Perspective on Geometric Analysis:

Geometric Structures

I

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The subject of geometric analysis evolves according to our understanding of geometry and analysis. However, one should say that ideas of algebraic geometry and representation theory have been extremely powerful in both global and local geometry.

In fact, the spectacular idea of using geometry to understand Diophantine problem has already widened our concept of space. The desire to find suitable geometry to accommodate unified field theory in physics would certainly drastically change the scope of geometry in the near future.

In the following lectures, we shall focus on an important branch of geometric analysis: the construction of geometric structures over a given topological space.

There are many kinds of geometric structures; most of them can be classified through the theory of groups and their representations. Some of their structures are motivated by physical science.

The idea of classifying geometric structures through group theory dated back to the famous Erlangen Program of Felix Klein and the later work of E. Cartan.

Most geometric structures are defined by a family of special coordinate charts such that the coordinate transformations or the Jacobian of the coordinate transformations respect some algebraic structure, such as a complex structure, an affine structure, a projective structure or a foliated structure.

Special coordinate systems give connections on natural bundles such as the tangent bundles or some bundles construct from tangent bundles. (Projective structure is related to tangent bundle plus the trivial line bundle, for example.) Connections provide ways to covariantly differentiate vector fields along any curve. For any closed loop at a fixed point, parallel transportation along the loop gives rise to a linear transformation of the tangent space at the point to itself.

The totality of such transformations forms a group called the holonomy group of the connection. This group reflects the algebraic aspects of the geometric structure. Therefore, a necessary condition for the geometric structure to exist is the existence of a connection with a special holonomy group on some natural bundle.

On the other hand, connections give rise to a torsion tensor. In order for the existence of a connection with special holonomy group to become sufficient condition for existence of preferred coordinate systems, we usually require the torsion tensor of the connection to be trivial.

In fact, Cartan-Kähler developed an extensive theory of exterior differential systems to provide proofs that, in the real analytic category, existence of a torsion-free connection with special holonomy group is indeed sufficient for the existence of local coordinate systems for most geometric structures.

The smooth version of Cartan-Kähler theory has not been established in general. The most spectacular work to date was due to Newlander-Nirenberg on the existence of an integrable complex structure, assuming the complex Frobenius condition. In accordance with our previous discussion, Newlander-Nirenberg proved that if the tangent bundle admits a connection with holonomy group $U(n)$ and the torsion form equal to zero, then the manifold admits a complex structure.

The paper of Newlander-Nirenberg is the first application of nonlinear partial differential equations to constructing geometric structures. Cartan-Kähler theory and Newlander-Nirenberg theory are key contributions to the local theory of geometry.

The global theory of geometric structures is quite complicated and is far from being completed. Deformation theory of global structure was initiated by Kodaira-Spencer. Calabi-Vessiniti and Kuranishi studied deformations of complex structures based on Hodge theory. Calabi, Weil, Borel, Matsushima and others studied deformations of geometric structures on deformation of discrete group which eventually lead to the global rigidity theorems of Mostow and Margulius for locally symmetric spaces.

The approaches of using periods of holomorphic forms (Toreli) and geometric invariant theory (Mumford) to study global algebraic structures are very powerful. Geometric invariant theory has modern interpretations in terms of moment maps of group actions on symplectic manifold. Moment maps and symplectic reductions have important consequence on the theory of nonlinear differential equations.

The idea is that stability arose from actions of noncompact groups and based on this, I proposed the following point of view. If there is a noncompact group acting behind a system of nonlinear differential equations, the existence question of such system will be related to the question of the stability of some algebraic structure that defines this system of nonlinear system.

An important example is the existence of the Kähler-Einstein metric on Fano manifolds where I conjectured to be equivalent to the stability of the algebraic manifolds in the sense of geometric invariant theory.

I shall only touch on the part of geometric structures that can be studied by nonlinear differential equations. They are questions that I am fond of.

The basic idea is to use nonlinear differential equations to build geometric structures which in turn can be used to solve problems in topology or algebraic geometry.

Historically the first global question on geometric structure is the uniformization of conformal structure for domains in the plane. This question dates back to Riemann. It is still an important problem. For instance, we are still trying to understand the structure of moduli space of complex structures over manifolds.

For two dimensional domains, the uniformization theorem of conformal structure gives a description of canonical domains which are bounded by circular arcs. Any finitely connected domain must be conformal to such canonical domains. (The moduli space of such canonical domains can be described easily.)

On the other hand, we can say that any finitely connected domain admits a conformal metric which is flat and whose boundary has constant geodesic curvature. The question of uniformization is then reduced to proving existence and classifying such conformal metrics. Such differential geometric interpretations of problems in conformal geometry is the approach that we shall follow.

