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A general extension theorem for cohomology classes on non reduced analytic subspaces

In memory of Professor LU QiKeng (1927–2015)

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Abstract The main purpose of this paper is to generalize the celebrated L^2 extension theorem of Ohsawa and Takegoshi in several directions: The holomorphic sections to extend are taken in a possibly singular hermitian line bundle, the subvariety from which the extension is performed may be non reduced, the ambient manifold is Kähler and holomorphically convex, but not necessarily compact.

Keywords compact Kähler manifold, singular hermitian metric, coherent sheaf cohomology, Dolbeault cohomology, plurisubharmonic function, L^2 estimates, Ohsawa-Takegoshi extension theorem, multiplier ideal sheaf

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1 Introduction and preliminaries

The purpose of this paper is to generalize the celebrated L^2 extension theorem of Ohsawa and Takegoshi [18] under the weakest possible hypotheses, along the lines of [5, 16]. Especially, the ambient complex manifold X is a Kähler manifold that is only assumed to be *holomorphically convex*, and is not necessarily compact; by the Remmert reduction theorem, this is the same as a Kähler manifold X that admits a proper holomorphic map $\pi : X \to S$ onto a Stein complex space S. This allows in particular to consider relative situations over a Stein base. We consider a holomorphic line bundle $E \to X$ equipped with a singular hermitian metric h, namely a metric which can be expressed locally as $h = \mathrm{e}^{-\varphi}$ where φ is a *quasi-psh* function, i.e., a function that is locally the sum $\varphi = \varphi_0 + u$ of a plurisubharmonic function φ_0 and of a smooth function u. Such a bundle admits a curvature current

$$\Theta_{E,h} := \mathrm{i}\partial\overline{\partial}\varphi = \mathrm{i}\partial\overline{\partial}\varphi_0 + \mathrm{i}\partial\overline{\partial}u, \tag{1.1}$$

which is locally the sum of a positive $(1,1)$-current $\mathrm{i}\partial\overline{\partial}\varphi_0$ and a smooth $(1,1)$-form $\mathrm{i}\partial\overline{\partial}u$. Our goal is to extend sections that are defined on a (non necessarily reduced) complex subspace $Y \subset X$, when the

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structure sheaf $\mathcal{O}_Y := \mathcal{O}_X/\mathcal{I}(\mathrm{e}^{-\psi})$ is given by the multiplier ideal sheaf of a quasi-psh function ψ with *neat analytic singularities*, i.e., locally on a neighborhood V of an arbitrary point $x_0 \in X$ we have

$$\psi(z) = c \log \sum |g_j(z)|^2 + v(z), \quad g_j \in \mathcal{O}_X(V), \quad v \in C^\infty(V). \tag{1.2}$$

Let us recall that the multiplier ideal sheaf $\mathcal{I}(\mathrm{e}^{-\varphi})$ of a quasi-psh function φ is defined by

$$\mathcal{I}(\mathrm{e}^{-\varphi})_{x_0} = \left\{ f \in \mathcal{O}_{X,x_0}; \exists\, U \ni x_0, \int_U |f|^2 \mathrm{e}^{-\varphi} d\lambda < +\infty \right\} \tag{1.3}$$

with respect to the Lebesgue measure λ in some local coordinates near x_0. As usual, we also denote by $K_X = \Lambda^n T_X^*$ the canonical bundle of an n-dimensional complex manifold X. As is well known, $\mathcal{I}(\mathrm{e}^{-\varphi}) \subset \mathcal{O}_X$ is a coherent ideal sheaf (see [3]). Our main result is given by the following general statement.

Theorem 1.1. *Let E be a holomorphic line bundle over a holomorphically convex Kähler manifold X. Let h be a possibly singular hermitian metric on E, ψ a quasi-psh function with neat analytic singularities on X. Assume that there exists a positive continuous function $\delta > 0$ on X such that*

$$\Theta_{E,h} + (1 + \alpha\delta) \mathrm{i}\partial\overline{\partial}\psi \geqslant 0 \quad \textit{in the sense of currents,} \quad \textit{for all } \alpha \in [0, 1]. \tag{1.4}$$

Then the morphism induced by the natural inclusion $\mathcal{I}(h\mathrm{e}^{-\psi}) \to \mathcal{I}(h)$,

$$H^q(X, K_X \otimes E \otimes \mathcal{I}(h\mathrm{e}^{-\psi})) \to H^q(X, K_X \otimes E \otimes \mathcal{I}(h)) \tag{1.5}$$

is injective for every $q \geqslant 0$. In other words, the morphism induced by the natural sheaf surjection $\mathcal{I}(h) \to \mathcal{I}(h)/\mathcal{I}(h\mathrm{e}^{-\psi})$,

$$H^q(X, K_X \otimes E \otimes \mathcal{I}(h)) \to H^q(X, K_X \otimes E \otimes \mathcal{I}(h)/\mathcal{I}(h\mathrm{e}^{-\psi})) \tag{1.6}$$

is surjective for every $q \geqslant 0$.

Remark 1.2. If h is smooth, we have $\mathcal{I}(h) = \mathcal{O}_X$ and $\mathcal{I}(h)/\mathcal{I}(h\mathrm{e}^{-\psi}) = \mathcal{O}_X/\mathcal{I}(\mathrm{e}^{-\psi}) := \mathcal{O}_Y$, where Y is the zero subvariety of the ideal sheaf $\mathcal{I}(\mathrm{e}^{-\psi})$. Then for $q = 0$, the surjectivity statement can be interpreted as an extension theorem for holomorphic sections, with respect to the restriction morphism

$$H^0(X, K_X \otimes E) \to H^0(Y, (K_X \otimes E)_{|Y}). \tag{1.7}$$

In general, the quotient sheaf $\mathcal{I}(h)/\mathcal{I}(h\mathrm{e}^{-\psi})$ is supported in an analytic subvariety $Y \subset X$, which is the zero set of the quotient ideal

$$\mathcal{J}_Y := \mathcal{I}(h\mathrm{e}^{-\psi}) : \mathcal{I}(h) = \{ f \in \mathcal{O}_X; f \cdot \mathcal{I}(h) \subset \mathcal{I}(h\mathrm{e}^{-\psi}) \},$$

and (1.6) can be considered as a restriction morphism to Y. □

The crucial idea of the proof is to prove the results (say, in the form of the surjectivity statement), only up to approximation. This is done by solving a $\overline{\partial}$-equation

$$\overline{\partial} u_\varepsilon + w_\varepsilon = v,$$

where the right-hand side v is given and w_ε is an error term such that $\|w_\varepsilon\| = O(\varepsilon^a)$ as $\varepsilon \to 0$, for some constant $a > 0$. A twisted Bochner-Kodaira-Nakano identity introduced by Donnelly and Fefferman [7], and Ohsawa and Takegoshi [18] is used for that purpose, with an additional correction term. The version we need can be stated as follows.

Proposition 1.3 (See [5, Proposition 3.12])**.** *Let X be a complete Kähler manifold equipped with a (non necessarily complete) Kähler metric ω, and let (E, h) be a Hermitian vector bundle over X. Assume that there are smooth and bounded functions η, $\lambda > 0$ on X such that the curvature operator*

$$B = B^{n,q}_{E,h,\omega,\eta,\lambda} = [\eta\, \Theta_{E,h} - \mathrm{i}\partial\overline{\partial}\eta - \mathrm{i}\lambda^{-1} d\partial\eta \wedge \overline{\partial}\eta, \Lambda_\omega] \in C^\infty(X, \mathrm{Herm}(\Lambda^{n,q} T_X^* \otimes E))$$

satisfies $B + \varepsilon I > 0$ *for some* $\varepsilon > 0$ (*so that* B *can be just semi-positive or even slightly negative*; *here* I *is the identity endomorphism*). *Given a section* $v \in L^2(X, \Lambda^{n,q}T^*_X \otimes E)$ *such that* $\overline{\partial} v = 0$ *and*

$$M(\varepsilon) := \int_X \langle (B + \varepsilon I)^{-1} v, v\rangle \, dV_{X,\omega} < +\infty,$$

there exists an approximate solution $f_\varepsilon \in L^2(X, \Lambda^{n,q-1}T^*_X \otimes E)$ *and a correction term* $w_\varepsilon \in L^2(X, \Lambda^{n,q}T^*_X \otimes E)$ *such that* $\overline{\partial} u_\varepsilon = v - w_\varepsilon$ *and*

$$\int_X (\eta + \lambda)^{-1} |u_\varepsilon|^2 \, dV_{X,\omega} + \frac{1}{\varepsilon} \int_X |w_\varepsilon|^2 \, dV_{X,\omega} \leqslant M(\varepsilon).$$

Moreover, if v *is smooth, then* u_ε *and* w_ε *can be taken smooth.*

In our situation, the main part of the solution, namely u_ε, may very well explode as $\varepsilon \to 0$. In order to show that the equation $\overline{\partial} u = v$ can be solved, it is therefore needed to check that the space of coboundaries is closed in the space of cocycles in the Fréchet topology under consideration (here, the $L^2_{\rm loc}$ topology), in other words, that the related cohomology group $H^q(X, \mathcal{F})$ is Hausdorff. In this respect, the fact of considering $\overline{\partial}$-cohomology of smooth forms equipped with the C^∞ topology on the one hand, or cohomology of forms $u \in L^2_{\rm loc}$ with $\overline{\partial} u \in L^2_{\rm loc}$ on the other hand, yields the same topology on the resulting cohomology group $H^q(X, \mathcal{F})$. This comes from the fact that both complexes yield fine resolutions of the same coherent sheaf $\mathcal{F}$, and the topology of $H^q(X, \mathcal{F})$ can also be obtained by using Čech cochains with respect to a Stein covering $\mathcal{U}$ of X. The required Hausdorff property then comes from the following well-known fact.

