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On Calabi's conjecture for complex surfaces with positive first Chern class

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It is known by classification theory of complex surfaces that $CP^2 \# n\overline{CP^2} (0 \leq n \leq 8)$ and $CP^1 \times CP^1$ are only compact differential 4-manifolds on which there is a complex structure with positive first Chern class. In [TY], the authors proved that for any n between 3 and 8, there is a compact complex surface M diffeomorphic to $CP^2 \# n\overline{CP^2}$ such that $C_1(M) > 0$ and M admits a Kähler-Einstein metric. This paper is the continuation of my joint work with professor S.T. Yau [TY]. The main result of this paper is the following.

Main theorem. *Any compact complex surface M with $C_1(M) > 0$ admits a Kähler-Einstein metric if Lie* (Aut(M)) *is reductive.*

This theorem solves one of Calabi's conjectures in case of complex surfaces. The conjecture says that there is a Kähler-Einstein metric on any compact Kähler manifold with positive first Chern class and without holomorphic vector field. Our proof of the above theorem is based on a partial C^0-estimate of the solutions of some complex Monge-Ampére equations we will develop in this paper (Theorem 2.2, Theorem 5.1) and the previous work of the author in [T1] and the joint work with S.T. Yau in [TY].

Let M be a compact Kähler manifold with positive first Chern class and g be a Kähler metric with its associated Kähler class ω_g in $C_1(M)$. Then, the existence of a Kähler-Einstein metric on M is equivalent to the solvability of the following complex Monge-Ampére equations

$$\begin{cases} \left(\omega_g + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi\right)^n = e^{f - t\varphi}\omega_g^n \\ \left(\omega_g + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi\right) > 0 \end{cases} \quad \text{on } M \qquad (0.1)_t$$

where $n = \dim_C M$, $\varphi \in C^\infty(M, R^1)$, $0 \leq t \leq 1$ and f is a smooth function determined by equations

$$\mathrm{Ric}(g) - \omega_g = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}f \quad \text{and} \quad \int_M e^f \omega_g^n = \int_M \omega_g^n = C_1(M)^n$$

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Since Yau's solution of Calabi's conjecture for Kähler manifolds with vanishing first Chern class, it has been known that the solvability of $(0.1)_t$ for $0 \leqq t \leqq 1$ would follow from an a prior C^0-estimate of the solutions of $(0.1)_t$. The difficulty is that such a C^0-estimate does not exist in general due to those obstructions found by Matsushima [Ma], Futaki [Fu]. In [T1], the author reduces such a C^0-estimate to some integral bound on the solutions of $(0.1)_t$. There are two ways of obtaining such an integral bound on the solutions. One of them is to evaluate the optimal constant of some linearized versions of Moser-Trudinger inequalities for almost plurisubharmonic functions on M as the author did in [T1]. Another is to obtain more informations about the solutions of $(0.1)_t$ and relate the above integral bound of the solutions to the geometry on M, specially, the geometry of plurianticanonical divisors in M. This second approach is our major motivation in this paper to develop an a prior partial C^0-estimate for the solutions of $(0.1)_t$ in case of complex surfaces.

Our partial C^0-estimate for $(0.1)_t$ is based on the following observation. The solvability of $(0.1)_t$ for $0 \leqq t \leqq 1$ does not depend on the choice of a particular Kähler metric g, that is, there is some "gauge" group of the complex Monge-Ampére equations $(0.1)_t$. The author's belief is that such a "gauge" group should play a role in obtaining the C^0-estimate for the solutions of $(0.1)_t$. To understand this "gauge" group, we first recall some natural classes of Kähler metric with its Kähler form in $C_1(M)$. Note that the anticanonical line bundle K_M^{-1} is ample. Therefore, by Kodaira's embedding theorem, for m sufficiently large, the plurianticanonical line bundle K_M^{-1} is very ample, that is, any basis of the group $H^0(M, K_M^{-m})$ gives an embedding of M into some projective space CP^{N_m}, where $N_m + 1 = \dim_C H^0(M, K_M^{-m})$. Then we have a collection C_m of Kähler metrics consisting of the restrictions of the $\frac{1}{m}$-multiple of Fubini-Study metric on CP^{N_m} to the embeddings of M induced by the bases of $H^0(M, K_M^{-m})$. These Kähler metrics are parametrized by the group $PGL(N_m + 1)$. Thus one can consider $PGL(N_m + 1)$ for m large as a "gauge" group of $(0.1)_t$. Now let us see how this group plays a role in our partial C^0-estimate for solutions of $(0.1)_t$. The differences of the metrics in $\mathscr{C}_m$ from the fixed Kähler one g provide a natural set $\mathscr{C}'_m$ of smooth functions ψ in $C^\infty(M, R^1)$ with $\omega_g + \frac{\sqrt{-1}}{2\pi}\partial\bar\partial\psi > 0$ on M and $\sup_M \psi = 0$. One can regard the functions in C'_m as the generalized polynomials on M of degree m. For instance, if $M = CP^n$, then the sections in $H^0(M, K_M^{-m})$ correspond one-to-one to the homogeneous polynomials of degree m and any function in $\mathscr{C}'_m$ is determined by a basis of the linear space of all homogeneous m-polynomials. We now propose the following estimate for the solution of $(0.1)_t$: there is a $m > 0$, depending only on the geometry of M, such that for any solution φ of $(0.1)_t$, there is a function ψ in $\mathscr{C}'_m$ satisfying

$$\|\varphi - \sup_M \varphi - \psi\|_{C^0(M)} \leqq C \tag{0.2}$$

where C is a constant independent of φ, t. In particular, (0.2) implies that for any solution φ of $(0.1)_t$, there is a subvariety $V_\varphi \subset M$ away from which $\varphi - \sup_M \varphi$ is uniformly bounded and the degree V_φ with respect to K_M^{-1} is bounded independent of φ. Therefore, we can call (0.2) a partial C^0-estimate for $(0.1)_t$.

We further observe (cf. Lemma 2.2) that (0.2) is equivalent to the following

$$\log\left(\sum_{\nu=0}^{N_m} \|S_\nu^m\|_{g_t}^2\right) \geqq C' \tag{0.3}$$

where C' is a constant depending only on the geometry of M, the metric g_t is given by its Kähler form $\omega_g + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi$, $\|\cdot\|_{g_t}$ is the hermitian metric of K_M^{-m} with $\omega_g + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi$ as its curvature form and $\{S_\nu^m\}_{0\leqq\nu\leqq N_m}$ is an orthonormal basis with respect to the inner product induced by g_t and $\|\cdot\|_{g_t}$. To prove (0.3), it suffices to construct plurianticanonical sections on M with its norm bounded from below at any assigned point. In this paper, it is done for $t = 1$ and complex surfaces, i.e., $n = 2$, by using L^2-estimate for $\bar{\partial}$-operators, Gromov's compactness theorem and Uhlenbeck's theory for Yang-Mills connections (cf. Theorem 2.2, 5.1). Moreover, we can take m in either (0.2) or (0.3) to be less than 7 for $t = 1$ and on compact complex surfaces with positive first Chern class. In general, we believe that the estimate (0.3) is also true on higher dimensional Kähler manifolds with positive first Chern class. Note that the group $PGL(N_m + 1)$ contains the automorphism group $\mathrm{Aut}(M)$ of M. The obstructions of Matsushima and Futaki are from this latter group.

Next, we assume that M is always a complex surface with positive first Chern class. In order to prove the existence of Kähler-Einstein metric on M, we need to evaluate the supremum $\alpha_m(M)$ of those exponents α such that the function $\exp(-\alpha\psi)$ for ψ in $\mathscr{C}'_m$ are uniformly L^1-bounded, where $m \leqq 6$ appears in the partial C^0-estimate (0.2). If such an $\alpha_m(M)$ is strictly larger than 2/3, then by [T1] and the partial C^0-estimate (0.2) for solutions of $(0.1)_1$, there is a Kähler-Einstein metric on M. But sometimes the number $\alpha_m(M)$ could be exactly 2/3, then we need to further study the generalized polynomial functions ψ in the partial C^0-estimate (0.2) for the solutions of $(0.1)_1$ and improve the main theorem in [T1] (cf. section 2 for details). All these lead to the proof of our main theorem stated above.

The organization of this paper is as follows. In section 1, we give some preliminary discussions and reduce the proof of our main theorem to some a prior C^0-estimate for the solutions of complex Monge-Ampére equations. The arguments here are standard. In section 2, we prove our main theorem under the assumption of Theorem 2.2 (strong partial C^0-estimate). Some interesting improvements of the main result in [T1] are given. In sections 3 and 4, we begin our first step of the proof of the strong C^0-estimate, i.e., Theorem 2.2. Gromov's compactness theorem and Uhlenbeck's theory for Yang-Mills connections are applied for this purpose. In section 5, we use Hömmander's L^2-estimate for $\bar{\partial}$-operators on plurianticanonical line bundles to prove a weaker version of Theorem 2.2, i.e., partial C^0-estimate stated in either (0.2) or (0.3). We also apply L^2-estimate for $\bar{\partial}$-operators to making reductions of the singular points of some 2-dimensional Kähler-Einstein orbifolds. In particular, we prove that if a Kähler-Einstein orbifold is the limit of some sequence of Kähler-Einstein surfaces with positive scalar curvature, then it has at most some Hirzebruch-Jung singularities of special type besides rational double points (Theorem 5.2). In sections 6, 7, by studying the plurianticanonical divisors of some rational surfaces, we complete the proof of Theorem 2.2 (strong partial C^0-estimate). There are two appendices, in which we prove one lemma (Lemma 2.4) and one proposition (Proposition 2.1) stated in section 2. The lemma concerns the singularities of plurianticanonical divisors on a complex surface with positive first Chern class and diffeomorphic to either $CP^2 \# 5\overline{CP^2}$ or $CP^2 \# 6\overline{CP^2}$. The proposition should be a classical and elementary result. In the course of the proof of our main theorem, we also obtain some results on the degeneration of Kähler-Einstein surfaces (Theorem 7.1). We refer readers to the end of section 7 for details.

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1. Preliminaries

Let M be a complex surface with positive first Chern class $C_1(M)$. It is known (cf. [GH]) that M is of form either $CP^1 \times CP^1$ or $CP^2 \# n\overline{CP^2}$ $(0 \leqq n \leqq 8)$, i.e., the surface obtained by blowing up CP^2 at n generic points, where "generic" means that no three of these points are colinear and no six of them are on the same quadratic curve. As symmetric spaces, $CP^1 \times CP^1$ and CP^2 admit the standard invariant metrics. These invariant metrics are Kähler-Einstein metrics. An easy computation shows that for $n = 1$ or 2, $CP^2 \# n\overline{CP^2}$ has non-trivial holomorphic vector fields and the Lie algebra of these holomorphic vector fields is not reductive. Thus Matsushima's theorem [Ma] excludes the existence of Kähler-Einstein metrics on these $CP^2 \# n\overline{CP^2}$ $(n = 1, 2)$. The following theorem is proved in [TY] by estimating the lower bound of the holomorphic invariant introduced in [T1].

Theorem 1.1. *For each integer n between 3 and 8, there is a complex surface M of form* $CP^2 \# nCP^2$ *such that M admits a Kähler-Einstein metric with positive scalar curvature.*

Remark. By a completely different method, Professor Siu [Si] also proved the existence of Kähler-Einstein metrics on $CP^2 \# 3\overline{CP^2}$ and Fermat surface in CP^3. His proof requires that the manifolds considered must have many symmetries, while ours does not.

Denote by $\mathfrak{J}_n$ the collection of all complex surfaces of form $CP^2 \# n\overline{CP^2}$ with positive first Chern class, in other words, those complex structures on the underlying differential manifold $CP^2 \# n\overline{CP^2}$ such that the first Chern class is positive. One can easily prove that for $n = 3$ or 4, $\mathfrak{J}_n$ consists of only one element, i.e. the one on which there is a Kähler-Einstein metric constructed in Theorem 1.1. Hence, in order to prove our main theorem, we may assume that $5 \leqq n \leqq 8$.

Lemma 1.1. *$\mathfrak{J}_n$ is connected in the sense that for any M, M' in $\mathfrak{J}_n$, there is a family of $\{M_t\}_{0 \leqq n \leqq 1}$ such that $M_0 = M$, $M_1 = M'$, $M_t \in \mathfrak{J}_n$ and M_t depends smoothly on t.*

Proof. By induction, we may assume that M, M' are the surfaces obtained by blowing up CP^2 at generic points $p_1, \ldots, p_n$; $p'_1, \ldots, p'_n$, respectively and $p_i = p'_i$ for $i = 1, \ldots, n-1$. Let D be the union of lines $\overline{p_i p_j}$ in CP^2 for $1 \leqq i, j \leqq n-1$ and the quadratic curves in CP^2 passing through five of $p_1, \ldots, p_{n-1}$. $CP^2 \backslash D$ is connected. Now it is easy to see how to connect M to M' smoothly in $\mathfrak{J}_n$.

Lemma 1.2. *For $n \geqq 5$, each M in $\mathfrak{J}_n$ has no non-trivial holomorphic vector field.*

Proof. It suffices to prove that the identity component $\mathrm{Aut}_0(M)$ of the automorphism group is discrete. It is clear that

$$\mathrm{Aut}_0(M) = \{\sigma \in \mathrm{Aut}(CP^2) | \sigma \text{ interchanges } p_1, \ldots, p_n\}$$

where $p_1, \ldots, p_n$ are the blowing-up points of M in CP^2. Then the lemma follows from a straightforward calculation.

We fix a n between 5 and 8 and let M_0 be the complex surface with Kähler-Einstein metric g_0 in Theorem 1.1. Then M_0 is in $\mathfrak{J}_n$. In order to prove our main theorem, we pick an arbitrary smooth family $\{M_t\}_{0 \leqq t \leqq 1}$ from $\mathfrak{J}_n$. It suffices to prove that any M_t admits a Kähler-Einstein metric with positive scalar curvature. We will use the continuous method. Let

$$I = \{t \in [0, t] | M'_t \text{ admits Kähler-Einstein metric for } t' \leqq t\} .$$

Then I contains 0, in particular, I is nonempty. We need to prove that I is both open and closed. In this section, we will prove that I is open. The next several sections are devoted to the proof of the closedness. As in [Y1], [T1], etc., we first convert the existence of Kähler-Einstein metrics into solvability of some complex Monge-Amperé equation. Since the entire family $\{M_t\}$ is in $\mathfrak{J}_n$, there is a smooth family of Kähler metrics $\tilde{g}_t$ on M_t with $0 \leqq t \leqq 1$. Let $\omega_{\tilde{g}_t}$ be the associated Kähler form of $\tilde{g}_t$, then in local coordinates (z_1, z_2) of M_t,

$$\omega_{\tilde{g}_t} = \frac{\sqrt{-1}}{2\pi} \sum_{i,j=1}^{2} \tilde{g}_{ti\bar{j}} dz_i \wedge \overline{dz_j} \quad \text{where} \quad \tilde{g}_t = \sum_{i,j=1}^{2} \tilde{g}_{ti\bar{j}} dz_i \otimes \overline{dz_j}$$

We choose $\tilde{g}_t$ such that its Kähler form $\omega_{\tilde{g}_t}$ represents the first Chern class of M_t. Then by solving a family of elliptic equations, we can have a smooth family of functions $\{f_t\}_{0 \leqq t \leqq 1}$ such that

$$\mathrm{Ric}(\tilde{g}_t) = \omega_{\tilde{g}_t} + \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} f_t, \quad \int_{M_t} e^{f_t} dV_{\tilde{g}_t} = 9 - n$$

where $dV_{\tilde{g}_t} = \omega_{\tilde{g}_t}^2 = \omega_{\tilde{g}_t} \wedge \omega_{\tilde{g}_t}$ is the associated volume form of $\tilde{g}_t$ and $\mathrm{Ric}(\tilde{g}_t)$ is the Ricci form, i.e., in local coordinates (z_1, z_2), if we denote by $(R_{ti\bar{j}})$ the Ricci curvature tensor, then

$$\mathrm{Ric}(\tilde{g}_t) = \frac{\sqrt{-1}}{2\pi} \sum_{i,j=1}^{2} R_{ti\bar{j}} dz_i \wedge \overline{dz_j}$$

It is well-known that the existence of Kähler-Einstein metrics on M_t is equivalent to the solvability of the following complex Monge-Amperé equation on M_t (cf. [Y1]).

$$\begin{cases} \left(\omega_{\tilde{g}_t} + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi\right)^2 = e^{f_t - \varphi}\omega_{\tilde{g}_t}^2 \\ \left(\omega_{\tilde{g}_t} + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi\right) > 0 \end{cases} \qquad (1.1)_t$$

We remark that if $(1.1)_t$ has a solution φ_t, then the corresponding Kähler-Einstein metric has $\omega_{\tilde{g}_t} + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi_t$ as its Kähler form.

We first prove the openness of I by applying Implicit Function Theorem to the equation $(1.1)_t$.

Lemma 1.3. *Let $\{M_t\}$, I be defined as above. Then I is open.*

Proof. Let $(1.1)_{t_0}$ has a solution φ_{t_0}. The linearized operator of $(1.1)_t$ at $t = t_0$ is $L_{t_0} = \Delta_{t_0} - id$, i.e., for any smooth function v, $L_{t_0}v = \Delta_{t_0}v - v$, where Δ_{t_0} is the Laplacian associated to the Kähler-Einstein metric g_{t_0} on M_{t_0}.

By Lemma 1.2 and the standard Bochner's formula, one can prove that the first nonzero eigenvalue of Δ_{t_0} is strictly greater than one (cf. [Au]). It follows that the linearized operator L_{t_0} is invertible. Then this lemma follows from Implicit Function Theorem.

Therefore, it suffices to prove that I is closed in the interval $[0, 1]$. Without losing the generality, we may assume that $[0, 1) \in I$. By the standard elliptic theory, to show that $1 \in I$, it suffices to prove the uniform C^3-estimate of the solutions of the equations $(1.1)_t$.

Lemma 1.4. *There is a constant C independent of t such that for any solution φ of some equation $(1.1)_t$, we have*

$$\sup_{z \in M_t} \{\|\nabla_t \varphi\|_{\tilde{g}_t}(x), \|\nabla_t^2 \varphi\|_{\tilde{g}_t}(x), \|\nabla_t^3 \varphi\|_{\tilde{g}_t}(x)\} \leq C \sup_{z \in M_t} \{|\varphi(x)|\}$$

where ∇_t is the gradient with respect to the metric $\tilde{g}_t$ and $|\cdot|_{\tilde{g}_t}$ is the norm induced by $\tilde{g}_t$.

Proof. It follows from same computations as the corresponding ones in [Y1].

By this lemma, we only need to prove the uniform C^0-estimate of the solutions of equations $(1.1)_t$. This will be done in the following sections. We should mention that obtaining such an a prior C^0-estimate is the hardest part of solving this conjecture of E. Calabi.

2. The proof of main theorem

In this section, we will prove our main theorem under the assumption of Theorem 2.2. We postpone the proof of this theorem to the next several sections due to its length. Basically, this theorem provides us a strong partial C^0-estimate of the solutions of the complex Monge-Amperé equations $(1.1)_t$. Let $\{M_t\}_{0 \leqq t \leqq 1}$ be the smooth family in $\mathfrak{I}_n$ given in last section, where $5 \leqq n \leqq 8$. Then each M_t for $t < 1$ has a Kähler-Einstein metric g_t with $\mathrm{Ric}(g_t) = \omega_{g_t}$. It follows that each equation $(1.1)_t$ for $t < 1$ has a solution φ_t. Note that such a solution φ_t is unique by Bando and Mabuchi's uniqueness theorem [BM]. By the discussions in last section, in order to prove the main theorem, it suffices to show the uniform C^0-estimate of those solutions φ_t.

In [T1], the author reduces the C^0-estimate of the solution φ_t to the evaluation of some integral of φ_t. By evaluating this integral, the author proves in [T1] the existence of Kähler-Einstein metrics on Fermat hypersurfaces in CP^{n+1} of degree n or $n+1$ and the authors in [TY] prove Theorem 1.1. Here we develop an effective method to evaluate the integral posed in [T1] for the C^0-estimate of φ_t. We start with the following theorem, which is essentially the main theorem proved in [T1].

Theorem 2.1. *Let $\{M_t\}$ be the family in $\mathfrak{I}_n$ given as above. Then the Kähler manifold $M = M_1$ admits a Kähler-Einstein metric with positive scalar curvature if and only if one of the following holds.*

(1) *There are constants $\varepsilon, C > 0$ and a subsequence $\{t_i\}_{i \geqq 1}$ in the interval $[0, 1)$ with $\lim_{i\to\infty} t_i = 1$, such that for all i and any solution φ_{t_i} of $(1.1)_{t_i}$,*

$$\int_{M_{t_i}} e^{-(2/3+\varepsilon)(\varphi_{t_i} - \sup_{M_{t_i}} \varphi_{t_i})} dV_{\tilde{g}_{t_i}} \leqq C \tag{2.1}$$

(2) *There are constants $\varepsilon, C > 0$ and a subsequence $\{t_i\}_{i \geqq 1}$ in the interval $[0, 1)$ with $\lim_{i\to\infty} t_i = 1$, such that for all i and any solution φ_{t_i} of $(1.1)_{t_i}$,*

$$-\inf_{M_{t_i}} \varphi_{t_i} \leqq (2-\varepsilon) \sup_{M_{t_i}} \varphi_{t_i} + C \tag{2.2}$$

and moreover for any $\lambda < \frac{2}{3}$, there is a constant $C(\lambda)$, depending only on λ, such that

$$\int_{M_{t_i}} e^{-\lambda(\varphi_{t_i} - \sup_{M_{t_i}} \varphi_{t_i})} dV_{\tilde{g}_{t_i}} \leqq C(\lambda) \tag{2.3}$$

Proof. Consider complex Monge-Amperé equations

$$\begin{cases} \left(\omega_{\tilde{g}_t} + \dfrac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi\right)^2 = e^{f_t - s\varphi}\omega_{\tilde{g}_t}^2 \\ \left(\omega_{\tilde{g}_t} + \dfrac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi\right) > 0 \end{cases} \tag{2.4$_{t,s}$}$$

where $s \leqq 1$. By Lemma 1.2 and uniqueness theorem in [BM], the solution of equation $(1.1)_t$ for each t is unique. Thus by the same argument as in [BM] for the proof of the uniqueness, one can produce a family $\{\varphi_{t,s}\}_{0 \leqq t,s < 1}$ of smooth functions, such that the function $\varphi_{t,s}$ solves the equation $(2.4)_{t,s}$ and $\lim_{s \to 1} \varphi_{t,s} = \varphi_t$. Thus in case (1), the theorem follows from the proof of the main theorem in [T1].

In the case (2), choose λ between $\frac{2-\varepsilon}{3-\varepsilon}$ and $\frac{2}{3}$, then as in [T1], by the concavity of the logarithm function, we have

$$\sup_{M_{t_i}} \varphi_{t_i} \leqq \frac{1-\lambda}{\lambda} \int_{M_{t_i}} (-\varphi_{t_i}) dV_{g_{t_i}} + C'(\lambda) \tag{2.5}$$

where $C'(\lambda)$ is a constant depending only on λ. Combining this with (2.2), we obtain the uniform C^0-estimate of the solution φ_{t_i}. The theorem follows. We refer readers to [T1] for more details.

By this theorem, we see that, to prove the main theorem, it suffices to find a subsequence $\{t_i\}$ having the estimates either (2.1) or (2.2), (2.3).

For $t < 1$, we choose an orthonormal basis $\{S^t_{m\beta}\}_{0 \leqq \beta \leqq N_m}$ of the group $H^0(M_t, K^m_{M_t})$ with respect to the metric g_t, when $m \geqq 1$ and $N_m + 1$ is the dimension of $H^0(M_t, K^m_{M_t})$. By Kodaira's embedding theorem, those bases $\{S^t_{m\beta}\}_{0 \leqq \beta \leqq N_m}$ define embeddings $\Phi(t, m)$ from M_t into CP^{N_m} for m large. In fact, when $n = 5$, 6, the embeddings $\Phi(t, 1)$ are well-defined.

Theorem 2.2. (Strong partial C^0-estimate). *There are constants $c(n, m) > 0$, depending only on n, m, and a subsequence $\{t_i\}$ in the interval* $[0, 1)$ *with* $\lim_{i \to \infty} t_i = 1$, *such that for $m = 6k$ $(k \geqq 1)$ in case $n = 5$, 6, and $m = 2k$ $(k \geqq 1)$ in case $n = 7$, 8,*

$$\inf_{x \in M_{t_i}} \left\{ \sum_{\beta=0}^{N_m} \|S^{t_i}_{m\beta}\|^2_{g_{t_i}}(x) \right\} \geqq c(n, m) \tag{2.6}$$

where $\|\cdot\|_{g_t}$ is the norm on the line bundle $K^{-m}_{M_t}$ induced by the metric g_t.

Remark. In case $n = 5, 6, 7$, the estimate (2.6) should also hold for $m = 1$. This can be used to simplify the proof of our main theorem, but the simplification is not substantial.

The proof of Theorem 2.2 will be given in the following sections. We will first prove a weak version of this theorem, i.e., Theorem 5.1, in §5. Then we deduce Theorem 2.2 from that weak version.

Let us now see the implications of Theorem 2.2. For each $t \in [0, 1)$, we further choose an orthonormal basis $\{\tilde{S}^t_{m\beta}\}_{0 \leqq \beta \leqq N_m}$ of $H^0(M_t, K^{-m}_{M_t})$ with respect to the metric $\tilde{g}_t$. Such a basis gives an embedding $\Psi(t, m)$ of M_t into CP^{N_m} whenever the basis $\{S^t_{m\beta}\}$ does. Two embeddings $\Psi(t, m)$, $\Phi(t, m)$ are different by an automorphism $\sigma(t, m)$ in CP^{N_m}, i.e. $\sigma(t, m) \in PGL(N_{m+1})$ and

$$\Phi(t, m) = \sigma(t, m) \circ \Psi(t, m) \tag{2.7}$$

By changing the orthonormal bases $\{\tilde{S}^t_{m\beta}\}_{0 \leqq \beta \leqq N_m}$, $\{S^t_{m\beta}\}_{0 \leqq \beta \leqq N_m}$ if necessary, we may assume that each $\sigma(t, m)$ is represented by a diagonal matrix $\operatorname{diag}(\lambda_j(t))_{0 \leqq j \leqq N_m}$ with $0 < \lambda_0(t) \leqq \ldots \leqq \lambda_{N_m}(t) = 1$.

The following lemma is actually an observation on which the whole proof is based.

Lemma 2.1. *Let M_t, φ_t, $\{S^t_{m\beta}\}_{0\leqq\beta\leqq N_m}$, $\{\tilde{S}^t_{m\beta}\}_{0\leqq\beta\leqq N_m}$ be given as above, where $t<1$. Then for $m=6k(k\geqq 1)$ in case $n=5$, 6 or $m=2k$ $(k\geqq 1)$ in case $n=7$, 8,*

$$\varphi_t=f_t-\frac{1}{m}\log\left(\sum_{\beta=0}^{N_m}\|S^t_{m\beta}\|^2_{g_t}\right)+\frac{1}{m}\log\left(\sum_{\beta=0}^{N_m}|\lambda_\beta(t)|^2\|\tilde{S}^t_{m\beta}\|^2_{\tilde{g}_t}\right)+a(t) \tag{2.8}$$

where $a(t)$ is a constant depending only on t.

Proof. We remark that under the assumption on m, the group $H^0(M_t, K_{M_t}^{-m})$ is free of base point. Thus the left-handed side of (2.8) is a well-defined function on M_t. We denote it by φ'_t. One can check that φ'_t satisfies the equation

$$\omega_{g_t}=\omega_{\tilde{g}_t}+\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi'_t$$

On the other hand, the solution φ_t of $(1.1)_t$ also satisfies the same equation. The lemma follows.

From now on, in this section, we fix the subsequence $\{t_i\}$ in Theorem 2.2. For simplicity, in the following, we will replace t_i by i whenever t_i appear as subscripts or superscripts. The following lemma is a corollary of Theorem 2.2 and also explains why we call (2.6) in Theorem 2.2 "partial C^0-estimate".

Lemma 2.2. *Let $\{(M_i, g_i)\}$ be the subsequence of Kähler-Einstein manifolds given in Theorem 2.2. Define $v=6$ in case $n=5$, 6 and $v=2$ in case $n=7$, 8. Then there is a constant C independent of i, such that for any solution φ_i of $(1.1)_{t_i}$,*

$$\sup_{M_i}\left|\varphi_i-\sup_{M_i}\varphi_i-\frac{1}{v}\log\left(\sum_{\beta=0}^{N_v}|\lambda_\beta(i)|^2\|\tilde{S}^i_{m\beta}\|^2_{\tilde{g}_t}\right)\right|\leqq C \tag{2.9}$$

where $\{\lambda_\beta(i)\}_{0\leqq\beta\leqq N_v}$ are defined by those automorphisms $\sigma(t_i, v)$ in (2.7).

Proof. Note that both metrics $\tilde{g}_i$ (resp. functions f_i) converge to a smooth metric $\tilde{g}$ (resp. a smooth function f) equal to $\tilde{g}_t$(resp. f_t) with $t=1$. Then the lemma follows from Lemma 2.1 and Theorem 2.2.

Remark. This lemma implies that the normalizations $\varphi_i-\sup_{M_i}\varphi_i$ are uniformly bounded away from some subvarieties in M_i. One should be able to derive from it that $\varphi_i-\sup_{M_i}\varphi_i$ either are uniformly bounded or converge to a function G on $M\backslash D$, where $M=\lim M_i$ and D is a subvariety of M contained in anticanonical divisors, such that G satisfies the equation $\left(\omega_{\tilde{g}}+\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}G\right)^2=0$ outside D and has logarithmic singularity along D. This singular function G can be regarded as a Green function of the complex Monge-Amperé operator. This Green function must impose some analytic structures on M. Hopefully, by studying these structures, one can determine when φ_i converge to a bounded function on M.

Let v be defined as in the above lemma. We define rational integrals $I(\alpha, i)$ as follows,

$$I(\alpha, i)=\int_{M_i}\left(\sum_{\beta=0}^{N_v}|\lambda_\beta(i)|^2\|\tilde{S}^i_{v\beta}\|^2_{\tilde{g}_i}(z)\right)^{-\alpha/v}dV_{\tilde{g}_i}(z) \tag{2.10}$$

Then by Lemma 2.2, we see that the estimates (2.1) and (2.3) are equivalent to $I(\alpha, i) \leqq C$ for $\alpha = 2/3 + \varepsilon$ and λ, respectively. Thus we are bound to estimate the integrals in (2.10). This will be done in the following lemmas.

Let $\{\tilde{S}_\beta\}_{0 \leqq \beta \leqq N_v}$ be the limit of the bases $\{\tilde{S}^i_{v\beta}\}_{0 \leqq \beta \leqq N_v}$ as i goes to infinity. Then this limit is an orthonormal basis of $H^0(M, K_M^{-v})$ with respect to the metric $\tilde{g}$, where $M = M_t$ for $t = 1$. By taking the subsequence if necessary, we may assume that $\lim_{i \to \infty} \lambda_\beta(i) = \lambda_\beta \geqq 0$ for each β. Note that $\lambda_{N_v} = 1$ and $\lambda_0 \leqq \ldots \leqq \lambda_{N_v}$.

Lemma 2.3. *We adopt the notations given above. Let $N = N_v$. Then we have the following estimates,*

(i) *if $n = 8$, then $I(\alpha, i) \leqq C_\alpha$ for any $\alpha < \frac{5}{6}$.*
(ii) *if $n = 7$, then $I(\alpha, i) \leqslant C_\alpha$ for any $\alpha < \frac{3}{4}$.*

Proof. Since the proof for (ii) is identical to that for (i), we only prove (i) here. Put

$$\psi_i = \frac{1}{v}\log\left(\sum_{\beta=0}^{N} |\lambda_\beta(i)|^2 \|\tilde{S}^i_{v\beta}\|^2_{\tilde{g}_i}\right)$$

Denote by $\tilde{\omega}_i$ the Kähler form associated to the metric $\tilde{g}_i$. Then $\tilde{\omega}_i + \partial\bar{\partial}\psi_i$ define positive, d-closed, (1.1)-currents ω'_i on M_i. When i tends to infinity, we may assume that ω'_i converge weakly to a positive, d-closed, (1.1)-current $\frac{\sqrt{-1}}{2v\pi}\partial\bar{\partial}\log(|\tilde{S}_N|^2)$ denoted by ω'_∞, where $|\cdot|$ is the absolute value. Now $I(\alpha, i)$ is just the integral $\int_{M_i} e^{-\alpha\psi_i} dV_{\tilde{g}_i}$. By the discussion in §2 of [TY], we have

Fact (†). If $z_i \in M_i$ with $\lim_{i \to \infty} z_i = z \in M$ and the Lelong number $L_{\tilde{g}}(\omega'_\infty, z) \leqq 1$, then there is a $r > 0$ independent of i, such that

$$\int_{B_r(z_i, \tilde{g}_i)} e^{-\alpha\psi_i} dV_{\tilde{g}_i} \leqq C_\alpha \quad \text{for any} \quad \alpha < 1 \tag{2.11}$$

where $B_r(z_i, \tilde{g}_i)$ is the geodesic ball in M_i with radius r and the center at z_i. For more about the Lelong number, one can refer to [Le].

Let D be the zero divisor of $\tilde{S}_N$. If z is either in the complement of D or a point of D with multiplicity 2, the Lelong number $L_{\tilde{g}}(\tilde{\omega}_\infty, z) \leqq 1$. Therefore, we only need to estimate the integral in (2.11) near those points $z_i \in M_i$ with $\lim_{i \to \infty} z_i = z$ being a singular point of D with multiplicity $\geqq 3$.

Since $C_1(M)^2 = 1$, D has no singular point with the multiplicity greater than 2 if it is irreducible. It implies that the Lelong number $L_{\tilde{g}}(\tilde{\omega}_\infty, z) \leqq 1$ at every point z in M if D is irreducible. So we may write $\tilde{S}_N = S'_1 \cdot S'_2$. Since $C_1(M)^2 = 1$, one can easily derive (2.11) by Fact (†) so long as S'_1 is not colinear to S'_2. So we may assume that $S'_1 = S'_2$ and both S'_1, S'_2 are in $H^0(M, K_M^{-1})$. Also by Fact (†), it suffices to estimate the integral in (2.11) at the singular points of the divisor $\{x \in M \mid S'_1(x) = 0\}$. Let $\pi_i: M_i \mapsto CP^2$, $\pi: M \mapsto CP^2$ be the natural projections induced by blowing-ups. Then each $\pi_{i*}(\tilde{S}^i_N)$ is a sextic curve in CP^2, converging to $\pi_*(\tilde{S}_N) = \pi_*(S'_1)^2$ as i goes to infinity. Now $\pi_*(S'_1)$ is an irreducible singular cubic curve in CP^2. Each $\pi_*(S'_i)$ has only one singular point x, which is either an ordinary double point or a cusp, and is not one of blowing-up points. Without losing generality, we may assume that x is a cusp. Let U be a small neighborhood of x in

CP^2, determined later. Then

$$\int_{M_i} e^{-\alpha\psi_i} dV_{\tilde g_i} \leqq C\left(1 + \int_U \frac{1}{|f_i|^{2\alpha}} dV\right) \tag{2.12}$$

where f_i is the local holomorphic function defining $\pi_{i*}(\tilde S_N^i)$ in U, dV is the volume form of the euclidean metric on U and C is a constant depending only on U. We will always use C to denote a constant independent of i in this proof, although the quantity of this could be changed in different places. We normalize those f_i such that the limit $f = \lim_{i\to\infty} f_i$ exists and defines $\pi_*(\tilde S_N)$ in U. Choose local coordinates (z, w) with $x = (0, 0)$ and

$$f = (z^2 - w^3)^2 \tag{2.13}$$

where f is the local defining function of the divisor D in U. By holomorphic transformations of form $(z, w) \mapsto (z + b_1 + b_2 w + b_3 w^2, w)$, we may assume that

$$\begin{aligned} f_i &= f + \sum_{\substack{3k+2l<12 \\ k\neq 3}} a_{kl}(i) z^k w^l + \sum_{3k+2l\geqq 12} a_{kl}(i) z^k w^l \\ &= f + f_{iL} + f_{iR} \end{aligned} \tag{2.14}$$

where $\lim_{i\to\infty} f_{iL} = 0$, $\lim_{i\to\infty} f_{iR} = 0$. Define

$$\delta_i = 40 \max_{3k+2l<12} \left\{ |a_{kl}(i)|^{\frac{1}{12-3k-2l}} \right\} \tag{2.15}$$

We denote by J_i the integral on the right side of (2.12) and split J_i into three parts J_{i1}, J_{i2}, J_{i3} as follows,

$$\begin{aligned} J_{i1} &= \int_{\substack{|z|^4 \leqq |w|^6 \\ 1 \geqq |w| \geqq \delta_i^2}} \frac{dV}{|f_i|^{2\alpha}} \\ &= \int_{\substack{|w|\leqq 1 \\ |\xi|\leqq 1}} \frac{|w|^8 dV}{|f_i(w^3\xi, w^2)|^{2\alpha}} \\ &= \int_{\substack{|w|\leqq 1 \\ |\xi|\leqq 1}} \frac{|w|^{8-12\alpha} dw \wedge d\bar w \wedge d\xi \wedge d\bar\xi}{|(\xi^2 - 1)^2 + w^{-6}(f_{iL}(w^3\xi, w^2) + f_{iR}(w^3\xi, w^2)|^{2\alpha}} \end{aligned} \tag{2.16}$$

By the definition of f_{iL}, f_{iR} and δ_i, for i sufficiently large, one can easily see

$$|w^{-6}(f_{iL}(w^3\xi, w^2) + f_{iR}(w^3\xi, w^2))| \leqq \tfrac{1}{4} \quad \text{for} \quad \delta_i \leqq |w| \leqq 1, \quad |\xi| \leqq 1 \tag{2.17}$$

It follows that for any fixed w with $|w| \leqq \delta_i$, the holomorphic functions $(\xi^2 - 1) + w^{-6}(f_{iL} + f_{iR})$ have exactly four distinct zeroes in $\{|\xi| \leqq 1\}$ for all i, moreover, these four zeroes are disjoint from each other by a uniform distance. Therefore, we conclude that the integral J_{i1} is uniformly bounded independent of i for $\alpha < \frac{5}{6}$, i.e., $J_{i1} \leqq C_\alpha$. Similarly, we also have

$$J_{i2} = \int_{\substack{|z|^4 \geqq |w|^6 \\ 1 \geqq |z| \geqq \delta_i^3}} \frac{dV}{|f_i|^{2\alpha}} \leqq C_\alpha \quad \text{for any } \alpha < \frac{5}{6} \tag{2.18}$$

It remains to estimate J_{i3}, which is equal to $J_i - J_{i1} - J_{i2}$. By scaling $(z, w) \to (\delta_i^3 z, \delta_i^2 w)$, we have

$$J_{i3} = \int_{\substack{|z| \leq 1 \\ |w| \leq 1}} \frac{\delta_i^{10-12\alpha} dV}{|\delta_i^{-12} f_i(\delta_i^3 z, \delta_i^2 w)|^{2\alpha}}$$

Put $g_i = \delta_i^{-12} f_i(\delta_i^3 z, \delta_i^2 w)$, then by taking a subsequence, we may assume that g_i converge to a polynomial g. Note that by the expansion of f_i in (2.14), we have

$$g = (z^2 - w^3)^2 + \sum_{\substack{3k+2l<12 \\ k<3}} b_{kl} z^k w^l \tag{2.19}$$

where b_{kl} are constants and at least one of them is nonzero. This new polynomial g is less singular than the function f. For example, one can compute

$$\int_{\substack{|z| \leq 1 \\ |w| \leq 1}} \frac{dV}{|g|^{2\alpha}} < +\infty \quad \text{for any } \alpha < \frac{11}{24} \tag{2.20}$$

while

$$\int_{\substack{|z| \leq 1 \\ |w| \leq 1}} \frac{dV}{|f|^{2\alpha}} = +\infty \quad \text{for any } \alpha \geqq \frac{10}{24} = \frac{5}{12} \tag{2.21}$$

Also, the multiplicity of g at any point in $\{|z| < 1, |w| < 1\}$ is less than that of f at the origin. Therefore, by induction we can prove that $J_{i3} \leqq C_\alpha$ for $\alpha < \frac{5}{6}$. Case (i) is proved. The same arguments can be identically applied to case (ii). Then the lemma is proved.

