On the Mathematical Heritage of Henri Cartan

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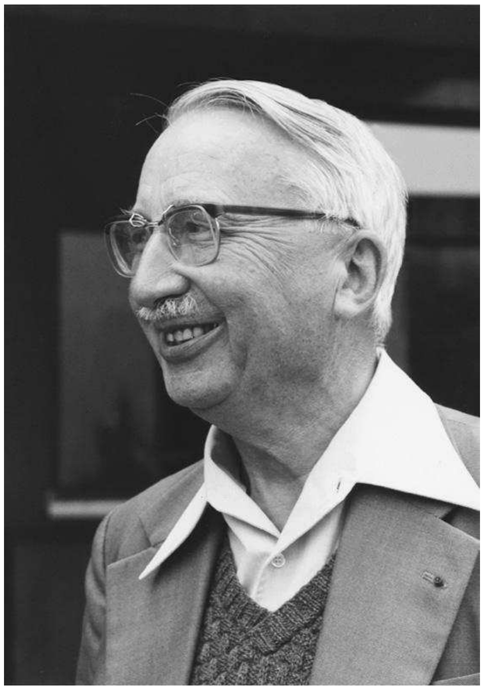
Henri Cartan left us on August 13, 2008 at the age of 104. His influence on generations of mathematicians worldwide has been considerable. In France especially, his role as a professor at École Normale Supérieure in Paris between 1948 and 1965 led him to supervise the PhD theses of Jean Pierre Serre (Fields Medal 1954), René Thom (Fields Medal 1958), and of many other prominent mathematicians such as Pierre Cartier, Jean Cerf, Adrien Douady, Roger Godement, Max Karoubi and Jean-Louis Koszul.

Henri Cartan during the 50's, while he was Professor at the Faculty of Sciences of Paris



Henri Cartan at Oberwolfach, September 3, 1981

However, rather than rewriting history which is well known to many people, I would like here to share lesser known facts about his career and work, especially those related to parts I have been involved with. It is actually quite surprising, in spite of the fact that I was born more than half a century later, how present Henri Cartan still was during my studies. My first mathematical encounter with Cartan was when I was about 12, in 1969. In the earlier years, my father had been a primary school teacher and had decided to go back to Lille University to try to become a math teacher in secondary education; there was a strong national effort in France to recruit teachers, due to the much increased access of pupils and students to higher education, along with a strong research effort in technology and science. I remember quite well that my father had a book with a mysterious title "Théorie élémentaire des fonctions analytiques d'une ou plusieurs complexes" (Hermann, 4th edition from 1961)" [Ca3], by Henri Cartan, which contained magical stuff like contour integrals and residues. I could then of course not understand much of it, but my father was quite absorbed with the book; I was equally impressed by the photograph of Cartan on the cover pages and by the style of the contents which brought obvious similarity with the "New Math" we started being taught at school – namely set theory and symbols like $\cup$, $\cap$, $\in$, $\subset$ ... My father explained to me that Henri Cartan was one of the leading French mathematicians, and that he was one of the founding members of the somewhat secretive Bourbaki group which had been the source of inspiration for the new symbolism and for the reform of education. In France, the leader of the reform commission was A. Lichnerowicz, at least as far as mathematics are concerned, and I got myself involved with the new curriculum in grade ten in 1970. Although overly zealous promoters of the "New Math" made the reform fail less than 15 years later, for instance by pushing abstract set theory even down to kindergarten – a failure which resulted into very bad counter reforms around 1985 – I would like to testify that in spite of harsh criticism sometimes geared towards the reform, what we were taught appeared well thought, quite rigorous and even very exciting. In the rather modest high school I was frequenting at the time, the large majority of my fellows in the science class was certainly enjoying the menu and taking a large benefit. The disaster came only later, from the great excess of reforms applied at earlier stages of education.

In any case, my father left me from that period three books by Henri Cartan, namely the one already described and two other texbooks "Differential calculus" and "Differential forms" [Ca4] (also by Hermann, Paris) which I never ceased using. These books are still widely used and are certainly among the primary references for the courses I have been delivering at the University of Grenoble since 1983. I find it actually quite remarkable that French secondary school teachers of the 1960-1990 era could be taught mathematics in the profound textbooks by mathematicians as Cartan, Dieudonné or Serre, especially in comparison with the general evolution of education in the last two or three decades in France, and other Western countries as well, about which it seems that one cannot be so optimistic ...

In 1975, I entered École Normale Supérieure in Paris and although Henri Cartan had left the École 10 years earlier, he was still very much in the backyard when I began learning holomorphic functions of one variable. His role was eminently stressed in the course proposed to first year students by Michel Hervé, who made great efforts to introduce sheaves to us, e.g. as a means to explain analytic continuation and the maximal domain of existence of a germ of holomorphic function.