Lemma 1.4. *Let* X *be a holomorphically convex complex space and* $\mathcal{F}$ *a coherent analytic sheaf over* X. *Then all cohomology groups* $H^q(X, \mathcal{F})$ *are Hausdorff with respect to their natural topology* (*induced by the Fréchet topology of local uniform convergence of holomorphic cochains*)[1)].

In fact, the Remmert reduction theorem implies that X admits a proper holomorphic map $\pi : X \to S$ onto a Stein space S, and Grauert's direct image theorem shows that all direct images $R^q\pi_*\mathcal{F}$ are coherent sheaves on S. Now, as S is Stein, Leray's theorem combined with Cartan's theorem B tells us that we have an isomorphism $H^q(X, \mathcal{F}) \simeq H^0(S, R^q\pi_*\mathcal{F})$. More generally, if $U \subset S$ is a Stein open subset, we have

$$H^q(\pi^{-1}(U), \mathcal{F}) \simeq H^0(U, R^q\pi_*\mathcal{F}) \tag{1.8}$$

and when $U \Subset S$ is relatively compact, it is easily seen that this is a topological isomorphism of Fréchet spaces since both sides are $\mathcal{O}_S(U)$ modules of finite type and can be seen as a Fréchet quotient of some direct sum $\mathcal{O}_S(U)^{\oplus N}$ by looking at local generators and local relations of $R^q\pi_*\mathcal{F}$. Therefore, $H^q(X, \mathcal{F}) \simeq H^0(S, R^q\pi_*\mathcal{F})$ is a topological isomorphism and the space of sections in the right-hand side is a Fréchet space. In particular, $H^q(X, \mathcal{F})$ is Hausdorff.

The isomorphism (1.8) shows that it is enough to prove Theorem 1.1 locally over X, i.e., we can replace X by $X' = \pi^{-1}(S') \Subset X$, where $S' \Subset S$. Therefore, we can assume that $\delta > 0$ is a constant rather than a continuous function.

2 Proof of the extension theorem

In this section, we give a proof of Theorem 1.1 based on a generalization of the arguments of [5, Theorem 2.14]. We start by proving the special case of the extension result for holomorphic sections ($q = 0$).

Theorem 2.1. *Let* (X, ω) *be a holomorphically convex Kähler manifold and* ψ *be a quasi-psh function with neat analytic singularities. Let* E *be a line bundle with a possibly singular metric* h, *and* Y *the*

1) It was pointed out to us by Professor Takeo Ohsawa that this result does not hold under the assumption that X is weakly pseudoconvex, i.e., if we only assume that X admits a smooth psh exhaustion. A counter-example can be derived from [13]. As a consequence, it is unclear whether the results of the present paper extend to the Kähler weakly pseudoconvex case, although the main L^2 estimates are still valid in that situation.

support of the sheaf $\mathcal{I}(h)/\mathcal{I}(he^{-\psi})$, *along with the structure sheaf* $\mathcal{O}_Y := \mathcal{I}(he^{-\psi}) : \mathcal{I}(h)$. *Assume that there is a continuous function* $\delta > 0$ *such that*

$$\mathrm{i}\Theta_{E,h} + (1+\alpha\delta)\mathrm{i}\partial\overline{\partial}\psi \geqslant 0 \quad \textit{in the sense of currents,} \quad \textit{for all } \alpha \in [0,1].$$

Then the restriction morphism

$$H^0(X, \mathcal{O}_X(K_X \otimes E) \otimes \mathcal{I}(h)) \to H^0(Y, \mathcal{O}_X(K_X \otimes E) \otimes \mathcal{I}(h)/\mathcal{I}(he^{-\psi})_{|Y})$$

is surjective.

Proof. (a) Let us first assume for simplicity that h is smooth. We will explain the general case later. Then $\mathcal{I}(h) = \mathcal{O}_X$ and $\mathcal{I}(h)/\mathcal{I}(he^{-\psi}) = \mathcal{O}_Y = \mathcal{O}_X/\mathcal{I}(e^{-\psi})$. After possibly shrinking X into a relatively compact holomorphically convex open subset $X' = \pi^{-1}(S') \Subset X$, we can suppose that $\delta > 0$ is a constant and that $\psi \leqslant 0$, after subtracting a large constant to ψ. Also, without loss of generality, we can assume that ψ admits a discrete sequence of "jumping numbers"

$$0 = m_0 < m_1 < \cdots < m_p < \cdots \quad \text{such that} \quad \mathcal{I}(m\psi) = \mathcal{I}(m_p\psi) \quad \text{for} \quad m \in [m_p, m_{p+1}[. \tag{2.1}$$

Since ψ is assumed to have analytic singularities, this follows from using a log resolution of singularities, thanks to the Hironaka desingularization theorem (by the much deeper result of [11] on the strong openness conjecture, one could even possibly eliminate the assumption that ψ has analytic singularities). We fix here p such that $m_p \leqslant 1 < m_{p+1}$, and in the notation of [5], we let $Y = Y^{(m_p)}$ be defined by the non necessarily reduced structure sheaf $\mathcal{O}_Y = \mathcal{O}_X/\mathcal{I}(e^{-\psi}) = \mathcal{O}_X/\mathcal{I}(e^{-m_p\psi})$.

Step 1 (Construction of a smooth extension). Take

$$f \in H^0(Y, \mathcal{O}_X(K_X \otimes E)_{|Y}) = H^0(X, \mathcal{O}_X(K_X \otimes E) \otimes \mathcal{O}_X/\mathcal{I}(e^{-m_p\psi})).$$

Let $\mathcal{U} = (U_i)$ be a Stein covering of X and let (ρ_i) be a partition of unity subordinate to (U_i). Thanks to the exact sequence

$$0 \to \mathcal{I}(e^{-\psi}) \to \mathcal{O}_X \to \mathcal{O}_X/\mathcal{I}(e^{-\psi}) \to 0, \tag{2.2}$$

we can find an $\widetilde{f}_i \in H^0(U_i, \mathcal{O}_X(K_X \otimes E))$ such that

$$\widetilde{f}_i\,|_{Y\cap U_i} = f\,|_{Y\cap U_i}.$$

Then (2.2) implies that

$$\widetilde{f}_i - \widetilde{f}_j \in H^0(U_i \cap U_j, \mathcal{O}_X(K_X \otimes E) \otimes \mathcal{I}(e^{-\psi})). \tag{2.3}$$

As a consequence, the smooth section $\widetilde{f} := \sum_i \rho_i \cdot \widetilde{f}_i$ is a smooth extension of f and satisfies $\overline{\partial}\widetilde{f} = \sum_i (\overline{\partial}\rho_i) \cdot (\widetilde{f}_i - \widetilde{f}_j)$ on U_j, hence

$$\int_X |\overline{\partial}\widetilde{f}|^2_{\omega,h} e^{-\psi} dV_{X,\omega} = \int_X \sum_j \rho_j \bigg| \sum_i (\overline{\partial}\rho_i) \cdot (\widetilde{f}_i - \widetilde{f}_j) \bigg|^2_{\omega,h} e^{-\psi} dV_{X,\omega} < +\infty. \tag{2.4}$$

Step 2 (L^2-estimates). We follow here the arguments of [5, proof of Theorem 2.14, p. 217]. Let $t \in \mathbb{Z}^-$ and let χ_t be the negative convex increasing function defined in [5, (5.8∗), p. 211]. Put $\eta_t := 1 - \delta \cdot \chi_t(\psi)$ and $\lambda_t := 2\delta \frac{(\chi_t^2(\psi))^2}{\chi_t''(\psi)}$. We set

$$\begin{aligned} R_t &:= \eta_t(\Theta_{E,h} + \mathrm{i}\partial\overline{\partial}\psi) - \mathrm{i}\partial\overline{\partial}\eta_t - \lambda_t^{-1} i\partial\eta_t \wedge \overline{\partial}\eta_t \\ &= \eta_t(\Theta_{E,h} + (1 + \delta\eta_t^{-1}\chi_t'(\psi))\mathrm{i}\partial\overline{\partial}\psi) + \frac{\delta \cdot \chi_t''(\psi)}{2} \mathrm{i}\partial\psi \wedge \overline{\partial}\psi. \end{aligned}$$

Note that $\chi_t''(\psi) \geqslant \frac{1}{8}$ on $W_t = \{t < \psi < t+1\}$. The curvature assumption (1.4) implies

$$\Theta_{E,h} + (1 + \delta\eta_t^{-1}\chi_t'(\psi))\,\mathrm{i}\partial\overline{\partial}\psi \geqslant 0 \quad \text{on} \quad X.$$

As in [5], we find

$$R_t \geqslant 0 \quad \text{on} \quad X \tag{2.5}$$

and

$$R_t \geqslant \frac{\delta}{16}\mathrm{i}\partial\psi\wedge\overline{\partial}\psi \quad \text{on} \quad W_t=\{t<\psi<t+1\}. \tag{2.6}$$