As a consequence of above lemma and Theorem 2.1, we have

Corollary 2.1. *If $n = 7$ or 8, then M admits a Kähler-Einstein metric with positive scalar curvature.*

In order to complete the proof of main theorem, it remains to consider the case that $n = 5$ or 6. In fact, by the same arguments in the proof of Lemma 2.3, one can have analogous estimates of the integrals $I(\alpha, i)$ in (2.10) (cf. Lemma 2.5 in the proof in the following). Of course, the involved computations are more complicated. Instead, we give an alternative discussion here.

Proposition 2.1. *Let $\{f_i\}$ be a sequence of holomorphic functions on the unit ball $B_1 = \{z \in C^2 \,|\, |z| < 1\}$ such that $\lim_{i\to\infty} = f, f \not\equiv 0$. Let $\beta > 0$ be such that the integral $\int_{|z| \leq 1} |f|^{2\beta} dV$ is finite, then for any $\alpha < \beta$, we have*

$$\int_{|z| \leq \frac{1}{2}} \frac{dV}{|f|^{2\alpha}} = \lim_{i \to \infty} \int_{|z| \leq \frac{1}{2}} \frac{dV}{|f_i|^{2\alpha}} \tag{2.22}$$

where dV is the standard volume form on C^2.

Remark. In fact, Lemma 2.3 is a corollary of Proposition 2.1 and some properties of plurianticanonical divisors. We gave a separated proof because it is much simpler and transparent. The above proposition should be a classical result. But since I could not find its proof in literature to my limited knowledge, I include a sketched proof in Appendix 2. Actually, the proof is based on the modification of the above arguments in that of Lemma 2.3. The key point is how to make induction in general as we did before. The induction, as well as the proof for the following lemma, is completed by means of Newton polyhedrons associated to holomorphic functions (see Appendix 1 for details).

Lemma 2.4. *Let $n = 5$ or 6 and S be a global section in $H^0(M, K_M^{-6})$. Then there is an $\varepsilon > 0$, such that* (i) *if $n = 6$, we have*

$$\int_M \|S\|^{-\frac{2+\varepsilon}{9}} dV_{\tilde{g}} < \infty \tag{2.23}$$

unless the reduced divisor $\{S = 0\}_{\text{red}}$ is an anticanonical divisor and the union of three lines on M intersecting at a common point, where by a line on M, we mean an irreducible curve of degree 1 *with respect to the anticanonical line bundle K_M^{-1}.* (ii) *if $n = 5$, we also have* (2.23) *unless either $\{S = 0\}$ contains a curve with multiplicity* 9 *or $\{S = 0\}_{\text{red}}$ is an anticanonical divisor and the union of two lines and a curve of degree* 2 *intersecting transversally at a common point.*

In order to avoid distracting the readers from the main stream in the proof of our main theorem, we postpone the proof of this lemma in Appendix 1.

Lemma 2.5. *Let $n = 5$ or 6 and $I(\alpha, i)$ be defined as in* (2.10). *Then there are constants $\varepsilon > 0$ and $C_\alpha > 0$ depending on α, where $0 < \alpha < \frac{2}{3}$, such that* (i) *For $\alpha < \frac{2}{3}$, we have*

$$I(\alpha, i) \leqq C_\alpha \tag{2.24}$$

(ii) *For $\alpha \geqq \frac{2}{3}$, we have*

$$I(\alpha, i) \leqq C_\alpha \cdot \lambda_{N-1}(i)^{-\frac{\alpha}{3}} \tag{2.25}$$

where $N = N_6$ as in Lemma 2.2. Note that ε, C_α are independent of i.

Proof. Choose $\varepsilon > 0$ such that 4ε is given in Lemma 2.4. First we assume that $n = 6$. As before, put $D = \{\tilde{S}_N = 0\}$. If D_{red} is not an anticanonical divisor consisting of three lines intersecting at a common point, then by Lemma 2.4(i), for $\alpha \leqq \frac{2}{3} + \varepsilon$

$$\int_M \|\tilde{S}_N\|^{-\frac{2\alpha}{6}} dV_{\tilde{g}} < \infty \tag{2.26}$$

By Proposition 2.1, we have for $\alpha \leqq \frac{2}{3} + \varepsilon$

$$\begin{aligned} \lim_{i\to\infty} I(\alpha, i) &\leqq \lim_{i\to\infty} \int_{M_i} \|\tilde{S}^i_{\nu N}\|^{-\frac{2\alpha}{6}} dV_{\tilde{g}_i} \\ &= \int_M \|\tilde{S}_N\|^{-\frac{2\alpha}{6}} dV_{\tilde{g}} < \infty \end{aligned} \tag{2.27}$$

Hence, there are constants C_α depending on α such that $I(\alpha, i) \leqq C_\alpha$ for all i and $\alpha \leqq \frac{2}{3} + \varepsilon$. Thus we may assume that $\tilde{S}_N = (\tilde{S})^6$, where $\tilde{S}$ is an anticanonical section which zero divisor is the union of three lines intersecting at a common point. One can easily check that for $\alpha < \frac{2}{3}$,

$$\int_M \|\tilde{S}_N\|^{\frac{-2\alpha}{6}} dV_{\tilde{g}} < \infty \tag{2.28}$$

Then (1) follows from (2.28) and Proposition 2.1 as above.

Let p be the intersection point of the three lines in $\{\tilde{S} = 0\}$, then for any open neighborhood U of p and $0 < \alpha \leqq \frac{2}{3} + \varepsilon$,

$$\int_{M \setminus U} \|\tilde{S}_N\|^{\frac{-2\alpha}{6}} dV_{\tilde{g}} < \infty \tag{2.29}$$

It is true simply because $\{\tilde{S} = 0\}$ is smooth outside p. On the other hand, by the fact that no four lines of M intersect at a common point and $\tilde{S}_{N-1}$ is linearly independent of $\tilde{S}_N$, one can easily show that if the neighborhood U of p is sufficiently small, then for $\alpha \leqq \frac{2}{3} + \varepsilon$,

$$\int_U \|\tilde{S}_{N-1}\|^{\frac{-2\alpha}{6}} < \infty \tag{2.30}$$

Now applying Proposition 2.1, we have for $\alpha \leqq \frac{2}{3} + \varepsilon$,

$$\begin{aligned} \lim_{i \to \infty} (\lambda_{N-1}^{\alpha/3} I(\alpha, i)) &< \lim_{i \to \infty} \int_{M_i} \frac{1}{(\|\tilde{S}^i_{\nu N-1}\|^2 + \|\tilde{S}^i_{\nu N}\|^2)^{\frac{\alpha}{6}}} dV_{\tilde{g}_i} \\ &\leqq \lim_{i \to \infty} \left(\int_{U_i} \|\tilde{S}^i_{\nu N-1}\|^{\frac{-\alpha}{3}} dV_{\tilde{g}_i} + \int_{M_i \setminus U_i} \|\tilde{S}^i_{\nu N}\|^{\frac{-\alpha}{3}} dV_{\tilde{g}_i} \right) \\ &= \int_U \|\tilde{S}_{N-1}\|^{\frac{-2\alpha}{6}} dV_{\tilde{g}} + \int_{M \setminus U} \|\tilde{S}_N\|^{\frac{-2\alpha}{6}} dV_{\tilde{g}} < \infty \end{aligned} \tag{2.31}$$

where U_i are open sets of M_i and converge to U as $i \to \infty$. Thus (i) is proved.

Next we assume that $n = 5$. If neither D contains a curve with multiplicity 9 nor D_{red} is an anticanonical divisor consisting of two lines and a curve of degree 2 intersecting at a common point, then both (i) and (ii) follows from Proposition 2.1 and Lemma 2.4 as before. Therefore, we may assume that $\tilde{S}_N$ is the section listed in Lemma 2.4(ii) as an exceptional case. If D contains a curve with multiplicity 9, then $D = 9L_1 + 3(L_2 + \ldots + L_6)$, where $L_i (1 \leqq i \leqq 6)$ are lines in M satisfying: $L_1 \cdot L_j = 1 (j \geqq 2)$, $L_i \cdot L_j = 0$ for $i, j \geqq 2$. It follows that there are exactly sixteen such divisors of K_M^{-6}. If D does not contain any curve with multiplicity 9, then $D = 6(L_1 + L_2 + E)$, where L_1, L_2 are lines in M and E is a curve of degree 2 satisfying: L_1, L_2, E intersect to each other at a common point. There are exactly 40 such sections. We will call a section described as above the one of special type in $H^0(M, K_M^{-6})$. Take any two different sections S'_1, S'_2 in $H^0(M, K_M^{-6})$ described as above, by the fact that each point of M can lie in at most two lines, one can easily check the following estimate $\alpha < \frac{3}{4}$,

$$\int_M (\|S'_1 S'_2\|^{\frac{-\alpha}{6}}) dV_{\tilde{g}} < \infty \tag{2.32}$$

If $\tilde{S}_{N-1}$ is not a section of special type in $H^0(M, K_M^{-6})$, then by lemma 2.4(ii) and Proposition 2.1, for $\alpha \leqq \frac{2}{3} + \varepsilon$

$$\lim_{i \to \infty} (\lambda_{N-1}(i))^{\frac{\alpha}{3}} I(\alpha, i)) \leqq \lim_{i \to \infty} \int_{M_i} \|\tilde{S}^i_{\nu N-1}\|^{\frac{-\alpha}{3}} dV_{\tilde{g}_i} = \int_M \|\tilde{S}_{N-1}\|^{\frac{-\alpha}{3}} dV_{\tilde{g}} < \infty \tag{2.33}$$

If $\tilde{S}_{N-1}$ is a section of special type, by Proposition 2.1 and (2.32), we have that for $\alpha \leqq \frac{2}{3} + \varepsilon < \frac{3}{4}$

$$\begin{aligned}\lim_{i \to \infty} (\lambda_{N-1}(i))^{\frac{\alpha}{3}} I(\alpha, i)) &\leqq \lim_{i \to \infty} \int_{M_i} (\|\tilde{S}^i_{\nu N-1}\|^2 + \|\tilde{S}^i_{\nu N}\|^2)^{\frac{-\alpha}{6}} dV_{\tilde{g}_i} \\ &\leqq 2^{\frac{-\alpha}{6}} \lim_{i \to \infty} \int_{M_i} \|\tilde{S}^i_{\nu N-1} \tilde{S}^i_{\nu N}\|^{\frac{-\alpha}{6}} dV_{\tilde{g}_i} \\ &= 2^{\frac{-\alpha}{6}} \int_M \|\tilde{S}_{N-1} \tilde{S}_N\|^{\frac{-\alpha}{6}} dV_{\tilde{g}_i} < \infty \end{aligned} \tag{2.34}$$

Now (ii) of this lemma follows from (2.33) and (2.34).

Lemma 2.6. *Suppose that M is one of complex surfaces of form either $CP^2 \# 5\overline{CP^2}$ or $CP^2 \# 6\overline{CP^2}$ with positive first Chern class. Then we have*

$$\sup_{M_i} \varphi_i \leqq -\delta^{-1} \log(\lambda_{N-1}(i)) + C \tag{2.35}$$

where δ, C are constants independent of i.

Proof. By Lemma 2.2, 2.4, there are two constants $\varepsilon > 0$ and $C' > 0$ such that for all i

$$\int_{M_i} e^{-\left(\frac{2}{3}+\varepsilon\right)\left(\varphi_i - \sup_{M_i} \varphi_i\right)} dV_{\tilde{g}_i} \leqq C' \lambda_{N-1}(i)^{-\frac{2}{9}-\frac{\varepsilon}{3}} \tag{2.36}$$

Using the equation $(1.1)_{t_i}$, we can rewrite (2.36) as

$$\int_{M_i} e^{\left(\frac{2}{3}+\varepsilon\right)\sup_{M_i} \varphi_i + \left(\frac{1}{3}-\varepsilon\right)\varphi_i} dV_{g_i} \leqq C' e^{\sup_{M_i} f_i} \cdot \lambda_{N-1}(i)^{-\frac{2}{9}-\frac{\varepsilon}{3}} \tag{2.37}$$

where $f_i = f_{t_i}$ are given in $(1.1)_{t_i}$. By the concavity of logarithmic functions, we obtain

$$(2 + 3\varepsilon)\sup_{M_i} \varphi_i \leqq -(1 - 3\varepsilon)\inf_{M_i} \varphi_i - (\tfrac{2}{3} + \varepsilon)\log \lambda_N(i) + C'' \tag{2.38}$$

where C'' is a constant independent of i. On the other hand, it is proved in [T1] (also cf. [T2] for stronger form) that

$$-\inf_{M_i} \varphi_i \leqq 2 \sup_{M_i} \varphi_i + C''' \tag{2.39}$$

where C''' is a constant independent of i. Then (2.35) follows from (2.38) and (2.39) by taking $\delta = (6\varepsilon)^{-1}(\frac{2}{3} + \varepsilon)$ and $C = \varepsilon^{-1}(C'' + C''')$.

Corollary 2.2. *Let M be given as in Lemma 2.5. If $\lambda_{N-1} \neq 0$, then M admits a Kähler-Einstein metric with positive scalar curvature.*

Proof. Since $\lambda_{N-1} \neq 0$, all numbers $-\log \lambda_{N-1}(i)$ are uniformly bounded independent of i. Then by Lemma 2.5, there are uniform C^0-estimates of the solutions φ_{t_i} of $(1.1)_{t_i}$. So the corollary follows from the discussion in §1.

Lemma 2.7. *Let M be as in Lemma 2.5 and $\lambda_{N-1} = 0$. Then there is an $\varepsilon' > 0$, such that for any solution ϕ_i of $(1.1)_{t_i}$, we have*

$$-\inf_{M_i} \phi_i \leqq (2 - \varepsilon')\sup_{M_i} + C \tag{2.40}$$

where C is a constant independent of i.

Proof. We define two functionals first considered by Aubin in [Au] (see also [BM], [T1]) as follows,

$$I_i(u) = \int_{M_i} u\left(\omega_{\tilde{g}_i}^2 - \left(\omega_{\tilde{g}_i} + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u\right)^2\right)$$

$$J_i(u) = \int_0^1 \frac{I(su)}{s}\,ds$$

where the metric $\tilde{g}_i$ is just $\tilde{g}_{t_i}$. Note that $\tilde{g}_i$ converge to a Kähler metric $\tilde{g}$ on M. Then by the proof for Proposition 2.3 in [T1], we have

$$\int_{M_i} (-\phi_i)\omega_{g_i}^2 \leqq I_i(\phi_i) - J_i(\phi_i) \tag{2.41}$$

where g_i is the unique Kähler-Einstein metric on M_i and $\omega_{g_i} = \omega_{\tilde{g}_i} + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\phi_i$.

As in the proof of Lemma 2.2 in [T1], we compute

$$\begin{aligned}(I_i - J_i)(\phi_i) &= \int_{M_i} \frac{\sqrt{-1}}{2\pi}\partial\phi_i \wedge \bar{\partial}\phi_i \wedge \left(\frac{1}{3}\omega_{\tilde{g}_i} + \frac{2}{3}\omega_{g_i}\right)\\ &\quad + \int_{M_i} \phi_i(\omega_{\tilde{g}_i} - \omega_{g_i})\left(\frac{1}{3}\omega_{\tilde{g}_i} + \frac{2}{3}\omega_{g_i}\right)\\ &= -\frac{2}{3}\int_{M_i} \phi_i\omega_{g_i}^2 + \frac{2}{3}\int_{M_i} \phi_i\omega_{\tilde{g}_i}^2 - \frac{1}{3}\int_{M_i} \frac{\sqrt{-1}}{2\pi}\partial\phi_i \wedge \bar{\partial}\phi_i \wedge \omega_{\tilde{g}_i}\end{aligned} \tag{2.42}$$

It follows from this and (2.41) that

$$\int_{M_i} (-\phi_i)\omega_{g_i}^2 \leqq 2\sup_{M_i}\phi_i - \frac{\sqrt{-1}}{2\pi}\int_{M_i} \partial\phi_i \wedge \bar{\partial}\phi_i \wedge \omega_{\tilde{g}_i} \tag{2.43}$$

Take a smooth point x_∞ on the zero divisor D of the section $\tilde{S}_N$ in M, where $N = N_v$ and $v = 6$. Let x_i be on the divisor $\{\tilde{S}_{vN}^i = 0\}$ in M_i such that $\lim_{i\to\infty} x_i = x_\infty$ as M_i converge to M. Let $\eta > 0$ be small. Then one can choose neighborhoods U_i of x_i in M_i, U_∞ of x_∞ in M and local coordinates (z_i, w_i) at x_i, (z, w) at x_∞ such that $x_i = (0,0) \in U_i = \{|z_i| < \eta, |w_i| < \eta\}$, $x_\infty = (0,0) \in U_\infty = \{|z| < \eta, |w| < \eta\}$,

$\lim_{i\to\infty} z_i = z$, $\lim_{i\to\infty} w_i = w$ and $w_i = \tilde{S}^i_{\nu N}$ in U_i, $w = \tilde{S}_N$ in U_∞. Then for i sufficiently large such that $\lambda_{N-1}(i) < \eta^4$, by the choice of the above (z_i, w_i), one can have the estimate

$$\frac{\sqrt{-1}}{2\pi}\int_{M_i} \partial \log\left(\sum_{\beta=0}^{N} \|\lambda_\beta(i)\tilde{S}^i_{\nu\beta}\|^2_{\tilde{g}_i}\right) \wedge \bar{\partial} \log\left(\sum_{\beta=0}^{N} \|\lambda_\beta(i)\tilde{S}^i_{\nu\beta}\|^2_{\tilde{g}_i}\right) \wedge \omega_{\tilde{g}_i}$$
$$\geqq \frac{\sqrt{-1}}{2\pi} \int_{\substack{|x_i|<\eta \\ \sqrt[3]{\lambda_{N-1}(i)} < |w_i| < \sqrt[4]{\lambda_{N-1}(i)}}} \frac{C^{-1}(|w_i|^2 - C\lambda_{N-1}(i))}{(|w_i|^2 + C|\lambda_{N-1}(i)|^2)^2} dw_i \wedge d\bar{w}_i \wedge dz_i \wedge d\bar{z}_i$$
$$\geqq -\varepsilon'' \log(\lambda_{N-1}(i))$$

where C, $\varepsilon'' > 0$ are some constants independent of i.

On the other hand, by Lemma 2.2, we have

$$\left|\frac{\sqrt{-1}}{2\pi}\int_{M_i}\left(\partial\phi_i \wedge \bar{\partial}\phi_i - \partial\log\left(\sum_{\beta=0}^{N} \|\lambda_\beta(i)\tilde{S}^i_{\nu\beta}\|^2_{\tilde{g}_i}\right)\right.\right.$$
$$\left.\left.\wedge\, \bar{\partial}\log\left(\sum_{\beta=0}^{N} \|\lambda_\beta(i) S^i_{\nu\beta}\|^2_{\tilde{g}_i}\right) \wedge \omega_{\tilde{g}_i}\right|$$
$$\leqq \int_{M_i} \left|\phi_i - \sup_{M_i}\phi_i - \log\left(\sum_{\beta=0}^{N} \|\lambda_\beta(i) S^i_{\nu\beta}\|^2_{\tilde{g}_i}\right)\right|$$
$$\times\left(\omega_{g_i} + 2\omega_{\tilde{g}_i} + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\left(\sum_{\beta=0}^{N} \|\lambda_\beta(i) S^i_{\nu\beta}\|^2_{\tilde{g}_i}\right)\right)$$
$$\leqq C \text{ for some constant independent of } i$$

Recall that $-\inf_{M_i}\phi_i$ is dominated by the average $\int_{M_i}(-\phi_i)\omega^2_{g_i}$ (cf. [T1]). Then it follows from (2.41) and the above inequalities

$$-\inf_{M_i}\phi \leqq 2\sup_{M_i}\phi_i + \varepsilon''\log(\lambda_{N-1}(i)) + C' \tag{2.44}$$

where C' is a constant independent of i. Now (2.40) follows from (2.44), Lemma 2.5.

Now our main theorem follows from Corollary 2.1, 2.2, Theorem 2.1 and Lemma 2.5(i) and Lemma 2.7.

3. An application of Gromov's compactness theorem

In this section, we will apply Gromov's compactness theorem ([GLP], [GW]) and Uhlenbeck's curvature estimate for Yang-Mills equation to studying the degeneration of Kähler-Einstein metrics for compact complex surfaces in $\mathfrak{J}_n (5 \leqq n \leqq 8)$. It is the first step towards the proof of Theorem 2.2 (strong partial C^0-estimate). The more general version of the result in this section, i.e., Proposition 3.2, is stated in [An] and [Na]. But for reader's convenience and our own sake, we include a complete and independent proof here for our special case of Kähler-Einstein metrics.

Let $\{(M_i, g_i)\}$ be a fixed sequence of compact Kähler surfaces in $\mathfrak{J}_n$ with positive first Chern class and Kähler-Einstein metric g_i, where $5 \leqq n \leqq 8$. We normalize each metric g_i such that $\mathrm{Ric}(g_i) = \omega_{g_i}$.

For each Kähler-Einstein manifold (M_i, g_i), one can define a tensor $T(i)$ on M_i, which measures the deviation of Kähler manifold (M_i, g_i) from being of constant holomorphic sectional curvature. In local coordinates (z_1, z_2) of M_i, define

$$T(i)_{\alpha\bar\beta\gamma\bar\delta} = R(i)_{\alpha\bar\beta\gamma\bar\delta} - \tfrac{1}{3}(g_{i\alpha\bar\beta}g_{i\gamma\bar\delta} + g_{i\alpha\bar\delta}g_{i\gamma\bar\beta})\,. \qquad (3.1)$$

$$(1 \leqq \alpha, \beta, \gamma, \delta \leqq 2)$$

where $R(i)$ denotes the bisectional curvature tensor of the metric g_i.

A straightforward computation as in [Y2] shows the following equality for each (M_i, g_i) with M_i in $\mathfrak{J}_n$ $(3 \leqq n \leqq 8)$,

$$\int_{M_i} \|T(i)\|_{g_i}^2 dV_{g_i} = \tfrac{4}{3}(3C_2(M_i) - C_1^2(M_i)) = \tfrac{8}{3}(9 - n)\,. \qquad (3.2)$$

where $\|T(i)\|_{g_i}$ is the norm of the tensor $T(i)$ with respect to g_i, that is in local coordinates (z_1, z_2),

$$\|T(i)\|_{g_i}^2(x) = g_i^{\alpha\bar\alpha'} g_i^{\beta'\bar\beta} g_i^{\gamma\bar\gamma'} g_i^{\delta'\bar\delta}\, T(i)_{\alpha\bar\beta\gamma\bar\delta}\, \overline{T(i)_{\alpha'\bar\beta'\gamma'\bar\delta'}}(x)\,. \qquad (3.3)$$

One may also see [Ban] for reference of (3.2), too.

In particular, it implies that the L^2-integral of $\|R(i)\|_{g_i}$ is uniformly bounded from above by a uniform constant.

Lemma 3.1. *Let (M_i, g_i) be a Kähler-Einstein surface given as above. Then there are uniform constants C', C'' such that for any f in $C^1(M_i, R)$*

$$C'\left(\int_{m_i} |f|^4 dV_{g_i}\right)^{\frac{1}{2}} - C'' \int_{M_i} |f|^2 dV_{g_i} \leqq \int_{M_i} |\nabla f|^2 dV_{g_i} \qquad (3.4)$$

where ∇f denotes the gradient of f.

Proof. Since $\mathrm{Ric}(g_i) = \omega_{g_i}$ and $\mathrm{Vol}_{g_i} M_i = 9 - n$ is a constant, the lemma follows from a combination of results in C. Croke [Cr] and P. Li [Li].

The following lemma is essentially due to K. Uhlenbeck [Uh2].

Lemma 3.2. *Let N be the integer $\left[\frac{50}{(C')^2}\right] + 1$, where C' is the Sobolev constant given in* (3.4), $[a]$ *denotes the integer part of the real number a. Then there is a universal constant $C \geqq 0$, such that for any $r \in (0, 1)$ and any Kähler-Einstein surface (M_i, g_i) given as above, there are finite many points $x_{i1}^r, \ldots, x_{in}^r$ in M_i such that*

$$\|R(i)\|_{g_i}(x) \leqq \frac{C}{r^2}\left(\int_{B_{\frac{r}{4}}(x, g_i)} \|R(i)\|_{g_i}^2(x) dV_{g_i}\right)^{\frac{1}{2}} \quad \text{for any} \quad x \in M_i \setminus \bigcup_{\beta=1}^{N} B_r(x_{i\beta}^r, g_i)\,. \qquad (3.5)$$

where $B_r(x_{i\beta}^r, g_i)$ is the geodesic ball with radius r and center at $x_{i\beta}^r$ and $\|R(i)\|_{g_i}$ is the norm of $R(i)$ with respect to g_i.

Proof. A straightforward computation shows

$$-\Delta_{g_i}(\|R(i)\|_{g_i}) \leqq \|R(i)\|_{g_i} + \mu(\|R(i)\|_{g_i})^2 \qquad (3.6)$$

where Δ_{g_i} is the laplacian of g_i and μ is a positive constant independent of i, which actual value is not important to us. Define

$$E_i = \left\{ x \in M_i \,\middle|\, \int_{B_{\frac{r}{4}}(x, g_i)} \|R(i)\|_{g_i}^2 dV_{g_i} \geqq \varepsilon \right\} \tag{3.7}$$

Then by (3.2) and the well-known covering lemma, E_i can be covered by N geodesic balls of radius $\frac{r}{2}$. Take $x_{i1}^r, \ldots, x_{iN}^r$ to be the centers of these balls. Then for any $x \in M_i \setminus \bigcup_{\beta=1}^N B_r(x_{i\beta}^r, g_i)$,

$$\int_{B_{\frac{r}{4}}(x, g_i)} \|R(i)\|_{g_i}^2 dV_{g_i} \geqq \varepsilon \tag{3.8}$$

Let $\eta: R_+^1 \to R_+^1 = \{t \in R^1 \mid t \geqq 0\}$ be a cut-off function satisfying $\eta \equiv 1$ for $t \leqq 1$; $\eta \equiv 0$ for $t \geqq 2$ and $|\eta'(t)| \leqq 1$.

For any $x \in M_i \setminus \bigcup_{\beta=1}^N B_r(x_{i\beta}^r, g_i)$, denote by $\rho_z(\cdot)$ the distance function on M_i from x.

Put $f = \|R(i)\|_{g_i}$. Multiplying $\eta^2\left(\frac{8\rho_x}{r}\right) f$ on both sides of (3.6) and then integrating by parts, one obtains

$$\int_{M-i} |\nabla(\eta f)|^2 dV_{g_i} \leqq \int_{M_i} \eta^2 f^2 dV_{g_i} + \int_{M_i} |\nabla\eta|^2 f^2 dV_{g_i} + \int_{M_i} \eta^2 f^4 dV_{g_i} \tag{3.9}$$

By Lemma 3.1 and Hölder inequality,

$$C'\left(\int_{M_i} |\eta f|^4 dV_{g_i}\right)^{\frac{1}{2}} - C'' \int_{M_i} |\eta f|^2 dV_{g_i} \leqq \int_{M_i} \left(\eta^2 + \frac{64|\eta'|^2}{r^2}\right)|f|^2 dV_{g_i} + \left(\int_{M_i} |\eta f|^4 dV_{g_i}\right)^{\frac{1}{2}} \left(\int_{B_{\frac{r}{4}}(x, g_i)} |f|^2 dV_{g_i}\right)^{\frac{1}{2}} \tag{3.10}$$

Therefore, for some universal constant $C \geqq 0$, we have

$$\left(\int_{B_{\frac{r}{4}}(x, g_i)} |f|^4 dV_{g_i}\right)^{\frac{1}{2}} \leqq \frac{C}{r^2(C' - \sqrt{\varepsilon})} \int_{B_{\frac{r}{4}}(x, g_i)} |f|^2 dV_{g_i} \tag{3.11}$$

Similarly, by multiplying $\eta^2 f^3$ on both sides of (3.6) and processing as above, we have

$$\left(\int_{B_{\frac{r}{16}}(x, g_i)} |f|^8 dV_{g_i}\right)^{\frac{1}{2}} \leqq \frac{C}{r^2(C' - \sqrt{\varepsilon})} \int_{B_{\frac{r}{8}}(x, g_i)} |f|^4 dV_{g_i} \tag{3.12}$$

Combining (3.11) and (3.12), and letting $\varepsilon \leqq \frac{1}{4}(C')^2$,

$$\left(\int_{B_{\frac{r}{16}}(x, g_i)} |f|^8 dV_{g_i}\right)^{\frac{1}{8}} \leqq \frac{C}{r^{3/2}} \left(\int_{B_{\frac{r}{4}}(x, g_i)} |f|^2 dV_{g_i}\right)^{\frac{1}{2}} \tag{3.13}$$

Then (3.3) follows from Moser's iteration as in the proof of Theorem 8.17 in [GT].

We further observe that we may take the set $\{x_{i1}^{\frac{r}{4}}, \ldots, x_{iN}^{\frac{r}{4}}\}$ contained in the union of the balls $B_r(x_{i\beta}^r, g_i)$. Let $\{r_j\}_{j\geqq 1}$ be a decreasing sequence of positive numbers such that $r_1 \leqq \frac{1}{4}$, $r_j \leqq \frac{r_{j-1}}{4}$ and if we write $x_{i\beta}^j$ as $x_{i\beta}^{r_j}$ and define

$$\Omega_i^j = M_i \bigcup_{\beta=1}^N B_{2r_j}(x_{i\beta}^j, g_i) \tag{3.14}$$

Then

$$\bar{\Omega}_i^j \subseteq \Omega_i^{j+1}\left(\frac{r_{j+1}}{8}\right), \quad \text{and} \quad \bigcup_{j\geqq 1} \Omega_i^j = M_i \backslash \{x_{i1}, \ldots, x_{iN}\}$$

where $x_{i\beta} = \lim_{j\to\infty} x_{i\beta}^j$ for any $1 \leqq \beta \leqq N$,
$\Omega_i^{j+1}(\varepsilon) = \{x \in \Omega_i^{j+1} | \mathrm{dist}_{g_i}(x, \partial\Omega_i^{j+1}) > \varepsilon\}$.

The following proposition is essentially a special case of the famous Gromov's compactness theorem (cf. [GP], [GW]).

Proposition 3.1. *Let $\{(X_i, h_i)\}$ be a sequence of n-dimensional Kähler-Einstein manifolds (maybe noncompact) and Ω_i be a sequence of domains in X_i with boundary $\partial\Omega_i$ for each $i \geqq 1$. Suppose that all i,*

(i) *The norm $\|R(h_i)\|_{h_i}(x)$ of the bisectional curvatures $R(h_i)$ are uniformly bounded for x in Ω_i.*
(ii) InjRad$(x) \geqq C$ *for $x \in \Omega_i$ and some uniform constant C.*
(iii) $0 \leqq C' \leqq \mathrm{Vol}_{h_i}(\Omega_i) \leqq C''$ *for some uniform constants C', C''.*

Then given any $\varepsilon > 0$, there is a subsequence $\{\Omega_{i_k}(\varepsilon), h_{i_k}\}_{k\geqq 1}$ of Kähler-Einstein manifolds $\{\Omega_i(\varepsilon), h_i\}_{i\geqq 1}$, where $\Omega_i(\varepsilon) = \{x \in \Omega_i | \mathrm{dist}_{h_i}(x, \partial\Omega_i) > \varepsilon\}$, and a Kähler-Einstein manifold $(\Omega_\infty(\varepsilon), h_\infty)$ such that for the compact subset $K \subset \Omega_\infty(\varepsilon)$, there is an $\varepsilon' > \varepsilon$ such that for k sufficiently large, there are diffeomorphisms ϕ_{i_k} of $\Omega_{i_k}(\varepsilon')$ into $\Omega_\infty(\varepsilon)$ satisfying
(1) *$K \subset \phi_{i_k}(\Omega_{i_k}(\varepsilon'))$ for any $k \geqq 1$.*
(2) *$(\phi_{i_k}^{-1})^* h_{i_k}$ converge uniformly to h_∞ on K.*
(3) *$(\phi_{i_k})_* \cdot J_{i_k} \cdot (\phi_{i_k}^{-1})_*$ converge uniformly to J_∞ on K, where J_{i_k}, J_∞ are the almost complex structures of Ω_i, $\Omega_\infty(\varepsilon)$, respectively.*

Proof. By some standard computations and the assumption that (X_i, h_i) are Kähler-Einstein manifolds, the bisectional curvature tensor $R(h_i)$ satisfies a quasi-linear elliptic system. The assumption (i), (ii), and (iii) imply that the Sobolev inequalities hold on $\Omega_i(\varepsilon)$ with uniform Sobolev constant. It follows from some well-known elliptic estimates (cf. [GT], [Uh1]) that

$$\|D^l R(h_i)\|_{h_i}(x) \leqq C(l), \quad l = 1, 2, \ldots, \infty\,. \tag{3.16}$$

where $D^l R(h_i)$ denotes the l^{th} covariant derivative of $R(h_i)$ on Ω_i and $C(l)$ are uniform constants depending only on l. Then by Gromov's compactness theorem ([GP], [GW]), there are a subsequence $\{(\Omega_{i_k}(\varepsilon), h_{i_k})\}$ and a Riemannian manifold $(\Omega_\infty(\varepsilon), h_\infty)$ such that the above (1) and (2) hold. Let K be any compact subset in $\Omega_\infty(\varepsilon)$ and ϕ_{i_k} defined as in the statement of this proposition. For the almost complex structure J_{i_k} on Ω_{i_k}, it is clear that $(\phi_{i_k})_* \cdot J_{i_k} \cdot (\phi_{i_k}^{-1})_*$ is an almost complex on K. By taking subsequence of $\{i_k\}$, we may assume that $(\phi_{i_k})_* \cdot J_{i_k} \cdot (\phi_{i_k}^{-1})$ converge on K. Since K is arbitrary, we obtain an almost complex structure J_∞ on $\Omega_\infty(\varepsilon)$. It is easy to check that this J_∞ is integrable and h_∞ is a Kähler-Einstein metric with respect to this J_∞.

Combining this proposition with lemma 3.2, we have the following corollary.

Lemma 3.3. *Let $\{(M_i, g_i)\}$ be the sequence of Kähler-Einstein surfaces in Theorem 3.1. By taking a subsequence of $\{(M_i, g_i)\}$, we may assume that $(M_i, \setminus \{x_{i\beta}\}_{1 \leqq \beta \leqq N}, g_i)$ converge to a Kähler-Einstein manifold (M_∞, g_∞) in the following sense: for any compact subset $K \subset M_\infty$, there is a $r > 0$ such that there are diffeomorphisms ϕ_i from $M_i \setminus \bigcup_{\beta=1}^N B_r(x_{i\beta}, g_i)$ into M_∞ with K in the images and satisfying:*

(1) $(\phi_i^{-1})^* g_i$ converge to g_∞ uniformly on K.

(2) $\phi_{i*} \circ J_i \circ (\phi_i^{-1})_*$ converge to J_∞ uniformly on K, where J_i, J_∞ are the almost complex structures of M_i, M_∞, respectively.

Moreover, the limit M_∞ has only finite many connected components and the curvature tensor $R(g_\infty)$ of g_∞ is L^2-bounded by a universal constant.

Proof. For any $j \geqq 1$, by Lemma 3.2, the curvature tensor $R(i)$ of g_i are uniformly bounded on the domains Ω_i^j in (3.14), and $\mathrm{Vol}_{g_i}(\Omega_i^j)$ uniformly approximte to $(9-n)\pi^2$. Since $\mathrm{Ric}(g_i) = \omega_{g_i}$ for all i, the diameters $\mathrm{diam}(M_i, g_i)$ are bounded from above by $\sqrt{3}\pi$. By Volume Comparison Theorem [Bi], we have for any $0 < r < \sqrt{3}\pi$ and $x \in M_i$,

$$\mathrm{Vol}(B_r(x, g_i)) \geqq cr^4 \tag{3.17}$$

where c is a uniformly constant. Thus by the estimate on injectivity radius in [CGT], we prove that those assumptions (i)–(iii) in Proposition 3.1 are satisfied by (Ω_i^j, g_i), $i, j \geqq 1$. Then by Proposition 3.1, we have a sequence of open Kähler-Einstein surfaces $(\Omega_\infty^j, g_\infty^j)$. By the properties (1)–(3) in that proposition, we can naturally identify Ω_∞^j with a domain in Ω_∞^{j+1} such that the restriction of g_∞^{j+1} to Ω_∞^j is just g_∞^j. Thus we glue $\{(\Omega_\infty^j, g_\infty^j)\}_{j \geqq 1}$ together to obtain the required (M_∞, g_∞) with properties (1) and (2) as stated in this lemma. It is clear by Fatou's lemma that $R(g_\infty)$ is L^2-bounded by the universal constant for the L^2-bounded of $R(g_i)$. The finiteness of the connected components of M_∞ follows from Lemma 3.4 we will prove in the following.

Let ρ_i be the distance function on $M_i \times M_i$ induced by the metric g_i, and ρ_∞ be the limit of ρ_i. Note that to make $\rho_\infty = \lim \rho_i$ meaningful, we may need to take the subsequence of $\{i\}$. Obviously, this function ρ_∞ is a Lipschitz function on $M_\infty \times M_\infty$. Also for each β between 1 and N, the function $\rho(x_{i\beta})$ converge to a Lipschitz function $\rho_{\infty\beta}$. According to [GLP], one may attach finite many points $x_{\infty 1}, \ldots, x_{\infty N}$ to M_∞ such that $M_\infty \cup \{x_{\infty 1}, \ldots, x_{\infty N}\}$ becomes a complete length space with length function ρ_∞ extending that ρ_∞ on $M_\infty \times M_\infty$, $\rho_\infty(x_{\infty\beta}, \cdot) = \rho_\infty(\cdot, x_{\infty\beta}) = \rho_{\infty\beta}$.

Lemma 3.4. *For any $r > 0$, put $E_\beta(r) = \{x \in M_\infty | \rho_{\infty\beta}(x) < r\}$, then there is a constant L independent of r such that the number of the connected components in $E_\beta(r)$ is less than L for any $1 \leqq \beta \leqq N$.*

Proof. By the construction of M_∞, it is easy to show that any point in $E_\beta(r)$ can be connected to a point in $E_\beta(r')$ by a path within $E_\beta(r)$ for any $r' > 0$. Thus it suffices to prove the lemma for sufficiently small r. Choose $r_0 > 0$ such that for $r \leqq r_0$, $E_\beta(r) \cap E'_\beta(r) = \emptyset$ for $\beta \neq \beta'$. By Volume Comparison Theorem [Bi] and positivity of the Ricci curvatures of (M_i, g_i), by taking limit of $B_r(x_{i\beta}, g_i)$, $B_{\frac{r}{4}}(x_{i\beta}, g_i)$ etc., as i goes to infinity, we have for $1 \leqq \beta \leqq N$,

$$\mathrm{Vol}_{g\infty}(E_\beta(r)) \leqq Cr^4 \tag{3.18}$$

$$\mathrm{Vol}_{g\infty}(B_{\frac{r}{4}}(x, g_\infty)) \geqq C^{-1}r^4 \quad \text{for} \quad x \in \partial E_\beta\left(\frac{r}{2}\right) \tag{3.19}$$

where C is some constant independent of r.

