Kähler-Einstein Metrics on Algebraic Manifolds

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1. Introduction

It is one of fundamental problems in differential geometry to find a distinguished metric on a smooth manifold. H. Poincaré's Uniformization theorem settles this problem for Riemann surfaces. That is, there is a unique metric with constant curvature in each Kähler class on a Riemann surface. Trying to generalize it to higher dimensions, E. Calabi conjectured in the 50s the existence of Kähler-Einstein metrics on a compact Kähler manifold with its first Chern class definite. A Kähler-Einstein metric is a Kähler metric with constant Ricci curvature.

In the middle of the 70s, this conjecture was solved by S.T. Yau in case the first Chern class is vanishing and Aubin and Yau, independently, in case the Chern class is negative (cf. [Y1]). The uniqueness in these two cases was done by E. Calabi himself in the 50s. Such Kähler-Einstein metrics were then applied to studying projective manifolds. For instance, Yau used these metrics to show the Miyaoka-Yau inequality on surface of general type, its generalized version in higher dimensions and the characterization of the quotients of the complex hyperbolic spaces (cf. [Y2]). We also refer readers to [CY, Ko, Ts, TY1] for the generalizations of these to quasi-projective manifolds.

However, this conjecture of Calabi still remains open in general in case the first Chern class is positive. In this paper, we will survey the recent progress on this part of Calabi's conjecture, including the uniqueness and the existence of Kähler-Einstein metrics with positive scalar curvature, the outline of the complete solution for Calabi's conjecture in case of complex dimension two, etc. Some related problems will also be discussed.

From now on, we always denote by M a compact Kähler manifold with positive first Chern class $C_1(M)$, that is, M is a smooth Fano variety. Then we can choose a Kähler metric g with its Kähler class ω_g representing $C_1(M)$. In local coordinates $(z_1, \cdots, z_n)$ of M with $\dim_C M = n$, if g is represented by positive hermitian metrices $\{g_{i\bar{j}}(z)\}_{1\le i,j,\le n}$,

$$\omega_g = \frac{\sqrt{-1}}{2\pi} \sum_{i,j=1}^{n} g_{i\bar{j}} dz_i \wedge d\bar{z}_j .$$

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It is well-known that the Ricci curvature form $Ric(g)$ also represents the first Chern class, therefore, there is a smooth function f in $C^\infty(M,R)$ such that

$$Ric(g) = \omega_g + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}f$$

and

$$\int_M (e^f - 1)\omega_g^n = 0$$

where $\omega_g^n = \omega_g \wedge \cdots \wedge \omega_g$. In local coordinates, the Ricci curvature has the following expression

$$R_{i\bar{j}} = \partial_i\partial_{\bar{j}}\log\det(g_{k\bar{l}})$$

$$Ric(g) = \frac{\sqrt{-1}}{2\pi}\sum_{i,j=1}^{n} R_{i\bar{j}}dz_i \wedge d\bar{z}_j .$$

It follows from this that the existence of a Kähler-Einstein metric on M is equivalent to the solvability of the following complex Monge-Amperé equations,

$$\begin{cases} \left(\omega_g + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi\right)^n = e^{f-t\varphi}\,\omega_g^n & \text{on } M \\ \left(\omega_g + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi\right) > 0 & \text{on } M \end{cases} \qquad (1.1)_t$$

where $n = \dim_C M$.

In case $(1.1)_1$ has a solution φ, then $\omega_g + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi$ gives the Kähler form of a Kähler-Einstein metric.

Theorem 1.1 (Bando and Mabuchi [BM]). *The solution of* $(1.1)_1$ *is unique modulo the connected component* $\mathrm{Aut}_0(M)$ *of* $\mathrm{Aut}(M)$ *containing the identity if it exists, where* $\mathrm{Aut}(M)$ *denotes the group of all holomorphic automorphisms of* M. *In particular, it implies the uniqueness of the Kähler-Einstein metric on M if it exists.*

Since Yau's solution of Calabi's conjecture in case of vanishing first Chern class more than fifteen years ago, it has been known that the solvability of $(1.1)_t$ follows from an a priori C^0-estimate for the solutions, namely, there is a uniform constant C depending only on M and g such that for any solution φ of $(1.1)_t$,

$$\sup_M |\varphi| \leq C. \qquad (1.2)$$

However, such an estimate (1.2) does not exist in general due to the analytic obstructions discussed in the next section.