Let $\theta : [-\infty,+\infty[\ \to [0,1]$ be a smooth non increasing real function satisfying $\theta(x)=1$ for $x\leqslant 0$, $\theta(x)=0$ for $x\geqslant 1$ and $|\theta'|\leqslant 2$. By applying the L^2 estimate (see Proposition 1.3), for every $\varepsilon>0$ we can find sections $u_{t,\varepsilon}$ and $w_{t,\varepsilon}$ satisfying

$$\overline{\partial}u_{t,\varepsilon}+w_{t,\varepsilon}=v_t:=\overline{\partial}(\theta(\psi-t)\cdot\widetilde{f}) \tag{2.7}$$

and

$$\int_X(\eta_t+\lambda_t)^{-1}|u_{t,\varepsilon}|^2_{\omega,h}\mathrm{e}^{-\psi}dV_{X,\omega}+\frac{1}{\varepsilon}\int_X|w_{t,\varepsilon}|^2_{\omega,h}\mathrm{e}^{-\psi}dV_{X,\omega}\leqslant\int_X\langle(R_t+\varepsilon I)^{-1}v_t,v_t\rangle\mathrm{e}^{-\psi}dV_{X,\omega}, \tag{2.8}$$

where

$$v_t=\overline{\partial}(\theta(\psi-t)\widetilde{f})=\theta'(\psi-t)\,\overline{\partial}\psi\wedge\widetilde{f}+\theta(\psi-t)\,\overline{\partial}\widetilde{f}. \tag{2.9}$$

Combining (2.5), (2.6), (2.8) and (2.9), we get $\int_X|u_{t,\varepsilon}|^2_{\omega,h}\mathrm{e}^{-\psi}dV_{X,\omega}<+\infty$ and

$$\int_X|w_{t,\varepsilon}|^2_{\omega,h}\mathrm{e}^{-\psi}dV_{X,\omega}\leqslant\frac{128\,\varepsilon}{\delta}\int_{\{t<\psi<t+1\}}|\widetilde{f}\,|^2_{\omega,h}\mathrm{e}^{-\psi}dV_{X,\omega}+2\int_{\{\psi<t+1\}}|\overline{\partial}\widetilde{f}\,|^2_{\omega,h}\mathrm{e}^{-\psi}dV_{X,\omega}. \tag{2.10}$$

We now estimate the right-hand side of (2.10). Since $\widetilde{f}$ is smooth, we have an obvious upper bound of the first term

$$\int_{\{t<\psi<t+1\}}|\widetilde{f}|^2_{\omega,h}\mathrm{e}^{-\psi}dV_{X,\omega}\leqslant C_1\mathrm{e}^{-t}, \tag{2.11}$$

where C_1 is the C^0 norm of $\widetilde{f}$. For the second term, thanks to (2.1), (2.3) and (2.4), we have

$$\int_X|\overline{\partial}\widetilde{f}|^2_{\omega,h}\mathrm{e}^{-(1+\alpha)\psi}dV_{X,\omega}<+\infty \tag{2.12}$$

for any $\alpha\in\,]0,m_{p+1}-1[$. As a consequence, we get

$$\int_{\{\psi<t+1\}}|\overline{\partial}\widetilde{f}|^2_{\omega,h}\mathrm{e}^{-\psi}dV_{X,\omega}\leqslant C_2\mathrm{e}^{\alpha t} \tag{2.13}$$

for some constant C_2 depending only on α. By taking $\varepsilon=\mathrm{e}^{(1+\alpha)t}$, (2.10), (2.11) and (2.13) imply

$$\int_X|w_{t,\varepsilon}|^2_{\omega,h}\mathrm{e}^{-\psi}dV_{X,\omega}\leqslant C_3\mathrm{e}^{\alpha t}=O(\varepsilon^{\frac{\alpha}{1+\alpha}}), \tag{2.14}$$

for some constant C_3, whence the error tends to 0 as $t\to-\infty$ and $\varepsilon\to 0$.

Step 3 (Final conclusion). Putting everything together and redefining $u_t=u_{t,\varepsilon}$ and $w_t=w_{t,\varepsilon}$ for simplicity of notation, we get

$$\overline{\partial}(\theta(\psi-t)\cdot\widetilde{f}-u_t)=w_t,\quad\int_X|u_t|^2_h\mathrm{e}^{-\psi}dV_{X,\omega}<+\infty \tag{2.15}$$

and

$$\lim_{t\to-\infty}\int_X|w_t|^2_{\omega,h}\mathrm{e}^{-\psi}dV_{X,\omega}=0. \tag{2.16}$$

After shrinking X, we can assume that we have a finite Stein covering $\mathcal{U}=(U_i)$, where the U_i are biholomorphic to bounded pseudoconvex domains. The standard Hörmander L^2 estimates then provide L^2 sections $s_{t,j}$ on U_j such that $\overline{\partial}s_{t,j}=w_t$ on U_j and

$$\lim_{t\to-\infty}\int_{U_j}|s_{t,j}|^2_{\omega,h}\mathrm{e}^{-\psi}dV_{X,\omega}=0. \tag{2.17}$$

Then

$$\overline{\partial}\bigg(\theta(\psi-t)\cdot\widetilde{f}-u_t-\sum_j\rho_j s_{t,j}\bigg)=-\sum_j(\overline{\partial}\rho_j)\cdot s_{t,j}\quad\text{on } X$$
$$=-\sum_j(\overline{\partial}\rho_j)\cdot(s_{t,j}-s_{t,i})\quad\text{on } U_i. \tag{2.18}$$

As $\overline{\partial}(s_{t,j}-s_{t,i})=0$ on $U_i\cap U_j$, the difference is holomorphic and the right-hand side of (2.18) is smooth. Moreover, (2.17) shows that these differences converge uniformly to 0, hence the right-hand side of (2.18) converges to 0 in C^∞ topology. The left-hand side implies that this is a coboundary in the C^∞ Dolbeault resolution of $\mathcal{O}_X(K_X\otimes E)$. By applying Lemma 1.4, we conclude that there is a C^∞ section σ_t of $K_X\otimes E$ converging uniformly to 0 on compact subsets of X as $t\to-\infty$, such that $\overline{\partial}\sigma_t=\sum_j(\overline{\partial}\rho_j)\cdot s_{t,j}$ on X. This implies that

$$\widetilde{f}_t:=\theta(\psi-t)\cdot\widetilde{f}-u_t-\sum_j\rho_j s_{t,j}+\sigma_t$$

is holomorphic on X. Hörmander's L^2 estimates also produce local smooth solutions $\sigma_{t,i}$ on U_i with the additional property that $\lim_{t\to-\infty}\int_{U_i}|\sigma_{t,i}|^2_{\omega,h}\mathrm{e}^{-\psi}dV_{X,\omega}=0$. Therefore,

$$\widetilde{f}_{t,i}:=\theta(\psi-t)\cdot\widetilde{f}-u_t-\sum_j\rho_j s_{t,j}+\sigma_{t,i}$$

is holomorphic on U_i and $\widetilde{f}_t-\widetilde{f}_{t,i}$ converges uniformly to 0 on compact subsets of U_i. However, by construction, $\widetilde{f}_{t,i}-\widetilde{f}_i$ is a holomorphic section on U_i that satisfies the L^2 estimate with respect to the weight $\mathrm{e}^{-\psi}$, hence $\widetilde{f}_{t,i}-\widetilde{f}_i$ is a section of $\mathcal{O}_X(K_X\otimes E)\otimes\mathcal{I}(\mathrm{e}^{-\psi})$ on U_i, in other words the image of $\widetilde{f}_{t,i}$ in

$$H^0(U_i,\mathcal{O}_X(K_X\otimes E)\otimes\mathcal{O}_X/\mathcal{I}(\mathrm{e}^{-\psi}))$$

coincides with $f_{|U_i}$. As a consequence, the image of $\widetilde{f}_t$ in

$$H^0(X,\mathcal{O}_X(K_X\otimes E)\otimes\mathcal{O}_X/\mathcal{I}(\mathrm{e}^{-\psi}))=H^0(Y,(K_X\otimes E)_{|Y})$$

converges to f. By the direct image argument used in the preliminary section, this density property implies the surjectivity of the restriction morphism to Y.

(b) We now prove the theorem for the general case when $h=\mathrm{e}^{-\varphi}$ is not necessarily smooth. We can reduce ourselves to the case when ψ has divisorial singularities (see [5] or the next section for a more detailed argument). Let us pick a section

$$f\in H^0(X,\mathcal{O}_X(K_X\otimes E)\otimes\mathcal{I}(h)/\mathcal{I}(h\mathrm{e}^{-\psi})).$$

By using the same reasoning as in Step 1, we can find a smooth extension $\widetilde{f}\in\mathcal{C}^\infty(X,K_X\otimes E)$ of f such that

$$\int_X|\overline{\partial}\widetilde{f}|^2_{\omega,h}\mathrm{e}^{-\psi}dV_{X,\omega}<+\infty. \tag{2.19}$$

For every $t\in\mathbb{Z}^-$ fixed, as ψ has divisorial singularities, we still have

$$\Theta_{E,h}+(1+\delta\eta_t^{-1}\chi_t'(\psi))(\mathrm{i}\partial\overline{\partial}\psi)_{\mathrm{ac}}\geqslant0\quad\text{on}\quad X,$$

where $(\mathrm{i}\partial\overline{\partial}\psi)_{\mathrm{ac}}$ is the absolutely continuous part of $\mathrm{i}\partial\overline{\partial}\psi$. The regularization techniques of [6] and [4, Theorem 1.7, Remark 1.11] (see also the next section) produce a family of singular metrics $\{h_{t,\varepsilon}\}_{k=1}^{+\infty}$, which are smooth in the complement $X\setminus Z_{t,\varepsilon}$ of an analytic set, such that $\mathcal{I}(h_{t,\varepsilon})=\mathcal{I}(h)$, $\mathcal{I}(h_{t,\varepsilon}\mathrm{e}^{-\psi})=\mathcal{I}(h\mathrm{e}^{-\psi})$ and

$$\Theta_{E,h_{t,\varepsilon}}+(1+\delta\eta_t^{-1}\chi_t'(\psi))\,\mathrm{i}\partial\overline{\partial}\psi\geqslant-\frac{1}{2}\varepsilon\omega\quad\text{on}\quad X.$$