Put $L = [C^2] + 2$. We claim that the number of connected components in $E_\beta(r)$ is less than L. In fact, if not, by taking r smaller, we may have $y_1, \ldots, y_L$ in different components of $\partial E_\beta\left(\frac{r}{2}\right)$ such that $B_{\frac{r}{4}}(y_j, g_\infty) \cap B_{\frac{r}{4}}(y_{j'}, g_\infty) = \emptyset$ for $j \neq j'$ and $B_{\frac{r}{4}}(y_i, g_\infty) \subset E_\beta(r)$. Thus by (3.18) and (3.19), we have

$$C^{-1}r^4 L \leqq \sum_{j=1}^{L} \mathrm{Vol}_{g\infty}(B_{\frac{r}{4}}(y_j, g_\infty)) \leqq \mathrm{Vol}_{g\infty}(E_\beta(r)) \leqq Cr^4$$

i.e. $L \leqq C^2$. A contradiction. Therefore, our claim is true and the lemma is proved.

Lemma 3.5. *There is a decreasing positive function* $\varepsilon(r)$, *satisfying* $\lim_{r\to 0}\varepsilon(r) = 0$, *such that for any point* x *in* M_∞, *we have*

$$\|R(g_\infty)\|(x) \leqq \frac{\varepsilon(r(x))}{r^2(x)}$$

where $r(x) = \min_{1 \leqq j \leqq N}\{\rho_\infty(x_{\infty j}, x)\}$.

Proof. It simply follows from Lemma 3.2 by taking limit on i and using Lemma 3.3.

Denote by $\Delta\left(\frac{1}{k}, k\right)$ the truncated ball $\left\{z \,\middle|\, \frac{1}{k} < |z| < k\right\}$ in euclidean space C^2. Put $\Delta_r^* = \bigcup_{r \geqq r' > 0}\Delta(\frac{1}{2}r', 2r')$, then Δ_r^* is the punctured ball in C^2 with radius r. Also denote by g_F the standard euclidean metric on C^2.

Lemma 3.6. *Let* E *be one of connected components in* $\bigcup_{\beta=1}^N E_\beta(r_0)$, *where* r_0 *is chosen as in the proof of Lemma* 3.4 *such that* $E_\beta(r_0) \cap E_{\beta'}(r_0) = \emptyset$ *for* $\beta \neq \beta'$. *Then there are a* $\tilde{r} > 0$ *and a diffeomorphism* ϕ *from* $\Delta_{\tilde{r}}^*$ *into the universal covering* $\tilde{E}$ *of* $E \cap \bigcup_{\beta=1}^N E_\beta(\tilde{r})$ *such that the covering map* $\pi_E: \tilde{E} \to E$ *is finite and for* $r \geqq \tilde{r}$,

$$\max_{\Delta_r^*} |(\pi_E \circ \phi)^* g_\infty - g_F|_{g_F} \leqq \varepsilon_1(r) \tag{3.21}$$

where $\varepsilon_1(r)$ *is a decreasing function of* r *with* $\lim_{r\to\infty}\varepsilon_1(r) = 0$.

Proof. For any integer $k \geqq 2$, $kr \leqq r_0$, we define an open manifold $D(r, k)$ with

Kähler metric $g(r, k)$,

$$D(r, k) = E \cap \bigcup_{\beta=1}^{N} \left(E_\beta(kr) \setminus E_\beta\left(\frac{r}{k}\right)\right).$$

$$g(r, k) = \frac{1}{r^2} g_\infty |_{D(r,k)}$$

Then by Lemma 3.5,

$$\|R(g(r, k))\|_{g(r,k)} \leqq k^2 \varepsilon(kr) \tag{3.22}$$

Claim 1. For any fixed integer $k > 0$ and any sequence $\{r(i)\}_{i \geqq 1}$ with $kr(i) \leqq r_0$ and $\lim r(i) = 0$, there is a subsequence $\{i_j\}$ of $\{i\}$ such that Kähler-Einstein manifolds $(D(r(i_j), k), g(r(i_j), k))$ converge to a flat Kähler manifold $D_{\infty,k}$. Here the meaning of convergence is as that in Lemma 3.3.

This is simply a consequence of Proposition 3.1 and (3.22) and the fact that each $(D(r, k), g(r, k))$ is Kähler-Einstein.

For simplicity, we assume that the subsequence $\{r(i_j)\}$ is just $\{r(i)\}$ in the above claim. By the diagonal method and taking subsequence of $\{r(i)\}$ if necessary, we may assume that for any $k' \geqq 2$, $(D(r(i), k'), g(r(i), k'))$ converge to a flat Kähler manifold $D_{\infty,k'}$. We can naturally identify $D_{\infty,k'}$ as an open set of $D_{\infty,k''}$ if $k' < k''$. Thus each manifold $D_{\infty,k}$ in claim 1 is contained as open subset in a flat Kähler manifold $D_\infty = \bigcup_{k' \geqq 2} D_{\infty,k'}$. As before, the distance function $\frac{1}{r(i)^2}\rho_\infty$ of the dilated metric $\frac{1}{r(i)^2} g(r(i), k)$ converge to the distance function ρ_F of the flat metric g_F on D_∞.

Let E be in $E_\beta(r_0)$, then $\frac{1}{r(i)^2}\rho_\beta$ also converge to a Lipschitz function, formally denoted by $\rho_F(o, \cdot)$. One may think o as an attached point to D_∞. Note that $D_{\infty,k} = \left\{ z \in D_\infty \,\middle|\, \frac{1}{k} < \rho_F(0, z) < k \right\}$. Let $\tilde{D}_\infty$ be the universal covering of D_∞. Since $\tilde{D}_\infty$ is flat and simple-connected, we may assume that $\tilde{D}_\infty$ is an open subset in C^2.

Claim 2. The fundamental group $\pi_1(D_\infty)$ is finite. In fact, the number of elements in $\pi_1(D_\infty)$ is bounded by a uniform constant.

By the construction of D_∞, it is easy to find a point y in D_∞ with $\rho_F(0, y) = 1$ and a geodesic ray γ in D_∞ such that $\gamma(1) = y$ and $\rho_F(0, \gamma(t)) = t$. For any $R > 0$, define a modified geodesic ball of radius $R > 0$ by

$$B_R^m(\gamma(3R), g_F) = \{ z \in D_\infty \mid \rho_F(\gamma(3R), z) < R \text{ and } \exists \text{ a unique geodesic jointing } z \text{ to } \gamma(3R)\} \tag{3.22}$$

Then the closure of $B_R^m(\gamma(3R), g_F)$ is the limit of the balls $B_R\left(y_{iR}, \frac{1}{r(i)^2} g_\infty\right)$ in $E \subset E_\beta(r_0)$, where all y_{iR} lie on a geodesic $\gamma_0 \subset E$, whose dilations by $r(i)^{-1}$ converge to the previous ray γ as i goes to infinity. It follows from Volume

Comparison Theorem [Bi] that

$$\mathrm{Vol}_{gF}(B_R^m(\gamma(3R), g_F)) = \mathrm{Vol}_{gF}(\overline{B_R^m(\gamma(3R), g_F)})$$

$$= \lim_{i\to\infty} \mathrm{Vol}_{g\infty}\left(B_R^m\left(y_{iR}, \frac{1}{r(i)^2} g_\infty\right)\right) \geqq CR^4$$

where C is a uniform constant. Let $\tilde{\gamma}$ be a lifting of γ to $\tilde{D}_\infty$ such that $\pi(\tilde{\gamma}(1)) = \gamma(1) = y$, where $\pi: \tilde{D}_\infty \to D_\infty$ is natural projection. Also $\pi(\tilde{\gamma}(3R)) = \gamma(3R)$. Since $B_R^m(\gamma(3R), g_F)$ is contractible by definition, there is an open subset $\tilde{B}_R \subset \tilde{D}_\infty$ such that $\pi|_{\tilde{B}_R}$ is an isometry from $\tilde{B}_R$ onto $B_R^m(\gamma(3R), g_F)$. In particular, $\mathrm{Vol}_{gF}(\tilde{B}_R) \geqq CR^4$ with C given above. Any element σ in $\pi_1(D_\infty)$ can be considered as a deck transformation on $\tilde{D}_\infty$. By definition of $\tilde{B}_R$, we have $\sigma(\tilde{B}_R) \cap \tilde{B}_R = \emptyset$. Denote by $B_{R'}(\tilde{\gamma}(1))$ the standard ball in C^2 with center at $\tilde{\gamma}(1)$. Then

$$(36\pi)^2 R^4 = \mathrm{Vol}_{gF}(B_{6R}(\tilde{\gamma}(1))) \geqq \sum_{\mathrm{dist}(\sigma(\tilde{\gamma}(1)), \tilde{\gamma}(1)) < R} \mathrm{Vol}_{gF}(\sigma(\tilde{B}_R))$$

$$\geqq CR^4 \cdot \#\{\sigma \in \pi_1(D_\infty) | \mathrm{dist}(\sigma(\tilde{\gamma}(1)), \tilde{\gamma}(1)) < R\}$$

By letting R go to infinity, we conclude

$$\#\{\sigma \in \pi_1(D_\infty)\} \leqq \frac{(36\pi)^2}{C} < \infty .$$

Claim 2 is proved.

Any element in $\pi_1(D_\infty)$ is an isometry of the open subset $\tilde{D}_\infty$ in C^2. Thus $\pi_1(D_\infty)$ can be considered as a subgroup in the linear automorphism group of C^2. Since $\pi_1(D_\infty)$ is finite, we may further assume that $\pi_1(D_\infty)$ is in the unitary group $U(2)$.

Claim 3. The Kähler manifold $\tilde{D}_\infty \subset C^2$ coincides with $C^2 \backslash \{o\}$. Moreover, the pullback $\pi^* \rho_F(o, \cdot)$ coincides with the euclidean distance function from the origin in C^2.

We adopt the notations in the proof of Claim 2. Let $q = \lim_{t\to+0} \tilde{\gamma}(t)$. Let $\overline{\tilde{D}_\infty}$ be the closure of $\tilde{D}_\infty$ in C^2, and $p \in \overline{\tilde{D}_\infty} \backslash \tilde{D}_\infty$. Then there is a sequence $\{p_j\} \subset \tilde{D}_\infty$ such that $p = \lim p_j$ in C^2. It is clear that $\{\pi(p_j)\}$ has no limit point in D_∞, so $\lim_{j\to\infty} \rho_F(o, \pi(p_j)) = 0$. Each $\pi(p_j)$ can be connected to a $\gamma(t_j)$ by a path γ_j' with length $l(\gamma_j')$ and $\lim_{j\to\infty} l(\gamma_j') = 0$. Thus for each j, there is another lifting $\tilde{p}_j$ of $\pi(p_j)$ in $\tilde{D}_\infty$ such that $\lim_{j\to\infty} \tilde{p}_j = q$ in C^2. For each j, there is a $\sigma_j \in \pi_1(D_\infty)$ such that $p_j = \sigma_j(\tilde{p}_j)$. By taking a subsequence of $\{j\}$ if necessary, we may assume that all σ_j are the same, denoted by σ. Then $p = \sigma(q)$. Thus we proved that $\tilde{D}_\infty = C^2 \backslash \pi_1(D_\infty) \cdot q$. Let o be the origin of C^2, then $o \notin \tilde{D}_\infty$ since $\pi_1(D_\infty) \subset U(2)$ acts on $\tilde{D}_\infty$ freely. It follows that $q = 0$ and $\tilde{D}_\infty = C^2 \backslash \{o\}$.

Claim 4. For any $\varepsilon \in (0, 1)$, there is a $r_\varepsilon > 0$ such that for any $r > r_\varepsilon$, there is a diffeomorphism ϕ_r from $\Delta(\frac{1}{2}r, 2r)$ into $\pi_E^{-1}(D(r, 2))$ with its image containing $\pi_E^{-1}(D(r, 2-\varepsilon))$ and

$$\max\left\{ \|\phi_r^* \pi_E^* g_\infty - g_F\|_{g_F}(x) | x \in \Delta\left(\frac{1}{2}r, 2r\right)\right\} \leqq \varepsilon \qquad (3.23)$$

where g_F is induced by the euclidean metric on C^2.

We prove it by contradiction. Suppose that the claim is not true, then there is a sequence $\{r(i)\}$ with $\lim r(i) = 0$ such that for any $r(i)$ no diffeomorphism with the above properties exists. But by previous three claims, we can easily find a subsequence of $\{r(i)\}$, for simplicity, say $\{r(i)\}$ itself, such that $\left(D(2, r(i)), \frac{1}{r(i)^2} g_\infty\right)$ converge to $\Delta(\frac{1}{2}, 2)/\Gamma$ in $D_\infty = C^2 \backslash \{0\}/\Gamma$ for some finite group $\Gamma \in U(2)$ with $\# \Gamma \leqq C$, where C is a uniform constant given in Claim 2. Since the estimate (3.23) is invariant under scaling, by the definition of the convergence given in Proposition 3.1, we may have the diffeomorphisms ϕ_i from $\Delta(\frac{1}{2}r(i), 2r(i))$ into $\pi_E^{-1}(D(r(i), 2))$ for i large satisfying (3.23) above. A contradiction. We proved this claim.

This above claim implies immediately that there is a decreasing function $\varepsilon'(r)$ on r with $\lim_{r \to \infty} \varepsilon'(r) = 0$, such that for any $r < r_{\frac{1}{2}}$, we can replace ε by $\varepsilon'(r)$ in (3.23) of Claim 4.

It remains to glue all ϕ_r together to obtain the required local diffeomorphism ϕ in the statement of our lemma. Put $r_1 = \frac{1}{2}\tilde{r}$, $r_i = \frac{1}{2} r_{i-1}$ for $i \geqq 2$, where $\tilde{r}$ is sufficiently small. Let $\phi_i = \phi_{r_i}$ be the diffeomorphism from $\Delta(\frac{1}{2}r_i, 2r_i)$ into $\pi_E^{-1}(D(r_i, 2))$ given by Claim 4. Then for any $i \geqq 2$, the composition $\phi_{i-1}^{-1} \circ \phi_i$ is a diffeomorphism from $(\pi_E \circ \phi_i)^{-1}(D(r_i, 2) \cap (r_{i-1}, 2))$ onto $(\pi_E \circ \phi_{i-1})^{-1}$ $(D(r_i, 2) \cap D(r_{i-1}, 2))$ and satisfies

$$\sup\{\|(\phi_{i-1}^{-1} \circ \phi_i)^* g_F - g_F\|_{g_F}(x) | x \in (\pi \circ \phi_i)^{-1}(D(r_i, 2) \cap D(r_{i-1}, 2))\} \leqq 4\varepsilon'(r_{i-1}) \tag{3.24}$$

$$\sup\{\|(\phi_i^{-1} \circ \phi_{i-1})^* g_F - g_F\|_{g_F}(x) | x \in (\pi_E \circ \phi_{i-1})^{-1}(D(r_i, 2) \cap D(r_{i-1}, 2))\} \leqq 4\varepsilon'(r_{i-1}) \tag{3.25}$$

By (3.24) and (3.25) and letting $\tilde{r}$ small enough, one can easily modify $\phi_{i-1}^{-1} \circ \phi_i$ to be a smooth diffeomorphism ψ_i from $(\pi_E \circ \phi_i)^{-1}(D(r_i, 2) \cap D(r_{i-1}, 2))$ into $\Delta_{\tilde{r}}^*$ such that

$$\psi_i = \begin{cases} \phi_{i-1}^{-1} \circ \phi_i & \text{in } (\pi_E \circ \phi_i)^{-1}((D(r_{i-1}, 2) \cap D(\frac{3}{10} r_{i-1}, 2)) \\ \mathrm{Id} & \text{in } (\pi_E \circ \phi_i)^{-1}(D(r_i, 2) \cap D(\frac{5}{3} r_{i-1}, 2)) \end{cases}$$

and the estimates

$$\sup\{\|\psi_i^* g_F - g_F\|_{g_F} | x \in (\pi_E \cdot \phi_i)^{-1}(D(r_i, 2) \cap D(r_{i-1}, 2))\} \leqq 400\varepsilon(r_{i-1}) \tag{3.26}$$

Now we define a diffeomorphism $\phi: \Delta_{\tilde{r}}^* \to \tilde{E}$ by

$$\phi|_{\Delta(\frac{5}{6}r_1, 2r_1)} = \phi_1, \quad \phi|_{\Delta(\frac{5}{6}r_i, \frac{6}{5}r_i)} = \phi_i (i \geqq 2)$$

$$\phi|_{\Delta(\frac{5}{3}r_{i+1}, \frac{5}{6}r_i)} = \phi_i \circ \psi_{i+1} \quad \text{for} \quad i \geqq 1 .$$

Then ϕ satisfies (3.21) for some decreasing function $\varepsilon(r)$ with $\lim_{r \to 0} \varepsilon_1(r) = 0$. The finiteness of π_E follows directly from Claim 2. The lemma is proved.

By this lemma, we can compactify M_∞ topologically by adding a point $x_{\infty\beta}$ to $E_\beta(\tilde{r})$ for each β between 1 and N. Denote by $\bar{M}_\infty$ the compactification of M_∞. Then $\bar{M}_\infty$ has the following properties: for any $x_{\infty\beta}$, there is a neighborhood U_β of $x_{\infty\beta}$ in $\bar{M}_\infty$ such that any connected component $U_{\beta j} (1 \leqq j \leqq l_\beta)$ of $U_\beta \cap M_\infty$ is covered by a smooth manifold $\tilde{U}_{\beta j}$ with the covering group $\Gamma_{\beta j}$ isomorphic to a finite group in $U(2)$, and $\tilde{U}_{\beta j}$ is diffeomorphic to a punctured ball $\Delta_{\tilde{r}}^*$ in C^2. Let

$\phi_{\beta j}$ be the diffeomorphism from $\Delta_{\tilde{r}}^*$ onto $\tilde{U}_{\beta j}$ and $\pi_{\beta j}$ be the covering map from $\tilde{U}_{\beta j}$ onto $U_{\beta j}$. Then by Lemma 3.6, the pull-back metric $\phi_{\beta j}^* \circ \pi_{\beta j}^*(g_\infty)$ extends to a C^0-metric on the ball $\Delta_{\tilde{r}}$ with the estimate (3.21). Note that the metric $\phi_{\beta j}^* \circ \pi_{\beta j}^*(g_\infty)$ is an Einstein one outside the origin.

Such a $\bar{M}_\infty$ with the metric g_∞ is called a generalized topological orbifold with C^0-metric g_∞. The previous discussions in this section yields.

Proposition 3.2. *Let $\{(M_i, g_i)\}$ be a sequence of compact Kähler-Einstein surface in $\mathfrak{J}_n(5 \leqq n \leqq 8)$ as given at the beginning of this section. Then by taking a subsequence if necessary, we may assume that (M_i, g_i) converge to an open Kähler-Einstein surface (M_∞, g_∞) with $M_\infty = \bar{M}_\infty \backslash \{x_{\infty\beta}\}_{1 \leqq \beta \leqq N}$ in the sense of Lemma 3.3, where $\bar{M}_\infty$ is a generalized topological orbifold such that g_∞ can be extended to be a C^0-metric on $\bar{M}_\infty$ described as above.*

The differential structure on M_∞ can be extended to $\bar{M}_\infty$ and the extension may not be unique. But there is at most one with which the metric g_∞ on M_∞ can be extended smoothly to $\bar{M}_\infty$. In next section, we will prove that there is such an extension by using Uhlenbeck's theory of removing singularities of Yang-Mills connections. We would like to point out that Proposition 3.2 also holds for a sequence of real 4-dimension Einstein manifolds as claimed and proved in [An] and [Na]. We refer readers to these papers.

4. Removing isolated singularities of Kähler-Einstein metrics

In [Uh1], K. Uhlenbeck invented a beautiful theory of removing isolated singularities of Yang-Mill connections on real 4-dimensional manifolds. The purpose of this section is to apply this theory of Uhlenbeck to the Kähler-Einstein metric g_∞ on M_∞ constructed in the last section and prove that g_∞ can be smoothly extended to the generalized topological orbifold $\bar{M}_\infty$ with some differential structure (cf. Proposition 3.2 for details). The latter $\bar{M}_\infty$ is considered as a compactification of M_∞ and the complement $\bar{M}_\infty \backslash M_\infty$ consists of finitely many points $\{x_{\infty\beta}\}_{1 \leqq \beta \leqq N}$.

Let $U_{\beta j}$ be any connected component in $U_\beta \cap M_\infty (1 \leqq \beta \leqq N, 1 \leqq j \leqq l_\beta)$, where U_β is a small neighborhood of $x_{\infty\beta}$ in $\bar{M}_\infty$. Recall that each $U_{\beta j}$ is covered by $\Delta_{\tilde{r}}^*$ in C^2 with the covering group $\Gamma_{\beta j}$ isomorphic to a finite group in $U(2)$ and $\pi_{\beta j}^* g_\infty$ extends to a C^0-metric on the ball $\Delta_{\tilde{r}}$ with the estimate (3.21). The smooth extension of g_∞ to $\bar{M}_\infty$ is local in nature. Therefore, we may

Fix β and j $(1 \leqq \beta \leqq N, 1 \leqq j \leqq l_\beta)$ and denote $\pi_{\beta j}^* g_\infty$ by g for simplicity. We need to construct a homeomorphism ψ of $\Delta_{\tilde{r}}$ into itself, such that the restriction of ψ to $\Delta_{\tilde{r}}^*$ has its image in $\Delta_{\tilde{r}}^*$ and is C^∞-smooth and $\psi^* g$ extends smoothly across the origin in $\Delta_{\tilde{r}}^*$.

The first step towards this goal is to prove the boundedness of the curvature tensor $R(g)$. The proof here is identical to that for Yang-Mill connections in [Uh1] with some modifications. However, for reader's convenience, we include a sketched proof here. We will just consider $\Delta_{\tilde{r}}^*$ as a real 4-dimensional manifold for being, where $\tilde{r}$ is given in Lemma 3.6. In the proof of Lemma 3.6, we observe that by the definition of the metric g, we may choose the diffeomorphism $\phi = \phi_{\beta j}$ properly such that for $\tilde{r}$ sufficiently small, the following estimate hold,

$$\begin{cases} \|dg_{ij}\|_{gF}(x) \leqq \dfrac{\varepsilon_1(r(x))}{r(x)}, & x \in \Delta_{\tilde{r}}^*,\ 1 \leqq i, j, \leqq 4 \\ \left\| d\left(\dfrac{\partial g_{ij}}{\partial x_k}\right)\right\|_{gF}(x) \leqq \dfrac{\varepsilon_1(r(x))}{r(x)^2}, & x \in \Delta_{\tilde{r}}^*,\ 1 \leqq i, j, k \leqq 4 \\ \left\| d\left(\dfrac{\partial^2 g_{ij}}{\partial x_k \partial x_l}\right)\right\|_{gF}(x) \leqq \dfrac{\varepsilon(r(x))}{r(x)^3}, & x \in \Delta_{\tilde{r}}^*,\ 1 \leqq i, j, k, l \leqq 4 \end{cases} \tag{4.1}$$

where d is the exterior differential on $C^2 = R^4$, $\|\cdot\|_{gF}$ is the norm on T^1R^4 with respect to the euclidean metric g_F and $g_{ij} = g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)$ for the standard coordinates (x_1, x_2, x_3, x_4) for R^4.

Let $\tilde{A}$ be the connection form uniquely associated to the metric g on $\Delta_{\tilde{r}}^*$, that is, $\tilde{D} = d + \tilde{A}$ is the covariant derivative with respect to g. Clearly, we can regard $\tilde{A}$ as a function in $C^{1,\alpha}(\Delta_{\tilde{r}}^*, so(4) \times R^4)$ for $\alpha \in (0, 1)$.

The following lemma is essentially Theorem 2.8 in [Uh1].

Lemma 4.1. *Let $\tilde{r}$ be sufficiently small. Then there is a gauge transformation u in $C^\infty(\overline{\Delta(r, 2r)}, so(4))$ satisfying: if $D = e^{-u}\cdot\tilde{D}\cdot e^u = d + A$, then $d^*A = 0$ on $\Delta(r, 2r)$, $d_\psi^* A_\psi = 0$ on $\partial\Delta(r, 2r)$ and $\int_{\Delta(r,2r)} A(\nabla_F r(x))dV_g = 0$, where d^*, d_ψ^* are the adjoint operators of the exterior differentials on $\Delta(r, 2r)$ or $\partial\Delta(r, 2r)$ with respect to g, respectively, and ∇_F denotes the standard gradient, dV_g is the volume form of g. Moreover, we have*

$$\sup_{\Delta(r,\,2r)} (\|A\|_g(x)) \leqq \frac{\varepsilon_2(r)}{r} \tag{4.2}$$

where $\varepsilon_2(r)$ is a decreasing function on r with $\lim_{r\to 0}\varepsilon_2(r) = 0$.

Proof. As in [Uh1], the proof follows from an application of Implicit Function Theorem. For reader's convenience, we sketch a proof here.

By scaling, we may take $r = 1$ and $\bar{\Omega} = \overline{\Delta(1, 2)}$ with the scaled metric $\frac{1}{r^2}g$ is sufficiently close to the flat metric g_F on $\bar{\Omega}$ if r is small enough. Let $C^{2,\alpha}(\bar{\Omega}, S^2T_{\bar{\Omega}}^*)$ be the collection of $C^{2,\alpha}$-smooth covariant symmetric tensors on $\bar{\Omega}$. Then for any $h \in C^{2,\alpha}(\bar{\Omega}, S^2T_{\bar{\Omega}}^*)$ sufficiently close to the zero tensor, we have a new metric $g_F + h = g_h$, consequently, it induces a unique $so(4)$-connection A_h on $\bar{\Omega}$. As we have pointed out in the above (4.1), the difference of the scaled metric $\frac{1}{r^2}g$ from g_F is small in $C^{2,\alpha}(\bar{\Omega}, S^2T_{\bar{\Omega}}^*)$ whenever r is small. We define operators

$$Q: C_0^{2,\alpha}(\bar{\Omega}, so(4)) \times C^{2,\alpha}(\bar{\Omega}, S^2T_{\bar{\Omega}}^*) \to C^{0,\alpha}(\bar{\Omega}, so(4)) \times C_0^{0,\alpha}(\partial\bar{\Omega}, so(4))$$
$$(u, h) \to (d_h^*(e^{-u}de^u + e^{-u}A_h e^u), d_{h\psi}^*(e^{-u}d_\psi e^u + e^{-u}A_{h\psi}e^u))$$
$$f: C_0^{2,\alpha}(\bar{\Omega}, so(4)) \times C^{2,\alpha}(\bar{\Omega}, S^2T_{\bar{\Omega}}^*) \to so(4)$$
$$(u, h) \to \int_{\bar{\Omega}} (e^{-u}de^u + e^{-u}A_h e^u)(\nabla r)dV_{g_h}$$

where $0 < \alpha < 1$, the subspace $C_0^{k,\alpha}(\cdot,\cdot)$ consists of all functions orthogonal to constant ones ($k = 0, 2$), and d_h^*, $d_{h\psi}^*$ are the adjoint operators of d, d_ψ with respect to the metric g_h, respectively.

To prove the lemma, it suffices to find a u such that $(Q,f)\left(u, \frac{1}{r^2}g - g_F\right) = 0$. One can easily check that the partial derivative of the operator (Q,f) with respect to u is an isomorphism at the point $(0, 0)$, so the lemma follows from Implicit Function Theorem.

Lemma 4.2. *Let A be the connection for given in Lemma* 4.1, *then for r small, we have*

$$\sup_{\Delta(r,2r)} \|A\|_g(x) \leqq Cr \sup_{\Delta(r,2r)} \|R_A\|_g(x) \tag{4.3}$$

$$\int_{\Delta(r,2r)} \|A\|_g^2(x)dV_g \leqq Cr^2 \int_{\Delta(r,2r)} \|R_A\|_h^2(x)dV_g \tag{4.4}$$

Proof. Both (4.3) and (4.4) are invariant under scaling. So it suffices to prove the lemma on $\Omega = \Delta(1, 2)$ with the scaling metric $\frac{1}{r^2}g$. By Lemma 3.6, the metric $\frac{1}{r^2}g$ converge uniformly to g_F on $\bar{\Omega}$ as r goes to zero. Thus by the proof of Corollary 2.9 in [Uh1], we conclude that

$$\lambda(r) = \inf\left\{ \frac{\int_{\bar{\Omega}} \|df\|^2_{\frac{1}{r^2}g} dV_g}{\int_{\bar{\Omega}} \|f\|^2_{\frac{1}{r^2}g} dV_g} \;\middle|\; f \in C^2(\bar{\Omega}, T^*_{\mathbb{R}^4} \otimes so(4)),\ d^*f = 0,\ d_\psi^* f\Big|_{\partial\Omega} = 0,\ \int_{\bar{\Omega}} f(\nabla r(x))dV_{\frac{1}{r^2}g} = 0 \right\}$$

has a uniform lower bound λ independent of r. By the equation $dA + [A, A] = DA = R_A$, where D is the covariant derivative associated to A, we have

$$\lambda \int_{\Delta(1,2)} \|A\|_g^2 dV_g \leqq \int_{\Delta(1,2)} \|dA\|_g^2 dV_g \leqq 2 \int_{\Delta(1,2)} \|R_A\|_g^2 dV_g + 2 \int_{\Delta(1,2)} \|[A, A]\|_g^2 dV_g$$

By the estimate on $\|A\|_g$ in Lemma 4.1, the last integral is bounded by $C\varepsilon_2(r)\int_{\Delta(1,2)} \|A\|_g^2 dV_g$ for some constant C independent of r. Then (4.4) follows when r is sufficiently small.

On the other hand, by Lemma 4.1 and the equation $dA + [A, A] = R_A$, we have

$$\|dA\|_g(x) \leqq \|R_A\|_g(x) + C\varepsilon_2(r)\|A\|_g(x) \tag{4.5}$$

where C is some uniform constant. Then (4.3) follows easily from (4.4), (4.5) and the fact that $d^*A = 0$ and the scaled metric $\frac{1}{r^2}g$ is close to the flat metric g_F on $\bar{\Omega}$ if r is sufficiently small.

Lemma 4.3. *Given any* $1 > \delta > 0$, *there is a* $r(\delta) > 0$, $r(\delta) < \tilde{r}$ *such that*

$$\|R(g)\|_g(x) \leqq \frac{1}{r(x)^\delta} \quad \text{on} \quad \Delta^*_{r(\delta)} \tag{4.6}$$

Proof. The proof is essentially same as that of Proposition 4.7 in [Uh1]. So we just sketch the proof here. Choose a small $r < \tilde{r}$, put $r_1 = \frac{r}{2}, r_2 = \frac{r_1}{2}, \dots, r_i = \frac{r_{i-1}}{2}, \dots$. Let A_i be the connection on $\Delta(r_i, r_{i-1})$ given by Lemma 4.1. Then $d^*_\psi A_{i\psi}|_{\partial\Delta(r_i, r_{i-1})} = 0$, $d^*_\psi A_{i-1\psi}|_{\partial\Delta(r_i, r_{i-1})} = 0$, it follows that the restrictions $A_{i\psi}$ and $A_{i-1\psi}$ to $\partial\Delta_{r_{i-1}}$ are distinct by a constant gauge on $\partial\Delta_{r_{i-1}}$. So we may assume that $A_{i\psi}|_{\partial\Delta r_{i-1}} = A_{i-1\psi}|_{\partial\Delta r_{i-1}}$. Put $\Omega_i = \Delta(r_i, r_{i-1})$, then we have

$$\begin{aligned}
\int_{\Omega_i} \|R_{A_i}\|^2_g dV_h &= \int_{\Omega_i} \langle dA_i + [A_i, A_i], R_{A_i}\rangle_g dV_g \\
&= \int_{\Omega_i} \langle D_i A_i - [A_i, A_i], R_{A_i}\rangle_g dV_g \\
&= -\int_{\Omega_i} \langle [A_i, A_i], R_{A_i}\rangle_g dV_g - \int_{\Omega_i} \langle A_i, D_i^* R_{A_i}\rangle_g dV_g \\
&\quad - \int_{S_i} \langle A_{i\psi}, (R_{A_i})_{r\psi}\rangle_g d\sigma_g + \int_{S_{i-1}} \langle A_{i\psi}, (R_{A_i})_{r\psi}\rangle_g d\sigma_g
\end{aligned} \tag{4.7$_i$}$$

where $D_i = d + A_i$, $S_i = \partial\Delta_{r_i}$ and $d\sigma$ is the induced volume form on S_i, S_{i-1}, etc. Because the metric g is Einstein and A_i is equivalent to $\tilde{A}$ by gauge transformation, we have $D^* R_{A_i} = 0$. On the other hand, by (4.1) and Lemma 4.1, one can easily show that $\|A_i\|_g(x) \leqq \frac{C\varepsilon_2(r(x))}{r(x)}$, $\|R_{A_i}\|_g(x) \leqq \frac{C\varepsilon_2(r(x))}{r(x)^2}$ for some constant C independent of i and x in Ω_i. Thus by summing equations $(4.7)_i$ over $i \geqq 1$ and observing that $(R_{A_i})_{r\psi} = (E_{A_{i+1}})_{r\psi}$ on S_i, we obtain

$$\begin{aligned}
\int_{\Delta_r} \|R(g)\|^2_g dV_g &= \sum_{i=1}^{\infty} \int_{\Omega_i} \|R(g)\|^2_g dV_g \\
&= \sum_{i=1}^{\infty} \int_{\Omega_i} \|R_{A_i}\|^2_g dV_g \\
&= -\sum_{i=1}^{\infty} \int_{\Omega_i} \langle R_{A_i}, [A_i, A_i]\rangle_g dV_g + \int_{\partial\Delta_r} \langle A_{1\psi}, (R_{A_i}) r\psi\rangle_g dV_g
\end{aligned} \tag{4.8}$$

By Lemma 4.2 and the previous estimate on $\|R_A\|_g(x)$, we have

$$\begin{aligned}
\left\| \int_{\Omega_i} \langle R_{A_i}, [A_i, A_i]\rangle_g dV_g \right\| &\leqq C' \sup_{\Omega_i}(\|R_{A_i}\|_g(x)) \int_{\Omega_i} \|A_i\|^2_g dV_g \\
&\leqq C\varepsilon_2(r_i) \int_{\Omega_i} \|R_{A_i}\|^2_g dV_g
\end{aligned} \tag{4.9}$$

where C, C' are some constants independent of i.

By Lemma 3.6, one can see that $\left(\partial\Delta_r, \frac{1}{r^2}g\right)$ converge uniformly on (S^3, g_F), i.e. the unit sphere with the standard metric. Then by the proof of corollary 2.6 in [Uh1], one can find a decreasing function $\varepsilon'(r)$ on r with $\lim_{r\to 0}\varepsilon'(r) = 0$ such that

$$\int_{\partial\Delta_r} \|A_{1\psi}\|_g^2 dV_g \leqq (2-\varepsilon'(r))^{-2} r^2 \int_{\partial\Delta_r} \|(F_{A_1})_{\psi\psi}\|_g^2 dV_g \tag{4.10}$$

Combining (4.8), (4.9) and (4.10), we have

$$\int_{\Delta_r} \|R(g)\|_g^2 dV_g \leqq (2-\varepsilon'(r))^{-1}(1 + C\varepsilon_2(r))r\left(\int_{\partial\Delta_r} \|(R_{A_1})_{\psi\psi}\|_g^2 dV_g\right)^{1/2}$$

$$\left(\int_{\partial\Delta_r} \|(R_{A_1})_{r\psi}\|_g^2 dV_g\right)^{1/2} \leqq \frac{(2+\varepsilon'(r))^{-1}(1 + C\varepsilon_2(r))r}{2} \int_{\partial\Delta_r} \|R(g)\|_g^2 dV_g$$

$$\leqq \frac{r}{4}\left(1 + \frac{\delta}{2}\right) \int_{\partial\Delta_r} \|R(g)\|_g^2 dV_g.$$

whenever $r(\delta)$ is sufficiently small and $r \leqq r(\delta)$.

Then it is standard to conclude from above inequality that

$$\int_{\Delta_r} \|R(g)\|_g^2 dV_g \leqq r^{4-\frac{\delta}{4}} \quad \text{for } r \leqq r(\delta) \tag{4.11}$$

The estimate (4.6) follows from (4.11) and the fact that g is Einstein and close to the flat metric on $\Delta_{\tilde r}$.

With help of Lemma 4.3, we can now regularize the extension of the metric g across the origin in $\Delta_{\tilde r} \subset C^2$.

Lemma 4.4. *Let g, $\Delta_{\tilde r}$, g_F have the meanings as above. Then if $\tilde r$ is sufficiently small, there is a self-diffeomorphism ψ of $\Delta_{\tilde r}^*$ such that ψ extends to be a homeomorphism of $\Delta_{\tilde r}$* and

$$\|\psi^* g - g_F\|_{gF}(x) \leqq r(x)^{\frac{3}{2}} \quad x \in \Delta_{\tilde r}^* \tag{4.12}$$

$$\|d(\psi^* g)\|_{gF}(x) \leqq r(x)^{\frac{1}{2}} \quad x \in \Delta_{\tilde r}^* \tag{4.13}$$

Proof. We first construct ψ_r with properties analogous to (4.12) and (4.13) in the annulus $\Delta((\frac{3}{2}-\varepsilon)r, (\frac{3}{2}+\varepsilon)r)$ for some small $\varepsilon > 0$ independent of r. By scaling, we may construct ψ on $\Delta(\frac{3}{2}-\varepsilon, \frac{3}{2}+\varepsilon) \subset \Omega = \Delta(1, 2)$ with metric $\frac{1}{r^2}g$, still denoted by g for simplicity. Then by previous lemma, if $r < \tilde r$ is small, we have

$$\sup_{\Omega}(\|R(g)\|_g(x)) \leqq r^{\frac{7}{4}} \tag{4.14}$$

and also $\sup_\Omega\{\|R(g)\|_g(x)\} \leqq \varepsilon'$, where ε' can be very small if we want.

For any point $x \in \partial\Delta_{\frac{3}{2}}$, using the harmonic coordinates constructed in [Jo], one can find a diffeomorphism ψ_x from the euclidean ball $B_{\frac{1}{4}}(x)$ into Ω such that

$\psi_x(B_{\frac{1}{4}}(x))$ is very close to $B_{\frac{1}{4}}(x)$ in Hausdorff distance and

$$\sup\{\|\psi_x^* g - g_F\|_{g^F}(y),\quad \|d\psi_x^* g\|_{g^F}(y) | y \in B_{\frac{1}{4}}(x)\} \leqq Cr^{\frac{7}{4}} \tag{4.15}$$

where C is a constant independent of r and x.

