THE YAU-TIAN-DONALDSON CONJECTURE FOR GENERAL POLARIZATIONS

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ABSTRACT. In this paper, assuming that a polarized algebraic manifold (X, L) is strongly K-stable in the sense of [8], we shall show that the class $c_1(L)_{\mathbb{R}}$ admits a constant scalar curvature Kähler metric. Since strong K-stability implies asymptotic Chow-stability (cf. [11]), we have a sequence $\{\omega_i\}$ of balanced metrics in the class $c_1(L)_{\mathbb{R}}$. Replace the sequence by its suitable subsequence if necessary. Then if $\{\omega_i\}$ were not convergent, the associated sequence $\{\mu_i\}$ of polarized test configurations would satisfy the inequality

$$F_1(\{\mu_i\}) \geq 0$$

in contradiction to strong K-stability for (X, L). Hence the sequence $\{\omega_i\}$ converges to a constant scalar curvature Kähler metric in $c_1(L)_{\mathbb{R}}$.

1. INTRODUCTION

By a *polarized algebraic manifold* (X, L), we mean a pair of a nonsingular irreducible projective algebraic variety X, defined over $\mathbb{C}$, and a very ample line bundle L over X. Replacing L by its positive integral multiple if necessary, we may assume that

$$H^q(X, L^{\otimes \ell}) = \{0\}, \qquad \ell = 1, 2, \dots; \, q = 1, 2, \dots, n,$$

where n is the complex dimension of X. In this paper, we fix once for all such a pair (X, L). For the affine line $\mathbb{A}^1 := \{z \in \mathbb{C}\}$, let the algebraic torus $T := \mathbb{C}^*$ act on $\mathbb{A}^1$ by multiplication of complex numbers

$$T \times \mathbb{A}^1 \to \mathbb{A}^1, \qquad (t, z) \mapsto tz.$$

By fixing a Hermitian metric h for L such that $\omega := c_1(L; h)$ is Kähler, we endow the space $V_\ell := H^0(X, L^{\otimes \ell})$ of holomorphic sections for $L^{\otimes \ell}$ with the Hermitian metric ρ_ℓ defined by

$$\langle \sigma', \sigma'' \rangle_{\rho_\ell} := \int_X (\sigma', \sigma'')_h \, \omega^n, \qquad \sigma', \sigma'' \in V_\ell,$$

where $(\sigma', \sigma'')_h$ denotes the pointwise Hermitian inner product of σ' and σ'' by the ℓ-multiple of h. For the Kodaira embedding $\Phi_\ell : X \hookrightarrow \mathbb{P}^*(V_\ell)$

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associated to the complete linear system $|L^{\otimes \ell}|$ on X, we put $X_\ell := \Phi_\ell(X)$. Let $\psi : \mathbb{C}^* \to \mathrm{GL}(V_\ell)$ be an algebraic group homomorphism such that the compact subgroup $S^1 \subset \mathbb{C}^* \,(= T)$ acts isometrically on (V_ℓ, ρ_ℓ). Take the irreducible algebraic subvariety $\mathcal{X}^\psi$ of $\mathbb{A}^1 \times \mathbb{P}^*(V_\ell)$ obtained as the closure of $\cup_{z\in\mathbb{C}^*}\mathcal{X}^\psi_z$ in $\mathbb{A}^1 \times \mathbb{P}^*(V_\ell)$. Here we set

$$\mathcal{X}^\psi_z := \{z\} \times \psi(z)\Phi_\ell(X), \qquad z \in \mathbb{C}^*,$$

and $\psi(z)$ in $\mathrm{GL}(V_\ell)$ acts naturally on the space $\mathbb{P}^*(V_\ell)$ of all hyperplanes in V_ℓ passing through the origin. We then consider the map

$$\pi : \mathcal{X}^\psi \to \mathbb{A}^1$$

induced by the projection of $\mathbb{A}^1 \times \mathbb{P}^*(V_\ell)$ to the first factor $\mathbb{A}^1$. Moreover, for the hyperplane bundle $\mathcal{O}_{\mathbb{P}^*(V_\ell)}(1)$ on $\mathbb{P}^*(V_\ell)$, we consider the pullback

$$\mathcal{L}^\psi \;:=\; \mathrm{pr}_2^*\,\mathcal{O}_{\mathbb{P}^*(V_\ell)}(1)_{|\mathcal{X}^\psi},$$

where $\mathrm{pr}_2 : \mathbb{A}^1 \times \mathbb{P}^*(V_\ell) \to \mathbb{P}^*(V_\ell)$ denotes the projection to the second factor. For the dual space V_ℓ^* of V_ℓ, the $\mathbb{C}^*$-action on $\mathbb{A}^1 \times V_\ell^*$ defined by

$$\mathbb{C}^* \times (\mathbb{A}^1 \times V_\ell^*) \to \mathbb{A}^1 \times V_\ell^*, \quad (t, (z, p)) \mapsto (tz, \psi(t)p),$$

naturally induces the $\mathbb{C}^*$-action on $\mathbb{A}^1 \times \mathbb{P}^*(V_\ell)$ and $\mathcal{O}_{\mathbb{P}^*(V_\ell)}(-1)$, where $\mathrm{GL}(V_\ell)$ acts on V_ℓ^* by the contragradient representation. This then induces $\mathbb{C}^*$-actions on $\mathcal{X}^\psi$ and $\mathcal{L}^\psi$, and $\pi : \mathcal{X}^\psi \to \mathbb{A}^1$ is a $\mathbb{C}^*$-equivariant projective morphism with relative very ample line bundle $\mathcal{L}^\psi$ such that

$$(\mathcal{X}^\psi_z, \mathcal{L}^\psi_z) \;\cong\; (X, L^{\otimes \ell}), \qquad z \neq 0,$$

where $\mathcal{L}^\psi_z$ is the restriction of $\mathcal{L}^\psi$ to $\mathcal{X}^\psi_z := \pi^{-1}(z)$. Then a triple $(\mathcal{X}, \mathcal{L}, \psi)$ is called a *test configuration for* (X, L), if we have both $\mathcal{X} = \mathcal{X}^\psi$ and $\mathcal{L} = \mathcal{L}^\psi$. Here ℓ is called the *exponent* of $(\mathcal{X}, \mathcal{L}, \psi)$. From now on until the end of Step 1 of Section 4, for $(\mathcal{X}, \mathcal{L}, \psi)$ to be a test configuration, we make an additional assumption that ψ is written in the form

$$\psi : \mathbb{C}^* \to \mathrm{SL}(V_\ell).$$

Then $(\mathcal{X}, \mathcal{L}, \psi)$ is called *trivial*, if ψ is a trivial homomorphism. We now consider the set $\mathcal{M}$ of all sequences $\{\mu_j\}$ of test configurations

$$\mu_j \;=\; (\mathcal{X}_j, \mathcal{L}_j, \psi_j), \qquad j = 1, 2, \dots,$$

for (X, L) such that for each j, the exponent ℓ_j of the test configuration μ_j satisfies the following condition:

$$\ell_j \to \infty, \quad \text{as } j \to \infty.$$

In [8], for each $\{\mu_j\} \in \mathcal{M}$, we defined the Donaldson-Futaki invariant $F_1(\{\mu_j\}) \in \mathbb{R} \cup \{-\infty\}$. Then we have the strong version of K-stability and K-semistability as follows:

Definition 1.1. (1) The polarized algebraic manifold (X, L) is called *strongly K-semistable*, if $F_1(\{\mu_j\}) \leq 0$ for all $\{\mu_j\} \in \mathcal{M}$.

(2) A strongly K-semistable polarized algebraic manifold (X, L) is called *strongly K-stable*, if for every $\{\mu_j\} \in \mathcal{M}$ satisfying $F_1(\{\mu_j\}) = 0$, there exists a j_0 such that μ_j are trivial for all j with $j \geq j_0$.

