

The Mabuchi Geometry of Finite Energy Classes

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Abstract

We introduce different Finsler metrics on the space of smooth Kähler potentials that will induce a natural geometry on various finite energy classes $\mathcal{E}_{\bar{\chi}}(X, \omega)$. Motivated by questions raised by R. Berman, V. Guedj and Y. Rubinstein, we characterize the underlying topology of these spaces in terms of convergence in energy and give applications of our results to existence of Kähler-Einstein metrics on Fano manifolds.

1 Introduction and Main Results

Suppose (X, ω) is a compact connected Kähler manifold and \mathcal{H} is the space of Kähler potentials associated to ω . We have three major goals in this paper. First, in hopes of unifying the treatment of different geometries/topologies on \mathcal{H} already present in the current literature, we introduce different Finsler structures on this space. As motivation we single out two well known examples. The path length metric d_2 associated to the L^2 Riemannian structure induces the much studied Mabuchi geometry of \mathcal{H} [Ma]. As it turns out, by introducing the analogous L^1 metric and the associated path length metric d_1 , one recovers the extremely useful strong topology introduced in [BBEGZ], whose study lead to many important applications in the study of weak solutions to complex Monge–Ampère equations (see [BBEGZ, BBGZ, EGZ]). In this part we point that one can treat these different structures in a unified manner, by working with certain very general Orlicz–Finsler type metrics on \mathcal{H} .

Second, with applications in mind, one would like to characterize the metric geometry of these Orlicz–Finsler structures using very concrete terms. Until now, the lack of tools to give a satisfactory answer to this problem presented a formidable roadblock in drawing geometric conclusions using the L^2 geometry of \mathcal{H} . With this in mind V. Guedj [G, Section 4.3] conjectured the following pluripotential theoretic characterization:

$$\begin{aligned} \frac{1}{C} \left(\int_X (u_1 - u_0)^2 \omega_{u_0}^n + \int_X (u_1 - u_0)^2 \omega_{u_1}^n \right) &\leq d_2(u_0, u_1)^2 \\ &\leq C \left(\int_X (u_1 - u_0)^2 \omega_{u_0}^n + \int_X (u_1 - u_0)^2 \omega_{u_1}^n \right), \quad u_0, u_1 \in \mathcal{H}, \end{aligned}$$

for some $C > 1$. We prove this result in the setting of the more general Orlicz-Finsler structures mentioned above. As a consequence we note that convergence in these very general spaces implies convergence in capacity of the potentials in the sense of [K] and that $\sup_X u$ is always controlled by $d_2(0, u)$ for any $u \in \mathcal{H}$.

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Third, we give some immediate applications of the tools developed here. Parallelling the analogous results for the Calabi metric in [CR] and answering questions posed in this same paper, we connect the existence of Kahler–Einstein metrics on Fano manifolds with stability of the Kähler–Ricci flow and properness of Ding’s \mathcal{F} –functional measured in these general geometries. We defer other applications to a future correspondence.

1.1 Finsler Metrics on the Space of Kähler Potentials

Given (X^n, ω) , a connected compact Kähler manifold, the space of smooth Kähler potentials \mathcal{H} is the set

$$\mathcal{H} = \{u \in C^\infty(X) \mid \omega_u := \omega + i\partial\bar{\partial}u > 0\}.$$

Clearly, \mathcal{H} is a Fréchet manifold as an open subset of $C^\infty(X)$. For $v \in \mathcal{H}$ one can identify $T_v\mathcal{H}$ with $C^\infty(X)$. Given a normalized Young weight $\chi : \mathbb{R} \rightarrow \mathbb{R}^+ \cup \{+\infty\}$ (convex, even, lower semi–continuous and satisfying the normalizing conditions $\chi(0) = 0$, $1 \in \partial\chi(1)$) one can introduce a Finsler metric on \mathcal{H} associated to the Orlicz norm of χ (see Section 2.1):

$$\|\xi\|_{\chi,u} = \inf \left\{ r > 0 : \frac{1}{\text{Vol}(X)} \int_X \chi\left(\frac{\xi}{r}\right) \omega_u^n \leq \chi(1) \right\}, \quad \xi \in T_v\mathcal{H}. \quad (1)$$

If $\chi(l) = \chi_p(l) = l^p/p$, then $\|\xi\|_{\chi,u}$ is just the L^p norm of ξ with respect to the volume ω_u^n , hence (1) generalizes the much studied Mabuchi Riemannian metric, initially investigated in [Ma, Se, Do], which corresponds to the case $p = 2$. As we shall see later, the path length metric associated to the case $p = 1$ is tied up with the strong topology of \mathcal{H} introduced in [BBEGZ, Section 2.1].

Unfortunately, the weights $\chi_p(l) = l^p/p$ are not twice differentiable for $1 \leq p < 2$. This will cause problems as we shall see, and an approximation with smooth weights is necessary to carry out even the most basic geometric arguments. For this reason, one needs to work with a fairly big class of Young weights, even if one only wants to study the L^p geometry of \mathcal{H} . The classes \mathcal{W}_p^+ , $p \geq 1$ are sufficient for this purpose. These weights are finite and satisfy the growth condition:

$$l\chi'(l) \leq p\chi(l), \quad l > 0.$$

Some of our results below hold for even more general weights, however the proofs would be much more complicated, and it is unlikely that greater generality will find applications in Kähler geometry.

As usual, the length of a smooth curve $[0, 1] \ni t \rightarrow \alpha_t \in \mathcal{H}$ is computed by the formula:

$$l_\chi(\alpha) = \int_0^1 \|\dot{\alpha}_t\|_{\chi, \alpha_t} dt. \quad (2)$$

The distance $d_\chi(u_0, u_1)$ between $u_0, u_1 \in \mathcal{H}$ is the infimum of the length of smooth curves joining u_0 and u_1 . In case of the L^2 metric X.X. Chen proved that $d_2(u_0, u_1) = 0$ if and only if $u_0 = u_1$, thus (\mathcal{H}, d_2) is a metric space [C1]. Unfortunately, smooth geodesics don’t run between the points of \mathcal{H} [LV], however Chen proved (with complements by Blocki [Bl1]) that for $u_0, u_1 \in \mathcal{H}$ there exists a curve

$$[0, 1] \ni t \rightarrow u_t \in \mathcal{H}_\Delta = \text{PSH}(X, \omega) \cap \{\Delta u \in L^\infty(X)\} \quad (3)$$

that satisfies the geodesic equation in a certain weak sense, and it also holds that

$$d_2(u_0, u_1) = \|\dot{u}_t\|_{2, u_t}, \quad t \in [0, 1].$$

With our first theorem we extend these results to the more general Finsler geometric setting:

Theorem 1 (Theorem 3.5). *If $\chi \in \mathcal{W}_p^+$, $p \geq 1$ then (\mathcal{H}, d_χ) is a metric space and for any $u_0, u_1 \in \mathcal{H}$ the weak geodesic from (3) satisfies*

$$d_\chi(u_0, u_1) = \|\dot{u}_t\|_{\chi, u_t}, \quad t \in [0, 1].$$

The proof of this result follows Chen's original arguments and involves a careful differential analysis of Orlicz norms. Although the metric spaces (\mathcal{H}, d_χ) are not equivalent (they have different metric completion, see below), it is remarkable that the weak geodesic of (3) is able to see all the different metrics d_χ . Topping this, as we will see shortly, this same curve is an honest geodesic in the metric completion each space (\mathcal{H}, d_χ) .

Building on [Da1], which considers the L^2 metric, we want to describe the metric completion of (\mathcal{H}, d_χ) , $\chi \in \mathcal{W}_p^+$. Before this, we need to recall a few facts about finite energy classes. Given $u \in \text{PSH}(X, \omega)$, as explained in [GZ1], one can define the non-pluripolar measure ω_u^n that coincides with the usual Bedford-Taylor volume when u is bounded. We say that ω_u^n has full volume ($u \in \mathcal{E}(X, \omega)$) if $\int_X \omega_u^n = \int_X \omega^n$. To be compatible with the notation in [GZ1], to all $\chi \in \mathcal{W}_p^+$ we attach another weight $\tilde{\chi}$, defined as follows:

$$\tilde{\chi}(l) = \begin{cases} -\chi(l) & \text{if } l \leq 0 \\ 0 & \text{if } l > 0. \end{cases} \quad (4)$$

Given $v \in \mathcal{E}(X, \omega)$, we say that $v \in \mathcal{E}_{\tilde{\chi}}(X, \omega)$ if

$$E_{\tilde{\chi}}(v) = \int_X \tilde{\chi}(v) \omega_v^n > -\infty,$$

Following [GZ1], one can introduce finite energy classes for more general weights. We will focus only on the special classes introduced above which already received attention in [GZ1, Section 3], where they are referred to as classes of high energy. For a quick review of all the results needed about finite energy classes in our study we refer to [Da1, Section 2.3].

Next we introduce a geodesic metric space structure on $\mathcal{E}_{\tilde{\chi}}(X, \omega)$. Suppose $u_0, u_1 \in \mathcal{E}_{\tilde{\chi}}(X, \omega)$. Let $\{u_0^k\}_{k \in \mathbb{N}}, \{u_1^k\}_{k \in \mathbb{N}} \subset \mathcal{H}$ be sequences decreasing pointwise to u_0 and u_1 respectively. By [BK] it is always possible to find such approximating sequences. We define the metric $d_\chi(u_0, u_1)$ as follows:

$$d_\chi(u_0, u_1) = \lim_{k \rightarrow \infty} d_\chi(u_0^k, u_1^k). \quad (5)$$

Clearly, one has to argue that this limit exists and is independent of the approximating sequences, and we do this in Section 4 below. Let us also define geodesics in this space. Let $u_t^k : [0, 1] \rightarrow \mathcal{H}_\Delta := \text{PSH}(X, \omega) \cap \{\Delta u \in L^\infty\}$ be the weak geodesic joining u_0^k, u_1^k , whose existence was proved by Chen (see Section 2.2). We define $t \rightarrow u_t$ as the decreasing limit:

$$u_t = \lim_{k \rightarrow +\infty} u_t^k, \quad t \in (0, 1). \quad (6)$$

The curve $t \rightarrow u_t$ is well defined and we additionally have $u_t \in \mathcal{E}_{\tilde{\chi}}(X, \omega)$, $t \in (0, 1)$, as follows from [Da1, Theorem 6].

Theorem 2 (Theorem 4.17). *If $\chi \in \mathcal{W}_p^+$, $p \geq 1$ then $(\mathcal{E}_{\tilde{\chi}}(X, \omega), d_\chi)$ is a geodesic metric space, which is the metric completion of (\mathcal{H}, d_χ) . Additionally, the curve defined in (6) is a geodesic connecting u_0, u_1 .*

Although not specifically pursued there, from the ideas developed in [G, Section 4], the toric version of this theorem easily follows. Using the Riemannian metric of Calabi [Clb, Clm], the completion of the space of Kähler metrics was first studied in [CR], and in Section 7.2 of this paper the authors proposed to study the completion with respect to other Riemannian metrics. The above theorem fits into this framework and in a future publication we will compare the Calabi and Mabuchi geometries.

The proof of Theorem 2 is given by adapting to our general setting the arguments of [Da1, Theorem 1], which deals with the particular case of the L^2 metric. As a result of the condition $\chi \in \mathcal{W}_p^+$, few things need to be altered, and we will indicate only the necessary changes. In [Da1] it is additionally proved that $(\mathcal{E}^2(X, \omega), d_2)$ is a CAT(0) space, hence geodesics connecting different points in this space are unique. As we shall see, none of these properties hold for general χ (see discussion after Theorem 4.17).