Our ψ_r is obtained by gluing finitely many $\psi_{x_i}(x_i \in \partial\Delta_{\frac{3}{2}})$. We just sketch this gluing process here. Let (y_1, y_2, y_3, y_4) be the euclidean coordinates, then $\partial\Delta_{\frac{3}{2}} = \left\{(y_1, \ldots, y_4)\} | \sqrt{\sum_{i=1}^{4} |y_i|^2} = \frac{3}{2}\right\}$. For any fixed a, b with $a^2 + b^2 < \frac{5}{4}$, we choose finitely many points $x_1, \ldots, x_N$ on the circle $S(a, b) \subset \partial\Delta_{\frac{3}{2}}$ consisting of all points (y_1, y_2, y_3, y_4) with $(y_3, y_4) = (a, b)$ given above, such that $B_{\frac{1}{4}}(x_i) \cap B_{\frac{1}{4}}(x_j) = \emptyset$ if $|i - j| > 1$ and the union of $B_{\frac{1}{4}}(x_i)(1 \leqq i \leqq N)$ contains the neighborhood $B_{\frac{1}{8}}(S(a, b))$ of $S(a, b)$. Note that the number N of $\{x_i\}$ is bounded independent of (y_3, y_4) and r. Now we glue $\psi_{x_1}, \ldots, \psi_{x_N}$ together as in the proof of Lemma 3.6 and obtain a diffeomorphism $\psi_{(a,b)}$ from $B_{\frac{1}{8}}(S(a, b))$ into Ω such that the estimate (4.15) holds for this diffeomorphism on $B_{\frac{1}{8}}(S(a, b))$. Note that the constant C may be different, but still independent of r. Next, fix $b < \frac{5}{4}$, choose $a_1, \ldots, a_{N'}$ such that $a_1 = -\sqrt{\frac{9}{4} - b^2} < a_2 < \ldots < a_{N'} = \sqrt{\frac{9}{4} - b^2}$ and $B_{\frac{1}{4}}(0, 0, a_1, b) \cap B_{\frac{1}{8}}(S(a_2, b)) = \emptyset$ for $j \geqq 3$; $B_{\frac{1}{8}}(S(a_i, b)) \cap B_{\frac{1}{8}}(S(a_j, b)) = \emptyset$ for $|i - j| \geqq 2$; $B_{\frac{1}{8}}(S(a_j, b)) \cap B_{\frac{1}{4}}(0, 0, a_{N'}, b) = \emptyset$ for $j \leqq N - 2$, also the union of them covers the neighborhood $B_{\frac{1}{16}}(S(b))$ of $S(b) = \{(y_1, y_2, y_3, y_4) \in \partial\Delta_{\frac{3}{2}} | y_4 = b\}$. Then we glue $\psi_{(0,0,a_1,b)}$, $\psi_{(0,0,a_N,b)}$ and those $\psi_{(a_i,b)}(2 \leqq i \leqq N - 1)$ construct in step one to obtain a diffeomorphism ψ_b from $B_{\frac{1}{16}}(S(b))$ into Ω such that (4.15) holds for ψ_b on $B_{\frac{1}{16}}(s(b))$. Finally, we choose finitely many points $b_1, \ldots, b_{N''}$ in the interval $[-\frac{3}{2}, \frac{3}{2}]$ and repeat the above gluing process for $B_{\frac{1}{4}}((0, 0, 0, -\frac{3}{2}))$, $B_{\frac{1}{4}}((0, 0, 0, \frac{3}{2}))$ and $B_{\frac{1}{16}}(b_i)$ for $2 \leqq i \leqq N'' - 1$. We then have the diffeomorphism ψ_r from $\Delta(\frac{47}{32}, \frac{49}{32})$ into Ω. By scaling, we may consider ψ_r as a diffeomorphism from $\Delta(\frac{47}{32}r, \frac{49}{32}r)$ into Ω such that the image contains $\Delta((\frac{47}{32} + \tilde{\varepsilon}(r))r, (\frac{49}{32} - \tilde{\varepsilon}(r))r)$ in ω and

$$\sup\left\{\frac{1}{r}\|\psi_r^* g - g_F\|_{g^F}(x), \|d\psi_r^* g\|_{g^F}(x) | x \in \Delta\left(\frac{47}{32}r, \frac{49}{32}r\right)\right\} \leqq Cr^{\frac{3}{4}} \tag{4.16}$$

where $\tilde{\varepsilon}(r)$ is a decreasing function with $\lim_{r\to 0}\tilde{\varepsilon}(r) = 0$.

Now let $r_1 = \tilde{r}$, $r_{i+1} = \frac{24}{25}r_i$ for $i \geqq 1$. We have constructed diffeomorphisms ψ_i from $\Delta(\frac{47}{49}r_i, r_i)$ into Ω with properties described above. Then the required ψ is obtained by gluing these ψ_i properly as in the proof of Lemma 3.6. The estimates (4.12) and (4.13) follows from (4.16) if $\tilde{r}$ is sufficiently small.

Lemma 4.5. *Let g, $\Delta_{\tilde{r}}$ be as in Lemma 4.4. Then there is a diffeomorphism ψ from $\Delta_{\tilde{r}}^*$ into $\Delta_{\tilde{r}}^*$ such that $\psi^* g$ extends to be a C^∞-metric on $\Delta_{\tilde{r}}$. Moreover, if J is the almost complex structure on $\Delta_{\tilde{r}}^*$ such that g is Kähler with respect to J, then*

$(\psi^{-1})_* \circ J \circ \psi_*$ *extends to be an integrable almost complex structure on* $\Delta_{\tilde r}$ *such that* $\psi_* g$ *is Kähler-Einstein with respect to it.*

Proof. By Lemma (4.4), we may assume

$$\|g - g_F\|_{gF}(x) \leqq r(x)^{\frac{3}{2}}, \quad \|dg\|_{gF}(x) \leqq r(x)^{\frac{1}{2}} \quad \text{on } \Delta_{\tilde r}^* \tag{4.17}$$

Then g is a $C^{1,\frac{1}{2}}$-metric on $\Delta_{\tilde r}$. Let $x_1, \ldots, x_4$ be the euclidean coordinate functions on $\Delta_{\tilde r}$ and

$$|\Delta_g x_i|(x) \leqq Cr(x)^{1/2}, \quad x \in \Delta_{\tilde r} \tag{4.18}$$

where Δ_g is the Laplacian of the metric g and C is a constant independent of x, $\tilde r$. Solving Dirichlet problems for k_i,

$$\begin{cases} \Delta_g k_i = -\Delta_g x_i & \text{on } \Delta_{\tilde r} \\ k_i|_{\partial \Delta_{\tilde r}} = \delta_i \end{cases} \tag{4.19}$$

where δ_i are constants of order $O(\tilde r^{\frac{1}{2}})$ such that $k_i(0) = 0$. By the standard elliptic theory [GT], we have $C^{2,\frac{1}{2}}$-solutions k_i of (4.19) such that $\sup_{\Delta_{\tilde r}}(\|dk_i\|_{gF}) = O(\tilde r^{\frac{1}{2}})$. It follows that $(l_1, \ldots, l_4)$, where $l_i = x_i - k_i$, is a harmonic coordinate system in $\Delta_{\tilde r}$ with respect to the metric g when $\tilde r$ is sufficiently small. Let $\psi: \Delta_{\tilde r} \to \Delta_{2\tilde r}$ be the diffeomorphism by mapping $(x_1, \ldots, x_4)$ to $(l_1(x), \ldots, l_4(x))$ and $\{g_{ij}\}$ be the tensor representing the metric $\psi^* g$. Then $\{g_{ij}\}$ are $C^{1,\frac{1}{2}}$-smooth and as in [KT], the Einstein condition on g implies

$$-\frac{1}{2}\sum_{r,s} g^{r,s}\frac{\partial^2 g_{ij}}{\partial l_r \partial l_s} + \text{lower order term} = -g_{ij} \text{ on } \Delta_{\tilde r} \tag{4.20}$$

By elliptic regularity theory [GT], we conclude that $\{g_{ij}\}$ are C^∞-smooth. In fact, $\{g_{ij}\}$ are real analytic. Now it is clear that $\psi_*^{-1} \circ J \circ \psi_*$ is extendable and $\psi^* g$ is a Kähler-Einstein metric with respect to the extended complex structure on $\Delta_{\tilde r}$.

Recall that $\bar M_\infty$ is obtained by adding finitely many points to the limit (M_∞, g_∞) of the sequence of Kähler-Einstein manifolds $\{(M_i, g_i)\}$ in Lemma 3.3. Summarizing the above discussions and using Proposition 3.2, we have actually proved that the compactification $\bar M_\infty$ of (M_∞, g_∞) has the properties: for any added point $x_\infty \in \bar M_\infty \backslash M_\infty$, there is a neighbourhood U of x_∞ in $\bar M_\infty$ such that any connected component U_j of $U \cup M_\infty$ is covered by a punctured ball $\Delta_{\tilde r}^*$ in C^2 with the covering group isomorphic to a finite group in $U(2)$. Moreover, if $\pi_j: \Delta_{\tilde r}^* \to U_j$ is the covering map, then $\pi_j^* g_\infty$ extends to be a Kähler-Einstein metric on $\Delta_{\tilde r}$ in C^2 with respect to the standard complex structure. Therefore, we have

Proposition 4.2. *Let* $\{(M_i, g_i)\}$ *be the sequence of compact Kähler-Einstein manifolds given at beginning of this section. Then by taking a subsequence if necessary, we may assume that* (M_i, g_i) *converge to a Kähler-Einstein manifold* $(M_\infty \backslash \mathrm{Sing}(M_\infty), g_\infty)$ *in the sense of Lemma* 3.3, *where* (M_∞, g_∞) *is a connected Kähler-Einstein orbifold (maybe reducible) and* $\mathrm{Sing}(M_\infty)$ *is the finite set of singular points of* M_∞.

Here a complex orbifold is defined in the general sense as described before this proposition. Moreover, from now on, we will say that the sequence of Kähler-Einstein manifolds converge to a Kähler-Einstein orbifold (M_∞, g_∞) if the conclusion in the above proposition is true for $\{(M_i, g_i)\}$.

5. Application of L^2-estimate for $\bar{\partial}$-operators on Kähler-Einstein orbifolds

Let $\{(M_i, g_i)\}_{i \geq 1}$ be a sequence of Kähler-Einstein surfaces in $\mathfrak{J}_n$ with $\mathrm{Ric}(g_i) = \omega_{g_i}$. In previous two sections, it is proved that some subsequence of $\{(M_i, g_i)\}_{i \geq 1}$ converge to a Kähler-Einstein orbifold (M_∞, g_∞) in the sense of Proposition 4.2. In this section, we will apply L^2-estimate for $\bar{\partial}$-operators to studying the properties of this limiting orbifold M_∞. We will prove that the plurianticanonical group $H^0(M_i, K_{M_i}^{-m})$ converge to $H^0(M_\infty, K_{M_\infty}^{-m})$ for any integer $m > 0$ as (M_i, g_i) converge to (M_∞, g_∞) and M_∞ is an irreducible normal surface with only rational double points and some special Hirzebruch-Jung singularities as singular points (cf. [BPV]).

We first recall the definition of a line bundle on M_∞ (cf. [Bai]).

Definition 5.1. A line bundle on the complex orbifold M_∞ is a line bundle L on the regular part $M_\infty \backslash \mathrm{Sing}(M_\infty)$ such that for each local uniformization $\pi_p: \tilde{U}_p \to M_\infty$ of a singular point p, the pull-back $\pi_p^* L$ on $\tilde{U}_p \backslash \pi_p^{-1}(p)$ can be extended to the whole $\tilde{U}_p$.

On the Kähler-Einstein orbifold M_∞, we have natural line bundles in the sence of Definition 5.1 such as pluricanonical line bundles $K_{M_\infty}^m$ and plurianticanonical line bundles $K_{M_\infty}^{-m} (m \in Z_+)$. A global section of $K_{M_\infty}^{-m}$ is an element in $H^0(M_\infty \backslash \mathrm{Sing}(M_\infty), K_{M_\infty}^{-m})$. Let $H^0(M_\infty, K_{M_\infty}^{-m})$ denote the linear space of all these global sections of $K_{M_\infty}^{-m}$. Note that the metric g_∞ induces natural hermitian orbifold metrics h_∞^m on $K_{M_\infty}^{-1}$.

Lemma 5.1. *Let $\{(M_i, g_i)\}$ be the sequence of Kähler-Einstein surfaces given at the beginning of this section and S^i be a global holomorphic section in $H^0(M_i, K_{M_i}^{-m})$ with $\int_{M_i} \|S^i\|_{g_i}^2 dV_{g_i} = 1$, where m is a fixed positive integer. Then there is a subsequence $\{i_k\}$ of $\{i\}$ such that the sections S^{i_k} converge to a global holomorphic section S^∞ in $H^0(M_\infty, K_{M_\infty}^{-m})$. In particular, if $\{S_\beta^i\}_{0 \leq \beta \leq N_m}$ is an orthonormal basis of $H^0(M_i, K_{M_i}^{-m})$ with respect to the induced inner product by g_i, then by taking a subsequence, we may assume that $\{S_\beta^i\}_{0 \leq \beta \leq N_m}$ converge to an orthonormal basis of a subspace in $H^0(M_\infty, K_{M_\infty}^{-m})$, where $N_m + 1 = \dim_C H^0(M_i, K_{M_i}^{-m})$.*

Remark. Before we prove this lemma, we should justify the meaning of the convergence of $\{S^i\}$ in the above lemma since these sections are no longer on a same Kähler manifold. Recall that Lemma 3.3 says: for any compact subset $K \subset M_\infty \backslash \mathrm{Sing}(M_\infty)$, there are diffeomorphisms ϕ_i from compact subsets $K_i \subset M_i$ onto K such that $(\phi_i^{-1})^* g_i$ and $\phi_{i*} \circ J_i \circ (\phi_i^{-1})_*$ converge to g_∞ and J_∞ on K, respectively. Now with ϕ_i as above, we can push the sections S^i down to the sections $\phi_{i*}(S^i)$ of $\otimes^m(\Lambda^2(T_c M \otimes \overline{T_c M}))$ on K. The convergence in Lemma 5.1 means that for any compact subset K of $M_\infty \backslash \mathrm{Sing}(M_\infty)$ and ϕ_i as above, the sections $\phi_{i_k *}(S^{i_k})$ converge to a section S^∞ of $K_{M_\infty}^{-1}$ on K in C^∞-topology. Note that the limit S^∞ is automatically holomorphic.

Proof of Lemma 5.1. Let Δ_i be the laplacian of the metric g_i, then by a direct computation, we have

$$\Delta_i(\|S^i\|_{g_i}^2)(x) = \|D_i S^i\|_{g_i}^2(x) - 2m \|S^i\|_{g_i}^2(x) \tag{5.1}$$

where D_i is the covariant derivative with respct to g_i. Since $\int_{M_i} \|S^i\|^2_{g_i}(x)dV_{g_i} = 1$, by Lemma 3.1 and applying Moser's iteration to (5.1), there is a constant $C(m)$ depending only on m such that

$$\sup_{M_i}(\|S^i\|^2_{g_i}(x)) \leqq C(m) \tag{5.2}$$

Let K be a compact subset in $M_\infty \backslash \mathrm{Sing}(M_\infty)$ and ϕ_i be diffeomorphism from K_i onto K as in the above remark. To prove the lemma, it suffices to show (*): for any integer $l > 0$, the l^{th} convariant derivatives of $\phi_{i*}(S^i)$ with respect to g_∞ are bounded in K by a constant C_l' depending only on l and K. There is a $r > 0$, depending only on K, such that for any point x in K_i, the geodesic ball $B_r(x, g_i)$ is uniformly biholomorphic to an open subset in C^2. On each $B_r(x, g_i)$, the section S_i is represented by a holomorphic function $f_{i,x}$. By (5.1), the function $f_{i,x}$ are uniformly bounded. Therefore, by the well-known Cauchy integral formula, one can easily prove that at x the l^{th} covariant derivative of S^i are uniformly bounded by a constant depending only on l, K. It follows (*) since $(\phi_i^{-1})^* g_i$ uniformly converge to g_∞ in K. The lemma is proved.

Remark. One can easily prove the existence of hermitian orbifold metrics on a line bundle as above by unit partition. The following proposition can be easily proved by modifying the proof of ([Ho] p. 92, Theorem 4.4.1) with the use of the Bochner-Kodaira Laplacian formula (see e.g. [KM]).

Proposition 5.1. *Suppose that (X, g) be a complete Kähler orbifold of complex dimension n, L be a line bundle on X with the hermitian orbifold metric h, and ψ be a function on X, which can be approximated by a decreasing sequence of smooth function $\{\psi_l\}_{1 \leqq l < +\infty}$. If, for any tangent vector v of type* (1, 0) *at any point of X and for each l,*

$$\left\langle \partial\bar\partial\psi_l + \frac{2\pi}{\sqrt{-1}}(\mathrm{Ric}(h) + \mathrm{Ric}(g)), v \wedge \bar v \right\rangle_g \geqq C\|v\|_g^2 \tag{5.3}$$

where C is a constant independent of l and $\langle , \rangle_g$ is the inner product induced by g. Then for any C^∞ L-valued (0, 1)*-form w on X with $\bar\partial w = 0$ and $\int_X \|w\|^2 e^{-\psi} dV_g$ finite, there exists a C^∞ L-valued function u on X such that $\bar\partial u = w$ and*

$$\int_X \|u\|^2 e^{-\psi} dV_g \leqq \frac{1}{C}\int_X \|w\|^2 e^{-\psi} dV_g \tag{5.4}$$

where $\|\cdot\|$ is the norm induced by h and g.

Lemma 5.2. *Any section S in $H^0(M_\infty, K_{M_\infty}^{-m})$ is the limit of some sequence $\{S^i\}$ with S^i in $H^0(M_i, K_{M_i}^{-m})$. In particular, it implies that the dimension of $H^0(M_\infty, K_{M_\infty}^{-m})$ is same as that of $H^0(M_i, K_{M_i}^{-m})$, that is, plurianticanonical dimensions are invariant under the degeneration of Kähler-Einstein manifolds with positive scalar curvature.*

Proof. We may assume that $\int_{M_\infty} \|S\|^2_{g_\infty}(x)dV_{g_\infty} = 1$. Let $\{r_i\}$ be a sequence of positive numbers with $\lim_{i\to\infty} r_i = 0$ such that for each i, there is a diffeomorphism ϕ_i from $M_i \backslash \bigcup_{\beta=1}^N B_{r_i}(x_{i\beta}, g_i)$ into $M_\infty \backslash \mathrm{Sing}(M_\infty)$ as given in Lemma 3.3, where

N is defined in Lemma 3.2 and $(x_{i\beta})$ are defined in (3.14). Then ϕ_i satisfy (1) $\lim_{i\to\infty}(\mathrm{Im}(\phi_i))$ is just $M_\infty \setminus \mathrm{Sing}(M_\infty)$; (2) $(\phi_i^{-1})^* g_i$ uniformly converge to g_∞ on any compact subset of $M_\infty \setminus \mathrm{Sing}(M_\infty)$ in C^∞-topology; (3) $\phi_{i*} \circ J_i \circ (\phi_i^{-1})_*$ converge to J_∞, where J_i, J_∞ are almost complex structures on M_i, M_∞, respectively. Define a cut-off function η: $R^1 \to R^1_+$ satisfying: $\eta(t) = 0$ for $t \leqq 1$; $\eta(t) = 1$ for $t \geqq 2$ and $|\eta'| \leqq 1$. Also let π_i be the natural projection from the bundle $\otimes^m(\Lambda^2(TM_i \otimes \overline{TM_i})$ onto $K_{M_i}^{-m} = \otimes^m(\Lambda^2 TM_i)$. For each i, we have a smooth section $v_i = \eta\left(\frac{\rho_i(x)}{2r_i}\right) \cdot \pi_i((\phi_i^{-1})_* S)$ of $K_{M_i}^{-m}$ on M_i, where $\rho_i(x)$ is a Lipschitz function defined by $\rho_i(x) = \min_{1 \leqq \beta \leqq N}\{\mathrm{dist}_{g_i}(x, x_{i\beta})\}$. Then by the facts (2) and (3) above, there is a decreasing function $\varepsilon_3(r)$ on r with $\lim_{r\to 0}\varepsilon_3(r) = 0$ such that

$$\sup\{\|\bar\partial_i \pi_i((\phi_i^{-1})_* S)\|_{g_i}(x) \mid x \in M_i \setminus \bigcup_{\beta=1}^{N} B_{2r_i}(x_{i\beta}, g_i)\} \leqq \varepsilon_3(r_i) \tag{5.5}$$

$$\left|\int_{M_i} \|v_i\|_{g_i}^2(x) dV_{g_i} - 1\right| \leqq \varepsilon_3(r_i) \tag{5.6}$$

where $\bar\partial_i$ is the corresponding $\bar\partial$-operator on M_i.

By (5.5), we have

$$\begin{aligned} &\int_{M_i} \|\bar\partial_i v_i\|_{g_i}^2(x) dV_{g_i} \\ &\leqq \varepsilon_3(r_i)\mathrm{Vol}_{g_i}(M_i) + \sum_{\beta=1}^{N} \int_{B_{4r_i}(x_{i\beta}, g_i)} \left\|\bar\partial_i\left(\eta\left(\frac{\rho_i}{2r_i}\right)\right)\cdot \pi_i((\phi_i^{-1})_* S)\right\|_{g_i}^2 (x) dV_{g_i} \\ &\leqq \sum_{\beta=1}^{N} \frac{1}{4r_i^2} \mathrm{Vol}(B_{4r_i}(x_{i\beta}, g_i)) \cdot \sup\{\|(\phi_i^{-1})_* S\|_{g_i}^2(x) \mid x \in M_i \\ &\setminus \bigcup_{\beta=1}^{N} B_{2r_i}(x_{i\beta}, g_i)\} + \varepsilon_3(r_i)\mathrm{Vol}_{g_i}(M_i) \end{aligned} \tag{5.7}$$

As in the proof of Lemma 5.1, one may bound $\sup_{M_\infty}(\|S\|_{g_\infty}^2(x))$ by the constant $C(m)$ in (5.2). Thus by (5.7), Volume Comparison Theorem and the convergence of $(\phi_i^{-1})^* g_i$ in the above fact (2), there is a constant C independent of i such that

$$\int_{M_i} \|\bar\partial_i v_i\|_{g_i}^2(x) dV_{g_i} \leqq C(r_i^2 + \varepsilon_3(r_i)) \,. \tag{5.8}$$

Now applying Proposition 5.1 i.e., L^2-estimate of $\bar\partial$-operators, we have a C^∞-smooth $K_{M_i}^{-m}$-valued function u_i such that

$$\begin{cases} \bar\partial u_i = \bar\partial v_i, & (5.9) \\ \int_{M_i} \|u_i\|_{g_i}^2(x) dV_{gi} \leqq \frac{1}{m+1}\int_{M_i} \|\bar\partial_i v_i\|_{g_i}^2(x) dV_{gi} \leqq \frac{C}{m+1}(r_i^2 + \varepsilon(r_i)) & (5.10) \end{cases}$$

By (5.9), the norm function $\|u_i\|_{g_i}^2$ for each i satisfies an elliptic equation

$$\Delta_i(\|u_i\|_{g_i}^2(x)) = \|D_i u_i\|_{g_i}^2(x) - 2m\|u_i\|_{g_i}^2(x) + 2\mathrm{Re}(h_i^m(u_i, \bar\partial_i^* \bar\partial_i v_i))(x) \tag{5.11}$$

where $\bar{\partial}_i^*$ is the adjoint operator of $\bar{\partial}_i$ on $K_{M_i}^{-m}$-valued function with respect to g_i. As in (5.5), we also have

$$\sup\{\|\bar{\partial}_i^*\bar{\partial}_i v_i\|_{g_i}^2(x) | x \in M_i \setminus B_{4r_i}(x_{i\beta}, g_i)\} \to 0 \quad \text{as} \quad i \to \infty \tag{5.12}$$

Combining (5.9) and (5.10), we see that u_i converge uniformly to zero in the sense of the remark after Lemma 5.1 as i goes to infinity. Put

$$S^i(x) = \frac{(v_i(x) - u_i(x))}{\left(\int_{M_i} \|v_i - u_i\|_{g_i}^2(x) dV_{g_i}\right)^{\frac{1}{2}}} \tag{5.13}$$

Then $\{S^i\}$ is the required sequence.

Lemma 5.3. *Let $\{(M_i, g_i)\}$ and (M_∞, g_∞) be given in Proposition 4.2. For each integer $m > 0$, we have othonormal bases $\{S_{m\beta}^i\}_{0 \leqq \beta \leqq N_m}$ (resp. $\{S_{m\beta}^\infty\}$) of $H^0(M_i, K_{M_i}^{-m})$ (resp. $H^0(M_\infty, K_{M_\infty}^{-m})$). Then*

$$\lim_{i \to \infty}\left(\inf_{M_i}\left\{\sum_{\beta=0}^{N_m} \|S_{m\beta}^i\|_{g_i}^2(x)\right\}\right) \geqq \inf_{M_\infty}\left\{\sum_{\beta=0}^{N_m} \|S_{m\beta}^\infty\|_{g_\infty}^2(x)\right\}. \tag{5.14}$$

Proof. By direct computations, we have

$$\Delta_i(\|D_i S_{m\beta}^i\|_{g_i}^2)(x) = \|D_i D_i S_{m\beta}^i\|_{g_i}^2(x) - (4m-1)\|D_i S_{m\beta}^i\|_{g_i}^2(x) \tag{5.15}$$

where Δ_i (resp. D_i) is laplacian (resp. covariant derivative) with respect to g_i. Then by (5.1), Lemma 2.1 and standard Moser's iteration, there is a constant $C'(m)$ depending only on m such that

$$\sup\{\|D_i S_{m\beta}^i\|_{g_i}^2(x) | 0 \leqq \beta \leqq N_m,\ x \in M_i\} \leqq C'(m). \tag{5.16}$$

Combining it with (5.2), we conclude that the first derivatives of $\sum_{\beta=0}^{N_m} \|S_{m\beta}^i\|_{g_i}^2(x)$ are uniformly bounded independent of i. Then (5.14) follows from this and Lemma 5.1, 5.2.

As a corollary of this lemma, we have the following weak partial C^0-estimates of the solution of $(1.1)_t$.

Theorem 5.1. *There are a universal integer $m_0 > 0$ and a universal constant $C > 0$ such that for any Kähler-Einstein surface (M', g') in $\mathfrak{J}_n (5 \leqq n \leqq 8)$, we have*

$$\inf_{M'}\left\{\sum_{\beta=0}^{N_m} \|S'_\beta\|_{g'}^2\right\} \geqq C > 0 \tag{5.17}$$

where $N_m + 1$ is the complex dimension of $H^0(M', K_{M'}^{-m_0})$ and $\{S'_\beta\}_{0 \leqq \beta \leqq N}$ is an orthonormal basis of $H^0(M', K_{M'}^{-m_0})$ with respect to the inner product induced by g'.

Proof. It suffices to prove that for any sequence of Kähler-Einstein surface $\{(M_i, g_i)\}$ converging to a Kähler-Einstein orbifold (M_∞, g_∞) in the sense of Proposition 4.2, there are $m_0 > 0$ and $C > 0$ such that (5.17) holds for these (M_i, g_i). By Lemma 5.3, it is sufficient to find a large m such that

$$\inf\left\{\sum_{\gamma=0}^{N_m} \|S_{m\gamma}^\infty\|^2(x) | x \in M_\infty\right\} > 0 \tag{5.18}$$

where $\{S^{\infty}_{m\gamma}\}$, N_m are given as in Lemma 5.3. It is equivalent to that for any point x in M_{∞}, there is a holomorphic global section S in $H^0(M_{\infty}, K^{-m}_{M_{\infty}})$ such that $S(x) \neq 0$. The latter can be achieved by the application of L^2-estimate (Proposition 5.1) as follows. Let $x_{\infty 1}, \ldots, x_{\infty N}$ be the singular points of M_{∞}. There is a small positive number r independent of β such that for any $x_{\infty\beta}$ in M_{∞}, the closure of each connected component in $B_r(x_{\infty\beta}, g_{\infty})\backslash\{x_{\infty\beta}\}$ is locally uniformized by a neighborhood $\tilde{U}_{\beta j}(1 \leqq j \leqq l_{\beta})$ of the origin o in C^2 with finite uniformization group Γ_{β}. Let $\pi_{\beta j}: \tilde{U}_{\beta j} \to B_r(x_{\infty\beta}, g_{\infty})$ be the natural projection with $\pi_{\beta j}(o) = x_{\infty\beta}$, and $q = \prod_{1\leqq\beta\leqq N}(\prod_{1\leqq j\leqq R_{\beta}} q_{\beta j})$, where $q_{\beta j}$ is the order of the finite group $\Gamma_{\beta j}$. Let $m = pq$. We will choose p later. We may take r to be sufficiently small such that the function $\rho\beta = \mathrm{dist}(\cdot, x_{\infty i})^2$ is smooth on $B_r(x_{\infty\beta}, g_{\infty})\backslash\{x_{\infty\beta}\}$ for any β. Now fix a $x_{\infty\beta}$ and $\tilde{U}_{\beta j}$.

Let (z_1, z_2) be a coordinate system on $\tilde{U}_{\beta j}$, define an n-anticanonical section v by

$$v(y) = \sum_{\sigma\in\Gamma_{\beta j}} \sigma^*\left(\left(\frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial z_2}\right)^q\right)(y), \quad y \in \tilde{U}_{\beta j}\,.$$

By the definition of q, we have $v(o) \neq 0$. Let $\eta: R^1 \to R^1_+$ be a cut-off function such that $\eta(t) = 1$ for $t \leqq 1$; $\eta(t) = 0$ for $t \geqq 2$. $|\eta'(t)| \leqq 1$. Then $w = \eta\left(\frac{4\rho\beta}{r^2}\right)(\pi_{\beta j})_*(v^p)$ is a C^{∞}-global section of the line bundle $K^{-m}_{M_{\infty}}$. Choose a large p depending only on r such that for tangent vector v of type $(1, 0)$,

$$\left\langle \partial\bar{\partial}\left(8\eta\left(\frac{4\rho\beta}{r^2}\right)\log\left(\frac{\rho\beta}{r^2}\right)\right) + \frac{2\pi p}{\sqrt{-1}}\omega_{g\infty}, v \wedge \bar{v}\right\rangle_{g\infty} \geqq \|v\|^2_{g\infty}\,. \tag{5.19}$$

Applying Proposition 5.1, we obtain a C^{∞} smooth $K^{-m}_{M_{\infty}}$-valued function u satisfying $\bar{\partial}u = \bar{\partial}w$ and

$$\int_{M_{\infty}} \|u\|^2_{g\infty} e^{-8\eta\log\left(\frac{\rho\beta}{r^2}\right)} dV_{g_{\infty}} \leqq \int \|\bar{\partial}w\|^2_{g\infty} e^{-8\eta\log\left(\frac{\rho\beta}{r^2}\right)} dV_{g_{\infty}} < +\infty\,.$$

It follows that the pull-back $\pi^*_{\beta j}u$ of u vanishes up to order 3 at the origin in $\tilde{U}_{\beta j} \subset C^2$. Put

$$S_{\beta j} = \frac{w - u}{\left(\int_{M_{\infty}} \|w - u\|^2_{g\infty} dV_{g_{\infty}}\right)^{\frac{1}{2}}} \tag{5.20}$$

then $S_{\beta j} \in H^0(M_{\infty}, K^m_{M_{\infty}})$ and $\inf_{\tilde{U}_{\beta j}}\{\pi^*_{\beta j}\|S_{\beta j}\|_{g\infty}(x)\} > 0$. By the same arguments as in the proof of Lemma 5.3, one can bound the first derivatives of these $S_{\beta j}$ by a uniform constant. So if r is taken sufficiently small, we have

$$\begin{aligned}
&\inf\left\{\sum_{\gamma=0}^{N_m} \|S^{\infty}_{m\gamma}\|^2_{g\infty}(x) \,|\, x \in B_r(x_{\infty\beta}, g_{\infty}),\ 1 \leqq \beta \leqq N_m\right\}\\
&\geqq \inf\{\|S_{\beta j}\|^2_{g\infty}(x) \,|\, x \in \pi_{\beta j}(\tilde{U}_{\beta j}),\ 1 \leqq \beta \leqq N_m,\ 1 \leqq j \leqq l_{\beta}\}\\
&> 0.
\end{aligned}$$

For any point x in $M_\infty \setminus \bigcup_{\beta=1}^{N} B_r(x_{\infty\beta}, g_\infty)$, define $\rho_x = \mathrm{dist}(\cdot, x)^2$. As above, applying Proposition 5.1 to $K_{M_\infty}^{-m}$-valued $\bar\partial$-equation with the weight function $8\eta\left(\frac{4\rho_x}{r^2}\right)\log\left(\frac{\rho_x}{r^2}\right)$, one can easily construct a holomorphic section S_x in $H^0(M_\infty, K_{M_\infty}^{-m})$ such that $S_x(x) \neq 0$. Thus the inequality (5.18) is proved. So is Theorem 5.1.

Proposition 5.2. *The Kähler-Einstein orbifold (M_∞, g_∞) in Proposition 4.2 is locally irreducible, that is, for any $r > 0$ and any point x in M_∞, the punctured ball $B_r(x, g_\infty)\setminus\{x\}$ is connected. In particular, the orbifold M_∞ is irreducible.*

Proof. It suffices to check the local irreducibility at a singular point x, say $x = x_{\infty 1}$, in M_∞. Suppose that M_∞ is not locally irreducible at $x = x_{\infty 1}$, then we have open subsets $\tilde U_{11}, \ldots, \tilde U_{1l_1}$ $(l_1 \geqq 2)$ uniforming the closures of the connected components in $B_r(x, g_\infty)\setminus\{x\}$. In the above proof of Theorem 5.1, we construct a S in $H^0(M_\infty, K_{M_\infty}^{m})$ for $m = m_0$ such that it can be decomposed into $v + u$ in $B_{\frac{r}{2}}(x, g_\infty)\setminus\{x\}$. Both v, u are holomorphic in $B_{\frac{r}{2}}(x_\infty, g_\infty)\setminus\{x_{\infty 1}\}$ and satisfy (1) $v = 0$ on $\pi_{1j}(\tilde U_{1j})(j \geqq 2)$, and $\inf\{\|v(y)\| \,|\, y \in \pi_{11}(\tilde U_{11})) \geqq c' > 0$; (2) $\|u\|_{g\infty}(y) \leqq C\,\mathrm{dist}(y, x_1)^4$ for a constant C. In particular, it implies that for any sufficient small $r' > 0$, we have a uniform lower bound $\frac{c'}{2}$ of the oscillation,

$$\omega_{r'}(S, g_\infty) = \sup\{|\|S\|_{g_\infty}(y) - \|S\|_{g_\infty}(z)| \mid y, z \in \partial B_{r'}(x_1, g_\infty)\} \geqq \frac{c'}{2}$$

By Lemma 5.2, there is a sequence $\{S^i\}$ with S^i in $H^0(M_i, K_{M_i}^{-m})$ such that S^i converge to S as M_i converge to M_∞. Then for any fixed small r',

$$\lim_{i\to\infty} \omega_{r'}(S^i, g_i) = \omega_{r'}(S, g_\infty) \geqq \frac{c'}{2}\,. \tag{5.21}$$

On the other hand, in the proof of Lemma 5.3, we proved that there is a constant $C(m)$ depending only on m, such that

$$\sup_{M_i}(\| \nabla_i(\|S^i\|_{g_i})\|_{g_i}) \leqq C(m)\,.$$

It follows that $\omega_{r'}(S', g_i) \leqq 2C(m)r'$. It certainly contradicts to (5.21) when r' is sufficiently small. So M_∞ is locally irreducible at $x_{\infty 1}$, similarly at $x_{\infty j}$ $(2 \leqq j \leqq l_1)$. The proposition is proved.

Remark. Let $x_{i\beta} \in M_i (1 \leqq \beta \leqq N)$ be given in (3.14), and $\tilde B_r(x_{i\beta}, g_i)$ be the universal covering of the ball $B_r(x_{i\beta}, g_i)$ in M_i. Then, as Proposition 4.2, for any fixed β, these $\tilde B_r(x_{i\beta}; g_i)$ converge to an open Kähler-Einstein orbifold $\tilde B_\infty(r)$. The above arguments also prove that this $\tilde B_\infty(r)$ is locally irreducible, in particular irreducible. As a consequence, it implies that $\partial\tilde B_r(x_{i\beta}, g_i)$ is connected if r is small and the fundamental group $\pi_1(B_r(x_{i\beta}, g_i))$ is a quotient group of $\pi_1(\partial B_r(x_{i\beta}, g_i))$ by some normal subgroups.

Now we come to study the singularities $\{x_{\infty\beta}\}_{1\leqq\beta\leqq N}$ of the Kähler-Einstein orbifold (M_∞, g_∞) in Proposition 4.2, where N is given as in Lemma 3.2. Precisely, we want to make reduction of those local uniformization groups Γ_β for $\{x_{\infty\beta}\}_{1\leqq\beta\leqq N}$.

Choose a small $\tilde{r}>0$, such that for any β between 1 and N, the ball $B_{\tilde{r}}(x_{\infty\beta}, g_\infty)$ is geodesically convex and is locally uniformized by an open subset $\tilde{U}_\beta$ in C^2 with the local uniformization group Γ_β. We identify Γ_β with the induced action of Γ_β on $T_o\tilde{U}_\beta$, where o is the preimage of $x_{\infty\beta}$ under the natural projection $\pi_\beta: \tilde{U}_\beta \to B_{\tilde{r}}(x_{\infty\beta}, g_\infty)$. Then Γ_β can be considered as a finite subgroup in $U(2)$. Let $\{r_i\}$ be a decreasing sequence with $\lim_{i\to\infty} r_i = 0$, they by Proposition 4.2, there are diffeomorphisms ϕ_i from $M_i/\bigcup_{\beta=1}^N B_{2r_i}(x_{\infty\beta}, g_\infty)$ and $(\phi_i^{-1})^* g_i$ converge pointwisely to g_∞ in C^∞-topology. We may assume that all $r_i<\frac{\tilde{r}}{2}$.

Fix a singular point $x=x_{\infty\beta}$, say $\beta=1$ for simplicity, we define $S_r(i)$ to be $\phi_i^{-1}(\partial B_r(x, g_\infty))$ for $2r_i\leqq r\leqq\tilde{r}$. Then each $S_r(i)$ is isomorphic to a generalized lens space S^3/Γ with $\Gamma=\Gamma_1$ and covers a domain $B_r(i)$ such that $B_r(i)$ converge to $B_r(x, g_\infty)$ as i goes to infinity in the sense of Proposition 4.2. By taking $\tilde{r}$ smaller, we may assume that each $B_r(i)$ is geodesically convex. As mentioned in the above remark, by the same proof as that for Theorem 5.1, one can prove that for any $r>0$ and i, the fundamental group $\pi_1(B_r(i))$ is the quotient group of Γ_1 by its normal subgroup. In particular, the group $\pi_1(B_r(i))$ is a finite group with its order uniformly bounded. We may assume that the orders of these $\pi_1(B_r(i))$ are all same.

