A CONVERSE TO THE
ANDREOTTI-GRAUERT THEOREM

by

Jean-Pierre Demailly

Abstract. — The goal of this paper is to show that there are strong relations between certain Monge-Ampère integrals appearing in holomorphic Morse inequalities, and asymptotic cohomology estimates for tensor powers of holomorphic line bundles. Especially, we prove that these relations hold without restriction for projective surfaces, and in the special case of the volume, i.e. of asymptotic 0-cohomology, for all projective manifolds. These results can be seen as a partial converse to the Andreotti-Grauert vanishing theorem.

Résumé. — Le but de ce travail est de montrer qu'il y a des relations fortes entre certaines intégrales de Monge-Ampère apparaissant dans les inégalités de Morse holomorphes, et les estimations asymptotiques de cohomologie pour les fibrés holomorphes en droites. En particulier, nous montrons que ces relations sont satisfaites sans restriction pour toutes les surfaces projectives, et dans le cas particulier du volume, c'est-à-dire de la 0-cohomologie asymptotique, pour toutes les variétés projectives. Ces résultats peuvent être vus comme une réciproque partielle au théorème d'annulation d'Andreotti-Grauert.

Dedicated to Professor Nguyen Thanh Van

1. Main results

Throughout this paper, X denotes a compact complex manifold, $n = \dim_{\mathbb{C}} X$ its complex dimension and $L \to X$ a holomorphic line bundle. In order to estimate the growth of cohomology groups, it is interesting to consider appropriate "asymptotic cohomology functions". Following partly notation and concepts introduced by A. Küronya [Kür06, FKL07], we introduce

(1.1) Definition.

(i) *The q-th asymptotic cohomology functional is defined as*

$$\widehat{h}^q(X, L) := \limsup_{k \to +\infty} \frac{n!}{k^n} h^q(X, L^{\otimes k}).$$

(ii) *The q-th asymptotic holomorphic Morse sum of L is*

$$\widehat{h}^{\leq q}(X, L) := \limsup_{k \to +\infty} \frac{n!}{k^n} \sum_{0 \leq j \leq q} (-1)^{q-j} h^j(X, L^{\otimes k}).$$

When the lim sup's are limits, we have the obvious relation

$$\widehat{h}^{\leq q}(X, L) = \sum_{0 \leq j \leq q} (-1)^{q-j} \widehat{h}^j(X, L).$$

Clearly, Definition 1.1 can also be given for a $\mathbb{Q}$-line bundle L or a $\mathbb{Q}$-divisor D, and in the case $q = 0$ one gets what is usually called the volume of L, namely

(1.2) $$\mathrm{Vol}(X, L) = \widehat{h}^0(X, L) = \limsup_{k \to +\infty} \frac{n!}{k^n} h^0(X, L^{\otimes k}).$$

(see also [DEL00], [Bou02], [Laz04]). It has been shown in [Kür06] for the projective case and in [Dem10] in general that the $\widehat{h}^q$ functional induces a continuous map

$$\mathrm{DNS}_{\mathbb{R}}(X) \ni \alpha \mapsto \widehat{h}^q(X, \alpha)$$

defined on the "divisorial Neron-Severi space" $\mathrm{DNS}_{\mathbb{R}}(X) \subset H^{1,1}_{\mathrm{BC}}(X, \mathbb{R})$, i.e. the vector space spanned by real linear combinations of classes of divisors in the real Bott-Chern cohomology group of bidegree $(1, 1)$. Here $H^{p,q}_{\mathrm{BC}}(X, \mathbb{C})$ is defined as the quotient of d-closed (p, q)-forms by $\partial\overline{\partial}$-exact (p, q)-forms, and there is a natural conjugation $H^{p,q}_{\mathrm{BC}}(X, \mathbb{C}) \to H^{q,p}_{\mathrm{BC}}(X, \mathbb{C})$ which allows us to speak of real classes when $q = p$. The $\widehat{h}^q$ functional is in fact locally Lipschitz continous on $\mathrm{DNS}_{\mathbb{R}}(X)$, and can be obtained as a limit (not just a limsup) on all those classes. Notice that $H^{p,q}_{\mathrm{BC}}(X, \mathbb{C})$ coincides with the usual Dolbeault cohomology group $H^{p,q}(X, \mathbb{C})$ when X is Kähler, and that $\mathrm{DNS}_{\mathbb{R}}(X)$ coincides with the usual Néron-Severi space

$$\mathrm{NS}_{\mathbb{R}}(X) = \mathbb{R} \otimes_{\mathbb{Q}} \left(H^2(X, \mathbb{Q}) \cap H^{1,1}(X, \mathbb{C}) \right)$$

when X is projective. It follows from holomorphic Morse inequalities (cf. [Dem85], [Dem91]) that asymptotic cohomology can be compared with certain Monge-Ampère integrals.