1.2 Characterization of the Path Length Metric Topology

One would like to characterize the topology of $(\mathcal{E}_{\tilde{\chi}}(X, \omega), d_\chi)$ in very concrete terms. With this goal in mind, given $u_0, u_1 \in \mathcal{E}_{\tilde{\chi}}(X, \omega)$ we introduce the following energy functional:

$$I_\chi(u_0, u_1) = \|u_1 - u_0\|_{\chi, u_0} + \|u_1 - u_0\|_{\chi, u_1}.$$

This functional was first proposed in [G] in the particular case of the L^2 metric. We say that $\{u_j\}_{j \in \mathbb{N}} \subset \mathcal{E}_{\tilde{\chi}}(X, \omega)$ converges to $u \in \mathcal{E}_{\tilde{\chi}}(X, \omega)$ in χ -energy if $\lim_j I_\chi(u_j, u) = 0$. When $\chi(l) = \chi_1(l) = |l|$, this is equivalent to the notion of strong convergence introduced in [BBEGZ, Section 2.1] (see Proposition 5.9). The next theorem says that convergence in χ -energy and d_χ -convergence are the same:

Theorem 3 (Theorem 5.5). *Given $\chi \in \mathcal{W}_p^+$, $p \geq 1$, there exists $C(p) > 1$ such that*

$$\frac{1}{C} I_\chi(u_0 - u_1) \leq d_\chi(u_0, u_1) \leq C I_\chi(u_0 - u_1), \quad u_0, u_1 \in \mathcal{E}_{\tilde{\chi}}(X, \omega). \quad (7)$$

We note the following immediate consequence:

Corollary 4. *Given $\chi \in \mathcal{W}_p^+$ and $u \in \mathcal{E}_{\tilde{\chi}}(X, \omega)$ there exists $C(p) > 1$ such that*

$$\sup_X u \leq C d_\chi(0, u) + C.$$

Proof. As all Orlicz norms dominate the L^1 norm [RR], the d_1 metric is dominated by all d_χ metrics and also $\mathcal{E}_{\tilde{\chi}}(X, \omega) \subset \mathcal{E}^1(X, \omega)$. Hence, it is enough to prove the result for d_1 . It is known that for all $u \in \text{PSH}(X, \omega)$ there exists $C' > 0$ such that $\sup_X u - C' \leq \int_X u \omega^n$. On the other hand we have $\int_X u \omega^n \leq I_1(0, u) \leq C d_1(0, u)$ for $u \in \mathcal{E}^1(X, \omega)$ by the above theorem. \square

From the last theorem and the techniques developed in [BBGZ] and [BBEGZ] it follows that d_χ -convergence implies more classical notions of convergence:

Theorem 5 (Corollary 5.7, Theorem 5.8). *Suppose $\chi \in \mathcal{W}_p^+$ and $u_k, u \in \mathcal{E}_{\tilde{\chi}}(X, \omega)$ with $d_\chi(u_k, u) \rightarrow 0$. Then the following holds:*

(i) $u_k \rightarrow u$ in capacity in the sense of [K]. In particular, $\omega_{u_k}^n \rightarrow \omega_u^n$ weakly.

(ii) For any $v \in \mathcal{E}_{\tilde{\chi}}(X, \omega)$ we have $\|u_k - u\|_{\chi, v} \rightarrow 0$.

Fixing $v \in \mathcal{E}_{\tilde{\chi}}(X, \omega)$, by [GZ1, Proposition 3.6] it follows that $(\mathcal{E}_{\tilde{\chi}}(X, \omega), d_{\tilde{\chi}}) \subset L^X(\omega_v^n)$. The content of Theorem 5(ii) is that this inclusion is continuous as a map between the following metric spaces:

$$(\mathcal{E}_{\tilde{\chi}}(X, \omega), d_{\tilde{\chi}}) \rightarrow (L^X(\omega_v^n), \|\cdot\|_{\chi, v}).$$

In the particular case of the L^2 metric, the last three results answer questions raised by R. Berman [Brm2] and V. Guedj [G, Section 4.3].

1.3 The d_{χ} -convergence of the Kähler-Ricci trajectories

Lastly, we give some immediate applications of our results to the stability of the Kähler-Ricci flow and Kähler-Einstein metrics. Let us assume that (X, J, ω) is a Fano manifold. We introduce the following "totally geodesic" hypersurfaces of \mathcal{H} and $\mathcal{E}^1(X, \omega)$ (see [BBGZ, Proposition 6.2]):

$$\begin{aligned} \mathcal{H}_{AM} &= \mathcal{H} \cap \{AM(\cdot) = 0\}, \\ \mathcal{E}_{AM}^1(X, \omega) &= \mathcal{E}^1(X, \omega) \cap \{AM(\cdot) = 0\}, \end{aligned}$$

where $AM : \mathcal{E}^1(X, \omega) \rightarrow \mathbb{R}$ is the Aubin-Mabuchi energy functional defined by the formula:

$$AM(v) = \frac{1}{(n+1)\text{Vol}(X)} \sum_{j=0}^n \int_X v \omega^j \wedge (\omega + i\partial\bar{\partial}v)^{n-j}.$$

As we will see (Lemma 4.15), $AM(\cdot)$ restricted to $\mathcal{E}_{\chi}(X, \omega)$ is d_{χ} -continuous for any $\chi \in \mathcal{W}_p^+$. Additionally, this functional is also linear along geodesics.

A smooth metric $\omega_{u_{KE}}$ is Kähler-Einstein if $\omega_{u_{KE}} = \text{Ric } \omega_{u_{KE}}$. One can study such metric by looking at the long time asymptotics of Hamilton's Kähler-Ricci flow:

$$\begin{cases} \frac{d\omega_{r_t}}{dt} = -\text{Ric } \omega_{r_t} + \omega_{r_t}, \\ r_0 = v. \end{cases} \quad (8)$$

As proved by Cao [Cao], for any $v \in \mathcal{H}_{AM}$, this PDE has a smooth solution $[0, 1) \ni t \rightarrow r_t \in \mathcal{H}_{AM}$. It follows from a theorem of Perelman and work of Chen-Tian, Tian-Zhu and Phong-Song-Sturm-Weinkove, that whenever a Kähler-Einstein metric cohomologous to ω exists, then ω_{r_t} converges exponentially fast to one such metric (see [CT, TZ, PSSW]).

We remark that our choice of normalization is different from the alternatives used in the literature (see [BEG, Chapter 6]). We choose to work with the normalization $AM(\cdot) = 0$, as this seems to be the most natural one from the point of view of Mabuchi geometry. As we shall see in Section 6, from the point of view of long time asymptotics, this choice is equivalent to all other alternatives.

Fix $\chi \in \mathcal{W}_p^+$. Motivated by the above, we would like to study the stability of Kähler-Ricci trajectories from the point of view of the d_{χ} -geometry. In analogy with [CR, Definition 6.2], we say that a Fano manifold (X, J, ω) is χ -unstable if there exists a starting point $v \in \mathcal{H}_{AM}$ and a corresponding Kähler-Ricci trajectory $[0, \infty) \ni t \rightarrow r_t \in \mathcal{H}_{AM}$ that diverges with respect to d_{χ} . Otherwise, we say that (X, J, ω) is χ -stable.

Digressing slightly, we recall the definition of Ding's \mathcal{F} -functional and the \mathcal{J} -functional, which are as follows: $\mathcal{F}, \mathcal{J} : \mathcal{E}_{AM}^1(X, \omega) \rightarrow \mathbb{R}$,

$$\mathcal{F}(u) = -AM(u) - \log \int_X e^{-u+h} \omega^n = -\log \int_X e^{-u+h} \omega^n, \quad (9)$$

$$\mathcal{J}(u) = \frac{1}{\text{Vol}(X)} \int_X u \omega^n - AM(u) = \frac{1}{\text{Vol}(X)} \int_X u \omega^n. \quad (10)$$

Here $h \in C^\infty(X)$ is the Ricci potential of ω , i.e. $\text{Ric } \omega = \omega + i\partial\bar{\partial}h$. By Theorem 5 above, both of these functionals are continuous with respect to d_χ . By a result of Tian [T1, Theorem 1.6], if (X, J) admits no non-trivial holomorphic vector-fields, then (X, J, ω) admits a Kähler-Einstein metric if and only if \mathcal{F} is \mathcal{J} -proper, meaning that $\mathcal{J}(u_k) \rightarrow \infty$ implies $\mathcal{F}(u_k) \rightarrow \infty$. One would like to understand this condition from a more geometric point of view. Connecting all of the above, we give the following result:

Theorem 6 (Theorem 6.1, Theorem 6.2). *Suppose (X, J, ω) is a Fano manifold with no non-trivial holomorphic vector fields. The following are equivalent:*

- (i) *There exists a Kähler-Einstein metric cohomologous to ω .*
- (ii) *(X, J, ω) is χ -stable for any $\chi \in \mathcal{W}_p^+$.*
- (iii) *\mathcal{F} is d_1 -proper on $\mathcal{E}_{AM}^1(X, \omega)$, i.e. sublevel sets of \mathcal{F} are d_1 -bounded.*

With the help of Theorem 5, the equivalence between (i) and (ii) can be given along the same lines as the analogous result for the Calabi metric [CR, Theorem 6.3]. As we shall see, the assumption that (X, J) has no non-trivial holomorphic vector fields is not needed for this equivalence. In addition to this, whenever a Kähler-Einstein metric exists, not only is (X, J, ω) χ -stable, but each Ricci trajectory $t \rightarrow r_t$ converges to one such metric exponentially fast. This strengthens results of McFeron [Mc] and answers questions raised in [R, Section 4.3.4]. We refer to this last expository work for connections with other conjectures in Kähler geometry.

The d_1 -metric always dominates the Ding functional, i.e. there exists $A, B > 0$ such that $\mathcal{F}(u) \leq Ad_1(0, u) + B$ (Remark 6.3). Because of this and the fact the each d_χ dominates d_1 , it is not possible to replace d_1 -properness in (iii) with the more general d_χ -properness, as d_1 and d_χ are not equivalent in general.

In (46) we will give the following explicit formula for the d_1 metric:

$$d_1(u_0, u_1) = AM(u_0) + AM(u_1) - 2AM(P(u_0, u_1)), \quad u_0, u_1 \in \mathcal{E}^1(X, \omega),$$

where $P(u_0, u_1) = \sup\{u : u \in \text{PSH}(X, \omega), u \leq u_0, u_1\} \in \mathcal{E}^1(X, \omega)$. This will immediately imply that \mathcal{J} -properness is equivalent to d_1 -properness, which in turn gives the equivalence between (i) and (iii) by the above mentioned result of Tian.

Lastly, we mention that existence of Kähler-Einstein metrics on Fano manifolds is also known to be equivalent to the \mathcal{J} -properness of the Mabuchi K-energy functional restricted to \mathcal{H} . With a better understanding of the extension of this functional to $\mathcal{E}_{AM}^1(X, \omega)$ one could hope for analogous results in terms of this functional instead of the Ding energy. For results in this direction we refer to [BrmBrn2, St1, St2].

Future work. Continuing to explore the d_χ -geometry of the Kähler–Ricci flow, in a future correspondence we will construct d_χ -geodesic rays that are weakly asymptotic to diverging Kähler–Ricci trajectories, giving an affirmative answer to questions raised in [R, Conjecture 4.10]. These geodesic rays will have special properties, in particular the Ding energy is always decreasing along them. In case of the L^2 metric, this will help settle questions raised in [H2], where in analogy with Donaldson’s original program on constant scalar curvature metrics [Do], the equivalence between existence of Kähler–Einstein metrics and properness of the Ding energy along d_2 -geodesic rays was proposed. As pointed out to us by R. Berman, his paper [Brm1] and the results of [CDS1, CDS2, CDS3, T2] already imply this last equivalence. Continuing the trend of the present work, our methods will be purely analytical and will allow us to avoid the recently established equivalence between K-stability and existence of Kähler–Einstein metrics on Fano manifolds.