Lemma 5.4. *Let $pr_i: \tilde{B}_{\tilde{r}}(i) \to B_{\tilde{r}}(i)$ be the universal covering and $\tilde{g}_i = pr_i^* g_i$. Then $(\tilde{B}_{\tilde{r}}(i), \tilde{g}_i)$ converge to an open Kähler-Einstein orbifold $(\tilde{B}_{\tilde{r}}(\infty), \tilde{g})$ in the sense of Proposition 4.2. Moreover, there is a natural projection $p_{r_\infty}: \tilde{B}_{\tilde{r}}(g_\infty) \to B_{\tilde{r}}(x, \infty)$ of order $\#\pi_1(B_{\tilde{r}}(i))$ and $\tilde{g}_\infty = p_{r_\infty}^* g_\infty$. and $\tilde{B}_{\tilde{r}}(\infty)$ has a rational double point as the only singular point.*

Proof. The convergence of $\{(\tilde{B}_{\tilde{r}}(i), \tilde{g}_i)\}$ follows from Proposition 4.2 and the definitions of $\tilde{B}_{\tilde{r}}(i)$ and $\tilde{B}_{\tilde{r}}(i)$. It is clear that $\tilde{B}_{\tilde{r}}(\infty)$ has only one singular point. So it suffices to prove that this singular point is a rational double point. By Theorem 5.1, we have holomorphic sections S^i in $H^0(M_i, K_{M_i}^{-m_0})$ such that when $r>0$ is sufficiently small, there is a positive number $c>0$,

$$0<c\leqq\|S^i\|_{g_i}^2(x)\leqq 1 \quad \text{for } x\in B_r(i) \tag{5.21}$$

Then each $pr_i^* S^i$ is a holomorphic m_0-anticanonical section on $\tilde{B}_{\tilde{r}}(i)$. Since the preimage $(p_{r_i})^{-1}(B_r(i))$ is simply-connected, the m_0-root of $p_{r_i}^* S^i$ exists as a holomorphic anticanonical section on $((pr_i)^{-1}(B_r(i))$, denoted by $\tilde{S}^i$. Then

$$0<c^{\frac{1}{m_0}}\leqq\|\tilde{S}^i\|_{\tilde{g}_i}^2(x)\leqq 1 \quad \text{for } x\in(p_{r_i})^{-1}(B_r(i)) \tag{5.22}$$

By (5.22), we may assume that $\tilde{S}^i$ converge to a nonvanishing holomorphic anticanonical section $\tilde{S}$ in $(p_{r_\infty})^{-1}(B_r(x, g_\infty))$. The local uniformization $\tilde{U}_1$ of x in M_∞ is also the one for $(p_{r_\infty})^{-1}(x)$ in $\tilde{B}_{\tilde{r}}(\infty)$, and we have the following commutative diagram,

$$\begin{array}{ccc} \tilde{U}_1 & \xrightarrow{\tilde{\pi}_1} & \tilde{B}_{\tilde{r}}(\infty) \\ & {}_{\pi_1}\searrow \quad \swarrow_{p_{r_\infty}} & \\ & B_{\tilde{r}}(x, g_\infty) & \end{array}$$

Let Γ' be the local uniformization group of $(p_{r_\infty})^{-1}(x)$ in $\tilde{B}_{\bar{r}}(\infty)$. Then $\tilde{\pi}_1^* \tilde{S}$ is a Γ'-invariant holomorphic anticanonical section on $\pi_1^{-1}(B_r(x, g_\infty)) \subset U_1$. Since $\tilde{\pi}_1^* \tilde{S} \neq 0$, the induced action of Γ' on $\Lambda^2(T_o \tilde{U}_1)$ is trivial. This means that Γ' is a finite group in $SU(2)$ if we identify Γ' with its induced group on $T_o \tilde{U}_1 \cong C^2$. So the singular point $p_{r_\infty}^{-1}(x)$ is a rational double point. The lemma is proved.

Lemma 5.5. *The induced group of Γ on $T_o \tilde{U}_1 \cong C^2$ is either a finite subgroup in $SU(2)$ or one of the following cyclic group $Z_{l,p,q}$ defined as follows, where p, q are coprime, let (z_1, z_2) be the euclidean coordinates in C^2, define*

$$\sigma_{l,p,q}: C^2 \to C^2,$$

$$\sigma_{l,p,q}((z_1, z_2)) = \left(e^{\frac{2\pi\sqrt{-1}}{lp^2}} z_1, e^{-\frac{2\pi\sqrt{-1}}{lp^2} + \frac{2\pi q\sqrt{-1}}{p}} z_2\right)$$

then $Z_{l,p,q}$ is generated by $\sigma_{l,p,q}$.

Proof. Let ϕ_i be the diffeomorphisms given before Lemma 5.4. There is a decreasing sequence $\{\varepsilon_i\}_{i \geqq 1}$ with $\lim_{i\to\infty} \varepsilon_i = 0$ such that

$$\sup_{\substack{1 \leqq k \leqq 5 \\ y \in M_\infty \setminus \bigcup_{\beta=1}^N B_{r_i}(x_{\infty\beta}, g_\infty)}} \{ \|(\phi_i^{-1})^* g_i - g_\infty\|_{g_\infty}(y), \|D^k(\phi_i^{-1})^*(g_i)\|_{g_\infty}(y)\} \leqq \varepsilon_i \,. \tag{5.23}$$

where D is the covariant derivative with respect to g_∞. Put $\mu_i = \frac{1}{\sqrt{r_i}}$. Then

$$\sup_{\substack{1 \leqq k \leqq N \\ y \in B_{\bar{r}}(x, g_\infty) \setminus B_{r_i}(x, g_\infty)}} \{\|(\phi_i^{-1})^*(\mu_i g_i) - \mu_i g_\infty\|_{\mu_i g_\infty}(y), \|D_\mu^k(\phi_i^{-1})^*(\mu_i g_i)\|_{\mu_i g_\infty}(y)\} \leqq \varepsilon_i \tag{5.24}$$

where D_μ is the covariant derivative with respect to g_∞. Since the curvature tensor $R(g_\infty)$ of g_∞ is uniformly bounded on M_∞, the dilated manifolds $(B_{\bar{r}}(x, g_\infty), \mu_i g_\infty)$ converge to the flat cone C^2/Γ_1 with complete flat metric g_F. So it follows from (5.23) that $(B_{\bar{r}}(i), \mu_i g_i)$ converge to $(C^2/\Gamma_1, g_F)$. Similarly, the Kähler-Einstein manifolds $(B_{\bar{r}}(i), \mu_i pr_i^* g_i)$ converge to the cone $(C^2/\Gamma', g_F)$ with $\Gamma' \subset SU(2)$. The fundamental groups $\pi_1(B_{\bar{r}}(i))$ can be regarded as the finite isometry groups on $\tilde{B}_{\bar{r}}(i)$. Then the actions of these $\pi_1(B_{\bar{r}}(i))$ converge to the linear action of $\tilde{\Gamma} = \Gamma_1/\Gamma'$ on C^2/Γ'. Note that Γ' is a normal subgroup of Γ.

It is classical result by Klein that C^2/Γ' is one of the following normal hypersrfaces in C^3 with euclidean coordinates (w_1, w_2, w_3),

$$\begin{aligned} A_{n-1}(n \geqq 2)&: w_1 w_3 + w_2^n = 0 \text{ or } w_3^2 + w_1^2 + w_1^2 = 0 \\ D_n(n \geqq 4)&: w_3^2 + w_2(w_1^2 + w_2^{n-1}) = 0 \\ E_6&: w_3^2 + w_1^3 + w_2^4 = 0 \\ E_7&: w_3^2 + w_1(w_1^2 + w_2^3) = 0 \\ E_8&: w_3^2 + w_1^3 + w_2^5 = 0 \end{aligned} \tag{5.25}$$

Claim 1. The finite group $\tilde{\Gamma}$ has a faithful representation in $SL(3, C)$ such that the action of $\tilde{\Gamma}$ on C^2/Γ' coincides with its restriction to the hypersurface corresponding to C^2/Γ'.

Although this claim should be the special case of a more general theorem, we give an elementary proof here.

As before, let (z_1, z_2) be the euclidean coordinates of C^2. Let $C_{\Gamma'}[z_1, z_2]$ be the algebra of all Γ'-invariant polynomials in $C[z_1, z_2]$. Then the result of Klein actually says that $C_{\Gamma'}[z_1, z_2]$ is generated by three homogeneous polynomials, i.e. induced ones by restriction to C^2/Γ' of coordinates w_i on C^3. We still denote them by w_1, w_2, w_3. Since Γ' is a normal subgroup of Γ, any $\sigma \in \Gamma$ preserves the subalgebra $C_{\Gamma'}[z_1, z_2]$ of the polynomial algebra $C[z_1, z_2]$. Thus Γ has a holomorphic action on C^3 with its restriction to C^2/Γ equal to the original action of Γ. We need to prove that this induced action σ^* on C^3 is linear. By the explicit forms of the defining polynomials in (5.25), one can easily prove that as polynomials of (z_1, z_2),

$$\deg(w_2) \leqq \deg(w_1) \leqq \deg(w_3) \deg(w_3) < 2\deg(w_1) \tag{5.26}$$

If $\deg(w_3) = \deg(w_2)$, then $\sigma^* w_i$ is a linear combination of w_1, w_2 and w_3 for any $\sigma \in \Gamma$. Thus we may assume that $\deg(w_3) > \deg(w_2)$. If $\deg(w_1) = \deg(w_2)$, then for any $\sigma \in \Gamma$, $\sigma^* w_i$ depends linearly on w_1, w_2 for any $i = 1, 2$. On the other hand, $\sigma^* w_3$ can not contain w_1, w_2 since σ^* preserves the defining equations in (5.25), so σ^* is linear. If $\deg(w_1) < \deg(w_3)$, then by the second inequality in (5.26), the polynomial $\sigma^*(w_3)$ does not depend on w_1 and $\sigma^*(w_i)$ does not depend on w_3 for $i = 1, 2$. Thus by the fact that σ^* preserves one of the equations in (5.25), one can easily see that $\sigma^*(w_j) = \lambda_j w_j$ for some constants λ_j $(j = 1, 2, 3)$, in particular, the action of Γ is linear. The final case is that $\deg(w_1) = \deg(w_3)$, $\deg(w_2) < \deg(w_3)$. In this case, the group Γ' is of type $A_n (n \geqq 2)$, i.e., the hypersurface C^2/Γ' is defined by A_n-type polynomials in (5.25). Then $w_1 = z_1^n$, $w_2 = z_1 z_2$, $w_3 = z_2^n$. By the Γ'-invariance of $\sigma^* w_j$, one can easily prove that σ^* is linear. The faithfulness can be easily proved. The claim is proved.

Note that Γ' is the subgroup of Γ consisting of all elements with determinant one, so $\tilde{\Gamma} = \Gamma/\Gamma'$ is a cyclic group. Let σ be one of its generators. By the above proof of Claim1, we actually proved that the induced action σ^* on C^3 is diagonal except that Γ' is of type A_1. But if Γ' is of type A_1, all w_i have the same degree on (z_1, z_2), so by a linear transformation of (w_1, w_2, w_3), we may also assume that the action of Γ is a diagonal and the defining equation of C^2/Γ' is of form given in (5.25).

Write $\sigma = \operatorname{diag}\left(e\left(\frac{p_1}{p}\right), e\left(\frac{p_2}{p}\right), e\left(\frac{p_3}{p}\right)\right)$, where $e\left(\frac{p_i}{p}\right) = e^{\frac{2\pi\sqrt{-1}}{p}}$. Consider the representation of Γ in $S' \in C^*$ defined as follows:

$$\rho : \tau \in \Gamma \to \det(\tau) \in S^1 \subset C^*$$

where the determinant is taken with respect to (z_1, z_2). Then $\ker(\rho) = \Gamma'$ and the induced representation of ρ on Γ/Γ' is faithful. We still denote it by ρ. We may assume that $\rho(\sigma)\left(\frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial z_2}\right) = e\left(-\frac{1}{p}\right)\left(\frac{\partial}{\partial z_2} \wedge \frac{\partial .}{\partial z_2}\right)$. So $u_j = w_j\left(\frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial z_2}\right)^{p_j}$

$(j = 1, 2, 3)$ are Γ-invariant holomorphic local sections of $K_{C^2/\Gamma'}^{-p_j}$ on C^2/Γ'.

Let ϕ_i be the diffeomorphisms as those at the beginning of this proof. Put $\rho_i(y) = \rho_\infty \circ \phi_i(y)$, where ρ_∞ is the square of the standard distance function on C^2. Then for i sufficiently large, the function ρ_i is plurisubharmonic on $B_{4\sqrt{r_i}}(i)/B_{4r_i}(i)$, and these ρ_i converge to ρ_∞.

As in the proof of Lemma 5.3, we let $\bar\partial_i$ be the $\bar\partial$-operator associated to M_i and π_i be the projection from $\otimes^{p_i}(\Lambda^2(TM_i \oplus \overline{TM_i}))$ onto $K_{M_i}^{-p_i}$. Define

$$v_{ij} = \bar\partial_i\left(\eta\left(\frac{\rho_i}{2r_i}\right)\pi_i((\phi_i^{-1})_*(u_j))\right)$$

where η is a cut-off function on R^1 with $\eta(t) = 0$ for $t \leqq 1$ and $\eta(t) = 1$ for $t \geqq 1$, $|\eta'(t)| \leqq 1$. Let ψ be an increasing function on $(-\infty, 4)$ such that $\psi(t) = 0$ for $t \leqq 1$ and $\psi(t)$ goes to $+\infty$ very fast as $t \to 4$. As in the proof of Lemma 5.3, we apply L^2-estimate of $\bar\partial$-operators with weight function $\psi \circ \rho_i$ (Proposition 5.1) to the equations $\bar\partial_i u = v_{ij}$ on $\phi_i^{-1}(B_{2\sqrt{r_i}}(x_{\infty 1}, g_\infty)) \cup B_{4r_i}(i)$ and obtain local holomorphic anticanonical sections u_{ij} on $(B_{\sqrt{r_i}}(i), \mu_i g_i)$ $(j = 1, 2, 3)$ such that u_{ij} converge to u_j as $(B_{\sqrt{r_i}}(i), \mu_i g_i)$ converge to $(\{\rho_\infty < 1\}, g_F) \subset (C^2/\Gamma, g_F)$. Recall that $pr_i : \tilde B_{\sqrt{r_i}}(i) \to B_{\sqrt{r_i}}(i)$ are universal coverings and there are nonvanishing holomorphic sections $\tilde S^i$ of $K_{\tilde B_{\bar r}(i)}^{-1}$ (cf. the proof of Lemma 5.4). Without losing generality, we may assume that $\tilde S^i$ converge to $\frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial z_2}$ on C^2/Γ' as i goes to infinity. Thus the holomorphic functions $w_{ij} = pr_i^*(u_{ij})/(S^i)^{p_j}$ on $(\tilde B_{\sqrt{r_i}}(i), \mu_i g_i)$ converge to w_j on C^2/Γ'.

Claim 2. For any sufficiently large i, the functions $\{w_{ij}\}_{1 \leqq j \leqq 3}$ give an embedding Ψ_i of $\tilde B_{\sqrt{r_i}}(i)$ into C^3 satisfying:

(1) A generator σ in $\pi_1(B_{\bar r}(i))$ can be taken such that it acts on C^3 by a diagonal matrix $\sigma^*(w_{ij}) = e\left(\frac{p_j}{p}\right)w_{ij}$. Note that $\pi_1(B_{\bar r}(i))$ is a cyclic group.

(2) $\Psi_i(\tilde B_{\sqrt{r_i}}(i))$ converge to the open subset $\{\rho_\infty < 1\}$ in C^2/Γ in Hausdorff topology.

The map Ψ_i is defined by assigning y in $\tilde B_{\sqrt{r_i}}(i)$ to $(w_{i1}(y), w_{i2}(y), w_{i3}(y))$ in C^3. Since $\{(\phi_i^{-1})^* w_{ij}\}_{1 \leqq j \leqq 3}$ converge uniformly to $\{w_j\}_{1 \leqq j \leqq 3}$ on $\{\varepsilon \leqq \rho_\infty < 1\}$ for any $\varepsilon > 0$, the map Ψ_i is an embedding on $\tilde B_{\sqrt{r_i}}(i)/\tilde B_{\varepsilon\sqrt{r_i}}(i)$ for the sufficiently large i. Because $K_{M_i}^{-1}$ is ample and there is a section in $H^0(M_i, K_{M_i}^{-m})$ vanishing nowhere in $\tilde B_{\sqrt{r_i}}(i)$ for a large m (Theorem 5.1), there is no complete holomorphic curve in $\tilde B_{\sqrt{r_i}}(i)$. It follows that Ψ_i fails to be injective at most at finitely many points in $\tilde B_{\sqrt{r_i}}(i)$. If Ψ_i is not injective, then there are two points y_1, y_2 in $\tilde B_{\sqrt{r_i}}(i)$ such that $\Psi_i(y_1) = \Psi_i(y_2)$. Choose a very small $r > 0$, such that the geodesic balls $B_r(y_i, pr_i^*, \dot g_i)$ do not intersect to each other and Ψ_i is an embedding at any point of these balls except y_1, y_2. Let f_i be the local defining function of $\mathrm{Im}(\Psi_i)$ at $\Psi_i(y_1)$. Then f_i is reducible. Let f_i' (resp. f_i'') be the irreducible component of f_i such that $\{f_i' = 0\}$ (resp. $\{f_i'' = 0\}$) corresponds to $\Psi_i(B_r(y_1, pr_i^* g_i))$ (resp. $\Psi_i(B_r(y_2, pr_i^* g_i))$). Then $\{f_i' = 0\}$ (resp. $\{f_i'' = 0\}$) is smooth outside $\Psi_i(y_1)$ (resp. $\Psi_i(y_2)$). Let D be the curve defined by $f_i' = 0$ and $f_i'' = 0$. Then Ψ_i fails to be injective along $\Psi_i^{-1}(D)$. A contradiction. Therefore, Ψ_i is an embedding from $\tilde{B}_{\sqrt{r_i}}(i)$ into C^3.

In order to prove the statement (1), it suffices to check that for any r in $\pi_1(B_{\bar{r}}(i))$, we have $r^*(w_{ij}) = \lambda_{ij} w_{ij}$ for some real numbers $\lambda_{ij} (1 \leqq j \leqq 3)$. By definition, we can write $w_{ij} = pr_i^*(u_{ij})/(\tilde{S}_i)^{p_j}$. Since each $pr_i^*(u_{ij})$ is Γ-invariant, we have

$$\tau^* w_{ij} = \frac{pr_i^*(u_{ij})}{(\tau^* \tilde{S}^i)^{p_j}} = \left(\frac{\tilde{S}^i}{\tau^* \tilde{S}^i}\right)^{p_j} w_{ij} = \lambda_{ij} w_{ij}$$

where λ_{ij} are holomorphic functions on $\tilde{B}_{\sqrt{r_i}}(i)$. Because τ is an isometric of the metric $pr_i^* g_i$ and $\tilde{S}^i$ is the m_0-root of a Γ-invariant m_0-anticanonical section, the absolute values of λ_{ij} are identically one, so λ_{ij} are constants.

The statement (2) is then trivially true. The claim is proved.

Now we can complete the proof of this lemma. We identify each $\pi_1(B_{\bar{r}}(i))$ with Γ/Γ' such that $\sigma = \operatorname{diag}\left(e\left(\frac{p_1}{p}\right), e\left(\frac{p_2}{p}\right), e\left(\frac{p_3}{p}\right)\right)$ as a linear transformation on C^3. Let f_i, f_∞ be the defining equations of $\Psi_i(\tilde{B}_{\sqrt{r_i}}(i))$ and C^2/Γ in C^3, respectively. Then f_∞ is one of the polynomials in (5.25) and $\lim_{i\to\infty} f_i = f_\infty$. Since σ preserves the hypersurfaces $\{f_i = 0\}$ and $\{f_\infty = 0\}$, we have that $\sigma^* f_i = \lambda f_i$, $\sigma^* f_\infty = \lambda f_\infty$, where λ is a nonzero constant. If $\lambda \neq 1$, then any f_i has no constant term, in particular, the origin of C^3 is in $\{f_i = 0\}$. It follows that the group $\pi_1(B_{\bar{r}}(i)) = \Gamma_1/\Gamma'$ does not act on $\tilde{B}_{\sqrt{r_i}}(i)$ freely. A contradiction. So $\lambda = 1$ and each f_i must have nonzero constant term. It follows that $\{f_i = 0, w_j = 0\}$ is non empty for any $i \geqq 1$ and $1 \leqq j \leqq 3$. We remark that $\pi_1(B_{\bar{r}}(i))$ acts on $\{f_i = 0\}$ freely, so if $kp_{j'} \equiv 0$ for (mod p) for some j', then $kp_{j'} \equiv 0 \pmod p$ for all j, where k is an integer. Thus we may assume that $0 < p_j < p$ for all $j = 1, 2, 3$. If Γ' is of type other than A_n, then $2p_3 \equiv 0 \pmod p$, so by the above remark, we have $2p_j = p$ for all j. It follows that $\sigma = \operatorname{diag}(-1, -1, -1)$. It is certainly impossible since the latter diagonal matrix does not preserve the equation of type D_n, E_6, E_7, E_8 in (5.25). Therefore, Γ' must be of type A_n.

If the defining polynomial f_∞ is of form $w_3^2 + w_1^2 + w_2^2 = 0$, then the same argument as above shows that $\sigma = \operatorname{diag}(-1, -1, -1)$.

Now we assume that f_∞ is of form $w_1 w_3 + w_2^n = 0$.

Claim 3. The element σ in $\pi_1(B_{\bar{r}}(i))$ is one of $\sigma_{l,n,q}$ described in the statement of this lemma. In particular, the lemma follows from this claim.

By the fact that the action of $\pi_1(B_{\bar{r}}(i))$ is free of fixed point, we may assume that $p_1 = 1$ and p, p_2 are coprime. Then $n = pn'$ by $\sigma^* f_\infty = f_\infty$. Recall that $w_1 = z_1^n$, $w_3 = z_2^n$, $w_2 = z_1 z_2$ for euclidean coordinates (z_1, z_2) in C^2. As an element in $U(2)$, we can write $\sigma = \operatorname{diag}\left(\frac{1 + pq_1}{np}, -\frac{1 + pq_1}{np} + \frac{p_2}{p}\right)$. Let m be the largest common factor of $1 + pq_1$ and np, then m, p are coprime. Write $1 + pq_1 = ml_1$, $np = n'p^2 = mlp^2$, where l_1 is coprime to both l and p. Let m_1, m_2 be such that $m_1 l_1 + m_2 lp^2 = 1$ and m_1 is coprime to l, p. So $\sigma^{m_1} = \operatorname{diag}\left(\frac{1}{lp^2}, -\frac{1}{lp^2} + \frac{m_1 p_2}{p}\right)$. Thus we may assume that $\sigma = \sigma_{l,p,q}$. The claim is proved, and so is the lemma.

We summarize the above discussion in the following

Theorem 5.2. *Let $\{(M_i, g_i)\}$ be a sequence of Kähler-Einstein surfaces in $\mathfrak{J}_n$ with $\mathrm{Ric}(g_i) = \omega_{g_i}$. Then by taking a subsequence if necessary, we may have that $(M_i g_i)$ converge to an irreducible Kähler-Einstein orbifold (M_∞, g_∞) (in the sense of Proposition 4.2) satisfying*

(1) *For all integers $m > 0$, $h^0(M_\infty, K_{M_\infty}^{-m}) = h^0(M_i, K_{M_i}^{-m}) = 1 + \frac{m(m+1)}{2}(9-n)$.*

(2) *M_∞ has finitely many isolated singularities. Each of these singularities is either a rational double point (cf. [BPV], p87) or a singular point of type $C^2/Z_{l,p,q}$ with a cyclic group $Z_{l,p,q}$ defined in Lemma 5.5. Moreover, $lp^2 < \frac{48}{9-n}$. The latter singular point is a Hirzebruch-Jung singularity (cf. [BPV], p80).*

Proof. By Proposition 4.2, Lemma 5.2 and Lemma 5.5, it suffices to prove the upper bound of lp^2 required in the above (2).

By Bonnet-Myers Theorem ([CE]), we have

$$\text{diameter of } (M_i, g_i) \leqq \sqrt{3}\pi \text{ for all } i \tag{5.27}$$

It follows

$$\text{diameter of } (M_\infty, g_\infty) \leqq \sqrt{3}\pi. \tag{5.28}$$

For any fixed i and $x_i \in M_i$, by Bishop's Volume Comparison Theorem [Bi], we have

$$\frac{\mathrm{Vol}_{g_i}(B_r(x_i, g_i))}{9\,\mathrm{Vol}(B_{\frac{r}{\sqrt{3}}}(o, g_{S^4}))} > \frac{\mathrm{Vol}_{g_i}(M_i)}{9\,\mathrm{Vol}(S^4)} \text{ for } r \text{ small} \tag{5.29}$$

where (S^4, g_4) is the sphere with standard metric g_{S^4}, and o is the north pole in S^4.

Taking x_i in M_i such that $\lim_{i\to\infty} x_i = x_\infty$ is a singular point of type $C^2/Z_{l,p,q}$, then it follows from (5.29) that

$$\frac{1}{lp^2} > \frac{(9-n)}{18\,\mathrm{Vol}(S^4)} = \frac{9-n}{48}$$

i.e.

$$lp^2 < \frac{48}{9-n}. \tag{5.30}$$

The theorem is proved.

Remark.

(i) In case $n = 5$, there are three possible (l, p, q) for $Z_{l,p,q}$ in Theorem 5.1, i.e., $(l, p, q) = (1, 2, 1), (2, 2, 1), (1, 3, 1)$. Note that $Z_{(1,3,1)} \cong Z_{(1,3,2)}$.

(ii) In case $n = 6$, there are four possibilities: $(l, p, q) = (1, 2, 1), (2, 2, 1), (3, 2, 1), (1, 3, 1)$.

(iii) In case $n = 7$, the triple (l, p, q) could be $(l, 2, 1)$ for $1 \leqq l \leqq 5$, $(l, 3, 1)$ for $1 \leqq l \leqq 2$ and $(1, 4, 1)$.

(iv) In case $n = 8$, the triple (l, p, q) could be $(l, 2, 1)$ for $1 \leqq l \leqq 11$, $(l, 3, 1)$ for $1 \leqq l \leqq 5$, $(l, 4, 1)$ for $1 \leqq l \leqq 2$, $(1, 5, 1)$, $(1, 5, 2)$, and $(1, 6, 1)$.

6. Anticanonical divisors on some Kähler-Einstein orbifolds

We still denote by (M_∞, g_∞) the irreducible Kähler-Einstein orbifold in Proposition 4.2. Then this M_∞ is a normal surface with finitely many singular points. Each of these singularities is either a rational double point or a Hirzebruch-Jung singularity (cf. [BPV]). The purpose of this section is to study the plurianticanonical divisors on M_∞. Although the results here should hold in more general situation, we will confine our discussions to our special case.

Lemma 6.1. (Poincaré Duality Formula) *Let (M_∞, g_∞) be as above and ω_{g_∞} be the Kähler form associated to the metric g_∞. Then*

(1) *For any pluri-anti-canonical section $S \in H^0(M_\infty, K_{M_\infty}^{-m})$, we have*

$$\int_{\{S=0\}} \omega_{g_\infty}^2 = m \int_{M_\infty} \omega_{g_\infty}^2 = (9-n)m \tag{6.1}$$

(2) *Let D be a divisor in M_∞, $S \in H^0(M_\infty, K_{M_\infty}^{-m})$ and D_S is the divisor defined by the section S such that D and D_S have no common component, then*

$$m \int_D \omega_{g_\infty} = \sum_{x \in M_\infty} \frac{1}{\deg(\pi_x)} i_o(\pi_x^* D, \pi_x^* D_S) \tag{6.2}$$

where $\pi_x: \tilde{U}_x \to M_\infty$ is a local uniformization of x with $\pi_x(o) = x$ and $i_o(\pi_x^* D, \pi_x^* D_S)$ is the intersection multiplicity of $\pi_x^* D$ and $\pi_x^* D_S$ at the origin (cf. [BPV]). Note that x is smooth iff $\deg(\pi_x) = 1$.

Proof. The proof is standard. For example, in the case (1),

$$m \int_{M_\infty} \omega_{g_\infty}^2 = \lim_{\varepsilon \to 0} \int_{M_\infty} \omega_{g_\infty} \wedge \left(m\omega_{g_\infty} + \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log(\varepsilon + \|S\|_{g_\infty}^2) \right).$$

One can easily check that the right-handed side is just $\int_{\{S=0\}} \omega_{g_\infty}$. The case (2) can be similarly proved.

Let $\pi: \tilde{M}_\infty \to M_\infty$ be the minimal resolution. Then for each singular point x in M_∞, the exceptional curve $\pi^{-1}(x)$ is either an $A-D-E$ curve or a Hizebruch-Jung string according to whether x is a rational double point or not (cf. [BPV]). In particular, any singular point is rational. It is easy to show that $h^0(\tilde{M}_\infty, K_{\tilde{M}_\infty}^{-m}) = h^0(M_\infty, K_{M_\infty}^{-m})$ for any m. For any integer $m > 0$,

$$K_{\tilde{M}_\infty}^{-m} = \tilde{B}_m + \tilde{D}_m, \; K_{M_\infty}^{-m} = B_m + D_m \tag{6.3}$$

where $\tilde{B}_m$(resp. B_m) consists of all one dimensional components in the base locus of $K_{\tilde{M}_\infty}^{-m}$(resp. $K_{M_\infty}^{-m}$). Then $\pi(\tilde{B}_m) = B_m$, $\pi(\tilde{D}_m) = D_m$. We denote by $|\tilde{D}_m|$ the linear system of the divisor $\tilde{D}_m$, etc.

We will first prove that the generic divisor in $|\tilde{D}_1|$ is irreducible. If $n = 8$, then $\dim|\tilde{D}_1| = h^0(\tilde{M}_\infty, \tilde{D}_1) - 1 = 1$, it follows that the generic divisor in $|D_1|$ is irreducible.

Lemma 6.2. *Let $n = 5, 6, 7$. Then if the generic divisor in $|\tilde{D}_1|$ is reducible, we can write $\tilde{D}_1 = (9-n)E$ with $E^2 = 0$ and $h^0(\tilde{M}_\infty, E) = 2$.*

Proof. We can write $\tilde{D}_1 = \sum_{i=1}^{l} \alpha_i E_i$ such that E_i is not linearly equivalent to E_j for $i \neq j$, any $|E_i|$ is free of one dimensional component in its base locus and the addition map: $\prod_{i=1}^{l} |E_i|^{\alpha_i} \to |\tilde{D}_1|$ is generically surjective. We need to prove that $l = 1$. Suppose that $l \geqq 2$. We may assume that E_1 can not be decomposed into the sum of E_2 and an effective divisor. We remark that the addition map: $|E_1| \times |E_2| \to |E_1 + E_2|$ is also generically surjective. Choose an irreducible divisor in $|E_1|$, say E_1 for simplicity. Then for any point x in E_1 outside the base locus of $|E_1 + E_2|$, we have a divisor in $|E_2|$ intersecting E_1 at x. Thus $E_1 \cdot E_2 \geqq 1 + \#\{\text{base points of } |E_1 + E_2|\}$ (count multiplicity). By Bertini's theorem (cf. [GH]), the generic divisor E in $|E_1 + E_2|$ is smooth outside the base locus of $|E_1 + E_2|$. So E can not be written as the sum of divisors in $|E_1|$ and $|E_2|$. A contradiction. Thus $l = 1$, $\tilde{D}_1 = \alpha E$ $(\alpha \geqq 2)$. The above arguments also show that $h^0(\tilde{M}_\infty, E) = 2$. Since $h^0(\tilde{M}_\infty, \tilde{D}_1) = 10 - n$ and the generic divisor in $|\tilde{D}_1|$ can be written as the sum of α divisor in $|E|$, we must have $\alpha = 9 - n$. Then by Riemann-Roch Theorem [GH], $10 - n \geqq 1 + \alpha^2 E^2$, so $E^2 = 0$. The lemma is proved.

Denote by $IP(l, p, 1)$ be the germ of all holomorphic functions f at the origin of C^2 such that $\sigma_{l,p,1}^* f = e\left(\frac{1}{p}\right) f$. One can easily compute

$$IP(l, p, 1) = \Big\{ f \in C^2\{z_1, z_2\} \,|\, f = \sum_{k=0}^{pl-2} \Big(z_1^k \sum_{j=0}^{\infty} f_j^k(z_1^{pl-1}, z_2) \Big) \tag{6.4}$$

$$f_j^k \text{ are homogeneous polynomials of degree } j_k + jp^2 l \Big\}$$

where $j_0 = (p-1)pl$, $j_{k+1} \equiv j_k + (pl+1) \pmod{p^2 l}$ and $0 < j_k < p^2 l$ *for* $0 \leqq k \leqq pl - 2$.

The significance of $IP(l, p, 1)$ in our context is the following. If x is a singularity in M_∞ of type $C^2/Z_{l,p,1}$, let $\pi_x : \tilde{U}_x \to U_x \subset M_\infty$ be a local uniformization, where $\tilde{U}_x \subset C^2$, $\tilde{U}_x(o) = x$, then the local holomorphic sections of $K_{M_x}^{-1}$ in U_x correspond to one-to-one the functions in $IP(l, p, 1)$.

We list some simple lemmas in the following.

Lemma 6.3. *Let* $IP(l, p, 1)$ *be defined as in* (6.4). *Then*

(1) *The monomials in* $IP(l, 2, 1)$ *of degree* $\leqq 2l$ *are* $z_1^{2l}, z_2^{2l}, z_1 z_2, z_1^3 z_2^3, \ldots,$ $z_1^{2k+1} z_2^{2k+1}$, *where* $k = \left[\frac{l-1}{2}\right]$.

(2) *If* $l = 1, 2$, *then the monomials in* $IP(l, 3, 1)$ *of degree* $\leqq \frac{9l}{2}$ *are among* $z_1^{3l}, z_1 z_2, z_1^4 z_2^4$.

(3) *The monomials in* $IP(1, 4, 1)$ *of degree* $\leqq 8$ *are* $z_1 z_2$, z_2^4, $z_1^2 z_2^6$.

In particular, if $p^2 l < 24$, *then for each* (λ, μ) *being either* $(4, 0)$, *or* $(0, 4)$ *or* $\lambda \geqq 2$, $\mu \geqq 2$, *there are at most two monomials in* $IP(l, p, 1)$ *with degree* $\leqq \frac{p^2 l}{2}$ *and containing the factor* $z_1^\lambda z_2^\mu$.

This follows directly from (6.4) and some simple computations.

Lemma 6.4. *Let f, g be holomorphic functions at $o \in C^2$ and have no common component. We further assume that $\sigma^*_{l,p,1} f = cf$ and $\{f = 0\}$ is smooth at 0. Then we have*

$$i_o(\{f=0\},\{g=0\}) \geqq \begin{cases} \inf\{\lambda(p^2 l - pl + 1) + \mu \mid z_1^\lambda z_2^\mu \text{ is in } g\}, & \text{if } f = z_1 + O(|z|^2) \\ \inf\{\lambda + \mu(pl - 1) \mid z_1^\lambda z_2^\mu \text{ is in } g\}, & \text{if } f = z_2 + O(|z|^2) \end{cases} \tag{4.13}$$

where $i_o(\cdot,\cdot)$ is the intersection multiplicity (cf. [BPV]) and $|z|^2 = |z_1|^2 + |z_2|^2$.

Proof. We assume that $f = z_1 + O(|z|^2)$. The proof for the other case is same. Write $f = z_1 f_1(z_1, z_2) + f_2(z_2)$, then $f_1(0,0) \neq 0$ and $\mathrm{ord}_o(f_2) \geqq 2$. By the assumption that $\sigma^*_{l,p,1} f = cf$, we have $\mathrm{ord}_o(f_2) \geqq p^2 l - pl + 1$. Put $w_1 = z_1 f_1(z_1, z_2) + f_2(z_2)$, $w_2 = z_2$, then $z_1 = w_1 \tilde{f}_1(w_1, w_2) + \tilde{f}_2(w_2)$ with $\mathrm{ord}_o(\tilde{f}_2) \geqq p^2 l - pl + 1$. Now by the definition of the intersection multiplicity, we have

$$i_0(\{f=0\},\{g=0\}) = \mathrm{ord}_o(g|_{\{f=0\}}) = \mathrm{ord}_o(g(\tilde{f}_2(w_2), w_2)) .$$

Then lemma is proved.

We now come to apply these two lemmas to studying the properties the anticanonical divisors on M_∞.

Lemma 6.5. *Let $\tilde{D}_1$ be defined as in* (6.3). *Then the generic divisor in the linear system $|\tilde{D}_1|$ is irreducible unless $n = 7$, $B_1 = \varnothing$ and M_∞ has exactly two singular points of type $C^2/Z_{1,2,1}$.*

Proof. By some results in [De], we may assume that $n \leqq 7$ and M_∞ has at least one singular point besides rational double points. Let $x_1, \ldots, x_l$ ($l \geqq 1$) be those singular points in M_∞ other than rational double points, and $F_1, \ldots, F_l$ be the exceptional curves in the minimal resolution $\tilde{M}_\infty$ over $x_1, \ldots, x_\infty$. Suppose that the generic divisor in $|\tilde{D}_1|$ is reducible. By Lemma 6.2, we write $\tilde{D}_1 = (9-n)E$.

Claim 1. The divisor E must intersect one of F_j, say F_1 for simplicity. Moreover, $E \cdot F_1 = 1$.

If $E \cdot F_1 = 0$ for any j, let H be a divisor of $K_{M_\infty}^{-m}$ for some $m > 0$ such that $x_j \notin H$ ($j = 1, 2, \ldots, l$). Then $H \cdot \pi(E) = K_{M_\infty}^{-m} \cdot E \geqq m$. By Lemma 6.1 (1), we have

$$\begin{aligned} 9 - n &= \int_{B_1} \omega_{g_\infty} + (9-n) \int_{\pi(E)} \omega_{g_\infty} \\ &= \int_{B_l} \omega_{g_\infty} + \frac{(9-n) H \cdot \pi(E)}{m} . \end{aligned}$$

Therefore, $B_1 = \varnothing$. It follows that all x_j are rational double points. A contradiction. So we may assume that $E \cdot F_1 \geqq 1$. Let F_{11} be the component in F_1 with $F_{11} \cdot E \geqq 1$. In case that $9 - n \geqq 3$ or there is another component in $\tilde{B}_1$ intersecting F_{11} at some point, the divisor F_{11} has multiplicity two in $\tilde{B}_1$ by $-K_{\tilde{M}_\infty} \cdot F_{11} = F_{11}^2 + 2$. It follows that $E \cdot F_{11} + E \cdot F_1 \leqq \tilde{B}_1 \cdot E \leqq 2$, i.e. $E \cdot F_1 = 1$. Thus we may assume that x_1 is of type $C^2/Z_{1,2,1}$ and $F_1 \cdot (\tilde{B}_1 - F_1) = 0$. By adjunction formula, we have $-2 = K_{\tilde{M}_\infty}^{-1} \cdot F_1 \geqq F_1^2 + 2F_1 \cdot E = -4 + 2F_1 \cdot E$, i.e., $F_1 \cdot E = 1$. The claim is proved.

Claim 2. If x_1 is of type $C^2/Z_{1,2,1}$, then $n = 7$, $B_1 = \emptyset$ and M_∞ has exactly two singular points of type $C^2/Z_{1,2,1}$.

We first prove that E must intersect another exceptional curve F_j other than F_1. In fact, if it is not true, then F_1 has multiplicity two in $\tilde{B}_1$. By $-K_{\tilde{M}_\infty} \cdot F_1 = F_1^2 + 2$, there are exactly six curves in a generic anticanonical divisor on $\tilde{M}_\infty$ intersecting F_1. Thus for any S in $H^0(M_\infty, K_{M_\infty}^{-1})$, the pull-back $\pi_1^* S$ is locally represented by a holomorphic function f_S in $IP(1, 2, 1)$ with vanishing order at least 6 at o, where $\pi_1 : \tilde{U}_1 \to U_1$ is the local uniformization of x_1 in M_∞. By (6.2) in Lemma 6.1, if $\{S = 0\}$ has no common component with $\pi(E)$, we have

$$9 - n = \int_{B_1} \omega_{g_\infty} + (9 - n) \int_{\pi(E)} \omega_{g_\infty} \geqq \frac{(9-n)}{4} \operatorname{mult}_o(\{S = 0\}) \geqq \frac{3}{2}(9 - n) .$$

A contradiction. Therefore, the divisor E intersects with another exceptional curve, say F_2. Then $n = 7$ and $F_2 \cdot E = 1$, so by $K_{\tilde{M}_\infty}^{-1} \cdot F_2 = F_2^2 + 2$, the singular point x_2 must be of type $C^2/Z_{1,2,1}$. As above, by using Lemma 6.1, one can prove that $B_1 = \emptyset$. Thus there is no other singular point in M_∞ besides rational double points. Claim 2 is proved.