(1.3) Theorem ([Dem85]). *For every holomorphic line bundle L on a compact complex manifold X, one has the "weak Morse inequality"*

(i) $$\widehat{h}^q(X,L) \leq \inf_{u \in c_1(L)} \int_{X(u,q)} (-1)^q u^n$$

where u runs over all smooth d-closed $(1,1)$-forms which belong to the coho-mology class $c_1(L) \in H^{1,1}_{\mathrm{BC}}(X,\mathbb{R})$, and $X(u,q)$ is the open set

$$X(u,q) := \big\{ z \in X \, ; \, u(z) \text{ has signature } (n-q,q) \big\}.$$

Moreover, if $X(u,\leq q) := \bigcup_{0 \leq j \leq q} X(u,j)$, one has the "strong Morse inequa-lity"

(ii) $$\widehat{h}^{\leq q}(X,L) \leq \inf_{u \in c_1(L)} \int_{X(u,\leq q)} (-1)^q u^n.$$

It is a natural problem to ask whether the inequalities (1.3) (i) and (1.3) (ii) might not always be equalities. These questions are strongly related to the Andreotti-Grauert vanishing theorem [AG62]. A well-known variant of this theorem says that if for some integer q and some $u \in c_1(L)$ the form $u(z)$ has at least $n - q + 1$ positive eigenvalues everywhere (so that $X(u, \geq q) = \bigcup_{j \geq q} X(u,j) = \emptyset$), then $H^j(X, L^{\otimes k}) = 0$ for $j \geq q$ and $k \gg 1$. We are asking here whether conversely the knowledge that cohomology groups are asymptotically small in a certain degree q implies the existence of a hermitian metric on L with suitable curvature, i.e. no q-index points or only a very small amount of such.

The first goal of this note is to prove that the answer is positive in the case of the volume functional (i.e. in the case of degree $q = 0$), at least when X is projective algebraic.

(1.4) Theorem. *Let L be a holomorphic line bundle on a projective algebraic manifold. then*

$$\mathrm{Vol}(X,L) = \inf_{u \in c_1(L)} \int_{X(u,0)} u^n.$$

The proof relies mainly on five ingredients: (a) approximate Zariski decom-position for a Kähler current $T \in c_1(L)$ (when L is big), i.e. a decomposition $\mu^* T = [E] + \beta$ where $\mu : \widetilde{X} \to X$ is a modification, E an exceptional divisor and β a Kähler metric on $\widetilde{X}$; (b) the characterization of the pseudoeffective cone ([BDPP04]), and the orthogonality estimate

$$E \cdot \beta^{n-1} \leq C \big(\mathrm{Vol}(X,L) - \beta^n \big)^{1/2}$$

proved as an intermediate step of that characterization; (c) properties of so-lutions of Laplace equations to get smooth approximations of $[E]$; (d) log concavity of the Monge-Ampère operator ; and finally (e) birational invariance of the Morse infimums. In the case of higher cohomology groups, we have been able to treat only the case of projective surfaces :

(1.5) Theorem. *Let $L \to X$ be a holomorphic line bundle on a complex projective surface. Then both weak and strong inequalities* (1.3) (i) *and* (1.3) (ii) *are equalities for $q = 0,\ 1,\ 2$, and the $\limsup$'s involved in $\widehat{h}^q(X,L)$ and $\widehat{h}^{\leq q}(X,L)$ are limits.*

Thanks to the Serre duality and the Riemann-Roch formula, the (in)equality for a given q is equivalent to the (in)equality for $n - q$. Therefore, on surfaces, the only substantial case which still has to be checked in addition to Theorem 1.4 is the case $q = 1$: this is done by using Grauert' criterion that the intersection matrix $(E_i \cdot E_j)$ is negative definite for every exceptional divisor $E = \sum c_j E_j$. Our statements are of course trivial on curves since the curvature of any holomorphic line bundle can be taken to be constant with respect to any given hermitian metric.

(1.6) Remark. It is interesting to put these results in perspective with the algebraic version of holomorphic Morse inequalities proved in [Dem94] (see also [Siu93] and [Tra95] for related ideas, and [Ang94] for an algebraic proof). When X is projective, the algebraic Morse inequalities used in combination with the birational invariance of the Morse integrals (cf. section 2) imply the inequalities

(i) $$\inf_{u \in c_1(L)} \int_{X(u,q)} (-1)^q u^n \leq \inf_{\mu^*(L) \simeq \mathcal{O}(A-B)} \binom{n}{q} A^{n-q} B^q \,,$$

(ii) $$\inf_{u \in c_1(L)} \int_{X(u, \leq q)} (-1)^q u^n \leq \inf_{\mu^*(L) \simeq \mathcal{O}(A-B)} \sum_{0 \leq j \leq q} (-1)^{q-j} \binom{n}{j} A^{n-j} B^j \,,$$