The additional error term $-\frac{1}{2}\varepsilon\omega$ is irrelevant when we use Proposition 1.3, as it is absorbed by taking the hermitian operator $B+\varepsilon I$. Therefore for every $t\in\mathbb{Z}^-$, with the adjustment $\varepsilon=\mathrm{e}^{\alpha t}$, $\alpha\in\,]0,m_{k+1}-1[$, we can find a singular metric $h_t = h_{t,\varepsilon}$, which is smooth in the complement $X \setminus Z_t$ of an analytic set, such that $\mathcal{I}(h_t) = \mathcal{I}(h)$, $\mathcal{I}(h_t \mathrm{e}^{-\psi}) = \mathcal{I}(h\mathrm{e}^{-\psi})$ and $h_t \uparrow h$ as $t \to -\infty$, and approximate solutions of the $\overline{\partial}$-equation such that

$$\overline{\partial}(\theta(\psi - t) \cdot \widetilde{f} - u_t) = w_t, \quad \int_X |u_t|^2_{\omega,h_t} \mathrm{e}^{-\psi} dV_{X,\omega} < +\infty$$

and

$$\lim_{t\to-\infty} \int_X |w_t|^2_{\omega,h_t} \mathrm{e}^{-\psi} dV_{X,\omega} = 0.$$

Proposition 1.3 can indeed be applied since $X \setminus Z_t$ is complete Kähler (at least after we shrink X a little bit as $X' = \pi^{-1}(S')$, see [1]). The theorem is then proved by using the same argument as in Step 3; it is enough to notice that the holomorphic sections $s_{t,j} - s_{t,i}$ and $\widetilde{f}_{t,i} - \widetilde{f}_i$ satisfy the L^2-estimate with respect to (h_t, ψ) (instead of the expected (h, ψ)), but the multiplier ideal sheaves involved are unchanged. The Hausdorff property is applied to the cohomology group $H^1(X, \mathcal{O}_X(K_X \otimes E) \otimes \mathcal{I}(h))$ instead of $H^1(X, K_X \otimes E)$, and the density property to the morphism of direct image sheaves

$$\pi_*(\mathcal{O}_X(K_X \otimes E) \otimes \mathcal{I}(h)) \to \pi_*(\mathcal{O}_X(K_X \otimes E) \otimes \mathcal{I}(h)/\mathcal{I}(h\mathrm{e}^{-\psi}))$$

over the Stein space S. □

Proof of the extension theorem for degree q cohomology classes. The reasoning is extremely similar, so we only explain the few additional arguments needed. In fact, Proposition 1.3 can be applied right away to arbitrary (n, q)-forms with $q \geqslant 1$, and the twisted Bochner-Kodaira-Nakano inequality yields exactly the same estimates. Any cohomology class in

$$H^q(Y, \mathcal{O}_X(K_X \otimes E) \otimes \mathcal{I}(h)/\mathcal{I}(h\mathrm{e}^{-\psi}))$$

is represented by a holomorphic Čech q-cocycle with respect to the Stein covering $\mathcal{U} = (U_i)$, say

$$(c_{i_0\cdots i_q}), \quad c_{i_0\cdots i_q} \in H^0(U_{i_0} \cap \cdots \cap U_{i_q}, \mathcal{O}_X(K_X \otimes E) \otimes \mathcal{I}(h)/\mathcal{I}(h\mathrm{e}^{-\psi})).$$

By the standard sheaf theoretic isomorphisms with Dolbeault cohomology (see [2]), this class is represented by a smooth (n, q)-form

$$f = \sum_{i_0,\ldots,i_q} c_{i_0\cdots i_q} \rho_{i_0} \overline{\partial}\rho_{i_1} \wedge \cdots \wedge \overline{\partial}\rho_{i_q}$$

by means of a partition of unity (ρ_i) subordinate to (U_i). This form is to be interpreted as a form on the (non reduced) analytic subvariety Y associated with the ideal sheaf $\mathcal{J} = \mathcal{I}(h\mathrm{e}^{-\psi}) : \mathcal{I}(h)$ and the structure sheaf $\mathcal{O}_Y = \mathcal{O}_X/\mathcal{J}$. We get an extension as a smooth (no longer $\overline{\partial}$-closed) (n, q)-form on X by taking

$$\widetilde{f} = \sum_{i_0,\ldots,i_q} \widetilde{c}_{i_0\cdots i_q} \rho_{i_0} \overline{\partial}\rho_{i_1} \wedge \cdots \wedge \overline{\partial}\rho_{i_q},$$

where $\widetilde{c}_{i_0\cdots i_q}$ is an extension of $c_{i_0\cdots i_q}$ from $U_{i_0} \cap \cdots \cap U_{i_q} \cap Y$ to $U_{i_0} \cap \cdots \cap U_{i_q}$. Again, we can find approximate L^2 solutions of the $\overline{\partial}$-equation such that

$$\overline{\partial}(\theta(\psi - t) \cdot \widetilde{f} - u_t) = w_t, \quad \int_X |u_t|^2_{\omega,h_t} \mathrm{e}^{-\psi} dV_{X,\omega} < +\infty$$

and

$$\lim_{t\to-\infty} \int_X |w_t|^2_{\omega,h_t} \mathrm{e}^{-\psi} dV_{X,\omega} = 0.$$

The difficulty is that L^2 sections cannot be restricted in a continuous way to a subvariety. In order to overcome this problem, we play again the game of returning to Čech cohomology by solving inductively $\overline{\partial}$-equations for w_t on $U_{i_0} \cap \cdots \cap U_{i_k}$, until we reach an equality

$$\overline{\partial}(\theta(\psi - t) \cdot \widetilde{f} - \widetilde{u}_t) = \widetilde{w}_t := - \sum_{i_0,\ldots,i_{q-1}} s_{t,i_0\cdots i_q} \overline{\partial}\rho_{i_0} \wedge \overline{\partial}\rho_{i_1} \wedge \cdots \wedge \overline{\partial}\rho_{i_q} \tag{2.20}$$

with holomorphic sections $s_{t,I} = s_{t,i_0\cdots i_q}$ on $U_I = U_{i_0} \cap \cdots \cap U_{i_q}$, such that

$$\lim_{t\to-\infty} \int_{U_I} |s_{t,I}|^2_{\omega,h_t} \mathrm{e}^{-\psi} dV_{X,\omega} = 0.$$

Then the right-hand side of (2.20) is smooth, and more precisely has coefficients in the sheaf $\mathcal{C}^\infty \otimes_{\mathcal{O}} \mathcal{I}(h\mathrm{e}^{-\psi})$, and $\widetilde{w}_t \to 0$ in C^∞ topology. A priori, $\widetilde{u}_t$ is an L^2 (n,q)-form equal to u_t plus a combination $\sum \rho_i s_{t,i}$ of the local solutions of $\overline{\partial} s_{t,i} = w_t$, plus $\sum \rho_i s_{t,i,j} \wedge \overline{\partial}\rho_j$ where $\overline{\partial} s_{t,i,j} = s_{t,j} - s_{t,i}$, plus etc., and is such that

$$\int_X |\widetilde{u}_t|^2_{\omega,h_t} \mathrm{e}^{-\psi} dV_{X,\omega} < +\infty.$$

Since $H^q(X, \mathcal{O}_X(K_X \otimes E) \otimes \mathcal{I}(h\mathrm{e}^{-\psi}))$ can be computed with the L^2_{loc} resolution of the coherent sheaf, or alternatively with the $\overline{\partial}$-complex of $(n,\bullet)$-forms with coefficients in $\mathcal{C}^\infty \otimes_{\mathcal{O}} \mathcal{I}(h\mathrm{e}^{-\psi})$, we may assume that $\widetilde{u}_t \in \mathcal{C}^\infty \otimes_{\mathcal{O}} \mathcal{I}(h\mathrm{e}^{-\psi})$, after playing again with Čech cohomology. Lemma 1.4 yields a sequence of smooth (n,q)-forms σ_t with coefficients in $\mathcal{C}^\infty \otimes_{\mathcal{O}} \mathcal{I}(h)$, such that $\overline{\partial}\sigma_t = \widetilde{w}_t$ and $\sigma_t \to 0$ in C^∞-topology. Then $\widetilde{f}_t = \theta(\psi - t) \cdot \widetilde{f} - \widetilde{u}_t - \sigma_t$ is a $\overline{\partial}$-closed (n,q)-form on X with values in $\mathcal{C}^\infty \otimes_{\mathcal{O}} \mathcal{I}(h) \otimes \mathcal{O}_X(E)$, whose image in $H^q(X, \mathcal{O}_X(K_X \otimes E) \otimes \mathcal{I}(h)/\mathcal{I}(h\mathrm{e}^{-\psi}))$ converges to $\{f\}$ in C^∞ Fréchet topology. We conclude by a density argument on the Stein space S, by looking at the coherent sheaf morphism

$$R^q\pi_*(\mathcal{O}_X(K_X \otimes E) \otimes \mathcal{I}(h)) \to R^q\pi_*(\mathcal{O}_X(K_X \otimes E) \otimes \mathcal{I}(h)/\mathcal{I}(h\mathrm{e}^{-\psi})). \qquad \square$$

3 An alternative proof based on injectivity theorems

We give here an alternative proof based on injectivity theorems, in the case when X is compact Kähler. The case of a holomorphically convex manifold is entirely similar, so we will content ourselves to indicate the required additional arguments at the end.