Now we may assume that x_1 is not of type $C^2/Z_{1,2,1}$, then F_1 has multiplicity two in $\tilde{B}_1$ and E intersects with no other $F_j (j \geqq 2)$. By (6.2) in Lemma 6.1, we have

$$\int_{\pi(E)} \omega_{g_\infty} \leqq 1 \quad \text{and} \quad < 1 \text{ if } B_1 \neq \emptyset . \tag{6.5}$$

Let $\pi_1 : \tilde{U}_1 \to U_1$ be the local uniformization of x_1 with uniformization group Γ_1, then as above, the section S in $H^0(M_\infty, K_{M_\infty}^{-1})$ is locally represented by a f_S in $IP(l, p, 1)$, where x_1 is of type $C^2/Z_{l,p,1}$. It follows from (6.5) and Lemma 6.1,

$$i_o(\{f_S = 0\}, \pi_1^{-1}(\pi(E))) \leqq p^2 l \quad \text{and} \quad < p^2 l \text{ if } B_1 \neq \emptyset \tag{6.6}$$

where $i_o(\cdot, \cdot)$ is the intersection multiplicity at the origin in $\tilde{U}_1$.

Claim 3. For generic E in the linear system $|E|$, the pull-back $\pi_1^{-1}(\pi(E))$ is smooth at the origin.

Let (z_1, z_2) be the local coordinates of $\tilde{U}_1$ such that the generator $\sigma_{l,p,1}$ of $\Gamma_1 \cong Z_{l,p,1}$ is diagonal in (z_1, z_2). Also let h_E be the defining function of $\pi_1^{-1}(\pi(E))$ in $\tilde{U}_1$, then $\sigma_{l,p,1}^* h_E = c h_E$ for some constant c. First we prove that $\operatorname{ord}_o(h_E) \leqq 2$ for generic E. If this is not true, then $\operatorname{ord}_o(f_S) \geqq 6$ for any section S in $H^0(M_\infty, K_{M_\infty}^{-1})$. By Lemma 6.3 and the fact that $h^0(M_\infty, K_{M_\infty}^{-1}) \geqq 3$, one can easily find a section S in $H^0(M_\infty, K_{M_\infty}^{-1})$, the local representation f_S vanishes at o of order $> \frac{p^2 l}{3}$ for those $(l, p, 1)$ with $p^2 l < 24$. Now we choose a generic E such that E has no common component with $\{S = 0\}$, then we have

$$i_o(\{f_s = 0\}, \pi_1^{-1}(\pi(E))) \geqq \operatorname{ord}_o(h_E) \cdot \operatorname{ord}_o(f_S) > p^2 l .$$

It contradicts to (6.6). So $\operatorname{ord}_o(h_E) \leqq 2$. Using the invariance $\sigma_{l,p,1}^* h_E = c h_E$ and the fact that $\sigma_{l,p,1}$ has distinct eigenvalues, we conclude that $h_E(z_1, z_2) = z_j^\lambda + O(|z|^{\lambda+1})$ or $z_1 z_2 + O(|z|^3)$, where $\lambda = 1, 2$. In the former case, if $\lambda = 2$, then for any S in $H^0(M_\infty, K_{M_\infty}^{-1})$, the lowest order term of f_S has the fact z_j^4, so by Lemma 5.3 again, we can find such a S that $\mathrm{ord}_o(f_S) > \frac{p^2 l}{2}$. As above, we can deduce a contradiction to (6.6). Some arguments exclude the possibility of $h_E = z_1 z_2 + O(|z|^3)$. Therefore $\lambda = 1$ and the claim is proved.

We now want to apply the formula in Lemma 6.4 to our cases and conclude the proof of this lemma. It suffices to find a S in $H^0(M_\infty, K_{M_\infty}^{-1})$ such that for any term $z_1^\lambda z_2^\mu$ in f_S, either $\lambda(p^2 l - pl + 1) + \mu > p^2 l$ for $j = 1$, where $h_E = z_1 + O(|z|^2)$, or $\lambda + \mu(pl - 1) > p^2 l$ for $j = 2$, where $h_E = z_2 + O(|z|^2)$. Since there is only one monomial $z_1^\lambda z_2^\mu$ in $IP(l, p, 1)$ such that $\lambda + \mu \leqq p^2 l$ and $\lambda \leqq 1$, we may find a S in $H^0(M_\infty, K_{M_\infty}^{-1})$ such that any term $z_1^\lambda z_2^\mu$ in f_S has either $\lambda + \mu > p^2 l$ or $\lambda \geqq 2$. So by the fact that $2(p^2 l - pl + 1) > p^2 l$ for $p \geqq 2$, we may assume that $h_E = z_2 + O(|z|^2)$. By (6.4), any monomial $z_1^\lambda z_2^\mu$ in $IP(l, p, 1)$ can be written $z_1^{k + (pl-1)\lambda'} z_2^\mu$ with $\lambda' + \mu = j_k$, thus

$$\lambda + \mu(pl - 1) = k + (pl - 1) j_k .$$

One can easily check that for those $(l, p, 1)$ with $p^2 l < 24$, the monomials $z_1 z_2$, z_1^{pl} are only ones in $IP(l, p, 1)$ with $\lambda + \mu(pl - 1) \leqq p^2 l$. Then by the fact that $h^0(M_\infty, K_{M_\infty}^{-1}) \geqq 3$, we may find a S in $H^0(M_\infty, K_{M_\infty}^{-1})$ such that f_S does not contain $z_1 z_2$ and z_1^{pl}. It follows from Lemma 6.4,

$$i_0(\{h_E = 0\}, \{f_S = 0\}) > p^2 l \tag{6.7}$$

Therefore, we obtain a contradiction from (6.6) and (6.7). The lemma is proved.

Proposition 6.1. *Let $\{(M_i, g_i)\}$, (M_∞, g_∞) be given as in Proposition 4.2. Assume that $n = 7$. Then M_∞ has only rational double points as singular points unless M_∞ has exactly two singular points of type $C^2/Z_{1,2,1}$, and $|K_{M_\infty}^{-1}|$ is free of one dimensional components in its base locus. Moreover, the linear system $|K_{M_\infty}^{-2}|$ is always free of base point.*

Proof. It is well-known that each M_i is branched double covering of CP^2, in particular, each M_i admits a nontrivial involution τ_i (cf. [De]). One can easily check that the fixed point set A_i of τ_i is a connected smooth divisor in $|2K_{M_i}^{-1}|$ and τ_i preserves any anticanonical divisor. These τ_i converge to a nontrivial involution τ_∞ of M_∞ as M_i converge to M_∞ in the sense of Proposition 4.2. The fixed point set A_∞ of τ_∞ is the limit of A_i and then is 2-anticanonical divisor in $|2K_{M_\infty}^{-1}|$.

Let $\pi: \tilde{M}_\infty \to M_\infty$ be the minimal resolution as above and $\tilde{B}_1$, $\tilde{D}_1$, B_1, D_1 defined as in (6.3). We first assume that the generic divisor in $|D_1|$ is irreducible. Choose such an irreducible one, say D_1 for simplicity. Fix a regular point x of M_∞ in $D_1 \backslash (A_\infty \cup B_1)$. Since $h^0(M_\infty, K_{M_\infty}^{-1}) = 3$, we can find another divisor D_1' in $|D_1|$ such that D_1 and D_1' have no common component and $x \in D_1 \cap D_1'$. Since both D_1 and D_1' are stabilized by τ_∞, their intersection $D_1 \cap D_1'$ also contains $\tau_\infty(x)$. By Lemma 6.1, we have

$$\begin{aligned} 2 &= \int_{B_1} \omega_{g_\infty} + \int_{D_1} \omega_{g_\infty} \\ &\geqq \int_{B_1} \omega_{g_\infty} + \sum_{y \in \mathrm{Sing}(M_\infty)} \frac{1}{\deg(\pi_y)} i_o(\pi_y^* D_1, \pi_y^* D_1') + i_x(D_1, D_1') + i_{\tau_\infty(x)}(D_1, D_1') \end{aligned}$$

where $\pi_y: \tilde{U}_y \to U_y$ is the local uniformization of y in the singular set $\mathrm{Sing}(M_\infty)$. It follows that $B_1 = \emptyset$ and none of singular points in M_∞ is in the base locus of $|K_{M_\infty}^{-1}|$. The latter implies that all singular points of M_∞ are rational double points (cf. [BPV]).

It remains to conside the case that the generic divisor in $|D_1|$ is reducible. Then by Lemma 6.5, it suffices to prove that the base locus of $|2K_{M_\infty}^{-1}|$ does not contain those points of type $C^2/Z_{1,2,1}$. Now we have a natural divisor A_∞ in $|2K_{M_\infty}^{-1}|$. It is smooth and irreducible since it is the fixed point set of τ_∞ and τ_∞ is an isometry of (M_∞, g_∞). We claim that A_∞ does not contain an singular point of type $C^2/Z_{1,2,1}$. In fact, if the claim is not true, both singular points p_1, p_2 of type $C^2/Z_{1,2,1}$ are in A_∞. By our assumption on $|D_1|$, we can write $D_1 = 2E$ and the generic divisor in $|E|$ is an irreducible rational curve passing through p_1, p_2. On the other hand, since τ_∞ preserves $|D_1|$, it also preserves $|E|$. Choose a generic divisor in $|E|$, say E for simplicity, such that E intersects with A_∞ at a point outside p_1, p_2. Note that two generic divisors in $|E|$ do not intersect to each other outside p_1, P_2. Thus τ_∞ must stabilize E, then it fixes E since it fixes three points on E and E is rational. It follows that τ_∞ fixes the generic points in M_∞, i.e., τ_∞ is an identity. A contradiction! Therefore, $p_1, p_2 \notin A_\infty$ and $|2K_{M_\infty}^{-1}|$ is free of base point. The proposition is proved.

Remark. One can also construct local nonvanishing sections of $2K_{M_\infty}^{-1}$ at the above p_1, p_2 and then use L^2-estimate of $\bar{\partial}$-operators (Proposition 5.1) with weight function $a\log(\sum_{\beta=0}^{2} \|S_\beta^\infty\|_{g_\infty}^2)$ to produce a section of $|2K_{M_\infty}^{-1}|$ which is nonzero at p_1, p_2, where $\{S_\beta^\infty\}_{0 \leqq \beta \leqq 2}$ is an orthonormal basis of $H^0(M_\infty, K_{M_\infty}^{-1})$ with respect to the inner product induced by g_∞. In particular, it implies that $|2K_{M_\infty}^{-1}|$ is free of base point.

Proposition 6.2. *Let $\{(M_i, g_i)\}$ and (M_∞, g_∞) be as in Proposition 4.2. Assume that $n = 8$. Then M_∞ has at most one singular point of type $C^2/Z_{l,2,1}$ $(2 \leqq l \leqq 7)$ besides rational double points. Moreover, the linear system $|2K_{M_\infty}^{-1}|$ is free of base point.*

Proof. It is known (cf. [De]) that each M_i is a branched double covering over a quadratic cone in CP^3. It implies that each M_i admits a nontrivial involution τ_i. Also, this τ_i preserves both anticanonical divisors and 2-anticanonical divisors. These τ_i converge to a nontrivial involution τ_∞ of M_∞ as M_i converge to M_∞ in the sense of Proposition 4.2.

Let $\tilde{M}_\infty$ be minimal resolution of M_∞ and $\tilde{B}_m, \tilde{D}_m, B_m, D_m$ be defined as in (6.3). The involution τ_∞ can be lifted to $\tilde{M}_\infty$, still denoted by τ_∞ for simplicity (cf. [La]). Then τ_∞ stabilizes $B_m, D_m, \tilde{B}_m, \tilde{D}_m$ $(m = 1, 2)$, respectively. We assume that M_∞ has a singular point other than rational double points.

Claim 1. $\tilde{D}_1^2 = 0$

By Riemann-Roch Theorem and the fact that $h^0(\tilde{M}_\infty, \tilde{D}_1) = 2$, we have either $\tilde{D}_1^2 = 0$ or $\tilde{D}_1^2 = 1$ and $\tilde{B}_1 \cdot \tilde{D}_1 = 0$. If $\tilde{D}_1^2 = 1$, then there are two irreducible divisors $\tilde{D}'_1$, $\tilde{D}''_1$ in $|\tilde{D}_1|$ such that $\tilde{D}'_1$ intersects $\tilde{D}''_1$ at a point outside $\tilde{B}_1$. By Lemma 6.1,

$$1 = \int_{B_1} \omega_{g_\infty} + \int_{D_1} \omega_{g_\infty} \geqq \int_{B_1} \omega_{g_\infty} + 1$$

It follows that $B_1 = \emptyset$. Thus all singular points are rational double points. A contradiction. The claim is proved.

The same arguments also prove that D_1 does not intersect B_1 outside singular points. Now $|\tilde{D}_1|$ induces a fibration $\pi_f: \tilde{M}_\infty \to CP^1$. We claim that the generic divisor in $|\tilde{D}_2|$ has a horizontal component. In fact, if it is not true, then by $h^0(\tilde{M}_\infty, \tilde{D}_2) = 4$, we have $\tilde{D}_2 = 3\tilde{D}_1$. It follows that $\tilde{D}_1 + \tilde{B}_2 = 2\tilde{B}_1$. Since $\tilde{B}_2 \subset 2\tilde{B}_1$, the divisor $2\tilde{B}_1 - \tilde{B}_2$ is effective. By $\tilde{D}_1^2 = 0$, we conclude that $2\tilde{B}_1 - \tilde{B}_2$ is vertical with respect to the fibration $\pi_f: \tilde{M}_\infty \to CP^1$. It is easy to prove that $2\tilde{B}_1 - \tilde{B}_2$ is connected since $\tilde{D}_1$ is. Let E be the reduced divisor supporting $2\tilde{B}_1 - \tilde{B}_2$, then E is a proper subset of one fiber in $\tilde{M}_\infty$ and $2\tilde{B}_1 - \tilde{B}_2 - E = \tilde{D}_1 - E$. Since $2\tilde{B}_1 - B_2 - E$ and $\tilde{D}_1 - E$ have no common irreducible component, it follows that $(\tilde{D}_1 - E)^2 \geqq 0$. On the other hand, the divisor $\tilde{D}_1 - E$ is a proper subset of a fiber, then $(\tilde{D}_1 - E)^2 < 0$. We get a contradiction. Therefore, the claim is proved.

Now choose a generic $\tilde{D}_2$ in $|\tilde{D}_2|$ such that the generic divisor in $|\tilde{D}_1|$ intersects $\tilde{D}_2$ at at least one point outside $\tilde{B}_1$. Since τ_∞ preserves divisors in $|\tilde{D}_1|$ and $|\tilde{D}_2|$, we have that $\pi(\tilde{D}_2)$ intersects the generic divisor in $|D_1|$ at at least two smooth points in M_∞. By Lemma 6.1, we can conclude that $B_1 = \emptyset$ and $\pi(\tilde{D}_2)$ does not pass through the singularities of M_∞ other than rational double points. Then there are at most two singular points of type $C^2/Z_{l,2,1}$ in M_∞ besides rational double points. By adjunction formula, one can easily prove that there is at most one singular point in M_∞ besides rational double points. The above arguments also show that $|2K_{M_\infty}^{-1}|$ is free of base point.

Corollary 6.1. *Let $n = 7$ or 8. There are constants $c(n, k)$ depending only on n, $k \geqq 1$ such that for any Kähler-Einstein surface (M', g') in $\mathfrak{Z}_n$, we have*

$$\inf_{M'} \left\{ \sum_{\beta=0}^{N_m} \|S'_\beta\|_{g'}^2 \right\} \geqq c(n, k) \tag{6.8}$$

where $m = 2k$, $N_m + 1 = \dim_C H^0(M', K_{M'}^{-m})$ and $\{S'_\beta\}_{0 \leqq \beta \leqq N_m}$ is an orthonormal basis of $H^0(M', K_{M'}^{-m})$ with respect to the inner product induced by g'.

Proof. It follows from Lemma 5.3, Propositions 6.1 and 6.2 (cf. the proof of Theorem 5.1).

7. Completion of the proof for strong partial C^0-estimate

In this section, we will complete the proof of Theorem 2.2, i.e., the strong partial C^0-estimate stated in section 2. By Corollary 6.1, Lemma 5.3 and the arguments in the proof of Theorem 5.1, Theorem 2.2 will follows from the following proposition.

Proposition 7.1. *Let $n = 5$ or 6, and (M_∞, g_∞) be the irreducible Kähler-Einstein orbifold in Proposition* 4.2 *or Theorem* 5.1. *Then the linear system $|6K_{M_\infty}^{-1}|$ is free of the base point.*

The rest of this section is devoted to the proof of this proposition. The basic tools are still the Poincaré duality formula (Lemma 6.1) and adjunction formula we have used in last section.

We fix the Kähler-Einstein orbifold (M_∞, g_∞) as in Proposition 7.1 with $C_1(M_\infty)^2 = 9 - n$, where $n = 5$ or 6. Let $\pi: \tilde{M}_\infty \to M_\infty$ be the minimal resolution and $x_1, \ldots, x_k$ be the singular points of M_∞ besides rational double points. The corresponding exceptional divisors in M_∞ are denoted by $F_1, \ldots, F_k$. Define D_m, B_m, $\tilde{D}_m$, $\tilde{B}_m$ as in (6.3). If M_∞ has only rational double points as singular points, then by the results in [De], the linear system $|K_{\tilde{M}_\infty}^{-1}|$ is free of base point. So we suppose that $k \geqq 1$. As in the proof of Proposition 6.1, etc, we may assume that F_1 intersects $\tilde{D}_1$. We collect some simple facts either built up before or that can be easily proved by using Riemann-Roch Theorem and adjunction formula (cf. [BPV], [GH]).

(F1) The generic divisor in $|\tilde{D}_1|$ is a smooth rational curve and $\tilde{D}_1^2 = 8 - n$, $K_{\tilde{M}_\infty}^{-1} \cdot \tilde{D}_1 = \tilde{D}_1^2 + 2$.

(F2) For $8 - n$ generic distinct points $\{y_j\}_{1 \leqq j \leqq 8-n}$ in $\tilde{M}_\infty$ outside exceptional curves, there is a pencil of divisors in $|\tilde{D}_1|$, denoted by $|(\tilde{D}_1, \{y_j\}_{1 \leqq j \leqq 8-n})|$ such that the generic divisor in this pencil is a smooth rational curve and $|(\tilde{D}_1, \{y_j\}_{1 \leqq j \leqq 8-n})|$ is free of base point outside $\{y_j\}_{1 \leqq j \leqq 8-n}$.

(F3) Let E be an exceptional, irreducible curve, i.e. $E^2 < 0$, then if $E \not\subset \tilde{B}_1$ and $\tilde{B}_1 \cdot E > 0$, then E is of first kind, i.e. $E^2 = -1$. Moreover, this E intersects exactly one irreducible component in $\tilde{B}_1$.

(F4) Each singular point $x_j (1 \leqq j \leqq k)$ is of type either $C^2/Z_{l,2,1} (1 \leqq l \leqq 3)$ or $C^2/Z_{1,3,1}$. Thus the corresponding exceptional curve F_j is reduced and can be written as $F_{j1} + \ldots + F_{jk_j}$ such that $F_{ji} \cdot F_{ji'} = 1$ if $|i - i'| = 1$; $= 0$ if $|i - i'| > 1$ and F_{ji} is irreducible for each i between 1 and k_j, where $k_j = 1, 2, 3$. If $k_j = 1$, then $F_{ji}^2 = -4$ and x_j is of type $C^2/Z_{1,2,1}$. If $k_j = 2$, then either $F_{j1}^2 = F_{j2}^2 = -3$ or $F_{j1}^2 = -2$, $F_{j2}^2 = -5$ according to that x_j is of type either $C^2/Z_{2,2,1}$ or $C^2/Z_{1,3,1}$. If $k_j = 3$, then x_j is of type $C^2/Z_{3,2,1}$, and $F_{j1}^2 = F_{j3}^2 = -3$, $F_{j2}^2 = -2$.

By these facts, there are three cases of $\tilde{M}_\infty$ as follows.

Case 1. $F_1 \cdot \tilde{D}_1 = 2$, $F_j \cdot \tilde{D}_1 = 0$ for $j \geqq 2$.

Case 2. $F_1 \cdot \tilde{D}_1 = 1$, $F_j \cdot \tilde{D}_1 = 0$ for $j \geqq 2$, so the irreducible component in F_1 intersecting $\tilde{D}_1$ has multiplicity two in $\tilde{B}_1$.

Case 3. $F_1 \cdot \tilde{D}_1 = 1$ and $F_2 \cdot \tilde{D}_1 = 1$, $F_j \cdot \tilde{D}_1 = 0$ for $j \geqq 3$.

We will treat these cases separately in the following lemmas.

Lemma 7.1. *Let (M_∞, g_∞), F_j, $\tilde{D}_1$, etc. be given as in Proposition 7.1 and $F_1 \cdot \tilde{D}_1 = 2$, $F_j \cdot \tilde{D}_1 = 0$ for $j \geqq 2$. Then $B_1 = \emptyset$, $\tilde{B}_1 = F_1$ and $\tilde{D}_1$ intersects with F_{11} and F_{1k_1} at one point, respectively. Moreover, the linear system $|6K_{\tilde{M}_\infty}^{-1}|$ is free of base point.*

Proof. Let α_i be the multiplicity of the irreducible component F_{1i} $(1 \leqq i \leqq k_1)$ in $\tilde{B}_1$, then by adjunction formula and (F1), we have

$$2 \geqq \tilde{B}_1 \cdot \tilde{D}_1 \geqq \alpha_i F_{1i} \cdot \tilde{D}_1 + (\tilde{B}_1 - F_{1i}) \cdot \tilde{D}_1$$

$$\tilde{D}_1 \cdot F_{1i} + (\tilde{B}_1 - F_{1i}) \cdot F_{1i} = 2 \tag{7.1}$$

It follows from (7.1) and $F_1 \cdot \tilde{D}_1 = 2$ that $\tilde{D}_1 \cdot \tilde{F}_{1i} \leqq 1$ unless $k_1 = 1$, i.e., x_1 is of type

$C^2/Z_{1,2,1}$, and $D_1 \cdot F_{1i} = 1$ iff $(\tilde{B}_1 - F_{1i}) \cdot F_{1i} = 1$ and $\alpha_i = 1$. Therefore $\tilde{D}_1 \cdot F_{1i} = 1$ for $i = 1, k_1$ and $(\tilde{B}_1 - F_1) \cdot F_1 = 0$, i.e., $x_1 \notin B_1$. By Lemma 6.1 and the above (F2), the generic divisor D_1 does not intersect B_1 outside $x_1, \ldots, x_k$. On the other hand, since (M_∞, g_∞) is a limit of some sequence of Kähler-Einstein surfaces according to Proposition 4.2, any anticanonical divisor in $|K_{M_\infty}^{-1}|$ must be connected, so $D_1 \cap B_1$ contains some x_j for $j \geqq 2$ if $B_1 \neq \emptyset$. By our assumption, we have $x_j \notin D_1$ for $j \geqq 2$. It implies that $B_1 = \emptyset$ and $\tilde{B}_1 = F_1$.

It remains to prove that $|6K_{M_\infty}^{-1}|$ is free of base point. By Lemma 6.1 and the above (F2), one can easily prove that $|K_{M_\infty}^{-1}| = |D_1|$ is free of base point outside x_1. So we only need to construct a global section of $6K_{M_\infty}^{-1}$ not vanishing at x_1. It will be done by applying Proposition 5.1. Define

$$\psi = 6\log\left(\sum_{\beta=0}^{N_1} \|S_\beta^\infty\|_{g\infty}^2\right) \tag{7.2}$$

where $N_1 = \dim_C H^0(M_\infty, g_\infty) - 1$ and $\{S_\beta^\infty\}$ is an orthonormal basis with respect to the inner product on $H^0(M_\infty, K_{M_\infty}^{-1})$ induced by g_∞. Since the base locus of $K_{M_\infty}^{-1}$ is the point x_1, the function ψ is bounded and continuous outside x_1. As we remarked in §6, each section S_β^∞ is represented by a function in $IP(l, p, 1)$ in the local uniformization of x_1, where x_1 is of type $C^2/Z_{l,p,1}$. In particular, it follows that for any neighborhood U of x_1 in M_∞, we have

$$\int_U e^{-\psi} dV_{g_\infty} = +\infty \tag{7.3}$$

By the definition of ψ in (7.2), for any tangent vector v of type (1.0) at any point of X, we have

$$\left\langle \partial\bar{\partial}\psi + \frac{14\pi}{\sqrt{-1}}\omega_{g_\infty}, v \wedge \bar{v} \right\rangle_{g_\infty} \geqq \|v\|_{g_\infty}^2 \tag{7.4}$$

Thus by Proposition 5.1, in order to have a global section of $6\,K_{M_\infty}^{-1}$ nonvanishing at x_1, it suffices to construct a nonvanishing local section of $6K_{M_\infty}^{-1}$ in neighborhood of x_1. It is obviously possible since x_1 is a singular point of type $C^2/Z_{l,p,1}$ with $1 \leqq l \leqq 3$, $2 \leqq p \leqq 3$. Then the lemma is proved.

Lemma 7.2. *Let (M_∞, g_∞), F_j, $\tilde{D}_1$, etc. be given as in Proposition 7.1. Then the irreducible component in F_1 intersecting $\tilde{D}_1$ has the multiplicity one in $\tilde{B}_1$.*

Proof. We prove it by contradiction. Assume that the conclusion of this lemma is not true. First let F_1 be irreducible, i.e. x_1 is of type $C^2/Z_{1,2,1}$. Then by Lemma 4.10 and adjunction formula $K_{M_\infty}^{-1} \cdot F_1 = F_1^2 + 2$, there are at least five irreducible components $C_\alpha (1 \leqq \alpha \leqq 5)$ in $\tilde{B}_1$ such that $C_\alpha \cdot F_1 = 1$. Obverse that the pluri-anticanonical divisor $K_{M_\infty}^{-6}$ is an ample Cartier one, so

$$\sum_{\alpha=1}^{5} \int_{C_\alpha} \omega_{g_\infty} \geqq \sum_{\alpha=1}^{5} \frac{1}{6} = \frac{5}{6} \tag{7.5}$$

Choose two generic divisors $\tilde{D}_1, \tilde{D}_1'$ in $|\tilde{D}_1|$ such that $\tilde{D}_1$ intersects $\tilde{D}_1'$ at $8-n$ distinct points outside $\tilde{B}_1$. Let S' be the section in $H^0(M_\infty, K_{M_\infty}^{-1})$ such that

$\{S'=0\}$ is just the sum of $\pi(\tilde{D}'_1)$ and B_1. Thus one can compute that

$$\sum_{z\in M_\infty}\frac{1}{\deg(\pi_z)}i_o(\pi_z^{-1}\{S'=0\},\pi_z^{-1}(\pi(\tilde{D}_1)))\geqq 8-n+\frac{1}{2} \tag{7.6}$$

where $\pi_z:\tilde{U}_z\to U_z$ is the local uniformization of M_∞ at z with $\pi_z(o)=z$. Combining (7.6) with (7.5), we get a contradiction to Lemma 6.1. So x_1 can not be of type $C^2/Z_{1,2,1}$.

If x_1 is of type $C^2/Z_{2,2,1}$, then by the same arguments as above, we can prove that B_1 does not contain more than three irreducible components. Write $\tilde{B}_1=2F_{11}+\alpha F_{12}+\tilde{B}'_1$ where $\tilde{B}'_1$ is an effective divisor having no common component with F_1. Note that any irreducible component E in $\tilde{B}'_1$ with $E\cdot F_1>0$ can not be contracted to a point by the projection $\pi:\tilde{M}_\infty\to M_\infty$, in particular, $E^2=-1$. Moreover, if $\pi(E)$ does not pass through a singular point of type $C^2/Z_{1,3,1}$, then $\int_{\pi(E)}\omega_{g_\infty}\geqq\frac{1}{2}$ and B_1 contains at most one irreducible component. By adjunction formula, it contradicts to the fact that F_{11} has multiplicity two in $\tilde{B}_1$ and $F_1\cdot\tilde{D}_1=1$. So for any such an E, $\pi(E)$ must pass through a singular point of type $C^2/Z_{1,3,1}$. Then one can easily show that $\alpha=1$ and $B_1=3\pi(E)$, where E is an irreducible component intersecting F_{11}. Let $\pi_1:\tilde{U}_{x_1}\to U_{x_1}$ be the local uniformization of M_∞ at x_1. Then any section S in $H^0(M_\infty,K_{M_\infty}^{-1})$ is locally presented by a holomorphic function f_S on $\tilde{U}_{x_1}$ of form $(h_E)^3\tilde{f}_S$, where h_E is the defining function of $\pi_1^{-1}(E)$ in $\tilde{U}_{x_1}$ and $\deg_o(\tilde{f}_S)\geqq 1$, $\deg_o(h_E)\geqq 1$. Since f_S is also in $IP(2,2,1)$, there is at most one monomial term in f_S with degree less than 5. Thus we can choose two divisors $\tilde{D}_1,\tilde{D}'_1$ in $|\tilde{D}_1|$ such that $\pi(\tilde{D}_1)$ intersects $\{S'=0\}$ at $8-n$ distinct points outside B_1 and $\deg_o(f_{S'})\geqq 5$, where S' is the section in $H^0(M_\infty,K_{M_\infty}^{-1})$ with $\{S'=0\}=\pi(\tilde{D}'_1)\cup B_1$. By Lemma 6.1,

$$\begin{aligned}9-n&=3\int_{\pi(E)}\omega_{g_\infty}+\int_{\pi(\tilde{D}_1)}\omega_{g_\infty}\\&\geqq\frac{1}{2}+8-n+\frac{1}{\deg(\pi_1)}i_o(\pi_1^{-1}(\pi(\tilde{D}_1)),\{f_{S'}=0\})\\&\geqq\frac{1}{2}+8-n+\frac{5}{8}>9-n\end{aligned}$$

A contradiction! Thus x_1 is not of type $C^2/Z_{2,2,1}$, either.

Next, we assume that x_1 is of type $C^2/Z_{3,2,1}$, then $n=6$. Let $\tilde{B}_1=\alpha F_{11}+\beta F_{12}+\gamma F_{13}+\tilde{B}'_1$, where $F_{11}^2=F_{13}^2=-3$, $F_{12}^2=-2$ and $\tilde{B}'_1$ does not contain $F_{1j}(1\leqq j\leqq 3)$ any more.

Claim 1. The generic $\tilde{D}_1$ intersects F_{12}.

If the claim is not true, then we may assume that $\tilde{D}_1\cdot F_{11}=1$, so $\alpha=2,\beta\geqq 2$. If $\gamma=1$, then $\beta=2$. By adjunction formula, we have $\tilde{B}'_1\cdot F_{11}=2$, $\tilde{B}'_1\cdot F_{12}=1$, $\tilde{B}'_1\cdot F_{13}=0$. Let E' be the exceptional curve in $\tilde{B}'_1$ intersecting with F_{12}, then $E'\cdot(\tilde{B}_1-2F_{12})=0$. It follows that $\pi(E')$ does not pass through any singular point other than x_1 and rational double points, so $\int_{\pi(E')}\omega_{g_\infty}\geqq\frac{1}{2}$. By Lemma 6.1, it implies that $B_1=\pi(E')$ and then $\tilde{B}'_1=E'$. It contradicts to $\tilde{B}'_1\cdot F_{11}=2$. Thus $\gamma\geqq 2$. We observe that no E in $\tilde{B}'_1$ intersects two components of F_1 by adjunction formula or

Lemma 6.1. So if we let k_j be the number of irreducible components in $\tilde{B}'_1$ intersecting with $F_{1j}(1 \leqq j \leqq 3)$, then $k_1 + k_2 + k_3 \leqq 3$ by Lemma 6.1. By adjunction formula, we have

$$\begin{cases} 1 - 6 + \beta + k_1 = -1 \\ 2 - 2\beta + \gamma + k_2 = 0 \\ \beta - 3\gamma + k_3 = -1 \end{cases} \tag{7.7}$$

Summing these three equations, we derive $\gamma \leqq 1$. A contradiction. The claim is proved.

Then $\beta = 2$ and $\alpha + \gamma \leqq 3$. By the above arguments in excluding that x_1 is of type $C^2/Z_{2,2,1}$, one can prove that $\alpha = 1$, $\gamma = 1$. Thus $B_1 = F_{11} + 2F_{12} + F_{13} + E$, where E is an exceptional curve of first kind and $E \cdot F_{12} = 1$. There are now two methods to conclude a contradiction. One of them is to use Lemma 6.1. We can easily choose two divisors D_1, D'_1 in $|D_1|$ such that they intersect to each other exactly at $8 - n$ distinct points besides x_1 and $i_o(\pi_1^{-1}(D_1), \pi_1^{-1}(D'_1 + B_1)) \geqq 7$, where $\pi_1 : \tilde{U}_1 \to U_{x_1}$ is a local uniformization of x_1. It will contradict to formula (6.1) in Lemma 6.1. There is another method described as follows. Let $\tilde{M}^0_\infty$ be the surface obtained by blowing down E and then F_{12} in $\tilde{M}_\infty$, and F^0_{1j} be the images of F_{1j} ($j = 1, 3$) under the natural projection from $\tilde{M}_\infty$ onto $\tilde{M}^0_\infty$. Then $(F_{1j})^2 = -2(j = 1, 3)$. Inductively, let $\tilde{M}^k_\infty$ be the surface obtained by blowing down an exceptional curve intersecting either F^{k-1}_{11} or F^{k-1}_{13}. Put $F^k_{1j} = \pi'_k(F^{k-1}_{1j})(j = 1, 3)$, where $\pi'_k : \tilde{M}^{k-1}_\infty \to \tilde{M}^k_\infty$ is the natural projection. Let $\tilde{M}^m_\infty$ be the last surface obtained in such a process. Then $(K_{\tilde{M}^m_\infty})^2 = 4 + m$. By $h^0(\tilde{M}_\infty, K^{-1}_{\tilde{M}_\infty}) = 4$, we have $m \leqq 4$. It is well-known that the relatively minimal rational surface are exactly CP^2 and Hirzebruch surfaces $\sum_l (l \geqq 0)$ (cf. [BPV]). In particular, it follows that $m = 4$ and $(F^m_{1j})^2 = 0$ ($j = 1, 3$). So $M^m_\infty = \sum_0$. Now any anticanonical divisor of $K^{-1}_{\tilde{M}_\infty}$ descends to the one of $K^{-1}_{\Sigma_0}$ containing F^m_{11}, F^m_{13} and $F^m_{11} \cap F^m_{13}$ with multiplicity $\geqq 3$. It follows that $h^0(\tilde{M}_\infty, K^{-1}_{\tilde{M}_\infty}) \leqq 3$. This contradicts to the fact that $h^0(\tilde{M}_\infty, K^{-1}_{\tilde{M}_\infty}) = 4$.

Hence x_1 can only be of type $C^2/Z_{1,3,1}$. By the same arguments as in proving Claim 1, one can easily show that $F_{11} \cdot \tilde{D}_1 = 1$, $F_{12} \cdot \tilde{D}_1 = 0$ and $F^2_{11} = -2$. By adjunction formula, we have two exceptional curves E_1, E_2 intersecting F_{11} outside F_{12} (E_1 may coincide with E_2). In particular, it follows that any section S in $H^0(M_\infty, K^{-1}_{M_\infty})$ is locally represented by a holomorphic function $f_S \in IP(1, 3, 1)$ on $\tilde{U}_1$ with $\operatorname{ord}_o(f_S) \geqq 3$, where $\pi_1 : \tilde{U}_1 \to U_1$ is a local uniformization of M_∞ at x_1, moreover, we can write $f_S = h_{E_1} h_{E_2} \tilde{f}_S$, where $\deg_o(\tilde{f}_S) \geqq 1$ and h_{E_i} is the defining function of $\pi_1^{-1}(\pi(E_i))$ for $i = 1, 2$. Then we can choose two divisors $\tilde{D}_1$ and $\tilde{D}'_1$ in $|\tilde{D}_1|$ such that $\tilde{D}_1$ has no common component with $\tilde{D}'_1$ and $\tilde{B}_1$ and intersects with $\tilde{D}'_1$ at $8 - n$ points outside $\tilde{B}_1$, and the divisor $\pi(\tilde{D}'_1) + B_1$ defines a section S' in $H^0(M_\infty, K^{-1}_{M_\infty})$ with $\deg_o(f_{S'}) \geqq 7$. Thus by Lemma 6.1 and the fact that $6K^{-1}_{M_\infty}$ is a Cartier divisor,

$$\begin{aligned} 9 - n &= \int_{E_1} \omega_{g_\infty} + \int_{E_2} \omega_{g_\infty} + \int_{\tilde{D}_1} \omega_{g_\infty} \\ &\geqq \frac{1}{3} + \int_{D_1} \omega_{g_\infty} \end{aligned}$$

$$\geqq \frac{1}{3} + (8-n) + \frac{1}{\deg_o(\pi_1)} i_o(\pi_1^{-1}(\pi(\tilde{D}_1)), \{f_{S'} = 0\})$$

$$\geqq \frac{1}{3} + (8-n) + \frac{7}{9} > 9 - n$$

A contradiction! The lemma is proved.

Now we consider the last case that $F_1 \cdot \tilde{D}_1 = 1$, $F_2 \cdot \tilde{D}_1 = 1$ and $F_j \cdot \tilde{D}_1 = 0$ for $j \geqq 3$.

Lemma 7.3. *Let (M_∞, g_∞), F_j, $\tilde{D}_1$ be given as in Proposition 7.1. Suppose that $F_1 \cdot \tilde{D}_1 = 1$, $F_2 \cdot \tilde{D}_1 = 1$. Then M_∞ has exactly two singular points besides rational double points, one of them is of type $C^2/Z_{l,2,1}$ $(1 \leqq l \leqq 3)$ and another one is of type $C^2/Z_{1,3,1}$. Moreover,*

$$\tilde{B}_1 = F_1 + F_2 + E, \quad B_1 = \pi(E) \tag{7.8}$$

where E is an exceptional curve of first kind in minimal resolution $\tilde{M}_\infty$ of M_∞.

Proof. By adjunction formula, it can be proved that the connected component in $\tilde{B}_1$ containing F_1 is a chain of rational curves ending at F_2. Since the generic divisors in $|\tilde{D}_1|$ do not intersect $F_j (j \geqq 3)$ if such F_j exist and the anticanonical divisor in M_∞ is connected, we conclude that $\tilde{B}_1$ is a chain of rational curves with F_1 and F_2 as two ends. By Lemma 6.1, we can further conclude that $\tilde{B}_1 = F_1 + F_2 + E$ and E is an exceptional curve in $\tilde{M}_\infty$ of first kind. Also by Lemma 6.1, one can easily prove that at least one of x_1, x_2 in M_∞ must be of type $C^2/Z_{1,3,1}$. We assume that x_2 is of type $C^2/Z_{1,3,1}$.

Claim. The singular point x_1 can not be of type $C^2/Z_{1,3,1}$.