where the infimums on the right hand side are taken over all modifications $\mu : \widetilde{X} \to X$ and all decompositions $\mu^* L = \mathcal{O}(A - B)$ of $\mu^* L$ as a difference of two nef $\mathbb{Q}$-divisors A, B on $\widetilde{X}$. In case A and B are ample, the proof simply consists of taking positive curvature forms $\Theta_{\mathcal{O}(A),h_A}$, $\Theta_{\mathcal{O}(B),h_B}$ on $\mathcal{O}(A)$ and $\mathcal{O}(B)$, and evaluating the Morse integrals with $u = \Theta_{\mathcal{O}(A),h_A} - \Theta_{\mathcal{O}(B),h_B}$; the general case follows by approximating the nef divisors A and B by ample divisors $A + \varepsilon H$ and $B + \varepsilon H$ with H ample and $\varepsilon > 0$, see [Dem94]. Again, a natural question is to know whether these infimums derived from algebraic intersection numbers are equal to the asymptotic cohomology functionals $\widehat{h}^q(X,L)$ and $\widehat{h}^{\leq q}(X,L)$. A positive answer would of course automatically yield a positive answer to the equality cases in 1.3 (i) and 1.3 (ii). However, the Zariski decompositions involved in our proofs of the "analytic equality case" produces certain effective exceptional divisors which are not nef. It is unclear how to write those effective divisors as a difference of nef divisors. This fact raises a lot of doubts upon the sufficiency of taking merely differences of nef divisors in the infimums 1.6 (i) and 1.6 (ii). $\square$

I warmly thank Burt Totaro for stimulating discussions in connection with his recent work [Tot10].

2. Invariance by modification

It is easy to check that the asymptotic cohomology function is invariant by modification, namely that for every modification $\mu : \widetilde{X} \to X$ and every line bundle L we have

(2.1) $\widehat{h}^q(X, L) = \widehat{h}^q(\widetilde{X}, \mu^*L).$

In fact the Leray's spectral sequence provides an E_2 term

$$E_2^{p,q} = H^p(X, R^q\mu_* \mathcal{O}_{\widetilde{X}}(\mu^*L^{\otimes k})) = H^p(X, \mathcal{O}_X(L^{\otimes k}) \otimes R^q\mu_*\mathcal{O}_{\widetilde{X}}).$$

Since $R^q\mu_*\mathcal{O}_{\widetilde{X}}$ is equal to $\mathcal{O}_X$ for $q = 0$ and is supported on a proper analytic subset of X for $q \geq 1$, one infers that $h^p(X, \mathcal{O}_X(L^{\otimes k} \otimes R^q\mu_*\mathcal{O}_{\widetilde{X}}) = O(k^{n-1})$ for all $q \geq 1$. The spectral sequence implies that

$$h^q(X, L^{\otimes k}) - \widehat{h}^q(\widetilde{X}, \mu^*L^{\otimes k}) = O(k^{n-1}).$$

We claim that the Morse integral infimums are also invariant by modification.

(2.2) Proposition. *Let (X, ω) be a compact Kähler manifold, $\alpha \in H^{1,1}(X, \mathbb{R})$ a real cohomology class and $\mu : \widetilde{X} \to X$ a modification. Then*

(i) $\displaystyle \inf_{u \in \alpha} \int_{X(u,q)} (-1)^q u^n = \inf_{v \in \mu^*\alpha} \int_{X(v,q)} (-1)^q v^n,$

(ii) $\displaystyle \inf_{u \in \alpha} \int_{X(u,\leq q)} (-1)^q u^n = \inf_{v \in \mu^*\alpha} \int_{X(v,\leq q)} (-1)^q v^n.$

Proof. — . Given $u \in \alpha$ on X, we obtain Morse integrals with the same values by taking $v = \mu^*u$ on $\widetilde{X}$, hence the infimum on $\widetilde{X}$ is smaller or equal to what is on X. Conversely, we have to show that given a smooth representative $v \in \mu^*\alpha$ on $\widetilde{X}$, one can find a smooth representative $u \in X$ such that the Morse integrals do not differ much. We can always assume that $\widetilde{X}$ itself is Kähler, since by Hironaka [Hir64] any modification $\widetilde{X}$ is dominated by a composition of blow-ups of X. Let us fix some $u_0 \in \alpha$ and write

$$v = \mu^*u_0 + dd^c\varphi$$

where φ is a smooth function on $\widetilde{X}$. We adjust φ by a constant in such a way that $\varphi \geq 1$ on $\widetilde{X}$. There exists an analytic set $S \subset X$ such that $\mu : \widetilde{X} \smallsetminus \mu^{-1}(S) \to X \smallsetminus S$ is a biholomorphism, and a quasi-psh function

ψ_S which is smooth on $X \smallsetminus S$ and has $-\infty$ logarithmic poles on S (see e.g. [Dem82]). We define

$$(2.3) \quad \widetilde{u} = \mu^* u_0 + dd^c \max_{\varepsilon_0}(\varphi + \delta \, \psi_S \circ \mu \, , \, 0) = v + dd^c \max_{\varepsilon_0}(\delta \, \psi_S \circ \mu \, , \, -\varphi)$$

where $\max_{\varepsilon_0}$, $0 < \varepsilon_0 < 1$, is a regularized max function and $\delta > 0$ is very small. By construction $\widetilde{u}$ coincides with $\mu^* u_0$ in a neighborhood of $\mu^{-1}(S)$ and therefore $\widetilde{u}$ descends to a smooth closed $(1,1)$-form u on X which coincides with u_0 near S, so that $\widetilde{u} = \mu^* u$. Clearly $\widetilde{u}$ converges uniformly to v on every compact subset of $\widetilde{X} \smallsetminus \mu^{-1}(S)$ as $\delta \to 0$, so we only have to show that the Morse integrals are small (uniformly in δ) when restricted to a suitable small neighborhood of the exceptional set $E = \mu^{-1}(S)$. Take a sufficiently large Kähler metric $\widetilde{\omega}$ on $\widetilde{X}$ such that

$$-\frac{1}{2}\widetilde{\omega} \le v \le \frac{1}{2}\widetilde{\omega}, \quad -\frac{1}{2}\widetilde{\omega} \le dd^c \varphi \le \frac{1}{2}\widetilde{\omega}, \quad -\widetilde{\omega} \le dd^c \psi_S \circ \mu.$$

