EXTENSION THEOREMS, NON-VANISHING AND THE EXISTENCE OF GOOD MINIMAL MODELS

JEAN-PIERRE DEMAILLY, CHRISTOPHER D. HACON
AND MIHAI PĂUN

Abstract. We prove an extension theorem for effective plt pairs $(X, S+B)$ of non-negative Kodaira dimension $\kappa(K_X+S+B) \geq 0$. The main new ingredient is a refinement of the Ohsawa-Takegoshi L^2 extension theorem involving singular hermitian metrics.

1. Introduction

Let X be a complex projective variety with mild singularities. The aim of the minimal model program is to produce a birational map $X \dashrightarrow X'$ such that:

(1) If K_X is pseudo-effective, then X' is a good minimal model so that $K_{X'}$ is semiample; i.e. there is a morphism $X' \to Z$ and $K_{X'}$ is the pull-back of an ample $\mathbb{Q}$-divisor on Z.
(2) If K_X is not pseudo-effective, then there exists a Mori-Fano fiber space $X' \to Z$, in particular $-K_{X'}$ is relatively ample.
(3) The birational map $X \dashrightarrow X'$ is to be constructed out of a finite sequence of well understood "elementary" birational maps known as flips and divisorial contractions.

The existence of flips was recently established in [BCHM10] where it is also proved that if K_X is big then X has a good minimal model and if K_X is not pseudo-effective then there is a Mori-Fano fiber space. The focus of the minimal model program has therefore shifted to varieties (or more generally log pairs) such that K_X is pseudo-effective but not big.

Conjecture 1.1 (Good Minimal Models)**.** *Let (X, Δ) be an n-dimensional klt pair. If $K_X + \Delta$ is pseudo-effective then (X, Δ) has a good minimal model.*

Date: February 1, 2016.

The second author was partially supported by NSF research grant no: 0757897. During an important part of the preparation of this article, the third named author was visiting KIAS (Seoul); he wishes to express his gratitude for the support and excellent working conditions provided by this institute. We would like to thank F. Ambro, B. Berndtsson and C. Xu for interesting conversations about this article.

Note that in particular the existence of good minimal models for log pairs would imply the following conjecture (which is known in dimension ≤ 3 cf. [KMM94], [Kolláretal92]):

Conjecture 1.2 (Non-Vanishing)**.** *Let (X, Δ) be an n-dimensional klt pair. If $K_X + \Delta$ is pseudo-effective then $\kappa(K_X + \Delta) \geq 0$.*

It is expected that (1.1) and (1.2) also hold in the more general context of log canonical (or even semi-log canonical) pairs (X, Δ). Moreover, it is expected that the Non-Vanishing Conjecture implies existence of good minimal models. The general strategy for proving that (1.2) implies (1.1) is explained in [Fujino00]. One of the key steps is to extend pluri-log canonical divisors from a divisor to the ambient variety. The key ingredient is the following.

Conjecture 1.3 (DLT Extension)**.** *Let $(X, S+B)$ be an n-dimensional dlt pair such that $\lfloor S+B \rfloor = S$, K_X+S+B is nef and $K_X+S+B \sim_{\mathbb{Q}} D \geq 0$ where $S \subset \mathrm{Supp}(D) \subset \mathrm{Supp}(S+B)$. Then*

$$H^0(X, \mathcal{O}_X(m(K_X + S + B))) \to H^0(S, \mathcal{O}_S(m(K_X + S + B)))$$

is surjective for all $m > 0$ sufficiently divisible.

We remark that (1.3) follows from (1.1). In fact, $(X, \Delta = S+B-\epsilon D)$ is a klt pair for any $0 < \epsilon \ll 1$. By (1.1) $K_X+\Delta \sim (1-\epsilon)(K_X+S+B)$ is semiample. Let $f : X \to Z$ be the corresponding morphism. If $\dim Z = 0$, then $D = 0$ and there is nothing to prove. We may thus assume that $\dim Z > 0$. As $S \subset \mathrm{Supp}(D)$, it follows that $f_*S = 0$ and hence by [Fujino09, 6.3.i], the short exact sequence

$$\begin{aligned} 0 \to \mathcal{O}_X(m(K_X + S + B) - S) &\to \mathcal{O}_X(m(K_X + S + B)) \\ &\to \mathcal{O}_S(m(K_X + S + B)) \to 0 \end{aligned}$$

remains exact after pushing forward by f. By [Fujino09, 6.3.ii], we have that $H^1(Z, f_*\mathcal{O}_X(m(K_X + S + B) - S)) = 0$ and hence

$$H^0(Z, f_*\mathcal{O}_X(m(K_X + S + B))) \to H^0(Z, f_*\mathcal{O}_S(m(K_X + S + B)))$$

is surjective and (1.3) follows. The assumption $\mathrm{Supp}(D) \subset \mathrm{Supp}(S+B)$ is necessary for technical reasons. It is expected that it is not a necessary assumption. In practice it is often easily satisfied (see for example the proof of (7.1)).

In practice, one would like to use (1.3) to prove (1.1). We prove the following result (cf. (7.1) for a more precise statement).

Theorem 1.4. *Assume $(1.3)_n$ holds and that $(1.2)_n$ holds for all semi-log canonical pairs. Then $(1.1)_n$ holds (i.e. (1.1) holds in dimension n).*

The main purpose of this article is to prove that Conjecture 1.3 holds under the additional assumption that $(X, S + B)$ is plt, see Theorem 1.7 below.

Remark 1.5. *Property* (1.2) *is known to hold in dimension* ≤ 3 *cf.* [Kawamata92], [Miyaoka88], [KMM94], [Fujino00] *and when* $K_X + \Delta$ *is nef and* $\kappa_\sigma(K_X + \Delta) = 0$ *cf.* [Nakayama04]. *See also* [Ambro04] *and* [Fukuda02] *for related results. A proof of the case when* X *is smooth and* $\Delta = 0$ *has been announced in* [Siu09] *(this is expected to imply the general case cf.* (8.8)*).*

The existence of a good minimal models is known for canonical pairs $(X, 0)$ *where* K_X *is nef and* $\kappa(K_X) = \nu(K_X)$ *cf.* [Kawamata85a], *when* $\kappa(K_X) = \dim(X)$ *by* [BCHM10] *and when the general fiber of the Iitaka fibration has a good minimal model by* [Lai10].

Birkar has shown that (1.2) *implies the existence of minimal models (resp. Mori-Fano fiber spaces) and the existence of the corresponding sequence of flips and divisorial contractions cf.* [Birkar09, 1.4]. *The existence of minimal models for klt* 4*-folds is proven in* [Shokurov09].

We also recall the following important consequence of (1.1) (cf. [Birkar09]).

Corollary 1.6. *Assume* $(1.1)_n$. *Let* (X, Δ) *be an* n*-dimensional klt pair and* A *an ample divisor such that* $K_X + \Delta + A$ *is nef. Then any* $K_X + \Delta$*-minimal model program with scaling terminates.*

Proof. If $K_X + \Delta$ is not pseudo-effective, then the claim follows by [BCHM10]. If $K_X + \Delta$ is pseudo-effective, then the result follows from [Lai10]. $\square$

We now turn to the description of the main result of this paper (cf. (1.7)) which we believe is of independent interest.

Let X be a smooth variety, and let $S + B$ be a $\mathbb{Q}$-divisor with simple normal crossings, such that $S = \lfloor S + B \rfloor$,

$$K_X + S + B \in \mathrm{Psef}(X) \qquad \text{and} \quad S \not\subset N_\sigma(K_X + S + B).$$

We consider $\pi : \widetilde{X} \to X$ a log-resolution of $(X, S + B)$, so that we have

$$K_{\widetilde{X}} + \widetilde{S} + \widetilde{B} = \pi^\star(K_X + S + B) + \widetilde{E}$$

where $\widetilde{S}$ is the proper transform of S. Moreover $\widetilde{B}$ and $\widetilde{E}$ are effective $\mathbb{Q}$-divisors, the components of $\widetilde{B}$ are disjoint and $\widetilde{E}$ is π-exceptional.

Following [HM10] and [Paun08], if we consider the *extension obstruction divisor*

$$\Xi := N_\sigma(\|K_{\widetilde{X}} + \widetilde{S} + \widetilde{B}\|_{\widetilde{S}}) \wedge \widetilde{B}|_{\widetilde{S}},$$

then we have the following result.

Theorem 1.7 (Extension Theorem)**.** *Let* X *be a smooth variety,* $S+B$ *a* $\mathbb{Q}$*-divisor with simple normal crossings such that*

(1) $(X, S+B)$ *is plt (i.e.* S *is a prime divisor with* $\mathrm{mult}_S(S+B) = 1$ *and* $\lfloor B \rfloor = 0$*),*

(2) *there exists an effective* $\mathbb{Q}$*-divisor* $D \sim_{\mathbb{Q}} K_X + S + B$ *such that* $S \subset \mathrm{Supp}(D) \subset \mathrm{Supp}(S + B)$*, and*
(3) S *is not contained in the support of* $N_\sigma(K_X + S + B)$ *(i.e., for any ample divisor* A *and any rational number* $\epsilon > 0$*, there is an effective* $\mathbb{Q}$*-divisor* $D \sim_{\mathbb{Q}} K_X + S + B + \epsilon A$ *whose support does not contain* S*).*

Let m *be an integer, such that* $m(K_X + S + B)$ *is Cartier, and let* u *be a section of* $m(K_X + S + B)|_S$*, such that*

$$Z_{\pi^\star(u)} + m\widetilde{E}|_{\widetilde{S}} \geq m\Xi,$$

where we denote by $Z_{\pi^\star(u)}$ *the zero divisor of the section* $\pi^\star(u)$*. Then* u *extends to a section of* $m(K_X + S + B)$*.*

The above theorem is a strong generalization of similar results available in the literature (see for example [Siu98], [Siu00], [Takayama06], [Takayama07], [HM07], [Paun07], [Claudon07], [EP07], [dFH09], [Var08], [Paun08], [Tsuji05], [HM10], [BP10]). The main and important difference is that we do not require any *strict positivity* from B. The positivity of B (typically one requires that B contain an ample $\mathbb{Q}$-divisor) is of great importance in the algebraic approach as it allows us to make use of the Kawamata-Viehweg Vanishing Theorem. It is for this reason that *so far* we are unable to give an algebraic proof of (1.7).

In order to understand the connections between (1.7) and the results quoted above, a first observation is that one can reformulate (1.7) as follows. *Let* $(X, S + B)$ *be a plt pair, such that there exists an effective divisor* $D \sim_{\mathbb{Q}} K_X + S + B$ *with the property that* $S \subset \mathrm{Supp}(D) \subset \mathrm{Supp}(S + B)$*. Let* u *be a section of the bundle* $m(K_X + S + B)|_S$*, such that* $u^{\otimes k} \otimes s_A$ *extends to* X*, for each* k *and each section* s_A *of a sufficiently ample line bundle* A*; then* u *extends to* X*.* This result will be extensively discussed in Section 4 and Section 5; here we mention very briefly a few key points.

We write as usual

$$m(K_X + S + B) = K_X + S + L$$

where $L := (m-1)(K_X+S+B)+B$, and is this context, we recall that the fundamental results (cf. [OT87], [Manivel93]) allows us to extend the section u of $(K_X+S+L)|_S$, provided that we can endow the bundles $\mathcal{O}(S)$ and L with metrics $h_S = e^{-\varphi_S}$ and $h_L = e^{-\varphi_L}$ respectively, such that the next inequalities are satisfied

$$(\star) \qquad \Theta_{h_L}(L) \geq 0, \quad \Theta_{h_L}(L) \geq \frac{1}{\alpha}\Theta_{h_S}(S),$$

(here $\alpha > 1$ is a real number) and such that u is L^2 with respect to $h_L|_S$.

Assume first that the $\mathbb{Q}$-bundle K_X+S+B admits a metric $h=e^{-\psi}$ with semi-positive curvature, and such that it is adapted to u

$$\psi|_S \geq \frac{1}{m}\log|u|^2 + C,$$

i.e. the restriction of the metric h to S is well defined, and less singular than the metric induced by u. Our first contribution to the subject is to show that the hypothesis

$$S \subset \mathrm{Supp}(D) \subset \mathrm{Supp}(S+B)$$

in (1.7) arising naturally from algebraic geometry *fits perfectly* with the analytic requirements ($\star$), i.e. this hypothesis together with the existence of the metric $h=e^{-\psi}$ can be combined in order to produce the metrics h_S, h_L as above. We refer to the Section 4 for details; we establish a generalization of the version of the Ohsawa-Takegoshi Theorem (cf. [OT87], [Ohsawa03], [Ohsawa04]) established in [Manivel93], [Var08], [MV07]. The overall idea is that the existence of the divisor D is used as a substitute for the usual strict positivity of B.

We remark next that the existence of the metric $h=e^{-\psi}$ is a-priori not clear; instead we know that $u^{\otimes k}\otimes s_A$ extends to X. The extension $U_A^{(km)}$ of the section $u^{\otimes k}\otimes s_A$ induces a metric on the bundle $K_X+S+B+\frac{1}{km}A$, and one could think of defining $e^{-\psi}$ by taking the limit as $k\to\infty$ of these metrics. Alas, this cannot be done directly, since we do not have uniform estimates for the L^2 norm of the section $U_A^{(km)}$ as $k\to\infty$. Prior to the limit process, we address this issue in Section 5: by using an iteration process we are able to convert the qualitative information about the (extension properties of the) sections $u^{\otimes k}\otimes s_A$ into a quantitative one. The iteration process is a bit too technical to be presented here, but very vaguely, the important idea is quite simple: assume that we know that some section of the Cartier divisor $(km(K_X+S+B)+A)|_S$ extends to X. Then (under some precise curvature conditions) the extension with minimal $\mathrm{L}^{2/km}$-norm verifies an "effective" estimate (cf. inequality (59) in Section 5).

In conclusion, we show that a family of extensions can be constructed with a very precise estimate of their norm, as $k\to\infty$. By a limit process justified by the estimates we have just mentioned together with the classical results in [Lelong69], we obtain a metric on K_X+S+B adapted to u, and then the extension of u follows by our version of Ohsawa-Takegoshi (which is applied several times in the proof of (1.7)).

In many applications the following corollary to (1.7) suffices.

Corollary 1.8. *Let K_X+S+B be a nef plt pair such that there exists an effective $\mathbb{Q}$-divisor $D\sim_{\mathbb{Q}} K_X+S+B$ with $S\subset \mathrm{Supp}(D)\subset \mathrm{Supp}(S+B)$.*

Then

$$H^0(X, \mathcal{O}_X(m(K_X+S+B))) \to H^0(S, \mathcal{O}_S(m(K_X+S+B)))$$

is surjective for all sufficiently divisible integers $m > 0$.

In particular, if $\kappa((K_X+S+B)|_S) \geq 0$, then the stable base locus of K_X+S+B does not contain S.

This paper is organized as follows: In Section 2 we recall the necessary notation, conventions and preliminaries. In Section 3 we give some background on the analytic approach and in particular we explain the significance of good minimal models in the analytic context. In Section 4 we prove an Ohsawa-Takegoshi extension theorem which generalizes a result of L. Manivel and D. Varolin. In Section 5 we prove the Extension Theorem 1.7. Finally, in Section 7 we prove Theorem 1.4.

2. Preliminaries

2.1. **Notation and conventions.** We work over the field of complex numbers $\mathbb{C}$.

Let $D = \sum d_i D_i$ and $D' = \sum d'_i D_i$ be $\mathbb{Q}$-divisors on a normal variety X, then the **round-down** of D is given by $\lfloor D \rfloor := \sum \lfloor d_i \rfloor D_i$ where $\lfloor d_i \rfloor = \max\{z \in \mathbb{Z} | z \leq d_i\}$. Note that by definition we have $|D| = |\lfloor D \rfloor|$. We let $D \wedge D' := \sum \min\{d_i, d'_i\} D_i$ and $D \vee D' := \sum \max\{d_i, d'_i\} D_i$. The $\mathbb{Q}$-Cartier divisor D is **nef** if $D \cdot C \geq 0$ for any curve $C \subset X$. The $\mathbb{Q}$-divisors D and D' are **numerically equivalent** $D \equiv D'$ if and only if $(D - D') \cdot C = 0$ for any curve $C \subset X$. The **Kodaira dimension** of D is

$$\kappa(D) := \mathrm{tr.deg}_{\mathbb{C}} \left(\oplus_{m \geq 0} H^0(X, \mathcal{O}_X(mD)) \right) - 1.$$

If $\kappa(D) \geq 0$, then $\kappa(D)$ is the smallest integer $k > 0$ such that $\liminf h^0(\mathcal{O}_X(mD))/m^k > 0$. We have $\kappa(D) \in \{-1, 0, 1, \ldots, \dim X\}$. If $\kappa(D) = \dim X$ then we say that D is **big**. If $D \equiv D'$ then D is big if and only if D' is big. If D is numerically equivalent to a limit of big divisors, then we say that D is **pseudo-effective**.

Let A be a sufficiently ample divisor, and D a pseudo-effective $\mathbb{Q}$-divisor, then we define

$$\kappa_\sigma(D) := \max\{k > 0 | \limsup_{m \to \infty} \frac{h^0(\mathcal{O}_X(mD + A))}{m^k} < +\infty\}.$$

It is known that $\kappa(D) \leq \kappa_\sigma(D)$ and equality holds when $\kappa_\sigma(D) = \dim X$.

Let $V \subset |D|$ be a linear series, then we let $\mathrm{Bs}(V) = \{x \in X | x \in \mathrm{Supp}(E)\ \forall\ E \in V\}$ be the **base locus of** V and $\mathrm{Fix}(V) = \wedge_{E \in V} E$ be the **fixed part** of $|V|$. In particular $|V| = |V - F| + F$ where $F = \mathrm{Fix}(V)$. If $V_i \subset |iD|$ is a sequence of (non-empty) linear series such that $V_i \cdot V_j \subset V_{i+j}$ for all $i, j > 0$, then we let $\mathbf{B}(V_\bullet) = \cap_{i>0} \mathrm{Bs}(B_i)$ be the **stable base locus** of $V_\bullet$ and $\mathbf{Fix}(V_\bullet) = \cap_{i>0}\mathrm{Supp}(\mathrm{Fix}(B_i))$ be the **stable fixed part** of $V_\bullet$. When $V_i = |iD|$ and $\kappa(D) \geq 0$, we will simply write $\mathbf{Fix}(D) = \mathbf{Fix}(V_\bullet)$ and $\mathbf{B}(D) = \mathbf{B}(V_\bullet)$. If D is pseudo-effective and A is an ample divisor on X, then we let $\mathbf{B}_-(D) = \bigcup_{\epsilon\in\mathbb{Q}_{>0}} \mathbf{B}(B+\epsilon A)$ be the **diminished stable base locus**. If C is a prime divisor, and D is a big $\mathbb{Q}$-divisor, then we let

$$\sigma_C(D) = \inf\{\mathrm{mult}_C(D')|D' \sim_{\mathbb{Q}} D,\ D' \geq 0\},$$

and if D is pseudo-effective then we let

$$\sigma_C(D) = \lim_{\epsilon\to 0} \sigma_C(D + \epsilon A).$$

Note that $\sigma_C(D)$ is independent of the choice of A and is determined by the numerical equivalence class of D. Moreover the set of prime divisors for which $\sigma_C(D) \neq 0$ is finite (for this and other details about $\sigma_C(D)$, we refer the reader to [Nakayama04]). One also defines the $\mathbb{R}$-divisor

$$N_\sigma(D) = \sum_C \sigma_C(D)C$$

so that the support of $N_\sigma(D)$ equals $\bigcup_{\epsilon\in\mathbb{Q}_{>0}} \mathbf{Fix}(B + \epsilon A)$.