Recall that these stabilities are independent of the choice of the Hermitian metric h for L (see [12]). The purpose of this paper is to show the following:

Main Theorem. *If (X, L) is strongly K-stable, then the class $c_1(L)_{\mathbb{R}}$ admits a constant scalar curvature Kähler metric.*

2. The Donaldson-Futaki invariant F_1 on $\mathcal{M}$

Definition 2.1. For a complex vector space V, let $\phi : T \to \mathrm{GL}(V)$ be an algebraic group homomorphism. For the real Lie subgroup

$$T_{\mathbb{R}} \;:=\; \{t \in T\,;\, t \in \mathbb{R}_+\}$$

of the algebraic torus $T = \{t \in \mathbb{C}^*\}$, we define the associated Lie group homomorphism $\phi^{\mathrm{SL}} : T_{\mathbb{R}} \to \mathrm{SL}(V)$ by

$$\phi^{\mathrm{SL}}(t) \;:=\; \frac{\phi(t)}{\det(\phi(t))^{1/N}}, \qquad t \in T_{\mathbb{R}},$$

where $N := \dim V$. Let $b_1, b_2, \ldots, b_N$ be the weights of the action by ϕ^{SL} on the dual vector space V^* of V, so that we have the equalities

$$\phi^{\mathrm{SL}}(t) \cdot \sigma_\alpha \;=\; t^{-b_\alpha}\sigma_\alpha, \qquad \alpha = 1, 2, \ldots, N,$$

for some basis $\{\sigma_1, \sigma_2, \ldots, \sigma_N\}$ of V. Then we define $\|\phi\|_1$ and $\|\phi\|_\infty$ by

$$\|\phi\|_1 \;:= \Sigma_{\alpha=1}^N\, |b_\alpha| \quad \text{ and } \quad \|\phi\|_\infty := \max\{\, |b_1|, |b_2|, \ldots, |b_N|\,\}.$$

Definition 2.2. Put $d := \ell^n c_1(L)^n[X]$. For (V_ℓ, ρ_ℓ) in the introduction, we define a space W_ℓ by

$$W_\ell \;:=\; \{\mathrm{Sym}^d(V_\ell)\}^{\otimes n+1},$$

where $\mathrm{Sym}^d(V_\ell)$ is the d-th symmetric tensor product of V_ℓ. Then the dual space W_ℓ^* of W_ℓ admits the Chow norm (cf. [16])

$$W_\ell^* \ni w \;\mapsto\; \|w\|_{\mathrm{CH}(\rho_\ell)} \in \mathbb{R}_{\geq 0},$$

associated to the Hermitian metric ρ_ℓ on V_ℓ. For the Kodaira embedding $\Phi_\ell : X \hookrightarrow \mathbb{P}^*(V_\ell)$ as in the introduction, let

$$0 \neq \hat{X}_\ell \in W_\ell^*$$

be the associated Chow form for $X_\ell = \Phi_\ell(X)$ viewed as an irreducible reduced algebraic cycle on the projective space $\mathbb{P}^*(V_\ell)$.

Let $\mu_j = (\mathcal{X}_j, \mathcal{L}_j, \psi_j)$, $j = 1, 2, \dots$, be a sequence of test configurations for (X, L). We then define $\|\mu_j\|_1$ and $\|\mu_j\|_\infty$ by

$$\|\mu_j\|_1 \;:=\; \|\psi_j\|_1/\ell_j^{n+1} \quad \text{and} \quad \|\mu_j\|_\infty := \|\psi_j\|_\infty/\ell_j, \tag{2.3}$$

where ℓ_j denotes the exponent of the test configuration μ_j. Let $\delta(\mu_j)$ be $\|\mu_j\|_\infty/\|\mu_j\|_1$ or 1 according as $\|\mu_j\|_\infty \neq 0$ or $\|\mu_j\|_\infty = 0$. If $\|\mu_j\|_\infty \neq 0$, we write $t \in T_{\mathbb{R}}$ as $t = \exp(s/\|\mu_j\|_\infty)$ for some $s \in \mathbb{R}$, while we require no relation between $s \in \mathbb{R}$ and $t \in T_{\mathbb{R}}$ if $\|\mu_j\|_\infty = 0$. Note that

$$\psi_j^{\mathrm{SL}} : T_{\mathbb{R}} \to \mathrm{SL}(V_{\ell_j})$$

is just the restriction of ψ_j to $T_{\mathbb{R}}$. Since the group $\mathrm{SL}(V_{\ell_j})$ acts naturally on $W_{\ell_j}^*$, we can define a real-valued function $f_j = f_j(s)$ on $\mathbb{R}$ by

$$f_j(s) \;:=\; \delta(\mu_j)\, \ell_j^{-n} \log \|\psi_j(t)\cdot \hat{X}_{\ell_j}\|_{\mathrm{CH}(\rho_{\ell_j})}, \qquad s \in \mathbb{R}. \tag{2.4}$$

Put $\dot{f}_j := df_j/ds$. Here, once h is fixed, the derivative $\dot{f}_j(0)$ is bounded from above by a positive constant C independent of the choice of j (see [8]). Hence we can define $F_1(\{\mu_j\}) \in \mathbb{R} \cup \{-\infty\}$ by

$$F_1(\{\mu_j\}) \;:=\; \lim_{s\to-\infty} \{ \varliminf_{j\to\infty} \dot{f}_j(s)\} \;\leq\; C, \tag{2.5}$$

since the function $\varliminf_{j\to\infty} \dot{f}_j(s)$ is non-decreasing in s by convexity of the function f_j (cf. [16]; see also [5], Theorem 4.5).

3. Test configurations associated to balanced metrics

Hereafter, we assume that the polarized algebraic manifold (X, L) is strongly K-stable. Then by [11], (X, L) is asymptotically Chow-stable, and hence for some $\ell_0 \gg 1$, for all $\ell \geq \ell_0$, there exists a Hermitian metric h_ℓ for L such that $\omega_\ell := c_1(L; h_\ell)$ is a *balanced* Kähler metric (cf. [1], [16]) on $(X, L^{\otimes \ell})$ in the sense that

$$|\sigma_1|^2_{h_\ell} + |\sigma_2|^2_{h_\ell} + \cdots + |\sigma_{N_\ell}|^2_{h_\ell} \;=\; N_\ell/c_1(L)^n[X], \tag{3.1}$$

where $\{\sigma_\alpha\,;\, \alpha = 1, 2, \dots, N_\ell\}$ is an arbitrarily chosen orthonormal basis for (V_ℓ, ρ_ℓ). Let $\hat{\rho}_\ell$ be the associated Hermitian metric on V_ℓ defined by

$$\langle \sigma', \sigma'' \rangle_{\hat{\rho}_\ell} := \int_X (\sigma', \sigma'')_{h_\ell}\, \omega_\ell^n, \qquad \sigma', \sigma'' \in V_\ell,$$

where $(\sigma', \sigma'')_{h_\ell}$ denotes the pointwise Hermitian inner product of σ and σ' by the ℓ-multiple of h_ℓ. Now we can find orthonormal bases

$$\{\sigma_{\ell,1}, \sigma_{\ell,2}, \dots, \sigma_{\ell,N_\ell}\} \quad \text{and} \quad \{\tau_{\ell,1}, \tau_{\ell,2}, \dots, \tau_{\ell,N_\ell}\}$$

for $(V_\ell, \hat{\rho}_\ell)$ and (V_ℓ, ρ_ℓ), respectively, such that

$$\sigma_{\ell,\alpha} \;=\; \lambda_{\ell,\alpha}\tau_{\ell,\alpha}, \qquad \alpha = 1, 2, \dots, N_\ell, \tag{3.2}$$

for some positive real numbers $\lambda_{\ell,\alpha}$. Multiplying h_ℓ by a positive real constant which possibly depends on ℓ, we may assume that

$$\Pi_{\alpha=1}^{N_\ell}\lambda_{\ell,\alpha} = 1.$$

Then for each $\ell \geq \ell_0$, we have a sequence of points $\hat{\gamma}_k = (\hat{\gamma}_{k;1}, \hat{\gamma}_{k;2}, \dots, \hat{\gamma}_{k;N_\ell})$, $k = 1, 2, \dots$, in $\mathbb{Q}^{N_\ell}$ such that $\Sigma_{\alpha=1}^{N_\ell}\hat{\gamma}_{k;\alpha} = 0$ for all k, and that

$$\text{(3.3)} \qquad \hat{\gamma}_k \to -(\log\lambda_{\ell,1}, \log\lambda_{\ell,2}, \dots, \log\lambda_{\ell,N_\ell}), \qquad \text{as } k \to \infty.$$

Let $a_{\ell,k}$ be the smallest positive integer such that $a_{\ell,k}\hat{\gamma}_k$ is integral. By rewriting $a_{\ell,k}\hat{\gamma}_k$ as $\gamma_k = (\gamma_{k;1}, \gamma_{k;2}, \dots, \gamma_{k;N_\ell})$ for simplicity, we now define an algebraic group homomorphism $\psi_{\ell,k} : T = \{\, t \in \mathbb{C}^* \} \to \mathrm{SL}(V_\ell)$ by setting

$$\psi_{\ell,k}(t)\cdot\tau_{\ell,\alpha} \;:=\; t^{-\gamma_{k;\alpha}}\tau_{\ell,\alpha}, \qquad \alpha = 1, 2, \dots, N_\ell,$$

for all $t \in \mathbb{C}^*$. Let $\{\tau^*_{\ell,\alpha}\,;\, \alpha = 1, 2, \dots, N_\ell\}$ be the basis for V^*_ℓ dual to $\{\tau_{\ell,\alpha}\,;\, \alpha = 1, 2, \dots, N_\ell\}$ defined by

$$\langle \tau_{\ell,\alpha}, \tau^*_{\ell,\beta}\rangle \;=\; \begin{cases} 1, & \text{if } \alpha = \beta, \\ 0, & \text{if } \alpha \neq \beta. \end{cases}$$