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2 Preliminaries

2.1 Orlicz spaces

We record here some facts about Orlicz spaces that will be used the most. For a thorough introduction we refer to [RR]. Suppose (Ω, Σ, μ) is a Borel measure space with $\mu(\Omega) = 1$ and (χ, χ^*) is a complementary pair of Young weights. This means that $\chi : \mathbb{R} \rightarrow \mathbb{R}^+ \cup \{\infty\}$ is convex, even, lower semi-continuous and satisfies the normalizing conditions $\chi(0) = 0, 1 \in \partial\chi(1)$. Such χ alone is called a normalized Young weight. The complement χ^* is just the Legendre transform of χ :

$$\chi^*(h) = \sup_{l \in \mathbb{R}} (lh - \chi(l)). \quad (11)$$

One can see that χ^* is also a normalized Young weight and (χ, χ^*) satisfies the Young identity and inequality:

$$\chi(\chi'(a)) + \chi^*(a) = a\chi'(a), \quad \chi(a) + \chi^*(b) \geq ab, \quad a, b \in \mathbb{R}. \quad (12)$$

The most typical example to keep in mind is the pair $\chi_p(l) = |l|^p/p$ and $\chi(l)_p^* = |l|^q/q$, where $p, q > 1$ and $1/p + 1/q = 1$. Let $L^\chi(\mu)$ be the following space of measurable functions:

$$L^\chi(\mu) = \left\{ f : \Omega \rightarrow \mathbb{R} \cup \{\infty, -\infty\} : \exists r > 0 \text{ s.t. } \int_\Omega \chi(rf) d\mu < \infty \right\}.$$

One can introduce the following (gauge) norm on $L^\chi(\mu)$:

$$\|f\|_{\chi, \mu} = \inf \left\{ r > 0 : \int_\Omega \chi\left(\frac{f}{r}\right) d\mu \leq \chi(1) \right\}. \quad (13)$$

The reference [RR] uses the notation $N_\chi^\mu(f)$ instead, however in our context this notation seems to be more cumbersome when carrying out computations. The set $\left\{ \int_\Omega \chi(f) d\mu \leq \right.$

$\chi(1)$ is convex and symmetric in $L^X(\mu)$, hence $\|\cdot\|_{\chi,\mu}$ is nothing but the Minkowski seminorm of this set. One can verify that $\|f\|_{\chi,\mu} = 0$ implies $f = 0$ a.e. with respect to μ , hence $\|\cdot\|_{\chi,\mu}$ is a norm that makes $L^X(\mu)$ a complete metric space (see [RR, Theorem 3.3.10]). The reason we work with a complementary pair of Young weights is the Hölder inequality which follows from (12):

$$\int_X fg d\mu \leq \|f\|_{\chi,\mu} \|g\|_{\chi^*,\mu}, \quad f \in L^X(\mu), \quad g \in L^{X^*}(\mu). \quad (14)$$

Coming back to examples, $L^{X^p}(\mu) = L^p(\mu)$ and $\|\cdot\|_{\chi^p,\mu}$ is just the usual L^p norm.

Motivated by the study in [GZ1, Section 3], we will be interested in normalized Young weights that are finite and satisfy the following growth estimate:

$$l\chi'(l) \leq p\chi(l), \quad l > 0, \quad (15)$$

for some $p \geq 1$. For such weights we write $\chi \in \mathcal{W}_p^+$ (see [GZ1]). We note that in the theory of Orlicz spaces the notation $\chi \in \Delta_2 = \bigcup_{p \geq 1} \mathcal{W}_p^+$ is used (see [RR]). The main point is the following estimate :

Proposition 2.1. *For $\chi \in \mathcal{W}_p^+$, $p \geq 1$ and $0 < \varepsilon < 1$ we have*

$$\varepsilon^p \chi(l) \leq \chi(\varepsilon l) \leq \varepsilon \chi(l), \quad l > 0. \quad (16)$$

Proof. The second estimate follows from convexity of χ . For the first estimate we notice that for any $\delta > 0$ we still have $\chi_\delta(l) = \chi(l) + \delta|l| \in \mathcal{W}_p^+$. As $\chi_\delta(h) > 0$ for $h > 0$, we can integrate $\chi'_\delta(h)/\chi_\delta(h) \leq p/h$ from εl to l to obtain:

$$\varepsilon^p \chi_\delta(l) \leq \chi_\delta(\varepsilon l).$$

Letting $\delta \rightarrow 0$ the desired estimate follows. □

Estimate (16) immediately implies that for $f \in L^X(\mu)$ we have

$$\|f\|_{\chi,\mu} = \alpha > 0 \text{ if and only if } \int_\Omega \chi\left(\frac{f}{\alpha}\right) d\mu = \chi(1). \quad (17)$$

To simplify future notation, we introduce the bijective increasing functions $M_p(l) = \max\{l, l^p\}$, $m_p(l) = \min\{l, l^p\}$ for $p > 0, l \geq 0$. Observe that $M_p \circ m_{1/p} = m_p \circ M_{1/p} = Id$. As a consequence of (16), we can establish the following very useful estimates:

Proposition 2.2. *If $\chi \in \mathcal{W}_p^+$ and $f \in L^X(\mu)$ then*

$$m_p(\|f\|_{\chi,\mu}) \leq \frac{\int_\Omega \chi(f) d\mu}{\chi(1)} \leq M_p(\|f\|_{\chi,\mu}). \quad (18)$$

$$m_{1/p}\left(\frac{\int_\Omega \chi(f) d\mu}{\chi(1)}\right) \leq \|f\|_{\chi,\mu} \leq M_{1/p}\left(\frac{\int_\Omega \chi(f) d\mu}{\chi(1)}\right). \quad (19)$$

Hence, given a sequence $\{f_j\}_{j \in \mathbb{N}}$, we have $\|f_j\|_{\chi,\mu} \rightarrow 0$ if and only if $\int_\Omega \chi(f_j) d\mu \rightarrow 0$ and $\|f_j\|_{\chi,\mu} \rightarrow N$ for $N > 0$ if and only if $\int_\Omega \chi(f_j/N) d\mu \rightarrow \chi(1)$.

In the future we will approximate Orlicz norms with weight in \mathcal{W}_p^+ with Orlicz norms having smooth weights. The following two approximation results, which are by no means optimal, will be very useful for us:

Proposition 2.3. *Suppose $\chi \in \mathcal{W}_p^+$ and $\{\chi_k\}_{k \in \mathbb{N}}$ is a sequence of normalized Young weights that converges uniformly on compacts to χ . Let f be a bounded μ -measurable function on X . Then $f \in L^X(\mu), L^{\chi_k}(\mu)$, $k \in \mathbb{N}$ and we have that*

$$\lim_{k \rightarrow +\infty} \|f\|_{\chi_k, \mu} = \|f\|_{\chi, \mu}.$$

Proof. Suppose $N = \|f\|_{\chi, \mu}$. If $N = 0$, then $f = 0$ a.e. with respect to μ implying that $\|f\|_{\chi_k, \mu} = \|f\|_{\chi, \mu} = 0$. So we assume that $N > 0$. As $\chi \in \mathcal{W}_p^+$, by (17), for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\int_X \chi\left(\frac{f}{(1+\varepsilon)N}\right) d\mu < \chi(1) - 2\delta < \chi(1) < \chi(1) + 2\delta < \int_X \chi\left(\frac{f}{(1-\varepsilon)N}\right) d\mu$$

As χ_k tends uniformly on compacts to χ , f is bounded and $\mu(X) = 1$, it follows from the dominated convergence theorem that for k greater than some k_0 we have

$$\int_X \chi_k\left(\frac{f}{(1+\varepsilon)N}\right) d\mu < \chi_k(1) - \delta < \chi_k(1) < \chi_k(1) + \delta < \int_X \chi_k\left(\frac{f}{(1-\varepsilon)N}\right) d\mu$$

This implies that $(1-\varepsilon)N \leq \|f\|_{\chi_k, \mu} \leq (1+\varepsilon)N$ for $k \geq k_0$, from which the conclusion follows. \square

Proposition 2.4. *Given $\chi \in \mathcal{W}_p^+$, there exists $\chi_k \in \mathcal{W}_{p_k}^+ \cap C^\infty(\mathbb{R})$, $k \in \mathbb{N}$ with $\{p_k\}_k$ possibly unbounded such that $\chi_k \rightarrow \chi$ uniformly on compacts.*

Proof. There are many ways to construct the sequence χ_k . First we smoothen and normalize χ , introducing the sequence $\tilde{\chi}_k$ in the process:

$$\tilde{\chi}_k(l) = (\delta_k \star \chi)(h_k l) - (\delta_k \star \chi)(0), \quad l \in \mathbb{R}, k \in \mathbb{N}^*.$$

Here δ is a typical choice of bump function with support in $(-1, 1)$, $\delta_k(\cdot) = k\delta(k\cdot)$, and $h_k > 0$ is chosen in such a way that $\tilde{\chi}_k$ becomes normalized ($\tilde{\chi}_k(0) = 0$ and $\tilde{\chi}'_k(1) = 1$). As χ is normalized it follows that $1 - 1/k \leq h_k \leq 1 + 1/k$, hence the $\tilde{\chi}_k$ are normalized Young weights that converge to χ uniformly on compacts.

As χ is strictly increasing, so is each $\tilde{\chi}_k$, but our construction does not seem to guarantee that $\tilde{\chi}_k \in \mathcal{W}_{p_k}^+$ for some $p_k \geq 1$. This can be fixed by changing smoothly the values of $\tilde{\chi}_k$ on the sets $|x| \leq 1/k$ and $|x| > k$ in such a manner that the possibly altered weights χ_k satisfy the estimate

$$l\chi'_k(l) \leq p_k\chi_k(l), \quad l > 0,$$

for some $p_k \geq 1$. The sequence we obtained satisfies the required properties. \square

2.2 Weak Geodesics in the Metric Space (\mathcal{H}, d_2)

We summarize some of the properties of the metric space (\mathcal{H}, d_2) that we will need later, for a thorough introduction we refer to [Bl2]. If in (1) we choose $\chi(l) = \chi_2(l) = l^2/2$, we obtain the Mabuchi Riemannian metric on \mathcal{H} . Given a smooth curve $(0, 1) \ni t \rightarrow v_t \in \mathcal{H}$ and a vector field $(0, 1) \ni t \rightarrow f_t \in C^\infty(X)$ along this curve, the covariant derivative $\nabla_{\dot{v}_t} f_t$ is as follows:

$$\nabla_{\dot{v}_t} f_t = \dot{f}_t - \frac{1}{2} \langle \nabla^{\omega_{v_t}} \dot{v}_t, \nabla^{\omega_{v_t}} f_t \rangle_{\omega_{v_t}} \quad (20)$$

Suppose $S = \{0 < \operatorname{Re} s < 1\} \subset \mathbb{C}$. Following [Se], one can compute that a smooth curve $[0, 1] \ni t \rightarrow u_t \in \mathcal{H}$ connecting $u_0, u_1 \in \mathcal{H}$ is a geodesic if it is the (unique) smooth solution of the following Dirichlet problem on $S \times X$:

$$\begin{aligned} (\pi^* \omega + i\partial\bar{\partial}u)^{n+1} &= 0 \\ u(t + ir, x) &= u(t, x) \quad \forall x \in X, t \in (0, 1), r \in \mathbb{R} \\ \lim_{t \rightarrow 0, 1} u_t &= u_{0,1} \text{ uniformly in } X, \end{aligned} \tag{21}$$

where $u(s, x) = u_{\operatorname{Re} s}(x)$ is the complexification of $t \rightarrow u_t$. Unfortunately the above problem does not usually have smooth solutions (see [LV, Da3]), but a unique solution in the sense of Bedford-Taylor does exist and with bounded Laplacian (see [C1]) and this regularity is essentially optimal (see [DL]). This curve $[0, 1] \ni t \rightarrow u_t \in \mathcal{H}_\Delta := \operatorname{PSH}(X, \omega) \cap \{\Delta v \in L^\infty\}$ is called the weak geodesic connecting u_0, u_1 , and it can be approximated by smooth ε -geodesics which are solutions of the following elliptic Dirichlet problem on $[0, 1] \times X$:

$$\begin{aligned} (\ddot{u}_t^\varepsilon - \frac{1}{2} \langle \nabla \dot{u}_t^\varepsilon, \nabla \dot{u}_t^\varepsilon \rangle) (\omega + i\partial\bar{\partial}u_t^\varepsilon)^n &= \varepsilon \omega^n, \quad t \in [0, 1]. \\ \lim_{t \rightarrow 0, 1} u_t &= u_{0,1} \text{ uniformly in } X, \end{aligned} \tag{22}$$