Then $\widetilde{u} \ge -\widetilde{\omega}$ and $\widetilde{u} \le \widetilde{\omega} + \delta \, dd^c \psi_S \circ \mu$ everywhere on $\widetilde{X}$. As a consequence

$$|\widetilde{u}^n| \le \big(\widetilde{\omega} + \delta(\widetilde{\omega} + dd^c \psi_S \circ \mu)\big)^n \le \widetilde{\omega}^n + n\delta(\widetilde{\omega} + dd^c \psi_S \circ \mu) \wedge \big(\widetilde{\omega} + \delta(\widetilde{\omega} + dd^c \psi_S \circ \mu)\big)^{n-1}$$

thanks to the inequality $(a+b)^n \le a^n + nb(a+b)^{n-1}$. For any neighborhood V of $\mu^{-1}(S)$ this implies

$$\int_V |\widetilde{u}^n| \le \int_V \widetilde{\omega}^n + n\delta(1+\delta)^{n-1} \int_{\widetilde{X}} \widetilde{\omega}^n$$

by Stokes formula. We thus see that the integrals are small if V and δ are small. The reader may be concerned that Monge-Ampère integrals were used with an unbounded potential ψ_S, but in fact, for any given δ, all the above formulas and estimates are still valid when we replace ψ_S by $\max_{\varepsilon_0}(\psi_S, -(M+2)/\delta)$ with $M = \max_{\widetilde{X}} \varphi$, especially formula (2.3) shows that the form $\widetilde{u}$ is unchanged. Therefore our calculations can be handled by using merely smooth potentials. $\square$

3. Proof of the infimum formula for the volume

We have to show that

$$(3.1) \qquad \inf_{u \in c_1(L)} \int_{X(u,0)} u^n \le \mathrm{Vol}(X, L)$$

Let us first assume that L is a big line bundle, i.e. that $\mathrm{Vol}(X, L) > 0$. Then it is known by [Bou02] that $\mathrm{Vol}(X, L)$ is obtained as the supremum of $\int_{X \smallsetminus \mathrm{Sing}(T)} T^n$ for Kähler currents $T = -\frac{i}{2\pi}\partial\overline{\partial}h$ with analytic singularities in $c_1(L)$; this means that locally $h = e^{-\varphi}$ where φ is a strictly plurisubharmonic function which has the same singularities as $c \log \sum |g_j|^2$ where $c > 0$ and the g_j are holomorphic functions. By [Dem92], there exists a blow-up $\mu : \widetilde{X} \to X$ such

that $\mu^* T = [E] + \beta$ where E is a normal crossing divisor on $\widetilde{X}$ and $\beta \geq 0$ smooth. Moreover, by [BDPP04] we have the orthogonality estimate

$$(3.2) \qquad [E] \cdot \beta^{n-1} = \int_E \beta^{n-1} \leq C\big(\operatorname{Vol}(X, L) - \beta^n\big)^{1/2},$$

while

$$(3.3) \qquad \beta^n = \int_{\widetilde{X}} \beta^n = \int_{X \smallsetminus \operatorname{Sing}(T)} T^n \quad \text{approaches } \operatorname{Vol}(X, L).$$

In other words, E and β become "more and more orthogonal" as β^n approaches the volume (these properties are summarized by saying that $\mu^* T = [E] + \beta$ defines an approximate Zariski decomposition of $c_1(L)$, cf. also [Fuj94]). By subtracting to β a small linear combination of the exceptional divisors and increasing accordingly the coefficients of E, we can achieve that the cohomology class $\{\beta\}$ contains a positive definite form β' on $\widetilde{X}$ (i.e. the fundamental form of a Kähler metric); we refer e.g. to ([DP04], proof of Lemma 3.5) for details. This means that we can replace T by a cohomologous current such that the corresponding form β is actually a Kähler metric, and we will assume for simplicity of notation that this situation occurs right away for T. Under this assumption, there exists a smooth closed $(1, 1)$-form v belonging to the Bott-Chern cohomology class of $[E]$, such that we have identically $(v - \delta\beta) \wedge \beta^{n-1} = 0$ where

$$(3.4) \qquad \delta = \frac{[E] \cdot \beta^{n-1}}{\beta^n} \leq C'(\operatorname{Vol}(X, L) - \beta^n)^{1/2}$$

for some constant $C' > 0$. In fact, given an arbitrary smooth representative $v_0 \in \{[E]\}$, the existence of $v = v_0 + i\partial\overline{\partial}\psi$ amounts to solving a Laplace equation $\Delta\psi = f$ with respect to the Kähler metric β, and the choice of δ ensures that we have $\int_X f\, \beta^n = 0$ and hence that the equation is solvable. Then $\widetilde{u} := v + \beta$ is a smooth closed $(1, 1)$-form in the cohomology class $\mu^* c_1(L)$, and its eigenvalues with respect to β are of the form $1 + \lambda_j$ where λ_j are the eigenvalues of v. The Laplace equation is equivalent to the identity $\sum \lambda_j = n\delta$. Therefore

$$(3.5) \qquad \sum_{1 \leq j \leq n} \lambda_j \leq C''(\operatorname{Vol}(X, L) - \beta^n)^{1/2}.$$