Then $\psi_{\ell,k}(t)\cdot\tau^*_{\ell,\alpha} \;=\; t^{\gamma_{k;\alpha}}\tau^*_{\ell,\alpha}$. Each $\vec{z} = (z_1, z_2, \dots, z_{N_\ell}) \in \mathbb{C}^{N_\ell}\setminus\{0\}$ sitting over $(z_1 : z_2 : \cdots : z_{N_\ell}) \in \mathbb{P}^{N_\ell - 1}(\mathbb{C}) = \mathbb{P}^*(V_\ell)$ is expressible as $\Sigma_{\alpha=1}^{N_\ell} z_\alpha \tau^*_{\ell,\alpha}$, and hence the action by $t \in \mathbb{C}^*$ on $\vec{z}$ is written in the form

$$(z_1, z_2, \dots, z_{N_\ell}) \;\mapsto\; (t^{\gamma_{k;1}}z_1, t^{\gamma_{k;2}}z_2, \dots, t^{\gamma_{k;N_\ell}}z_{N_\ell}).$$

We now identify X with the subvariety $X_\ell := \Phi_\ell(X)$ in the projective space $\mathbb{P}^*(V_\ell) = \mathbb{P}^{N_\ell-1}(\mathbb{C}) = \{(z_1 : z_2 : \cdots : z_{N_\ell})\}$ via the Kodaira embedding

$$\Phi_\ell(x) := (\tau_{\ell,1}(x) : \tau_{\ell,2}(x) : \cdots : \tau_{\ell,2}(x)), \qquad x \in X.$$

For each $\ell \geq \ell_0$, we observe that $\mathrm{SL}(V_\ell)$ acts naturally on W^*_ℓ. Then by considering the sequence of test configurations

$$\mu_{\ell,k} = (\mathcal{X}^{\psi_{\ell,k}}, \mathcal{L}^{\psi_{\ell,k}}, \psi_{\ell,k}), \qquad k = 1, 2, \dots,$$

associated to $\psi_{\ell,k}$, we define a real-valued function $f_{\ell,k} = f_{\ell,k}(s)$ on the real line $\mathbb{R} = \{-\infty < s < +\infty\}$ by

$$f_{\ell,k}(s) \;:=\; \delta(\mu_{\ell,k})\,\ell^{-n}\log\|\psi_{\ell,k}(t)\cdot\hat{X}_\ell\|_{\mathrm{CH}(\rho_\ell)}.$$

Here $s \in \mathbb{R}$ and $t \in \mathbb{R}_+$ are related by $t = \exp(s/\|\mu_{\ell,k}\|_\infty)$ for $\|\mu_{\ell,k}\|_\infty \neq 0$, while we require no relations between $s \in \mathbb{R}$ and $t \in \mathbb{R}_+$ if $\|\mu_{\ell,k}\|_\infty = 0$. Put $\dot{f}_{\ell,k} := df_{\ell,k}/ds$ and $\theta_{s;\ell,k} := (1/2\pi)\log\{(\Sigma_{\alpha=1}^{N_\ell}(n!/\ell^n)\,t^{2\gamma_{k;\alpha}}|\tau_{\ell,\alpha}|^2)^{1/\ell}\}$. Then on X_ℓ viewed also as X via Φ_ℓ, we can write

$$\text{(3.4)} \qquad \psi_{\ell,k}(t)^*(\omega_{\mathrm{FS}}/\ell) = \sqrt{-1}\partial\bar{\partial}\theta_{s;\ell,k},$$

where $\omega_{\mathrm{FS}} := (\sqrt{-1}/2\pi)\partial\bar\partial \log\{(\Sigma_{\alpha=1}^{N_\ell}(n!/\ell^n)|z_\alpha|^2)^{1/\ell}\}$, and $\psi_{\ell,k}(t)$ is regarded as a mapping from $X_\ell = (\mathcal{X}^{\psi_{\ell,k}})_1$ to $\psi_{\ell,k}(t)(X_\ell) = (\mathcal{X}^{\psi_{\ell,k}})_t$. In view of [16] (see also [5] and [13]), we obtain

$$(3.5)\qquad \dot f_{\ell,k}(s) \;=\; \ell\,\delta(\mu_{\ell,k})\int_X (\partial\theta_{s;\ell,k}/\partial s)\,(\sqrt{-1}\partial\bar\partial\theta_{s;\ell,k})^n.$$

Put $\nu_{\ell,k} := \|\mu_{\ell,k}\|_\infty/a_{\ell,k} = \max\{\,|\hat\gamma_{k;\alpha}|/\ell\,;\,\alpha = 1,2,\dots,N_\ell\,\}$, where for the time being, we vary ℓ and k independently. Then

$$(3.6)\qquad (\partial\theta_{s;\ell,k}/\partial s)_{\,|s=-\nu_{\ell,k}} \;=\; \frac{\Sigma_{\alpha=1}^{N_\ell}\,\hat\gamma_{k;\alpha}\,\exp(-2\hat\gamma_{k;\alpha})\,|\tau_{\ell,\alpha}|^2}{\pi\ell\,\nu_{\ell,k}\,\Sigma_{\alpha=1}^{N_\ell}\,\exp(-2\hat\gamma_{k;\alpha})\,|\tau_{\ell,\alpha}|^2}.$$

Now for each integer r, let $O(\ell^r)$ denote a function u satisfying the inequality $|u| \le C_0\ell^r$ for some positive constant C_0 independent of the choices of k, ℓ, and α. We now fix a positive integer $\ell \gg 1$. Then by (3.3), we obtain

$$(3.7)\qquad \lambda_{\ell,\alpha}^{-2}\,\exp(-2\hat\gamma_{k;\alpha}) \,-\, 1 \;=\; O(\ell^{-n-2}), \qquad k \gg 1.$$

Moreover, in view of (3.1) and (3.2), the Kähler form ω_ℓ is written as $(\sqrt{-1}/2\pi)\partial\bar\partial\log\{(\Sigma_{\alpha=1}^{N_\ell}(n!/\ell^n)\lambda_{\ell,\alpha}^2|\tau_{\ell,\alpha}|^2)^{1/\ell}\}$. Now by (3.3), as $k\to\infty$, we have $\sqrt{-1}\partial\bar\partial\theta_{s;\ell,k\,|\,s=-\nu_{\ell,k}} \to \omega_\ell$ in C^∞. In particular for $k \gg 1$, we can further assume that

$$(3.8)\qquad \|\sqrt{-1}\partial\bar\partial\theta_{s;\ell,k\,|\,s=-\nu_{\ell,k}} - \omega_\ell\,\|_{C^m(X)} \;=\; O(\ell^{-n-2}),$$

where we fix an arbitrary integer m satisfying $m \ge 5$. Hence for each $\ell \gg 1$, we can find a positive integer $k(\ell) \gg 1$ such that both (3.7) and (3.8) hold for $k = k(\ell)$. From now on, we assume

$$(3.9)\qquad k \;=\; k(\ell),$$

and $\nu_{\ell,k} = \nu_{\ell,k(\ell)}$ will be written as ν_ℓ for simplicity. Then, since $\ell\nu_\ell \ge |\hat\gamma_{k;\alpha}|$ for all α, we have $(\partial\theta_{s;\ell,k}/\partial s)_{|\,s=-\nu_\ell} = O(1)$ by (3.6). Hence

$$(3.10)\qquad \int_X (\partial\theta_{s;\ell,k}/\partial s)\,\{\,(\sqrt{-1}\partial\bar\partial\theta_{s;\ell,k})^n \,-\, \omega_\ell^n\,\}_{\,|\,s=-\nu_\ell} \;=\; O(\ell^{-n-2}).$$

Put $I_1 := \pi\ell\,\nu_\ell\,\Sigma_{\alpha=1}^{N_\ell}\,\lambda_{\ell,\alpha}^2\,|\tau_{\ell,\alpha}|^2$ and $I_2 := \pi\ell\,\nu_\ell\,\Sigma_{\alpha=1}^{N_\ell}\,\exp(-2\hat\gamma_{k;\alpha})\,|\tau_{\ell,\alpha}|^2$. Put also $J_1 := \Sigma_{\alpha=1}^{N_\ell}\,\hat\gamma_{k;\alpha}\lambda_{\ell,\alpha}^2|\tau_{\ell,\alpha}|^2$ and $J_2 := \Sigma_{\alpha=1}^{N_\ell}\hat\gamma_{k;\alpha}\exp(-2\hat\gamma_{k;\alpha})|\tau_{\ell,\alpha}|^2$. Then by (3.6), we obtain

$$(3.11)\qquad \int_X (\partial\theta_{s;\ell,k}/\partial s)_{\,|s=-\nu_\ell}\,\omega_\ell^n \;=\; A \,+\, B \,+\, P,$$

where $A := \int_X\{(J_2/I_2) - (J_2/I_1)\}\,\omega_\ell^n$, $B := \int_X\{(J_2/I_1) - (J_1/I_1)\}\,\omega_\ell^n$ and $P := \int_X (J_1/I_1)\,\omega_\ell^n$. Note that $J_2/I_2 = O(1)$ by $\ell\nu_\ell \ge |\hat\gamma_{k;\alpha}|$, while by (3.7),