Chen proved that the path length metric can be computed using ε -geodesics:

$$d_2(u_0, u_1) = \lim_{\varepsilon \rightarrow 0} l_\varepsilon(u^\varepsilon), \tag{23}$$

and there exists $C > 0$ independent of ε such that

$$\|\Delta u^\varepsilon\|_{L^\infty([0,1] \times X)} \leq C. \tag{24}$$

From this it follows that the ε -geodesics u^ε converge to the weak geodesic u in $C^{1,\alpha}(\bar{S} \times X)$, $0 < \alpha < 1$ as $\varepsilon \rightarrow 0$. Using this, one can relate $d_2(u_0, u_1)$ directly to the weak geodesic $t \rightarrow u_t$:

$$d_2(u_0, u_1) = \|\dot{u}_t\|_{2, u_t} = \sqrt{\frac{1}{\operatorname{Vol}(X)} \int_X \dot{u}_t^2 \omega_{u_t}^n}, \quad t \in [0, 1]. \tag{25}$$

This formula is geometrically justified, as in finite dimensional Riemannian geometry geodesics have constant speed. Although weak geodesics connecting points of \mathcal{H} leave this space, they are bona fide geodesics in the metric completion of (\mathcal{H}, d_2) (see [Da1]).

Given $u_0, u_1 \in \mathcal{H}$ one can introduce the rooftop-envelope $P(u_0, u_1)$:

$$P(u_0, u_1) = \sup\{v \in \operatorname{PSH}(X, \omega) \mid v \leq \min\{u_0, u_1\}\}.$$

As observed in [DR], one has $P(u_0, u_1) \in \mathcal{H}_\Delta$, and we have the following partition formula for the volume form:

$$\omega_{P(u_0, u_1)}^n = \mathbb{1}_{\{u_0 = P(u_0, u_1)\}} \omega_{u_0}^n + \mathbb{1}_{\{u_1 = P(u_0, u_1)\} \setminus \{u_0 = P(u_0, u_1)\}} \omega_{u_1}^n. \tag{26}$$

Moreover, these envelopes are intimately connected to the weak geodesic $[0, 1] \ni t \rightarrow u_t \in \mathcal{H}_\Delta$, joining u_0, u_1 in the following manner:

$$\inf_{t \in [0, 1]} (u_t - \tau t) = P(u_0, u_1 - \tau), \tag{27}$$

$$\{\dot{u}_0 \geq \tau\} = \{P(u_0, u_1 - \tau) = u_0\}, \quad (28)$$

for $\tau \in \mathbb{R}$. For the proof of these facts we refer to [Da1, Section 2.2 and Lemma 6.5] and [DR].

Developing (25) further, by a result of Berndtsson [Brn2], the pushforward measures $\dot{u}_{t*}\omega_{u_t}$ along the weak geodesic segment are the same for any $t \in [0, 1]$. As a consequence of this we obtain that weak geodesics have constant speed with respect to all χ -Finsler metrics:

Remark 2.5. *Suppose $[0, 1] \ni t \rightarrow u_t \in \mathcal{H}_\Delta$ is the weak geodesic connecting $u_0, u_1 \in \mathcal{H}$ and $\chi \in \mathcal{W}_p^+$. Then we have*

$$\|\dot{u}_0\|_{\chi, u_0} = \|\dot{u}_t\|_{\chi, u_t}, \quad t \in [0, 1].$$

Proof. Suppose $N = \|\dot{u}_0\|_{\chi, u_0}$. If $N = 0$ then $\dot{u}_0 = 0$, hence $d_2(u_0, u_1) = 0$ by (25). This in turn implies $u_0 = u_1$, finishing the proof in this case. If $N \neq 0$, then by the above mentioned result of Berndtsson it follows that

$$\chi(1) = \frac{1}{\text{Vol}(X)} \int_X \chi\left(\frac{\dot{u}_0}{N}\right) \omega_{u_0}^n = \frac{1}{\text{Vol}(X)} \int_X \chi\left(\frac{\dot{u}_t}{N}\right) \omega_{u_t}^n, \quad t \in [0, 1].$$

Hence, $N = \|u_0\|_{\chi, u_0} = \|u_t\|_{\chi, u_t}$. □

This remark suggests that weak geodesics have special role, not just in the L^2 geometry, but in the χ -Finsler geometry of \mathcal{H} as well. This is indeed the case, as we will see shortly.

3 The Metric Spaces (\mathcal{H}, d_χ)

In this section we give the proof of Theorem 1. Our method will follow Chen's original approach ([C1], see [Bl2, Section 6] for a recent survey) along with a careful analysis involving Orlicz norms. In hopes of easing the technical nature of future calculations, for Sections 3–5 we fix the volume normalizing condition

$$\text{Vol}(X) = \int_X \omega^n = 1.$$

Given a differentiable normalized Young weight χ , the differentiability of the associated gauge norm is well understood (see [RR, Chapter VII]). Adapted to our setting we prove the following result which is essentially contained in [RR, Theorem VII.2.3].

Proposition 3.1. *Suppose $\chi \in \mathcal{W}_p^+ \cap C^\infty(\mathbb{R})$. Given a smooth curve $(0, 1) \ni t \rightarrow u_t \in \mathcal{H}$, i.e. $u(t, x) := u_t(x) \in C^\infty((0, 1) \times X)$, and a non-vanishing vector field $(0, 1) \ni t \rightarrow f_t \in C^\infty(X)$ along this curve, the following formula holds:*

$$\frac{d}{dt} \|f_t\|_{\chi, u_t} = \frac{\int_X \chi' \left(\frac{f_t}{\|f_t\|_{\chi, u_t}} \right) \nabla_{\dot{u}_t} f_t \omega_{u_t}^n}{\int_X \chi' \left(\frac{f_t}{\|f_t\|_{\chi, u_t}} \right) \frac{f_t}{\|f_t\|_{\chi, u_t}} \omega_{u_t}^n}, \quad (29)$$

where ∇ is the covariant derivative from (20).

Proof. We introduce the smooth function $F : \mathbb{R}^+ \times (0, 1) \rightarrow \mathbb{R}$ given by

$$F(r, t) = \int_X \chi\left(\frac{f_t}{r}\right) \omega_{u_t}^n.$$

By our assumption on χ we have $\chi'(l) > 0$, $l > 0$. As $t \rightarrow f_t$ is non-vanishing, it follows that

$$\frac{d}{dr} F(r, t) = -\frac{1}{r^2} \int_X f_t \chi'\left(\frac{f_t}{r}\right) \omega_{u_t}^n < 0$$

for all $r < 0$, $t \in (0, 1)$. Using $F(\|f_t\|_{\chi, u_t}, t) = \chi(1)$, an application of the implicit function theorem yields that the map $t \rightarrow \|f_t\|_{\chi, u_t}$ is differentiable and the following formula holds:

$$\frac{d}{dt} \|f_t\|_{\chi, u_t} = \frac{\int_X \left[\dot{f}_t \chi'\left(\frac{f_t}{\|f_t\|_{\chi, u_t}}\right) + \|f_t\|_{\chi, u_t} \chi'\left(\frac{f_t}{\|f_t\|_{\chi, u_t}}\right) \Delta_{\omega_{u_t}} \dot{u}_t \right] \omega_{u_t}^n}{\int_X \frac{f_t}{\|f_t\|_{\chi, u_t}} \chi'\left(\frac{f_t}{\|f_t\|_{\chi, u_t}}\right) \omega_{u_t}^n}.$$

An integration by parts yields (29). \square

Corollary 3.2. *Suppose $\chi \in \mathcal{W}_p^+ \cap C^\infty(\mathbb{R})$ and $v_0, v_1 \in \mathcal{H}$, $v_0 \neq v_1$. Then there exists $\varepsilon_0(v_0, v_1) > 0$ such that for any $\varepsilon < \varepsilon_0$ and $u_0, u_1 \in \mathcal{H}$ satisfying $\|u_0 - v_0\|_{C^2(X)}, \|u_1 - v_1\|_{C^2(X)} \leq \varepsilon_0$, the ε -geodesic $[0, 1] \ni t \rightarrow u_t \in \mathcal{H}$ of (22), connecting u_0, u_1 satisfies:*

$$\frac{d}{dt} \|\dot{u}_t\|_{\chi, u_t} = \varepsilon \frac{\int_X \chi'\left(\frac{\dot{u}_t}{\|\dot{u}_t\|_{\chi, u_t}}\right) \omega^n}{\int_X \frac{\dot{u}_t}{\|\dot{u}_t\|_{\chi, u_t}} \chi'\left(\frac{\dot{u}_t}{\|\dot{u}_t\|_{\chi, u_t}}\right) \omega_{u_t}^n}, \quad t \in [0, 1]. \quad (30)$$

Proof. From [Bl2, Lemma 13] and the estimates of [Bl2, Theorem 12] it follows that there exists $\varepsilon_0(v_0, v_1) > 0$ such that for any $\varepsilon < \varepsilon_0$ and $u_0, u_1 \in \mathcal{H}$ satisfying $\|u_0 - v_0\|_{C^2(X)}, \|u_1 - v_1\|_{C^2(X)} \leq \varepsilon_0$ the ε -geodesic $(0, 1) \ni t \rightarrow u_t \in \mathcal{H}$ connecting u_0, u_1 satisfies the non-vanishing condition $\dot{u}_t \neq 0$, $t \in [0, 1]$. Formula (30) is now just a consequence of the prior proposition and (22). \square

Continuing to focus on smooth weights χ , we establish a concrete lower bound on the χ -length of tangent vectors along the ε -geodesics defined in (22). This is the analog of [Bl2, Lemma 13] in our more general setting.

Proposition 3.3. *Suppose $\chi \in \mathcal{W}_p^+ \cap C^\infty(\mathbb{R})$ and $v_0, v_1 \in \mathcal{H}$, $v_0 \neq v_1$. Then there exists $\varepsilon_0(\chi, v_0, v_1) > 0$ and $R(\chi, v_0, v_1) > 0$ such that for any $\varepsilon < \varepsilon_0$ and $u_0, u_1 \in \mathcal{H}$ satisfying $\|u_0 - v_0\|_{C^2(X)}, \|u_1 - v_1\|_{C^2(X)} \leq \varepsilon_0$, the ε -geodesic $[0, 1] \ni t \rightarrow u_t \in \mathcal{H}$ connecting u_0, u_1 satisfies:*

$$\int_X \chi(\dot{u}_t) \omega_{u_t}^n \geq \max\left\{ \int_X \chi(\min(u_1 - u_0, 0)) \omega_{u_0}^n, \int_X \chi(\min(u_0 - u_1, 0)) \omega_{u_1}^n \right\} - \varepsilon R > 0, \quad (31)$$

$t \in [0, 1]$. Additionally, there also exists $R_0(\varepsilon_0, \chi, v_0, v_1), R_1(\varepsilon_0, \chi, v_0, v_1) > 0$ such that

$$(i) \quad \|\dot{u}_t\|_{\chi, u_t} > R_0, \quad t \in [0, 1],$$

$$(ii) \quad \left| \frac{d}{dt} \|\dot{u}_t\|_{\chi, u_t} \right| \leq \varepsilon R_1, \quad t \in [0, 1].$$

Proof. Suppose $u_0, u_1 \in \mathcal{H}$. As $t \rightarrow u_t(x)$ is convex for any $x \in X$, on the set $\{u_0 \geq u_1\}$ the estimate $\dot{u}_0 \leq u_1 - u_0 \leq 0$ holds, hence

$$\int_X \chi(\dot{u}_0) \omega_{u_0}^n \geq \int_X \chi(\min(u_1 - u_0, 0)) \omega_{u_0}^n.$$

We can similarly deduce that

$$\int_X \chi(\dot{u}_1) \omega_{u_1}^n \geq \int_X \chi(\min(u_1 - u_0, 0)) \omega_{u_1}^n.$$

For $t \in [0, 1]$, using the fact that ∇ is a Riemannian connection and $t \rightarrow u_t$ is an ε -geodesic, we can write:

$$\left| \frac{d}{dt} \int_X \chi(\dot{u}_t) \omega_{u_t}^n \right| = \left| \int_X \chi'(\dot{u}_t) \nabla_{\dot{u}_t} \dot{u}_t \omega_{u_t}^n \right| = \varepsilon \left| \int_X \chi'(\dot{u}_t) \omega_{u_t}^n \right| \leq \varepsilon R(\chi, u_0, u_1),$$

where in the last estimate we have used that \dot{u}_t is uniformly bounded in terms of $\|u_0\|_{C^2}, \|u_1\|_{C^2}$. Putting together the last three estimates it is clear that for some small $\varepsilon_0(\chi, v_0, v_1)$ the estimate of (31) holds. Using Proposition 2.2, (i) follows from (31).