The inequality between arithmetic means and geometry means implies

$$\prod_{1 \leq j \leq n} (1 + \lambda_j) \leq \Big(1 + \frac{1}{n} \sum_{1 \leq j \leq n} \lambda_j\Big)^n \leq 1 + C_3(\operatorname{Vol}(X, L) - \beta^n)^{1/2}$$

whenever all factors $(1 + \lambda_j)$ are nonnegative. By 2.2 (i) we get

$$
\begin{aligned}
\inf_{u \in c_1(L)} \int_{X(u,0)} u^n &\leq \int_{\widetilde{X}(\widetilde{u},0)} \widetilde{u}^n \\
&\leq \int_{\widetilde{X}} \beta^n \big(1 + C_3 (\mathrm{Vol}(X,L) - \beta^n)^{1/2}\big) \\
&\leq \mathrm{Vol}(X,L) + C_4 (\mathrm{Vol}(X,L) - \beta^n)^{1/2}.
\end{aligned}
$$

As β^n approches $\mathrm{Vol}(X,L)$, this implies inequality (3.1).

We still have to treat the case when L is not big, i.e. $\mathrm{Vol}(X,L) = 0$. Let A be an ample line bundle and let $t_0 \geq 0$ be the infimum of real numbers such that $L + tA$ is a big $\mathbb{Q}$-line bundle for t rational, $t > t_0$. The continuity of the volume function implies that $0 < \mathrm{Vol}(X, L + tA) \leq \varepsilon$ for $t > t_0$ sufficiently close to t_0. By what we have just proved, there exists a smooth form $u_t \in c_1(L + tA)$ such that $\int_{X(u_t,0)} u_t^n \leq 2\varepsilon$. Take a Kähler metric $\omega \in c_1(A)$ and define $u = u_t - t\omega$. Then clearly

$$
\int_{X(u,0)} u^n \leq \int_{X(u_t,0)} u_t^n \leq 2\varepsilon,
$$

hence

$$
\inf_{u \in c_1(L)} \int_{X(u,0)} u^n = 0.
$$

Inequality (3.1) is now proved in all cases. $\square$

4. Estimate of the first cohomology group on a projective surface

We start with a projective non singular variety X of arbitrary dimension n, and will later restrict ourselves to the case when X is a surface. The proof again consists of using (approximate) Zariski decomposition, but now we try to compute more explicitly the resulting curvature forms and Morse integrals; this will turn out to be much easier on surfaces.

Assume first that L is a *big* line bundle on X. As in section 3, we can find an approximate Zariski decomposition, i.e. a blow-up $\mu : \widetilde{X} \to X$ and a current $T \in c_1(L)$ such $\mu^* T = [E] + \beta$, where E an effective divisor and β a Kähler metric on $\widetilde{X}$ such that

$$
(4.1) \qquad \mathrm{Vol}(X,L) - \eta < \beta^n < \mathrm{Vol}(X,L), \qquad \eta \ll 1.
$$

(On a projective surface, one could even get exact Zariski decomposition, but we want to remain general as long as possible). By blowing-up further, we may assume that E is a normal crossing divisor. We select a hermitian metric h on

$\mathcal{O}(E)$ and take

$$(4.2) \qquad u_\varepsilon = \frac{i}{2\pi} \partial\overline{\partial} \log(|\sigma_E|_h^2 + \varepsilon^2) + \Theta_{\mathcal{O}(E),h} + \beta \quad \in \quad \mu^* c_1(L)$$

where $\sigma_E \in H^0(\widetilde{X}, \mathcal{O}(E))$ is the canonical section and $\Theta_{\mathcal{O}(E),h}$ the Chern curvature form. Clearly, by the Lelong-Poincaré equation, u_ε converges to $[E] + \beta$ in the weak topology as $\varepsilon \to 0$. Straightforward calculations yield

$$u_\varepsilon = \frac{i}{2\pi} \frac{\varepsilon^2 D_h^{1,0}\sigma_E \wedge \overline{D_h^{1,0}\sigma_E}}{(\varepsilon^2 + |\sigma_E|^2)^2} + \frac{\varepsilon^2}{\varepsilon^2 + |\sigma_E|^2}\Theta_{E,h} + \beta.$$

The first term converges to $[E]$ in the weak topology, while the second, which is close to $\Theta_{E,h}$ near E, converges pointwise everywhere to 0 on $\widetilde{X} \smallsetminus E$. A simple asymptotic analysis shows that

$$\left(\frac{i}{2\pi} \frac{\varepsilon^2 D_h^{1,0}\sigma_E \wedge \overline{D_h^{1,0}\sigma_E}}{(\varepsilon^2 + |\sigma_E|^2)^2} + \frac{\varepsilon^2}{\varepsilon^2 + |\sigma_E|^2}\Theta_{E,h} \right)^p \to [E] \wedge \Theta_{E,h}^{p-1}$$

in the weak topology for $p \geq 1$, hence

$$(4.3) \qquad \lim_{\varepsilon \to 0} u_\varepsilon^n = \beta^n + \sum_{p=1}^n \binom{n}{p} [E] \wedge \Theta_{E,h}^{p-1} \wedge \beta^{n-p}.$$

