal{O}(1)$:

$$-K_{\mathbb{P}(T_X)} = \mathcal{O}(n),$$

where $n = \dim X$. Since T_X is nef, the top Segre class of T_X is non-negative: $s_n(X) \geq 0$ by [DPS94]. Equivalently, $c_1(\mathcal{O}(1))^{2n-1} \geq 0$. Thus, if $s_n(X) \neq 0$, then $s_n(X) > 0$, i.e. $\mathcal{O}(1)$ is big. We will show that under this bigness assumption, T_X will be spanned. The main idea here is to consider $T_X \oplus \mathcal{O}_X$ instead of T_X, combined with the trivial observation that the spannedness of some symmetric power $S^m(T_X \oplus \mathcal{O}_X)$ implies spannedness of T_X.

Notice that rational-homogeneous manifolds have indeed positive top Segre class; see e.g. [St76]; we will provide a short proof in sect. 4. Thus, in order to complete the proof of the main conjecture, it remains to show

1.2. Conjecture. *Let X be a Fano manifold fo dimension n. If T_X is nef, then $s_n(X) \neq 0$.*

2. Basic Notions

Recall that a vector bundle E over a projective manifold is nef if the "hyperplane bundle"

$$\mathcal{O}_{\mathbb{P}(E)}(1)$$

is nef. Here we take the projectivization in Grothendieck's sense (using hyperplanes).

Equivalently E is nef if and only the following holds. Given an irreducible curve C with normalization $\eta : \tilde{C} \to C$ and an epimorphism $\eta^*(E) \to Q \to 0$, the determinant $\det Q$ has non-negative degree.

The notion of a nef vector bundle can be defined on any compact complex manifold using the above definition; it suffices to say that a line bundle L on a compact manifold Z is nef, if $c_1(L)$ can be represented by a positive closed current on Z. For details and properties of nef bundles we refer to [DPS94]. In particular, it is shown in [DPS94] that all Segre classes $s_i(E)$ of a nef bundle are non-negative, in particular $s_n(E)$ is a non-negative integer, where $n = \dim X$.

A vector bundle E which is nef as well as its dual E^* is called numerically trivial. By [DPS94], E is numerically trivial if and only E is nef and $\det E^*$ is nef. All Chern classes of a numerically trivial bundle E vanish and E has a filtration by unitary flat bundles.

3. Spannedness of the tangent bundle

In this section we show that a Fano manifold with nef tangent bundle whose top Segre class does not vanish, must be rational-homogeneous.

3.1. Theorem. *Let X be a Fano manifold of dimension n. Suppose that its tangent bundle T_X is nef and that the top Segre class $s_n(X) \neq 0$, i.e. the anticanonical bundle of $\mathbb{P}(T_X)$ is big. Then T_X is spanned, hence X is rational-homogeneous.*

Proof. First observe that

$$s_n(T_X) = s_n(T_X \oplus \mathcal{O}_X),$$

(simply because s_n is a function of the Chern classes $c_1, \dots, c_n$). Therefore also the anticanonical bundle of $\mathbb{P} := \mathbb{P}(T_X \oplus \mathcal{O}_X)$ is big (and nef of course). Let

$$L = \mathcal{O}_{\mathbb{P}}(1),$$

so that $-K_{\mathbb{P}} = (n+1)L$. Then, by the base point freeness theorem, any large multiple mL is spanned. Let $f : \mathbb{P} \to Z$ be the associated morphism; hence $L = f^*(L')$ and Z is Gorenstein with at most canonical singularities.

In a first step, let $C \subset X$ be a smooth curve.We claim that

$$S^m(T_X|C \oplus \mathcal{O}_C) \text{ is spanned} \tag{3.1.1}$$

for $m \geq m_0(C)$. Once we know this, $T_X|C$ itself is spanned as a direct summand of $S^m(T_X|C \oplus \mathcal{O}_C)$.