To establish (ii) we shrink ε_0 further to satisfy the requirements of Corollary (3.2). Using the Young identity (12) we can write:

$$\begin{aligned} \left| \frac{d}{dt} \|\dot{u}_t\|_{\chi, u_t} \right| &= \varepsilon \frac{\left| \int_X \chi' \left(\frac{\dot{u}_t}{\|\dot{u}_t\|_{\chi, u_t}} \right) \omega_{u_t}^n \right|}{\int_X \frac{\dot{u}_t}{\|\dot{u}_t\|_{\chi, u_t}} \chi' \left(\frac{\dot{u}_t}{\|\dot{u}_t\|_{\chi, u_t}} \right) \omega_{u_t}^n} = \varepsilon \frac{\left| \int_X \chi' \left(\frac{\dot{u}_t}{\|\dot{u}_t\|_{\chi, u_t}} \right) \omega_{u_t}^n \right|}{\chi(1) + \int_X \chi^* \left(\chi' \left(\frac{\dot{u}_t}{\|\dot{u}_t\|_{\chi, u_t}} \right) \right) \omega_{u_t}^n} \leq \\ &\leq \frac{\varepsilon}{\chi(1)} \int_X \chi' \left(\frac{\dot{u}_t}{\|\dot{u}_t\|_{\chi, u_t}} \right) \omega_{u_t}^n. \end{aligned} \quad (32)$$

Using (i) and the fact that \dot{u}_t is uniformly bounded in terms of $\|u_0\|_{C^2}, \|u_1\|_{C^2}$ the estimate of (ii) follows. \square

With the help of (30) we can establish an estimate for ε -geodesics which is the analog of [Bl2, Theorem 14] in our more general setting.

Proposition 3.4. *Suppose $\chi \in \mathcal{W}_p^+ \cap C^\infty(\mathbb{R})$, $[0, 1] \ni s \rightarrow \psi_s \in \mathcal{H}$ is a smooth curve, $\phi \in \mathcal{H} \setminus \psi([0, 1])$ and $\varepsilon > 0$. We denote by $u \in C^\infty([0, 1] \times [0, 1] \times X)$ the smooth function for which $[0, 1] \ni t \rightarrow u(t, s, \cdot) \in \mathcal{H}$ is the ε -geodesic connecting ϕ and ψ_s , $s \in [0, 1]$. There exists $\varepsilon_0(\psi, \phi) > 0$ such that for any $\varepsilon \leq \varepsilon_0$ the following holds:*

$$l_\chi(u(\cdot, 0)) \leq l_\chi(\psi) + l_\chi(u(\cdot, 1)) + \varepsilon R,$$

for some $R(\phi, \psi, \chi, \varepsilon_0) > 0$ independent of $\varepsilon > 0$.

Proof. Fix $s \in [0, 1]$. To avoid cumbersome notation, derivatives in the t -direction will be denoted by dots, derivatives in the s -direction will be denoted by d/ds and sometimes we omit dependence on (t, s) . By Corollary 3.2 for $\varepsilon_0(\psi, \phi) > 0$ small enough we can write:

$$\frac{d}{ds} l_\chi(u(\cdot, s)) = \int_0^1 \frac{d}{ds} \|\dot{u}(t, s)\|_{\chi, u(t, s)} dt = \int_0^1 \frac{\int_X \chi' \left(\frac{\dot{u}}{\|\dot{u}\|_{\chi, u}} \right) \nabla_{\frac{d\dot{u}}{ds}} \dot{u} \omega_u^n}{\int_X \chi' \left(\frac{\dot{u}}{\|\dot{u}\|_{\chi, u}} \right) \frac{\dot{u}}{\|\dot{u}\|_{\chi, u}} \omega_u^n} dt$$

Using the Young identity (12) and the fact that ∇ is a Riemannian connection, we can continue:

$$\begin{aligned}
&= \int_0^1 \frac{\int_X \chi' \left(\frac{\dot{u}}{\|\dot{u}\|_{X,u}} \right) \nabla_{\frac{du}{ds}} \dot{u} \omega_u^n}{\chi(1) + \int_X \chi^* \left(\chi' \left(\frac{\dot{u}}{\|\dot{u}\|_{X,u}} \right) \right) \omega_u^n} dt \\
&= \int_0^1 \frac{\int_X \chi' \left(\frac{\dot{u}}{\|\dot{u}\|_{X,u}} \right) \nabla_{\dot{u}} \frac{du}{ds} \omega_u^n}{\chi(1) + \int_X \chi^* \left(\chi' \left(\frac{\dot{u}}{\|\dot{u}\|_{X,u}} \right) \right) \omega_u^n} dt \\
&= \int_0^1 \frac{\frac{d}{dt} \int_X \chi' \left(\frac{\dot{u}}{\|\dot{u}\|_{X,u}} \right) \frac{du}{ds} \omega_u^n - \int_X \frac{du}{ds} \nabla_{\dot{u}} \left(\chi' \left(\frac{\dot{u}}{\|\dot{u}\|_{X,u}} \right) \right) \omega_u^n}{\chi(1) + \int_X \chi^* \left(\chi' \left(\frac{\dot{u}}{\|\dot{u}\|_{X,u}} \right) \right) \omega_u^n} dt. \tag{33}
\end{aligned}$$

We make the following side computation:

$$\nabla_{\dot{u}} \left(\chi' \left(\frac{\dot{u}}{\|\dot{u}\|_{X,u}} \right) \right) \omega_u^n = \chi'' \left(\frac{\dot{u}}{\|\dot{u}\|_{X,u}} \right) \left(\frac{\nabla_{\dot{u}} \dot{u}}{\|\dot{u}\|_{X,u}} - \frac{1}{\|\dot{u}\|_{X,u}^2} \frac{d}{dt} \|\dot{u}\|_{X,u} \right) \omega_u^n \tag{34}$$

After possibly further shrinking $\varepsilon_0(\phi, \psi) > 0$, from Proposition 3.3(i)(ii) and (22) it follows that $\|\dot{u}\|_{X,u}$ is uniformly bounded away from zero and both $\nabla_{\dot{u}} \dot{u} \omega_u^n$ and $\frac{d}{dt} \|\dot{u}\|_{X,u}$ are of the form εR , where R is a uniformly bounded quantity for $\varepsilon < \varepsilon_0(\phi, \psi)$. Furthermore, it follows from Chen's arguments that \dot{u} and du/ds are uniformly bounded independently of ε (see again [Bl2, Theorem 12]). All of this implies that the quantity of (34) is also of the form εR . Building on this, the second term in the numerator of (33) can be estimated and we can continue to write:

$$= \int_0^1 \frac{\frac{d}{dt} \int_X \chi' \left(\frac{\dot{u}}{\|\dot{u}\|_{X,u}} \right) \frac{du}{ds} \omega_u^n}{\chi(1) + \int_X \chi^* \left(\chi' \left(\frac{\dot{u}}{\|\dot{u}\|_{X,u}} \right) \right) \omega_u^n} dt + \varepsilon R$$

As χ^* is the Legendre transform of χ , it follows that $\chi^{*'}(\chi'(l)) = l$, $l \in \mathbb{R}$. Using this, our prior observations and the chain rule, we obtain that the expression

$$\frac{d}{dt} \left(\chi(1) + \int_X \chi^* \left(\chi' \left(\frac{\dot{u}}{\|\dot{u}\|_{X,u}} \right) \right) \omega_u^n \right) = \int_X \frac{\dot{u}}{\|\dot{u}\|_{X,u}} \chi'' \left(\frac{\dot{u}}{\|\dot{u}\|_{X,u}} \right) \nabla_{\dot{u}} \left(\frac{\dot{u}}{\|\dot{u}\|_{X,u}} \right) \omega_u^n$$

is again of magnitude εR , hence in our sequence of calculations we can write

$$\begin{aligned}
&= \int_0^1 \frac{d}{dt} \frac{\int_X \chi' \left(\frac{\dot{u}}{\|\dot{u}\|_{X,u}} \right) \frac{du}{ds} \omega_u^n}{\chi(1) + \int_X \chi^* \left(\chi' \left(\frac{\dot{u}}{\|\dot{u}\|_{X,u}} \right) \right) \omega_u^n} dt + \varepsilon R \\
&= \frac{\int_X \chi' \left(\frac{\dot{u}(1,s)}{\|\dot{u}(1,s)\|_{X,\psi}} \right) \frac{d\psi(s)}{ds} \omega_\psi^n}{\chi(1) + \int_X \chi^* \left(\chi' \left(\frac{\dot{u}(1,s)}{\|\dot{u}(1,s)\|_{X,\psi}} \right) \right) \omega_\psi^n} + \varepsilon R \\
&\geq - \left\| \frac{d\psi(s)}{ds} \right\|_{X,\psi} + \varepsilon R, \tag{35}
\end{aligned}$$

where in the last line we have used the Young inequality (12) in the following manner:

$$\begin{aligned} \frac{\int_X \chi' \left(\frac{\dot{u}(1,s)}{\|\dot{u}(1,s)\|_{\chi,\psi}} \right) \frac{d\psi(s)}{ds} \omega_\psi^n}{\|d\psi/ds\|_{\chi,\psi}} &\geq - \int_X \left[\chi \left(\frac{d\psi/ds}{\|d\psi/ds\|_{\chi,\psi}} \right) + \chi^* \left(\chi' \left(\frac{\dot{u}(1,s)}{\|\dot{u}(1,s)\|_{\chi,\psi}} \right) \right) \right] \omega_\psi^n \\ &= - \left[\chi(1) + \int_X \chi^* \left(\chi' \left(\frac{\dot{u}(1,s)}{\|\dot{u}(1,s)\|_{\chi,\psi}} \right) \right) \omega_\psi^n \right]. \end{aligned}$$

Integrating estimate (35) with respect to s yields the desired inequality. \square

With all the ingredients of the proof in place we can establish the main result of this section:

Theorem 3.5. *Suppose $\chi \in \mathcal{W}_p^+$ and $[0, 1] \ni t \rightarrow u_t \in \mathcal{H}_\Delta$ is the weak geodesic of (22) joining $u_0, u_1 \in \mathcal{H}$. Then we have*

$$d_\chi(u_0, u_1) = l_\chi(u) = \|\dot{u}_t\|_{\chi, u_t}, \quad t \in [0, 1]. \quad (36)$$

Consequently, (\mathcal{H}, d_χ) is a metric space.