If S is a normal prime divisor on a normal variety X, P is a prime divisor on S and D is a divisor such that S is not contained in $\mathbf{B}_+(D)$ then we define

$$\sigma_P(||D||_S) = \inf\{\mathrm{mult}_P(D'|_S)|D' \sim_{\mathbb{Q}} D,\ D' \geq 0,\ S \not\subset \mathrm{Supp}(D')\}.$$

If instead D is a pseudo-effective divisor such that $S \not\subset \mathbf{B}_-(D)$ then we let

$$\sigma_P(||D||_S) = \lim_{\epsilon\to 0} \sigma_P(||D + \epsilon A||_S).$$

Note that $\sigma_P(||D||_S)$ is determined by the numerical equivalence class of D and independent of the choice of the ample divisor A. One can see that the set of prime divisors such that $\sigma_P(||D||_S) > 0$ is countable. For this and other details regarding $\sigma_P(||D||_S)$ we refer the reader to Section 9 of [HK10]. We now define $N_\sigma(||D||_S) = \sum_P \sigma_P(||D||_S)P$. Note that $N_\sigma(||D||_S)$ is a formal sum of countably many prime divisors on S with positive real coefficients.

2.2. **Singularities of the mmp.** If X is a normal quasi-projective variety and Δ is an effective $\mathbb{Q}$-divisor such $K_X + \Delta$ is $\mathbb{Q}$-Cartier, then we say that (X, Δ) is a **pair**. We say that a pair (X, Δ) is **log smooth** if X is smooth and the support of Δ has simple normal crossings. A **log resolution** of a pair (X, Δ) is a projective birational morphism $f : Y \to X$ such that Y is smooth, the exceptional set $\mathrm{Exc}(f)$ is a divisor with simple normal crossings support and $f^{-1}_*\Delta + \mathrm{Exc}(f)$ has simple normal crossings support. We will write $K_Y + \Gamma = f^*(K_X + \Delta) + E$ where Γ and E are effective with no common components. We say that (X, Δ) is **Kawamata log terminal** or **klt** (resp. **log canonical** or

lc) if there is a log resolution (equivalently for any log resolution) of (X, Δ) such that the coefficients of Γ are < 1 (resp. ≤ 1). We say that (X, Δ) is **divisorially log terminal** or **dlt** if the coefficients of Δ are ≤ 1 and there is a log resolution such that the coefficients of $\Gamma - f_*^{-1}\Delta$ are < 1. In this case, if we write $\Delta = S + B$ where $S = \sum S_i = \lfloor \Delta \rfloor$ then each component of a stratum $S_I = S_{i_1} \cap \ldots \cap S_{i_k}$ of S is normal and (S_I, Δ_{S_I}) is dlt where $K_{S_I} + \Delta_{S_I} = (K_X + \Delta)|_{S_I}$. If (X, Δ) is dlt and S is a disjoint union of prime divisors, then we say that (X, Δ) is **purely log terminal** or **plt**. This is equivalent to requiring that (S_i, Δ_{S_i}) is klt for all i. Often we will assume that S is prime.

2.3. **The minimal model program with scaling.** A proper birational map $\phi : X \dashrightarrow X'$ is a **birational contraction** if ϕ^{-1} contracts no divisors. Let (X, Δ) be a projective $\mathbb{Q}$-factorial dlt pair and $\phi : X \dashrightarrow X'$ a birational contraction to a normal $\mathbb{Q}$-factorial variety X', then ϕ is $K_X + \Delta$**-negative** (resp. **non-positive**) if $a(E, X, \Delta) < a(E, X', \phi_*\Delta)$ (resp. $a(E, X, \Delta) \leq a(E, X', \phi_*\Delta)$) for all ϕ-exceptional divisors. If moreover $K_{X'} + \phi_*\Delta$ is nef then ϕ is a **minimal model** for (X, Δ) (or equivalently a $K_X + \Delta$-minimal model). Note that in this case (by the Negativity Lemma), we have that $a(E, X, \Delta) \leq a(E, X', \phi_*\Delta)$ for all divisors E over X and $(X', \phi_*\Delta)$ is dlt. If moreover $K_X + \phi_*\Delta$ is semiample, then we say that ϕ is a **good minimal model** for (X, Δ). Note that if ϕ is a good minimal model for (X, Δ), then

$$\operatorname{Supp}(\mathbf{Fix}(K_X + \Delta)) = \operatorname{Supp}(N_\sigma(K_X + \Delta)) = \operatorname{Exc}(\phi) \tag{1}$$

is the set of ϕ-exceptional divisors. Another important remark is that if ϕ is a minimal model, then

$$H^0(X, \mathcal{O}_X(m(K_X + \Delta))) \cong H^0(Y, \mathcal{O}_Y(m(K_Y + \phi_*\Delta))).$$

More generally we have the following:

Remark 2.1. *If $\phi : X \dashrightarrow Y$ is a birational contraction such that $a(E, X, \Delta) \leq a(E, X', \phi_*\Delta)$ for every divisor E over X, then*

$$H^0(X, \mathcal{O}_X(m(K_X + \Delta))) \cong H^0(Y, \mathcal{O}_Y(m(K_Y + \phi_*\Delta)))$$

for all $m > 0$.

Let $f : X \to Z$ be a proper morphism surjective with connected fibers from a $\mathbb{Q}$-factorial dlt pair (X, Δ) such that $\rho(X/Z) = 1$ and $-(K_X + \Delta)$ is f-ample.

(1) If $\dim Z < \dim X$, we say that f is a **Fano-Mori contraction**.
(2) If $\dim Z = \dim X$ and $\dim \operatorname{Exc}(f) = \dim X - 1$, we say that f is a **divisorial contraction**.
(3) If $\dim Z = \dim X$ and $\dim \operatorname{Exc}(f) < \dim X - 1$, we say that f is a **flipping contraction**.

If f is a divisorial contraction, then $(Z, f_*\Delta)$ is a $\mathbb{Q}$-factorial dlt pair. If f is a flipping contraction, then by [BCHM10], the flip $f^+ : X^+ \to Z$ exists, it is unique and given by

$$X^+ = \mathrm{Proj}_Z \oplus_{m\geq 0} f_*\mathcal{O}_X(m(K_X + \Delta)).$$

We have that the induced rational map $\phi : X \dashrightarrow X^+$ is an isomorphism in codimension 1 and $(X^+, \phi_*\Delta)$ is a $\mathbb{Q}$-factorial dlt pair.

Let (X, Δ) be a projective $\mathbb{Q}$-factorial dlt pair (resp. a klt pair), and A an ample (resp. big) $\mathbb{Q}$-divisor such that $K_X + \Delta + A$ is nef. By [BCHM10], we may run the **minimal model program with scaling** of A, so that we get a sequence of birational contractions $\phi_i : X_i \dashrightarrow X_{i+1}$ where $X_0 = X$ and of rational numbers $t_i \geq t_{i+1}$ such that

(1) if $\Delta_{i+1} := \phi_{i*}\Delta_i$ and $H_{i+1} = \phi_{i*}H_i$, then (X_i, Δ_i) is a $\mathbb{Q}$-factorial dlt pair (resp. a klt pair) for all $i \geq 0$,
(2) $K_{X_i} + \Delta_i + tH_i$ is nef for any $t_i \geq t \geq t_{i+1}$,
(3) if the sequence is finite, i.e. $i = 0, 1, \ldots, N$, then $K_{X_N} + \Delta_N + t_N H_N$ is nef or there exists a Fano-Mori contraction $X_N \to Z$,
(4) if the sequence is infinite, then $\lim t_i = 0$.

If the sequence is finite, we say that the minimal model program with scaling terminates. Conjecturally this is always the case.

Remark 2.2. *Note that it is possible that $t_i = t_{i+1}$. Moreover, it is known that there exist infinite sequences of flops (cf.* [Kawamata97]*), i.e. $K_X + \Delta$ trivial maps.*

Remark 2.3. *Note that if $K_X + \Delta$ is pseudo-effective then the support of $N_\sigma(K_X + \Delta)$ contains finitely many prime divisors and it coincides with the support of $\mathbf{Fix}(K_X + \Delta + \epsilon A)$ for any $0 < \epsilon \ll 1$ (cf.* [Nakayama04]*). It follows that if the sequence of flips with scaling is infinite, then $N_\sigma(K_{X_i} + \Delta_i) = 0$ for all $i \gg 0$.*

Theorem 2.4. *If either*

(1) *no component of $\lfloor S \rfloor$ is contained in $\mathbf{B}_-(K_X+\Delta)$ (eg. if $K_X+\Delta$ is big and klt), or*
(2) *$K_X + \Delta$ is not pseudo-effective, or*
(3) *(X, Δ) has a good minimal model,*

then the minimal model program with scaling terminates.

Proof. See [BCHM10] and [Lai10]. □

Remark 2.5. *It is important to observe that in* [BCHM10] *the above results are discussed in the relative setting. In particular it is known that if (X, Δ) is a klt pair and $\pi : X \to Z$ is a birational projective morphism, then (X, Δ) has a good minimal model over Z. More precisely there exists a finite sequence of flips and divisorial contractions over Z giving rise to a birational contraction $\phi : X \dashrightarrow X'$ over Z such that $K_{X'} + \phi_*\Delta$ is semiample over Z (i.e. there is a projective* *morphism $q : X' \to W$ over Z such that $K_{X'} + \phi_*\Delta \sim_{\mathbb{Q}} q^*A$ where A is a $\mathbb{Q}$-divisor on W which is ample over Z).*

Remark 2.6. *It is known that the existence of good minimal models for pseudo-effective klt pairs is equivalent to the following conjecture (cf.* [GL10]*): If (X, Δ) is a pseudo-effective klt pair, then $\kappa_\sigma(K_X + \Delta) = \kappa(K_X + \Delta)$.*

Suppose in fact that (X, Δ) has a good minimal model say (X', Δ') and let $f : X' \to Z = \mathrm{Proj}R(K_{X'} + \Delta')$ so that $K_{X'} + \Delta' = f^*A$ for some ample $\mathbb{Q}$-divisor A on Z. Following Chapter V of [Nakayama04], *we have*

$$\kappa_\sigma(K_X + \Delta) = \kappa_\sigma(K_{X'} + \Delta') = \dim Z = \kappa(K_{X'} + \Delta') = \kappa(K_X + \Delta).$$

Conversely, assume that $\kappa_\sigma(K_X + \Delta) = \kappa(K_X + \Delta) \geq 0$. If $\kappa_\sigma(K_X + \Delta) = \dim X$, then the result follows from [BCHM10]. *If $\kappa_\sigma(K_X+\Delta) = 0$, then by* [Nakayama04, V.1.11], *we have that $K_X + \Delta$ is numerically equivalent to $N_\sigma(K_X + \Delta)$. By* [BCHM10] *(cf. (2.3)), after finitely many steps of the minimal model program with scaling, we may assume that $N_\sigma(K_{X'}+\Delta') = 0$ and hence that $K_{X'}+\Delta' \equiv 0$. Since $\kappa(K_X+\Delta) = 0$, we conclude that $K_{X'}+\Delta' \sim_{\mathbb{Q}} 0$ and hence (X, Δ) has a good minimal model. Assume now that $0 < \kappa_\sigma(K_X + \Delta) < \dim X$. Let $f : X \to Z = \mathrm{Proj}R(K_X + \Delta)$ be a birational model of the Iitaka fibration with very general fiber F. By Chapter V of* [Nakayama04], *we have that $\kappa_\sigma(K_X + \Delta) = \kappa_\sigma(K_F + \Delta|_F) + \dim Z$, but since $\dim Z = \kappa(K_X + \Delta)$, we have that $\kappa_\sigma(K_F + \Delta|_F) = 0$. Thus $(F, \Delta|_F)$ has a good minimal model. By* [Lai10], *(X, Δ) has a good minimal model.*

Recall the following result due to Shokurov known as Special Termination:

Theorem 2.7. *Assume that the minimal model program with scaling for klt pairs of dimension $\leq n - 1$ terminates. Let (X, Δ) be a $\mathbb{Q}$-factorial n-dimensional dlt pair and A an ample divisor such that $K_X + \Delta + A$ is nef. If $\phi_i : X_i \dashrightarrow X_{i+1}$ is a minimal model program with scaling, then ϕ_i is an isomorphism on a neighborhood of $\lfloor \Delta_i \rfloor$ for all $i \gg 0$.*

If moreover $K_X + \Delta \equiv D \geq 0$ and the support of D is contained in the support of $\lfloor \Delta \rfloor$ then the minimal model program with scaling terminates.

Proof. See [Fujino07]. □

We will also need the following standard results about the minimal model program.

Theorem 2.8. *[Length of extremal rays] Let (X, Δ) be a lc pair and (X, Δ_0) be a klt pair and $f : X \to Z$ be a projective morphism surjective with connected fibers such that $\rho(X/Z) = 1$ and $-(K_X+\Delta)$ is f-ample.*

Then there exists a curve Σ contracted by F such that

$$0 < -(K_X + \Delta) \cdot \Sigma \leq 2 \dim X.$$

Proof. See for example [BCHM10, 3.8.1]. □

Theorem 2.9. *Let $f : X \to Z$ be a flipping contraction and $\phi : X \dashrightarrow X^+$ be the corresponding flip. If L is a nef and Cartier divisor such that $L \equiv_Z 0$, then so is $\phi_* L$.*

Proof. Easy consequence of the Cone Theorem, see for example [KM98, 3.7]. □

2.4. **A few analytic preliminaries.** We collect here some definitions and results concerning (singular) metrics on line bundles, which will be used in the sections that follow. For a more detailed presentation and discussion, we refer the reader to [Demailly90].

Definition 2.10. *Let $L \to X$ be a line bundle on a compact complex manifold. A singular hermitian metric h_L on L is given in any trivialization $\theta : L|_\Omega \to \Omega \times \mathbb{C}$ by*

$$|\xi|^2_{h_L} := |\theta(\xi)|^2 e^{-\varphi_L(x)}, \quad \xi \in L_x$$

where $\varphi_L \in L^1_{\mathrm{loc}}(\Omega)$ is the local weight of the metric h_L and $h_L = e^{-\varphi_L}$.

The difference between the notions of smooth and singular metrics is that in the latter case the local weights are only assumed to verify a weak regularity property. The hypothesis $\varphi_L \in L^1_{\mathrm{loc}}(\Omega)$ is needed in order to define the *curvature current* of (L, h_L), as follows:

$$\Theta_{h_L}(L)|_\Omega := \frac{\sqrt{-1}}{2\pi} \partial\overline{\partial}\varphi_L.$$

If the local weights φ_L of h_L are *plurisubharmonic* ("psh" for short, see [Demailly90] and the references therein), then we have $\Theta_{h_L}(L) \geq 0$; conversely, if we know that if $\Theta_{h_L}(L) \geq 0$, then each φ_L coincides almost everywhere with a psh function.

We next state one of the important properties of the class of psh functions, which will be used several times in the proof of (1.7). Let β be a $\mathcal{C}^\infty$-form of (1, 1)-type, such that $d\beta = 0$. Let τ_1 and τ_2 be two functions in $L^1(X)$, such that

$$\beta + \sqrt{-1}\partial\overline{\partial}\tau_j \geq 0$$

on X, for each $j = 1, 2$. We define $\tau := \max\{\tau_1, \tau_2\}$, and then we have

$$\beta + \sqrt{-1}\partial\overline{\partial}\tau \geq 0$$

on X (we refer e.g. to [Demailly09] for the proof).

2.5. **Examples.** One of the best known and useful examples of singular metrics appears in the context of algebraic geometry: we assume that $L^{\otimes m}$ has some global holomorphic sections say $\{\sigma_j\}_{j\in J}$. Then there is a metric on L, whose local weights can be described by

$$\varphi_L(x) := \frac{1}{m}\log\sum_{j\in J}|f_j(x)|^2$$

where the holomorphic functions $\{f_j\}_{j\in J}\subset\mathcal{O}(\Omega)$ are the local expressions of the global sections $\{\sigma_j\}_{j\in J}$. The singularities of the metric defined above are of course the common zeroes of $\{\sigma_j\}_{j\in J}$. One very important property of these metrics is the *semi-positivity of the curvature current*

$$\Theta_{h_L}(L)\geq 0,$$

as it is well known that the local weights induced by the sections $\{\sigma_j\}_{j\in J}$ above are psh. If the metric h_L is induced by one section $\sigma\in H^0(X,L^{\otimes m})$ with zero set Z_σ, then we have that

$$\Theta_{h_L}(L)=\frac{1}{m}[Z_\sigma],$$

hence the curvature is given (up to a multiple) by the current of integration over the zero set of σ. From this point of view, the curvature of a singular hermitian metric is a natural generalization of an effective $\mathbb{Q}$-divisor in algebraic geometry.

A slight variation on the previous example is the following. Let L be a line bundle, which is *numerically equivalent* to an effective $\mathbb{Q}$-divisor

$$D=\sum_j\nu^jW_j.$$

Then D and L have the same first Chern class, hence there is an integer $m>0$ such that $L^{\otimes m}=\mathcal{O}_X(mD)\otimes\rho^{\otimes m}$, for some topologically trivial line bundle $\rho\in\mathrm{Pic}^0(X)$.

In particular, there exists a metric h_ρ on the line bundle ρ whose curvature is equal to zero (i.e. the local weights φ_ρ of h_ρ are real parts of holomorphic functions). Then the expression

$$\varphi_\rho+\sum_j\nu^j\log|f_j|^2$$

(where f_j is the local equation of W_j) is the local weight of a metric on L; we call it *the metric induced by* D (although it depends on the choice of h_ρ).

The following result is not strictly needed in this article, but we mention it because we feel that it may help to understand the structure of the curvature currents associated with singular metrics.

Theorem 2.11. [Siu74] *Let T be a closed positive current of $(1,1)$-type. Then we have*

$$T = \sum_{j\geq 1} \nu^j [Y_j] + \Lambda$$

where the ν^j are positive real numbers, and $\{Y_j\}$ is a (countable) family of hypersurfaces of X and Λ is a closed positive current whose singularities are concentrated along a countable union of analytic subsets of codimension at least two.

We will not make precise the notion of "singularity" appearing in the statement above. We just mention that it is the analog of the multiplicity of a divisor. By Theorem 2.11 we infer that if the curvature current of a singular metric is positive, then it can be decomposed into a divisor-like part (however, notice that the sum above may be infinite), together with a *diffuse* part Λ, which –very, very roughly–corresponds to a differential form.

As we will see in Section 4 below, it is crucial to be able to work with singular metrics in full generality: the hypothesis of all vanishing/extension theorems that we are aware of, are mainly concerned with the diffuse part of the curvature current, and not the singular one. Unless explicitly mentioned otherwise, all the metrics in this article are allowed to be singular.