In fact, if x_1 is of type $C^2/Z_{1,3,1}$, then by Lemma 6.1, one can easily show that $\tilde{D}_1 \cdot F_{l2} = 1$, $F_{l1} \cdot F_{l2} = 1$, $F_{l1} \cdot E = 1$, $F_{l1}^2 = -2$, $F_{l2}^2 = -5 (l = 1, 2)$. Let $\tilde{M}'_\infty$ be the surface obtained by blowing down E and F_{11} in $\tilde{M}_\infty$ successively. If $\pi_1 : \tilde{M}_\infty \to \tilde{M}'_\infty$ is the natural projection, then $\pi_1(\tilde{D}_1)^2 = \tilde{D}_1^2 = 8 - n$ and $\pi_1(F_{21})^2 = 0$, $\pi_1(F_{21}) \cdot \pi_1(\tilde{D}_1) = 0$. This $\pi_1(F_{21})$ induces a fibration of $\tilde{M}'_\infty$ over CP' with generic fiber CP^1 such that $\pi_1(\tilde{D}_1)$ is contained in fibers. It contradicts to the fact that the generic $\tilde{D}_1$ is irreducible and $\pi_1(\tilde{D}_1)^2 \geqq 2$. So the claim is true.

Thus x_1 is of type $C^2/Z_{l,2,1}$, where $1 \leqq l \leqq 3$ and $l = 3$ only if $n = 6$. The lemma is proved.

Lemma 7.4. *Let (M_∞, g_∞), E, F_1, F_2, $\tilde{D}_1$ be given as in last lemma and $\tilde{B}_1 = F_1 + F_2 + E$. Then $|6K_{M_\infty}^{-1}|$ is free of base point.*

Proof. We first remark that by Lemma 6.1, the generic divisor in $|D_1|$ does not intersect B_1 outside the singular points x_1, x_2. By Lemma 7.3, we have $B_1 = \pi(E)$, where E is defined there.

Claim 1. $B_2 = \varnothing$.

We prove this claim by contradiction. Suppose that $B_2 = \varnothing$, then $B_1 \subset B_2$. Fix an irreducible divisor in $|D_1|$, say D_1 for simplicity. Then a 2-anticanonical divisor in $|2K_{M_\infty}^{-1}|$ is the sum of two anticanonical divisors if it contains D_1. On the other hand, by Lemma 5.2, we have $h^0(M_\infty, K_{M_\infty}^{-1}) = 10 - n$,

$h^0(M_\infty, 2K_{M_\infty}^{-1}) = 28 - 3n$, where $n = 5$ or 6. Thus there is a global section S_2^∞ of $H^0(M_\infty, 2K_{M_\infty}^{-1})$ such that $\{S_2^\infty = 0\}$ does not contain D_1 and intersects with D_1 at $17 - 2n$ points outside B_1. By Lemma 7.3, we may assume that x_1 is the singular point of type $C^2/Z_{l,2,1}$ $(1 \leqq l \leqq 3)$. Let $\pi: \tilde{U}_{x_1} \to U_{x_1}$ be the local uniformization of M_∞ with $\pi_1(0) = x_1$ and f_2 be the holomorphic function locally representing $\pi_1^* S_2^\infty$ in $\tilde{U}_{x_1}$. Then f_2 is invariant under the action of $Z_{l,2,1}$, i.e., $\sigma^* f_2 = f_2$, where $\sigma \in Z_{l,2,1}$ is the generator. Then we can choose a local coordinate system (z_1, z_2) on $\tilde{U}_{x_1}$ such that $\pi_1^{-1}(D_1) = \{z_1 = 0\}$ and $\sigma = \sigma_{l,2,1}$ as defined in Lemma 5.5. It follows from σ-invariance of f_2 and f_2 is a holomorphic function on z_1^{4l}, z_2^{4l}, $(z_1 z_2)^2$, $z_1^{2l+1} z_2$, $z_1 z_2^{2l+1}$. Then one can easily deduce

$$i_0(\{f_2 = 0\}, \pi_1^{-1}(D_1)) \geqq 4l \tag{7.9}$$

By Lemma 6.1 and the above (7.9), we have

$$\begin{aligned} 18 - 2n &= 2\int_{B_1} \omega_{g_\infty} + 2\int_{D_1} \omega_{g_\infty} \\ &\geqq \frac{1}{3} + 17 - 2n + \frac{1}{4l} i_0(\{f_2 = 0\}, \pi_1^{-1}(D_1)) \\ &\geqq \frac{1}{3} + 18 - 2n \end{aligned} \tag{7.10}$$

A contradiction! The claim is proved.

The above claim implies that the base locus of $|2K_{M_\infty}^{-1}|$ consists of finitely many points. Define

$$\psi = 3\log\left(\sum_{\beta=0}^{N_2} \|S_{2\beta}^\infty\|_{g_\infty}^2 \right) \tag{7.11}$$

where $N_2 = h^0(M_\infty, 2K_{M_\infty}^{-1}) - 1$ and $\{S_{2\beta}^\infty\}$ is an orthonormal basis of $H^0(M_\infty, 2K_{M_\infty}^{-1})$ with respect to g_∞. Then ψ is smooth outside the base locus of $|2K_{M_\infty}^{-1}|$. The rest of the proof is exactly same as that from (7.2) to the end in the proof of Lemma 7.1.

Now Proposition 7.2 follows from Lemma 7.2, 7.2 and 7.4. Then the proof of Theorem 2.2 is completed.

The discussions in previous sections also yield the following result on the degeneration of Kähler-Einstein surfaces with positive scalar curvature.

Theorem 7.1. *Let $\{(M_i, g_i)\}$ be a sequence of Kähler-Einstein surfaces with $C_1(M_i)^2 = 9 - n$ $(5 \leqq n \leqq 8)$. Then by taking the subsequence, we may have that $(M - i, g_i)$ converge to a Kähler-Einstein orbifold (M_∞, g_∞) in the sense of Proposition 4.2 satisfying:*

(1) *if $n = 8$, then M_∞ has at most one singular point of type $C^2/Z_{l,2,1}$ $(2 \leqq l \leqq 7)$ besides rational double points;*
(2) *if $n = 7$, then M_∞ has either only rational double points or two singular points of type $C^2/Z_{1,2,1}$ besides rational double points;*
(3) *if $n = 5, 6$, M_∞ has at most two singular points of type $C^2/Z_{l,2,1}$ or $C^2/Z_{1,3,1}$*

$(1 \leqq l \leqq 3)$ *besides rational double points, moreover, in case* M_∞ *has two of such singular points, one of them must of type* $C^2/Z_{l,2,1}$, *while another one is of type* $C^2/Z_{1,3,1}$.

This theorem generalizes some results of M. Anderson [An] and Nakajima [Na] on the Hausdorff convergence of Einstein 4-manifolds with positive scalar curvature in case of complex geometry. Precisely, this theorem gives the reduction of quotient singular points in the limit Einstein orbifold which is the Hausdorff limit of a sequence of Kähler-Einstein surfaces. As we have already seen, such a reduction is, in general, completely nontrivial. In fact, we expect that M_∞ in Theorem 7.1 has only rational double points as singular points. If it is true, then we have stronger partial C^0-estimate than that in Theorem 2.2 and can simplify a lot of technical computations in section 2 and Appendices.

Appendix 1. Proof of Lemma 2.4

In this appendix, we will prove Lemma 2.4 stated in section 2. First we will prove a proposition concerning the evaluation of rational integrals.

Let f be a holomorphic function defined in the ball $B_R(o) \subset C^2$ with center at the origin o. For simplicity, take $R = 1$. For any $\varepsilon \in (0, \frac{1}{2})$, $x \in B_{\frac{1}{2}}(o)$ and $\alpha > 0$, we write

$$I_\varepsilon(f, \alpha, x) = \int_{B_\varepsilon(x)} \frac{dV}{|f|^{2\alpha}}$$

where dV denotes the standard euclidean volume form on C^2.

Then we can associate a local analytic invariant $\alpha_x(f)$ to f at any point x in $B_{\frac{1}{2}}(o)$ as follows,

$$\alpha_x(f) = \sup\{\alpha \mid \exists \varepsilon > 0, \text{ s.t. } I_\varepsilon(f, \alpha, x) < \infty\} \qquad \text{(A.1.1)}$$

Note that $\alpha_x(f)$ is independent of choices of local holomorphic coordinates at x.

We would like to evaluate $\alpha_x(f)$ in terms of the geometry of Z_f at x, where Z_f is the zero locus of f in $B_1(o)$. Obviously, if $x \notin Z_f$, then $\alpha_x(f) = +\infty$. Furthermore, by some elementary computations, one can easily check that if x is the smooth point of the reduced curve $(Z_f)_{\text{red}}$ of Z_f in $B_{\frac{1}{2}}(o)$ and m is the multiplicity of Z_f at x, then $\alpha_x(f) = \frac{1}{m}$. Note that $(Z_f)_{\text{red}}$ is defined as follows, write $Z_f = \alpha_1 Z_1 + \ldots + \alpha_k Z_k$, where $Z_i (1 \leqq i \leqq k)$ are distinct irreducible components of Z_f in $B_{\frac{1}{2}}(o)$, then $(Z_f)_{\text{red}} = Z_1 + \ldots + Z_k$. Therefore, it suffices to evaluate $\alpha_x(f)$ at the singular point of $(Z_f)_{\text{red}}$ in $B_{\frac{1}{2}}(o)$. Without losing generality, we may assume that o is the unique singular point of $(Z_f)_{\text{red}}$ in $B_{\frac{1}{2}}(o)$.

Given any local coordinates (z_1, z_2) of C^2 at o, one can expand f in a power series $\sum_{i,j \geqq 0} a_{ij} z_1^i z_2^j$ in a small neighborhood of o. Then we define the Newton polyhedron $N(f)$ of f as the convex hull of the set $\{(i,j) \in R_+ \times R_+, (0, +\infty), (+\infty, 1) \mid a_{ij} \neq 0\}$ in R^2. The boundary $\partial N(f)$ intersects with the line $\{x = y\}$ in R^2 at a point (x_f, y_f), where x, y are euclidean coordinates of R^2, $x_f = y_f$. This x_f is called in [AGV] the remotedness of $N(f)$, denoted by $r(N(f))$. Since $N(f)$ obviously depends on the choice of local coordinates C^2 at o, so does $r(N(f))$. However, we have

Proposition A.1.1. *Let f be given as above. Then there are a sequence of coordinate system (z_1^m, z_2^m) of C^2 at o such that for the associated Newton polyhedron $N_m(f)$ in the coordinate system (z_1^m, z_2^m), we have $r(N_m(f)) \leqq r(N_{m'}(f))$ if $m \leqq m'$ and $\alpha_o(f) = \lim_{m \to \infty} r(N_m(f))^{-1}$.*

Proof. It is clear that $\alpha_o(f) \leqq \frac{1}{m_o}$. Write $f = f_k + f_{k+1} + \ldots$, where each f_j is the homogeneous component of f with degree j, $k = \text{mult}_o(f)$. Obviously, $k_0 \geqq m_0$. Take $\alpha < m_0^{-1}$, then for any

$\varepsilon > 0$,

$$\int_{B_{\frac{1}{2}}(o)\setminus B_\varepsilon(o)} \frac{dV}{|f|^{2\alpha}} < +\infty \tag{A.1.2}$$

By a linear transformation, we may assume that $f_k = z_1^a z_2^b \prod_{i=1}^l (z_1 + \mu_i z_2)^{c_i}$ with $\mu_i \neq 0$ distinct, $a \geqq b \geqq c_i$ $(i = 1, 2, \ldots, l)$ and $a + b + \sum_{i=1}^l c_i = k$. Let $N(f)$ be the associated Newton polyhedron in R^2 of f in this coordinate system.

Case 1. $\partial N(f) \cap \{x = y\} \cap \{x + y = k\} \neq \emptyset$. Note that in this proof we always use x, y to denote the coordinates of R^2, z_1, z_2 to denote the coordinates of C^2.

In this case, we will prove that $\alpha_0(f) = 2/k$. In fact, for $\delta > 0$ sufficiently small, we have

$$I_\delta(f, \alpha, o) = \int_{B_\delta(o)} \frac{dV}{|f|^{2\alpha}} = \int_{\substack{|x| \leqq |y| \\ |x|^2 + |y|^2 \leqq \delta}} \frac{dV}{|f|^{2\alpha}} + \int_{\substack{|y| \leqq |x| \\ |x|^2 + |y|^2 \leqq \delta}} \frac{dV}{|f|^{2\alpha}} \tag{A.1.3}$$

$$\leqq \int_{\substack{|y| \leqq \delta \\ |\xi| \leqq 1}} \frac{dy \wedge d\bar{y} \wedge d\xi \wedge d\bar{\xi}}{|y|^{2k\alpha - 2} \left| \xi^a \prod^l (\xi + \mu_i)^{c_i} + \sum_{m \geqq k+1} y^{-k} f_m(y\xi, y) \right|^{2\alpha}}$$

$$+ \int_{\substack{|x| \leqq \delta \\ |\eta| \leqq 1}} \frac{dx \wedge d\bar{x} \wedge d\eta \wedge d\bar{\eta}}{|x|^{2k\alpha - 2} \left| \eta^b \prod_{i=1}^l (1 + \mu_i \eta)^{c_i} + \sum_{m \geqq k+1} x^{-k} f_m(x, x\eta) \right|^{2\alpha}}$$

It follows that $I_\delta(f, \alpha, o) < +\infty$ only if $\alpha < \frac{2}{k}$. On the other hand, by our assumption, $b + \sum_{i=1}^l c_i \geqq a$, $a \geqq b \geqq c_i$, so $\max\{c_i, b, a\} \leqq \frac{k}{2}$. Thus both polynomials $\xi^a \prod_{i=1}^l (\xi + \mu_i)^{c_i}$ and $\eta^b \prod_{i=1}^l (1 + \mu_i \eta)^{c_i}$ have roots with multiplicity $\leqq \frac{k}{2}$. We also have

$$\sum_{m \geqq k+1} y^{-k} f_m(y\xi, y) = 0 \quad \text{at } y = 0$$

$$\sum_{m \geqq k+1} x^{-k} f_m(x, x\eta) = 0 \quad \text{at } x = 0$$

Therefore, for δ sufficiently small, both integrals in (A.1.3) are finite if $\alpha < \frac{2}{k}$. So $\alpha_o(f) = \frac{2}{k}$. Actually, we have already proved that $I_\delta(f, \alpha, o)$ has a upper bound depending on δ, α and the upper bound of $|f|$ in $B_{\frac{1}{2}}(o)$.

Case 2. $\partial N(f) \cap \{x = y\} \cap \{x + y = k\} = \emptyset$.

Now $a > b + \sum_{i=1}^l c_i$. Let $L(u, v) = \{ux + vy = 1\}$ be the unique line containing the segment of $\partial N(f)$ having nonempty intersection with $\{x = y\}$ in R^2. Thus $\alpha_o(f) = r(N(f))^{-1}$. In the following, we may assume that $L(u, v)$ is not vertical. In fact, if $L(u, v)$ is vertical, then by Fubini theorem, one can easily check that $I_\varepsilon(f, \alpha, o)$ is finite iff $\alpha < r(N(f))^{-1}$. Note that $r(N(f))$ is the distance of the line $L(u, v)$ from y-axis in R^2. Let (i, j) and (i', j') be the two end points of $L(u, v) \cap \partial N(f)$ with $i > j$, $i' < j'$. We further have that $i > i'$, $j < j'$ and $i' + j' \geqq i + j$. A simple computation shows

$$u = \frac{j' - j}{ij' - ji'}, \quad v = \frac{i - i'}{ij' - ji'}$$

Then there are integers $\tilde{\alpha}$, $\tilde{\beta}$ $\tilde{\gamma}$ such that $u = \frac{\tilde{\alpha}}{\tilde{\gamma}}$, $v = \frac{\tilde{\beta}}{\tilde{\gamma}}$ and $\tilde{\alpha}$, $\tilde{\beta}$ are coprime.

Define a polynomial f_L by $\sum_{(k,l)\in L(u,v)\cap \partial N(f)} a_{kl} z_1^k z_2^l$. There are at most i terms in f_L. This polynomial f_L has the decomposition,

$$f_L(z_1, z_2) = c z_1^{i'} z_2^{j} \prod_{k=1}^{v} (z_1^{\tilde{\beta}} + d_k z_2^{\tilde{\alpha}})^{\mu k} \tag{A.1.4}$$

where $c \neq 0$ and $d_1, \ldots, d_v$ are distinct constants. Note that $\sum_{k=1}^{v} \tilde{\beta}\mu_k + i' = i$, $\sum_{k=1}^{v} \tilde{\alpha}\mu_k + j = j'$. For $\delta > 0$ sufficiently small, let $\delta_1 = \delta^{\tilde{\alpha}}$, $\delta_2 = \delta^{\tilde{\beta}}$, $\delta' = \min\{\delta_1, \delta_2\}$, then we have

$$\begin{aligned} I_{\delta'}(f, \alpha, o) &\leqq \int_{\substack{|z_1| \leqq \delta_1 \\ |z_2| \leqq \delta_2}} \frac{dV}{|f(z_1, z_2)|^{2\alpha}} = \tilde{\alpha}\tilde{\beta} \int_{\substack{|w_1| \leqq \delta \\ |w_2| \leqq \delta}} \frac{|w_1|^{2\tilde{\alpha}-2} |w_2|^{2\tilde{\beta}-2} dw_1 \wedge d\bar{w}_1 \wedge dw_2 \wedge d\bar{w}_2}{|f(w_1^{\tilde{\alpha}}, w_2^{\tilde{\beta}})|^{2\alpha}} \\ &= \tilde{\alpha} \int_{\substack{|w_1| \leqq \delta \\ |\eta| \leqq 1}} \frac{dw_1 \wedge d\bar{w}_1 \wedge d\eta \wedge d\bar{\eta}}{|w_1|^{2\alpha\tilde{\gamma} - 2\tilde{\alpha} - 2\tilde{\beta} + 2} \left| c\eta^j \prod_{k=1}^{v} (1 + d_k \eta)^{\mu_k} + w_1^{-\tilde{\gamma}} \tilde{f}(w_1^{\tilde{\alpha}}, w_1^{\tilde{\beta}} \eta) \right|^{2\alpha}} \\ &+ \tilde{\beta} \int_{\substack{|w_2| \leqq \delta \\ |\xi| \leqq 1}} \frac{dw_2 \wedge d\bar{w}_2 \wedge d\xi \wedge d\bar{\xi}}{|w_2|^{2\alpha\tilde{\gamma} - 2\tilde{\alpha} - 2\tilde{\beta} + 2} \left| c\xi^{i'} \prod_{k=1}^{v} (\xi + d_k)^{\mu_k} + w_1^{-\tilde{\gamma}} \tilde{f}(w_2^{\tilde{\alpha}} \xi, w_2^{\tilde{\beta}}) \right|^{2\alpha}} \end{aligned} \tag{A.1.5}$$

where $\tilde{f} = f - f_L$. It follows immediately from (A.1.5) that $I_{\delta'}(f, \alpha, 0) < +\infty$ only if $\alpha < \dfrac{\tilde{\alpha} + \tilde{\beta}}{\tilde{\gamma}}$.

By the definition of $r(N(f))$, we have $i', j \leqq r(N(f)) = \dfrac{\tilde{\gamma}}{\tilde{\alpha} + \tilde{\beta}}$. Since $w_1^{-\tilde{\gamma}} \tilde{f}(w_1^{\tilde{\alpha}}, w_1^{\tilde{\beta}} \eta) = 0$ at $w_1 = 0$ and $w_2^{-\tilde{\gamma}} \tilde{f}(w_2^{\tilde{\alpha}} \xi, w_2^{\tilde{\beta}}) =$ at $w_2 = 0$, one can easily show that $I_{\delta'}(\alpha, f, 0) < \infty$ if $\max_{1 \leqq k \leqq v}\{\mu_k\} < \dfrac{1}{\alpha}$ and $\alpha < r(N(f))^{-1}$. Thus if $\mu_k \leqq r(N(f))$ for all k, then $I_{\delta'}(\alpha, f, 0) < \infty$ iff $\alpha < (r(N(f)))^{-1}$, i.e., $\alpha_o(f) = r(N(f))^{-1}$ and the proposition is proved. Otherwise, there is a $\mu_k > r(N(f))$. For simplicity, say $k = 1$, i.e., $\mu_1 > r(N(f))$. Then by the fact that $\dfrac{\tilde{\gamma}}{\tilde{\alpha} + \tilde{\beta}} > \dfrac{i}{2}$, we derive $\tilde{\beta} = 1$, $\tilde{\alpha} \geqq 2$. Note that other μ_k for $k \geqq 2$ are all less than $r(N(f))$.

Making a transformation $(z_1, z_2) \to (z_1 + \mu_1 z_2^{\tilde{\alpha}}, z_2)$, we obtain a new local holomorphic function g at o with f_k as the first homogeneous term g_k. Also we observe that all points on $\partial N(f)$ below (i, j) are unchanged and are still the ones on $\partial N(g)$ below $(i, j) \in \partial N(g)$. Put $N_0(f) = N(f)$, $N_1(f) = N(g)$. Note that g is just the function f in the new coordinate system. The above process can be carried out successively to obtain Newton polyhedrons $N_0(f), N_1(f), \ldots$, unless for some m, $\alpha_0(f) = r(N_m(f))^{-1}$ and the proposition is proved. Suppose that such a m does not exists, then we have a sequence of Newton polyhedrons $N_0(f), N_1(f), \ldots$ in R^2. Moreover, the parts of $\partial N_m(f)$ $(0 \leqq m < \infty)$ below the line $\{x = y\}$ are all same. Then one can easily check that $\lim_{m\to\infty} r(N_m(f)) = i$. By previous discussions, $I_\delta(f, \alpha, 0) < +\infty$ for sufficiently small $\delta > 0$ if $\alpha < i^{-1}$, and $\alpha_o(f) \leqq r(N_m(f))^{-1}$ for all m. Thus $\alpha_o(f) = i^{-1}$, the proposition is proved.

Lemma A.1.2. *Let M be a smooth complete intersection of two quadratic polynomials in CP^4 and S be a global section in $H^0(M, K_M^{-6})$ such that its zero locus $Z(S)$ contains no curve with multiplicity 9 and $Z(S)_{\mathrm{red}}$ is not a union of two lines and a curve of degree 2 intersecting at one point. Then there is an $\varepsilon > 0$ such that for any $\alpha \leqq \frac{1}{9} + \varepsilon$,*

$$\int_M \frac{dV_{\tilde{g}}}{\|S\|_{\tilde{g}}^{2\alpha}} < +\infty \tag{A.1.6}$$

where $\tilde{g}$ is any fixed Kähler metric on M.

Proof. By our assumption, there are finitely many points $x_1, \ldots, x_l \in M$ such that for any $\delta > 0$,

$\alpha < \frac{3}{4}$ (cf. Fact(†) in the proof of Lemma 2.3),

$$\int_{M\setminus\bigcup_{i=1}^{l} B_\delta(x_i, \tilde g)} \frac{dV_{\tilde g}}{\|S\|_{\tilde g}^{\alpha/3}} < +\infty \tag{A.1.7}$$

where $B_\delta(x_i, \tilde g)$ is the geodesic ball at x_i with respect to $\tilde g$. At each point x_i, the section S is locally represented by a holomorphic function f_i, we define $\alpha_{x_i}(S) = \alpha_{x_i}(f_i)$. This $\alpha_{x_i}(S)$ is in fact independent of the choice of local representations of S. Then one can easily see that the lemma is equivalent to $\alpha_{x_i}(S) > \frac{1}{9} + 2\varepsilon$ for $1 \leqq i \leqq l$, Suppose that the lemma is false, then there is a x_i, say x_1, such that $\alpha_{x_i}(S) \leqq \frac{1}{9}$. We will derive a contradiction to our assumption on S.

Let (z_1, z_2) be the local holomorphic coordinates of M at x_1, and S be represented by a local holomorphic function f_S, then

$$\mathrm{mult}_o(f_S) \geqq \alpha_o(f_S)^{-1} = \alpha_{x_1}(S)^{-1} = 9\,. \tag{A.1.8}$$

Claim 1. There are at most one line of M through the point x_1.

We prove it by contradiction. Suppose that there are two lines L_1, L_2 of M through x_1, then there is a unique anticanonical divisor $D = L_1 + L_2 + E$, here E is a curve of degree 2 with respect to K_M^{-1} and $x_1 \in E$. Note that by smoothness of M, L_1, L_2, E must intersect to each other transversally at x_1. By $K_M^{-1}\cdot L_i = 6$ for $i = 1, 2$, we have that $2L_1 + 2L_2 \subset Z(S)$. Let l_1, l_2 be local defining functions of L_1, L_2, then $f_S = l_1^2 l_2^2 h$. By Hölder inequality, the facts that $\alpha_{x_1}(S) \leqq \frac{1}{9}$ and $\alpha_0(l_1 l_2) = 1$, we derive $\mathrm{mult}_o(h) \geqq 7$. Thus $8 = (K_M^{-6} - 2L_i)\cdot L_i \geqq 9$ unless $3L_i \subset Z(S)$ for $i = 1, 2$. Hence, $3L_1 + 3L_2 \subset Z(S)$, i.e., $f_S = (l_1 l_2)^3 h_1$. Now $\mathrm{mult}_o(h) \geqq 6$. We claim that $\mathrm{mult}_0(h) \geqq 7$. In fact, if not, then by Proposition A.1.1 and $\alpha_{x_1}(S) \leqq \frac{1}{9}$, for any $p > 0$, there is a local coordinate system (z_1, z_2) such that $h_1(z_1, z_2) = \sum_{i,j\geqq 0} a_{ij} z_1^i z_2^j$ at $(0, 0)$ and $a_{ij} = 0$ if either $\frac{i}{6} + \frac{j}{p} < 1$ or $\frac{i}{6} + \frac{j}{p} = 1, j \geqq 1$. In particular, the lowest homogeneous term f_{12} of f_S is of form $l_1^3 l_2^3 z_1^6$. Thus by Proposition A.1.1, $\alpha_0(f) \geqq \frac{1}{6}$ unless one of L_1, L_2 is tangent to $\{z_1 = 0\}$ at x_1. Assume that L_1 does so. If $4L_1 \not\subset Z(S)$, then

$$\begin{aligned} 9 = (6K_M^{-1} - 3L_1)\cdot L_1 &= 3 + \{h_1 = 0\}\cdot L_1 \\ &\geqq 3 + \inf\{2i + j \mid a_{ij} \neq 0\} \\ &\geqq 3 + 13 = 16 \end{aligned}$$

A contradiction! Therefore, $4L_1 \subset Z(S)$. One can actually prove that $5L_1 \subset Z(S)$. Choose local coordinates (z_1, z_2) such that $L_i = \{z_i = 0\}$ for $i = 1, 2$. By Proposition A.1.1 and $\alpha_o(S) \leqq \frac{1}{9}$, if we write $f_S = z_1^k z_2^3 h_2(z_1, z_2)$ where $8 \geqq k \geqq 5$, then $h_2(z_1, z_2) = (z_1 + z_2^\beta)^{9-k} + \tilde h_2(z_1, z_2)$ and $\tilde h_2$ does not have terms $z_1^i z_2^j$ with $i + j/\beta \leqq 9 - k$. By some direct computations, we can obtain $\alpha_o(S) > \frac{1}{9}$, a contradiction! Thus we must have that $\mathrm{mult}_o(h_1) \geqq 7$, so $\mathrm{mult}_o(f_S) \geqq 13$. Since $K_M^{-6}\cdot E = 12$, we conclude that $E \subset Z(S)$, so $S = S'\cdot S_5$, where S_5 is a global section of K_M^{-5}. By Hölder inequality, $\alpha_{x_1}(S_5) \leqq \frac{2}{15}$. Repeat the above arguments for S_5, we can conclude that $S_5 = S'\cdot S_4$ with $\alpha_{x_1}(S_4) \leqq \frac{1}{6}$. Inductively, we finally obtain $S = (S')^6$. By the definition of S', it contradicts to our assumption on S. Therefore, Claim 1 is proved.

Claim 2. There is no line of M through x_1.

If not, by Claim 1, there is exactly one line L_1 of M with $x_1 \in L_1$. As before, $2L_1 \subset Z(S)$. By the above arguments in the proof of Claim 1, it follows from $\alpha_{x_1} \leqq \frac{1}{9}$ that $\mathrm{mult}_o(f_S) \geqq 10$, where f_S is the local holomorphic representation of S at x_1.

Let f_k be the lowest homogeneous term of f_S at x_1. We may assume that there are k_1, k_2 with $k_1 + k_2 = k$ satisfying

$$k_1 = \max\{l_1, l_2 \mid l_1 + l_2 = k,\ z_1^{l_1} z_2^{l_2} \text{ is in } f_k\} \tag{A.1.9}$$

In particular, $k_1 \geqq k_2$. Give a partial order on monomials $z_1^{l_1}z_2^{l_2} : z_1^{l_1}z_2^{l_2} \leqq z_1^{l_1'}z_2^{l_2'}$ if $l_1 \leqq l_1'$. Let $z_1^{j_1}z_2^{j_2}$ be the smallest term in f_k with respect to the above partial order. If $j_1 \leqq j_2$, then by the proof of Proposition A.1.1 and $\alpha_{x_1}(S) \leqq \frac{1}{9}$, $k \geqq 18$. Choose an anticanonical divisor S' such that $Z(S') = L_1 + D$ and D is tangent to L_1 at x_1. By Claim 1 and the properties of M, the divisor D is irreducible and smooth at x_1. If $Z(S)$ does not contain D, then

$$18 = Z(S)\cdot D \geqq 16 + 2L_1 \cdot D = 20$$

A contradiction! So $Z(S') \subset Z(S)$. Inductively, we can prove $S = (S')^6$. But $\mathrm{mult}_{x_1}((S')^6) = 12$. It contradicts to that $k \geqq 18$. Thus $j_1 > j_2$. In particular, $k_1 > k_2$. Note that the same arguments as above show that $k \leqq 14$. By the geometric properties of M, one can easily check that there are exactly five curves $D_i (1 \leqq i \leqq 5)$ of degree 2 through the point x_1 and with distinct tangent directions at x_1. If $k = 14$, then $Z(S) = 4L_1 + 2\sum_{i=1}^5 D_1$. It implies that $\alpha_{x_1}(S) \geqq \frac{1}{7}$, impossible! If $k = 13$, then $4L_1 + \sum_{i=1}^5 D_1 \subset Z(S)$. One can also show that it will be against our assumption $\alpha_{x_1}(S) \leqq \frac{1}{9}$. Thus $k \leqq 12$. Since $\alpha_{x_1}(S) \leqq \frac{1}{9}, j_1 \geqq 9$ and $j_2 \leqq 3$, where $z_1^{j_1} z_2^{j_2}$ is the smallest term in f_k.

Choose an anticanonical section S'' such that $Z(S'') = L_1 + D'$, where D' is tangent to $\{z_1 = 0\}$ at x_1.

By Proposition A.1.1 and $\alpha_{x_1}(S) \leqq \frac{1}{9}$, we can choose (z_1, z_2) such that the Taylor expansion of f_S at x_1 does not have terms $z_1^i z_2^j$ with $i(9 - j_2 - \delta) + j(j_1 - 9 + \delta) < (9 - \delta)(j_1 - j_2)$, where $\delta > 0$ is sufficiently small. Then if $D' \not\subset Z(S)$, we have

$$\begin{aligned} Z(S)\cdot D' &\geqq \min\{2i + j \mid z_1^i z_2^j \text{ is in } f_S\} \\ &\geqq \min\{2i + j \mid z_1^i z_2^j \text{ is in } f_S,\ i \leqq 8\} \\ &\geqq 18 + \frac{27 - 2j_1 - j_2}{j_1 - 9}(9 - i) - \delta'\,, \end{aligned} \tag{A.1.10}$$

where δ' is small and depends on δ. First we assume that D' is irreducible, then by (A.1.10), $D' \subset Z(S)$, so $S = S'' \cdot S_5$, where S_5 is a global section of K_M^{-5}. One can compute that

$$\int_M \frac{dV_{\tilde g}}{\|S''\|_{\tilde g}^{2\beta}} < \infty \quad \text{if } \beta < \frac{3}{4}\,. \tag{A.1.11}$$

Then by $\alpha_{x_1}(S) \leqq \frac{1}{9}$, we have $\alpha_{x_1}(S_5) \leqq \frac{3}{23}$. In fact, one can easily prove that L_1 is also tangent to $\{z_1 = 0\}$. Inductively, we will conclude that $S = (S'')^6$. It implies that $\alpha_{x_1}(S) \geqq \frac{1}{6}\alpha_{x_1}(S'') = \frac{1}{8}$. A contradiction! Therefore, D' is reducible. It will have distinct tangent direction from that of L_1 at x_1. Then we have that $j_1 = 9, j_2 = 3$ and $L_1 = \{z_2 = 0\}$. By $K_M^{-6}\cdot D' = 12$, Proposition A.1.1 and D' is tangent to $\{z_1 = 0\}$, one can deduce that $5D' + 2L_1 \subset Z(S)$. Then by the arguments in the proof of Claim 1, we will have either $\alpha_{x_1}(S) > \frac{1}{9}$ or $9D' \subset Z(S)$. Since both cases are impossible, we complete the proof of Claim 2.

From now on, we may assume that no line of M passes through x_1. Then there is a pencil of anticanonical divisors such that the generic one of it is irreducible and vanishes at x_1 of order 2. In particular, it implies that $k \leqq 12$.

By Proposition A.1.1 and $\alpha_{x_1}(S) \leqq \frac{1}{9}$, we can choose local coordinates (z_1, z_2) such that the Taylor expansion of f_S at x_1 does not have terms $z_1^i z_2^j$ with $(9 - \delta - j_1)i + (j_1 - 9 + \delta)_j \leqq (9 - \delta)(j_1 - j_2)$, where j_1, j_2 are given in the proof of Claim 2, δ is sufficiently small number. On the other hand, we have an anticanonical section S' such that its local holomorphic representation $h_{S'}$ is of form $z_1 \tilde h_1(z_1, z_2) + z_2^3 \tilde h_2(z_2)$ at x_1.

Case 1. $Z(S')$ is irreducible.

If $Z(S)$ does not contain $Z(S')$, then

$$\begin{aligned} 24 &= Z(S)\cdot Z(S') \\ &\geqq \min\{2i + j \mid z_1^i z_2^j \text{ is in } f_S\} + \min\{i + j \mid z_1^i z_2^j \text{ is in } f_S\} \\ &\geqq 18 + 9 = 27 \end{aligned}$$

A contradiction! Thus $S = S' \cdot S_5$. An easy computation shows that $\alpha_{x_1} \geqq \frac{3}{4}$, so $\alpha_{x_1}(S_5) \leqq \frac{3}{23}$.

Case 2. $Z(S')$ is reducible.

By Claim 2, the divisor $Z(S')$ consists of two curves D_1, D_2 of degree 2 such that both D_1 and D_2 are smooth at x_1, D_1 is tangent to $\{z_1 = 0\}$. As in Case 1, we have $D_1 \subset Z(S)$. Let g_1 be the defining equation of D_1 at x_1, then $\alpha_0(g_1) = 1$. So if we decompose $f_S = g_1 \tilde f$, then $\alpha_0(\tilde f) \leqq \frac{1}{8}$,

$\text{mult}_o(f) \leqq 11$. Then one can deduce $2D_1 \subset Z(S)$. In fact, the same arguments show that $3D_1 \subset Z(S)$. On the other hand, arguments in the proof of Claim 1, one can show that $k = \text{mult}_o(f_S) \geqq 10$. Thus if $D_2 \not\subset Z(S)$, we would have

$$12 = D_2 \cdot Z(S) = 3D_2 \cdot D_1 + D_2(Z(S) - 3D_1))$$
$$\geqq 6 + 7 = 13$$

A contradiction! Therefore $D_2 \subset Z(S)$, i.e., $S = S'S_5$. Also $\alpha_{x_1}(S) \leqq \frac{3}{23}$.

Inductively, we can prove that $S = (S')^6$, then $\alpha_{x_1}(S) \geqq \frac{1}{8}$. It contradicts to our assumption. Thus the lemma is proved.

Lemma A.1.2. *Let M be a smooth cubic surface in CP^3 and S be a section of K_M^{-6}. Assume that $Z(S)_{\text{red}}$ is not an anticanonical divisor consisting of three lines intersecting at a common point. Then there is an $\varepsilon > 0$ such that for $\alpha \leqq \frac{2}{3} + \varepsilon$,*

$$\int_M \frac{dV_{\tilde{g}}}{\|S\|_{\tilde{g}}^{\alpha/3}} < \infty \tag{A.1.12}$$

where $\tilde{g}$ is any given Kähler metric on M.

Proof. As in the proof of last lemma, it suffices to prove that $\alpha_{x_i}(S) > \frac{1}{9}$ for $1 \leqq i \leqq l$, while $\alpha_x(S) > \frac{1}{9}$ for any $x \neq x_i$ $(1 \leqq i \leqq l)$. Assume that $\alpha_{x_1}(S) \leqq \frac{1}{9}$, we will derive a contradiction. Since the proof is identical to that of last lemma, we just sketch it. Note that there are at most two lines through x_1.

Case 1. There are exactly two lines L_1, L_2 of M through x_1.

In this case, there is an anticanonical section S' of K_M^{-1} with $Z(S') = L_1 + L_2 + L_3$, where L_3 is a line of M not through x_1. Then as in the proof of Claim 1 in Lemma A.1.1, either $4L_1 + 3L_2$ or $3L_1 + 4L_2$ is in $Z(S)$. But $L_1 \cdot L_3 = 1$, $L_2 \cdot L_3 = 1$, so $L_3 \subset Z(S)$, i.e., $S = S' \cdot S_5$. One can directly compute that $\alpha_{x_1}(S') = 1$, so $\alpha_{x_1}(S_5) \leqq \frac{1}{8}$. Inductively, one can show that $S = (S')^6$, then $\alpha_{x_1}(S) \geqq \frac{1}{6}$. A contradiction.

Case 2. There is exactly one line L_1 of M through x_1.

Let S' be the section of K_M^{-1} with $Z(S') = L_1 + D$, where D is an irreducible curve of degree 2 containing x_1. As before, $2L_1 \subset Z(S)$. Let f_S be the local holomorphic representation of S in some local coordinates (z_1, z_2), and l_1 be the defining equation of L_1, then $f_S = l_1^2 h$, $\text{mult}_o(h_1) \geqq 7$ and $\alpha_o(h_1) \leqq \frac{1}{7}$. Let f_k, h_{k-2} be the lowest homogeneous terms of f_S, h, respectively. Then we may assume that $h_{k-2} = z_1^{j_1} z_2^{j_2} + \dots$ and any term $z_1^i z_2^j$ in h_{k-2} has the property: $i \geqq j_1$. If L_1 is not tangent to $\{z_1 = 0\}$, then $f_k = \lambda z_1^{j_1} z_2^{j_2+2} + \dots$ satisfying: $\lambda \neq 0$ and no $z_1^i z_2^j$ in f_k with $i < j_1$. By the proof of Proposition A.1.1, $j_1 \geqq 9$, so $k \geqq 11$ and $3L_1 \subset Z(S)$. Consequently, it follows from $L_1 \cdot D = 2$ that $D \subset Z(S)$, i.e., $S = S' \cdot S_5$. If L_1 is indeed tangent to $\{z_1 = 0\}$, then we can prove that $4L_1 \subset Z(S)$, so $D \subset Z(S)$, otherwise,

$$12 = D \cdot Z(S) \geqq 4L_1 \cdot D + (k - 4) \geqq 8 + 5 = 13$$

A contradiction!

Therefore, we always have $S = S' \cdot S_5$. Then inductively, one can actually prove that $S = (S')^6$, so $\alpha_{x_1}(S) = \frac{1}{6}\alpha_{x_1}(S') \geqq \frac{1}{8}$. A contradiction.

Case 3. There is no line of M through x_1.