Proof. As the smooth ε -geodesics u^ε connecting u_0, u_1 converge to u in $C^{1,\alpha}(\bar{S} \times X)$ we have

$$\lim_{\varepsilon \rightarrow 0} l_\chi(u^\varepsilon) = l_\chi(u),$$

hence $d_\chi(u_0, u_1) \leq l_\chi(u)$.

For the other inequality, we assume first that $\chi \in \mathcal{W}_p^+ \cap C^\infty$. We have to prove that

$$l_\chi(\phi) \geq l_\chi(u) \quad (37)$$

for all smooth curves $[0, 1] \ni t \rightarrow \phi_t \in \mathcal{H}$ connecting u_0, u_1 . We can assume that $u_1 \notin \phi[0, 1]$ and let $h \in [0, 1)$. Letting $\varepsilon \rightarrow 0$ in the previous result we obtain that

$$l_\chi(v) \leq l_\chi(\phi|_{[0,h]}) + l_\chi(w^h),$$

where $(0, 1) \ni t \rightarrow v_t, w_t^h \in \mathcal{H}_\Delta$ are the weak geodesic segments joining u_1, u_0 and u_1, ϕ_h respectively. As $h \rightarrow 1$ we have $l_\chi(w^h) \rightarrow 0$ and we obtain (37). For general $\chi \in \mathcal{W}_p^+$ by Proposition 2.4 there exists a sequence $\chi_k \in \mathcal{W}_{p_k}^+ \cap C^\infty(\mathbb{R})$ such that χ_k converges to χ uniformly on compacts. From what we just proved it follows that

$$\int_0^1 \|\dot{\phi}_t\|_{\chi_k, \phi_t} dt = l_{\chi_k}(\phi) \geq l_{\chi_k}(u) = \int_0^1 \|\dot{u}_t\|_{\chi_k, u_t} dt.$$

Using Proposition 2.3 and the dominated convergence theorem ($\dot{\phi}_t, \dot{u}_t$ are uniformly bounded), we can take the limit in this last estimate to conclude (37). Formula (36) follows now from Remark 2.5.

Finally, if $u_0 \neq u_1$ then $\dot{u}_0 \neq 0$, hence $d_\chi(u_0, u_1) = \|\dot{u}_0\|_{\chi, u_0} > 0$. This implies that (\mathcal{H}, d_χ) is a metric space. \square

4 The Metric Spaces $(\mathcal{E}_{\tilde{\chi}}(X, \omega), d_{\chi})$

For the entire section we fix $\chi \in \mathcal{W}_p^+$. Our first result is the analog of [G, Proposition 2.16] in our more general setting:

Lemma 4.1. *Suppose $u_0, u_1 \in \mathcal{H}$ with $u_0 \leq u_1$. We have:*

$$\max\{2^{-n-2}\|u_1 - u_0\|_{\chi, u_0}, \|u_1 - u_0\|_{\chi, u_1}\} \leq d_{\chi}(u_0, u_1) \leq \|u_1 - u_0\|_{\chi, u_0}.$$

Proof. Suppose $(0, 1) \ni t \rightarrow u_t \in \mathcal{H}_{\Delta}$ is the weak geodesic segment joining u_0 and u_1 . By (25) we have

$$d_{\chi}(u_0, u_1) = \|\dot{u}_1\|_{\chi, u_1} = \|\dot{u}_0\|_{\chi, u_0}.$$

Since $u_0 \leq u_1$, we have that $u_0 \leq u_t$. Since $(t, x) \rightarrow u_t(x)$ is convex in the t -variable, it results that $0 \leq \dot{u}_0 \leq u_1 - u_0 \leq \dot{u}_1$ and

$$\|u_1 - u_0\|_{\chi, u_1} \leq d_{\chi}(u_0, u_1) \leq \|u_1 - u_0\|_{\chi, u_0} \quad (38)$$

follows.

We introduce $N := \|u_1 - u_0\|_{\chi, u_0}$. Using $\omega_{u_0}^n \leq 2^n \omega_{(u_0+u_1)/2}^n$ and the convexity of χ we have

$$\begin{aligned} \chi(1) &= \int_X \chi\left(\frac{u_1 - u_0}{N}\right) \omega_{u_0}^n \leq \int_X \chi\left(\frac{u_1 - (u_0 + u_1)/2}{N/2}\right) 2^n \omega_{(u_0+u_1)/2}^n \\ &\leq \int_X \chi\left(\frac{u_1 - (u_0 + u_1)/2}{N/2^{n+1}}\right) \omega_{(u_0+u_1)/2}^n. \end{aligned}$$

From the definition of the Orlicz norm it follows that

$$\frac{\|u_1 - u_0\|_{\chi, u_0}}{2^{n+1}} \leq \left\| u_1 - \frac{u_0 + u_1}{2} \right\|_{\chi, \frac{u_0+u_1}{2}} = \left\| u_0 - \frac{u_0 + u_1}{2} \right\|_{\chi, \frac{u_0+u_1}{2}}.$$

The first estimate of (38) allows us to continue and obtain:

$$\frac{\|u_1 - u_0\|_{\chi, u_0}}{2^{n+1}} \leq d_{\chi}\left(\frac{u_0 + u_1}{2}, u_0\right).$$

But we have $d_{\chi}((u_0+u_1)/2, u_0) \leq d_{\chi}(u_0, u_1) + d_{\chi}((u_0+u_1)/2, u_1)$, and $d_{\chi}((u_0+u_1)/2, u_1) \leq d_{\chi}(u_0, u_1)$ as follows from the lemma below. This implies the desired estimate. \square

Lemma 4.2. *Suppose $u, v, w \in \mathcal{H}$ with $u \geq v \geq w$. Then we have $d_{\chi}(u, v) \leq d_{\chi}(u, w)$.*

Proof. We notice that the weak geodesic $[0, 1] \ni t \rightarrow \alpha_t, \beta_t \in \mathcal{H}_{\Delta}$ connecting u, v and u, w respectively are both decreasing, satisfy $\alpha \geq \beta$ by the comparison principle, and $\alpha_0 = \beta_0$. From this it follows that $0 \geq \dot{\alpha}_0 \geq \dot{\beta}_0$. Using this, (36) yields the desired estimate. \square

Our next result is the analog of [Da1, Lemma 6.3]:

Lemma 4.3. *Suppose $\{u_k\}_{k \in \mathbb{N}} \subset \mathcal{H}$ is a sequence decreasing pointwise to $u \in \mathcal{E}_{\tilde{\chi}}(X, \omega)$. Then $d_{\chi}(u_l, u_k) \rightarrow 0$ as $l, k \rightarrow \infty$.*

Proof. Our argument is just a small modification of the original proof. Suppose that $l \leq k$. Then $u_k \leq u_l$, hence by the previous result and Proposition 2.2 we have:

$$d_\chi(u_l, u_k) \leq \|u_l - u_k\|_{\chi, u_k} \leq M_p \left(\int_X \chi(u_k - u_l) \omega_{u_k}^n / \chi(1) \right).$$

We clearly have $u - u_l, u_k - u_l \in \mathcal{E}_{\bar{\chi}}(X, \omega + i\partial\bar{\partial}u_l)$ and $u - u_l \leq u_k - u_l \leq 0$. Hence, applying [GZ1, Lemma 3.5] for the class $\mathcal{E}_{\bar{\chi}}(X, \omega + i\partial\bar{\partial}u_l)$ we obtain that there exists $C = C(p) > 0$ such that

$$d_\chi(u_l, u_k) \leq M_p \left(\int_X \chi(u_k - u_l) (\omega_{u_k})^n / \chi(1) \right) \leq CM_p \left(\int_X \chi(u - u_l) \omega_u^n \right). \quad (39)$$

As u_l decreases to $u \in \mathcal{E}_{\bar{\chi}}(X, \omega)$, it follows from the dominated convergence theorem that $d_\chi(u_l, u_k) \rightarrow 0$ as $l, k \rightarrow \infty$. \square

The next result is the analog of [Da1, Lemma 6.4] and its proof is the same as the original.

Lemma 4.4. *Given $u_0, u_1 \in \mathcal{E}_{\bar{\chi}}(X, \omega)$, the limit in (5) is finite and independent of the approximating sequences $u_0^k, u_1^k \in \mathcal{H}$.*

To conclude that d_χ is a metric on $\mathcal{E}_{\bar{\chi}}(X, \omega)$ all we need is that $d_\chi(u_0, u_1) = 0$ implies $u_0 = u_1$. This result is analogous to [Da1, Lemma 6.7] and using Proposition 2.2 its proof is carried out the same way:

Lemma 4.5. *Suppose $u_0, u_1 \in \mathcal{E}_{\bar{\chi}}(X, \omega)$ and $d_\chi(u_0, u_1) = 0$. Then $u_0 = u_1$.*

By [Da1, Theorem 6(i)] it follows that given $u_0, u_1 \in \mathcal{E}_{\bar{\chi}}(X, \omega)$, for the weak geodesic segment $t \rightarrow u_t$ connecting u_0, u_1 , as defined in (6), we have $u_t \in \mathcal{E}_{\bar{\chi}}(X, \omega)$, $t \in (0, 1)$. Next we show that this weak geodesic is an actual geodesic segment in $(\mathcal{E}_\chi(X, \omega), d_\chi)$ in the sense of metric spaces. One needs a technical lemma generalizing [Da1, Lemma 6.8]:

Lemma 4.6. *Suppose $v_0, v_1 \in \mathcal{H}_0 = \text{PSH}(X, \omega) \cap L^\infty$ and $\{v_1^j\}_{j \in \mathbb{N}} \subset \mathcal{H}_0$ is sequence decreasing to v_1 . By $(0, 1) \ni t \rightarrow v_t, v_t^j \in \mathcal{H}_0$ we denote the bounded weak geodesic segments connecting v_0, v_1 and v_0, v_1^j respectively. As we have convexity in the t -variable, we can define $\dot{v}_0 = \lim_{t \rightarrow 0} (v_t - v_0)/t$ and $\dot{v}_0^j = \lim_{t \rightarrow 0} (v_t^j - v_0)/t$. The following holds:*

$$\lim_{j \rightarrow \infty} \|\dot{v}_0^j\|_{\chi, v_0} = \|\dot{v}_0\|_{\chi, v_0}.$$

Proof. By an observation of Berndtsson (see Section 2.1 [Brn1]), there exists $C > 0$ such that $\|\dot{v}_0\|_{L^\infty(X)}, \|\dot{v}_0^j\|_{L^\infty(X)} \leq C$. We also have $v \leq v^j$, $j \in \mathbb{N}$ by the comparison principle. As all v^j are t -convex and share the same starting point, it also follows that $\dot{v}_0^j \searrow \dot{v}_0$ pointwise. Proposition 2.2 and the dominated convergence theorem implies now that $\|\dot{v}_0^j - \dot{v}_0\|_{\chi, v_0} \rightarrow 0$. \square

Given a metric space (M, ρ) , a curve $(0, 1) \ni t \rightarrow h_t \in M$ is a (parametrized) geodesic, if there exists $c > 0$ such that

$$\rho(h_l, h_s) = c|l - s|, \quad l, s \in (0, 1).$$

Using the last lemma, the proof of the next result is carried out the same way the analogous result in [Da1].

Lemma 4.7. *Suppose $u_0, u_1 \in \mathcal{E}_{\bar{\chi}}(X, \omega)$ and $(0, 1) \ni t \rightarrow u_t \in \mathcal{E}_{\bar{\chi}}(X, \omega)$ is the weak geodesic segment connecting u_0, u_1 defined in (6). Then $t \rightarrow u_t$ is a geodesic segment in $(\mathcal{E}_{\bar{\chi}}(X, \omega), d_{\chi})$ in the sense of metric spaces.*

Another technical lemma is needed:

Lemma 4.8. *Suppose $u, v \in \mathcal{E}_{\bar{\chi}}(X, \omega)$ satisfies $u \leq v$. Then:*

$$d_{\chi}(u, v) \leq CM_p \left(\int_X \chi(v - u) \omega_u^n \right)$$

where $C > 0$ only depends on $\dim X$ and $p \geq 1$.