2.6. **Construction of metrics.** We consider now the following set-up. Let L be a $\mathbb{Q}$-line bundle, such that:

(1) L admits a metric $h_L = e^{-\varphi_L}$ with semi-positive curvature current $\Theta_{h_L}(L)$.
(2) The $\mathbb{Q}$-line bundle L is numerically equivalent to the effective $\mathbb{Q}$-divisor

$$D := \sum_{j\in J} \nu^j W_j$$

where $\nu^1 > 0$ and the restriction of h_L to the generic point of W_1 is well-defined (i.e. not equal to ∞). We denote by h_D the metric on L induced by the divisor D.
(3) Let h_0 be a non-singular metric on L; then we can write

$$h_L = e^{-\psi_1} h_0, \quad h_D = e^{-\psi_2} h_0$$

where ψ_j are global functions on X. Suppose that we have

$$\psi_1 \geq \psi_2.$$

Working locally on some coordinates open set $\Omega \subset X$, if we let φ_L be the local weight of the metric h_L, and for each $j \in J$ we let f_j be an equation of $W_j \cap \Omega$, then the above inequality is equivalent to

$$\varphi_L \geq \varphi_\rho + \sum_{j \in J} \nu^j \log |f_j|^2 \tag{$\dagger$}$$

(cf. the above discussion concerning the metric induced by a $\mathbb{Q}$-divisor numerically equivalent to L).

In this context, we have the following simple observation.

Lemma 2.12. *Let $\Omega \subset X$ be a coordinate open set. Define functions $\varphi_{W_1} \in L^1_{\mathrm{loc}}(\Omega)$ which are the local weights of a metric on $\mathcal{O}_X(W_1)$, via the equality*

$$\varphi_L = \nu^1 \varphi_{W_1} + \varphi_\rho + \sum_{j \in J \setminus 1} \nu^j \log |f_j|^2.$$

Then ($\dagger$) is equivalent to the inequality

$$|f_1|^2 e^{-\varphi_{W_1}} \leq 1$$

at each point of Ω.

Proof. It is a consequence of the fact that $\log |f_j|^2$ are the local weights of the singular metric on $\mathcal{O}_X(W_j)$ induced by the tautological section of this line bundle, combined with the fact that L and $\mathcal{O}_X(D)$ are numerically equivalent. The inequality above is equivalent to ($\dagger$). □

2.7. **Mean value inequality.** We end this subsection by recalling a form of the mean value inequality for psh functions, which will be particularly useful in Section 5.

Let α be a smooth $(1,1)$ form on X, such that $d\alpha = 0$, and let $f \in L^1(X)$ be such that

$$\alpha + \sqrt{-1}\partial\overline{\partial} f \geq 0. \tag{+}$$

We fix the following quantity

$$I(f) := \int_X e^f dV_\omega$$

where dV_ω is the volume element induced by a metric ω on X. We have the following well-known result.

Lemma 2.13. *There exists a constant $C = C(X, \omega, \alpha)$ such that for any function $f \in L^1(X)$ verifying the condition (+) above we have*

$$f(x) \leq C + \log I(f), \quad \forall x \in X.$$

Proof. We consider a coordinate system $z := \{z^1, \dots, z^n\}$ defined on $\Omega \subset X$ and centered at some point $x \in X$. Let $B_r := \{\|z\| < r\}$ be the Euclidean ball of radius r, and let $d\lambda$ be the Lebesgue measure corresponding to the given coordinate system. Since X is a compact manifold, we may assume that the radius r is independent of the particular point $x \in X$.

By definition of $I(f)$ we have

$$I(f) \geq \frac{1}{\mathrm{Vol}(B_r)} \int_{z \in B_r} e^{f(z)+C(X,\omega)} d\lambda$$

where $C(X,\omega)$ takes into account the distortion between the volume element dV_ω and the local Lebesgue measure $d\lambda$, together with the Euclidean volume of B_r.

We can assume the existence of a function $g_\alpha \in \mathcal{C}^\infty(\Omega)$ such that $\alpha|_\Omega = \sqrt{-1}\partial\overline{\partial} g_\alpha$. By (+), the function $f + g_\alpha$ is psh on Ω. We now modify the inequality above as follows

$$I(f) \geq \frac{1}{\mathrm{Vol}(B_r)} \int_{z \in B_r} e^{f(z)+g_\alpha(z)+C(X,\omega,\alpha)} d\lambda.$$

By the concavity of the logarithm, combined with the mean value inequality applied to $f + g_\alpha$ we infer that

$$\log I(f) \geq f(x) - C(X, \omega, \alpha)$$

where the (new) constant $C(X,\omega,\alpha)$ only depends on the geometry of (X,ω) and on a finite number of potentials g_α (because of the compactness of X). The proof of the lemma is therefore finished. A last remark is that the constant "$C(X,\omega,\alpha)$" is uniform with respect to α: given $\delta > 0$, there exists a constant $C(X,\omega,\alpha,\delta)$ such that we can take $C(X,\omega,\alpha') := C(X,\omega,\alpha,\delta)$ for any closed $(1,1)$-form α' such that $\|\alpha - \alpha'\| < \delta$.

$\square$

3. Finite generation of modules

According to Remark 2.6, in order to establish the existence of good minimal models for pseudo-effective, klt pairs it suffices to show that

$$\kappa_\sigma(K_X + \Delta) = \kappa(K_X + \Delta).$$

In this section we will provide a direct argument for the equality above in the case where Δ is big. Even if this result is well known to experts and implicit in some of the literature, our point of view is slightly different (see however [CL10] for a related point of view), and it turns out to be very useful as a guiding principle for the arguments that we will invoke in order to prove Theorem 1.7.

Let X be a smooth, projective variety, and let Δ be a big $\mathbb{Q}$-divisor, such that (X,Δ) is klt. Analytically, this just means that Δ can be endowed with a metric $h_\Delta = e^{-\varphi_\Delta}$ whose associated curvature current dominates a metric on X, and such that $e^{-\varphi_\Delta} \in L^1_{\mathrm{loc}}(X)$. To be precise, what we really mean at this point is that the line bundle associated to $d_0\Delta$ can be endowed with a metric whose curvature current is greater than a Kähler metric, and whose $d_0 -$ th root is h_Δ.

Let $A \subset X$ be an ample divisor. We consider the following vector space

$$\mathcal{M} := \bigoplus_{m \in d_0\mathbb{N}} H^0\big(X, \mathcal{O}_X(m(K_X + \Delta) + A)\big)$$

which is an $\mathcal{R}$-module, where $\mathcal{R} := \bigoplus_{m \in d_0\mathbb{N}} H^0\big(X, \mathcal{O}_X(m(K_X + \Delta))\big)$.

In this section we will discuss the following result.

Proposition 3.1. *$\mathcal{M}$ is a finitely generated $\mathcal{R}$-module.*

Since the choice of the ample divisor A is arbitrary, the above proposition implies that we have $\kappa_\sigma(K_X + \Delta) = \kappa(K_X + \Delta)$.

We provide a sketch of the proof (3.1) below. As we have already mentioned, the techniques are well-known, so we will mainly highlight the features relevant to our arguments. The main ingredients are the finite generation of $\mathcal{R}$, coupled with the extension techniques originated in [Siu98] and Skoda's division theorem [Skoda72].

Sketch of the proof of Proposition 3.1. We start with some reductions. First, we may assume that $\kappa_\sigma(K_X + \Delta) \geq 0$ and hence that $\kappa(K_X + \Delta) \geq 0$ cf. [BCHM10]. Next, we can assume that:

• There exists a finite set of normal crossing hypersurfaces $\{Y_j\}_{j \in J}$ of X, such that

$$\Delta = \sum_{j \in J} \nu^j Y_j + A_\Delta \tag{2}$$

where $0 \leq \nu^j < 1$ for any $j \in J$, and A_Δ is an ample $\mathbb{Q}$-divisor. This can be easily achieved on a modification of X.

• Since the algebra $\mathcal{R}$ is generated by a finite number of elements, we may assume that it is generated by the sections of $m_0(K_X + \Delta)$, where m_0 is sufficiently large and divisible. The corresponding metric of $K_X + \Delta$ (induced by the generators of $\mathcal{R}$) is denoted by $h_{\min} = e^{-\varphi_{\min}}$ (cf. the construction recalled in Subsection 2.5; see [Demailly09] for a more detailed presentation). Hence, we may assume that

$$\Theta_{h_{\min}}(K_X + \Delta) = \sum a^j_{\min}[Y_j] + \Lambda_{\min} \tag{3}$$

(after possibly replacing X by a further modification). In relation (3) above, we can take the set of $\{Y_j\}_{j \in J}$ to coincide with the one in (2) (this is why we must allow some of the coefficients $\nu^j, a^j_{\min}$ to be equal to zero). $\Lambda_{\min}$ denotes a non-singular, semi-positive $(1,1)$-form.

For each integer m divisible enough, let Θ_m be the current induced by (the normalization of) a basis of sections of the divisor $m(K_X+\Delta)+A$; it belongs to the cohomology class associated to $K_X+\Delta+\frac{1}{m}A$. We can decompose it according to the family of hypersurfaces $\{Y_j\}_{j\in J}$ as follows

$$\Theta_m = \sum_{j\in J} a^j_m[Y_j] + \Lambda_m \tag{4}$$

where $a^j_m \geq 0$, and Λ_m is a closed positive current, which may be singular, despite the fact that Θ_m is less singular than $\Theta_{h_{\min}}(K_X+\Delta)$. Note that $a^j_m \leq a^j_{\min}$.

An important step in the proof of Proposition 3.1 is the following statement.

Claim 3.2. *We have*

$$\lim_{m\to\infty} a^j_m = a^j_{\min} \tag{5}$$

for each $j \in J$ (and thus Λ_m converges weakly to $\Lambda_{\min}$).

Proof. We consider an element $j \in J$. The sequence $\{a^j_m\}_{m\geq 1}$ is bounded, and can be assumed to be convergent, so we denote by a^j_∞ its limit. We observe that we have

$$a^j_\infty \leq a^j_{\min} \tag{6}$$

for each index j. Arguing by contradiction, we assume that at least one of the inequalities (6) above is strict.

Let Λ_∞ be any weak limit of the sequence $\{\Lambda_m\}_{m\geq 1}$. We note that in principle Λ_∞ *will be singular* along some of the Y_j, even if the Lelong number of each Λ_m at the generic point of Y_j is equal to zero, for any j: the reason is that any weak limit of $\{\Theta_m\}_{m\geq 1}$ it is expected to be at least as singular as $\Theta_{\min}$.

In any case, we remark that given any positive real number $t \in \mathbb{R}$, we have the following numerical identity

$$K_X + \sum_{j\in J} \big(\nu^j + t(a^j_{\min} - a^j_\infty) - a^j_\infty\big)Y_j + A_\Delta + t\Lambda_{\min} \equiv (1+t)\Lambda_\infty. \tag{7}$$

By using the positivity of A_Δ to tie-break, we can assume that for all $j \in J$ such that $a^j_{\min} \neq a^j_\infty$, the quantities

$$t^j := \frac{1-\nu^j + a^j_\infty}{a^j_{\min} - a^j_\infty} \tag{8}$$

are distinct, and we moreover assume that the minimum is achieved for $j = 1$. The relation (7) with $t = t^1$ becomes

$$K_X + Y_1 + \sum_{j\in J, j\neq 1} \tau^j Y_j + A_\Delta \equiv (1+t^1)\Lambda_\infty \tag{9}$$

where $\tau^j := \nu^j + t^1(a^j_{\min} - a^j_\infty) - a^j_\infty < 1$ are real numbers, which can be assumed to be positive (since we can "move" the negative ones on the right hand side). But then we have the following result (implicit in [Paun08]).

Theorem 3.3. *There exists an effective* $\mathbb{R}$*-divisor*

$$D := a^1_\infty Y_1 + \Xi$$

linearly equivalent to $K_X + \Delta$ *and such that* Y_1 *does not belong to the support of* Ξ.

We will not reproduce here the complete argument of the proof, instead we highlight the main steps of this proof.

• Passing to a modification of X, we can assume that the hypersurfaces $\{Y_j\}_{j\neq 1}$ are mutually disjoint and A_Δ is semi-positive (instead of ample), such that $A_\Delta - \sum_{j\neq 1} \varepsilon^j Y_j$ is ample (for some $0 < \varepsilon^j \ll 1$ where the corresponding Y_j are exceptional divisors). We denote $S := Y_1$.

• We can assume that $[\Lambda_m] = [\Lambda_\infty]$ i.e. the cohomology class is the same for any m, but for "the new" Λ_m we only have

$$\Lambda_m \geq -\frac{1}{m}\omega_{A_\Delta}. \tag{10}$$

• The restriction $\Lambda_m|_S$ is well-defined, and can be written as

$$\Lambda_m|_S = \sum_{j\neq 1} \rho^j_m Y_j|_S + \Lambda_{m,S}.$$

• By induction, we obtain an effective $\mathbb{R}$-divisor D_S *linearly equivalent to* $(K_X + S + \sum_{j\in J, j\neq 1} \tau^j Y_j + A_\Delta)|_S$, whose order of vanishing along $Y_j|_S$ is at least $\min\{\tau^j, \rho^j_\infty\}$. We note that in [Paun08], we only obtain an effective $\mathbb{R}$-divisor D_S which is *numerically equivalent to* $(K_X + S + \sum_{j\in J, j\neq 1} \tau^j Y_j + A_\Delta)|_S$. We may however assume that D_S is $\mathbb{R}$*-linearly equivalent to* $(K_X + S + \sum_{j\in J, j\neq 1} \tau^j Y_j + A_\Delta)|_S$ by an argument due to [CL10].

• The $\mathbb{R}$-divisor D_S extends to X, by the "usual" procedure, namely Diophantine approximation and extension theorems (see e.g. [Paun08] and [HM10]). This last step ends the discussion of the proof of Theorem 3.3.

Remark 3.4. *The last bullet above is the heart of the proof of the claim. We stress the fact that the factor* A_Δ *of the boundary is essential, even if the coefficients* τ^j *and the divisor* D_S *are rational.*

An immediate Diophantine approximation argument shows that the divisor D produced by Theorem 3.3 should not exist: its multiplicity along Y_1 is *strictly smaller than* $a^1_{\min}$, and this is a contradiction. $\square$

The rest of the proof is based of the following global version of the H. Skoda division theorem (cf. [Skoda72]), established in [Siu08].

Let G be a divisor on X, and let $\sigma_1, \dots, \sigma_N$ be a set of holomorphic sections of $\mathcal{O}_X(G)$. Let E be a divisor on X, endowed with a possibly singular metric $h_E = e^{-\varphi_E}$ with positive curvature current.

Theorem 3.5. [Skoda72] *Let u be a holomorphic section of the divisor $K_X + (n+2)G + E$, such that*

$$\int_X \frac{|u|^2 e^{-\varphi_E}}{\big(\sum_j |\sigma_j|^2\big)^{n+1}} < \infty \tag{11}$$

(we notice that the quantity under the integral sign is a global measure on X). Then there exists sections $u^1, \dots, u^N$ of $K_X + nG + E$ such that

$$u = \sum_j u^j \sigma_j. \tag{12}$$

This result together with Claim 3.2 above prove the finite generation of $\mathcal{M}$, along the same lines as in [Demailly90]; we provide next the details.

Let m be a sufficiently big and divisible integer (to be specified in a moment), and let u be a section of $m(K_X + \Delta) + A$. We recall that m_0 denotes a positive integer, such that the metric on $K_X + \Delta$ induced by the sections $\{\sigma_1, \dots, \sigma_N\}$ of $m_0(K_X + \Delta)$ is equivalent to $\varphi_{\min}$. We have

$$m(K_X + \Delta) + A = K_X + (n+1)G + E$$

where

$$G := m_0(K_X + \Delta)$$

and

$$E := \Delta + \big(m - (n+1)m_0 - 1\big)\Big(K_X + \Delta + \frac{1}{m}A\Big) + \frac{m_0(n+1)+1}{m}A$$

are endowed respectively with the metrics φ_G induced by the sections $\{\sigma_1, \dots, \sigma_N\}$ above, and

$$\varphi_E := \varphi_\Delta + \big(m - (n+1)m_0 - 1\big)\varphi_m + \frac{m_0(n+1)+1}{m}\varphi_A.$$

Here we denote by φ_m the metric on $K_X + \Delta + \frac{1}{m}A$ induced by the global sections of $\mathcal{O}_X(m(K_X + \Delta) + A)$. We next check that condition (11) is satisfied. Notice that

$$\int_X \frac{|u|^2}{(\sum_j |\sigma_j|^2)^{n+1}} e^{-\varphi_E} \leq C \int_X e^{(m_0(n+1)+1)\varphi_m - (n+1)m_0\varphi_{\min} - \varphi_\Delta}$$

since we clearly have $|u|^2 \leq C e^{m\varphi_m}$ (we skip the non-singular weight corresponding to A in the expression above). The fact that (X, Δ) is klt, together with Claim 3.2 implies that there exists some fixed index m_1 such that we have

$$\int_X e^{(m_0(n+1)+1)\varphi_m-(n+1)m_0\varphi_{\min}-\varphi_\Delta} d\lambda < \infty, \tag{13}$$

as soon as $m \geq m_1$. In conclusion, the relation (11) above holds true; hence, as long as $m \geq m_1$, Skoda's Division Theorem can be applied, and Proposition 3.1 is proved. □

Remark 3.6. As we have already mentioned, in the following sections we will show that a consistent part of the proof of (3.1) is still valid in the absence of the ample part A_Δ. Here we highlight the properties of $K_X + \Delta$ which will replace the strict positivity. We consider the following context. Let

$$\Delta \equiv \sum_{j\in J} \nu^j [Y_j] + \Lambda_\Delta \tag{14}$$

be a $\mathbb{Q}$-divisor, where $0 \leq \nu^j < 1$ and Λ_Δ is a semi-positive form of $(1,1)$-type. We assume as always that the hypersurfaces Y_j have simple normal crossings. The difference between this set-up and the hypothesis of (3.1) is that Δ is not necessarily big.

We assume that the $K_X+\Delta$ is $\mathbb{Q}$-effective. Recall that by [BCHM10], the associated canonical ring $R(K_X + \Delta)$ is finitely generated. The reductions performed at the beginning of the proof of (3.1) do not use A_Δ. However, difficulties arise when we come to the proof of Claim 3.2. Indeed, the assumption

$$a^j_\infty < a^j_{\min}$$

for some $j \in J$ implies that we will have

$$K_X + Y_1 + \sum_{j\in J, j\neq 1} \tau^j Y_j \equiv (1 + t^1)\Lambda_\infty \tag{15}$$

cf. (9), but in the present context, the numbers τ^j *cannot be assumed to be strictly smaller than* 1. Nevertheless we have that

(a) The $\mathbb{Q}$-divisor $K_X + Y_1 + \sum_{j\in J, j\neq 1} \tau^j Y_j$ is pseudo-effective, and

$$Y_1 \notin N_\sigma(K_X + Y_1 + \sum_{j\in J, j\neq 1} \tau^j Y_j).$$

(b) There exists an effective $\mathbb{R}$-divisor say $G := \sum_i \mu^i W_i$ which is $\mathbb{R}$-linearly equivalent to $K_X + Y_1 + \sum_{j\in J, j\neq 1} \tau^j Y_j$ such that

$$Y_1 \subset \mathrm{Supp}(G) \subset \{Y_j\}_{j\in J}.$$

The properties above are consequences of the fact that we have assumed that Claim 3.2 fails to hold. They indicate that the $\mathbb{Q}$-divisor

$$L := K_X + Y_1 + \sum_{j\in J, j\neq 1} \tau^j Y_j$$

has some kind of positivity: property (a) implies the existence of a sequence of metrics $h_m = e^{-\varphi_m}$ on the $\mathbb{Q}$-line bundle L such that

$$\Theta_{h_m}(L) = \sqrt{-1}\partial\overline{\partial}\varphi_m \geq -\frac{1}{m}\omega \tag{16}$$

and this combined with (b) shows that the line bundle $\mathcal{O}_X(Y_1)$ admits a sequence of singular metrics $g_m := e^{-\psi_{1,m}}$ such that

$$\varphi_m = \mu^1\psi_{1,m} + \sum_{i\neq 1}\mu^i \log|f_{W_i}|^2, \tag{17}$$

where we assume that $W_1 = Y_1$ (see Lemma 2.12). So the curvature of (L, h_m) is not just bounded from below by $-\frac{1}{m}\omega$, but we also have

$$\Theta_{h_m}(L) \geq \mu^1\Theta_{g_m}(Y_1) \tag{18}$$

as shown by (17) above.