Proof of Theorem 1.1. First of all, we reduce the proof of Theorem 1.1 to the case when ψ has divisorial singularities. Since ψ has analytic singularities, there exists a modification $\pi\colon X' \to X$ such that the pull-back $\pi^*\psi$ has divisorial singularities. For the singular hermitian line bundle $(E', h') := (\pi^*E, \pi^*h)$ and the quasi-psh function $\psi' := \pi^*\psi$, we can easily check that

$$\pi_*(K_{X'} \otimes E' \otimes \mathcal{I}(h'\mathrm{e}^{-\psi'})) = K_X \otimes E \otimes \mathcal{I}(h\mathrm{e}^{-\psi}),$$
$$\pi_*(K_{X'} \otimes E' \otimes \mathcal{I}(h')) = K_X \otimes E \otimes \mathcal{I}(h).$$

Hence we obtain the following commutative diagram:

$$\begin{array}{ccc} H^q(X, K_X \otimes E \otimes \mathcal{I}(h\mathrm{e}^{-\psi})) & \xrightarrow{\ f\ } & H^q(X, K_X \otimes E \otimes \mathcal{I}(h)) \\ \cong \big\downarrow \pi^* & \circlearrowleft & \big\downarrow \pi^* \\ H^q(X', K_{X'} \otimes E' \otimes \mathcal{I}(h'\mathrm{e}^{-\psi'})) & \xrightarrow{\ g\ } & H^q(X', K_{X'} \otimes E' \otimes \mathcal{I}(h')), \end{array}$$

where f and g are the morphisms induced by the natural inclusions and π^* is the natural edge morphism. It follows that the left edge morphism π^* is an isomorphism since the curvature of the singular hermitian metric $h'\mathrm{e}^{-\psi'}$ on E' is semi-positive by the assumption. Indeed, even if h' does not have analytic singularities, we can see that

$$R^q\pi_*(K_{X'} \otimes E' \otimes \mathcal{I}(h'\mathrm{e}^{-\psi'})) = 0 \quad \text{for every} \quad q > 0$$

by [16, Corollary 1.5]. (In the case of X being a projective variety, a relatively easy proof can be found in [10].) If Theorem 1.1 can be proven when ψ has divisorial singularities, it follows that the morphism g in the above diagram is injective since $(E', h') = (\pi^*E, \pi^*h)$ and $\psi' = \pi^*\psi$ satisfy the assumptions in Theorem 1.1 and ψ' has divisorial singularities. Therefore, the morphism f is also injective by the commutative diagram.

Now we explain the idea of the proof of Theorem 1.1. If we can obtain equisingular approximations h_ε of h satisfying the following properties:

$$\Theta_{E,h_\varepsilon} + \mathrm{i}\partial\overline{\partial}\psi \geqslant -\varepsilon\omega \quad \text{and} \quad \Theta_{E,h_\varepsilon} + (1+\delta)\mathrm{i}\partial\overline{\partial}\psi \geqslant -\varepsilon\omega,$$

then a proof similar to [10] works, where ω is a fixed Kähler form on X. In the case when ψ has divisorial singularities, we can attain either of the above curvature properties, but we do not know whether we can attain them at the same time. For this reason, we look for an essential curvature condition arising from the assumptions on the curvatures in Theorem 1.1, in order to use the "twisted" Bochner-Kodaira-Nakano identity.

From now on, we consider a quasi-psh ψ with divisorial singularities. Then there exist an effective $\mathbb{R}$-divisor D and a smooth $(1,1)$-form γ on X such that

$$\frac{\mathrm{i}}{2\pi}\partial\overline{\partial}\psi = [D] + \frac{1}{2\pi}\gamma$$

in the sense of $(1,1)$-currents, where $[D]$ denotes the current of the integration over D. For the irreducible decomposition $D = \sum_{i=1}^{N} a_i D_i$ and the defining section t_i of D_i, we can take a smooth hermitian metric b_i on D_i such that

$$\mathrm{e}^{\psi} = |s|_b^2 := |t_1|_{b_1}^{2a_1}|t_2|_{b_2}^{2a_2}\cdots|t_N|_{b_N}^{2a_N} \quad \text{and} \quad -\gamma = \Theta_b(D) := \sum_{i=1}^{N} a_i\Theta_{b_i}(D_i).$$

For a positive number $0 < c \ll 1$, we define the continuous functions σ and η on X by

$$\sigma = \sigma_c := \log(|s|_b^2 + c) \quad \text{and} \quad \eta = \eta_c := \frac{1}{c} - \chi(\sigma),$$

where $\chi(t) := t - \log(-t)$.

Remark 3.1. (i) We may assume that $|s|_b^2 < 1/5$ by subtracting a positive constant from ψ. Furthermore, we may assume that $\sigma < \log(1/5)$ and $1 < \chi'(\sigma) < 7/4$ by choosing a sufficiently small $c > 0$.

(ii) Furthermore, the function η is a continuous function on X with $\eta > 1/c$. The function η is smooth on $X \setminus D$, but it need not be smooth on X since $|s|_b^2$ is not smooth in the case when $0 < a_i < 1$ for some i.

Throughout the proof, we fix a Kähler form ω on X. The following proposition gives a suitable approximation of a singular hermitian metric h on E, which enables us to use the twisted Bochner-Kodaira-Nakano identity. The proof is based on the argument in [9,17] and the equisingular approximation theorem in [6, Theorem 2.3].

Proposition 3.2. *There exist singular hermitian metrics $\{h_\varepsilon\}_{0<\varepsilon\ll 1}$ on E with the following properties*:

(a) *h_ε is smooth on $X \setminus Z_\varepsilon$, where Z_ε is a proper subvariety on X.*

(b) *$h_{\varepsilon'} \leqslant h_{\varepsilon''} \leqslant h$ holds on X for $\varepsilon' > \varepsilon'' > 0$.*

(c) *$\mathcal{I}(h) = \mathcal{I}(h_\varepsilon)$ and $\mathcal{I}(h\mathrm{e}^{-\psi}) = \mathcal{I}(h_\varepsilon\mathrm{e}^{-\psi})$ on X.*

(d) *$\eta(\Theta_{h_\varepsilon}(E) + \gamma) - \mathrm{i}\partial\overline{\partial}\eta - \eta^{-2}\mathrm{i}\partial\eta \wedge \overline{\partial}\eta \geqslant -\varepsilon\omega$ on $X \setminus D$.*

(e) *For arbitrary $t > 0$, by taking a sufficiently small $\varepsilon > 0$, we have*

$$\int \mathrm{e}^{-t\phi} - \mathrm{e}^{-t\phi_\varepsilon} < \infty,$$

where ϕ (resp. ϕ_ε) is a local weight of h (resp. h_ε).

Proof. We fix a sufficiently small c with $7c/4 \leqslant \delta$. Then, by Remark 3.1, we can easily check that

$$\frac{\chi'(\sigma)|s|_b^2}{\eta(|s|_b^2 + c)} \leqslant \frac{7}{4\eta} \leqslant \frac{7}{4}c \leqslant \delta.$$

In particular, it follows that

$$\Theta_h + \left(1 + \frac{\chi'(\sigma)|s|_b^2}{\eta(|s|_b^2 + c)}\right)\gamma \geqslant 0 \quad \text{on} \quad X$$

since ψ has divisorial singularities and satisfies the assumptions in Theorem 1.1. By applying the equisingular approximation theorem (see [6, Theorem 2.3]) to h, we can take singular hermitian metrics $\{h_\varepsilon\}_{0<\varepsilon\ll 1}$ on E satisfying Properties (a), (b), (e), the former conclusion of (c), and the following curvature property:

$$\Theta_{h_\varepsilon} + \left(1 + \frac{\chi'(\sigma)|s|_b^2}{\eta(|s|_b^2 + c)}\right)\gamma \geqslant -\varepsilon\omega \quad \text{on} \quad X.$$

Now we check Property (d) from the above curvature property. The function η may not be smooth on X, but it is smooth on $X \setminus D$. Therefore the same computation as in [9, 17] works on $X \setminus D$. In particular, from a complicated but straightforward computation, we obtain

$$-\mathrm{i}\partial\overline{\partial}\eta = -\frac{\chi'(\sigma)|s|_b^2}{|s|_b^2 + c}\Theta_b(D) + \left(\frac{c}{\chi'(\sigma)|s|_b^2} + \frac{\chi''(\sigma)}{\chi'(\sigma)^2}\right)\mathrm{i}\partial\eta \wedge \overline{\partial}\eta$$

on $X \setminus D$ (see [9] for the precise computation). Then, by $-\gamma = \Theta_b(D)$ on $X \setminus D$, we can see that

$$\begin{aligned}
&\eta(\Theta_{h_\varepsilon}(E) + \gamma) - \mathrm{i}\partial\overline{\partial}\eta - \frac{1}{\eta^2}\mathrm{i}\partial\eta \wedge \overline{\partial}\eta \\
&= \left(\frac{c}{\chi'(\sigma)|s|_b^2} + \frac{\chi''(\sigma)}{\chi'(\sigma)^2} - \frac{1}{\eta^2}\right)\mathrm{i}\partial\eta \wedge \overline{\partial}\eta + \eta\left(\Theta_{h_\varepsilon}(E) + \left(1 + \frac{\chi'(\sigma)|s|_b^2}{\eta(|s|_b^2 + c)}\right)\gamma\right) \\
&\geqslant \left(\frac{c}{\chi'(\sigma)|s|_b^2} + \frac{\chi''(\sigma)}{\chi'(\sigma)^2} - \frac{1}{\eta^2}\right)\mathrm{i}\partial\eta \wedge \overline{\partial}\eta - \varepsilon\eta\omega
\end{aligned}$$

on $X \setminus D$. A straightforward computation yields that $\chi''(\sigma)/\chi'(\sigma)^2 \geqslant 1/\eta^2$, and thus the first term is semi-positive. Since η is bounded above, we infer that property (d) holds.