In order to prove Claim 3.1.1, set

$$\mathbb{P}_C = \mathbb{P}(T_X|C \oplus \mathcal{O}_C),$$

and $L_C = L|\mathbb{P}_C$. Let $g : \mathbb{P}_C \to Z_C$ be the morphism (with connected fibers) associated to $|mL_C|$. Notice that there is a map $Z_C \to f(\mathbb{P}_C)$, therefore we can write $L_C = g^*(L'_C)$ with some ample line bundle L'_C on Z_C. The projection $\mathbb{P}_C \to C$ is again denoted by π; let $F = \pi^{-1}(x)$ be a fiber of π. We need to prove that there exists a number m_0 (a priori depending on F), such that

$$H^0(\mathbb{P}_C, mL_C) \to H^0(F, mL_C|F) \simeq H^0(\mathbb{P}_n, \mathcal{O}(m)) \tag{3.1.2}$$

is surjective for $m \geq m_0$. This shows that $S^m(T_X|C \oplus \mathcal{O}_C)$ is spanned at x for $m \geq m_0(x)$. Then spannedness will be true also in an open neighborhood of x in C, and therefore a compactness argument shows that $S^m(T_X|C \oplus \mathcal{O}_C)$ is spanned everywhere for $m \gg 0$ proving Claim 3.1.1.

Claim 3.1.2 is equivalent to proving the injectivity of the natural map

$$H^1(\mathbb{P}_C, \mathcal{I}_F \otimes mL_C) \to H^1(\mathbb{P}_C, mL_C).$$

Now

$$H^q(Z_C, g_*(\mathcal{I}_F \otimes mL_C)) = H^q(Z_C, g_*(\mathcal{I}_F) \otimes mL'_C) = 0$$

and

$$H^q(Z_C, g_*(mL_C)) = H^q(Z_C, mL'_C) = 0$$

for $m \gg 0$ and $q \geq 1$, since L'_C is ample. Therefore by the Leray spectral sequence

$$H^1(\mathbb{P}_C, \mathcal{I}_F \otimes mL_C) = E^{\infty}_{0,1} = E^2_{0,1} =$$

$$= H^0(Z_C, R^1 g_*(\mathcal{I}_F \otimes mL_C)) = H^0(Z_C, R^1 g_*(\mathcal{I}_F) \otimes mL'_C)$$

and similarly

$$H^1(Z_C, mL_C) = H^0(Z_C, R^1 g_*(\mathcal{O}_{\mathbb{P}_C}) \otimes mL'_C).$$

Hence we are reduced to verify the injectivity of the map

$$\alpha : H^0(Z_C, R^1 g_*(\mathcal{I}_F) \otimes mL'_C) \to H^0(Z_C, R^1 g_*(\mathcal{O}_{\mathbb{P}_C}) \otimes mL'_C).$$

Consider the exact sequence

$$0 \to \mathcal{I}_F \to \mathcal{O}_{\mathbb{P}_C} \to \mathcal{O}_F \to 0$$

and apply g_*. Now $g_*(\mathcal{I}_F) = \mathcal{I}_{g(F)}$ and moreover $g_*(\mathcal{O}_F) = \mathcal{O}_{g(F)}$ by Lemma 3.2, observing that $g|F : F \to g(F)$ is biholomorphic. Hence the canonical map

$$R^1 g_*(\mathcal{I}_F) \to R^1 g_*(\mathcal{O}_{\mathbb{P}_C})$$

is injective, and so does the map α, establishing Claim 3.1.1.

Having proved Claim 3.1.1, we know that $T_X|C$ is spanned for all smooth curves C. In particular, this is true for C a smooth complete intersection curve

$$C = D_1 \cap \ldots \cap D_{n-1},$$

where $D_i \in |m_i H_i|$ with H_i ample and $m_i \gg 0$. Since through every point $x \in X$, there are plenty of these curves, it suffices to lift sections from C to X. But this is guaranteed by the vanishing

$$(*) \qquad H^1(X, \mathcal{I}_C \otimes T_X) = 0,$$

having in mind that $m_i \gg 0$. In fact, let

$$\mathcal{E} = \bigoplus_{i=1}^{n-1} \mathcal{O}(m_i H_i)$$

and consider the Koszul complex

$$0 \to \bigwedge^{n-1} \mathcal{E}^* \to \ldots \to \mathcal{E}^* \to \mathcal{I}_C \to 0.$$

Choosing N so large that

$$H^q(X, \mathcal{O}(-NH_i) \otimes T_X) = 0$$

for all i and $1 \leq q \leq n-1$, and furthermore choosing all $m_i \geq N$, we obtain the vanishing (*).