$(I_1 - I_2)/I_1 = O(\ell^{-n-2})$. Then

$$(3.12) \qquad A \;=\; \int_X \frac{J_2}{I_2}\cdot\frac{I_1 - I_2}{I_1}\,\omega_\ell^n \;=\; O(\ell^{-n-2}).$$

On the other hand by (3.7), $J_2 - J_1 \;=\; O(\ell^{-n-2})(\Sigma_{\alpha=1}^{N_\ell}|\hat{\gamma}_{k;\alpha}|\,\lambda_{\ell,\alpha}^2\,|\tau_{\ell,\alpha}|^2)$. From this together with $\ell\nu_\ell \geq |\hat{\gamma}_{k;\alpha}|$, we obtain

$$(3.13) \qquad B \;=\; \int_X \frac{J_2 - J_1}{I_1}\,\omega_\ell^n \;=\; O(\ell^{-n-2}).$$

By (3.2), $I_1 = \pi\ell\,\nu_\ell\,\Sigma_{\alpha=1}^{N_\ell}|\sigma_{\ell,\alpha}|^2$ and $J_1 := \Sigma_{\alpha=1}^{N_\ell}\hat{\gamma}_{k;\alpha}|\sigma_{\ell,\alpha}|^2$. Note also that $a_0 := \delta(\mu_{\ell,k})$ satisfies $0 < a_0 \leq \ell^n$. Put $a_1 := c_1(L)^n[X]$. In view of (3.1) and (3.5), by adding up (3.10), (3.11), (3.12) and (3.13), we obtain

$$(3.14) \qquad \begin{cases} \dot{f}_{\ell,k}(-\nu_\ell) \;=\; \ell\,a_0\,\int_X\{\,(\partial\theta_{s;\ell,k}/\partial s)\,(\sqrt{-1}\partial\bar{\partial}\theta_{s;\ell,k})^n\}_{|s=-\nu_\ell} \\ =\; a_0\,\{\,\ell\,P \,+\, O(\ell^{-n-1})\,\} = \int_X \frac{a_0\Sigma_{\alpha=1}^{N_\ell}\hat{\gamma}_{k;\alpha}|\sigma_{\ell,\alpha}|^2_{h_\ell}}{\pi\,\nu_\ell\,\Sigma_{\alpha=1}^{N_\ell}|\sigma_{\ell,\alpha}|^2_{h_\ell}}\,\omega_\ell^n \,+\, O(\ell^{-1}) \\ =\; a_0\,a_1\,(\Sigma_{\alpha=1}^{N_\ell}\hat{\gamma}_{k;\alpha})\,(\pi\,\nu_\ell\,N_\ell)^{-1} \,+\, O(\ell^{-1}) \;=\; O(\ell^{-1}), \end{cases}$$

where in the last line, we used the equality $\Sigma_{\alpha=1}^{N_\ell}\hat{\gamma}_{k;\alpha} \;=\; 0$. In the next section, the sequence of test configurations $\mu_{\ell,k(\ell)} = (\mathcal{X}^{\psi_{\ell,k(\ell)}}, \mathcal{L}^{\psi_{\ell,k(\ell)}}, \psi_{\ell,k(\ell)})$, $\ell \geq \ell_0$, for (X, L) will be considered.

4. Proof of Main Thorem

In this section, under the same assumption as in the previous section, we shall show that $c_1(L)$ admits a constant scalar curvature Kähler metric. Put

$$\nu_\infty \;:=\; \sup_\ell\;\nu_\ell,$$

where the supremum is taken over all positive integers ℓ satisfying $\ell \geq \ell_0$. Then the following cases are possible:

Case 1: $\nu_\infty \;=\; +\infty$. Case 2: $\nu_\infty \;<\; +\infty$.

Step 1. If Case 1 occurs, then an increasing subsequence $\{\,\ell_j\,;\,j = 1, 2, \dots\}$ of $\{\,\ell \in \mathbb{Z}\,;\,\ell \geq \ell_0\,\}$ can be chosen in such a way that $\{\nu_{\ell_j}\}$ is a monotone increasing sequence satisfying

$$(4.1) \qquad \lim_{j\to\infty}\,\nu_{\ell_j} \;=\; +\infty.$$

For simplicity, the functions $f_{\ell_j,k(\ell_j)}$ will be written as f_j, while we write the test configurations

$$\mu_{\ell_j,k(\ell_j)} = (\mathcal{X}^{\psi_{\ell_j,k(\ell_j)}}, \mathcal{L}^{\psi_{\ell_j,k(\ell_j)}}, \psi_{\ell_j,k(\ell_j)}), \qquad j = 1, 2, \dots,$$

as $\mu_j = (\mathcal{X}_j, \mathcal{L}_j, \psi_j)$. Now by (3.14), there exists a positive constant C independent of j such that

$$-C/\ell_j \;\le\; \dot{f}_j(-\nu_{\ell_j})$$

for all j. On the other hand, for all positive integers j' satisfying $j' \ge j$, we have $-\nu_{\ell_{j'}} \le -\nu_{\ell_j}$ by monotonicity. Since the function $\dot{f}_{j'}(s)$ in s is non-decreasing, we obtain

$$-C/\ell_{j'} \;\le\; \dot{f}_{j'}(-\nu_{\ell_{j'}}) \;\le\; \dot{f}_{j'}(-\nu_{\ell_j}). \tag{4.2}$$

We here observe that $-C/\ell_{j'} \to 0$ as $j' \to \infty$. It now follows from (4.2) that, for each fixed j,

$$\underline{\lim}_{j'\to\infty} \dot{f}_{j'}(-\nu_{\ell_j}) \;\ge\; 0.$$

Since the function $\underline{\lim}_{j'\to\infty} \dot{f}_{j'}(s)$ in s is non-decreasing, we therefore obtain

$$\underline{\lim}_{j'\to\infty} \dot{f}_{j'}(s) \;\ge\; 0 \;\text{ for all }\; s \ge -\nu_{\ell_j},$$

while this holds for all positive integers j. Then by (4.1), $\underline{\lim}_{j'\to\infty} \dot{f}_{j'}(s)$ is a nonnegative function in s on the whole real line $\mathbb{R}$. Hence

$$F_1(\{\mu_j\}) \;=\; \lim_{s\to-\infty}\{\, \underline{\lim}_{j'\to\infty} \dot{f}_{j'}(s)\} \;\ge\; 0.$$

Now by the strong K-stability of (X, L), we obtain $F_1(\{\mu_j\}) = 0$, so that μ_j are trivial for all $j \gg 1$. Then $\psi_{\ell_j, k(\ell_j)}$ are trivial for all $j \gg 1$. This usually gives us a contradiction. Even if not, however, by assuming the triviality of μ_j for all $j \gg 1$, we proceed as follow. By (3.4), for all $s \in \mathbb{R}$, we obtain

$$\sqrt{-1}\partial\bar{\partial}\theta_{s;\ell_j,k(\ell_j)} \;=\; (\omega_{\mathrm{FS}}/\ell_j)_{|X_{\ell_j}} \;=\; \Phi^*_{\ell_j}(\omega_{\mathrm{FS}}/\ell_j), \qquad j \gg 1,$$

by identfying X_{ℓ_j} with X via Φ_{ℓ_j}, where by [15], $\|\Phi^*_{\ell_j}(\omega_{\mathrm{FS}}/\ell_j) - \omega\|_{C^5(X)} = O(\ell_j^{-2})$. From this together with (3.8), we obtain

$$\|\,\omega \,-\, \omega_{\ell_j}\|_{C^m(X)} \;=\; O(\ell_j^{-2}), \qquad j \gg 1. \tag{4.3}$$

Let S_ω be the scalar curvature function for ω. Then by [4] (see also [15]), we obtain the following asymptotic expansion:

$$1 + (S_\omega/2)\ell_j^{-1} + O(\ell_j^{-2}) \;=\; \Sigma_{\alpha=1}^{N_{\ell_j}} (n!/\ell_j^n)\, |\tau_{\ell_j,\alpha}|_h^2 \;=\; B_{\ell_j}(\omega), \tag{4.4}$$

where for every Kähler form θ in $c_1(L)_{\mathbb{R}}$, $B_{\ell_j}(\theta)$ denotes the ℓ_j-th asymptotic Bergman kernel for (X, θ). On the other hand, for $\ell \gg 1$, we observe that N_ℓ is a polynomial in ℓ. Since each ω_{ℓ_j} is balanced, by setting $\ell = \ell_j$ in (3.1) and dividing both sides of the equality by $\ell_j^n/n!$, we obtain (cf. [7], (1.4))

$$1 \,+\, C_0\,\ell_j^{-1} \,+\, O(\ell_j^{-2}) \;=\; \Sigma_{\alpha=1}^{N_{\ell_j}} (n!/\ell_j^n)\, |\sigma_{\ell_j,\alpha}|^2_{h_{\ell_j}} \;=\; B_{\ell_j}(\omega_{\ell_j}), \tag{4.5}$$

where C_0 is a real constant independent of the choice of j. In view of (4.3), by comparing (4.4) with (4.5), we now conclude that $S_\omega/2 = C_0$. Hence ω is a constant scalar curvature Kähler metric in the class $c_1(L)_{\mathbb{R}}$.