2. Some Obstructions

Let $\eta(M)$ be the Lie algebra of the automorphism group $\mathrm{Aut}(M)$, that is, $\eta(M)$ consists of all holomorphic vector fileds on M. In [Fu], Futaki defined an analytic invariant $F : \eta(M) \to C$ as follows,

$$F(X) = \int_M X(f)\omega_g^n, \qquad X \in \eta(M).$$

Theorem 2.1 *Suppose that (M, g) admits a Kähler-Einstein metric with positive scalar curvature. Then*

1. *(Matsushima, 1957 [Ma]) The Lie algebra $\eta(M)$ is reductive;*
2. *(Futaki, 1983, [Fu]) The Futaki invariant F is identically zero on $\eta(M)$.*

Blowing up a one or two points in CP^2, we obtain a complex surface with positive first Chern class and non-reductive Lie algebra of holomorphic vector fields. Therefore, by the result of Matsushima, such a surface does not admit a Kähler-Einstein metric. In [Fu], Futaki constructed a Fano 3-fold with reductive Lie algebra of holomorphic vector fields, but nonvanishing Futaki invariant. These obstructions indicate that there is no a priori C^0-estimate for the solution of $(1.1)_t$ in general.

Both obstructions of Matsushima and Futaki come from holomorphic vector fields. One might expect that the absence of holomorphic vector field, as is the case with most Fano variaties, would exclude these obstructions. However, there is a very disturbing example constructed in [T1]. Before we give this example, we first remark that Calabi's conjecture can be generalized to compact Kähler orbifolds with positive Chern class and the previous discussions are still effective. Using Volume Comparison Theorem [Bi], one can show

Theorem 2.2 [T1]. *Let (M, g) be a n-dimensional Kähler-Einstein orbifold with positive first Chern class. Then for any singular point x in M,*

$$\sharp(G_x) < \frac{(2n-1)^n \,\mathrm{Vol}(S^{2n})}{C_1(M)^n} \tag{2.1}$$

where G_x is the local uniformization group of x in M, $C_1(M)^n$ is the volume of M with respect to any Kähler metric with the associated Kähler form in $C_1(M)$ and $\mathrm{Vol}(S^{2n})$ is the volume of S^{2n} with respect to the standard metric. In particular, in case $n = 2$, (2.1) becomes

$$\sharp(G_x) < \frac{48}{C_1(M)^2}. \tag{2.2}$$

Example 2.1 [T1]. By blowing up CP^2 successively at [0,0,1], [0,1,0], [1,0,0], [0,1,1] seven times, one can obtain a rational surface $\tilde{M}$ without holomorphic vector fields and containing a Hirzebruch-Jung string $D = E_1 + E_2 + E_3 + E_4$ with E_i smooth rational curves and $E_1^2 = E_4^2 = -3$, $E_2^2 = E_3^2 = -2$, $E_i \cdot E_j = 1$ if $i - j = \pm 1$; $= 0$ otherwise.

Let M be the 2-dimensional Kähler orbifold obtained from $\tilde{M}$ by collapsing the Hirzebruch string D. Then M has positive first Chern class $C_1(M)$ with $C_1(M)^2 = 3$ and an orbifold singular point x of type C^2/σ, where $\sigma : (z_1, z_2) \to$

$(e(\frac{1}{16})z_1, e_2(\frac{7}{16})z_2)$ for the Euclidean coordinates z_1, z_2 on C^2, $e(\frac{l}{m}) = \exp(\frac{2\pi l\sqrt{-1}}{m})$. In particular, $\sharp(G_x) = 16$ in equal to $48/C_1(M)^2$. Therefore, by Theorem 2.2, M does not admit any Kähler-Einstein orbifold metric. On the other hand, M has no holomorphic vector filed since $\tilde{M}$ does not.