The important remark is that the relations (16) and (18) are very similar to the curvature requirement in the geometric version of the Ohsawa-Takegoshi-type theorem, due to L. Manivel (cf. [Manivel93] and [Demailly09]). In the following section, we will establish the relevant generalization. As for the *tie-breaking* issue (cf. (15)), we are unable to bypass it with purely analytic methods.

4. A version of the Ohsawa-Takegoshi extension theorem

The main building block of the proof of the "invariance of plurigenera" (cf. [Siu98], [Siu00]) is given by the Ohsawa-Takegoshi theorem cf. [OT87] and [Berndtsson96]. In this section, we will prove a version of this important extension theorem, which will play a fundamental role in the proof of Theorem 1.7.

Actually, our result is a slight generalization of the corresponding statements in the articles quoted above, adapted to the set-up described in Remark 3.6. For clarity of exposition, we will change the notations as follows.

Let X be a projective manifold, and let $Y \subset X$ be a non-singular hypersurface. We assume that there exists a metric h_Y on the line bundle $\mathcal{O}_X(Y)$ associated to Y, denoted by $h_Y = e^{-\varphi_Y}$ with respect to any local trivialization, such that:

(i) If we denote by s the tautological section associated to Y, then

$$|s|^2 e^{-\varphi_Y} \leq e^{-\alpha} \tag{19}$$

where $\alpha \geq 1$ is a real number.

(ii) There exist two semi-positively curved hermitian $\mathbb{Q}$-line bundles, say $(G_1, e^{-\varphi_{G_1}})$ and $(G_2, e^{-\varphi_{G_2}})$, such that

$$\varphi_Y = \varphi_{G_1} - \varphi_{G_2} \tag{20}$$

(compare to (17) above).

Let $F \to X$ be a line bundle, endowed with a metric h_F such that the following curvature requirements are satisfied

$$\Theta_{h_F}(F) \geq 0, \quad \Theta_{h_F}(F) \geq \frac{1}{\alpha}\Theta_{h_Y}(Y). \tag{21}$$

Moreover, we assume the existence of real numbers $\delta_0 > 0$ and C such that

$$\varphi_F \leq \delta_0 \varphi_{G_2} + C; \tag{22}$$

that is to say, the poles of the metric which has the "wrong" sign in the decomposition (20) are part of the singularities of h_F. Since φ_{G_2} is locally bounded from above, we can always assume that $\delta_0 \leq 1$.

We denote by $\overline{h}_Y = e^{-\overline{\varphi}_Y}$ a non-singular metric on the line bundle corresponding to Y. We have the following result.

Theorem 4.1. *Let u be a section of the line bundle $\mathcal{O}_Y(K_Y + F|_Y)$, such that*

$$\int_Y |u|^2 e^{-\varphi_F} < \infty, \tag{23}$$

and such that hypotheses (19)–(22) *are satisfied. Then there exists a section U of the line bundle $\mathcal{O}_X(K_X + Y + F)$ with $U|_Y = u \wedge ds$, such that for every $\delta \in (0, 1]$ we have*

$$\int_X |U|^2 e^{-\delta\varphi_Y - (1-\delta)\overline{\varphi}_Y - \varphi_F} \leq C_\delta \int_Y |u|^2 e^{-\varphi_F}, \tag{24}$$

where the constant C_δ is given explicitly by

$$C_\delta = C_0(n)\delta^{-2}\big(\max_X |s|^2 e^{-\overline{\varphi}_Y}\big)^{1-\delta} \tag{25}$$

for some numerical constant $C_0(n)$ depending only on the dimension (*but not on δ_0 or C in* (22)).

Perhaps the closest statement of this kind in the literature is due to D. Varolin, cf. [Var08]; in his article, the metric h_Y is allowed to be singular, but the weights of this metric are assumed to be *bounded from above*. This hypothesis is not verified in our case; however, the assumption (22) plays a similar role in the proof of Theorem 4.1.

Proof. We will closely follow the "classical" arguments and show that the proof goes through (with a few standard modifications) in the more general setting of Theorem 4.1. The main issue which we have to address is the regularization procedure. Although the technique is more or less standard, since this is the key new ingredient, we will provide a complete treatment.

4.1. **Regularization procedure.** Let us first observe that every line bundle B over X can be written as a difference $B = \mathcal{O}_X(H_1 - H_2)$ of two very ample divisors H_1, H_2. It follows that B is trivial upon restriction to the complement $X \setminus (H_1' \cup H_2')$ for any members $H_1' \in |H_1|$ and $H_2' \in |H_2|$ of the corresponding linear systems. Therefore, one can find a finite family $H_j \subset X$ of very ample divisors, such that $Y \not\subset H_j$ and each of the line bundles under consideration F, $\mathcal{O}_X(Y)$ and $G_i^{\otimes N}$ (choosing N divisible enough so that $G_i^{\otimes N} \in \mathrm{Pic}(X)$) is trivial on the affine Zariski open set $X \setminus H$, where $H = \bigcup H_j$. We also fix a proper embedding $X \setminus H \subset \mathbb{C}^m$ in order to regularize the weights φ_F and φ_Y of our metrics on $X \setminus H$. The L^2 estimate will be used afterwards to extend the sections to X itself.

The arguments which follow are first carried out on a fixed affine open set $X \setminus H$ selected as above. In this respect, estimate (22) is then to be understood as valid only with a uniform constant $C = C(\Omega)$ on every relatively compact open subset $\Omega \subset\subset X \setminus H$. In order to regularize all of our weights φ_F and $\varphi_Y = \varphi_{G_1} - \varphi_{G_2}$ respectively, we invoke the following well known result which enables us to employ the usual convolution kernel in Euclidean space.

Theorem 4.2. [Siu76] *Given a Stein submanifold V of a complex analytic space M, there exist an open, Stein neighborhood $W \supset V$ of V, together with a holomorphic retract $r : W \to V$.*

In our setting, the above theorem shows the existence of a Stein open set $W \subset \mathbb{C}^m$, such that $X \setminus H \subset W$, together with a holomorphic retraction $r : W \to X \setminus H$. We use the map r in order to extend the objects we have constructed on $X \setminus H$; we define

$$\widetilde{\varphi}_F := \varphi_F \circ r, \quad \widetilde{\varphi}_{G_i} := \varphi_{G_i} \circ r, \quad \widetilde{\varphi}_Y := \widetilde{\varphi}_{G_1} - \widetilde{\varphi}_{G_2}. \tag{26}$$

Next we will use a standard convolution kernel in order to regularize the functions above. We consider exhaustions

$$W = \bigcup_k W_k, \qquad X \setminus H = \bigcup X_k \tag{27}$$

of W by *bounded* Stein domains (resp. of $X \setminus H$ by the relatively compact Stein open subsets $X_k := (X \setminus H) \cap W_k$). Let

$$\varphi_{F,\varepsilon} := \widetilde{\varphi}_F * \rho_\varepsilon : W_k \to \mathbb{R}$$

be the regularization of $\widetilde{\varphi}_F$, where for each k we assume that $\varepsilon \le \varepsilon(k)$ is small enough, so that the function above is well-defined. We use similar notations for the regularization of the other functions involved in the picture.

We show next that the normalization and curvature properties of the functions above are preserved by the regularization process. In first place, the assumption $\Theta_{h_F}(F) \geq 0$ means that φ_F is psh, hence $\varphi_{F,\varepsilon} \geq \varphi_F$ and we still have

$$(19_\varepsilon) \qquad |s|^2 e^{-\varphi_{Y,\varepsilon}(z)} \leq |s|^2 e^{-\varphi_Y(z)} \leq e^{-\alpha} \leq e^{-1}$$

on X_k. (Here of course $|s|$ means the absolute value of the section s viewed as a complex valued function according to the trivialization of $\mathcal{O}_X(Y)$ on $X \setminus H$). Further, (20)-(21) implies that all functions

$$(21_\varepsilon) \qquad z \to \varphi_{F,\varepsilon}(z), \quad z \to \varphi_{G_i,\varepsilon}(z), \quad z \to \varphi_{F,\varepsilon}(z) - \frac{1}{\alpha}\varphi_{Y,\varepsilon}(z)$$

are psh on X_k, by stability of plurisubharmonicity under convolution. Finally, (22) leads to

$$(22_\varepsilon) \qquad \varphi_{F,\varepsilon} \leq \delta_0 \varphi_{G_2,\varepsilon} + C(k) \quad \text{on } W_k,$$

by linearity and monotonicity of convolution.

In conclusion, the hypothesis of (4.1) are preserved by the particular regularization process we have described here. We show in the following subsection that the "usual" Ohsawa-Takegoshi theorem applied to the regularized weights allows us to conclude.

4.2. **End of the proof of Theorem 4.1.** We view here the section u of $\mathcal{O}_Y(K_Y + F|_Y)$ as a $(n-1)$-form on Y with values in $F|_Y$. Since F is trivial on $X \setminus H$, we can even consider u as a complex valued $(n-1)$-form on $Y \cap (X \setminus H)$. The main result used in the proof of (4.1) is the following technical version of the Ohsawa-Takegoshi theorem.

Theorem 4.3. [Demailly09] *Let M be a weakly pseudoconvex n-dimensional manifold, and let $f : M \to \mathbb{C}$ be a holomorphic function, such that $\partial f \neq 0$ on $f = 0$. Consider two smooth functions φ and ρ on M, such that*

$$\sqrt{-1}\partial\overline{\partial}\varphi \geq 0, \quad \sqrt{-1}\partial\overline{\partial}\varphi \geq \frac{1}{\alpha}\sqrt{-1}\partial\overline{\partial}\rho$$

and such that $|f|_\rho^2 := |f|^2 e^{-\rho} \leq e^{-\alpha}$, where $\alpha \geq 1$ is a constant. Then given a $n-1$ form γ on $M_f := \{f = 0\}$, there exists a n-form Γ on M such that

(a) $\Gamma|_{M_f} = \gamma \wedge df$;

(b) *We have*

$$\int_M \frac{|\Gamma|^2 e^{-\rho-\varphi}}{|f|_\rho^2 \log^2 |f|_\rho^2} \leq C_0(n) \int_{M_f} |\gamma|^2 e^{-\varphi}$$

where $C_0(n)$ is a numerical constant depending only on the dimension.

We apply the above version of the Ohsawa-Takegoshi theorem in our setting: for each k and for each $\varepsilon \leq \varepsilon(k)$ there exists a holomorphic n-form $U_{k,\varepsilon}$ on the Stein manifold X_k, such that

$$\int_{X_k} \frac{|U_{k,\varepsilon}|^2 e^{-\varphi_{Y,\varepsilon}-\varphi_{F,\varepsilon}}}{|s|^2 e^{-\varphi_{Y,\varepsilon}} \log^2\left(|s|^2 e^{-\varphi_{Y,\varepsilon}}\right)} \leq C_0(n) \int_{Y\cap X_k} |u|^2 e^{-\varphi_{F,\varepsilon}} \tag{28}$$

and such that $U_{k,\varepsilon}|_{Y\cap X_k} = u \wedge ds$. Notice that $\varphi_{F,\varepsilon} \geq \widetilde{\varphi}_F = \varphi_F$ on $Y \cap X_k$, hence we get the (ε, k)-uniform upper bound

$$\int_{Y\cap X_k} |u|^2 e^{-\varphi_{F,\varepsilon}} \leq \int_{Y\cap(X\setminus H)} |u|^2 e^{-\varphi_F}. \tag{29}$$

Our next task is to take the limit for $\varepsilon \to 0$ in the relation (28), while keeping k fixed at first. To this end, an important observation is that

$$\int_{X_k} |U_{k,\varepsilon}|^2 e^{-\delta\varphi_{Y,\varepsilon}-(1-\delta)\overline{\varphi}_Y-\varphi_{F,\varepsilon}} \leq C_\delta \int_{Y\cap(X\setminus H)} |u|^2 e^{-\varphi_F} \tag{30}$$

for any $0 < \delta \leq 1$. Indeed, the function $t \to t^\delta \log^2(t)$ is bounded from above by $e^{-2}(2/\delta)^2 \leq \delta^{-2}$ when t belongs to the fixed interval $[0, e^{-\alpha}] \subset [0, e^{-1}]$, so that $t(\log t)^2 \leq \delta^{-2} t^{1-\delta}$, and hence by (19_ε) we have

$$\frac{e^{-\varphi_{Y,\varepsilon}-\varphi_{F,\varepsilon}}}{|s|^2 e^{-\varphi_{Y,\varepsilon}} \log^2\left(|s|^2 e^{-\varphi_{Y,\varepsilon}}\right)} \geq \delta^2 \frac{e^{-\varphi_{Y,\varepsilon}-\varphi_{F,\varepsilon}}}{|s|^{2(1-\delta)} e^{-(1-\delta)\varphi_{Y,\varepsilon}}}. \tag{31}$$

We further observe that by compactness of X the continuous function $z \to |s(z)|^{2(1-\delta)} e^{-(1-\delta)\overline{\varphi}_Y}$ is bounded from above on X by $M^{1-\delta} = \max_X(|s|^2 e^{-\overline{\varphi}_Y})^{1-\delta} < +\infty$ and we can take $C_\delta = C_0(n) M^{1-\delta}\delta^{-2}$. Therefore, (30) follows from (28) and (31). Let us choose $\delta \leq \delta_0$ for the moment. Since $\varphi_Y = \varphi_{G_1} - \varphi_{G_2}$, we see that (22_ε) implies

$$\delta\varphi_{Y,\varepsilon} + (1-\delta)\overline{\varphi}_Y + \varphi_{F,\varepsilon} \leq \delta\varphi_{G_1,\varepsilon} + (\delta_0-\delta)\varphi_{G_2,\varepsilon} + (1-\delta)\overline{\varphi}_Y + C(k). \tag{32}$$

For $i \in \{1, 2\}$ the function $\varphi_{G_i,\varepsilon}$ is psh, thus in particular uniformly bounded from above on X_k by a constant independent of ε. Hence $\delta\varphi_{Y,\varepsilon} + (1-\delta)\overline{\varphi}_Y + \varphi_{F,\varepsilon}$ is uniformly bounded from above by a constant $C_3(k, \delta_0)$ if we fix e.g. $\delta = \delta_0$. By (30) the *unweighted* norm of $U_{k,\varepsilon}$ admits a bound

$$\int_{X_k} |U_{k,\varepsilon}|^2 \leq C_4(k, \delta_0).$$

We stress here the fact that the constant is independent of ε. Therefore we can extract a subsequence that is uniformly convergent on all compact subsets of X_k. Indeed, this follows from the classical Montel theorem, which in turn is a consequence of the Cauchy integral formula to prove equicontinuity (this is where we use the fact that $C_4(k, \delta_0)$ is independent of ε), coupled with the Arzelà-Ascoli theorem. Let U_k

be the corresponding limit. Fatou's lemma shows that we have the estimate

$$\int_{X_k} \frac{|U_k|^2 e^{-\varphi_Y-\varphi_F}}{|s|^2 e^{-\varphi_Y}\log^2(|s|^2 e^{-\varphi_Y})} \le C_0(n)\int_{Y\cap(X\setminus H)} |u|^2 e^{-\varphi_F}. \tag{33}$$

If we now let $k\to+\infty$, we get a convergent subsequence U_k with limit $U=\lim U_k$ on $X\setminus H=\bigcup X_k$, which is uniform on all compact subsets of $X\setminus H$, and such that

$$\int_{X\setminus H} \frac{|U|^2 e^{-\varphi_Y-\varphi_F}}{|s|^2 e^{-\varphi_Y}\log^2(|s|^2 e^{-\varphi_Y})} \le C_0(n)\int_{Y\cap(X\setminus H)} |u|^2 e^{-\varphi_F}. \tag{34}$$

We can reinterpret U as a section of $(K_X+Y+F)|_{X\setminus H}$ satisfying the equality

$$U|_{Y\cap(X\setminus H)} = u\wedge ds. \tag{35}$$

Then the estimate (34) is in fact an intrinsic estimate in terms of the hermitian metrics (i.e. independent of the specific choice of trivialization we have made, especially since H is of measure zero with respect to the L^2 norms). The proof of (30) also shows that

$$\int_{X\setminus H} |U|^2 e^{-\delta\varphi_Y-(1-\delta)\overline{\varphi}_Y-\varphi_F} \le C_\delta\int_{Y\cap(X\setminus H)} |u|^2 e^{-\varphi_F} \tag{36}$$

for every $\delta\in(0,1]$. In a neighborhood of any point $x_0\in H$, the weight $\delta\varphi_Y+(1-\delta)\overline{\varphi}_Y+\varphi_F$ expressed with respect to a local trivialization of F near x_0 is locally bounded from above by (32), if we take $\delta\le\delta_0$. We conclude that U extends holomorphically to X and Theorem 4.1 is proved. □

Remark 4.4. In the absence of hypothesis (22), the uniform bound arguments in the proof of (4.1) collapse. In particular the limit U might acquire poles along H.

The way we are using the parameter δ_0 during the proof above may be a little bit confusing, so we offer here a few more explanations. Indeed, the presence of the denominator in the L^2-norm (28) of the local solutions $U_{k,\varepsilon}$ shows that the unweighted norm of this family of sections is bounded by a constant which *may depend on* δ_0, but it is independent of ε, k. Therefore, the said family is normal so that we can extract a limit. Fatou lemma combined with the original estimate (28) show that the inequality obtained for the limit (34) *is independent of* δ_0. Hence the exact value of δ_0 does not matter quantitatively, even if its existence is of fundamental importance for our arguments.

We also remark that in most of the applications, the denominator in Theorem 4.3 is simply ignored; in our proof, its presence is essential.

5. Proof of Theorem 1.7

In the present section, we will prove Theorem 1.7. Our proof relies heavily on Theorem 4.1. However, in order to better understand the relevance of the technical statements which follow (see Theorem 5.3 below), we first consider a particular case of (1.7).

Theorem 5.1. *Let $\{S, Y_j\}$ be a set of hypersurfaces of a smooth, projective manifold X having normal crossings. Assume also that there exists rational numbers $0 < b^j < 1$ such that $K_X + S + B$ is hermitian semi-positive, where $B = \sum_j b^j Y_j$ and that there exists an effective $\mathbb{Q}$-divisor $D := \sum_j \nu^j W_j$ on X, numerically equivalent to $K_X + S + B$, such that $S \subset \operatorname{Supp}(D) \subset \operatorname{Supp}(S + B)$. Let m_0 be a positive integer, such that $m_0(K_X + S + B)$ is Cartier. Then every section u of the line bundle $\mathcal{O}_S(m_0(K_S + B|_S))$ extends to X.*

Proof. Let $h_0 = e^{-\varphi_0}$ be a smooth metric on $K_X + S + B$, with semi-positive curvature. As in the introductory Subsection 2.3, we have

$$\mathcal{O}_X(N(K_X + S + B)) \cong \mathcal{O}_X(ND) \otimes \rho^N$$

where N is a sufficiently divisible positive integer, and ρ is a line bundle on X, which admits a metric $h_\rho = e^{-\varphi_\rho}$ whose curvature form is equal to zero.