Step 2. Suppose that Case 2 occurs. Put $\hat{\lambda}_{\ell,\alpha} := -(1/\ell)\log\lambda_{\ell,\alpha}$. Then by (3.3), we may assume that $k = k(\ell)$ in (3.9) is chosen in such a way that

$$(4.6) \qquad \hat{\gamma}_{k(\ell);\alpha} - 1 \;\leq\; \ell\,\hat{\lambda}_{\ell,\alpha} \;\leq\; \hat{\gamma}_{k(\ell);\alpha} + 1, \qquad \alpha = 1, 2, \dots, N_\ell,$$

for all ℓ with $\ell \geq \ell_0$. Then for each ℓ, by using the notation in Definition 5.3 in Appendix, we have an ℓ-th root

$$(\mathcal{Y}^{(\ell)}, \mathcal{Q}^{(\ell)}, D^{(\ell)}, \varphi_\ell), \qquad \ell \geq \ell_0,$$

of the test configuration $\mu_{\ell,k(\ell)}$ in Section 3. Let $\chi_{\ell,\beta}$, $\beta = 1, 2, \dots, N_1$, be the weights of the $T_{\mathbb{R}}$-action via $\varphi_\ell^{\mathrm{SL}}$ on V_1^*, where $V_1 := H^0(X, L)$. Put $\hat{\chi}_{\ell,\beta} := \chi_{\ell,\beta}/a_{\ell,k(\ell)}$. For ℓ with $\ell \geq \ell_0$, let α and β be arbitrary integers satisfying $1 \leq \beta \leq N_1$ and $1 \leq \alpha \leq N_\ell$. By (4.6) together with the definition of $\nu_{\ell,k}$, we easily see from the inequality $\nu_\infty < +\infty$ that

$$(4.7) \qquad |\hat{\lambda}_{\ell,\alpha}| \leq C_1 \quad \text{and} \quad |\hat{\chi}_{\ell,\beta}| \leq C_1,$$

where C_1 is a positive real constant independent of the choices of ℓ, α and β (see [12] for the second inequality of (4.7); see also [10]). Let $Z_\ell := (\varphi_\ell)_*(t\partial/\partial t) \in \mathfrak{sl}(V_1)$ be the infinitesimal generator for the one-parameter group $\varphi_\ell^{\mathrm{SL}}$. Then by setting $\hat{Z}_\ell := Z_\ell/a_{\ell,k(\ell)}$, we obtain

$$\hat{Z}_\ell \cdot \kappa_{\ell,\beta} \;=\; -\,\hat{\chi}_{\ell,\beta}\kappa_{\ell,\beta}, \qquad \beta = 1, 2, \dots, N_1,$$

for a suitable orthonormal basis $\{\kappa_{\ell,1}, \kappa_{\ell,2}, \dots, \kappa_{\ell,N_1}\}$ for (V_1, ρ_1). For the sequence $\{\, \hat{Z}_\ell\,;\, \ell \geq \ell_0 \,\}$, by choosing its suitable subsequence

$$\{\, \hat{Z}_{\ell_j}\,;\, j = 1, 2, \dots \},$$

we obtain real numbers $\hat{\chi}_{\infty,\beta} \in \mathbb{R}$, $\beta = 1, 2, \dots, N_1$, and an orthonormal basis $\{\kappa_{\infty,1}, \kappa_{\infty,2}, \dots, \kappa_{\infty,N_1}\}$ for V_1 such that, for all β,

$$\kappa_{\ell_j,\beta} \to \kappa_{\infty,\beta} \quad \text{and} \quad \hat{\chi}_{\ell_j,\beta} \to \hat{\chi}_{\infty,\beta},$$

as $j \to \infty$. Hence we can define $\hat{Z}_\infty \in \mathfrak{sl}(V_1)$ such that $\hat{Z}_\infty \cdot \kappa_{\infty,\beta} = -\,\hat{\chi}_{\infty,\beta}\kappa_{\infty,\beta}$ for all β. Then we have the following convergence in C^∞:

$$(4.8) \qquad \hat{Z}_{\ell_j} \to \hat{Z}_\infty, \qquad \text{as } j \to \infty.$$

For each ℓ, in view of the relation $t = \exp(s/\|\mu_{\ell,k(\ell)}\|_\infty)$, $s = -\nu_\ell$ corresponds to $t = \hat{t}_\ell$, where $\hat{t}_\ell := \exp(-\nu_\ell/\|\mu_{\ell,k(\ell)}\|_\infty) = \exp(-1/a_{\ell,k(\ell)})$. Until the end of this section, test configurations $\mu_{\ell,k(\ell)}$ for (X, L) will be written simply as

$$\mu_\ell = (\mathcal{X}^{(\ell)}, \mathcal{L}^{(\ell)}, \psi_\ell), \qquad \ell \geq \ell_0.$$

For the test configuration μ_ℓ, each $t \in T$ not as a complex number but as an element of the group T of transformations on μ_ℓ will be written as $g_{\mu_\ell}(t)$. For the Kodaira embedding $\Phi_\ell : X \hookrightarrow \mathbb{P}^{N_\ell - 1}(\mathbb{C})$ in Section 3, we consider $\mathbb{C}^{N_\ell} \setminus \{0\} = \{(z_1, z_2, \dots, z_{N_\ell}) \neq 0\}$ over $\mathbb{P}^{N_\ell}(\mathbb{C})$, so that $z = (z_1, z_2, \dots, z_{N_\ell})$ sits over $[z] = (z_1 : z_2 : \cdots : z_{N_\ell})$. Since the restriction of $\mathcal{O}_{\mathbb{P}^{N_\ell-1}}(\mathbb{C})$ to X_ℓ is viewed as L by identifying X with its image $X_\ell := \Phi_\ell(X)$, we can write

$$z_{\alpha|X_\ell} \;=\; \tau_{\ell,\alpha}, \qquad \alpha = 1, 2, \dots, N_\ell,$$

for the orthonormal basis $\{\tau_{\ell,1}, \tau_{\ell,2}, \dots, \tau_{\ell,N_\ell}\}$ of (V_ℓ, ρ_ℓ). We now define a Hermitian metric ϕ_ℓ for L^{-1} by setting, for all $[z] = \Phi_\ell(x)$ in X_ℓ,

$$\phi_\ell([z]) \;:=\; \{\, (n!/\ell^n)\Sigma_{\alpha=1}^{N_\ell} |z_\alpha|^2 \,\}^{1/\ell} \;=\; \{\, (n!/\ell^n)\Sigma_{\alpha=1}^{N_\ell} |\tau_{\ell,\alpha}(x)|^2 \,\}^{1/\ell},$$

where the line bundle $L^{-\ell}$ on X is viewed as the dual $\{\mathcal{L}^{(\ell)}{}_{|X_\ell}\}^{-1}$ of the line bundle $\mathcal{L}^{(\ell)}$ restricted to $\mathcal{X}_1^{(\ell)}\,(= X_\ell)$. Let $\mathcal{K}_t$, $t \neq 0$, denote the set of all Hermitian metrics on the line bundle $\{\mathcal{L}^{(\ell)}{}_{|\mathcal{X}_t^{(\ell)}}\}^{-1}$. Then the action by $g_{\mu_\ell}(t)$ takes $\mathcal{K}_1$ to $\mathcal{K}_t$. For instance, $g_{\mu_\ell}(t)$ takes the point $z = (z_1, z_2, \dots, z_{N_\ell})$ to $g_{\mu_\ell}(t) \cdot z = (t^{\gamma_{k(\ell),1}} z_1, t^{\gamma_{k(\ell),2}} z_2, \dots, t^{\gamma_{k(\ell),N_\ell}} z_{N_\ell})$, while for each $[z] \in X_\ell$, $\phi_\ell([z])$ is mapped to the point $g_{\mu_\ell}(t) \cdot \phi_\ell([z])$ defined by

$$\{\, (n!/\ell^n)\Sigma_{\alpha=1}^{N_\ell} |g_{\mu_\ell}(t) \cdot z_\alpha|^2 \,\}^{1/\ell} \;=\; \{\, (n!/\ell^n)\Sigma_{\alpha=1}^{N_\ell} |t|^{2\gamma_{k(\ell),\alpha}} |z_\alpha|^2 \,\}^{1/\ell},$$

and this defines $g_{\mu_\ell}(t) \cdot \phi_\ell \in \mathcal{K}_t$. Now by [15], $u_\ell := (1/2\pi)\log(\phi_\ell/h^*)$ viewed as a function on X can be estimated in the form

$$\|u_\ell\|_{C^{m+2}(X)} = O(\ell^{-2}), \tag{4.9}$$

where the dual h^* of h is viewed as a Hermitian metric for the line bundle L^{-1}. Put $\omega(\ell, t) := (\sqrt{-1}/2\pi)\partial\bar\partial \log(g_{\mu_\ell}(t)^*\{g_{\mu_\ell}(t) \cdot h^*\})$, $t \neq 0$. For the Fubini-Study form $\omega_{\rm FS}$ in Section 3, its restriction to $X_\ell\,(= X)$ is written as

$$\omega_{{\rm FS}\,|X_\ell} \;=\; (\sqrt{-1}\,\ell/2\pi)\partial\bar\partial \log \phi_\ell.$$