Finally, we check the last conclusion of Property (c) by proving the following lemma, which can be obtained from the strong openness theorem (see [11, 12, 14]) and Property (e).

Lemma 3.3. *For a quasi-psh function φ, we have $\mathcal{I}(h\mathrm{e}^{-\varphi}) = \mathcal{I}(h_\varepsilon \mathrm{e}^{-\varphi})$. In particular, we obtain the last conclusion of Property* (c).

Proof. We have the inclusion $\mathcal{I}(h\mathrm{e}^{-\varphi}) \subset \mathcal{I}(h_\varepsilon\mathrm{e}^{-\varphi})$ by $h_\varepsilon \leqslant h$. To get the converse inclusion, we consider a local holomorphic function g such that $|g|^2\mathrm{e}^{-\varphi-\phi_\varepsilon}$ is integrable, where ϕ_ε (resp. ϕ) is a local weight of h_ε (resp. h). Then Hölder's inequality yields

$$\begin{aligned}
\int |g|^2\mathrm{e}^{-\phi-\varphi} &= \int |g|^2\mathrm{e}^{-\varphi-\phi_\varepsilon}\mathrm{e}^{-\phi+\phi_\varepsilon} \\
&\leqslant \left(\int |g|^{2p}\mathrm{e}^{-p(\varphi+\phi_\varepsilon)}\right)^{1/p} \cdot \left(\int \mathrm{e}^{-q(\phi-\phi_\varepsilon)}\right)^{1/q},
\end{aligned}$$

where p and q are real numbers such that $1/p + 1/q = 1$ and $p > 1$. By the strong openness theorem, the function $|g|^{2p}\mathrm{e}^{-p(\varphi+\phi_\varepsilon)}$ is integrable when p is sufficiently close to one. On the other hand, we have

$$\int \mathrm{e}^{-q(\phi-\phi_\varepsilon)} - 1 = \int \mathrm{e}^{q\phi_\varepsilon}(\mathrm{e}^{-q\phi} - \mathrm{e}^{-q\phi_\varepsilon}) \leqslant \sup \mathrm{e}^{q\phi_\varepsilon}\int(\mathrm{e}^{-q\phi} - \mathrm{e}^{-q\phi_\varepsilon}).$$

The right-hand side is finite for a sufficiently small ε by Property (e). □

This concludes the proof of Proposition 3.2. □

From now on, we proceed to prove Theorem 1.1 by using Proposition 3.2. In the same way as in [10, Section 5], one constructs a family of complete Kähler forms $\{\omega_{\varepsilon,\delta}\}_{0<\delta\ll 1}$ on $Y_\varepsilon := X \setminus (Z_\varepsilon \cup D)$ with the following properties:

(A) $\omega_{\varepsilon,\delta}$ is a complete Kähler form on $Y_\varepsilon := X \setminus (Z_\varepsilon \cup D)$ for every $\delta > 0$.

(B) $\omega_{\varepsilon,\delta} \geqslant \omega$ on Y_ε for every $\delta \geqslant 0$.

(C) For every point p in X, there exists a bounded function $\Psi_{\varepsilon,\delta}$ on an open neighborhood B_p such that $\omega_{\varepsilon,\delta} = \mathrm{i}\partial\overline{\partial}\Psi_{\varepsilon,\delta}$ on B_p and $\Psi_{\varepsilon,\delta}$ converges uniformly to a bounded function that is independent of ε.

For simplicity, we put $H := h\mathrm{e}^{-\psi}$ and $H_\varepsilon := h_\varepsilon \mathrm{e}^{-\psi}$. We consider a cohomology class $\beta \in H^q(X, K_X \otimes E \otimes \mathcal{I}(H))$ such that $\beta = 0 \in H^q(X, K_X \otimes E \otimes \mathcal{I}(h))$. By the De Rham-Weil isomorphism

$$H^q(X, K_X \otimes E \otimes \mathcal{I}(H)) \cong \frac{\operatorname{Ker}\overline{\partial} : L^{n,q}_{(2)}(E)_{H,\omega} \to L^{n,q+1}_{(2)}(E)_{H,\omega}}{\operatorname{Im}\overline{\partial} : L^{n,q-1}_{(2)}(E)_{H,\omega} \to L^{n,q}_{(2)}(E)_{H,\omega}},$$

the cohomology class β can be represented by a $\overline{\partial}$-closed E-valued (n,q)-form u with $\|u\|_{H,\omega} < \infty$ (i.e., $\beta = \{u\}$). Here, $L^{n,\bullet}_{(2)}(E)_{H,\omega}$ is the L^2-space of E-valued $(n,\bullet)$-forms on X with respect to the L^2-norm $\|\bullet\|_{H,\omega}$ defined by

$$\|\bullet\|^2_{H,\omega} := \int_X |\bullet|^2_{H,\omega}\, dV_\omega,$$

where $dV_\omega := \omega^n/n!$ and $n := \dim X$. For the L^2-norm $\|\bullet\|_{H_\varepsilon,\omega_{\varepsilon,\delta}}$ defined by

$$\|\bullet\|^2_{\varepsilon,\delta} := \|\bullet\|^2_{H_\varepsilon,\omega_{\varepsilon,\delta}} := \int_X |\bullet|^2_{H_\varepsilon,\omega_{\varepsilon,\delta}}\, dV_{\omega_{\varepsilon,\delta}},$$

one can easily check that

$$\|u\|_{\varepsilon,\delta} \leqslant \|u\|_{H,\omega_{\varepsilon,\delta}} \leqslant \|u\|_{H,\omega} < \infty. \tag{3.1}$$

Indeed, the first inequality is obtained from Property (b), and the second inequality is obtained from Property (B) for $\omega_{\varepsilon,\delta}$ (for example see [10, Lemma 2,4]). In particular, we see that u belongs to the L^2-space

$$L^{n,q}_{(2)}(E)_{\varepsilon,\delta} := L^{n,q}_{(2)}(Y_\varepsilon, E)_{H_\varepsilon,\omega_{\varepsilon,\delta}}$$

of E-valued (n,q)-forms on Y_ε (not X) with respect to $\|\bullet\|_{\varepsilon,\delta}$. By the orthogonal decomposition (see for example [15, Proposition 5.8])

$$L^{n,q}_{(2)}(F)_{\varepsilon,\delta} = \operatorname{Im}\overline{\partial} \oplus \mathcal{H}^{n,q}_{\varepsilon,\delta}(F) \oplus \operatorname{Im}\overline{\partial}^*_{\varepsilon,\delta},$$

the E-valued form u can be decomposed as follows:

$$u = \overline{\partial} w_{\varepsilon,\delta} + u_{\varepsilon,\delta} \quad \text{for some} \quad w_{\varepsilon,\delta} \in \operatorname{Dom}\overline{\partial} \subset L^{n,q-1}_{(2)}(E)_{\varepsilon,\delta}, \quad \text{and} \quad u_{\varepsilon,\delta} \in \mathcal{H}^{n,q}_{\varepsilon,\delta}(E). \tag{3.2}$$

Here, $\overline{\partial}^*_{\varepsilon,\delta}$ is (the maximal extension of) the formal adjoint of the $\overline{\partial}$-operator and $\mathcal{H}^{n,q}_{\varepsilon,\delta}(E)$ is the space of harmonic forms on Y_ε, i.e.,

$$\mathcal{H}^{n,q}_{\varepsilon,\delta}(E) := \{w \in L^{n,q}_{(2)}(E)_{\varepsilon,\delta} \,|\, \overline{\partial} w = 0 \text{ and } \overline{\partial}^*_{\varepsilon,\delta} w = 0\}.$$

Proposition 3.4 (resp. Proposition 3.5) can be proved by the same method as in [10, Propositions 5.4, 5.6 and 5.7] (resp. [10, Proposition 5.9, 5.10]), so we omit the proofs here.

Proposition 3.4. *If we have*

$$\varliminf_{\varepsilon\to 0} \varliminf_{\delta\to 0} \|u_{\varepsilon,\delta}\|_{K,h_\varepsilon,\omega_{\varepsilon,\delta}} = 0,$$

for every relatively compact set $K \Subset X \setminus D$, then the cohomology class β is zero in $H^q(X, K_X \otimes E \otimes \mathcal{I}(H))$. Here, $\|\bullet\|_{K,h_\varepsilon,\omega_{\varepsilon,\delta}}$ denotes the L^2-norm on K with respect to h_ε (not H_ε) and $\omega_{\varepsilon,\delta}$.