We can assume that

$$\varphi_0 \geq \varphi_\rho + \sum_j \nu^j \log |f_{W_j}|^2 \tag{37}$$

i.e., that h_0 is less singular that the metric induced by the divisor $\sum_j \nu^j W_j$, simply by adding to the local weights of h_0 a sufficiently large constant.

Assume that $S = W_1$. We define a metric φ_S on the line bundle corresponding to S such that the following equality holds

$$\varphi_0 = \nu^1 \varphi_S + \varphi_\rho + \sum_{j \neq 1} \nu^j \log |f_{W_j}|^2.$$

In order to apply Theorem 4.1, we write

$$m_0(K_X + S + B) = K_X + S + B + (m_0 - 1)(K_X + S + B)$$

and we endow the line bundle corresponding to $F := B + (m_0 - 1)(K_X + S + B)$ with the metric $\varphi_F := \varphi_B + (m_0 - 1)\varphi_0$.

Then, we see that the hypothesis of (4.1) are verified, as follows.

- We have $|f_S|^2 e^{-\varphi_S} \leq 1$ by the inequality (37) above.
- We have $\Theta_{h_F}(F) \geq 0$, as well as $\Theta_{h_F}(F) \geq \dfrac{1}{\alpha}\Theta_{h_S}\big(\mathcal{O}_X(S)\big)$, for any $\alpha \geq \dfrac{1}{(m_0 - 1)\nu^1}$.

In order to apply Theorem (4.1), we define

$$\alpha := \max\left\{1, \frac{1}{(m_0-1)\nu^1}\right\}$$

and we rescale the metric h_S by a constant, as follows

$$\varphi_S^\alpha := \varphi_S + \alpha.$$

Then we have $|f_S|^2 e^{-\varphi_S^\alpha} \leq e^{-\alpha}$ thanks to the first bullet above, and moreover the curvature conditions

$$\Theta_{h_F}(F) \geq 0, \quad \Theta_{h_F}(F) \geq \frac{1}{\alpha}\Theta_{h_S^\alpha}\big(\mathcal{O}_X(S)\big)$$

are satisfied.

• The (rescaled) weight φ_S^α can be written as the difference of two psh functions, $\varphi_{G_1} - \varphi_{G_2}$ where $\varphi_{G_1} := \alpha + \delta_0\varphi_0$ and $\varphi_{G_2} := \delta_0\big(\varphi_\rho + \sum_{j\neq 1}\nu^j \log|f_{W_j}|^2\big)$ and $\delta_0 = 1/\nu^1$. We have

$$C + \delta_0\big(\varphi_\rho + \sum_{j\neq 1}\nu^j\log|f_{W_j}|^2\big) \geq \varphi_F$$

by the assumption concerning the support of D.

We also have $\displaystyle\int_S |u|^2 e^{-\varphi_F} < \infty$, since (X, B) is klt and h_0 is non-singular. Therefore, by Theorem 4.1 the section u extends to X. □

Remark 5.2. *The norm of the extension we construct by this procedure will depend only on C_δ computed in Theorem 4.1, the rescaling factor $e^{\delta\alpha}$ where $\alpha := \max\left\{1, \frac{1}{(m_0-1)\nu^1}\right\}$ and $\displaystyle\int_S |u|^2 e^{-\varphi_F}$.*

5.1. **Construction of potentials for adjoint bundles.** As one can see, the hypothesis (3) of (1.7) is much weaker than the corresponding one in (5.1) (i.e. the hermitian semi-positivity of $K_X + S + B$), and this induces many complications in the proof. The aim of Theorem 5.3 below is to construct a substitute for the smooth metric h_0, and it is the main technical tool in the proof of (1.7).

Theorem 5.3. *Let $\{S, Y_j\}$ be smooth hypersurfaces of X with normal crossings. Let $0 < b^j < 1$ be rational numbers, such that:*

1. *We have $K_X + S + \sum_j b^j Y_j \equiv \sum_j \nu^j W_j$, where ν^j are positive rational numbers, and $\{W_j\} \subset \{S, Y_j\}$.*
2. *Let m_0 be a positive integer such that $m_0(K_X + S + \sum_j b^j Y_j)$ is Cartier, and there exists a non-identically zero section u of $\mathcal{O}_S(m_0(K_S + \sum_j b^j Y_j|_S))$.*

(3) *Let h be a non-singular, fixed metric on the $\mathbb{Q}$-line bundle $K_X + S + \sum_j b^j Y_j$, then there exists a sequence $\{\tau_m\}_{m\geq 1} \subset L^1(X)$, such that $\Theta_h(K_X+S+\sum_j b^j Y_j)+\sqrt{-1}\partial\overline{\partial}\tau_m \geq -\frac{1}{m}\omega$ as currents on X, the restriction $\tau_m|_S$ is well defined and we have*

$$\tau_m|_S \geq C(m) + \log|u|^{\frac{2}{m_0}} \tag{38}$$

where $C(m)$ is a constant, which is allowed to depend on m.

Then there exists a constant $C < 0$ independent of m, and a sequence of functions $\{f_m\}_{m\geq 1} \subset L^1(X)$ such that:

(i) *We have $\sup_X f_m = 0$, and moreover $\Theta_h(K_X + S + \sum_j b^j Y_j) + \sqrt{-1}\partial\overline{\partial} f_m \geq -\frac{1}{m}\omega$ as currents on X.*

(ii) *The restriction $f_m|_S$ is well-defined, and we have*

$$f_m|_S \geq C + \log|u|^{\frac{2}{m_0}}. \tag{39}$$

The proof of Theorem 5.3 follows an iteration scheme, that we now explain. We start with the potentials $\{\tau_m\}$ provided by the hypothesis (3) above; then we construct potentials $\{\tau_m^{(1)}\}$ such that the following properties are satisfied.

(a) We have $\sup_X \tau_m^{(1)} = 0$, and moreover

$$\Theta_h(K_X + S + \sum_j b^j Y_j) + \sqrt{-1}\partial\overline{\partial}\tau_m^{(1)} \geq -\frac{1}{m}\omega$$

in the sense of currents on X.

(b) The restriction $\tau_m^{(1)}|_S$ is well-defined, and there exists a constant C independent of m such that

$$\tau_m^{(1)} \geq C + \log|u|^{\frac{2}{m_0}} + \rho \sup_S \tau_m,$$

at each point of S (where $0 < \rho < 1$ is to be determined).

The construction of $\{\tau_m^{(1)}\}$ with the pertinent curvature and uniformity properties (a) and (b) is possible by Theorem 4.1. Then we repeat this procedure: starting with $\{\tau_m^{(1)}\}$ we construct $\{\tau_m^{(2)}\}$, and so on. Thanks to the uniform estimates we provide during this process, the limit of $\{\tau_m^{(p)}\}$ as $p \to \infty$ will satisfy the requirements of (5.3). We now present the details.

Proof. Let $B := \sum_j b^j Y_j$. By hypothesis, there exists $\{\tau_m\} \subset L^1(X)$, such that

$$\max_X \tau_m = 0, \quad \Theta_h(K_X + S + B) + \sqrt{-1}\partial\overline{\partial}\tau_m \geq -\frac{1}{m}\omega \tag{40}$$

on X. We denote by $D = \sum \nu^j W_j$ the $\mathbb{Q}$-divisor provided by hypothesis (1) of (5.3); and let $\tau_D := \log |D|^2_{h\otimes h_\rho^{-1}}$ (by this we mean the norm of the $\mathbb{Q}$-section associated to D, measured with respect to the metric $h \otimes h_\rho^{-1}$, cf. the proof of (5.1)) be the logarithm of its norm. We can certainly assume that $\tau_D \leq 0$. By replacing τ_m by $\max\{\tau_m, \tau_D\}$, the relations (40) above are still satisfied (cf. §2.4) and in addition we can assume that we have

$$\tau_m \geq \tau_D \tag{41}$$

at each point of X.

Two things can happen: either S belongs to the set $\{W_j\}$, or not. In the later case there is nothing to prove as the restriction $\tau_m|_S$ is well defined, so we assume that $S = W_1$. After the normalization indicated above, we define a metric $e^{-\psi_{S,m}}$ on $\mathcal{O}_X(S)$ which will be needed in order to apply (4.1). Let

$$\varphi_{\tau_m} := \varphi_h + \tau_m$$

be the local weight of the metric $e^{-\tau_m}h$ on $K_X + S + B$. The metric $\psi_{S,m}$ is defined so that the following *equality* holds

$$\varphi_{\tau_m} = \nu^1 \psi_{S,m} + \varphi_\rho + \sum_{j\neq 1} \nu^j \log |f_{W_j}|^2 \tag{42}$$

(cf. Lemma 2.12) where f_{W_j} is a local equation for the hypersurface W_j. We use here the hypothesis (1) of Theorem 5.3. Then the functions $\psi_{S,m}$ given by equality (42) above are the local weights of a metric on $\mathcal{O}_X(S)$.

The norm/curvature properties of the objects constructed so far are listed below.

(a) *The inequality $|f_S|^2 e^{-\psi_{S,m}} \leq 1$ holds at each point of X.* Indeed, this is a direct consequence of the relations (41) and (42) above (cf. (2.12)).

(b) *We have $\Theta_{\varphi_{\tau_m}}(K_X + S + B) \geq -\dfrac{1}{m}\omega$.*

(c) *We have $\Theta_{\varphi_{\tau_m}}(K_X + S + B) \geq \nu^1 \Theta_{\psi_{S,m}}(S)$.* This inequality is obtained as a direct consequence of (42), since the curvature of h_ρ is equal to zero.

(d) *For each m there exists a constant $C(m)$ such that*

$$\tau_m|_S \geq C(m) + \log |u|^{\frac{2}{m_0}}.$$

Indeed, for this inequality we use the hypothesis (3) of Theorem 5.3, together with the remark that the renormalization and the maximum we have used to insure (41) preserve this hypothesis.

As already hinted, we will modify each element of the sequence of functions $\{\tau_m\}_{m\geq 1}$ by using some "estimable extensions" of the section u (given by hypothesis 2) and its tensor powers, multiplied by a finite number of auxiliary sections of some ample line bundle. Actually, we will concentrate our efforts on one single index e.g. $m = km_0$, and focus on understanding the uniformity properties of the constants involved in the computations.

In order to simplify the notations, let $\tau := \tau_{km_0}$ and denote by ψ_S the metric on $\mathcal{O}_X(S)$ defined by the equality

$$\varphi_\tau = \nu^1 \psi_S + \varphi_\rho + \sum_{j\neq 1} \nu^j \log |f_{W_j}|^2. \tag{43}$$

Even if this notation does not make it explicit, we stress here the fact that the metric ψ_S depends on the function τ we want to modify.

We consider a non-singular metric $h_S = e^{-\varphi_S}$ on $\mathcal{O}_X(S)$ which is independent of τ, and for each $0 \leq \delta \leq 1$, we define the convex combination metric

$$\psi_S^\delta := \delta\psi_S + (1-\delta)\varphi_S. \tag{44}$$

The parameter δ will be fixed at the end, once we collect all the requirements we need it to satisfy.

We assume that the divisor A is sufficiently ample, so that the metric ω in (b) above is the curvature of the metric h_A on $\mathcal{O}_X(A)$ induced by its global sections say $\{s_{A,i}\}$.

Now we consider the section $u^{\otimes k} \otimes s_A$ of the line bundle

$$\mathcal{O}_S(km_0(K_S + B|_S) + A|_S),$$

where $s_A \in \{s_{A,i}\}$, and we define the set

$$\mathcal{E} := \big\{U \in H^0\big(X, \mathcal{O}_X(km_0(K_X + S + B) + A)\big) : U|_S = u^{\otimes k} \otimes s_A\big\}.$$

A first step towards the proof of Theorem 5.3 is the following statement (see e.g. [BP10]).

Lemma 5.4. *The set $\mathcal{E}$ is non-empty; moreover, there exists an element $U \in \mathcal{E}$ such that the following integral is convergent*

$$\|U\|^{\frac{2}{km_0}(1+\delta)} := \int_X |U|^{\frac{2}{km_0}(1+\delta)} e^{-\delta\varphi_\tau - \psi_S^\delta - \varphi_B} e^{-\frac{1+\delta}{km_0}\varphi_A} < \infty \tag{45}$$

as soon as δ is sufficiently small.

Proof. We write the divisor $km_0(K_X + S + B) + A$ in adjoint form as follows

$$km_0(K_X + S + B) + A = K_X + S + F$$

where

$$F := B + (km_0 - 1)(K_X + S + B) + A.$$

We endow the line bundle $\mathcal{O}_X(F)$ with the metric whose local weights are

$$\varphi_F := \varphi_B + (km_0 - 1)\varphi_\tau + \varphi_A. \tag{46}$$

By property (b) and our choice of A, the curvature of this metric is greater than $\dfrac{1}{km_0}\omega$, and the section $u^{\otimes k} \otimes s_A$ is integrable with respect to it, by property (d).

The classical Ohsawa-Takegoshi theorem shows the existence of a section U corresponding to the divisor $km_0(K_X + S + B) + A$, such that $U|_S = u^{\otimes k} \otimes s_A$, and such that the following integral is convergent

$$\int_X |U|^2 e^{-\varphi_B-(km_0-1)\varphi_\tau-\varphi_A-\varphi_S} < \infty. \tag{47}$$

By the Hölder inequality we obtain

$$\int_X |U|^{\frac{2}{km_0}(1+\delta)} e^{-\delta\varphi_\tau-\psi_S^\delta-\varphi_B} e^{-\frac{1}{km_0}(1+\delta)\varphi_A} \leq CI^{\frac{km_0-1-\delta}{km_0}} \tag{48}$$

where we denote by I the following quantity

$$I := \int_X e^{\varphi_\tau - \frac{km_0}{km_0-1-\delta}\psi_S^\delta - \varphi_B}, \tag{49}$$

and C corresponds to the integral (47) raised to the power $\dfrac{1+\delta}{km_0}$. In the above expression we have skipped a non-singular metric corresponding to φ_S. By relation (43), the integral I will be convergent, provided that

$$\frac{\delta}{\nu^1} \leq \frac{1}{2} \tag{50}$$

so that the lemma is proved. □

We consider next an element $U \in \mathcal{E}$ for which the semi-norm (45) is *minimal*. Let $\{U_p\}_{p\geq 1} \subset \mathcal{E}$ such that the sequence

$$n_p := \int_X |U_p|^{\frac{2}{km_0}(1+\delta)} e^{-\delta\varphi_\tau-\psi_S^\delta-\varphi_B} e^{-\frac{1}{km_0}(1+\delta)\varphi_A}$$

converges towards the infimum say n_∞ of the quantities (45) when $U \in \mathcal{E}$. From this, we infer that $\{U_p\}_{p\geq 0}$ has a convergent subsequence, and obtain our minimizing section as its limit. This can be justified either by Hölder inequality, or by observing that we have

$$\psi_S^\delta + \varphi_B \leq C + \frac{\delta}{\nu^1}\varphi_\tau - \delta\sum_{j\neq 1}\frac{\nu^j}{\nu^1}\log|f_{Y_j}|^2 + \sum_{j\neq 1} b^j \log|f_{Y_j}|^2,$$

where the last term is bounded from above, as soon as δ verifies the inequalities

$$\delta\frac{\nu^j}{\nu^1} \leq b^j \tag{51}$$

for all j.

In conclusion, some element $U_{\min}^{(km_0)} \in \mathcal{E}$ with minimal semi-norm exists, by the usual properties of holomorphic functions, combined with the Fatou Lemma. Next, we will show that the semi-norm of $U_{\min}^{(km_0)}$ is bounded in a very precise way (again, see [BP10] for similar ideas).

To this end, we first construct an extension V of $u^{\otimes k} \otimes s_A$ by using $U_{\min}^{(km_0)}$ *as a metric*; this will be done by using (4.1), so we first write

$$km_0(K_X + S + B) + A = K_X + S + F$$

where

$$F := B + (km_0 - 1 - \delta)(K_X + S + B + \frac{1}{km_0}A) + \delta(K_X + S + B) + \frac{1+\delta}{km_0}A.$$

We endow the line bundle $\mathcal{O}_X(F)$ with the metric whose local weights are

$$\varphi_F := \varphi_B + \frac{km_0 - 1 - \delta}{km_0}\log|U_{\min}^{(km_0)}|^2 + \delta\varphi_\tau + \frac{1+\delta}{km_0}\varphi_A \tag{52}$$

and the line bundle corresponding to S with the metric $e^{-\psi_S}$ previously defined in (43). Prior to applying (4.1) we check here the hypothesis (19)-(22). Thanks to (41) and (43) we have

$$|s|^2 e^{-\psi_S} \leq 1. \tag{53}$$

Moreover, we see (cf. (43)) that we have

$$\psi_S = \frac{1}{\nu^1}\big(\varphi_\tau - \varphi_\rho - \sum_{j\neq 1}\nu^j \log|f_{W_j}|^2\big)$$

hence the requirement (22) amounts to showing that we have

$$C_1(k,\delta) + \sum_{j\neq 1}\frac{\nu^j}{\nu^1}\log|f_{W_j}|^2 \geq C_2(k,\delta)\varphi_F \tag{54}$$

where $C_j(k,\delta)$ are sufficiently large constants, depending (eventually) on the norm of the section $U_{\min}^{(km_0)}$, and on δ. The existence of such quantities is clear, because of the hypothesis (1) of 5.3 and of the presence of φ_B in the expression (52).

The curvature hypothesis required by (4.1) are also satisfied, since we have

$$\Theta_{h_F}(F) \geq 0, \quad \Theta_{h_F}(F) \geq \delta\nu_1\Theta_{h_S}(S) \tag{55}$$

by relations (52) and (43) (the slightly negative part of the Hessian of φ_τ is compensated by the Hessian of $\frac{1}{km_0}\varphi_A$).

Therefore, we are in position to apply Theorem 4.1 and infer the existence of an element $V_\delta \in \mathcal{E}$ such that

(56)
$$\int_X \frac{|V_\delta|^2}{|U_{\min}^{(km_0)}|^{2\frac{km_0-1-\delta}{km_0}}} e^{-\delta\varphi_\tau - \frac{1+\delta}{km_0}\varphi_A - \varphi_B - \psi_S^\delta} \leq C(\delta) \int_S |u|^{\frac{1+\delta}{m_0}} e^{-\varphi_B - \delta\varphi_\tau}.$$

The constant $C(\delta)$ in (56) above is obtained by Theorem 4.1, via the rescaling procedure described in the proof of (5.1) adapted to the current context; we stress on the fact that this constant in completely independent of k and φ_τ.

We will show next that it is possible to choose $\delta := \delta_1$ small enough, independent of k and τ, such that the right hand side of (56) is smaller than $Ce^{-\delta_1 \sup_S(\tau)}$. Here and in what follows we will freely interchange τ and φ_τ, as they differ by a function which only depends on the fixed metric h on $K_X + S + B$; in particular, the difference $\tau - \varphi_\tau$ equals a quantity independent of the family of potentials we are trying to construct.