Since $\psi_\ell(t)^*(\omega_{\rm FS}/\ell) = (\sqrt{-1}/2\pi)\partial\bar\partial \log(g_{\mu_\ell}(t)^*\{g_{\mu_\ell}(t) \cdot \phi_\ell\})$ on X_ℓ (see (3.4)), we can rewrite it in the form (see [9])

$$\psi_\ell(t)^*(\omega_{\rm FS}/\ell)_{\,|X_\ell} \;=\; \omega(\ell, t) \,+\, \sqrt{-1}\,\partial\bar\partial u_\ell. \tag{4.10}$$

Let us consider the test configuration $\bar\mu_\ell := (\mathcal{Y}^{(\ell)}, \mathcal{Q}^{(\ell)}, \varphi_\ell)$ for (X, L) of exponent 1. Each $t \in T$, not as a complex number but as an element of the group T of transformations on $\bar\mu_\ell$, will be denoted by $g_{\bar\mu_\ell}(t)$. Then by (5.8) in Appendix, we also have the expression

$$\omega(\ell, t) \;=\; (\sqrt{-1}/2\pi)\partial\bar\partial \log(g_{\bar\mu_\ell}(t)^*\{g_{\bar\mu_\ell}(t) \cdot h^*\}), \qquad t \in T_{\mathbb{R}}, \tag{4.11}$$

since for each such t, the action of $g_{\mu_\ell}(t)$ on $|\mathcal{L}|^{2/\ell}$ coincides with the action of $g_{\bar\mu_\ell}(t)$ on $|\mathcal{Q}|^2$ up to constant scalar multiplication, where constant scalar multiplication arises from the action on the factor $|\zeta|^{2/\ell}$. Since $\omega(\ell, t)$ doesn't change even if $g_{\bar{\mu}_\ell}(t)\cdot h^*$ in (4.11) is replaced by $C(t)g_{\bar{\mu}_\ell}(t)\cdot h^*$ for a positive real constant $C(t)$ possibly depending on t. Hence we may consider the action by $g_{\bar{\mu}_\ell}(t)$ on $|\mathcal{L}|^{2/\ell}$ modulo constant scalar multiplication. In this sense, for each $t \in T_{\mathbb{R}}$, the action by $g_{\bar{\mu}_\ell}(t)$ in (4.11) is induced by the action by the element $\varphi_\ell^{\mathrm{SL}}(t)$ in $\mathrm{SL}(V_1)$. In particular for $t = \hat{t}_\ell$,

$$(4.12) \qquad g_{\bar{\mu}_\ell}(\hat{t}_\ell)\text{'s action is induced by } \varphi_\ell^{\mathrm{SL}}(\hat{t}_\ell) \;=\; \exp(-\hat{Z}_\ell) \;\in\; \mathrm{SL}(V_1).$$

For $\theta_{s;\ell,k(\ell)} := (1/2\pi)\log\{(\Sigma_{\alpha=1}^{N_\ell}(n!/\ell^n)\, t^{2\gamma_{k(\ell);\alpha}}|\tau_{\ell,\alpha}|^2)^{1/\ell}\}$ in Section 3, at the point $s = -\nu_\ell$, we see from (3.4) that

$$(4.13) \qquad \sqrt{-1}\partial\bar{\partial}\theta_{s;\ell,k(\ell)\,|s=-\nu_\ell} \;=\; \psi_\ell(\hat{t}_\ell)^*(\omega_{\mathrm{FS}}/\ell)_{\,|X_\ell}.$$

Then by (3.8), (4.10) and (4.13),

$$(4.14) \qquad \|\omega_\ell - \omega(\ell,\hat{t}_\ell) - \sqrt{-1}\partial\bar{\partial}u_\ell\|_{C^m(X)} \;=\; O(\ell^{-n-2}).$$

For the element $\hat{Z}_\infty$ of $\mathfrak{sl}(V_1)$ in (4.8), we now define a subset $\mathcal{Y}_{\mathbb{R}}^{(\infty)}$ of $\mathbb{R}\times\mathbb{P}^*(V_1)$ as the closure of

$$\bigcup_{s\in\mathbb{R}} \{\pm\exp s\}\times\exp(s\hat{Z}_\infty)(X_1)$$

in the real manifold $\mathbb{R}\times\mathbb{P}^*(V_1)$, where X_1 is the image $\Phi_1(X)$ of X under the Kodaira embedding

$$\Phi_1 : X \to \mathbb{P}^*(V_1)$$

associated to the complete linear system $|L|$ on X. By the projection of $\mathbb{R}\times\mathbb{P}^*(V_1)$ to the first factor $\mathbb{R}$, we see that $\mathcal{Y}_{\mathbb{R}}^{(\infty)}$ has a natural structure of a fiber space over $\mathbb{R}$. Let $\mathcal{Q}^{(\infty)}$ denote the restriction to $\mathcal{Y}_{\mathbb{R}}^{(\infty)}$ of the pullback $\mathrm{pr}_2^*\,\mathcal{O}_{\mathbb{P}^*(V_1)}(1)$, where $\mathrm{pr}_2 : \mathbb{R}\times\mathbb{P}^*(V_1)\to\mathbb{P}^*(V_1)$ is the projection to the second factor. Then the $T_{\mathbb{R}}$-action on $\mathcal{Y}_{\mathbb{R}}^{(\infty)}$ induced by

$$T_{\mathbb{R}}\times(\mathbb{R}\times\mathbb{P}^*(V_1)) \to \mathbb{R}\times\mathbb{P}^*(V_1), \quad (\exp s,(r,x)) \mapsto (\,(\exp s)r, \exp(s\hat{Z}_\infty)\cdot x\,),$$

naturally lifts to a $T_{\mathbb{R}}$-action on $\mathcal{Q}^{(\infty)}$. This action is induced by the Lie group homomorphism $\varphi_\infty : \mathbb{R}_+ \to \mathrm{SL}(V_1)$ defined by

$$\varphi_\infty(t) := \exp(\,(\log t)\hat{Z}_\infty\,), \qquad t\in\mathbb{R}_+.$$

For $\bar{\mu}_\infty := (\mathcal{Y}_{\mathbb{R}}^{(\infty)}, \mathcal{Q}^{(\infty)}, \varphi_\infty)$, each $t\in T_{\mathbb{R}}$ not as a real number but as an element of the group $T_{\mathbb{R}}$ of transformations on $\bar{\mu}_\infty$ will be written as $g_{\bar{\mu}_\infty}(t)$. Consider the action by $g_{\bar{\mu}_\infty}(\hat{t}_\ell)$ on $|\mathcal{Q}^{(\infty)}|^2$ modulo constant scalar multiplication. For $\hat{t}_\infty := 1/e$, we have $\varphi_\infty(\hat{t}_\infty) \;=\; \exp(-\hat{Z}_\infty)$, and hence

$$(4.15) \quad g_{\bar{\mu}_\infty}(\hat{t}_\infty)\text{'s action is induced by } \varphi_\infty(\hat{t}_\infty) \;=\; \exp(-\hat{Z}_\infty) \;\in\; \mathrm{SL}(V_1).$$

Put $\omega_\infty := (\sqrt{-1}/2\pi)\partial\bar{\partial}\log(g_{\bar{\mu}_\infty}(\hat{t}_\infty)^*\{g_{\bar{\mu}_\infty}(\hat{t}_\infty)\cdot h^*\})$ (cf. Remark 5.9). By (4.11), (4.12) and (4.15), it follows from (4.8) that

$$(4.16) \qquad \omega(\ell_j,\hat{t}_{\ell_j}) \to \omega_\infty \text{ in } C^\infty, \qquad \text{as } j\to\infty.$$

Then by (4.9), (4.14) and (4.16),

$$\omega_{\ell_j} \to \omega_\infty \text{ in } C^m, \quad \text{as } j \to \infty. \tag{4.17}$$

By (4.17), given a sufficiently small $\varepsilon > 0$, there exists a $j_0 \gg 1$ such that $\|S_{\omega_{\ell_j}} - S_{\omega_\infty}\|_{C^0(X)} \leq \varepsilon$ for all $j \geq j_0$. Hence by [4] (see also [15]),

$$|\,\ell_j\{B_{\ell_j}(\omega_{\ell_j}) - \hat{N}_{\ell_j}\} - \ell_j\{B_{\ell_j}(\omega_\infty) - \hat{N}_{\ell_j}\}\,| \;\leq\; \varepsilon/2 \,+\, O(1/\ell_j), \quad j \geq j_0,$$

where $\hat{N}_{\ell_j} := (n!/\ell_j^n)N_{\ell_j}/c_1(L)^n[X]$. On the other hand, since each ω_{ℓ_j} is balanced, we have $B_{\ell_j}(\omega_{\ell_j}) = \hat{N}_{\ell_j}$ for all j. It then follows that

$$|\,\ell_j\{B_{\ell_j}(\omega_\infty) - \hat{N}_{\ell_j}\}\,| \;\leq\; \varepsilon/2 \,+\, O(1/\ell_j), \quad j \geq j_0.$$

Hence, since $\hat{N}_{\ell_j} = 1 + C_0\ell_j^{-1} + O(\ell_j^{-2})$ for a real constant C_0 independent of j, again by [4] (see also [15]) applied to ω_∞, we obtain

$$|\,(S_{\omega_\infty}/2) \,-\, C_0\,| \;\leq\; \varepsilon/2 \,+\, O(1/\ell_j), \qquad j \geq j_0,$$

so that by letting $j \to \infty$, we have $|\,(S_{\omega_\infty}/2) \,-\, C_0\,| \;\leq\; \varepsilon/2$. Since $\varepsilon > 0$ can be chosen arbitrarily, we obtain $S_{\omega_\infty} = 2C_0$, as required.