This example indicates that holomorphic vector fields may not be only obstructions in solving Calabi's conjecture. The author believes that the existence of Kähler-Einstein metric with positive scalar curvature should be closely related to the geometry of pluri-anti-canonical divisors.

3. Some Existence Theorems

In this section, let (M, g) be a compact Kähler manifold with the Kähler form ω_g in $C_1(M)$ as given in the first section. We will give some existence results. First of all, let us mention two known results among others, for special Kähler manifolds with positive first Chern class, in particular, those manifolds having a lot of holomorphic automorphisms. In [Ma2], Matsushima showed that every simply-connected homogeneous Kähler manifold admits a Kähler-Einstein metric. In case M is a P^1-bundle of certain type with C^*-actions which restrict to C^*-actions on the fibers P^1, Koiso and Sakane [KS] showed that M admits a Kähler-Einstein metric iff the Futaki invariant is zero. The manifold M in all these examples has very high homogeneity. In fact, until 1986, Siu [Si] and the author [T2] provided the first examples of compact Kähler-Einstein manifolds with positive first Chern class and without holomorphic vector fields. Precisely, Siu showed the existence of Kähler-Einstein metrics on the 2-dimensional Fermat surface and the surface obtained by blowing up CP^2 at three generic points, while the author showed the existence of Kähler-Einstein metrics on n-dimensional Fermat hypersurfaces of degree n or $n+1$. Their proofs are completely different. In fact, the examples of the author are just corollaries of a general existence theorem proved in the same paper [T2]. Let us first describe this result.

Let G be any compact subgroup in Aut(M). This group G preserves the first Chern class $C_1(M)$, therefore we may take g, f to be G-invariant. Define

$$P_G(M,g) = \left\{ \phi \in C^\infty(M,g) \;\middle|\; \phi \text{ is } G\text{-invariant on } M, \omega_g + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\phi > 0,\ \sup_M \phi = 0 \right\}$$

$$\alpha(M) = \sup\left\{\alpha \mid \exists C_\alpha > 0 \text{ s.t. } \int_M e^{-\alpha\phi} dV_g \le C_\alpha \text{ for all } \phi \in P_G(M,g)\right\}.$$

It is not hard to check that $\alpha(M)$ is independent of the particular choice of G, g, that is, $\alpha(M)$ is a holomorphic invariant.

Theorem 3.1 [T2]. *Let M be a compact Kähler manifold with $C_1(M) > 0$ as above. Then M admits a Kähler-Einstein metric whenever $\alpha(M) > n/n+1$.*

Theorem 3.1 is proved by deriving an a priori C^0-estimate for the solutions of $(1.1)_t$.

Remark. There is another proof of Theorem 3.1 given by W. Ding later in [Di]. His approach is interesting by itself and was based on an inequality proposed by T. Aubin.

Example 3.1. Let M be a smooth complete intersection in CP^{pq} defined by Fermat polynomials $\sum_{i=0}^{pq} a_{ji} z_i^p = 0$ for $1 \leq j \leq q$, where $[z_0, \cdots, z_{pq}]$ is the homogeneous coordinate of CP^{pq} and $p \geq 2, q \geq 1$ are positive integers. Then by Adjunction Formula (cf. [GH]), one can easily see that $C_1(M)$ is just the restriction of the hyperplane section on CP^{pq}. and the maximal compact subgroup G in $\mathrm{Aut}(M)$ contains all transformations of form $\sigma_j : [z_0, \cdots, z_{pq}] \to [z_0, \cdots, e(\frac{1}{p}) z_j, \cdots, z_{pq}]$ for $1 \leq j \leq pq$. The dimension n of M is $(p-1)q$. One can show (cf. [T3])

$$\alpha(M) \geq 1 .$$

In particular, these complete intersections have Kähler-Einstein metrics with positive scalar curvature.