□

3.2. Lemma. *Let C be a smooth compact curve and $\mathcal{E}$ a vector bundle over C. Let $L = \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ and assume that mL is spanned for some positive m. Let $\phi : \mathbb{P}(\mathcal{E}) \to W$ be the associated morphism with connected fibers. Assume furthermore that L is not ample and that $L = \phi^*(L')$ with some ample line bundle L'. Then for any fiber F of π, the restriction $\phi|F$ is biholomorphic (and all fibers of ϕ are sections of π).*

Proof. Let r be the rank of E. Observe that the bundle $\mathcal{E}$ is nef and consider the maximal ample subbundle $\mathcal{F} \subset \mathcal{E}$, [PW00, 2.3]. Then the quotient bundle

$$Q = \mathcal{E}/\mathcal{F}$$

is nef with $c_1(Q) = 0$, hence numerically flat in the sense of [DPS94].
If $Q = \mathcal{E}$ i.e., $\mathcal{F} = 0$, then $\mathcal{E}$ itself is numerically flat. Since $c_1(L)^r = c_1(\mathcal{E}) = 0$, the line bundle L is not big and ϕ is a fibration with $\dim W = r - 1$. Since the fibers F of the projection $\mathbb{P}(\mathcal{E}) \to C$ dominate W and since $L = \phi^*(L')$, it follows that $\phi|F$ is an isomorphism.

If $\mathcal{F} \neq 0$, then $c_1(L)^r = c_1(\mathcal{E}) > 0$, so L is big and ϕ is birational. The exceptional locus is exactly $\mathbb{P}(Q)$; let $W' = \phi(\mathbb{P}(Q)$. Then W' is normal, since $\phi|\mathbb{P}(Q)$ has connected fibers. We now simply apply the previous arguments to the numerically flat bundle Q to conclude that $W' = \mathbb{P}_{s-1}$ with s the rank of Q and that $\phi|F'$ is biholomorphic for any fiber F' of the projection $\mathbb{P}(Q) \to C$ (notice simply that $L|\mathbb{P}(Q) = \mathbb{P}_{\mathbb{P}(Q)}(1)$). Since $F' = F \cap \mathbb{P}(Q)$, we conclude. □

3.3. Remark. One might wonder whether the reduction to curves is really necessary. A direct argument could be as follows using the notations of the proof of Theorem 3.1. In order to show that

$$H^0(\mathbb{P}, mL) \to H^0(F, mL|F)$$

is surjective, we need to show that

$$H^1(Z, R^1f_*(\mathcal{I}_F) \otimes mL') \to H^1(Z, R^1f' * (\mathcal{O}_{\mathbb{P}}) \otimes mL')$$

is injective and therefore that

$$R^1f_*(\mathcal{I}_F) \to R^1f_*(\mathcal{O}_{\mathbb{P}})$$

is injective. This requires to know that $f|F$ is biholomorphic, which it is not at all clear a priori ($f|F$ is definitely finite and birational, but could be a normalization map).

We generalize Theorem 3.1 as follows.

3.4. Theorem. *Let X be a projective (or compact Kähler) manifold. Assume that $\mathcal{O}_{\mathbb{P}(T_X)}(1)$ is semi-ample (so in particular T_X is nef). Then T_X is spanned, i.e. X is homogeneous.*

Proof. We use the notations of the proof of Theorem 3.1; so $\mathbb{P} = \mathbb{P}(T_X \oplus \mathcal{O}_X)$ and $L = \mathcal{O}_{\mathbb{P}}(1)$. Again we are going to show that $S^m(T_X \oplus \mathcal{O}_X)$ is spanned for some m. We only need to show that L is semi-ample; then the arguments of the proof of Theorem 3.1 work. The spannedness will follow from

$$\kappa(L) = \nu(L) \tag{3.4.1}$$

by [Ka85], resp. [Na87], [Fn08] in the Kähler case. Now the numerical dimension $\nu(L)$ is computed by

$$\nu(L) = n + k,$$

where k is the largest number such that the Segre class

$$s_k(T_X \oplus \mathcal{O}_X) \neq 0.$$

Since

$$s_k(T_X \oplus \mathcal{O}_X) = s_k(T_X) = s_k(X)$$

and since

$$\nu(\mathcal{O}_{\mathbb{P}(T_X)}(1)) = n - 1 + k,$$

we conclude

$$\nu(L) = \nu(\mathcal{O}_{\mathbb{P}(T_X)}(1)) + 1.$$

In order to compute $\kappa(L)$ we use the decompostion

$$H^0(S^m(T_X \oplus \mathcal{O}_X)) = H^0(S^mT_X) \oplus H^0(S^{m-1}(T_X)) \oplus \ldots \oplus H^0(T_X) \oplus H^0(\mathcal{O}_X).$$