In arbitrary dimension, the signature of u_ε is hard to evaluate, and it is also non trivial to decide the sign of the limiting measure $\lim u_\varepsilon^n$. However, when $n = 2$, we get the simpler formula

$$\lim_{\varepsilon \to 0} u_\varepsilon^2 = \beta^2 + 2[E] \wedge \beta + [E] \wedge \Theta_{E,h}.$$

In this case, E can be assumed to be an exceptional divisor (otherwise some part of it would be nef and could be removed from the poles of T). Hence the matrix $(E_j \cdot E_k)$ is negative definite and we can find a smooth hermitian metric h on $\mathcal{O}(E)$ such that $(\Theta_{E,h})_{|E} < 0$, i.e. $\Theta_{E,h}$ has one negative eigenvalue everywhere along E.

(4.4) Lemma. *One can adjust the metric h of $\mathcal{O}(E)$ in such a way that $\Theta_{E,h}$ is negative definite on a neighborhood of the support $|E|$ of the exceptional divisor, and $\Theta_{E,h} + \beta$ has signature $(1,1)$ there. (We do not care about the signature far away from $|E|$).*

Proof. — . At a given point $x_0 \in X$, let us fix coordinates and a positive quadratic form q on $\mathbb{C}^2$. If we put $\psi_\varepsilon(z) = \varepsilon\chi(z)\log(1 + \varepsilon^{-1}q(z))$ with a suitable cut-off function χ, then the Hessian form of ψ_ε is equal to q at x_0 and decays rapidly to $O(\varepsilon \log \varepsilon)|dz|^2$ away from x_0. In this way, after multiplying h with $e^{\pm\psi_\varepsilon(z)}$, we can replace the curvature $\Theta_{E,h}(x_0)$ with $\Theta_{E,h}(x_0) \pm q$ without substantially modifying the form away from x_0. This allows to adjust $\Theta_{E,h}$

to be equal to (say) $-\frac{1}{4}\beta(x_0)$ at any singular point $x_0 \in E_j \cap E_k$ in the support of $|E|$, while keeping $\Theta_{E,h}$ negative definite along E. In order to adjust the curvature at smooth points $x \in |E|$, we replace the metric h with $h'(z) = h(z)\exp(-c(z)|\sigma_E(z)|^2)$. Then the curvature form $\Theta_{E,h}$ is replaced by $\Theta_{E,h'}(x) = \Theta_{E_h}(x) + c(x)|d\sigma_E|^2$ at $x \in |E|$ (notice that $d\sigma_E(x) = 0$ if $x \in \mathrm{Sing}|E|$), and we can always select a real function c so that $\Theta_{E,h'}$ is negative definite with one negative eigenvalue between $-1/2$ and 0 at any point of $|E|$. Then $\Theta_{E,h'} + \beta$ has signature $(1,1)$ near $|E|$. $\square$

With this choice of the metric, we see that for $\varepsilon > 0$ small, the sum

$$\frac{\varepsilon^2}{\varepsilon^2 + |\sigma_E|^2}\Theta_{E,h} + \beta$$

is of signature $(2,0)$ or $(1,1)$ (or degenerate of signature $(1,0)$), the non positive definite points being concentrated in a neighborhood of E. In particular the index set $X(u_\varepsilon, 2)$ is empty, and also

$$u_\varepsilon \leq \frac{i}{2\pi}\frac{\varepsilon^2 D_h^{1,0}\sigma_E \wedge \overline{D_h^{1,0}\sigma_E}}{(\varepsilon^2 + |\sigma_E|^2)^2} + \beta$$

on a neighborhood V of $|E|$, while u_ε converges uniformly to β on $\widetilde{X} \smallsetminus V$. This implies that

$$\beta^2 \leq \liminf_{\varepsilon \to 0}\int_{X(u_\varepsilon,0)}u_\varepsilon^2 \leq \limsup_{\varepsilon \to 0}\int_{X(u_\varepsilon,0)}u_\varepsilon^2 \leq \beta^2 + 2\beta \cdot E.$$

Since $\int_{\widetilde{X}}u_\varepsilon^2 = L^2 = \beta^2 + 2\beta \cdot E + E^2$ we conclude by taking the difference that

$$-E^2 - 2\beta \cdot E \leq \liminf_{\varepsilon \to 0}\int_{X(u_\varepsilon,1)}-u_\varepsilon^2 \leq \limsup_{\varepsilon \to 0}\int_{X(u_\varepsilon,1)}-u_\varepsilon^2 \leq -E^2.$$