Let S' be the anticanonical section vanishing at x_1 of order 2. Then $Z(S')$ is irreducible. If $Z(S') \subset Z(S)$, then $S = S' \cdot S_5$, where S_5 is a global section of K_M^{-5}. One can directly check that $\alpha_{x_1}(S') = \frac{5}{6}$, so $\alpha_{x_1}(S_5) \leqq \frac{1}{39}$. In particular, S_5 vanishes at x_1 of order at least 8. Thus by $Z(S_5) \cdot Z(S') = 15$, we conclude $Z(S') \subset Z(S_5)$. Inductively, we have $S = (S')^6$, so $\alpha_{x_1}(S') = \frac{1}{6}\alpha_{x_1}(S') > \frac{1}{9}$. A contradiction.

Now $Z(S') \not\subset Z(S)$. It implies that $k = \text{mult}_o(f_S) = 9$. By Proposition A.1.1 and $\alpha_{x_1}(S) \leqq \frac{1}{9}$, we can choose local coordinates (z_1, z_2) such that f_S does not have terms $z_1^i z_2^j$ in its power series at x_1 if $\frac{i}{9} + \frac{j}{p} < 1$, where p is a sufficiently large integer. In particular, the lowest homogeneous term f_k of f_S is z_1^9. By using the holomorphic transformation of form $(z_1, z_2) \to (z_1 + \sum_{k \geqq 1} b_k z_2^k, z_2)$, we can eliminate the monomials $z_1^8 z_2^l$ for $l \geqq 1$ in the Taylor expansion of f_S at o. Then the proof of proposition A.1.1 implies that either $\alpha_{x_i}(S) > \frac{1}{9}$ or f_S contains a curve with multiplicity 9. Both cases are impossible! Thus we also derive a contradiction in Case 3. The lemma is proved.

The Lemma 2.4 in section 2 obviously follows from Lemma A.1.1 and A.1.2.

Appendix 2. The Proof of Proposition 2.1

Let $\{f_i\}_{i \geq 1}$ be a sequence of holomorphic function in the unit ball $B_1(o) \subset C^2$ with $\lim_{i\to\infty} f_i = f \neq 0$. We want to prove that for $\alpha < \inf_{x \in B_{\frac{1}{2}}(o)}\{\alpha_x(f)\}$,

$$\lim_{i\to\infty} \int_{B_{\frac{1}{2}}(o)} \frac{dV}{|f_i|^{2\alpha}} = \int_{B_{\frac{1}{2}}(o)} \frac{dV}{|f|^{2\alpha}} \tag{A.2.1}$$

where dV is the standard euclidean volume form.

As before, we denote by $Z(f)$ the zero locus of f and $Z(f)_{\rm red}$ the sum of distinct irreducible components in $Z(f)$. Without losing the generality, we may assume that $Z(f)_{\rm red}$ is smooth outside the origin o. Then $\alpha_x(f) = \frac{1}{m_x}$ for $x \neq 0$, where m_x is the multiplicity in $Z(f)$ of the irreducible component of $Z(f)_{\rm red}$ containing x, $m_x = 0$ if $x \notin Z(f)_{\rm red}$. On the other hand, m_x is the Lelong number of (1,1)-positive current $\partial\bar\partial \log|f|^2$ at x. By the estimate in [TY], for any small neighborhood U of o, and any $\beta < \min_{x \in B_{\frac{1}{2}}(o)}\{\alpha_x(f)\}$, there is a constant $C_{U,\beta}$ independent of i such that

$$\int_{B_{\frac{1}{2}}(o)\setminus U} \frac{dV}{|f_i|^{2\beta}} \leqq C_{U,\beta} \tag{A.2.2}$$

It implies that

$$\lim_{i\to\infty} \int_{B_{\frac{1}{2}}(o)\setminus U} \frac{dV}{|f_i|^{2\beta}} = \int_{B_{\frac{1}{2}}(o)\setminus U} \frac{dV}{|f|^{2\alpha}} \tag{A.2.3}$$

therefore, by the famous Fatou's lemma, in order to prove (A.2.1), it suffices to find a small neighborhood U of o such that

$$\lim_{i\to\infty} \int_U \frac{dV}{|f_i|^{2\alpha}} \leqq \int_U \frac{dV}{|f|^{2\alpha}} \tag{A.2.4}$$

Lemma A.2.1. *Suppose that for sequence $\{F_i\}_{i \geq 1}$ of holomorphic functions in a neighborhood of o with $\lim_{i\to\infty} F_i = F \neq 0$, there is a constant C_α independent of i such that for $\alpha < \alpha_o(F)$*

$$\int_U \frac{dV}{|F_i|^{2\alpha}} \leqq C_\alpha < \infty \tag{A.2.5}$$

where U is a fixed small neighborhood of o. Then (A.2.4) is valid.

Proof. Since $\alpha < \alpha_o(f)$, by taking U smaller if necessary and using Proposition A.1.1, we can choose local coordinates (z_1, z_2) at o such that we have the following expansion of f

$$\begin{aligned} f(z_1, z_2) &= z_1^{i_1} z_2^{j_2} \prod_{\nu=1}^{n(f)} (z_1^{j_1} + \lambda_\nu z_2^{i_2})^{p_\nu} + f_R(z_1, z_2) \\ &= f_\Delta(z_1, z_2) + f_R(z_1, z_2) \end{aligned} \tag{A.2.6}$$

where

(1) $i_1, j_2, p_\nu < \frac{1}{\alpha}$, $\nu = 1, 1, \ldots, n(f)$.

(2) Δ is a line segment in the first quadrant R_+^2 of R^2 and f_Δ contains exactly terms $z_1^k z_2^l$ in the expansion of f with $(k, l) \in \Delta$.

(3) For any term $z_1^k z_2^l$ in the expansion of f_R (k, l) lies on the right side of the line in R^2 containing Δ.
(4) $\alpha < r(N(f))^{-1}$, where $r(N(f))$ is the remotedness of the associated Newton polyhedron of f in local coordinates (z_1, z_2).

Define $\varepsilon_i = K \max_{\bar{U}} \{|f_i - f|\}$, where $K \geqq 2$ is a constant determined later. Then

$$\lim_{i\to\infty} \int_U \frac{dV}{|f_i|^{2\alpha}} \tag{A.2.7}$$

$$\leqq \lim_{i\to\infty} \left\{ \frac{1}{(1-\varepsilon_i)^{2\alpha}} \int_U \frac{dV}{|f|^{2\alpha}} + \int_{\substack{|f| \leqq \varepsilon_i \\ U}} \frac{dV}{|f_i|^{2\alpha}} \right\}$$

$$= \int_U \frac{dV}{|f|^{2\alpha}} + \lim_{i\to\infty} \int_{\substack{|f| \leqq \varepsilon_i \\ U}} \frac{dV}{|f_i|^{2\alpha}}$$

Therefore, it suffices to prove that the last integral in (A.2.7) tends to zero as i goes to infinity.

We may take U to be $\{(z_1, z_2) \in C^2 \,|\, |z_1| \leqq \delta^{i_2}, |z_2| \leqq \delta^{j_1}\}$. In the following, all integrals are taken on the subsets of U as specified.

We decompose the last integral in (A.2.7) into three parts $J_{i1}(\varepsilon_i)$, $J_{i2}(\varepsilon_i)$, $J_{i3}(\varepsilon_i)$ and estimate them individually. First we deal with $J_{i1}(\varepsilon_i)$. For that, we put $z_2 = w^{j_1}$, $z_2 = w^{i_2}\xi$, $m = i_1 i_2 + j_1 j_2 + i_2 j_1 \sum_{v=1}^{l} p_v$, $\delta_i = \varepsilon_i^{\frac{1}{m}}$, then

$$J_{i1}(\varepsilon_i) = \int_{\substack{|f| \leqq \varepsilon_i \\ |z_1|^{j_1} \leqq |z_2|^{i_2} \\ \sqrt[j_1]{|z_2|} \geqq \delta_i}} \frac{dV}{|f_i|^{2\alpha}} \tag{A.2.8}$$

$$= j_1 \int_{\substack{|f| \leqq \varepsilon_i \\ \delta_i \leqq |w|, |x'| \leqq 1}} \frac{|w|^{2j_1 - 2 + 2i_2} dw \wedge d\bar{w} \wedge d\xi \wedge d\bar{\xi}}{|f_i(w^{i_2}\xi, w^{j_1})|^{2\alpha}}$$

$$= j_1 \int_{\substack{|f| \leqq \varepsilon_i \\ \delta_i \leqq |w|, |\xi| \leqq 1}} \frac{|w|^{2(j_1 + i_2 - 1 - m\alpha)} dw \wedge d\bar{w} \wedge d\xi \wedge d\bar{\xi}}{|\xi|^{i_1} \prod_{v=1}^{l} (\xi + \lambda_v)^{p_v} + w^{-m} f_R(w_{\xi}^{i_2} w^{j_1}) + (f_i - f)|^{2\alpha}}$$

By the definitions of δ_i and f_R, for $\delta_i \leqq |w| \leqq \delta$

$$|w^{-m} f_R(w^{i2}\xi, w_{j_1}) + w^{-m}(f_i - f)| \leqq C\delta + K^{-1} \tag{A.2.9}$$

where C is a constant depending only on f_R. Since $r(N(f)) = \dfrac{m}{j_1 + j_2}$, we have that $\alpha i_1 < 1$, $p_v \cdot \alpha < 1$ and $m\alpha - j_1 + 1 - i_2 < 1$. Thus if we choose K sufficiently large and δ sufficiently small, by Fubini theorem and (A.2.8), (A.2.9), one can easily see $\lim_{i\to\infty} J_{i1}(\varepsilon_i) = 0$. Similarly, one has $\lim_{i\to\infty} J_{i2}(\varepsilon_i) = 0$. Note that

$$J_{i2} = \int_{\substack{|f| \leqq \varepsilon_i \\ |z_1|^{j_1} \leqq |z_2|^{i_2} \\ \sqrt[i_2]{|z_1|} \geqq \delta_i}} \frac{dV}{|f_i|^{2\alpha}} \tag{A.2.10}$$

Next, we estimate $J_{i3}(\varepsilon_i)$, which is dominated by the following integral

$$I_{\varepsilon_i}(f_i, \alpha) = \int_{\substack{|z_1| \leqq \delta_i^{i_2} \\ |z| \leqq \delta_i^{j_1}}} \frac{dV}{|f_i|^{2\alpha}} \left(\delta_i = \varepsilon_i^{\frac{1}{m}} \right) \tag{A.2.11}$$

Define holomorphic functions F_i, F by

$$F_i(z_1, z_2) = \delta_i^{-\frac{m}{2}} f_i\left(\delta_i^{\frac{i_2}{2}} z_1, \delta_i^{\frac{i_1}{2}} z_2\right)$$

$$F(z_1, z_2) = f_\Delta(z_1, z_2) \tag{A.2.12}$$

For i sufficiently large, those functions are well-defined on U with $\lim_{i\to\infty} F_i = F$ and $\alpha < \alpha_0(F) = r(N(f))^{-1}$. By the definition in (A.2.11) and (A.2.12), we have

$$I_{\varepsilon_i}(f_i, \alpha) = \delta_i^{\frac{i_2 + j_1 - m\alpha}{2}} I_{\sqrt{\varepsilon_i}}(F_i, \alpha) \tag{A.2.13}$$

It follows from our assumption of the lemma that $\lim_{i\to\infty} I_{\varepsilon_i}(f_i, \alpha) = 0$. Then, the lemma is proved.

It remains to verify the assumption in Lemma A.2.1. We will complete it by induction. For simplicity of notations, we assume that $F_i = f_i$, $F = f$. By the proof of Proposition A.1.1, there is a local coordinate system (z_1, z_2) such that either $\alpha_0(f) = r(N(f))^{-1}$, where $r(N(f))$ is the remotedness of the associated Newton polyhedron $N(f)$, or $f(z_1, z_2) = z_1^{i_1} z_2^{j_1} + f_R(z_1, z_2)$ with $\alpha_0(f) = i_1^{-1} \leqq j_1^{-1}$ and $pl + l > pi_1 + j_1$ for any term $z_1^k z_2^l$ in f_R, where p is a sufficiently large integer. First we assume that f is in the second case. Then we can write $f_i = f_{iL} + z_1^{i_1} z_2^{j_1} + f_{iR}$, where f_{iL} consists of all terms $z_1^k z_2^l$ in f_i with $pk + l < pi_1 + j_1$. By the holomorphic transformation of form $(z_1, z_2) \to (z_1 + \sum_{k \geqq 1} z_2^k, z_2)$, we may further assume that f_{iL} does not have the term $z_1^{i_1 - 1} z_2^l$ with $l \geqq j_1 + 1$. Note that $\lim_{i\to\infty} f_{iL} = 0$, $\lim_{i\to\infty} f_{iR} = f_R$. Furthermore, by holomorphic transformation, we may assume that each f_{iL} does not contain any term $z_1^{i_1 - 1} z_2^l$ with $l > j_1$.

If $f_{iL} = \sum_{pk + l < pi_1 + j_1} a_{kl}(i) z_1^k z_2^l$, we define

$$\varepsilon_i = K \max \left\{ |a_{kl}(i)|^{\frac{1}{pi_1 + j_1 - pk - l}} \right\} \tag{A.2.14}$$

where K is a large constant independent of i. We decompose

$$\int_U \frac{dV}{|f_i|^{2\alpha}} = \left(\int_{\substack{|z_1| \leqq |z_2|^p \\ |z_2| \geqq \varepsilon_i}} + \int_{\substack{|z_1| \leqq |z_2^p| \\ |z_2| \geqq \varepsilon_i^p}} + \int_{\substack{|z_1| \leqq \varepsilon_i^p \\ |z_2| \geqq \varepsilon_i}} \right) \frac{dV}{|f_i|^{2\alpha}}$$

$$= J_{1\alpha}(f_i, \varepsilon_i) + J_{2\alpha}(f_i, \varepsilon_i) + I_{\varepsilon_i}(f_i, \alpha) \tag{A.2.15}$$

It is easy to show (cf. the proof of Lemma A.2.1) that both $J_{1\alpha}(f_i, \varepsilon_i)$ and $J_{2\alpha}(f_i, \varepsilon_i)$ are uniformly bounded if K is sufficiently large. Put $g_i = \varepsilon_i^{-(pi_1 + j_1)} f_i(\varepsilon_i^p z_1, \varepsilon_i z_2)$. Without losing generality, we may assume that $\lim_{i\to\infty} g_i = g$. Obviously, the holomorphic function g is the sum of $z_1^{i_1} z_2^{j_1}$ and some monomials $b_{kl} z_1^k z_2^l$ with $pk + l < pi_1 + j_1$.

Claim 1. For any point x in $B_1(o)$, we have $\alpha_x(g) \geqq \alpha_o(f)$.

We may take x to be o since the translations on C^2 preserve the property that g is a sum of $z_1^{i_1} z_2^{j_1}$ and some monomials $b_{kl} z_1^k z_2^l$ with $pk + l < pi_1 + j_1$ and $k < i_1 - 1$ or $k = i_1 - 1$, $l \leqq j_1$). Let $N_o(g)$ be the Newton polyhedron associated to g and coordinates (z_1, z_2), let Δ' be the line segment in $\partial N_o(g)$ intersecting with the diagonal line $\{x = y\}$ in R^2. Define

$$g_{\Delta'} = \sum_{(k, l) \in \Delta'} b_{kl} z_1^k z_2^l \text{ where } g = \sum_{pk + l \leqq pi_1 + j_1} b_{kl} z_1^k z_2^l \tag{A.2.16}$$

Then we can write

$$g_{\Delta'} = z_1^{i'_1} z_1^{j'_2} \prod_{v=1}^{n(g)} (z_1^{j'_1} + \lambda'_v z_2^{i'_2})^{p'_v} \tag{A.2.17}$$

By the proof of Proposition A.1.1, we have

$$\alpha_o(g) \geqq \left(\max_{1 \leqq v \leqq n(g)} \{ r(N_o(g)), i'_1, j'_2, p'_v \} \right)^{-1} \tag{A.2.18}$$

On the other hand, we may assume p so large that there is no integer pair (k, l) in R^2 with $pk + l < pi_1 + j_1$, $k > i_1$ and $k \geqq l$. In particular, it implies that $r(N_o(g)), i_1', j_2', p_v' \leqq i_1$. It follows that $\alpha_o(g) \geqq \alpha_o(f)$. The claim is proved.

We also observe that $\mathrm{mult}_o(g_{\Delta'}) \leqq i_1 + j_1$ and $\alpha_o(g) > \alpha_o(f)$ whenever $\mathrm{mult}_o(g) = i_1 + j_1$. Thus by induction on $\alpha_o(f)$, $i_1, i_1 + j_1$, etc., we can verify the assumption of Lemma A.2.1 in this special case. Note that i_1 and $i_1 + j_1$ are determined by the lower endpoint of the line segment Δ (cf. (A.2.6)) of $\partial N(f)$ for a more coordinate system.

Next, we may assume that in the local coordinates (z_1, z_2), $\alpha_0(f) = r(N(f))^{-1}$ for the associated Newton polyhedron $N(f)$. Let Δ be line segment in $\partial N(f)$ intersecting with $\{x = y\}$ in R^2 with the properties stated as in (A.2.6) of the proof of Lemma A.2.1. Note that $n(f) \geqq 1$. Furthermore, if $j_1 \geqq 2$, then f_Δ does not have any monomials $z_1^k z_2^l$ with $k = i_1 + j_1 \sum_{v=1}^{n(f)} p_v - 1$; if $j_1 = 1$, by the transformation of form $(z_1, z_2) \to (z_1 + cz_2^{i_2}, z_2)$ and using the fact that i_1, j_2 and all p_v are less than $r(N(f))^{-1}$, we may also assume that f_Δ does not have any monomial $z_1^k z_2^l$ with $k = i_1 + j_1 \sum_{v=1}^{n(f)} p_v - 1$.

We decompose $f_i = f_{iL} + f_\Delta + f_{iR}$, where f_{iL} is the part of f_i consisting of all terms $z_1^k z_2^l$ with $L_\Delta(k, l) < 0$ and L_Δ denotes the defining equation of the line containing Δ. By scaling, we may take U to be $D_1 \times D_1$ in C^2, where $D_r (r > 0)$ denotes the disk in C of radius r, and f to be f_Δ. It follows that $\lim_{i \to \infty} f_{iR} = 0$.

Claim 2. There are local biholomorphisms $\phi_i = D_{\frac{3}{4}} \times D_{\frac{3}{4}} \to D_1 \times D_1$ such that (i) converge uniformly to the identity as i goes to infinity; (ii) the Taylor expansions of $f_i \circ \phi_i$ at o do not contain terms $z_1^k z_2^l$ with either $L_\Delta(k, l) < 0$ and $k > k_1 = i_1 + j_1 \sum_{v=1}^{l} p_v$ or $k = k_1 - 1$, $l > j_2$ and $L_\Delta(k, l) \neq 0$; (iii) $\lim_{i \to \infty} f_i \circ \phi_i = f$ on $D_{\frac{3}{4}} \times D_{\frac{3}{4}}$.

Note that (k_1, j_2) is the lower end point of Δ with $k_1 \geqq j_2$. We define ϕ_i by equations

$$\phi_i'(z_1, z_2) = \left(z_1, z_2 + \eta_i + \sum_{k=1}^{n_1} \sum_{l=1}^{j_2-1} b_{kl}(i) z_1^k z_2^l\right) \tag{A.2.19}$$

where n_1 is a large integer. Then one computes

$$\begin{aligned} &z_1^s \left(z_2 + \eta_i + \sum_{k=1}^{n_1} \sum_{l=0}^{j_2-1} b_{kl}(i) z_1^k z_2^l\right)^t \\ &= z_1^s (z_2 + \eta_i)^t + t z_1^s \sum_{k=1}^{n_1} z_1^k \sum_{l=0}^{j_2-1} b_{kl}(i) z_2^l (z_2 + \eta_i)^{t-1} + O(|b|^2) \end{aligned} \tag{A.2.20}$$

where $O(|b|^2)$ denotes a quantity bounded by $C \sum_{k,l \geqq 0} |b_{kl}|^2$. Let $f_i = \sum_{k,l \geqq 0} a_{kl}(i) z_1^k z_2^l$ and $f_i \circ \phi_i = \sum_{k,l \geqq 0} c_{kl}(i) z_1^k z_2^l$ be the Taylor expansions, then $\lim_{i \to \infty} a_{kl}(i) = 0$ whenever $L_\Delta(k, l) \neq 0$. Moreover, for any (k, l) with $l < j_2$ and $n_1 > k > k_1$, it follows from (A.2.20) that

$$c_{kl}(i) = \sum_{s,t \geqq 0} a_{st}(i) t \sum_{j=0}^{l} b_{k-sj}(i) \binom{t-1}{j} \eta_i^{t-j-1} + O(|b|^2) + O(|a|) \tag{A.2.21}$$

where $O(|a|)$ denotes a quantity bounded by $C \sum_{L_\Delta(k,l) \neq 0} |a_{kl}|$ and we let b_{k-sj} be zero if $k - s \leqq 0$. By Cauchy formula, one can show that those $(\frac{1}{2})^{t+s} a_{st}(i)$ with $L_\Delta(s, t) \neq 0$ converge to zero uniformly as i goes to infinity. We choose $\eta_i \leqq \frac{1}{4}$ satisfying

$$\lim_{i \to \infty} \eta_i = 0, \qquad \sup_{L_\Delta(s,t) \neq 0} \left\{ \left| \left(\frac{1}{2}\right)^{s+t} a_{st}(i) \right| \right\} \ll \eta_i \tag{A.2.22}$$

By either using Implicit Function Theorem or an iteration, it follows from (A.2.21) that there are $\{b_{kl}(i)\}$ with $\lim_{i \to \infty} b_{kl}(i) = 0$ such that $c_{kl}(i) = 0$ for those (k, l) with $k_1 < k < n_1$ and $l < j_2$. We use these $\{b_{kl}(i)\}$ in the definition (A.2.19) of ϕ_i', then these ϕ_i' satisfy (i), (iii) in the statement of Claim 2 and the Taylor expansions of $f_i \circ \phi_i'$ do not contain terms $z_1^k z_2^l$ with $n > k > k_1$, $l \leqq j_2$. Next we construct a local biholomorphisms ϕ_1'' of torm $(z_1, z_2) \to (z_1 + \sum_{k \geqq 1} d_k z_2^k, z_2)$ to eliminate terms $z_1^{k_1-1} z_2^l$ in $f_i \circ \phi_i'$ with $l > j_2$ and $L_\Delta(k_1 - 1, l) \neq 0$. Then our biholomorphisms ϕ_i are the compositions $\phi_i' \circ \phi_i''$. The claim is proved.

In particular, Claim 2 implies that the Jacobians $\mathrm{Jac}(\phi_i)$ of ϕ_i are uniformly bounded on $D_{\frac{1}{2}} \times D_{\frac{1}{2}}$. Therefore, it suffices to prove that $\int_{D_{\frac{1}{2}} \times D_{\frac{1}{2}}} |f_i \circ \phi_i|^{-2\alpha} dV$ are uniformly bounded. For simplicity, we still denote $f_i \circ \phi_i$ by f_i. Thus each f_{iL} does not contain any term $z_1^k z_2^l$ with $L_\Delta(k, l) < 0$ and either $k > k_1$ or $k = k_1 - 1$, $l > j_2$.

Since $\lim_{i \to \infty} f_{iR} = 0$, without losing generality, we may assume that

$$|f_{iR}|_{C^0(D_{\frac{1}{2}} \times D_{\frac{1}{2}})} \leqq \delta \quad \text{for all } i \tag{A.2.23}$$

where $\delta > 0$ is a sufficiently small number determined later.

Define $m = i_1 i_2 + j_1 j_2 + (\sum_{\nu=1}^{l} p_\nu) i_2 j_1$ and

$$\varepsilon_i = K \max\{|a_{kl}(i)|^{\frac{1}{m - i_2 k - j_1 l}} \mid L_\Delta(k, l) < 0\} \tag{A.2.24}$$

Lemma A.2.2. *Let $h = \sum_{k, l \geqq 0} b_{kl} z_1^k z_2^l$ be a holomorphic function on $D_1 \times D_1 \subset C^2$ such that $b_{kl} = 0$ if $L_\Delta(k, l) < 0$. Then we have*

(i) *for* $|\xi| \leqq 1$, $|w|^{j_1} \leqq \frac{1}{2}$, $|w^{i_2}\xi| \leqq \frac{1}{2}$,

$$|h(w_{i_2}\xi, w^{j_1})| \leqq (2|w|)^m |h|_{C^0(D_{\frac{1}{2}} \times D_{\frac{1}{2}})} \tag{A.2.25}$$

(ii) *for* $|\xi| \leqq 1$, $|w|^{i_2} \leqq \frac{1}{2}$, $|w^{j_1}\xi| \leqq \frac{1}{2}$,

$$|h(w^{i_2}\xi, w^{j_1}\xi)| \leqq (2|w|)^m |h|_{C^0(D_{\frac{1}{2}} \times D_{\frac{1}{2}})} \tag{A.2.26}$$

Proof. We just prove (i) here. The other case is analogous. For any fixed ξ with $|\xi| \leqq 1$, by our assumption on h_1, the function $w^{-m} h(w^{i_2}\xi, w^{j_1})$ is holomorphic in the domain $E_\xi = \{|w|^{j_1} \leqq \frac{1}{2}, |w^{i_2}\xi| \leqq \frac{1}{2}\}$. Thus maximum principle implies that for

$$\begin{aligned} \sup_{E_\xi} |w^{-m} h(w_{i_2}\xi, w^{j_1})| &\leqq \sup_{\partial E_\xi} |w^{-m} h(w^{i_2}\xi, w^{j_1})| \\ &\leqq 2^m |h|_{C^0(D_{\frac{1}{2}} \times D_{\frac{1}{2}})} \end{aligned} \tag{A.2.27}$$

The inequality (A.2.20) follows from it. Similarly, we can prove (A.2.21).

We then compute

$$\begin{aligned} \int_{D_{\frac{1}{2}} \times D_{\frac{1}{2}}} \frac{dV}{|f_i|^{2\alpha}} &= \left(\int_{\substack{\sqrt[i_2]{|z_1|} \leqq \sqrt[j_1]{|z_2|} \\ \varepsilon_i \leqq \sqrt[j_1]{|z_2|}}} + \int_{\substack{\sqrt[i_2]{|z_1|} \geqq \sqrt[j_1]{|z_2|} \\ \sqrt[i_2]{|z_1|} \geqq \varepsilon_i}} + \int_{\substack{|z_1| \leqq \varepsilon_i^{i_2} \\ |z_2| \leqq \varepsilon_i^{j_1}}} \right) \frac{dV}{|f_i|^{2\alpha}} \\ &\leqq j_1 \int_{\substack{\varepsilon_i \leqq |w| \leqq (\frac{1}{2})^{\frac{1}{j_1}} \\ |\xi| \leqq 1, |w^{i_2}\xi| \leqq \frac{1}{2}}} \frac{|w|^{2i_2 + 2j_1 - 2 - 2m\alpha} dw \wedge d\bar{w} \wedge d\xi \wedge d\bar{\xi}}{\left| \xi^{i_1} \prod_{\nu=1}^{l} (\xi + \lambda_\nu)^{p_\nu} + w^{-m}(f_{iL}(w^{i_2}\xi, w^{j_1}) + f_{iR}(w^{i_2}\xi, w^{j_1})) \right|^{2\alpha}} \\ &\quad + i_2 \int_{\substack{\varepsilon_i \leqq |w| \leqq (\frac{1}{2})^{\frac{1}{i_2}} \\ |\xi| \leqq 1, |w^{j_1}\xi| \leqq \frac{1}{2}}} \frac{|w|^{2i_2 + 2j_1 - 2 - 2m\alpha} dw \wedge d\bar{w} \wedge d\xi \wedge d\bar{\xi}}{\left| \xi^{i_2} \prod_{\nu=1}^{l} (1 + \lambda_\nu \xi)^{p_\nu} + w^{-m}(f_{iL}(w^{i_2}, w^{j_1}\xi) + f_{iR}(w^{i_2}, w^{j_1}\xi)) \right|^{2\alpha}} \\ &\quad + I_{\varepsilon_i}(f_i, \alpha) \\ &= J_{1\varepsilon_i}(f_i, \alpha) + J_{2\varepsilon_i}(f_i, \alpha) + I_{\varepsilon_i}(f_i, \alpha) \end{aligned} \tag{A.2.28}$$

Since the nonzero roots $\lambda_1, \lambda_2, \ldots, \lambda_l$ are distinct, we have

$$\lambda = \min\{|\lambda_\mu - \lambda_\nu|, |\lambda_\mu - \lambda_\nu|^{-1}, |\lambda_\mu|, |\lambda_\nu|, |\lambda_\mu|^{-1}, |\lambda_\nu|^{-1} \mid 1 \leqq \mu < \nu \leqq l\} > 0 \tag{A.2.29}$$

By (A.2.19) and Lemma A.2.2, for $(\frac{1}{2})^{\frac{1}{j_1}} \geqq |w| \geqq \varepsilon_i$, $|\xi| \leqq 1$ and $|w^{i_2}\xi| \leqq \frac{1}{2}$,

$$|w^{-m}(f_{iL}(w^{i_2}\xi, w^{j_1}) + f_{iR}(w^{i_2}\xi, w^{j_1}))| \leqq n_\Delta \cdot K^{-1} + 2^m \delta \tag{A.2.30}$$

where n_Δ is the number of the integer pairs (k, l) with $L_\Delta(k, l) < 0$. Choose K, δ such that $n_\Delta K^{-1} < \frac{1}{10}\lambda^m$ and $2^{m+1}\delta < \frac{1}{10}\lambda^m$. Then there is a constant $C_{\lambda,\alpha}$, depending only on λ, α, such that $J_{1\varepsilon_i}(f_i, \alpha) \leqq \frac{1}{2}C_{\lambda,\alpha}$. Similarly, $J_{2\varepsilon_i}(f_i, \alpha) \leqq \frac{1}{2}C_{\lambda,\alpha}$. It follows that

$$\int_{D_{\frac{1}{2}} \times D_{\frac{1}{2}}} \frac{dV}{|f_i|^{2\alpha}} \leqq (i_2 + j_1)C_{\lambda,\alpha} + I_{\varepsilon_i}(f_i, \alpha) \tag{A.2.31}$$

Put $g_i = \varepsilon_i^{-m} g_i(\varepsilon_i^{i_2} z_1, \varepsilon_i^{j_1} z_2)$, then by taking a subsequence, we may assume that $\lim_{i\to\infty} g = g$. Note that g is a sum of $f = f_\Delta$ and some monomials $b_{kl}z_1^k z_2^l$ with $L_\Delta(k, l) < 0$ and $k \leqq k_1 = i_1 + j_1 \sum_{\nu=1}^{l} p_\nu$, $l \leqq j_2$ if $k = k_1 - 1$.

Lemma A.2.3. *Let f, g be given as above. Then*

$$\inf_{x \in D_1 \times D_1} \{\alpha_x(g)\} \geqq \alpha_o(f) \tag{A.2.32}$$

Proof. Since f does not contain any monomial $z_1^{k_1-1} z_2^l$, any composition of g with a translation in C^2 is still a sum of f_Δ and some monomials $b_{kl}z_1^k z_2^l$ with $L_\Delta(k, l) < 0$ and $k \leqq k_1$, $l \leqq j_2$ if $k = k_1 - 1$. Hence, we may take x to be the origin in (A.2.32). Recall

$$f_\Delta = z_1^{i_1} z_2^{j_2} \prod_{\nu=1}^{n(f)} (z_1^{j_1} + \lambda_\nu z_2^{i_2})^{p_\nu}, \; i_2 \geqq j_1$$

Let Δ' be the line segment in $\partial N(g)$ intersecting with the line $\{x = y\}$ in R^2 and $g_{\Delta'}$ be the polynomial consisting of all monomials $b_{kl}z_1^k z_2^l$ of g with $(k, l) \in \Delta'$. As above, we can write

$$g_{\Delta'} = z_1^{i'_1} z_2^{j'_2} \prod_{\nu=1}^{n(g)} (z_1^{j'_1} + \lambda'_\nu z_2^{i'_2})^{p'_\nu} \tag{A.2.33}$$

Then $i'_1, j'_2 \leqq r(N(f)) = \alpha_o(f)^{-1}$ and $r(N(g)) \leqq r(N(f))$.

By the proof of Proposition A.1.1, we have the estimate

$$\alpha_o(g) \geqq \min\{(i'_1)^{-1}, (j'_2)^{-1}, r(N(g))^{-1}, (p'_\nu)^{-1}\} \tag{A.2.34}$$

Suppose that one of p'_ν, say p'_1 for simplicity, is greater than $\alpha_0(f)^{-1}$. By (A.2.32) and the definition of g, we have

$$i'_1 + j'_1 \sum_{\nu=1}^{n(g)} p'_\nu \leqq i_1 + j_1 \sum_{\nu=1}^{n(f)} p_\nu \leqq 2r(N(f)) \tag{A.2.35}$$

where the equality holds iff $j_2 = i_1 + j_1 \sum_{\nu=1}^{n(f)} p_\nu = r(N(f))$. It follows that $j'_1 = 1$ and $i'_2 \geqq j'_1 = 1$.

By local holomorphic transformations at o, we may further assume that $i'_2 \geqq \dfrac{i_2}{j_1}$.

In case $j_1 = 1$, we claim that $i'_2 = i_2$. Suppose that $i'_2 \geqq i_2 + 1$. Note that g contains the monomial $z_1^{i'_1} z_2^{l'_2}$ with $l'_2 = j'_2 + i'_2 \sum_{\nu=1}^{n(f)} p'_\nu$. Then by $L_\Delta(i'_1, j'_2 + i'_2 \sum_{\nu=1}^{n(g)} p'_\nu) \leqq 0$, we have

$$\begin{aligned} i_1 i_2 + j_2 + i_2 \sum_{\nu=1}^{n(f)} p_\nu &\geqq i_2 i'_1 + \left(j'_2 + i'_2 \sum_{\nu=1}^{n(g)} p'_\nu\right) \\ &\geqq i'_2 p'_1 > i'_2 \alpha_o(f) \\ &= \frac{i'_2}{i_2 + 1}\left(i_1 i_2 + j_2 + i_2 \sum_{\nu=1}^{n(f)} p_\nu\right) \end{aligned}$$

A contradiction! Thus $i'_2 = i_2$. We define a new local coordinate system $(\tilde z_1, \tilde z_2)$, by setting $\tilde z_1 = z_1 + \lambda'_1 z_2^{i_2}$, $\tilde z_2 = z$, then

$$f_\Delta(\tilde z_1, \tilde z_2) = (\tilde z_1 - \lambda'_1 \tilde z_2^{i_2})^{i_1} \tilde z_2^{j_2} \prod_{\nu=1}^{n(f)} (\tilde z_1 + (\lambda_\nu - \lambda'_1)\tilde z_2^{i_2})^{p_\nu}$$

If either none of λ_ν is equal to λ'_1 or some λ_{ν_0} is equal to λ'_1 and $i_1 + j_2 + \sum_{\nu \neq \nu_0} p_\nu > p_{\nu_0}$, the function $f_\Delta(\tilde{z}_1, \tilde{z}_2)$ has the same properties of f_Δ in (z_1, z_2). The above arguments shows that $\alpha_o(g) \geqq \alpha_o(f)$. If some λ_{ν_0} is equal to λ'_1, $i_1 + j_2 + \sum_{\nu \neq \nu_0} p_\nu \leqq p_{\nu_0}$, then by the proof of the Proposition A.1.1, we have

$$\alpha_o(g) \geqq p_{\nu_0}^{-1} \geqq \alpha_0(f)$$

The lemma is proved in case $j_1 = 1$. The proof of the case $j_2 \geqq 2$ is analogous and a little bit complicated. We omit the details and make the following remarks. If $\alpha_o(g) < \alpha_o(f)$, then one can choose $\lambda_{k_1}, \ldots, \lambda_{k_n} \neq 0$ with $k_n > k_{n-1} > \ldots > k_1 \geqq 2$ such that in local coordinates $(\tilde{z}_1, \tilde{z}_2)$ with $\tilde{z}_1 = z_1 + \sum_{s=1}^{n} \lambda_k z_2^{k_s}$ and $\tilde{z}_2 = z_2$, the associated Newton polyhedron $N_1(g)$ lies entirely outside the triangle $D_{L'}$ defined by $\tilde{x}_1$-axis, $\tilde{x}_2$-axis and the line L' through $(\alpha_o(f)^{-1}, \alpha_o(f)^{-1})$ and an integer point (k', l') in $R_+ \times R_+$ with $k' > \alpha_0(f)^{-1}$ and $L_\Delta(k', l') \leqq 0$. That is, the polynomial $g(\tilde{z}_1, \tilde{z}_2)$ does not contain the monomials $z_1^k z_2^l$ with $(k, l) \in D_{L'}$. On the other hand, one can easily show that the triangle $D_{L'}$ contains strictly more integer points than the triangle D_Δ, defined by z_1-axis, z_2-axis and the line containing Δ, does. Moreover, the fact that $g(\tilde{z}_1, \tilde{z}_2)$ does not contain $\tilde{z}_1^k \tilde{z}_2^l$ with $(k, l) \in D_{L'}$ imposes sufficiently many independent equations on the coefficients in $g(z_1, z_2)$. In particular, it implies that $g(z_1, z_2) = f_\Delta(z_1, z_2)$. A contradiction! Therefore, we always have $\alpha_o(g) \geqq \alpha_o(f)$.

Let h be a holomorphic function in U, $x \in U$, we define a quantity $\mu_x(h)$ as follows. If there is a local coordinate system such that $\alpha_x(h) = r(N_x(f))^{-1}$ for the associated Newton polyhedron $N_x(f)$, we define $i_1(h_{\tilde{\Delta}})$ to be the smaller one of the x-component of the lower endpoint of $\tilde{\Delta}$ and the y-component of the upper endpoint of $\tilde{\Delta}$, where $\tilde{\Delta}$ is the line segment in $N_x(f)$ intersecting with $\{x = y\}$ in R^2. Then $\mu_x(h)$ is the infimum of such $i_1(h_{\tilde{\Delta}})$ among all positive local coordinate systems such that $\alpha_x(h)$ is the remotedness of the associated Newton polyhedron. Otherwise, there is a local coordinate system (z_1, z_2) such that $h = z_1^i h_1(z_1, z_2) + O((|z_1|^2 + |z_2|^2)^p)$, where $h_1(0, z_2) \neq 0$ and p is sufficiently large. Then we define $\mu_x(h) = i$.

Now we complete our verification of the assumption in Lemma A.2.1. Obviously, we have $\mu_x(g) \leqq \mu_o(f)$ for any $x \in D_1 \times D_1$ and in case $\mu_x(g) = \mu_o(f)$, for simplicity, say $x = o$, then by the assumption that $g_{\Delta'}$ does not have term $z_1^{k_1 - 1} z_2^l$ with $l > j_2$, we have $\alpha_0(g) \geqq \alpha_0(f) + \tilde{\varepsilon}$, where $\tilde{\varepsilon}$ is a positive number depending only on the upper bound of $\alpha_o(f)^{-1}$ and $\mathrm{mult}_o(f)$. Therefore by induction, one can easily see that the assumption in Lemma A.2.1 holds.

The proof of Proposition 2.1 is completed.

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Note added in proof

After submitting this paper, the author found V.N. Karpushkin's work on uniform estimates of oscillatory integrals in R^2 (J. of Soviet Math., **35**, 2809–2826 (1986)). A much simpler proof can be given for the main result in Appendix 2 by using this work, in particular, Theorem 3.1 in the above reference.