Prior to this, we recall the following basic result, originally due to L. Hörmander for open sets in $\mathbb{C}^n$, and to G. Tian in the following form.

Lemma 5.5. [Tian87] *Let M be a compact complex manifold, and let α be a real, closed $(1,1)$-form on M. We consider the family of normalized potentials*

$$\mathcal{P} := \left\{ f \in L^1(X) : \sup_M(f) = 0, \quad \alpha + \sqrt{-1}\partial\overline{\partial} f \geq 0 \right\}.$$

Then there exist constants $\gamma_H > 0$ and $C_H > 0$ such that

$$\int_M e^{-\gamma_H f} dV \leq C_H \tag{57}$$

for any $f \in \mathcal{P}$. In addition, the numbers γ_H and C_H are uniform with respect to α.

We will use the previous lemma as follows: first we notice that we have

$$\int_S |u|^{\frac{1+\delta}{m_0}} e^{-\varphi_B - \delta\varphi_\tau} = \int_S |u|^{\frac{1+\delta}{m_0}} e^{-\varphi_B - \delta\varphi_h - \delta\tau}$$

simply because by definition the equality $\varphi_\tau = \varphi_h + \tau$ holds true on X.

The pair $(S, B|_S)$ is klt, hence there exists a positive real number $\mu_0 > 0$ such that $e^{-(1+\mu_0)\varphi_{B|_S}} \in L^1_{\mathrm{loc}}(S)$. By the Hölder inequality, we have

$$\int_S |u|^{\frac{1+\delta}{m_0}} e^{-\varphi_B - \delta\varphi_h - \delta\tau} \leq C \Big(\int_S e^{-\frac{(1+\mu_0)}{\mu_0}\delta\tau} dV \Big)^{\frac{\mu_0}{1+\mu_0}}$$

where dV is a non-singular volume element on S, and the constant C above is independent of τ and of δ, provided that δ belongs to a fixed compact set (which is the case here, since $0 \leq \delta \leq 1$).

Next, in order to obtain an upper bound of the right hand side term of the preceding inequality, by the above lemma, we can write

$$\Big(\int_S e^{-\frac{(1+\mu_0)}{\mu_0}\delta\tau}dV\Big)^{\frac{\mu_0}{1+\mu_0}} = e^{-\delta\sup_S(\tau)}\Big(\int_S e^{-\frac{(1+\mu_0)}{\mu_0}\delta(\tau-\sup_S(\tau))}dV\Big)^{\frac{\mu_0}{1+\mu_0}}.$$

We fix now $\delta := \delta_1$ small enough such that the following conditions are satisfied:

- We have $\dfrac{\delta_1}{\nu^1} < \dfrac{1}{2}$ and $\delta_1\nu^j \leq \nu^1 b^j$ for all $j \neq 1$ (we recall that $S = Y_1$).
- The inequality below holds

$$\delta_1\frac{1+\mu_0}{\mu_0} \leq \gamma_H,$$

where γ_H is given by Lemma 5.5 applied to the following data: $M = S$ and $\alpha := \Theta_h(K_S + B) + \dfrac{1}{km_0}\omega$ (where h here is the restriction of the metric in (3) of (5.3) to S). We notice that γ_T, C_T can be assumed to be uniform with respect to k, precisely because of the uniformity property mentioned at the end of Lemma 5.5.

The conditions we imposed on δ_1 in the two bullets above are independent of the particular potential τ we choose. Hence, in the proof we first fix δ_1 as above, then construct the minimal element $U_{\min}^{(km_0)}$ and after that, we produce an element $V \in \mathcal{E}$ such that

(58)
$$I(V) := \int_X \frac{|V|^2}{|U_{\min}^{(km_0)}|^{2\frac{km_0-1-\delta_1}{km_0}}} e^{-\delta_1\varphi_\tau - \frac{1+\delta_1}{km_0}\varphi_A - \varphi_B - \psi_S^{\delta_1}} \leq Ce^{-\delta_1\sup_S(\tau)},$$

as a consequence of (56), Lemma 5.5 and the above explanations.

On the other hand, we claim that we have

$$\int_X |U_{\min}^{(km_0)}|^{\frac{2}{km_0}(1+\delta_1)} e^{-\delta_1\varphi_\tau - \psi_S^{\delta_1} - \varphi_B} e^{-\frac{1}{km_0}(1+\delta_1)\varphi_A} \leq I(V). \tag{59}$$

Indeed, this is a consequence of Hölder inequality: if relation (59) fails to hold, then it is easy to show that the quantity

$$\int_X |V|^{\frac{2}{km_0}(1+\delta_1)} e^{-\delta_1\varphi_\tau - \psi_S^{\delta_1} - \varphi_B} e^{-\frac{1}{km_0}(1+\delta_1)\varphi_A}$$

is *strictly smaller* than the left hand side of (59)–and this contradicts the the minimality property of the $U_{\min}^{(km_0)}$.

In conclusion, we have the inequality

$$\int_X |U_{\min}^{(km_0)}|^{\frac{2}{km_0}(1+\delta_1)} e^{-\delta_1\varphi_\tau - \psi_S^{\delta_1} - \varphi_B} e^{-\frac{1}{km_0}(1+\delta_1)\varphi_A} \leq Ce^{-\delta_1\sup_S(\tau)}, \tag{60}$$

by combining (58) and (59).

Let $\{s_{A,i}\}_{i=1,\ldots,M}$ be the finite set of sections of A which were fixed at the beginning of the proof. Then we can construct an extension

$U^{(km_0)}_{\min,i}$ of $u^{\otimes k} \otimes s_{A,i}$, with bounded $L^{\frac{2}{km_0}}$ norm as in (60). We define the function

$$\widetilde{\tau} := \frac{1}{km_0} \log \sum_i |U^{(km_0)}_{\min,i}|^2_{h^{\otimes km_0} \otimes h_A} \tag{61}$$

and we observe that we have

$$\int_X e^{(1+\delta_1)\widetilde{\tau}} dV \leq C e^{-\delta_1 \sup_S(\tau)}, \tag{62}$$

as a consequence of (60), since the function τ (or if one prefers, φ_τ) is *negative*, and the part of $\psi_S^{\delta_1}$ having the "wrong sign" is absorbed by φ_B.

Next, as a consequence of the mean inequality for psh functions, together with the compactness of X (see Lemma (2.13)) we infer from (62) that

$$\widetilde{\tau}(x) \leq C - \frac{\delta_1}{1+\delta_1} \sup_S(\tau) \tag{63}$$

for any $x \in X$. Again, the constant "C" has changed since (62), but in a manner which is universal, i.e. independent of τ.

We also remark that the restriction to S of the functions $\widetilde{\tau}$ we have constructed is completely determined by

$$\widetilde{\tau}|_S = \log |u|_h^{\frac{2}{m_0}}.$$

In order to re-start the same procedure, we introduce the functions

$$\tau^{(1)} := \widetilde{\tau} - \sup_X(\widetilde{\tau}) \tag{64}$$

and we collect their properties below:

(N) $\sup_X \tau^{(1)} = 0$;

(H) $\Theta_h(K_X + S + B) + \sqrt{-1}\partial\overline{\partial}\tau^{(1)} \geq -\dfrac{1}{km_0}\omega$ as currents on X;

(R) $\tau^{(1)}|_S \geq \dfrac{\delta_1}{1+\delta_1} \sup_S \tau - C + \dfrac{1}{m_0} \log |u|^2_h$; in particular, we have

$$\sup_S \tau^{(1)} \geq \frac{\delta_1}{1+\delta_1} \sup_S \tau - C. \tag{65}$$

In order to see that the last inequality holds, we fix *any* point $x_0 \in S$, such that u is does not vanish at this point. After possibly rescaling u, we can assume that $\dfrac{1}{m_0} \log |u(x_0)|^2_h = 0$ and then we have

$$\tau^{(1)}(x_0) \geq \frac{\delta_1}{1+\delta_1} \sup_S \tau - C + \frac{1}{m_0} \log |u(x_0)|^2_h.$$

We certainly have $\sup_S \tau^{(1)} \geq \tau^{(1)}(x_0)$; hence (65) follows.

We remark next that the restriction properties of $\tau^{(1)}$ have improved: modulo the function

$$x \to -C + \frac{1}{m_0} \log |u(x)|_h^2$$

which is independent of τ, we "gain" a factor $\dfrac{\delta_1}{1+\delta_1} < 1$.

Remark 5.6. *It is of paramount importance to realize that the constant "C" appearing in the expression of the function above is universal: it only depends on the geometry of (X, S), the metric h and the norm of u, and also on the form which compensate the negativity of the Hessian of τ (cf. (56), Tian's lemma 5.5 and the comments after that). Moreover, if these quantities are varying in a uniform manner (as is the case for our initial sequence $\{\tau_m\}$), then the constant C can be assumed to be independent of m.*

Therefore, it is natural to repeat this procedure with the sequence of functions $\tau^{(1)}$ as input, so that we successively produce the potentials $\{\tau^{(p)}\}$ with the properties (N), (H) and (R) as above, for each $p \geq 1$. We again stress the fact that the quantities C, δ_1 have two crucial uniformity properties: with respect to p (the number of iterations) and to m (the index of the sequence in 5.3).

Proceeding by induction we show that the following two equations hold:

$$\tau^{(p)}|_S \geq \Big(\frac{\delta_1}{1+\delta_1}\Big)^p \sup_S \tau - C\delta_1\Big(1 - \Big(\frac{\delta_1}{1+\delta_1}\Big)^{p-1}\Big) - C + \log |u|_h^{\frac{2}{m_0}}, \tag{66}$$

$$\sup_S \tau^{(p)} \geq \Big(\frac{\delta_1}{1+\delta_1}\Big)^p \sup_S \tau - C(1+\delta_1)\Big(1 - \Big(\frac{\delta_1}{1+\delta_1}\Big)^p\Big). \tag{67}$$

The relation (67) is obtained by successive applications of (65) of the relation (R), since we have

$$\sup_S \tau^{(p)} \geq \frac{\delta_1}{1+\delta_1} \sup_S \tau^{(p-1)} - C$$

for any $p \geq 1$. The lower bound (66) is derived as a direct consequence of (67) and (65).

The functions required by Theorem 5.3 are obtained by standard facts of pluripotential theory (see e.g. [Lelong69], [Lelong71], [Klimek91]): some subsequence $\{\tau_{km_0}^{(p_\nu)}\}$ of $\{\tau_{km_0}^{(p)}\}$ will converge in L^1 to the potential f_{km_0}, as upper regularized limits

$$f_{km_0}(z) := \limsup_{x \to z} \lim_{\nu \to \infty} \tau_{km_0}^{(p_\nu)}(x).$$

By letting $p \to \infty$, we see that the *non-effective* part of the estimate (66) tends to zero, so Theorem 5.3 is proved.

5.2. **Proof of the Extension Theorem.** In this last part of the proof of (1.7), we first recall that by the techniques originating in Siu's seminal article [Siu00], for any fixed section $s_A \in H^0(X, \mathcal{O}_X(A))$, the section $u^{\otimes k} \otimes s_A$ extends to X, for any $k \geq 1$, (cf. [HM10, 6.3]). Indeed, by assumption we have

$$Z_{\pi^\star(u)} + m_0 \widetilde{E}|_{\widetilde{S}} \geq m_0 \Xi$$

(cf. notations in the introduction) and then the extension of $u^{\otimes k} \otimes s_A$ follows.

These extensions provide us with the potentials $\{\tau_m\}_{m\geq 1}$ (simply by letting $\tau_m = \frac{1}{m}\log|U_k|^2$ where U_k is the given extension of $u^{\otimes k} \otimes s_A$) and then Theorem 5.3 converts this into a quantitative statement: there exists another family of potentials $\{f_m\}_{m\geq 1}$ such that

$$\max_X f_m = 0, \quad \Theta_h(K_X + S + B) + \sqrt{-1}\partial\overline{\partial} f_m \geq -\frac{1}{m}\omega \tag{68}$$

together with

$$f_m|_S \geq C + \log|u|^{\frac{2}{m_0}}$$

where C is a constant independent of m. Under these circumstances, we invoke the same arguments as at the end of the preceding paragraph to infer that some subsequence $\{f_{m_\nu}\}$ of $\{f_m\}$ will converge in L^1 to the potential f_∞, as an upper regularized limit

$$f_\infty(z) = \limsup_{x\to z} \lim_{\nu\to\infty} f_{m_\nu}(x)$$

for every $z \in X$.

The properties of the limit f_∞ are listed below:

$$f_\infty|_S \geq C + \log|u|^{\frac{2}{m_0}}, \quad \Theta_h(K_X + S + B) + \sqrt{-1}\partial\overline{\partial} f_\infty \geq 0. \tag{69}$$

We remark at this point that the metric $e^{-f_\infty}h$ constructed here plays in the proof of Theorem 5.3 the same role as the metric h_0 in the arguments we have provided for (5.1).

The rest of the proof is routine: we write

$$m_0(K_X + S + B) = K_X + S + F$$

where we use the following notation

$$F := B + (m_0 - 1)(K_X + S + B). \tag{70}$$

We endow the line bundles $\mathcal{O}_X(F)$ and $\mathcal{O}_X(S)$ respectively with the metrics

$$\varphi_F := \varphi_B + (m_0 - 1)\varphi_\infty$$

and

$$\varphi_\infty = \nu^1 \varphi_S + \varphi_\rho + \sum_{j\neq 1} \nu^j \log|f_{W_j}|^2$$

where $h_\infty = e^{-\varphi_\infty}$ is the metric given by $e^{-\widetilde{f}_\infty}h$; here we denote

$$\widetilde{f}_\infty := \max(f_\infty, \tau_D)$$

so that (as usual) we assume that

$$\varphi_\infty \geq \varphi_\rho + \sum_j \nu^j \log |f_{W_j}|^2 \tag{71}$$

and $S = Y_1$.

Then the requirements of (4.1) are easily checked, as follows:

• We have $|s|^2 e^{-\varphi_S} \leq 1$ by relation (71) above, and moreover we have the equality

$$\varphi_S = \frac{1}{\nu_1}\big(\varphi_\infty - \varphi_\rho - \sum_{j\neq 1} \nu^j \log |f_{W_j}|^2\big)$$

from which one can determine the hermitian bundles $(G_i, e^{-\varphi_{G_i}})$.

• There exists $\delta_0 > 0$ and C such that

$$\varphi_F \leq \delta_0 (\sum_{j\neq 1} \nu^j \log |f_{W_j}|^2) + C$$

because of the presence of the term φ_B in the expression of φ_F; hence (22) is satisfied.

• $\Theta_{h_F}(F) \geq 0$ by property (69), and for $\alpha > 1/\nu^1$ we have

$$\Theta_{h_F}(F) - \frac{1}{\alpha}\Theta_{h_S}(S) \geq \Big(\frac{\alpha\nu^1 - 1}{\alpha\nu^1}\Big)\Theta_{h_F}(F), \tag{72}$$

and we remark that the right hand side curvature term is greater than 0.

• We have

$$\int_S |u|^2 e^{-\varphi_B - (m_0 - 1)\varphi_\infty} \leq C$$

by relation (69). Indeed, as a consequence of (69) we have

$$|u|^2 e^{-\varphi_B - (m_0-1)\varphi_\infty} \leq |u|^{\frac{2}{m_0}} e^{-\varphi_B}$$

so the convergence of the preceding integral is due to the fact that $(S, B|_S)$ is klt.

Therefore, we can apply Theorem (4.1) and obtain an extension of u. Theorem 1.7 is proved. □

Remark 5.7. In fact, the metric (52) of the line bundle $\mathcal{O}_X(F)$ has strictly positive curvature, but the amount of positivity this metric has is $\frac{1}{km_0}\omega$, and the estimates for the extension we obtain under these circumstances are not useful, in the sense that the constant $C(\delta)$ in (56) becomes something like Ck^2.

6. Further consequences, I

In this section we derive a few results which are related to Theorem 1.7. Up to a few details (which we will highlight), their proof is similar to that of (1.7), so our presentation will be brief.

We first remark that the arguments in Sections 4 and 5 have the following consequence.

Theorem 6.1. *Let $\{S, Y_j\}$ be a finite set of hypersurfaces having normal crossings. Let $B = \sum b^j Y_j$ where $0 < b^j < 1$ is a set of rational numbers, such that:*

(i) *The bundle $K_X + S + B$ is pseudo-effective, and $S \notin N_\sigma(K_X + S + B)$.*
(ii) *We have $K_X + S + B \equiv \sum_j \nu^j W_j$, where $\nu^j > 0$ and $S \subset \mathrm{Supp}(\sum \nu^j W_j) \subset \mathrm{Supp}(S + B)$.*

Then $K_X + S + B$ admits a metric $h = e^{-\phi}$ with positive curvature and well-defined restriction to S.

As one can see, the only modification we have to operate for the proof of (1.7) is to replace the family of sections $u^{\otimes k} \otimes s_A$ with a family of sections approximating a closed positive current on S, whose existence is insured by the hypothesis (i).

The next statement of this section is an $\mathbb{R}$-version of (1.7).

Theorem 6.2. *Let $\{S, Y_j\}$ be a finite set of hypersurfaces having normal crossings. Let $0 < b^j < 1$ be a set of real numbers. Consider the $\mathbb{R}$-divisor $B := \sum_j b^j Y_j$, and assume that the following properties are satisfied.*

(a) *The $\mathbb{R}$-bundle $K_X + S + B$ is pseudo-effective, and $S \notin N_\sigma(K_X + S + B)$.*
(b) *There exists an effective $\mathbb{R}$-divisor $\sum_j \nu^j W_j$, numerically equivalent with $K_X + S + B$, such that $S \subset \{W_j\} \subset \mathrm{Supp}(S + B)$.*
(c) *The bundle $K_S + B|_S$ is $\mathbb{R}$-linearly equivalent to an effective divisor say $D := \sum_j \mu^j Z_j$, such that $\pi^\star(D) + \widetilde{E}|_{\widetilde{S}} \geq \Xi$ (we use here the notations and conventions of (1.7)).*

Then $K_X + S + B$ is $\mathbb{R}$-linearly equivalent to an effective divisor whose support do not contain S.

Proof. By a Diophantine approximation argument presented in detail in [HM10] (see also [Paun08, 1.G], [CL10]), we deduce the following fact. For any $\eta > 0$, there exists rational numbers b^j_η, ν^j_η and μ^j_η such that the following relations are satisfied.

- We have $K_X + S + B_\eta \equiv \sum_j \nu^j_\eta W_j$, where $B_\eta := \sum_j b^j_\eta Y_j$;
- The bundle $K_S + B_\eta|_S$ is $\mathbb{Q}$-linearly equivalent to $\sum_j \mu^j_\eta Z_j$;

- Let q_η be the common denominator of b^j_η, ν^j_η and μ^j_η; then we have

$$q_\eta\|b_\eta - b\| < \eta, \quad q_\eta\|\nu_\eta - \nu\| < \eta, \quad q_\eta\|\mu_\eta - \mu\| < \eta. \tag{73}$$

Let $u_\eta \in H^0\big(S, \mathcal{O}_S(q_\eta(K_S + B_\eta|_S))\big)$ be the section associated to the divisor $\sum_j \mu^j_\eta Z_j$. We invoke again the extension theorems in [HM10] (see also [Paun08, 1.H, 1.G]): as a consequence, the section

$$u_\eta^{\otimes k_\eta} \otimes s_{A,i}^{q_\eta}$$

of the bundle $k_\eta q_\eta(K_S+B_\eta)+q_\eta A$ extends to X, where k_η is a sequence of integers such that $k_\eta \to \infty$ as $\eta \to 0$.