Let us recall that $\beta \cdot E \leq C(\mathrm{Vol}(X,L) - \beta^2)^{1/2} = 0(\eta^{1/2})$ is small by (4.1) and the orthogonality estimate. The asymptotic cohomology is given here by $\widehat{h}^2(X,L) = 0$ since $h^2(X,L^{\otimes k}) = H^0(X, K_X \otimes L^{\otimes -k}) = 0$ for $k \geq k_0$, and we have by Riemann-Roch

$$\widehat{h}^1(X,L) = \widehat{h}^0(X,L) - L^2 = \mathrm{Vol}(X,L) - L^2 = -E^2 - \beta \cdot E + O(\eta).$$

Here we use the fact that $\frac{n!}{k^n}h^0(X,L^{\otimes k})$ converges to the volume when L is big. All this shows that equality occurs in the Morse inequalities (1.3) when we pass to the infimum. By taking limits in the Neron-Severi space $\mathrm{NS}_{\mathbb{R}}(X) \subset H^{1,1}(X,\mathbb{R})$, we further see that equality occurs as soon as L is pseudo-effective, and the same is true if $-L$ is pseudo-effective by Serre duality.

It remains to treat the case when neither L nor $-L$ are pseudo-effective. Then $\widehat{h}^0(X,L) = \widehat{h}^2(X,L) = 0$, and asymptotic cohomology appears only in degree 1, with $\widehat{h}^1(X,L) = -L^2$ by Riemann-Roch. Fix an ample line bundle

A and let $t_0 > 0$ be the infimum of real numbers such that $L + tA$ is big for t rational, $t > t_0$, resp. let $t_0' > 0$ be the infimum of real numbers t' such that $-L + t'A$ is big for $t' > t_0'$. Then for $t > t_0$ and $t' > t_0'$, we can find a modification $\mu : \widetilde{X} \to X$ and currents $T \in c_1(L + tA)$, $T' \in c_1(-L + t'A)$ such that

$$\mu^* T = [E] + \beta, \qquad \mu^* T' = [F] + \gamma$$

where β, γ are Kähler forms and E, F normal crossing divisors. By taking a suitable linear combination $t'(L + tA) - t(-L + t'A)$ the ample divisor A disappears, and we get

$$\frac{1}{t + t'} \Big(t'[E] + t'\beta - t[F] - t\gamma \Big) \in \mu^* c_1(L).$$

After replacing E, F, β, γ by suitable multiples, we obtain an equality

$$[E] - [F] + \beta - \gamma \in \mu^* c_1(L).$$

We may further assume by subtracting that the divisors E, F have no common components. The construction shows that $\beta^2 \leq \mathrm{Vol}(X, L + tA)$ can be taken arbitrarily small (as well of course as γ^2), and the orthogonality estimate implies that we can assume $\beta \cdot E$ and $\gamma \cdot F$ to be arbitrarily small. Let us introduce metrics h_E on $\mathcal{O}(E)$ and h_F on $\mathcal{O}(F)$ as in Lemma 4.4, and consider the forms

$$u_\varepsilon = + \frac{i}{2\pi} \frac{\varepsilon^2 D_{h_E}^{1,0} \sigma_E \wedge \overline{D_{h_E}^{1,0} \sigma_E}}{(\varepsilon^2 + |\sigma_E|^2)^2} + \frac{\varepsilon^2}{\varepsilon^2 + |\sigma_E|^2} \Theta_{E,h_E} + \beta$$

$$- \frac{i}{2\pi} \frac{\varepsilon^2 D_{h_F}^{1,0} \sigma_F \wedge \overline{D_{h_F}^{1,0} \sigma_F}}{(\varepsilon^2 + |\sigma_F|^2)^2} - \frac{\varepsilon^2}{\varepsilon^2 + |\sigma_F|^2} \Theta_{F,h_F} - \gamma \quad \in \quad \mu^* c_1(L).$$

Observe that u_ε converges uniformly to $\beta - \gamma$ outside of every neighborhood of $|E| \cup |F|$. Assume that $\Theta_{E,h_E} < 0$ on $V_E = \{|\sigma_E| < \varepsilon_0\}$ and $\Theta_{F,h_F} < 0$ on $V_F = \{|\sigma_F| < \varepsilon_0\}$. On $V_E \cup V_F$ we have

$$u_\varepsilon \leq \frac{i}{2\pi} \frac{\varepsilon^2 D_{h_E}^{1,0} \sigma_E \wedge \overline{D_{h_E}^{1,0} \sigma_E}}{(\varepsilon^2 + |\sigma_E|^2)^2} - \frac{\varepsilon^2}{\varepsilon^2 + |\sigma_F|^2} \Theta_{F,h_F} + \beta + \frac{\varepsilon^2}{\varepsilon_0^2} \Theta_{E,h_E}^+$$

where Θ_{E,h_E}^+ is the positive part of Θ_{E,h_E} with respect to β. One sees immediately that this term is negligible. The first term is the only one which is not uniformly bounded, and actually it converges weakly to the current $[E]$. By squaring, we find

$$\limsup_{\varepsilon \to 0} \int_{X(u_\varepsilon, 0)} u_\varepsilon^2 \leq \int_{X(\beta - \gamma, 0)} (\beta - \gamma)^2 + 2\beta \cdot E.$$