Remark 4.18. The $(1,1)$-form ω_∞ on X is positive-definite as follows: For each $t \in T_{\mathbb{R}}$ viewed as a real number, the fiber of $\mathcal{Y}_{\mathbb{R}}^{(\infty)}$ over $t \in \mathbb{R} \setminus \{0\}$ will be denoted by $\mathcal{Y}_t$, where $\mathcal{Y}_t \cong X$ biholomorphically. For simplicity, the fiber $(\mathcal{Q}^{(\infty)})_t$ of $\mathcal{Q}^{(\infty)}$ over t will be written as $\mathcal{Q}_t$, and $g_{\bar{\mu}_\infty}(\hat{t}_\infty)$ will be written as g. Then g takes $\mathcal{Y}_1$ holomorphically onto $\mathcal{Y}_{\hat{t}_\infty}$. Hence

$$\omega_\infty = (\sqrt{-1}/2\pi)g^*\partial\bar{\partial}\log(g \cdot h^*). \tag{4.19}$$

Moreover $g : \mathcal{Y}_1 \to \mathcal{Y}_{t'}$ lifts holomorphically to a map, denoted also by g by abuse of terminology, of $\mathcal{Q}_1$ onto $\mathcal{Q}_{\hat{t}_\infty}$. By choosing a local base b for $\mathcal{Q}_1$ on an open subset U of $\mathcal{Y}_1$, we can write h^* as $Hb\bar{b}$ for some positive real-valued function H on U. Since $\omega = c_1(L;h)$ is Kähler, $\sqrt{-1}\partial\bar{\partial}\log H$ is positive-definite on U. Then by $g \cdot h^* = \,(H \circ g^{-1})\, g(b)\overline{g(b)}$, we see that

$$\sqrt{-1}\partial\bar{\partial}\log(g \cdot h^*) \;=\; \sqrt{-1}\partial\bar{\partial}\log(H \circ g^{-1})$$

is positive-definite on $g(U)$. From this together with (4.19), we now conclude that ω_∞ is positive-definite.

5. Appendix

In this appendix, we consider a test configuration $\mu = (\mathcal{X}, \mathcal{L}, \psi)$ for (X, L), and let $\pi : \mathcal{X} \to \mathbb{A}^1$ be the associated T-equivariant projective morphism. For the exponent ℓ of μ, ψ is an algebraic group homomorphism

$$\psi : \mathbb{C}^* \to \mathrm{GL}(V_\ell),$$

and by choosing a Hermitian metric h for L, we endow $V_\ell := H^0(X, L^{\otimes \ell})$ with the Hermitian metric ρ_ℓ as in the introduction.

Definition 5.1. A pair $(\hat{\mathcal{X}}, \hat{\mathcal{L}})$ of a non-singular irreducible algebraic variety $\hat{\mathcal{X}}$ and an invertible sheaf $\hat{\mathcal{L}}$ over $\hat{\mathcal{X}}$ is called a *T-equivariant desingularization* of $(\mathcal{X}, \mathcal{L})$, if there exists a T-equivariant proper birational morphism $\iota : \hat{\mathcal{X}} \to \mathcal{X}$, isomorphic over $\mathcal{X} \setminus \mathcal{X}_0$, such that $\hat{\mathcal{L}} = \iota^* \mathcal{L}$.

Theorem 5.2. *There exist a T-equivariant desingularization $(\hat{\mathcal{X}}, \hat{\mathcal{L}})$ of $(\mathcal{X}, \mathcal{L})$ and a test configuration $(\mathcal{Y}, \mathcal{Q}, \varphi)$ for (X, L) of exponent 1 such that*

$$\hat{\mathcal{L}} \;=\; \mathcal{O}_{\hat{\mathcal{X}}}(\hat{D}) \otimes \eta^* \mathcal{Q}^{\otimes \ell},$$

where $\eta : \hat{\mathcal{X}} \to \mathcal{Y}$ is a T-equivariant proper birational morphism, isomorphic over $\mathcal{Y} \setminus \mathcal{Y}_0$, and $\hat{D}$ is a divisor on $\hat{\mathcal{X}}$ sitting in $\hat{\mathcal{X}}_0$ set-theoretically.

Definition 5.3. Taking the $\mathbb{Q}$-divisor $D := \hat{D}/\ell$ on $\hat{\mathcal{X}}$, we call the quadruple $(\mathcal{Y}, \mathcal{Q}, D, \varphi)$ an *ℓ-th root* of the test configuration $(\mathcal{X}, \mathcal{L}, \psi)$.

Proof: Consider the relative Kodaira embedding

$$\mathcal{X} \;\hookrightarrow\; \mathbb{A}^1 \times \mathbb{P}^*(V_\ell)$$

whose restriction $\mathcal{X}_z \hookrightarrow \{z\} \times \mathbb{P}^*(V_\ell)$ over each $z \in \mathbb{A}^1$ is the Kodaira embedding of $\mathcal{X}_z$ by the complete linear system $|\mathcal{L}_z|$. Let H be a general member in the complete linear system $|L|$ for the line bundle L on X. By the identification $X = \mathcal{X}_1$, we view H as a divisor on $\mathcal{X}_1$. Then on the projective bundle $\mathbb{A}^1 \times \mathbb{P}^*(V_\ell)$, a T-invariant irreducible reduced divisor δ can be chosen as a projective subbundle such that

$$\delta \cdot \mathcal{X}_1 \;=\; \ell H,$$

where ℓH is viewed as a member of the complete linear system $|\mathcal{L}_1| = |L^{\otimes \ell}|$ on $\mathcal{X}_1 = X$. For $\mathcal{X}$, we choose its proper T-equivariant desinguralization

$$\iota : \hat{\mathcal{X}} \to \mathcal{X}$$

isomorphic over $\mathcal{X} \setminus \mathcal{X}_0$. Put $\hat{\pi} := \pi \circ \iota$. Consider the T-invariant irreducible reduced divisor $\mathcal{H}$ on $\hat{\mathcal{X}}$ obtained as the closure in $\hat{\mathcal{X}}$ of the preimage of

$$\bigcup_{t \in \mathbb{C}^*} \{t\} \times \psi(t) H$$

under the mapping ι, where H on X is viewd as a subset $\mathbb{P}^*(V_\ell)$ via the Kodaira embedding $X \subset \mathbb{P}^*(V_\ell)$ associated to the complete linear system $|L^{\otimes \ell}|$. Then we have the following equality of divisors on $\hat{\mathcal{X}}$:

$$\iota^*(\delta \cdot \mathcal{X}) \;=\; \hat{D} + \ell \mathcal{H}, \tag{5.4}$$

where $\hat{D}$ is an effective divisor on $\mathcal{X}$ with support sitting in $\mathcal{X}_0$ set-theoretically. Since $\mathcal{H}$ is a T-invariant divisor on $\hat{\mathcal{X}}$, the T-action on $\hat{\mathcal{X}}$ lifts to a T-linearization of $\hat{\mathcal{Q}} := \mathcal{O}_{\hat{\mathcal{X}}}(\mathcal{H})$. Since $\mathcal{L} = \mathcal{O}_{\mathcal{X}}(\delta \cdot \mathcal{X})$, by (5.4), we obtain

$$\hat{\mathcal{L}} \;=\; \mathcal{O}_{\hat{\mathcal{X}}}(\hat{D}) \otimes \hat{\mathcal{Q}}^{\otimes \ell}. \tag{5.5}$$