We use the corresponding extensions $\{U_i^{(k_\eta,q_\eta)}\}_{i=1,\dots,M_\eta}$ in order to define a metric h_η on

$$K_X + S + B_\eta + \frac{1}{k_\eta}A,$$

with semi-positive curvature current, and whose restriction to S is equivalent with $\log|u_\eta|^{\frac{2}{q_\eta}}$.
The proof of Theorem 5.3 shows that for each $\eta > 0$, there exists a function $f_\eta \in L^1(X)$ such that

(1_η) We have $\max_X f_\eta = 0$, as well as $\Theta_h(K_X+S+B)+\sqrt{-1}\partial\overline{\partial} f_\eta \geq -\dfrac{2}{k_\eta}\omega$.

(2_η) The restriction $f_\eta|_S$ is well-defined, and we have

$$f_\eta|_S \geq C + \log|u_\eta|^{\frac{2}{q_\eta}}. \tag{74}$$

Passing to a subsequence we may assume that there is a limit of f_η; hence we infer the existence of a function f_∞, such that $\Theta_h(K_X + S + B) + \sqrt{-1}\partial\overline{\partial} f_\infty \geq 0$, and such that

$$f_\infty|_S \geq C + \log\Big(\prod_j |f_{Z_j}|^{2\mu^j}\Big). \tag{75}$$

We will next use the metric $h_\infty := e^{-f_\infty}h$ in order to extend the section u_η above, as soon as η is small enough. We write

$$q_\eta(K_X+S+B_\eta) = K_X+S+B_\eta+(q_\eta-1)(K_X+S+B)+(q_\eta-1)(B_\eta-B). \tag{76}$$

Consider a metric on $F_\eta := B_\eta+(q_\eta-1)(K_X+S+B)+(q_\eta-1)(B_\eta-B)$ given by the following expression

$$\sum_j \big(q_\eta b^j_\eta - (q_\eta-1)b^j\big)\log|f_{Y_j}|^2 + (q_\eta-1)\varphi_{f_\infty}. \tag{77}$$

In the expression above, we denote by φ_{f_∞} the local weight of the metric h_∞. By the maximum procedure used e.g. at the end of the proof of (5.3), we can assume that

$$\varphi_{f_\infty} \geq \log\Big(\prod_j |f_{W_j}|^{2\nu^j}\Big). \tag{78}$$

The metric in (77) has positive curvature, and one can easily check that the other curvature hypothesis are also verified (here we assume that $\eta \ll 1$, to insure the positivity of $q_\eta b^j_\eta - (q_\eta - 1)b^j$ for each j). The integrability requirement (23) is satisfied, since we have

$$\int_S \frac{\prod_j |f_{Z_j}|^{2q_\eta \mu^j_\eta}}{\prod_j |f_{Y_j}|^{2q_\eta b^j_\eta - 2(q_\eta-1)b^j} \prod_j |f_{Z_j}|^{2(q_\eta-1)\mu^j}} < \infty \tag{79}$$

for all $\eta \ll 1$, by the Dirichlet conditions at the beginning of the proof. Hence each section u_η extends to X, and the proof of (6.2) is finished by the usual convexity argument, namely $K_X + S + B$ belongs to the convex envelope of the effective bundles $(K_X + S + B_\eta)$. □

Our last statement concerns a version of (1.7) whose hypothesis are more analytic; the proof is obtained *mutatis mutandis*.

Theorem 6.3. *Let $\{S, Y_j\}$ be a finite set of hypersurfaces having normal crossings. Let $0 < b^j < 1$ be a set of rational numbers. Consider the $\mathbb{Q}$-divisor $B := \sum_j b^j Y_j$, and assume that the following properties are satisfied.*

(a) *The bundle $K_X + S + B$ is pseudo-effective, and $S \not\in N_\sigma(K_X + S + B)$.*

(b) *There exists a closed positive current $T \in \{K_X + S + B\}$ such that*

  (b.1) *We have $T = \nu^1[S] + \Lambda_T$, with $\nu_1 > 0$ and Λ_T is positive.*

  (b.2) *The following inequality holds*

$$\delta_0 \varphi_{\Lambda_T} \geq \varphi_B - C \tag{80}$$

  where δ_0 and C are positive real numbers.

(c) *The bundle $m_0(K_S + B)$ has a section u, whose zero divisor D satisfies the relation $\pi^\star(D) + \widetilde{E}|_{\widetilde{S}} \geq \Xi$ (we use here the notations and conventions in* (1.7)*).*

Then the section u extends to X; in particular, the bundle $K_X + S + B$ is $\mathbb{Q}$-effective, and moreover it has a section non-vanishing identically on S.

We remark here that in the proof of statement (6.3) we are using the full force of Theorem 4.1. The hypothesis above corresponds to the fact that $\{W_j\} \subset \{S, Y_j\}$ in (1.7).

7. Proof of Theorem 1.4

We begin by proving (1.8):

Proof of (1.8). Let $f : X' \to X$ be a log resolution of $(X, S+B)$ and write $K_{X'} + S' + B' = f^*(K_X + S + B) + E$ where S' is the strict transform of S, B' and E are effective $\mathbb{Q}$-divisors with no common components. Then $(X', S' + B' + \epsilon E)$ is also a log smooth plt pair for some rational number $0 < \epsilon \ll 1$. We may also assume that the components of B' are disjoint.

Since $K_X + S + B$ is pseudo-effective, so is $K_{X'} + S' + B' + \epsilon E$. Since $K_X + S + B$ is nef, $N_\sigma(K_X + S + B) = 0$ and so

$$N_\sigma(K_{X'} + S' + B' + \epsilon E) = N_\sigma(f^*(K_X + S + B) + (1+\epsilon)E) = (1+\epsilon)E.$$

In particular S' is not contained in $N_\sigma(K_{X'} + S' + B' + \epsilon E)$ and $N_\sigma(||K_{X'} + S' + B' + \epsilon E||_{S'}) = (1+\epsilon)E|_{S'}$ so that

$$\Xi = (B' + \epsilon E)|_{S'} \wedge (1+\epsilon)E|_{S'} = \epsilon E|_{S'}.$$

Since there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} K_X + S + B$ such that $S \subset \operatorname{Supp}(D) \subset \operatorname{Supp}(S+B)$, then $D' = f^*D + (1+\epsilon)E \sim_{\mathbb{Q}} K_{X'} + S' + B' + \epsilon E$ is an effective $\mathbb{Q}$-divisor such that $S' \subset \operatorname{Supp}(D') \subset \operatorname{Supp}(S' + B' + \epsilon E)$. By (1.7),

$$|m(K_{X'} + S' + B' + \epsilon E)|_{S'} \supset |m(K_{S'} + B'|_{S'})| + m\epsilon E$$

for any $m > 0$ sufficiently divisible. Let $\sigma \in H^0(S, \mathcal{O}_S(m(K_S + B_S)))$ and $\sigma' \in H^0(S', \mathcal{O}_{S'}(m(K_{S'} + B'|_{S'} + \epsilon E|_{S'})))$ be the corresponding section. By what we have seen above, this section lifts to a section $\tilde{\sigma}' \in H^0(X', \mathcal{O}_{X'}(m(K_{X'} + S' + B' + \epsilon E)))$. Let $\tilde{\sigma} = f_*\tilde{\sigma}' \in H^0(X, \mathcal{O}_X(m(K_X + S + B)))$, then $\tilde{\sigma}|_S = \sigma$. □

Theorem 1.4 is an immediate consequence of the following:

Theorem 7.1. *Assume* $(1.3)_n$ *and assume* $(1.2)_{n-1}$ *for semi-dlt pairs. Let* (X, Δ) *be an* n*-dimensional klt pair such that* $\kappa(K_X + \Delta) \geq 0$ *then* (X, Δ) *has a good minimal model.*

Proof. We proceed by induction on the dimension. In particular we may also assume that $(1.1)_{n-1}$ holds.

If $\kappa(K_X + \Delta) = \dim X$, then (X, Δ) has a good minimal model by [BCHM10].

If $0 < \kappa(K_X + \Delta) < \dim X$, then (X, Δ) has a good minimal model by [Lai10].

We may therefore assume that $\kappa(K_X + \Delta) = 0$. We write $K_X + \Delta \sim_{\mathbb{Q}} D \geq 0$. Passing to a resolution, we may assume that $(X, \Delta + D)$ is log smooth. We will need the following.

Lemma 7.2. *If* $D = \sum_{i \in I} d_i D_i$*, then it suffices to show that* (X, Δ') *has a good minimal model where* Δ' *is any* $\mathbb{Q}$*-divisor of the form* $\Delta' = \Delta + \sum g_i D_i$ *such that* $g_i \geq 0$ *are positive rational numbers and either*

(1) *(X, Δ') is klt, or*
(2) *(X, Δ') is dlt and $g_i > 0$ for all $i \in I$.*

Proof. If $g_i > 0$ for all $i \in I$, then for any rational number $0 < \epsilon \ll 1$, we have $(1-\epsilon)(K_X + \Delta') \sim_{\mathbb{Q}} K_X + \Delta' - \epsilon(D + \sum g_i D_i)$ where

$$\Delta \leq \Delta'' := \Delta' - \epsilon(D + \sum g_i D_i) \leq \Delta'$$

and (X, Δ'') is klt. Since $K_X + \Delta'' \sim_{\mathbb{Q}} (1-\epsilon)(K_X + \Delta')$, then (X, Δ') has a good minimal model if and only if (X, Δ'') has a good minimal model. Replacing Δ' by Δ'', we may therefore assume that (X, Δ') is klt.

Note that

$$K_X + \Delta \leq K_X + \Delta' \leq K_X + \Delta + gD \sim_{\mathbb{Q}} (1+g)(K_X + \Delta)$$

for some rational number $g > 0$. Therefore, $\kappa(K_X + \Delta') = 0$. In particular,

$$\mathbf{Fix}(K_X + \Delta') = \mathrm{Supp}(D + \Delta' - \Delta) = \mathrm{Supp}(D).$$

Suppose that (X, Δ') has a good minimal model $\phi : X \dashrightarrow X'$. Passing to a resolution, we may assume that ϕ is a morphism and that

$$\mathrm{Supp}(D) = \mathbf{Fix}(K_X + \Delta') = \mathrm{Exc}(\phi)$$

where **Fix** denotes the support of the divisors contained in the stable base locus.

We now run a $K_X + \Delta$-minimal model program with scaling over X'. By [BCHM10] (cf. (2.4) and (2.5)), this minimal model program terminates. Therefore, we may assume that $\mathbf{Fix}(K_X + \Delta/X') = 0$. Since D is exceptional over X', if $D \neq 0$, then by [BCHM10, 3.6.2.1], there is a component F of D which is covered by curves Σ such that $D \cdot \Sigma < 0$. This implies that $\mathbf{Fix}(K_X + \Delta/X') = \mathbf{Fix}(D/X')$ is non-empty; a contradiction as above. Therefore, $D = 0$ and hence $K_X + \Delta \sim_{\mathbb{Q}} 0$. ☐

We let $S = \sum S_i$ be the support of D and we let $S + B = S \vee \Delta$ (i.e. $\mathrm{mult}_P(S+B) = \max\{\mathrm{mult}_P(S), \mathrm{mult}_P(\Delta)\}$) and $G = D + S + B - \Delta$ so that $K_X + S + B \sim_{\mathbb{Q}} G \geq 0$ and $\mathrm{Supp}(G) = \mathrm{Supp}(S)$. By (7.2), it suffices to show that $(X, S+B)$ has a good minimal model. We now run a minimal model program with scaling of a sufficiently ample divisor. By $(1.6)_{n-1}$ and $(2.7)_n$, this minimal model terminates giving a birational contraction $\phi : X \dashrightarrow X'$ such $(X', S' + B' := \phi_*(S+B))$ is dlt and $K_{X'} + S' + B'$ is nef.

If $S' = 0$, then $K_{X'} + S' + B' \sim_{\mathbb{Q}} 0$ and we are done by (7.2). Therefore, we may assume that $S' \neq 0$. By $(1.2)_{n-1}$, we have $H^0(S', \mathcal{O}_{S'}(m(K_{S'} + B'_{S'}))) \neq 0$ for all sufficiently divisible integers $m > 0$. By $(1.3)_n$, the sections of $H^0(S', \mathcal{O}_{S'}(m(K_{S'} + B'_{S'})))$ extend to $H^0(X', \mathcal{O}_{X'}(m(K_{X'} + S' + B'))) = H^0(X', \mathcal{O}_{X'}(mG'))$ and hence $S' \not\subset \mathrm{Bs}(G')$ contradicting the fact that $\kappa(G') = 0$.

□

8. Further remarks

The goal of this section is to show that, one can reduce (1.2) to the following weaker conjecture:

Conjecture 8.1. *Let X be a smooth projective variety. If K_X is pseudo-effective then $\kappa(K_X) \geq 0$.*

We will need the following results.

Theorem 8.2 ([HMX12, 1.5])**.** *Let $d \in \mathbb{N}$ and $I \subset [0,1]$ be a set satisfying the DCC. Then there is a finite subset $I_0 \subset I$ such that if*

1. *X is a projective variety of dimension d,*
2. *(X, Δ) is a log canonical pair,*
3. *$\Delta = \sum \delta_i \Delta_i$ where $\delta_i \in I$,*
4. *$K_X + \Delta \equiv 0$,*

then $\delta_i \in I_0$.

Remark 8.3. *Recall that a set satisfies the DCC (i.e. the descending chain condition) if any non increasing sub-sequence is eventually constant.*

Theorem 8.4 (ACC for LCTs, [HMX12, 1.1])**.** *Let $d \in \mathbb{N}$, $I \subset [0,1]$ and $J \subset \mathbb{R}_{\geq 0}$ such that I and J satisfy the DCC. Then the set*

$$\{\mathrm{lct}(D, X, \Delta) |\ (X, \Delta) \text{ is lc},\ \dim X = d,\ \Delta \in I,\ D \in J\}$$

satisfies the ACC. Here D is $\mathbb{R}$-Cartier and $\Delta \in I$ (resp. $D \in J$) means $\Delta = \sum \delta_i \Delta_i$ where $\delta_i \in I$ (resp. $D = \sum d_i D_i$ where $d_i \in J$) and $\mathrm{lct}(D, X, \Delta) = \sup\{t \geq 0 | (X, \Delta + tD) \text{ is lc}\}$.

Remark 8.5. *Following* [Birkar07] *it seems likely that* (1.2) *in dimension n and* (8.4) *in dimension $n-1$ imply the termination of flips for any pseudo-effective n-dimensional lc pair.*

Definition 8.6. *Let (X, Δ) be a projective klt pair and G an effective $\mathbb{Q}$-Cartier divisor such that $K_X + \Delta + tG$ is pseudo-effective for some $t \gg 0$.*

Then the pseudo-effective threshold $\tau = \tau(X, \Delta; G)$ is given by

$$\tau = \inf\{t \geq 0 | K_X + \Delta + tG \text{ is pseudo} - \text{effective}\}.$$

Proposition 8.7. *If $\tau = \tau(X, \Delta; G)$ is the pseudo-effective threshold, then τ is rational.*

Proof. We may assume that $\tau = \tau(X, \Delta; G) > 0$. Fix an ample divisor A on X and for any $0 \leq x \leq \tau$ let $y = y(x) = \tau(X, \Delta + xG; A)$. Then $y(x)$ is a continuous function such that $y(\tau) = 0$ and $y(x) \in \mathbb{Q}$ for all rational numbers $0 \leq x < \tau$; moreover, for all $0 \leq x < \tau$, the $K_X + \Delta + xG$ minimal model program with scaling ends with a

$K_X+\Delta+xG+yA$-trivial Mori fiber space $g\circ f : X \dashrightarrow Y_x \to Z_x$ (cf. [BCHM10] or (2.4)). Let $F = F_x$ be the general fiber of g, then

$$K_F + \Delta_F + xG_F + yA_F := (K_{Y_x} + f_*(\Delta + xG + yA))|_F \equiv 0$$

and $K_F+\Delta_F+\tau G_F$ is pseudo-effective. By (8.4), we may assume that $K_F+\Delta_F+\tau G_F$ is log canonical. If this is not the case, since $K_F+\Delta_F+xG_F$ is log canonical, then we would in fact have an increasing sequence $\tau_i = \mathrm{lct}(G_F, F, \Delta_F)$ with limit τ which would contradict (8.4). Therefore $K_F+\Delta_F+\eta G_F$ is log canonical and numerically trivial for some rational number $x < \eta = \eta(x) \le \tau$. By (8.2), we may assume that $\eta = \eta(x)$ is constant and hence $\eta = \tau$. In particular $\tau \in \mathbb{Q}$. □

Theorem 8.8. *Assume* $(1.1)_{n-1}$ *and* $(8.1)_n$. *Let* (X,Δ) *be an* n*-dimensional klt pair such that* $K_X+\Delta$ *is pseudo-effective. Then* $\kappa(K_X+\Delta) \ge 0$.

Proof. We may assume that K_X is not pseudo-effective and $\Delta \ne 0$. Replacing X by a birational model, we may assume that (X,Δ) is log smooth. Let $\tau = \tau(X, 0; \Delta)$. Note that $\tau > 0$. By (8.7) and its proof, there is a birational contraction $f : X \dashrightarrow Y$, a rational number $\tau > 0$ and a $K_Y + \tau f_*\Delta$-trivial Mori fiber space $Y \to Z$. (Here, for $0 < \tau - x \ll 1$, we have denoted f_x by f, Y_x by Y and Z_x by Z.)

Assume that $\dim X > \dim Z > 0$. After possibly replacing X by a resolution, we may assume that $f : X \to Y$ is a morphism. Let C be a general complete intersection curve on F the general fiber of $Y \to Z$. We may assume that $C \cap f(\mathrm{Exc}(f)) = \emptyset$ and so we have an isomorphism $C' = f^{-1}(C) \to C$. Thus

$$(K_X + \tau\Delta)\cdot C' = (K_Y + \tau f_*\Delta)\cdot C = 0.$$

In particular $K_X+\tau\Delta$ is not big over Z. Since $\dim(X/Z) < \dim X$, by $(1.1)_{n-1}$, we may assume that the general fiber of $(X, \tau\Delta)$ has a good minimal model over Z. By [Lai10], we have a good minimal model $h : X \dashrightarrow W$ for (X,Δ) over Z. Let

$$r : W \to V := \mathrm{Proj}_Z R(K_W + \tau h_*\Delta)$$

be the corresponding morphism over Z. By [Ambro05, 0.2], we may write $K_W + \tau h_*\Delta \sim_{\mathbb{Q}} r^*(K_V + B_V)$ where (V, B_V) is klt. By induction on the dimension $\kappa(K_V + B_V) \ge 0$. Therefore

$$\kappa(K_X+\Delta) \ge \kappa(K_X+\tau\Delta) = \kappa(K_V+B_V) \ge 0.$$

Therefore, we may assume that $\dim Z_x = 0$ for all $\tau_0 \le x < \tau$ where $0 < \tau - \tau_0 \ll 1$. We claim that we may assume that $f := f_{\tau_0}$ is $K_X+\tau\Delta$-non-positive. Since $\rho(Y) = 1$ and $K_Y+\tau f_*\Delta$ is pseudo-effective, it follows that $K_Y+\tau f_*\Delta$ is nef. Then by the Negativity Lemma, it is easy to see that $\kappa(K_X+\tau\Delta) = \kappa(K_Y+\tau f_*\Delta)$ (cf. (2.1)). But as $\rho(Y) = 1$, we have that $\kappa(K_Y+\tau f_*\Delta) \ge 0$ as required.