Proposition 3.5. *There exists $v_{\varepsilon,\delta} \in L^{n,q-1}_{(2)}(E)_{h_\varepsilon,\omega_{\varepsilon,\delta}}$ satisfying the following properties:*

$$\overline{\partial} v_{\varepsilon,\delta} = u_{\varepsilon,\delta} \quad \textit{and} \quad \varlimsup_{\delta\to 0} \|v_{\varepsilon,\delta}\|_{\varepsilon,\delta} \quad \textit{is bounded by a constant independent of } \varepsilon. \tag{3.3}$$

Remark 3.6. In general, we have $L^{n,\bullet}_{(2)}(E)_{\varepsilon,\delta} = L^{n,\bullet}_{(2)}(E)_{H_\varepsilon,\omega_{\varepsilon,\delta}} \subsetneqq L^{n,\bullet}_{(2)}(E)_{h_\varepsilon,\omega_{\varepsilon,\delta}}$, and thus $v_{\varepsilon,\delta}$ may not be L^2-integrable with respect to H_ε.

For the above solution $v_{\varepsilon,\delta}$ of the $\overline{\partial}$-equation, by using the density lemma, we can take a family of smooth E-valued forms $\{v_{\varepsilon,\delta,k}\}_{k=1}^{\infty}$ with the following properties:

$$v_{\varepsilon,\delta,k} \to v_{\varepsilon,\delta} \quad \text{and} \quad \overline{\partial} v_{\varepsilon,\delta,k} \to \overline{\partial} v_{\varepsilon,\delta} = u_{\varepsilon,\delta} \quad \text{in} \quad L^{n,\bullet}_{(2)}(E)_{h_\varepsilon,\omega_{\varepsilon,\delta}}. \tag{3.4}$$

Now we consider the level set $X_c := \{x \in X \mid -|s|^2_b < c\} \Subset X \setminus D$ for a negative number c. The set of the critical values of $|s|^2_b$ is of Lebesgue measure zero from Sard's theorem. Hence, for a given relatively compact $K \Subset X \setminus D$, we can choose $-1 \ll c < 0$ such that

$$K \Subset X_c := \{x \in X \mid -|s|^2_b < c\} \quad \text{and} \quad d|s|^2_b \neq 0 \quad \text{at every point in} \quad \partial X_c.$$

Then, by [16, Proposition 2.5 and Remark 2.6] (see also [8, Proposition (1.3.2)]), we obtain

$$\langle\!\langle \overline{\partial} v_{\varepsilon,\delta,k}, u_{\varepsilon,\delta}\rangle\!\rangle_{X_d,h_\varepsilon,\omega_{\varepsilon,\delta}} = \langle\!\langle v_{\varepsilon,\delta,k}, \overline{\partial}^*_{h_\varepsilon,\omega_{\varepsilon,\delta}} u_{\varepsilon,\delta}\rangle\!\rangle_{X_d,h_\varepsilon,\omega_{\varepsilon,\delta}} - (\!(v_{\varepsilon,\delta,k}, (\overline{\partial}|s|^2_b)^* u_{\varepsilon,\delta})\!)_{\partial X_d,h_\varepsilon,\omega_{\varepsilon,\delta}} \tag{3.5}$$

for almost all $d \in]c-a, c+a[$, where a is a sufficiently small positive number. Here, $\overline{\partial}^*_{h_\varepsilon,\omega_{\varepsilon,\delta}}$ is the formal adjoint of the $\overline{\partial}$-operator in $L^{n,\bullet}_{(2)}(E)_{h_\varepsilon,\omega_{\varepsilon,\delta}}$ and $(\!(\bullet,\bullet)\!)_{\partial X_d,h_\varepsilon,\omega_{\varepsilon,\delta}}$ is the inner product on the boundary ∂X_d defined by

$$(\!(a,b)\!)_{\partial X_d,h_\varepsilon,\omega_{\varepsilon,\delta}} := \int_{\partial X_d} \langle a,b\rangle_{h_\varepsilon,\omega_{\varepsilon,\delta}}\, dS_{\varepsilon,\delta},$$

for smooth E-valued forms a and b, where $dS_{\varepsilon,\delta}$ denotes the volume form on ∂X_d defined by $dS_{\varepsilon,\delta} := -*d|s|^2_b/\big|d|s|^2_b\big|_{h_\varepsilon,\omega_{\varepsilon,\delta}}$ and $*$ denotes the Hodge star operator with respect to $\omega_{\varepsilon,\delta}$. Note that $dV_{\varepsilon,\delta} = dS_{\varepsilon,\delta} \wedge d|s|^2_b$. One can easily see that

$$\lim_{k\to\infty} \langle\!\langle \overline{\partial} v_{\varepsilon,\delta,k}, u_{\varepsilon,\delta}\rangle\!\rangle_{X_d,h_\varepsilon,\omega_{\varepsilon,\delta}} = \langle\!\langle \overline{\partial} v_{\varepsilon,\delta}, u_{\varepsilon,\delta}\rangle\!\rangle_{X_d,h_\varepsilon,\omega_{\varepsilon,\delta}} = \langle\!\langle u_{\varepsilon,\delta}, u_{\varepsilon,\delta}\rangle\!\rangle_{X_d,h_\varepsilon,\omega_{\varepsilon,\delta}}$$

by (3.4), and thus it is sufficient to show that the right-hand side of (3.5) converges to zero. For this purpose, we first prove the following proposition.

Proposition 3.7. *The following holds*:

$$\lim_{\varepsilon\to 0}\lim_{\delta\to 0} \|(\overline{\partial}|s|^2_b)^* u_{\varepsilon,\delta}\|_{\varepsilon,\delta} = 0.$$

Proof. By Properties (d) and (B), we have

$$\eta(\Theta_{h_\varepsilon}(E) + \gamma) - \mathrm{i}\partial\overline{\partial}\eta \geqslant \eta^{-2}\mathrm{i}\partial\eta \wedge \overline{\partial}\eta - \varepsilon\omega \geqslant \eta^{-2}\mathrm{i}\partial\eta \wedge \overline{\partial}\eta - \varepsilon\omega_{\varepsilon,\delta}.$$

Since $u_{\varepsilon,\delta}$ is harmonic with respect to H_ε and $\omega_{\varepsilon,\delta}$, we have $\overline{\partial}^*_{\varepsilon,\delta} u_{\varepsilon,\delta} = 0$ and $\overline{\partial} u_{\varepsilon,\delta} = 0$. Furthermore, we have $\gamma = \mathrm{i}\partial\overline{\partial}\psi$ on $X \setminus D$. Therefore, we obtain

$$\begin{aligned}
0 &\geqslant -\|\sqrt{\eta} D'^* u_{\varepsilon,\delta}\|^2_{\varepsilon,\delta} \\
&= \|\sqrt{\eta}\overline{\partial} u_{\varepsilon,\delta}\|^2_{\varepsilon,\delta} + \|\sqrt{\eta}\overline{\partial}^*_{\varepsilon,\delta} u_{\varepsilon,\delta}\|^2_{\varepsilon,\delta} - \|\sqrt{\eta} D'^* u_{\varepsilon,\delta}\|^2_{\varepsilon,\delta} \\
&= \langle\!\langle (\eta\Theta_{H_\varepsilon} - \mathrm{i}\partial\overline{\partial}\eta)\Lambda u_{\varepsilon,\delta}, u_{\varepsilon,\delta}\rangle\!\rangle_{\varepsilon,\delta} + 2\mathrm{Re}\langle\!\langle \overline{\partial}\eta \wedge \overline{\partial}^*_{\varepsilon,\delta} u_{\varepsilon,\delta}, u_{\varepsilon,\delta}\rangle\!\rangle_{\varepsilon,\delta} \\
&= \langle\!\langle (\eta\Theta_{H_\varepsilon} - \mathrm{i}\partial\overline{\partial}\eta)\Lambda u_{\varepsilon,\delta}, u_{\varepsilon,\delta}\rangle\!\rangle_{\varepsilon,\delta} \\
&\geqslant \langle\!\langle (\eta^{-2}\mathrm{i}\partial\eta \wedge \overline{\partial}\eta)\Lambda u_{\varepsilon,\delta}, u_{\varepsilon,\delta}\rangle\!\rangle_{\varepsilon,\delta} - \varepsilon q\|u_{\varepsilon,\delta}\|^2_{\varepsilon,\delta}
\end{aligned}$$

from the twisted Bochner-Kodaira-Nakano identity (see [17, Lemma 2.1] or [9, Propositions 2.20 and 2.21]). On the other hand, one can easily check that

$$\begin{aligned}
&\langle\!\langle (\eta^{-2}\mathrm{i}\partial\eta \wedge \overline{\partial}\eta)\Lambda u_{\varepsilon,\delta}, u_{\varepsilon,\delta}\rangle\!\rangle_{\varepsilon,\delta} = \|\eta^{-1}(\overline{\partial}\eta)^* u_{\varepsilon,\delta}\|^2_{\varepsilon,\delta} = \|\eta^{-1} * \partial\eta * u_{\varepsilon,\delta}\|^2_{\varepsilon,\delta}, \\
&\partial\eta = -\chi'(\sigma)\partial\sigma = -\frac{\chi'(\sigma)}{(|s|^2_b + c)}\partial|s|^2_b.
\end{aligned}$$

By the above arguments, we conclude that

$$\varepsilon q\|u_{\varepsilon,\delta}\|^2_{\varepsilon,\delta} \geqslant \left\|\frac{\chi'(\sigma)}{\eta(|s|^2_b+c)}(\overline{\partial}|s|^2_b)^* u_{\varepsilon,\delta}\right\|^2_{\varepsilon,\delta}.$$

It follows that the left-hand side converges to zero from $\|u\|_{H,\omega} \geqslant \|u_{\varepsilon,\delta}\|_{\varepsilon,\delta}$. Furthermore, the function $\chi'(\sigma)/\eta(|s|^2_b+c)$ is bounded below since we have

$$\frac{1}{5} > |s|^2_b, \quad C > \eta, \quad \chi'(\sigma) > 1$$

for some constant C. This completes the proof. □

Finally, we prove the following proposition by using Proposition 3.7.