Proof. Let $u_k, v_k \in \mathcal{H}$ be sequences decreasing to u, v , with the additional property $u_k \leq v_k$, $k \geq 1$. By Lemma 4.1 and Remark 2.2 we have:

$$d_{\chi}(u_k, v_k) \leq \|v_k - u_k\|_{\chi, u_k} \leq CM_p \left(\int_X \chi(u_k - v_k) \omega_{u_k}^n \right).$$

We clearly have $u - v_k, u_k - v_k \in \mathcal{E}_{\bar{\chi}}(X, \omega + i\partial\bar{\partial}v_k)$ and $u - v_k \leq u_k - v_k \leq 0$. Applying [GZ1, Lemma 3.5] to $\mathcal{E}_{\bar{\chi}}(X, \omega + i\partial\bar{\partial}v_k)$ we can conclude:

$$d_{\chi}(u_k, v_k) \leq CM_p \left(\int_X \chi(u - v_k) \omega_u^n \right).$$

Letting $k \rightarrow \infty$, by the dominated convergence theorem we arrive at the desired estimate. \square

Monotone sequences in $\mathcal{E}_{\bar{\chi}}(X, \omega)$ converge with respect to d_{χ} . Using the last lemma, the proof of this result is the same as [Da1, Proposition 6.11]:

Proposition 4.9. *If $\{w_k\}_{k \in \mathbb{N}} \subset \mathcal{E}_{\bar{\chi}}(X, \omega)$ decreases (increases a.e.) to $w \in \mathcal{E}_{\bar{\chi}}(X, \omega)$ then $d_{\chi}(w_k, w) \rightarrow 0$.*

As pointed out in [Da1, Section 7], formula (36) fails already when u_0, u_1 are Lipschitz. Following [Da1, Theorem 7.2], an extension is nevertheless possible for $u_0, u_1 \in \mathcal{H}_{\Delta}$. For this, first we have to adapt a theorem of Berndtsson [Brn2] to our setting.

Lemma 4.10. *Suppose $u_0, u_1 \in \mathcal{H}_{\Delta}$ and $(0, 1) \ni t \rightarrow u_t \in \mathcal{H}_{\Delta}$ is the geodesic connecting them, as guaranteed by [BD, Corollary 4.7] (see also [H1]). Then for any $f \in C(\mathbb{R})$ the following holds:*

$$\int_X f(\dot{u}_0) \omega_{u_0}^n = \int_X f(\dot{u}_t) \omega_{u_t}^n, \quad t \in [0, 1]. \quad (40)$$

Proof. We need to prove the desired identity only for $t = 1$. By translation we can assume that $f(0) = 0$. Using approximation, we can also assume that $f \in C^1(\mathbb{R})$. To obtain (40) for $t = 1$ we have to prove the following two formulas:

$$\int_{\{\dot{u}_0 > 0\}} f(\dot{u}_0) \omega_{u_0}^n = \int_{\{\dot{u}_1 > 0\}} f(\dot{u}_1) \omega_{u_1}^n, \quad (41)$$

$$\int_{\{\dot{u}_0 < 0\}} f(\dot{u}_0) \omega_{u_0}^n = \int_{\{\dot{u}_1 < 0\}} f(\dot{u}_1) \omega_{u_1}^n. \quad (42)$$

As $\int_{\{\dot{u}_0 > 0\}} f(\dot{u}_0) \omega_{u_0}^n = \int_0^{\infty} f'(\tau) \omega_{u_0}^n(\{\dot{u}_0 \geq \tau\}) d\tau$, one can see that the same calculation that gave [Da1, formulas (46), (47)] (which deals with the particular case $f(l) = l^2$) also gives (41) and (42). \square

Lemma 4.11. *Suppose $u_0, u_1 \in \mathcal{H}_\Delta$ and $(0, 1) \ni t \rightarrow u_t \in \mathcal{H}_\Delta$ is the geodesic connecting them. Then we have:*

$$d_\chi(u_0, u_1) = \|\dot{u}_t\|_{\chi, u_t}, \quad t \in [0, 1]. \quad (43)$$

Proof. As usual, let $u_0^k, u_1^k \in \mathcal{H}$ be sequences of potentials decreasing to u_0, u_1 . Let $(0, 1) \ni t \rightarrow u_t^{kl} \in \mathcal{H}_\Delta$ be the geodesic joining u_0^k, u_1^l . By (25)

$$d_\chi(u_0^k, u_1^l) = \|\dot{u}_0^{kl}\|_{\chi, u_0^k}.$$

If we let $l \rightarrow \infty$, by Lemma 4.6 and Proposition 4.9 we obtain that

$$d_\chi(u_0^k, u_1) = \|\dot{u}_0^k\|_{\chi, u_0^k},$$

where $(0, 1) \ni t \rightarrow u_t^k \in \mathcal{H}_\Delta$ is the geodesic connecting u_0^k with u_1 . Using the previous lemma we can write:

$$\chi(1) = \int_X \chi\left(\frac{\dot{u}_0^k}{\|\dot{u}_0^k\|_{\chi, u_0^k}}\right) \omega_{u_0^k}^n = \int_X \chi\left(\frac{\dot{u}_1^k}{\|\dot{u}_0^k\|_{\chi, u_0^k}}\right) \omega_{u_1^k}^n.$$

Hence,

$$d_\chi(u_0^k, u_1) = \|\dot{u}_0^k\|_{\chi, u_0} = \|\dot{u}_1^k\|_{\chi, u_1}.$$

Letting $k \rightarrow \infty$, another application of Lemma 4.6 yields (43) for $t = 1$. The case $t = 0$ follows by symmetry, and for $0 < t < 1$ the result follows because a subarc of a geodesic is again a geodesic. \square

With this last lemma under our belt, we can prove that $P(\cdot, \cdot)$ is a contraction in both components with respect to d_χ , which is the analog of [Da1, Proposition 8.2]:

Proposition 4.12. *Given $\chi \in \mathcal{W}_p^+$ and $u, v, w \in \mathcal{E}_\chi(X, \omega)$ we have*

$$d_\chi(P(u, v), P(u, w)) \leq d_\chi(v, w).$$

Before we can give the proof, we need to generalize the Pythagorean formula of [Da1, Proposition 8.1].

Proposition 4.13. *Given $u_0, u_1 \in \mathcal{H}_\Delta$, we have $P(u_0, u_1) \in \mathcal{H}_\Delta$. Let $(0, 1) \ni t \rightarrow u_t, v_t, w_t \in \mathcal{H}_\Delta$ be the weak geodesics joining (u_0, u_1) , $(u_0, P(u_0, u_1))$ and $(u_1, P(u_0, u_1))$ respectively. Then we have*

$$\int_X f(\dot{u}_t) \omega_{u_t}^n = \int_X f(\dot{v}_t) \omega_{v_t}^n + \int_X f(\dot{w}_t) \omega_{w_t}^n, \quad t, l, h \in [0, 1], \quad (44)$$

where $f \in C(\mathbb{R})$ is arbitrary with $f(0) = 0$.

Proof. The fact that $P(u_0, u_1) \in \mathcal{H}_\Delta$ follows from [DR]. Using an approximation argument we can assume that $f \in C^1(\mathbb{R})$. To justify (44), by Lemma 4.10 it is enough to show that

$$\begin{aligned} \int_{\{\dot{u}_0 > 0\}} f(\dot{u}_0) \omega_{u_0}^n &= \int_X f(\dot{w}_0) \omega_{u_1}^n. \\ \int_{\{\dot{u}_0 < 0\}} f(\dot{u}_0) \omega_{u_0}^n &= \int_X f(\dot{v}_0) \omega_{u_0}^n \end{aligned}$$

As $\int_{\{\dot{u}_0 > 0\}} f(\dot{u}_0) \omega_{u_0}^n = \int_0^\infty f'(\tau) \omega_{u_0}^n(\{\dot{u}_0 \geq \tau\}) d\tau$, both of the above identities can be proved using the same arguments as in [Da1, formulas (49) and (50)], where the particular case $f(l) = l^2$ is considered. \square

Corollary 4.14. *For $u_0, u_1 \in \mathcal{E}^p(X, \omega)$ we have*

$$d_p(u_0, u_1)^p = d_p(u_0, P(u_0, u_1))^p + d_p(u_1, P(u_0, u_1))^p. \quad (45)$$

Consequently,

$$d_1(u_0, u_1) = AM(u_0) + AM(u_1) - 2AM(P(u_0, u_1)). \quad (46)$$

For general $\chi \in \mathcal{W}_p^+$ and $u_0, u_1 \in \mathcal{E}_{\tilde{\chi}}(X, \omega)$ we have

$$\begin{aligned} \frac{1}{2}(d_{\chi}(u_0, P(u_0, u_1)) + d_{\chi}(u_1, P(u_0, u_1))) &\leq d_{\chi}(u_0, u_1) \\ &\leq 2(d_{\chi}(u_0, P(u_0, u_1)) + d_{\chi}(u_1, P(u_0, u_1))). \end{aligned} \quad (47)$$

These results can also be seen as a generalization of the Pythagorean formula [Da1, Proposition 8.1] to our setting.

Proof. To begin, we note that is enough to prove each identity/estimate for $u_0, u_1 \in \mathcal{H}$, as in each case an approximation procedure yields the general result. To obtain (45), one just puts $f = \chi_p$ into the previous proposition.

Formula (46) follows from the fact that $d_1(u_0, u_1) = AM(u_0) - AM(u_1)$ when $u_0 \geq u_1$. Indeed, the geodesic $t \rightarrow u_t$ connecting u_0, u_1 satisfies $\dot{u}_0 \leq 0$, hence $d_1(u_0, u_1) = \|\dot{u}_0\|_{1, u_0} = -\int_X \dot{u}_0 \omega_{u_0}^n = AM(u_0) - AM(u_1)$, as the Aubin-Mabuchi energy is linear along geodesics.

Trying to imitate all of this for general $\chi \in \mathcal{W}_p^+$ we choose $D = d_{\chi}(u_0, u_1)$ and $f(l) = \chi(l/D)$. From (44) it follows that

$$\chi(1) = \int_X \chi\left(\frac{\dot{v}_0}{D}\right) \omega_{v_0}^n + \int_X \chi\left(\frac{\dot{w}_0}{D}\right) \omega_{w_0}^n,$$

with $t \rightarrow v_t$ and $t \rightarrow w_t$ as in the previous proposition. From this identity and Lemma 4.11 it immediately follows that $d_{\chi}(u_0, P(u_0, u_1)), d_{\chi}(u_1, P(u_0, u_1)) \leq D$ implying the first estimate in (47). For the second estimate, we observe that at least one of the terms in the above identity is bigger than $\chi(1)/2$. We can assume that this term is the first one and convexity of χ implies

$$\int_X \chi\left(\frac{2\dot{v}_0}{D}\right) \omega_{v_0}^n \geq \int_X 2\chi\left(\frac{\dot{v}_0}{D}\right) \omega_{v_0}^n \geq \chi(1),$$

from which it follows that $d_{\chi}(u_0, P(u_0, u_1)) \geq D/2$, implying the second estimate in (47). \square

Proof of Proposition 4.12. As the general case follows from an approximation argument via Proposition 4.9, it is enough to prove the estimate for $u, v, w \in \mathcal{H}_{\Delta}$. In this particular case we also have $P(u, v), P(u, w), P(u, v, w) \in \mathcal{H}_{\Delta}$ (see [DR]). We digress slightly to prove the following claim central to our argument:

Claim. *Suppose $v, w \in \mathcal{H}_{\Delta}$, $v \leq w$ and let $(0, 1) \ni t \rightarrow \phi_t, \psi_t \in \mathcal{H}_{\Delta}$ be the weak geodesic segments joining v, w and $P(u, v), P(u, w)$ respectively. For any increasing non-negative function $f \in C([0, \infty))$ one has*

$$\int_X f(\psi_l) \omega_{\psi_l}^n \leq \int_X f(\phi_h) \omega_{\phi_h}^n, \quad l, h \in [0, 1]. \quad (48)$$

Proof. As follows from Lemma 4.10 it is enough to prove the above estimate for $l = h = 0$. From (26) it follows that that

$$\int_X f(\psi_0)\omega_{P(u,v)}^n \leq \int_{\{P(u,v)=u\}} f(\psi_0)\omega_u^n + \int_{\{P(u,v)=v\}} f(\psi_0)\omega_v^n$$

We argue that the first term in this sum is zero. As $v \leq w$ and $P(u, v) \leq P(u, w)$, it is clear that $t \rightarrow \phi_t, \psi_t$ are increasing in t . By the maximum principle, it is also clear that $\psi_t \leq u$, $t \in [0, 1]$. Hence, if $x \in \{P(u, v) = u\}$ then $\psi_t(x) = u(x)$, $t \in [0, 1]$, implying $\psi_0|_{\{P(u,v)=u\}} \equiv 0$.