Two years later I started a PhD thesis under the supervision of Henri Skoda in Paris, and it is only at this period that I began realizing the full extent of Cartan's contributions to mathematics, in particular those on the theory of coherent analytic sheaves, and his fundamental work in homological algebra and in algebraic topology [CE, CS1]. Taking part of its inspiration from J. Leray's ideas and from the important work of K. Oka in Japan, the celebrated Cartan seminar [Ca2] ran from 1948 to 1964, and as an outcome of the work by its participants, especially H. Cartan, J.-P. Serre and A. Grothendieck, many results concerning topology and holomorphic functions of several variables received their final modern formulation. One should mention especially the proof of the coherence of the ring of holomorphic functions $\mathcal{O}_X$ in an arbitrary number of variables, after ideas of Oka, and the coherence of the ideal sheaf of an analytic set proved by Cartan in 1950. Another important result is the coherence of the sheaf of weakly holomorphic meromorphic functions, which leads to Oka's theorem on the existence of the normalization of any complex space. In this area of complex analysis, Henri Cartan had a long record of collaboration with German mathematicians, in particular H. Behnke and P. Thullen [CT] already before World War II, and after the dramatic events of the war during which Cartan's brother was beheaded, a new era of collaboration started with the younger German generation represented by K. Stein, H. Grauert and R. Remmert. These events were probably among the main reasons for Cartan's strong engagement in politics, especially towards human rights and the construction of Europe; at age 80, Henri Cartan even stood unsuccessfully for election to the European Parliament in 1984, as head of list for a party called "Pour les États-Unis d'Europe", declaring himself to be a European Federalist.

In 1960, pursuing ideas and suggestions of Cartan, Serre [CS2, Se] and Grothendieck [Gt], H. Grauert proved the coherence of direct images of coherent analytic sheaves under proper holomorphic morphisms [Gr]. Actually, a further important coherence theorem was to be discovered more than three decades later as the culmination of work on L^2 techniques by L. Hörmander, E. Bombieri, H. Skoda, Y.T. Siu, A. Nadel and myself : if φ is a plurisubharmonic function, for instance a function of the form $\varphi(z) = c \log |\sum g_j(z)|^2$ where $c > 0$ and the g_j are holomorphic on an open set Ω in $\mathbb{C}^n$, then the sheaf $\mathcal{I}(\varphi) \subset \mathcal{O}_\Omega$ of germs of holomorphic functions f such that $|f|^2 e^{-\varphi}$ is locally integrable is a coherent ideal sheaf [Na]. The sheaf $\mathcal{I}(\varphi)$ is now called the Nadel mutiplier ideal sheaf associated with φ; its algebraic counterpart plays a fundamental role in modern algebraic geometry. The main philosophical reason is probably that L^2 theory is a natural framework for duality and vanishing theorems. It turns out that I got the privilege of explaining this material to young students of École Normale Supérieure around 1992. It was therefore a considerable honor to me that Henri Cartan came to listen to this lecture along with the younger members of the audience. Although he was close to being 90 years old at that time, it was a rare experience for me to have somebody there not missing a word of what I was saying – and sometimes raising embarrassing questions about insufficiently explained points ! I remember that the lecture actually had to be expanded at least half an hour beyond schedule, just to satisfy Cartan's pressing demands ...

During the nineties, my mathematical interests went to the study of entire curves drawn on projective algebraic varieties, especially in the direction of the work of Green-Griffiths [GG] on the "Bloch theorem" – for which they had provided a new proof in 1979. Henri Cartan had also taken an eminent role in this area, which is actually the subject of his PhD thesis [Ca1] under the supervision of Paul Montel, although these achievements are perhaps not as widely known as his later work on sheaves. In any case, Cartan proved after A. Bloch [Bl] several important results in the then nascent Nevanlinna theory, which, in his own terms, can be stated by saying that sequences of entire curves contained in the complex projective n-space minus $(n+2)$ hyperplanes in general position form an "almost normal family": namely, they either have a subsequence which has a limit contained in the complement, or a subsequence which approaches more and more closely a certain union of the "diagonal" hyperplanes. These results were put much later in geometric form by Kobayashi and Kiernan [KK] in terms of the concepts of taut and hyperbolically embedded domains. Very recently, M. Ru-P.M. Wong [RW], E. Nochka and P. Vojta [Vo] found various generalizations and improvements with a more arithmetic flavor. It is remarkable that Cartan's early work already contains many important ingredients such as the use of Nevanlinna estimates for Wronskians, which are still at the heart of contemporary research on the subject, e.g. in the form of the study of the geometry of jet bundles [De1, De2]. I had once again the privilege of explaining some of these modern developments in front of Henri Cartan in 1997, still as vigilant as ever, on the occasion of a celebration of his work by the French Mathematical Society.

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