Proposition 3.8. (i) *For a relatively compact set $K \Subset X \setminus D$, we have*

$$\lim_{\varepsilon\to 0}\lim_{\delta\to 0}\overline{\lim_{k\to 0}}\,\langle\langle v_{\varepsilon,\delta,k}, \overline{\partial}^*_{h_\varepsilon,\delta} u_{\varepsilon,\delta}\rangle\rangle_{K,h_\varepsilon,\omega_{\varepsilon,\delta}} = 0.$$

(ii) *For almost all $d \in\,]c-a, c+a[$, we have*

$$\underline{\lim}_{\varepsilon\to 0}\,\underline{\lim}_{\delta\to 0}\,\underline{\lim}_{k\to 0}\,(\!(v_{\varepsilon,\delta,k}, (\overline{\partial}|s|^2_b)^* u_{\varepsilon,\delta})\!)_{\partial X_d,h_\varepsilon,\omega_{\varepsilon,\delta}} = 0.$$

Proof of Proposition 3.8. In general, we have the formula $D'_g u = G^{-1}\partial(Gu)$ for a smooth hermitian metric g, where G is a local function representing g. Let G_ε be a local function representing h_ε. We remark that $G_\varepsilon e^{-\psi}$ is a local function representing H_ε. By the definition of $\overline{\partial}^*_{\varepsilon,\delta} = \overline{\partial}^*_{H_\varepsilon,\omega_{\varepsilon,\delta}}$, we have

$$0 = \overline{\partial}^*_{\varepsilon,\delta} u_{\varepsilon,\delta} = -*D'_{H_\varepsilon} * u_{\varepsilon,\delta} = -*(G_\varepsilon e^{-\psi})^{-1}\partial(G_\varepsilon e^{-\psi} * u_{\varepsilon,\delta}),$$

and thus we obtain $\overline{\partial}^*_{h_\varepsilon,\omega_{\varepsilon,\delta}} u_{\varepsilon,\delta} = -*(G_\varepsilon)^{-1}\partial(e^{\psi}G_\varepsilon e^{-\psi} * u_{\varepsilon,\delta}) = -*\partial e^{\psi} * u_{\varepsilon,\delta}e^{-\psi} = -(\overline{\partial}|s|^2_b)^* u_{\varepsilon,\delta}e^{-\psi}$. Now we have

$$\begin{aligned}\lim_{k\to\infty}|\langle\langle v_{\varepsilon,\delta,k}, \overline{\partial}^*_{h_\varepsilon,\omega_{\varepsilon,\delta}} u_{\varepsilon,\delta}\rangle\rangle_{K,h_\varepsilon,\omega_{\varepsilon,\delta}}| &\leqslant \lim_{k\to\infty}\|v_{\varepsilon,\delta,k}\|_{K,h_\varepsilon,\omega_{\varepsilon,\delta}}\|\overline{\partial}^*_{h_\varepsilon,\omega_{\varepsilon,\delta}} u_{\varepsilon,\delta}\|_{K,h_\varepsilon,\omega_{\varepsilon,\delta}}\\ &= \|v_{\varepsilon,\delta}\|_{K,h_\varepsilon,\omega_{\varepsilon,\delta}}\|\overline{\partial}^*_{h_\varepsilon,\omega_{\varepsilon,\delta}} u_{\varepsilon,\delta}\|_{K,h_\varepsilon,\omega_{\varepsilon,\delta}}.\end{aligned}$$

Since $\overline{\lim}_{\delta\to 0}\|v_{\varepsilon,\delta}\|_{K,h_\varepsilon,\omega_{\varepsilon,\delta}}$ can be bounded by a constant, i.e., independent of ε, it is sufficient to show that $\lim_{\varepsilon\to 0}\lim_{\delta\to 0}\|\overline{\partial}^*_{h_\varepsilon,\omega_{\varepsilon,\delta}} u_{\varepsilon,\delta}\|_{K,h_\varepsilon,\omega_{\varepsilon,\delta}} = 0$. We have $e^{-\psi/2} = 1/|s|_b < C_K$ on K for some constants $C_K > 0$, since K is a relatively compact set in $X \setminus D$. Hence, we see that

$$\|\overline{\partial}^*_{h_\varepsilon,\omega_{\varepsilon,\delta}} u_{\varepsilon,\delta}\|_{K,h_\varepsilon,\omega_{\varepsilon,\delta}} = \|-(\overline{\partial}|s|^2_b)^* u_{\varepsilon,\delta}e^{-\psi}\|_{K,h_\varepsilon,\omega_{\varepsilon,\delta}} \leqslant C_K\|(\overline{\partial}|s|^2_b)^* u_{\varepsilon,\delta}\|_{K,\varepsilon,\delta}.$$

We obtain the first statement (i) since the right-hand side converges to zero by Proposition 3.7.

Now we prove the statement (ii). By Cauchy-Schwarz inequality, we have

$$|(\!(v_{\varepsilon,\delta,k}, (\overline{\partial}|s|^2_b)^* u_{\varepsilon,\delta})\!)_{\partial X_d,h_\varepsilon,\omega_{\varepsilon,\delta}}|^2 \leqslant (\!(v_{\varepsilon,\delta,k}, v_{\varepsilon,\delta,k})\!)_{\partial X_d,h_\varepsilon,\omega_{\varepsilon,\delta}}(\!((\overline{\partial}|s|^2_b)^* u_{\varepsilon,\delta}, (\overline{\partial}|s|^2_b)^* u_{\varepsilon,\delta})\!)_{\partial X_d,h_\varepsilon,\omega_{\varepsilon,\delta}}.$$

By Fubini's theorem, we obtain

$$\int_{d\in]c-a,c+a[}(\!(v_{\varepsilon,\delta,k}, v_{\varepsilon,\delta,k})\!)_{\partial X_d,h_\varepsilon,\omega_{\varepsilon,\delta}}dS_{\varepsilon,\delta} = \int_{c-a<-|s|^2_b<c+a}|v_{\varepsilon,\delta,k}|^2_{h_\varepsilon,\omega_{\varepsilon,\delta}}dV_{\varepsilon,\delta} \leqslant \|v_{\varepsilon,\delta}\|^2_{h_\varepsilon,\delta}.$$

By Fatou's lemma, we see that

$$\int_{d\in]c-a,c+a[}\underline{\lim}_{\varepsilon\to 0}\,\underline{\lim}_{\delta\to 0}\,\underline{\lim}_{k\to\infty}\,(\!(v_{\varepsilon,\delta,k}, v_{\varepsilon,\delta,k})\!)_{\partial X_d,h_\varepsilon,\omega_{\varepsilon,\delta}}dS_{\varepsilon,\delta} \leqslant \underline{\lim}_{\varepsilon\to 0}\,\underline{\lim}_{\delta\to 0}\,\|v_{\varepsilon,\delta}\|^2_{h_\varepsilon,\delta} < \infty.$$

Therefore, the integrand of the left-hand side is finite for almost all $d \in (c-a, c+a)$. On the other hand, by the same argument, we see that

$$\int_{d\in]c-a,c+a[}\underline{\lim}_{\varepsilon\to 0}\,\underline{\lim}_{\delta\to 0}\,(\!((\overline{\partial}|s|^2_b)^* u_{\varepsilon,\delta}, (\overline{\partial}|s|^2_b)^* u_{\varepsilon,\delta})\!)_{\partial X_d,h_\varepsilon,\omega_{\varepsilon,\delta}}dS_{\varepsilon,\delta} \leqslant \underline{\lim}_{\varepsilon\to 0}\,\underline{\lim}_{\delta\to 0}\,\|(\overline{\partial}|s|^2_b)^* u_{\varepsilon,\delta}\|_{h_\varepsilon,\omega_{\varepsilon,\delta}} = 0.$$

Therefore, the integrand of the left-hand side is zero for almost all $d \in (c-a, c+a)$. This completes the proof. □

Theorem 1.1 is now a consequence of Propositions 3.4, 3.8, and (3.5). □

Remark 3.9. In the case of a holomorphically convex manifold, a proof based on injectivity theorems can be obtained by a slight modification of the above proof. The only problem is that an E-valued differential form u representing a given cohomology class is not necessarily L^2-integrable but just locally L^2-integrable. Since X admits a holomorphic map $\pi : X \to S$ to a Stein space S, the form u is L^2-integrable with respect to the metric $he^{-\psi}e^{-\Phi}$ for a suitable psh exhaustion function Φ on X. Then it is not hard to check that our arguments still work by replacing h with $he^{-\Phi}$.

Remark 3.10. It would be interesting to know whether the hypothesis that ψ has analytic singularities is really needed. The main statement still makes sense when ψ has arbitrary analytic singularities, and one may thus guess that the result can be extended by performing a further regularization of ψ.

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