Since

$$H^0(mL) = H^0(S^m(T_X \oplus \mathcal{O}_X)),$$

it is already clear that $\kappa(L) \geq \kappa(\mathcal{O}_{\mathbb{P}(T_X)}(1))$. Choose m_0 such that $S^{m_0}(T_X)$ is spanned. Then

$$h^0(S^{km_0}(T_X \oplus \mathcal{O}_X)) \geq \sum_{j=0}^{k} h^0(S^{jm_0}T_X) \sim \sum_{j=0}^{k} (jm_0)^l \sim k^{l+1}m_0^l$$

asympotically, where $l = \nu(\mathcal{O}_{\mathbb{P}(T_X)}(1))$. Hence $\kappa(L) \geq l+1$, so $\kappa(L) = l+1$ and Equation 3.4.1 holds.

□

3.5. Corollary. *Let X be a projective or compact Kähler manifold. If there exists some positive integer m such that S^mT_X is spanned, then X is homogeneous.*

4. The top Segre of a rational-homogeneous manifold

Here we give a simple non-group theoretic proof of the following (classical, but not so well documented)

4.1. Theorem. *Let X be a rational-homogeneous manifold of dimension n. Then the top Segre class $s_n(X) \neq 0$.*

Of course, Theorem 4.1 follows again from Theorem 5.1.

Proof. As in the proof of Theorem 3.1, we consider

$$\mathbb{P} := \mathbb{P}(T_X \oplus \mathcal{O}_X)$$

with projection $\pi : \mathbb{P} \to X$ and need to show that the spanned line bundle

$$L = \mathcal{O}_{\mathbb{P}}(1)$$

is big. Let

$$\Sigma = \mathbb{P}(\mathcal{O}_X) \subset \mathbb{P}$$

and

$$D = \mathbb{P}(T_X) \subset \mathbb{P}.$$

Let

$$\psi : \mathbb{P} \to Z \subset \mathbb{P}_N$$

be the holomorphic map defined by $H^0(\mathbb{P}, L)$. We argue by contradiction and assume that L is not big so that

$$\dim Z < \dim \mathbb{P} = 2n-1.$$

We choose a basis $s_0, \ldots, s_N$ corresponding to a basis $t_0 \oplus 0, \ldots, t_{N-1} \oplus 0, 0 \oplus 1$, where the t_i form a basis of $H^0(X, T_X)$. Notice that $L_D := L|D = \mathcal{O}_{\mathbb{P}(T_X)}(1)$ and that $\mathcal{O}_{\mathbb{P}(D)} = L$. Thus we have an exact sequence

$$0 \to \mathcal{O}_X \to L \to L_D \to 0$$

yielding a sequence in cohomology

$$0 \to \mathbb{C} \to H^0(\mathbb{P}, L) \to H^0(D, L_D) \to H^1(\mathbb{P}, \mathcal{O}_{\mathbb{P}}) = 0.$$

The section s_N is constant and non-vanishing along Σ, whereas $s_j|\Sigma = 0$ for $0 \leq j \leq N-1$. Thus ψ maps Σ to the point $z_0 = [0 : \ldots : 1]$ and $\psi^{-1}(z_0) = \Sigma$, since Σ is exactly the common vanishing locus of the $s_j, 0 \leq j \leq N_1$. Notice also that $\psi(D) = Z \cap H$ with a hyperplane $H \subset \mathbb{P}_N$.
Now consider a general fiber F of ψ (resp. a connected component). Since $\psi|\pi^{-1}(x)$

is an isomorphism for all x, we conclude that $d := \dim F \leq n$. By adjunction we have

$$K_F = K_{\mathbb{P}}|F = \mathcal{O}_F,$$

hence $\dim H^d(F, \mathcal{O}_F) = 1$, and therefore

$$R^d\psi_*(\mathcal{O}_{\mathbb{P}})$$

has rank 1 generically. If $\hat{\Sigma}$ denotes the formal completion of $\mathbb{P}$ along Σ, the comparison theorem of Grauert implies that

$$H^d(\hat{\Sigma}, \mathcal{O}_{\hat{\Sigma}}) \neq 0.$$

Therefore $H^d(\Sigma, S^k N^*_{\Sigma/X}) \neq 0$ for some $k \geq 0$. Since $N_{\Sigma/X} \simeq T_X$, this contradicts the vanishing

$$H^d(X, S^k T_X) = 0$$

for $k \geq 0$ on a rational-homogeneous manifold. □

5. The Albanese map

If X is a compact Kähler manifold with nef tangent bundle T_X, then the Albanese map is a surjective submersion, as already mentioned. To be more precise, let $\tilde{q}(X)$ be the maximum of all irregularities $q(\tilde{X})$, where $\tilde{X} \to X$ is any finite étale cover. Since $q(X) \leq \dim X$, the generalized irregularity $\tilde{q}(X)$ is also bounded by $\dim X$ and we can always pass to a finite étale cover of X to achieve $q(X) = \tilde{q}(X)$. Then [DPS94] in combination with the main theorem of this paper proves the following