Besides those mentioned above, one can find in [TY2, T3, Na] etc. more examples of Kähler-Einstein manifolds with positive scalar curvature by using this theorem.

The same proof as that for Theorem 3.1 in [T2] also shows

Theorem 3.2 (cf. [TY2]). *Any compact Kähler manifold M with positive first Chern class admits a Kähler metric g_t with Kähler form ω_{g_t} in $C_1(M)$ and Ricci curvature $\geq t$ for any t between* $\min\{1, \frac{n+1}{n}\alpha(M)\}$ *and* 1.

By Bishop's Volume Comparison Theorem, one can derive from it

Corollary 3.1. *For any compact Kähler manifold M with $C_1(M) > 0$, we have*

$$C_1(M)^n \leq C_n \alpha(M)^{-n}$$

where C_n is a constant depending only on n.

In case $n = 2$, we have the following complete solution of Calabi conjecture.

Theorem 3.3 [T4]. *Let M be a complex surface with positive first Chern class, i.e., a Del-Pezzo surface. Then M admits a Kähler-Einstein metric iff $\eta(M)$ is reductive.*

We will outline the proof of this theorem in the next section. Next we briefly discuss the application of L^2-estimate for $\bar{\partial}$-operators to estimating $\alpha(M)$ from below.

Theorem 3.4 (Theorem 3.1 in [T2]). *Let $\{\phi_i\}$ be a sequence of functions in $P(M, g)$, λ be a positive number. Then there exists a subsequence $\{i_k\}$ of $\{i\}$ and a subvariety S_λ of M with* $\dim_C S_\lambda \leq \dim_C M - 1$ *such that*

1. $\forall\, z \in M - S_\lambda$, $\exists\, r > 0$, $C > 0$, *s.t.*

$$\int_{B_r(z)} e^{-\lambda \phi_{i_k}(w)} d V_g(w) \leq C$$

2. $\forall\, z \in S_\lambda$,

$$\lim_{k\to\infty} \int_{B_r(z)} e^{-\lambda\phi_{i_k}(w)} dV_g(w) = +\infty \quad \text{for all } r.$$

This theorem can be proved with help of Theorem 5.2.4 in Hörmander's book [Ho]. It implies that the solution φ_t of $(1.1)_t$ should behave well outside a subvariety. Note that the degree of this subvariety can be controlled in terms of the Chern numbers of M. Obviously, $\alpha(M) \geq \lambda_0$ is equivalent to $S_\lambda = \emptyset$ for $\lambda < \lambda_0$. Therefore, it is important to understand S_λ for $\lambda \leq 1$. Since there is no nonconstant holomorphic function on M, one can easily show that each S_λ is connected by the same arguments as in the proof of the Theorem 3.4 (cf. [T2]). In fact, recently, A.Nadel has obtained more information on such a subvariety S_λ for $\lambda < 1$ by Bochner identities for $(0,q)$-forms with $0 < q \leq n$. It can be explained as follows. Denote by I_λ the ideal sheaf of S_λ counting multiplicity, then for each open subset $U \subset M$, $I_\lambda(U)$ consists of those local holomorphic functions f such that $\int_U |f|^2 e^{-\lambda\varphi_{i_k}(w)} dV_g$ are uniformly bounded for all k. On the other hand, one can easily choose the metric g such that $Ric(g) + \frac{\sqrt{-1}}{2\pi}\partial\bar\partial\varphi_{i_k} > 0$ on M. Therefore $h^i(M, I_\lambda) = 0$ for all $i \geq 1$, $\lambda < 1$ by same argument as that in the proof of Kodaira's Vanishing Theorem (cf. [KM, Na]).

For more applications of L^2-estimates to bounding $\alpha(M)$ from below, the readers may refer [T5, TY2].