We will now prove the claim. Notice that

$$\mathrm{Supp}(N_\sigma(K_X + x\Delta + y(x)A)) = \mathrm{Supp}(N_\sigma(K_X + x\Delta + (y(x) + \alpha)A))$$

for some $\alpha > 0$, and

$$\mathrm{Supp}(N_\sigma(K_X + x\Delta + (y(x) + \alpha)A)) \subset \mathrm{Supp}(N_\sigma(K_X + x'\Delta + y(x')A))$$

as long as $(\alpha + y(x) - y(x'))A + (x - x')\Delta$ is ample. Thus the above inclusion holds for any $||x - x'|| \ll 1$. Since the number of components in $\mathrm{Supp}(N_\sigma(K_X + x\Delta + y(x)A))$ is $\rho(X) - \rho(Y_x) = \rho(X) - 1$, this number is independent of x and so the inclusion $\mathrm{Supp}(N_\sigma(K_X + x\Delta + y(x)A)) \subset \mathrm{Supp}(N_\sigma(K_X + x'\Delta + y(x')A))$ is an equality (for any $||x - x'|| \ll 1$). In other words, we have shown that for any $\tau_0 \le x < \tau$, there is an $\epsilon > 0$ such that

$$\mathrm{Supp}(N_\sigma(K_X + x\Delta + y(x)A)) = \mathrm{Supp}(N_\sigma(K_X + x'\Delta + y(x')A))$$

for all $|x - x'| < \epsilon$. It then follows easily that $\mathrm{Supp}(N_\sigma(K_X + x\Delta + y(x)A))$ is independent of x (for all $\tau_0 \le x < \tau$) and so $Y_x \dashrightarrow Y$ is an isomorphism in codimension 1 for any $\tau_0 \le x < \tau$. Since $K_{Y_x} + f_{x*}(x\Delta + y(x)A) \sim_{\mathbb{R}} 0$, it is easy to see that $K_Y + f_*(x\Delta + y(x)A) \sim_{\mathbb{R}} 0$. By the Negativity lemma, we have that $a(E, Y, f_*(x\Delta + y(x)A)) = a(E, Y_x, f_{x*}(x\Delta + y(x)A))$ for E any f-exceptional divisor for $f : X \dashrightarrow Y$. For any $\tau_0 \le x < \tau$, we then have

$$\begin{aligned} a(E, Y, f_*(x\Delta + y(x)A)) &= a(E, Y_x, f_{x*}(x\Delta + y(x)A)) \\ &\ge a(E, X, x\Delta + y(x)A). \end{aligned}$$

Passing to the limit $x \to \tau$, we obtain the required inequality

$$a(E, Y, f_*(\tau\Delta)) \ge a(E, X, \tau\Delta)$$

i.e. f is $K_X + \tau\Delta$-non-positive.

□

Finally we recall the following important application of the existence of good minimal models.

Theorem 8.9. *Assume* (1.1). *Let X be a smooth projective variety and Δ_i be effective $\mathbb{Q}$-divisors on X such that $\sum \Delta_i$ has simple normal crossings support and $\lfloor \sum \Delta_i \rfloor = 0$. Let $\mathcal{C} \subset \{\Delta = \sum t_i\Delta_i | 0 \le t_i \le 1\}$ be a rational polytope.*

Then there are finitely many birational contractions $\phi_i : X \dashrightarrow X_i$ and finitely many projective morphisms $\psi_{i,j} : X_i \to Z_{i,j}$ (surjective with connected fibers) such that if $\Delta \in \mathcal{C}$ and $K_X + \Delta$ is pseudo-effective, then there exists i such that $\phi_i : X \dashrightarrow X_i$ is a good minimal model of (X, Δ) and j such that $Z_{i,j}$ is the ample model of (X, Δ) (in particular if Δ is a $\mathbb{Q}$-divisor, then $=$ $\mathrm{Proj}R(K_X + \Delta)$). Moreover, the closures of the sets

$$\mathcal{A}_{i,j} = \{\Delta = \sum t_i\Delta_i | K_X + \Delta \sim_{\mathbb{R}} \psi^*_{i,j} H_{i,j}, \text{ with } H_{i,j} \text{ ample on } Z_{i,j}\}$$

are finite unions of rational polytopes.

Proof. The proof follows easily along the lines of the proof of [BCHM10, 7.1]. We include the details for the benefit of the reader. We may work locally in a neighborhood $\mathcal{C}$ of any Δ as above. Let $\phi : X \dashrightarrow Y$ be a good minimal model of $K_X + \Delta$ and $\psi : Y \to Z = \mathrm{Proj} R(K_X + \Delta)$ so that $K_Y + \phi_*\Delta \sim_{\mathbb{R},Z} 0$. Since ϕ is $K_X+\Delta$-negative, we may assume that the same is true for any $\Delta' \in \mathcal{C}$ (after possibly replacing $\mathcal{C}$ by a smaller subset). We may therefore assume that for any $\Delta' \in \mathcal{C}$, the minimal models of (X, Δ) and $(Y, \phi_*\Delta')$ coincide (cf. [BCHM10, 3.6.9, 3.6.10]). Therefore, we may replace X by Y and hence assume that $K_X + \Delta$ is nef. Let $K_X + \Delta \sim_{\mathbb{R}} \psi^* H$ where H is ample on Z and $\psi : X \to Z$. Note that there is a positive constant δ such that $H \cdot C \geq \delta$ for any curve C on Z. We claim that (after possibly further shrinking $\mathcal{C}$), for any $\Delta' \in \mathcal{C}$, we have that $K_X + \Delta'$ is nef if and only if it is nef over Z.

To this end, note that if $(K_X + \Delta') \cdot C < 0$ and $(K_X + \Delta) \cdot C > 0$, then $(K_X + \Delta^*) \cdot C < 0$ where $\Delta^* = \Delta + t(\Delta' - \Delta)$ for some $t > 0$ belongs to the boundary of $\mathcal{C}$. But then, by (2.8), we may assume that $-(K_X + \Delta^*) \cdot C \leq 2 \dim X$. Since $(K_X + \Delta) \cdot C = H \cdot \psi_* C \geq \delta$ it follows easily that this can not happen for Δ' in any sufficiently small neighborhood $\Delta \subset \mathcal{C}' \subset \mathcal{C}$. We may replace $\mathcal{C}$ by $\mathcal{C}'$ and the claim follows.

Therefore, it suffices to prove the relative version of the Theorem over Z cf. [BCHM10, 7.1]. By induction on the dimension of $\mathcal{C}$, we may assume the theorem holds (over Z) for the boundary of $\mathcal{C}$. For any $\Delta \neq \Delta' \in \mathcal{C}$ we can choose Θ on the boundary of $\mathcal{C}$ such that

$$\Theta - \Delta = \lambda(\Delta' - \Delta), \qquad 0 < \lambda.$$

Since $K_X + \Delta \sim_{\mathbb{R},Z} 0$, we have

$$K_X + \Theta \sim_{\mathbb{R},Z} \lambda(K_X + \Delta').$$

Therefore $K_X + \Theta$ is pseudo-effective over Z if and only if $K_X + \Delta'$ is pseudo-effective over Z and the minimal models over Z of $K_X + \Theta$ and $K_X + \Delta'$ coincide (cf. [BCHM10, 3.6.9, 3.6.10]). It is also easy to see that if $\psi' : X \to Z'$ is a morphism over Z, then $K_X + \Theta \sim_{\mathbb{R}} \psi'^* H'$ for some ample divisor H' on Z' if and only if $K_X + \Delta' \sim_{\mathbb{R}} \psi'^*(\lambda H')$. The theorem now follows easily. $\square$

Theorem 8.10. *Assume* (1.1). *Let X be a smooth projective variety and Δ_i be $\mathbb{Q}$-divisors on X such that $\sum \Delta_i$ has simple normal crossings support and $\lfloor \sum \Delta_i \rfloor = 0$. Then the adjoint ring*

$$R(X; K_X + \Delta_1, \ldots, K_X + \Delta_r)$$

is finitely generated.

Proof. This is an easy consequence of (8.9), see for example the proof of [BCHM10, 1.1.9]. $\square$

Corollary 8.11. *With the notation of* (8.9)*. Let P be any prime divisor on X and $\mathcal{C}^+$ the intersection of $\mathcal{C}$ with the pseudo-effective cone. Then the function $\sigma_P : \mathcal{C}^+ \to \mathbb{R}_{\geq 0}$ is continuous and piecewise rational affine linear.*

Proof. Immediate from (8.9). □

References

[Ambro04] F. Ambro, *Nef dimension of minimal models.* Math. Ann. 330 (2004), no. 2, 309322.

[Ambro05] F. Ambro, *The moduli bb-divisor of an lc-trivial fibration.* Compos. Math. 141 (2005), no. 2, 385403.

[Birkar07] C. Birkar, *Ascending chain condition for log canonical thresholds and termination of log flips.* Duke Math. J. 136 (2007), no. 1, 173–180.

[Birkar09] C. Birkar, *On existence of log minimal models II.* arXiv:0907.4170

[Berndtsson96] B. Berndtsson, *On the Ohsawa-Takegoshi extension theorem* Ann. Inst. Fourier (1996).

[BP10] B. Berndtsson, M. Păun, *Qualitative extensions of twisted pluricanonical forms and closed positive currents*, arXiv 2010.

[BCHM10] C. Birkar, P. Cascini, C. Hacon, J. M^{c}Kernan, *Existence of minimal models for varieties of log general type.* J. Amer. Math. Soc. 23 (2010), no. 2, 405–468.

[Claudon07] B. Claudon, *Invariance for multiples of the twisted canonical bundle,* Ann. Inst. Fourier (Grenoble) 57, 2007.

[CL10] A. Corti, V. Lazić, *New outlook on Mori theory, II.* ArXiv:1005.0614

[dFH09] T. de Fernex, C. Hacon, *Deformations of canonical pairs and Fano varieties.* arXiv:0901.0389

[Demailly90] J.-P. Demailly, *Singular hermitian metrics on positive line bundles,* Proc. Conf. Complex algebraic varieties (Bayreuth, April 26, 1990), edited by K. Hulek, T. Peternell, M. Schneider, F. Schreyer, Lecture Notes in Math., Vol. 1507, Springer-Verlag, Berlin, 1992.

[Demailly97] J.-P. Demailly, *On the Ohsawa-Takegoshi-Manivel extension theorem,* Proceedings of the Conference in honour of the 85th birthday of Pierre Lelong, Paris, September 1997.

[Demailly06] J.-P. Demailly, *Kähler manifolds and transcendental techniques in algebraic geometry,* Plenary talk and Proceedings of the Internat. Congress of Math., Madrid 2006, volume I.

[Demailly09] J.-P. Demailly, *Analytic methods in algebraic geometry,* on the web page of the author, December 2009.

[EP07] L. Ein, M. Popa, *Adjoint ideals and extension theorems,* personal communication june 2007, arXiv: 0811.4290.

[Fujino00] O. Fujino, *Abundance theorem for semi log canonical threefolds.* Duke Math. J. 102 (2000), no. 3, 513532.

[Fujino07] O. Fujino, *Special termination and reduction to pl flips.* Flips for 3-folds and 4-folds, 63–75, Oxford Lecture Ser. Math. Appl., 35, Oxford Univ. Press, Oxford, 2007.

[Fujino09] O. Fujino, *Fundamental theorems of the log minimal model program,* arXiv:0909.4445

[Fukuda02] S. Fukuda, *Tsujis numerically trivial fibrations and abundance,* Far East Journal of Mathematical Sciences, Volume 5, Issue 3, 2002, 247257.

[GL10] Y. Gongyo, B Lehmann, *Reduction maps and minimal model theory.* Preprint.

[HK10] C. Hacon, S. Kovacs, *Classification of higher dimensional algebraic varieties.* Oberwolfach Seminars, 41. Birkhäuser Verlag, Basel, 2010. x+208 pp.

[HM07] C. Hacon, J. M^{c}Kernan, *Boundedness of pluricanonical maps of varieties of general type.* Invent. Math. 166 (2006), no. 1, 125.

[HM10] C. Hacon, J. M^{c}Kernan, *Existence of minimal models for varieties of log general type. II.* J. Amer. Math. Soc. 23 (2010), no. 2, 469–490.

[HMX12] C. Hacon, J. M^{c}Kernan, C. Xu, *ACC for log canonical thresholds.* arXiv:1208.4150v1

[Kawamata85a] Y. Kawamata, *Pluricanonical systems on minimal algebraic varieties.* Invent. Math. 79 (1985), no. 3, 567588.

[Kawamata85b] Y. Kawamata, *The Zariski decomposition of log-canonical divisors,* Algebraic Geometry Bowdoin, Proc. Symp. Pure Math. 46 (1987), 1985, pp. 425–433.

[Kawamata92] Y. Kawamata, *Abundance theorem for minimal threefolds.* Invent. Math. 108 (1992), no. 2, 229246.

[Kawamata97] Y. Kawamata, *On the cone of divisors of Calabi-Yau fiber spaces.* Internat. J. Math. 8 (1997), no. 5, 665–687.

[Kawamata98] Y. Kawamata, *On the extension problem of pluricanonical forms.* Algebraic geometry: Hirzebruch 70 (Warsaw, 1998), Contemp. Math., vol. 241, Amer. Math. Soc., Providence, RI, 1999.

[Kawamata08] Y. Kawamata, *Flops connect minimal models.* Publ. Res. Inst. Math. Sci. 44 (2008), no. 2, 419423.

[KMM94] S. Keel, K. Matsuki, J M^{c}Kernan, *Log abundance theorem for threefolds.* Duke Math. J. 75 (1994), no. 1, 99–119.

[Kim08] D. Kim, *L^2 extension of adjoint line bundle sections,* arXiv:0802.3189, to appear in Ann. Inst. Fourier.

[Klimek91] M. Klimek, *Pluripotential Theory*, London Mathematical Society Monographs. New Series, 6. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1991, xiv+266 pp.

[KM98] J. Kollár and S. Mori, *Birational geometry of algebraic varieties,* Cambridge tracts in math- ematics, vol. 134, Cambridge University Press, 1998.

[Kolláretal92] J. Kollár et al., *Flips and abundance for algebraic threefolds,* Société Mathématique de France, Paris, 1992, Papers from the Second Summer Seminar on Algebraic Geometry held at the University of Utah, Salt Lake City, Utah, August 1991, Astérisque No. 211 (1992).

[Lai10] C.-J. Lai, *Varieties fibered by good minimal models* arXiv:0912.3012, to appear in Math. Ann.

[Lelong69] P. Lelong, *Plurisubharmonic Functions and Positive Differential Forms*, Gordon and Breach, Dunod, New York, Paris, 1969.

[Lelong71] P. Lelong, *Éléments extrémaux dans le cône des courants positifs fermés de type* (1, 1) *et fonctions plurisousharmoniques extrémales*, C. R. Acad. Sci. Paris Sér. A–B 273 (1971), A665–A667.

[Manivel93] L. Manivel, *Un théorème de prolongement L^2 de sections holomorphes d'un fibré hermitien* Math. Zeitschrift, 1993.

[MV07] J. McNeal, D. Varolin; *Analytic inversion of adjunction: L^2 extension theorems with gain*, Ann. Inst. Fourier (Grenoble) **57** (2007), no. 3, 703–718.

[Miyaoka88] Y. Miyaoka, *Abundance conjecture for minimal threefolds: $\nu = 1$ case.* Comp. Math. 68, 203220 (1988)

[Nakayama04] N. Nakayama, *Zariski-decomposition and abundance,* MSJ Memoirs, vol. 14, Mathematical Society of Japan, Tokyo, 2004.

[OT87] T. Ohsawa, K. Takegoshi, *On the extension of L^2 holomorphic functions* Math. Z., 1987.

[Ohsawa03] T. Ohsawa, *On the extension of L^2 holomorphic functions. VI. A limiting case,* Contemp. Math., Amer. Math. Soc., Providence, 2003.

[Ohsawa04] T. Ohsawa, *Generalization of a precise L^2 division theorem,* Complex analysis in several variables, Memorial Conference of Kiyoshi Oka's Centennial Birthday, 249–261, Adv. Stud. Pure Math., 42, Math. Soc. Japan, Tokyo, 2004.

[Paun07] M. Păun, *Siu's invariance of plurigenera: a one-tower proof,* J. Diff. Geom. 76 (2007), 485–493.

[Paun08] M. Păun, *Relative critical exponents, non-vanishing and metrics with minimal singularities.* arXiv:0807.3109.

[Shokurov09] V.V. Shokurov, *Letters of a bi-rationalist. VII. Ordered termination.* (Russian) Tr. Mat. Inst. Steklova 264 (2009), Mnogomernaya Algebraicheskaya Geometriya, 184–208.

[Siu74] Y.T. Siu, *Analyticity of sets associated to Lelong numbers and the extension of closed positive currents,* Invent. Math. 27 (1974), 53–156.

[Siu76] Y. T. Siu, *Every Stein subvariety admits a Stein neighborhood*, Invent. Math. 38 (1976) 89–100.

[Siu98] Y.-T. Siu, *Invariance of plurigenera.* Invent. Math. 134, 661–673 (1998).

[Siu00] Y.-T. Siu, *Extension of twisted pluricanonical sections with plurisubharmonic weight and invariance of semipositively twisted plurigenera for manifolds not necessarily of general type.* Complex geometry (Göttingen, 2000), 223–277, Springer, Berlin, 2002.

[Siu08] Y.-T. Siu, *Finite generation of canonical ring by analytic method.* Sci. China Ser. A 51 (2008), no. 4, 481–502.

[Siu09] Y.-T. Siu, *Abundance conjecture* arXiv:09120576, 2010.

[Skoda72] H. Skoda, *Applications des techniques L^2 à la théorie des idéaux d'une algèbre de fonctions holomorphes avec poids*, Ann. Scient. Ec. Norm. Sup. 4e Série, 5 (1972), 545–579.

[Takayama06] S. Takayama, *Pluricanonical systems on algebraic varieties of general type,* Invent. Math, 165 (2006), 551–587.

[Takayama07] S. Takayama, *On the invariance and lower semi-continuity of plurigenera of algebraic varieties,* J. Algebraic Geom, 16 (2007), 118.

[Tian87] G. Tian, *On Kähler-Einstein metrics on certain Kähler manifolds with* $c_1(M) > 0$, Invent. Math, 89 (1987), 225–246.

[Tsuji05] H. Tsuji, *Extension of log pluricanonical forms from subvarieties* math.CV/0511342.

[Var08] D. Varolin, *A Takayama-type extension theorem,* Compos. Math. 144 (2008), no. 2, 522–540.

Université de Grenoble I, Département de Mathématiques, Institut Fourier, 38402 Saint-Martin d'Hères, France

E-mail address: `demailly@fourier.ujf-grenoble.fr`

Department of Mathematics, University of Utah, 155 South 1400 East, JWB 233, Salt Lake City, UT 84112, USA
E-mail address: `hacon@math.utah.edu`

Institut Elie Cartan, Université Henri Poincaré, B. P. 70239, F-54506 Vandoeuvre-lès-Nancy Cedex, France
E-mail address: `Mihai.Paun@iecn.u-nancy.fr`