For surfaces with higher genus, there are natural conformal metrics that have constant negative curvature. Poincaré was the first to demonstrate that every metric can be conformally deformed to a unique metric with curvature equal to -1. The construction of the Poincaré metric has been fundamental in the understanding of the moduli space of Riemann surfaces.

The cotangent space of the moduli space are represented by holomorphic quadratic differentials. Using the Poincaré metric, one can define an inner product among such quadratic differentials and integrate the product over the surface. The resulting metric can be proved to be a Kähler metric called the Weil-Peterson metric.

On the Riemann surface, there are simple closed geodesics that will decompose the Riemann surface into a planar domain. The function defined by minus log of the sum of the length of these geodesic defines a convex function along geodesics of the Weil-Peterson geometry. In particular, it can be used to prove the universal cover of the moduli space is contractible and is a Stein manifold.

Holomorphic quadratic differential is very important in classical surface theory. For example, if a map from a Riemann surface into another manifold is harmonic (the map is a critical map of the energy), the pulled back metric $\sum h_{ij}dx^i dx^j$ gives rise to a holomorphic quadratic differential

$$h_{11} - h_{22} + 2\sqrt{-1}h_{12}.$$

This well known statements allows one to apply harmonic map to study the geometry of Teichmüller space. Michael Wolf made use of them to give a compactification which is equivalent to the Thurston compactification of the Teichmüller space, which depends on the theory of measured foliation.

Another interesting application of holomorphic quadratic differential is to solve the vacuum Einstein equation for spacetime with dimensional two plus one. Given a conformal structure on a Riemann surface and a holomorphic quadratic differential, the Einstein equation gives a path in the cotangent space of the Teichmüller space of the Riemann surface.

The Weil-Peterson metric is not complete in general. However, the negative of its Ricci tensor is complete. Liu, Sun and myself proved that it is equivalent to the Teichmüller metric which is obtained by considering extremal quasiconformal maps between Riemann surfaces. It is also equivalent to the canonical Kähler-Einstein metric that I shall discuss later.

There has been attempts to find a good representation of Teichmüller space or the moduli space of Riemann surfaces. For genus greater than 23, Harris-Mumford proved that moduli space is of general type. Hence there is no good parametrization of moduli space.

Teichmüller space has an embedding into $\mathbb{C}^{3g-3}$ due to Bers. However, it is not explicit and it is not known how smooth the boundary is. If a bounded domain is smooth, the curvature of the canonical Kähler-Einstein metric must be asymptotic to constant negative curvature in a neighborhood of the point where the domain is convex. This was observed by Cheng-Yau.

Since the moduli space of Riemann surfaces have a compactification where the divisor at infinity cannot be blown down to a point, the Kähler-Einstein metric cannot be asymptotic to constant negative curvature in any neighborhood. Hence there is no representation of the Teichmüller space as a smooth domain.

The question of how to represent a conformal structure on a Riemann surface is quite interesting. Of course one can compute periods of holomorphic differentials over cycles and Torelli theorem asserts that they can determine the conformal structure of a generic surface. However, how to construct the Riemann surface explicitly from the period is not clear. This is especially true if we want to recognize it in $\mathbb{R}^3$. Can we find canonical surfaces in three space that represent different conformal structures of the surface?

There is another important geometric structure over a two dimensional surface with higher genus. This is the projective structure. They are defined by coordinate neighborhoods whose coordinate transformations are given by projective transformations. There is a map from the universal cover of the surface to $\mathbb{R}P^2$ which preserves the projective structures. If the image is a convex domain, we call the projective structure convex. It turns out that convex projective structures are classified by Riemann surfaces with a cubic holomorphic differentials.

Since this classification is a good illustration of how we construct geometric structures, I shall discuss the construction little more detail.

Convex projective structure on a manifold has an invariant metric obtained in the following way:

The structure is obtained by the quotient of a bounded convex domain Ω in $\mathbb{R}^n$ quotiented by a discrete group of projective transformations. A projectively invariant metric on Ω is obtained by solving the following equation

$$\begin{cases} \det\left(\frac{\partial u}{\partial x_i \partial x_j}\right) = \left(-\frac{1}{u}\right)^{n+2} \\ u = 0 \qquad \text{on} \quad \partial\Omega. \end{cases}$$

The following metric

$$\sum\Big(-\frac{1}{u}\Big)\frac{\partial^2 u}{\partial x_i \partial x_j}dx^i \wedge dx^j$$

is observed by Tzitzeica (*Rend. Circ. Mat. Palermo*,25(1908)) and Loewner and Nirenberg (*Contributions to analysis*,1974) to be invariant under projective transformation. It generalizes the Hilbert model of the Poincarè disk.