4. Solution of Calabi's Conjecture for Surfaces

Theorem 3.3 gives the complete solution to Calabi's conjecture for complex surfaces. In this section, we describe briefly the three major steps in the proof. Let M be a complex surface with positive first Chern class. In the first step, we deform the complex structure on M to obtain a smooth family of Kähler surfaces $\{M_t\}_{0\leq t\leq 1}$ with positive first Chern class such that $M_1 = M$ and M_0 admits a Kähler-Einstein metric g_0. The existence of such a family follows from the main theorem in [TY2] and the classification theory of complex surfaces. Now we define $E = \{t \in [0,1] | M_{t'}$ has a Kähler-Einstein metric for any $t' \leq t\}$. Then E is nonempty. It is not hard to prove by the Implicit Function Theorem that E is open in [0,1]. Therefore, in order to show that $M = M_1$, has a Kähler-Einstein metric, it suffices to prove that E is closed. We should point out that the closedness is the most difficult part in the solution of Calabi's conjecture. We accomplish this in the following two steps. Without losing generality, we may assume that $E = [0,1)$. Let g_t $(0 \leq t < 1)$ be the Kähler-Einstein metrics on M_t. We need to show that g_t converges to a metric g_1 on M in C^2-topology. Since $\{M_t\}_{0\leq t\leq 1}$ is a smooth family of Kähler surfaces with positive first Chern class, there are Kähler metrics $\tilde g_t$ on M_t $(0 \leq t \leq 1)$ such that their Kähler forms $\omega_{\tilde g_t}$ are in $C_1(M_t) = C_1(M)$. Therefore, for each $t < 1$, $\omega_{g_t} = \omega_{\tilde g_t} + \frac{\sqrt{-1}}{2\pi}\partial\bar\partial\varphi_t$ for a unique function φ_t such that

$$\omega_{g_t}^2 = e^{\tilde f_t - \varphi_t}\, \omega_{\tilde g_t}^2 \quad \text{on } M_t \tag{4.1$_t$}$$

where $\tilde f_t$ are smooth functions determined by $\tilde g_t$ $(0 \leq t \leq 1)$, namely,

$$Ric(\tilde{g}_t) - \omega_{\tilde{g}_t} = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\tilde{f}_t \quad \text{on } M_t$$

and

$$\int_{M_t} (e^{\tilde{f}_t} - 1)\,\omega_{\tilde{g}_t}^2 = 0.$$

Now, in order to have g_t converge to a metric on M, it suffices to derive an a priori C^0-estimate of φ_t $(0 \le t \le 1)$.

In the second step in the solution of Calabi's conjecture, we give a partial C^0-estimate for φ_t.

Theorem 4.1. *For any positive integer $m \equiv 0 \bmod 6$, there is a constant C_m independent of φ_t such that there are an orthonormal basis $\{S_i(t)\}_{0\le i\le N_m}$ of $H^0(M, K_M^{-m})$ with respect to the inner product induced by $\tilde{g}_t$ and a sequence of positive numbers $0 < \lambda_0(t) \le \lambda_1(t) \le \cdots \le \lambda_{N_m}(t) = 1$ satisfying:*

$$\left\| \varphi_t - \sup_M \varphi_t - \frac{1}{m}\log\left(\sum_{i=0}^{N_m} \lambda_i(t)\|S_i(t)\|_{\tilde{g}_t}^2\right)\right\|_{C^0(M_t)} \le C_m \qquad (4.2)_t$$

where $\|\cdot\|_{\tilde{g}_t}$ are the norms of $K_{M_t}^{-m}$ induced by $\tilde{g}_t$.

Remark. It is interesting to know whether or not m has to be the multiple of 6 in $(4.1)_t$.

The estimate in $(4.1)_t$ in particular implies that each $\varphi_t - \sup_{M_t}\varphi_t$ is uniformly bounded in any compact subset outside the zero locus of $S_{N_m}(t)$. That is why $(4.2)_t$ is called a partial C^0-estimate of φ_t. Since there is no known local estimate for the solutions of complex Monge-Amperé equations, such a partial C^0-estimate is very precious. The key observation in the proof of $(4.2)_t$ is: by some computations and the maximum principle, one can show that $(4.2)_t$ is equivalent to

$$\log\left(\sum_{i=1}^{N_m} \|S_i'(t)\|_{g_t}^2\right) \ge C_m' \qquad (4.3)_t$$