For the direct image sheaf $F := \hat{\pi}_* \hat{\mathcal{Q}}$ over $\mathbb{A}^1$, let F_z be the fiber of F over each $z \in \mathbb{A}^1$. Then we have a T-equivariant rational map

$$\eta : \hat{\mathcal{X}} \to \mathbb{P}^*(F)$$

whose restriction over each $z \in \mathbb{A}^1 \setminus \{0\}$ is the Kodaira embedding $\eta_z : \hat{\mathcal{X}}_z \hookrightarrow \mathbb{P}^*(F_z)$ associated to the complete linear system $|\hat{\mathcal{Q}}_z|$ on $\hat{\mathcal{X}}_z$. Put $\mathcal{Y}_z := \eta_z(\hat{\mathcal{X}}_z)$. Then the open subset $\hat{\pi}^{-1}(\mathbb{A}^1 \setminus \{0\})$ of $\hat{\mathcal{X}}$ is naturally identified with the T-invariant subset

$$\mathcal{Y}^\circ := \bigcup_{0 \neq z \in \mathbb{A}^1} \mathcal{Y}_z$$

of $\mathbb{P}^*(F)$. Let $\mathcal{Y}$ be the T-invariant subvariety of $\mathbb{P}^*(F)$ obtained as the closure of $\mathcal{Y}^\circ$ in $\mathbb{P}^*(F)$, i.e., $\mathcal{Y}$ is the meromorphic image of $\hat{\mathcal{X}}$ under the rational map η. Then the restriction

$$\pi_{\mathcal{Y}} : \; \mathcal{Y} \;\to\; \mathbb{A}^1$$

to $\mathcal{Y}$ of the natural projection of $\mathbb{P}^*(F)$ onto $\mathbb{A}^1$ is a T-equivariant projective morphism with a relatively very ample invertible sheaf

$$\mathcal{Q} \;:=\; \mathcal{O}_{\mathbb{P}^*(F)}(1)_{|\mathcal{Y}}$$

on the fiber space $\mathcal{Y}$ over $\mathbb{A}^1$. Note that $\hat{\pi} = \pi_{\mathcal{Y}} \circ \eta$. The T-action on $\hat{\mathcal{Q}}$ naturally induces a T-action on F, and it then induces a T-action on $\mathcal{O}_{\mathcal{Y}/\mathbb{A}^1}(1)$ covering the T-action on $\mathcal{Y}$. By the affirmative solution of T-equivariant Serre's conjecture, we have a T-equivariant trivialization

$$F \;\cong\; \mathbb{A}^1 \times F_0,$$

where this isomorphism can be chosen in such a way that the Hermitian metric ρ_1 ($= \rho_{\ell\,|\ell=1}$) as in the introduction on

$$F_1 \;=\; V_1 \;=\; H^0(X, L)$$

is taken to a Hermitian metric on F_0 which is preserved by the action of the compact subgroup $S^1 \subset T$ (see [3]). By this trivialization, F_0 can be identified with F_1 ($= V_1$), so that the T-action on F_0 induces a representation

$$\varphi : \; T \to \mathrm{GL}(V_1).$$

Hence $(\mathcal{Y}, \mathcal{Q}, \varphi)$ is a test configuration for (X, L) of exponent 1. Since $\hat{\mathcal{Q}} = \mathcal{O}_{\hat{\mathcal{X}}}(\mathcal{H})$, the base point set B for the subspace of $H^0(\hat{\mathcal{X}}_0, \hat{\mathcal{Q}}_0)$ associated to F_0 contains no components of dimension n. However, replacing $\hat{\mathcal{X}}$ by its suitable birational model obtained from $\hat{\mathcal{X}}$ by a sequence of T-equivariant blowing-ups with centers sitting over B, we may assume without loss of generality that B is purely n-dimensional, i.e., $B = \emptyset$. Now the rational map $\eta : \hat{\mathcal{X}} \to \mathcal{Y} \subset \mathbb{P}^*(F)$ is holomorphic, and hence

$$\hat{\mathcal{Q}} \; = \; \eta^* \mathcal{Q},$$

as required. This together with (5.5) completes the proof of Theorem 5.2.

Remark 5.6. Note that the divisor $\hat{D}$ on $\hat{\mathcal{X}}$ is preserved by the T-action. Since $\mathcal{O}_{\hat{\mathcal{X}}}(\hat{D}) = \eta^* \mathcal{Q}^{\otimes \ell} \otimes \hat{\mathcal{L}}^{-1}$, the actions of $T\,(= \mathbb{C}^*)$ on $\mathcal{Q}$ and $\hat{\mathcal{L}}$ induce a T-action on the invertible sheaf $\mathcal{O}_{\hat{\mathcal{X}}}(\hat{D})$. Let ζ be a natural nonzero section for $\mathcal{O}_{\hat{\mathcal{X}}}(\hat{D})$ on $\hat{\mathcal{X}}$ having $\hat{D}$ as the divisor zero(ζ) of the zeroes. Then the action of each element t of T on the line $\mathbb{C}\zeta$ is written as

$$\zeta \; \mapsto \; t^{\alpha} \zeta,$$

where $\alpha \in \mathbb{Z}$ is the weight of the T-action on $\mathbb{C}\zeta$.

For test configurations μ and $\bar{\mu} := (\mathcal{Y}, \mathcal{Q}, \varphi)$ above, each $t \in T$ not as a complex number but as an element of the group T of transformation on μ and $\bar{\mu}$ will be written as $g_{\mu}(t)$ and $g_{\bar{\mu}}(t)$, repectively. Let $\mathrm{Aut}(\hat{\mathcal{L}})$ and $\mathrm{Aut}(\mathcal{Q})$ denote the groups of all biholomorphisms of the total spaces of $\hat{\mathcal{L}}$ and $\mathcal{Q}$, respectively. Then for φ in Theorem 5.2, the T-linearization of $\mathcal{Q}$ defines a T-action on the real line bundle $|\mathcal{Q}|^2 := \mathcal{Q} \otimes \bar{\mathcal{Q}}$ over $\mathcal{X}$ by

$$g_{\bar{\mu}}(t) \cdot |q|^2 := |g_{\bar{\mu}}(t) \cdot q|^2 = |\tilde{\varphi}(t)(q)|^2, \qquad (t, q) \in T \times \mathcal{Q},$$

where $\tilde{\varphi} : T \to \mathrm{Aut}(\mathcal{Q})$ denotes the homomorphism induced by φ. Note also that the T-linearization of $\hat{\mathcal{L}}$ induces a T-action on the real line bundle $|\hat{\mathcal{L}}|^2 := \hat{\mathcal{L}} \otimes \bar{\mathcal{L}}$ such that

$$g_{\mu}(t) \cdot |\sigma|^2 := |g_{\mu}(t) \cdot \sigma|^2 = |\tilde{\psi}(t)(\sigma)|^2, \qquad (t, \sigma) \in T \times \hat{\mathcal{L}},$$

where $\tilde{\psi} : T \to \mathrm{Aut}(\hat{\mathcal{L}})$ denotes the homomorphism induced by ψ. Note that both $g_{\bar{\mu}}(t)$ and $g_{\mu}(t)$ come from the same T-action. Then for $\hat{\mathcal{Q}} := \eta^* \mathcal{Q}$, by Theorem 5.2, we see that

$$|\hat{\mathcal{L}}|^{2/\ell} \; = \; |\zeta|^{2/\ell} \, |\hat{\mathcal{Q}}|^2, \tag{5.7}$$

where $T_{\mathbb{R}}$ acts on the real line $\mathbb{R}|\zeta|^{2/\ell}$ with weight $2\alpha/\ell$, so that $g_{\mu}(t) \cdot |\zeta|^{2/\ell} = t^{2\alpha/\ell} |\zeta|^{2/\ell}$ for all $t \in T_{\mathbb{R}}$. Since birational morphisms ι and η are isomorphic over $\mathbb{A}^1 \setminus \{0\}$, by restricting them to $\{z \neq 0\}$, we can identify the line bundles $\hat{\mathcal{L}}$ and $\hat{\mathcal{Q}}$ with $\mathcal{L}$ and $\mathcal{Q}$, respectively. Hence (5.7) restricts to

$$|\mathcal{L}|^{2/\ell} \; = \; |\zeta|^{2/\ell} |\mathcal{Q}|^2, \qquad z \neq 0. \tag{5.8}$$

Remark 5.9. The restriction of ζ to $z = 1$ gives a non-vanishing holomorphic section for $\mathcal{O}_{\hat{\mathcal{X}}}(\hat{D})_{|\hat{\mathcal{X}}_0}$. Define a Hermitian metric ρ for $\mathcal{O}_{\hat{\mathcal{X}}}(\hat{D})_{|\hat{\mathcal{X}}_0}$ by

$$|\zeta_{|\hat{\mathcal{X}}_0}|^2_\rho = 1$$

everywhere on $\hat{\mathcal{X}}_0$. Then by Theorem 5.2, when restricted to $z = 1$, we may assume that $\mathcal{L}$ and $\mathcal{Q}^{\otimes \ell}$ coincides holomorphically and metrically. In particular, any Hermitian metric for L can be viewed as a Hermitian metric for $\mathcal{Q}_{|\hat{\mathcal{X}}_0}$ via the identification of $\hat{\mathcal{X}}_0$ with X.

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