Notice that the term $- \frac{\varepsilon^2}{\varepsilon^2 + |\sigma_F|^2} \Theta_{F,h_F}$ does not contribute to the limit as it converges boundedly almost everywhere to 0, the exceptions being points of $|F|$,

but this set is of measure zero with respect to the current $[E]$. Clearly we have $\int_{X(\beta-\gamma,0)}(\beta-\gamma)^2 \le \beta^2$ and therefore

$$\limsup_{\varepsilon \to 0} \int_{X(u_\varepsilon,0)} u_\varepsilon^2 \le \beta^2 + 2\beta \cdot E.$$

Similarly, by looking at $-u_\varepsilon$, we find

$$\limsup_{\varepsilon \to 0} \int_{X(u_\varepsilon,2)} u_\varepsilon^2 \le \gamma^2 + 2\gamma \cdot F.$$

These lim sup's are small and we conclude that the essential part of the mass is concentrated on the 1-index set, as desired. $\square$

References

[AG62]   Andreotti, A., Grauert, H.: *Théorèmes de finitude pour la cohomologie des espaces complexes;* Bull. Soc. Math. France **90** (1962) 193–259.

[Ang95]   Angelini, F.: *An algebraic version of Demailly's asymptotic Morse inequalities;* arXiv: alg-geom/9503005, Proc. Amer. Math. Soc. **124** (1996) 3265–3269.

[Bou02]   Boucksom, S.: *On the volume of a line bundle;* Internat. J. Math. **13** (2002), 1043–1063.

[BDPP04]   Boucksom, S., Demailly, J.-P., Păun, M., Peternell, Th.: *The pseudo-effective cone of a compact Kähler manifold and varieties of negative Kodaira dimension;* arXiv: math.AG/0405285, see also Proceedings of the ICM 2006 in Madrid.

[Dem82]   Demailly, J.-P.: *Estimations L^2 pour l'opérateur $\overline{\partial}$ d'un fibré vectoriel holomorphe semi-positif au dessus d'une variété kählérienne complète;* Ann. Sci. École Norm. Sup. **15** (1982) 457–511.

[Dem85]   Demailly, J.-P.: *Champs magnétiques et inégalités de Morse pour la d''-cohomologie;* Ann. Inst. Fourier (Grenoble) **35** (1985), 189–229.

[Dem91]   Demailly, J.-P.: *Holomorphic Morse inequalities;* Lectures given at the AMS Summer Institute on Complex Analysis held in Santa Cruz, July 1989, Proceedings of Symposia in Pure Mathematics, Vol. **52**, Part 2 (1991), 93–114.

[Dem94]   Demailly, J.-P.: *L^2 vanishing theorems for positive line bundles and adjunction theory;* arXiv: alg-geom/9410022 ; Lecture Notes of the CIME Session "Transcendental methods in Algebraic Geometry", Cetraro, Italy, July 1994, Ed. F. Catanese, C. Ciliberto, Lecture Notes in Math., Vol. 1646, 1–9.

[Dem10]   Demailly, J.-P.: *Holomorphic Morse inequalities and asymptotic cohomology groups: a tribute to Bernhard Riemann;* Milan Journal of Mathematics **78** (2010) 265–277.

[Dem92]   Demailly, J.-P.: *Regularization of closed positive currents and Intersection Theory;* J. Alg. Geom. **1** (1992), 361–409.

[DEL00]   Demailly, J.-P., Ein, L., and Lazarsfeld, R.: *A subadditivity property of multiplier ideals;* Michigan Math. J. 48 (2000), 137-156.

[DP04]   Demailly, J.-P., Păun, M: *Numerical characterization of the Kähler cone of a compact Kähler manifold;* arXiv: math.AG/0105176 ; Annals of Math. **159** (2004) 1247–1274.

[FKL07]   de Fernex, T., Küronya, A., Lazarsfeld, R.: *Higher cohomology of divisors on a projective variety;* Math. Ann. **337** (2007) 443–455.

[Fuj94]   Fujita, T.: *Approximating Zariski decomposition of big line bundles;* Kodai Math. J. **17** (1994) 1-3.

[Hir64]   Hironaka, H.: *Resolution of singularities of an algebraic variety over a field of characteristic zero;* Ann. of Math. **79** (1964) 109–326.

[Kür06]   Küronya, A.: *Asymptotic cohomological functions on projective varieties;* Amer. J. Math. **128** (2006) 1475–1519.

[Laz04]   Lazarsfeld, R.: *Positivity in Algebraic Geometry I.-II;* Ergebnisse der Mathematik und ihrer Grenzgebiete, Vols. 48-49, Springer Verlag, Berlin, 2004.

[Siu93]   Siu, Y.T.: *An effective Matsusaka big theorem;* Ann. Inst. Fourier **43** (1993) 1387–1405.

[Tra95]   Trapani, S.: *Numerical criteria for the positivity of the difference of ample divisors;* Math. Zeitschrift **219** (1995) 387–401.

[Tot10]   Totaro, B.: *Line bundles with partially vanishing cohomology;* July 2010, arXiv: math.AG/1007.3955.

JEAN-PIERRE DEMAILLY, Université de Grenoble I, Institut Fourier, BP 74, 100 rue des Maths, 38402 Saint-Martin d'Hères, France
E-mail : demailly@fourier.ujf-grenoble.fr