where C_m' is a constant independent of t, $\|\cdot\|_{g_t}$ is the hermitian metric on $K_{M_t}^{-m}$ induced by g_t and $\{S_i'(t)\}_{0\le t\le N_m}$ is an orthonormal basis of $H^0(M, K_M^{-m})$ with respect to the inner product induced by g_t. In [T4], $(4.3)_t$ is proved by using Cheeger-Gromov's compactness Theorem, Uhlenbeck's Yang-Mills estimates and L^2-estimates for $\bar{\partial}$-operators.

In the last step of our solution for Calabi's conjecture, we first prove a numerical criterion for the existence of a Kähler-Einstein metric on M. To describe it, let us denote by $P_{G,m,k}(M, \tilde{g}_1)$ the collection of all G-invariant functions of the form $\frac{1}{m}\log(\sum_{i=1}^{k}\|S_i\|_{\tilde{g}_1}^2)$, where $\{S_i\}_{0\le t\le N_m}$ is an orthonormal basis of $H^0(M, K_M^{-m})$ with respect to the metric $\tilde{g}_1$, $0 \le k \le N_m$, $N_m = h^0(M, K_M^{-m}) - 1$. Define

$$\alpha_{m,k}(M) = \sup\left\{\alpha \,|\, \exists C_\alpha > 0,\ \text{s.t.} \int_M e^{-\alpha\varphi} dV_g \le C_\alpha,\ \text{for any } \varphi \text{ in } P_{G,m,k}(M, \tilde{g}_1)\right\}.$$

Theorem 4.2. *Fix an integer m such that* $(4.1)_t$ *hold for all* $t < 1$. *If either* $\alpha_{m,1}(M) > 2/3$ *or* $\alpha_{m,2}(M) > 2/3$ *and* $\alpha_{m,1}(M) > l_0/(l_0+1)$, *where* $l_0 = \max\{0, 2 - \frac{3\alpha_{m,2}(M)-2}{\alpha_{m,2}(M)}\}$, *then M admits a Kähler-Einstein metric.*

Remark. In general, if we replace 3 by $n+1$ and 2 by n in Theorem 4.2 and the partial C^0-estimate in Theorem 4.1 is valid for solutions of $(1.1)_t$, then the above conclusion is still true. It is just Theorem 6.1 in [T5].

In [T4], by direct computation, it is proved that $\alpha_{6,1}(M) \geq 2/3$, $\alpha_{6,2}(M) > 2/3$ for any complex surface M with positive first Chern class. Therefore, the proof for Theorem 3.3 is finished.

The lower bounds of these $\alpha_{m,k}(M)$ seem to be closly related to Mumford's stability of M in CP^{N_m} in the Chow variety. Therefore, it is quite possible that there is a connection between the existence of Kähler-Einstein metrics on M and Mumford's stability of M in the Chow variety.

5. Some Problems Related to Calabi's Conjecture in Higher Dimensions

According to the author's opinion, in order to solve Calabi's conjecture in higher dimensions, one has to understand further the behavior of the solutions of $(1.1)_t$ (cf. § 1). The difficulty is the lack of some fundamental estimates for complex Monge-Amperé equations. To get around it, we posed and verified a partial C^0-estimate for the solutions of $(1.1)_1$, on complex surfaces. Such an estimate is crucial in the solution of Calabi's conjecture for complex surfaces. The auther believes such an estimate holds on n-dimensional complex Kähler manifold (M, g) with positive first Chern class. For reader's convenience, we state its stronger version.