5.1. Theorem. *Let X be a compact Kähler manifold with T_X nef. Suppose that $q(X) = \tilde{q}(X)$. Then the Albanese map $\alpha : X \to A$ is a surjective submersion. The bundles $\alpha_*(-mK_X)$ are numerically flat. If $s_{n-q}(T_F) \neq 0$ for some fiber, then α is a fiber bundle with rational homogeneous fiber.*

The flatness of $\alpha_*(-mK_X)$ in the non-algebraic case is shown in [Cao12], Proof of 6.3; see also [Cao13, 4.4.1].

Here we prove the converse to Theorem 5.1.

5.2. Theorem. *Let A be a complex torus and $\alpha : X \to A$ a fiber bundle with rational homogeneous fiber. If $\alpha_*(-K_X)$ is a numerically flat bundle, then T_X is nef.*

Proof. Note that since $R^j\alpha_*(-K_X) = 0$ for $k \geq 1$, the sheaf

$$V = \alpha_*(-K_X)$$

is indeed locally free. Since $-K_X$ is relatively very ample, the fibers of α being rational homogeneous, we have an epimorphism

$$W := \alpha^*\alpha_*(-K_X) = \alpha^*(V) \to -K_X \to 0,$$

leading to an embedding

$$X \subset \mathbb{P}(W).$$

In order to show that T_X is nef, we prove equivalently that the relative tangent bundle $T_{X/A}$ is nef. Since the tangent bundles of the fibers of α are spanned, we have an epimorphism

$$\alpha^*\alpha_*(T_{X/A}) \to T_{X/A} \to 0,$$

consequently it suffices to show that

$$E = \alpha_*(T_{X/A})$$

is nef. Now notice that

$$\alpha_*(T_{\mathbb{P}(W)/A}) = (W^* \otimes W)/\mathcal{O},$$

hence $\alpha_*(T_{\mathbb{P}(W)/A})$ is nef, and actually numerically flat. Since $E = \alpha_*(T_{X/A})$ is a subbundle of $\alpha_*(T_{\mathbb{P}(W)/A})$, its dual E^* is nef. Consider the tangent bundle sequence

$$0 \to T_{X/A} \to T_{\mathbb{P}(W)/A} \to N_{X/\mathbb{P}(W)} \to 0,$$

where $N_{X/\mathbb{P}(W)} =: N$ denotes the normal bundle. We apply α_* to obtain

$$0 \to E \to \alpha_*(T_{\mathbb{P}(W)/A}) \to \alpha_*(N) \to 0,$$

having in mind that $H^1(F, T_F) = 0$ for the fibers F of α. We will show that $\det \alpha_*(N)$ is trivial, hence the last sequence shows that $\det E$ is trivial, too. Since E^* is nef, E will be numerically trivial, in particular nef, finishing the proof.
The normal bundle N is easily computed by

$$N = -K_X \otimes (W^*/K_X).$$

In order to control $\alpha_*(N)$, consider the exact sequence

$$0 \to K_X \to W^* \to W^*/K_X \to 0.$$

Tensorize by $-K_X$ to obtain

$$0 \to \mathcal{O}_X \to -K_X \otimes W^* \to N \to 0.$$

Applying α_* leads to

$$0 \to \mathcal{O}_A \to \alpha_*(-K_X \otimes W^*) \to \alpha_*(N) \to 0.$$

Since $\alpha^*(V^*) = \alpha^*(V)^*$, the sheaf V being locally free, we have

$$\alpha_*(-K_X \otimes W^*) = \alpha_*(-K_X \otimes \alpha^*(V^*)) = \alpha_*(-K_X) \otimes \alpha_*(-K_X)^* = V \otimes V^*,$$

hence

$$\det(\alpha_*(-K_X \otimes W^*)) = \mathcal{O}_A.$$

Thus $\det \alpha_*(N) = \mathcal{O}_A$, as claimed.

$\square$

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