Loewner-Nirenberg proved the existence and completeness of the metric for $n = 2$. The general case was proved by Cheng-Yau. The Ricci curvature of the metric can be proved to be negative.

A Legendre transformation will transform the graph $(x, u(x))$ to a new convex surface which is an affine sphere $\sum$. (Affine sphere is a hypersurface where all affine normals converge to a point. Affine normal is a vector transversal to the tangent space invariant under the affine group.) The discrete group of projective transformation become affine group of $\mathbb{R}^3$ acting on $\sum$. (This construction was observed by Calabi.)

The affine metric can be written as $e^v ds^2$ where $ds^2 = e^\phi \mid dz \mid^2$ is the hyperbolic metric on a Riemann surface.

Using the structure equation for affine sphere, Simon and Wang observed that the Pick cubic form in affine geometry is an holomorphic cubic differential Ψdz^3 on the Riemann surface defined by the affine metric so that

$$\triangle v + 4\exp(-2v) \parallel \Psi \parallel^2 - 2\exp(v) - 2K = 0,$$

where K is the Gauss curvature of the conformal metric. conversely, given a holomorphic cubic differential on a Riemann surface, a solution v of the above equation can be used to define an affine sphere in $\mathbb{R}^3$ which in turn gives rise to the projective structure.

The projective connection is in fact given by

$$\begin{pmatrix} \partial v + \partial \phi & v \exp(-v-\phi) d\bar{z} \\ v \exp(-v-\phi) dz & \bar{\partial} v + \bar{\partial} \phi \end{pmatrix}$$

with respect to the basis $\{\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}\}$.

Hence we have a good classification of convex projective structure over a Riemann surface. In general, there are projective structures which are not convex.

Choi has proved that projective structures on surfaces can be uniquely decomposed into several pieces. However, we do not have good understanding of the nonconvex part of the projective structure. The study of the moduli space of convex projective structure on surfaces was due to Hitchin, Goldman and Loftin using different approaches. The above approach relating it to affine spheres was due to Loftin.

Compact Riemann surface with higher genus cannot admit affine structures. But open surfaces may admit such a structure. In general, we are interested in affine structures over a compact manifold which may be singular along a codimensional two complex. The coordinate transformations are linear whose Jacobian has determinant equal to one.

Motivated by our study of real Monge-Ampere equations and Kähler geometry, S.Y. Cheng and I considered in 1979 affine manifolds which may support a metric which we called affine Kähler metric. This is a Riemannian metric which has the property that in each affine chart, there is a convex potential V_α where the metric can be written as

$$\sum \frac{\partial^2 V_\alpha}{\partial x_i \partial x_j} dx_i dx_j.$$

Note that the potentials are well defined up to a linear function.

The equation

$$\det\Big(\frac{\partial^2 u_\alpha}{\partial x_i \partial x_j}\Big) = 1$$

is well-defined and can be considered as an analogue of the corresponding equation for Calabi-Yau Kähler metrics. In fact, one simply introduces coordinates y_i and define $z_i = x_i + \sqrt{-1} y_i$. Then we can extend u_α to be a function on z_i and obtain a Kähler metric with zero Ricci curvature.

Note that the equation and the affine structures are defined only on the complement of a codimensional two complex. There is a monodromy associated to the equation. The study of existence for the equation with a given monodromy can be considered as an nonlinear analogue of the Riemann-Hilbert correspondence.

If the monodromy preserves some lattice structure, we can define a torus bundle over the affine manifold where the total space is a Ricci flat Kähler manifold. Strominger, Zaslow and myself conjectured that most Calabi-Yau manifold can be deformed to a complex manifold admitting a (singular) fibration structure, whose fibers are special Lagrangian torus. These are minimal Lagrangian submanifolds and were studied by Harvey and Lawson from different point of view.

There is another geometric structure that is of importance in surface theory. This is the line field structure (with singularity) on a surface. An important case is the line field defined by holomorphic quadratic differential and a polynomial vector field. The former case is used by Thurston to forma compactification of the Teichmüller space and the later case is related to the famous Hilbert sixteenth problem which asked the number of limit cycles associated to the vector field. The behavior of the singular points of the line field has practical importance also, e.g., in the study of finger print.

The attempts to generalize these structures on Riemann surfaces to higher dimensional manifolds have occupied the activities of geometric analysts in the past thirty years. The fact that there are much more freedom in higher dimensional manifolds mean that there are many different varieties of geometric structures.