Conjecture. *Given* $t > 0$ *and* $\mu > 0$. *There are constants* $m_0 < m_1$, C, *depending only on* n, t, μ, *satisfying: For any n-dimensional compact Kähler manifold* (M, g) *with* $Ric(g) \geq t\omega_g$ *and* $\sum_{i=1}^{2n} |b_i(M)| \leq \mu$, *where* $b_i(M)$ *is the ith betti number of M, there is an integer m in* $[m_0, m_1]$ *such that*

$$\inf_M \log\left(\sum_{i=0}^{N} \|S_i\|_g^2\right) \geq C \tag{5.1}$$

where $\|\cdot\|_g$ *is the hermitian metric on* K_M^{-m} *with* $m\omega_g$ *as its curvature form,* $\{S_i\}_{1\leq i\leq N}$ *is any orthonormal basis of* $H^0(M, K_M^{-m})$ *with respect to the inner product induced by g and* $\|\cdot\|_g$ *and* $N = \dim H^0(M, K_M^{-m}) - 1$.

The author is able to affirm this conjecture under two more assumptions: $\mathrm{Ricci}(g) \leq \mu\omega_g$ and $\int_M |\mathrm{Rm}|_g^n dV_g \leq \mu$, where Rm denotes the riemannian curvature tensor of g.

One may try to establish (5.2) by using Hörmander's L^2-estimate for $\bar\partial$-operator and constructing local nonvanishing sections of K_M^{-m} for m large, as we did in [T4] for complex surfaces. It boils down to understanding the local structure of a compact Kähler manifold with positive Ricci curvature. Naturally, it leads to the following problem.

Problem. Classify all complete Ricci-flat Kähler manifolds X with the following properties:

1) There is a point $x_0 \in X$ such that the geodesic balls $B_r(x_0)$ have Euclidean volume growth as r goes to infinity.

2) $\lim_{r \to \infty} r^{-2n+4} \int_{B_r(x_0)} \|R(g)\|^2 dV_g = 0$ where $R(g)$ denotes the curvature tensor of g.

In case $n = 2$, (X, g) is clearly the Euclidean space C^2. In case $n \geq 3$, we can obtain such (X, g) as follows. Define $X = X_1 \times \cdots \times X_l$. Each X_i is a resolution of C^{n_i}/Γ_i with $n_i \geq 3$ and $\Gamma_i \subset SU(n_i)$ such that the push-down to $(C^{n_i}\backslash\{0\})/\Gamma_i$ of the standard holomorphic n_i form on C^{n_i} can be nonvanishingly extended across X_i. In [T6], by using the method in [TY3], it is proved that there exists a complete Ricci flat Kähler metric g_i on X_i with Euclidean volume growth and $\int_{X_i} \|R(g_i)\|^{n_i} dV_{g_i} < +\infty$. Take g to be the product metric of g_i $(1 \leq i \leq l)$. Then (X, g) satisfies 1) and 2) above.

Are such (X, g) the only ones with 1) and 2)?

6. Degeneration of Kähler-Einstein Manifolds

In order to compactify the moduli space of Kähler-Einstein metrics with positive scalar curvature, we are bound to study Kähler-Einstein orbifold metrics as indicated by the work of M.Anderson [An] and Nakajima [Nk]. Such a metric is defined to be a Kähler orbifold metric on a normal Kähler orbifold such that its Ricci curvature form is a constant multiple of its Kähler form. On one hand, there are some analytic obstructions from holomorphic vector fields to the existence of Kähler-Einstein orbifold metrics analogous to those by Matsushima and Futaki in case of Kähler manifolds, on the other hand, the absence of holomorphic vector field does not assure the existence of a Kähler-Einstein orbifold metric as indicated by the example in § 2 (cf. § 2).

Problem 6.1. Determine when a Kähler orbifold $\tilde{M}$ with $C_1(\tilde{M}) > 0$ has a Kähler-Einstein metric.

It can be regarded as a generalized form of Calabi's conjecture. Of course, The discussions in previous sections, as well as those methods in [T2, TY2], etc., can be applied to this general case with slight modification. However, there are some additional difficulties due to the presence of singularities. For instance, the singularity may cause higher multiplicity of the anticanonical divisor, consequently, the invariant $\alpha(\tilde{M})$ gets smaller (cf. Section 3). Nevertheless, one can produce a lot of Kähler-Einstein orbifolds by means of the methods in [T2], etc. For example, we can prove that there is a Kähler-Einstein orbifold metric on the minimal model of the surface obtained by blowing up CP^2 at 7 or 8 points such that no six of them are on a common quadratic curve and exact three of them are collinear.

The simplest Kähler orbifold with positive first Chern class is the 2-dimensional projective variety with only rational double points as singularities. The minimal resolution of it is an almost Del-Pezzo surface, i.e., the rational surface with numerically positive anticanonical line bundle K^{-1} and $K^2 > 0$. It seems to be even nontrivial to solve Problem 6.1 in this particular case. The methods in [T2, T4] can be used for this purpose.

Not every Kähler-Einstein orbifold can be in the boundary of the moduli space of Kähler-Einstein metrics. Here the distance function on the moduli is defined in terms of some suitable limit, such as Hausdorff limit (cf. [Gr, T4]). In [T4], it is proved that the dimensions of all plurianticanonical divisors are preserved under Hausdorff limit for compact complex surfaces. In fact, the same proof also yields

Theorem 6.1. *Let $\{(M_i, g_i)\}$ be a sequence of Kähler-Einstein manifolds of complex dimension n. Suppose that (M_i, g_i) converge to a Kähler-Einstein orbifold (M_∞, g_∞) in the sense (cf. [Gr]): there are subsets V_i in M_i and V_∞ in M_∞ of real Hausdorff dimension less than $2n-2$ with respect to g_i and g_∞, respectively, such that for any $\varepsilon > 0$, there are diffeomorphisms ϕ_i from $M_\infty - B_{2\varepsilon}(V_\infty, g_\infty)$ into $M_i - B_\varepsilon(V_i, g_i)$ satisfying: (1) The image of ϕ_i contains the exterior of $B_{4\varepsilon}(V_i, g_i)$ for each i; (2) The pull-backs $\phi^* g_i$ converge to g_∞ in C^∞-topology. Then $H^0(M_i, K_{M_i}^{-m})$ converge to $H_0(M_\infty, K_{M_\infty}^{-m})$ for all m in Z. In particular, the dimensions of pluri-anticanonical or canonical sections stay same under the above convergence.*

Problem 6.2. Characterize the Kähler-Einstein orbifold in the boundary of the moduli space of Kähler-Einstein manifolds.

In fact, the author does not know an example of Kähler-Einstein orbifold with positive scalar curvature in the boundary of the moduli space such that its anticanonical ring is different from that of those Kähler-Einstein manifolds in the moduli. It is certainly an interesting question to be explored. We end this paper by a thereom on the degeneration of Kähler-Einstein surfaces (cf. Theorem 7.1 in [T4]).

Theorem 6.2. *Let $\{(M_i, g_i)\}$ be a sequence of Kähler-Einstein surfaces with $C_1(M_i)^2$ equal to $9-n$ $(5 \le n \le 8)$ and positive first Chern class. By taking a subsequence, we may assume that they converge to a Kähler-Einstein orbifold (M_∞, g_∞) (cf. [An], [Nk]). Then*

1. if $n = 8$, then M_∞ has either only rational double points or two singular points of type $C^2/Z_{l,2}$ besides rational points$(2 \le l \le 7)$;

2. if $n = 7$, then M_∞ has either only rational double points or two singular points of type $C^2/Z_{1,2}$ besides rational points;

3. if $n = 5, 6$, M_∞ has at most two singular points of type $C^2/Z_{l,2}$ or $C^2/Z_{1,3}$ $(1 \le l \le 3)$ besides rational double points. Moreover, in case M_∞ has two of such singular points, one of them must be of type $C^2/Z_{l,2}$, while the other is of type $C^2/Z_{1,3}$.

Here $Z_{p,q}$ denotes the finite cyclic group in $U(2)$ generated by diag $\left(e(\frac{1}{pq^2}),\right.$ $\left. e(-\frac{1}{pq^2} + \frac{1}{q})\right)$.

We in fact also have the explicit description of those M_∞ with singular points other than rational double points (cf. §6, §7 in [T4]). In fact, there are only a few of them.

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