At the same time, using the maximum principle again, it follows that $\psi_t \leq \phi_t$, $t \in [0, 1]$. This implies that $0 \leq \dot{\psi}_0|_{\{P(u,v)=v\}} \leq \dot{\phi}_0|_{\{P(u,v)=v\}}$, which in turn implies (48). \square

The proof is just an application of Proposition 4.13 and the claim just provided. Suppose $(0, 1) \ni t \rightarrow a_t, b_t, c_t, d_t, e_t, g_t \in \mathcal{H}_\Delta$ are the weak geodesics joining $(P(u, v), P(u, w))$, $(P(u, v), P(u, v, w))$, $(P(u, w), P(u, v, w))$, $(v, P(v, w))$, $(w, P(v, w))$, and (v, w) respectively. We introduce $D = \|\dot{a}_0\|_{\chi, P(u,v)}$, which by Lemma 4.11 is equals $d_\chi(P(u, v), P(u, w))$. Using Proposition 4.13 we can start writing

$$\chi(1) = \int_X \chi\left(\frac{\dot{a}_0}{D}\right)\omega_{P(u,v)}^n = \int_X \chi\left(\frac{\dot{b}_0}{D}\right)\omega_{P(u,v)}^n + \int_X \chi\left(\frac{\dot{c}_0}{D}\right)\omega_{P(u,w)}^n.$$

Now we use (48) for both of the terms in the above sum to continue:

$$\begin{aligned} &\leq \int_X \chi\left(\frac{\dot{d}_0}{D}\right)\omega_v^n + \int_X \chi\left(\frac{\dot{e}_0}{D}\right)\omega_w^n \\ &= \int_X \chi\left(\frac{\dot{g}_0}{D}\right)\omega_v^n, \end{aligned}$$

where in the last line we have used Proposition 4.13 again. From the estimate we obtained it follows that $d_\chi(v, w) = \|\dot{g}_0\|_v \geq D = d_\chi(P(u, v), P(u, w))$. \square

As an easy computation shows, for $u, v \in \mathcal{E}^1(X, \omega)$ we have

$$AM(u) - AM(v) = \frac{1}{n+1} \sum_{j=0}^n \int_X (u-v)(\omega + i\partial\bar{\partial}u)^j \wedge (\omega + i\partial\bar{\partial}v)^{n-j}. \quad (49)$$

We observe now that the Aubin-Mabuchi energy is Lipschitz continuous with respect to our path length metric:

Lemma 4.15. *Given $u_0, u_1 \in \mathcal{E}_{\bar{\chi}}(X, \omega)$, we have*

$$|AM(u_0) - AM(u_1)| \leq C(p)d_\chi(u_0, u_1).$$

Proof. By density we can suppose that $u_0, u_1 \in \mathcal{H}$. Let $(0, 1) \ni t \rightarrow u_t \in \mathcal{H}_\Delta$ be the geodesic connecting u_0, u_1 . By (36), (49) and the Hölder inequality we have:

$$\begin{aligned} AM(u_1) - AM(u_0) &= \int_0^1 \frac{dAM(u_t)}{dt} dt = \int_0^1 \int_X \dot{u}_t \omega_{u_t}^n dt \\ &\leq \int_0^1 \|1\|_{\chi^*, u_t} \|\dot{u}_t\|_{\chi, u_t} dt = \|1\|_{\chi^*, u_0} d_\chi(u_0, u_1), \end{aligned}$$

where we observed that all the norms $\|1\|_{\chi^*, u_t}$ are the same and are equal to $\|1\|_{\chi^*, u_0}$. \square

We are ready to prove completeness of $(\mathcal{E}_\chi(X, \omega), d_\chi)$. Roughly, the idea of the proof is to replace an arbitrary Cauchy sequence with an equivalent monotone Cauchy sequence which is much easier to deal with in light of the next result:

Lemma 4.16. *Suppose $\{u_k\}_{k \in \mathbb{N}} \subset \mathcal{E}_{\tilde{\chi}}(X, \omega)$ is a pointwise decreasing d_χ -bounded sequence. Then $u = \lim_{k \rightarrow \infty} u_k \in \mathcal{E}_{\tilde{\chi}}(X, \omega)$ and additionally $d_\chi(u, u_k) \rightarrow 0$.*

Proof. We can suppose that $u_k < 0$. First we assume that $\{u_k\}_{k \in \mathbb{N}} \subset \mathcal{H}$. By Lemmas 2.2 and 4.1 it follows that

$$m_p \left(\int_X \chi(u_j) \omega_{u_j}^n / \chi(1) \right) \leq \|u_j\|_{\chi, u_j} \leq 2^{n+1} d_\chi(u_j, 0) \leq D.$$

Now [GZ1, Proposition 5.6] says that

$$\int_X \tilde{\chi}(u) \omega_u^n < +\infty,$$

giving $u \in \mathcal{E}_{\tilde{\chi}}(X, \omega)$. The fact that $d_\chi(u_k, u) \rightarrow 0$ follows from Proposition 4.9. When elements of the sequence $\{u_k\}_{k \in \mathbb{N}}$ are non-smooth, the result follows from an approximation argument via Proposition 4.9 again. \square

The following theorem is the main result of this section. Its proof rests on Proposition 4.12 and the preceding two lemmas, and is carried out exactly the same way as the arguments in [Da1, Theorem 9.2].

Theorem 4.17. *If $\chi \in \mathcal{W}_p^+$, $p \geq 1$ then $(\mathcal{E}_{\tilde{\chi}}(X, \omega), d_\chi)$ is a geodesic metric space, which is the metric completion of (\mathcal{H}, d_χ) . Additionally, the curve defined in (6) is a geodesic connecting $u_0, u_1 \in \mathcal{E}_{\tilde{\chi}}(X, \omega)$.*

Unfortunately, geodesics connecting different points of $(\mathcal{E}_{\tilde{\chi}}(X, \omega), d_\chi)$ may not be unique. This is most easily seen for $(\mathcal{E}^1(X, \omega), d_1)$, as by (45) we have

$$d_1(u_0, u_1) = d_1(u_0, P(u_0, u_1)) + d_1(u_1, P(u_0, u_1)),$$

for all $u_0, u_1 \in \mathcal{E}^1(X, \omega)$. This formula implies that the geodesic connecting u_0 to $P(u_0, u_1)$ concatenated with the geodesic connecting $P(u_0, u_1)$ and u_1 has the same length as the geodesic connecting u_0, u_1 . The fact that geodesics connecting different points of $(\mathcal{E}_{\tilde{\chi}}(X, \omega), d_\chi)$ may not be unique implies that $(\mathcal{E}_{\tilde{\chi}}(X, \omega), d_\chi)$ is not a $CAT(0)$ space in general. This is in sharp contrast with the findings of [Da1] about $(\mathcal{E}^2(X, \omega), d_2)$.

5 Convergence in χ -Energy inside $\mathcal{E}_{\tilde{\chi}}(X, \omega)$

The first step in proving Theorem 3 is extending Lemma 4.1 for non-smooth potentials:

Lemma 5.1. *Suppose $\chi \in \mathcal{W}_p^+$ and $u_0, u_1 \in \mathcal{E}_{\tilde{\chi}}(X, \omega)$ with $u_0 \leq u_1$. The following holds:*

$$\max\{2^{-n-2} \|u_1 - u_0\|_{\chi, u_0}, \|u_1 - u_0\|_{\chi, u_1}\} \leq d_\chi(u_0, u_1) \leq \|u_1 - u_0\|_{\chi, u_0}$$

The proof of this result follows using approximation with smooth potentials via Proposition 4.9 and the next technical lemma:

Lemma 5.2. *Suppose $\chi \in \mathcal{W}_p^+$ and $\{u_k\}_{k \in \mathbb{N}}, \{v_k\}_{k \in \mathbb{N}}, \{w_k\}_{k \in \mathbb{N}} \subset \mathcal{E}_{\tilde{\chi}}(X, \omega)$ are decreasing sequences for which $u_k \leq v_k$, $u_k \leq w_k$ and also $u_k \searrow u \in \mathcal{E}_{\tilde{\chi}}(X, \omega)$, $v_k \searrow v \in \mathcal{E}_{\tilde{\chi}}(X, \omega)$ and $w_k \searrow w \in \mathcal{E}_{\tilde{\chi}}(X, \omega)$. Then we have*

$$\lim_{k \rightarrow \infty} \|u_k - v_k\|_{\chi, w_k} = \|u - v\|_{\chi, w} \quad (50)$$

Proof. We can suppose without loss of generality that all the functions involved are negative and $N = \|u - v\|_{\chi, w} > 0$. As $\chi \in \mathcal{W}_p^+$, by Proposition 2.2 it is enough to prove that

$$\int_X \chi\left(\frac{u_k - v_k}{N}\right) \omega_{w_k}^n - \chi(1) = \int_X \chi\left(\frac{u_k - v_k}{N}\right) \omega_{w_k}^n - \int_X \chi\left(\frac{u - v}{N}\right) \omega_w^n \rightarrow 0. \quad (51)$$

First we suppose that there exists $L > 1$ such that $-L < u, u_k, v, v_k, w, w_k < 0$ are uniformly bounded. Given $\varepsilon > 0$ one can find an open $O \subset X$ such that $\text{Cap}_X(O) < \varepsilon$ and u, u_k, v, v_k, w, w_k are all continuous on $X \setminus O$. We have

$$\int_X \chi\left(\frac{u_k - v_k}{N}\right) \omega_{w_k}^n - \int_X \chi\left(\frac{u - v}{N}\right) \omega_w^n = \int_O + \int_{X \setminus O} \left[\chi\left(\frac{u_k - v_k}{N}\right) - \chi\left(\frac{u - v}{N}\right) \right] \omega_{w_k}^n.$$

The integral on O is bounded by $2\varepsilon L^n \chi(2L/N)$. The second integral tends to 0 as on the closed set $X \setminus O$ we have $u_k \rightarrow u$ and $v_k \rightarrow v$ uniformly. We also have

$$\int_X \chi\left(\frac{u - v}{N}\right) \omega_{w_k}^n - \int_X \chi\left(\frac{u - v}{N}\right) \omega_w^n \rightarrow 0,$$

as the function $\chi\left(\frac{u-v}{N}\right)$ is quasi-continuous and bounded [BT, Theorem 3.2]. This all means that (51) follows when $-L < u, u_k, v, v_k, w, w_k < 0$. Now we argue that this last limit also holds when u, u_k, v, v_k, w, w_k are unbounded. For this we show that

$$\int_X \chi\left(\frac{u_k - v_k}{N}\right) \omega_{w_k}^n - \int_X \chi\left(\frac{u_k^L - v_k^L}{N}\right) \omega_{w_k^L}^n \rightarrow 0 \quad (52)$$

$$\int_X \chi\left(\frac{u - v}{N}\right) \omega_w^n - \int_X \chi\left(\frac{u^L - v^L}{N}\right) \omega_{w^L}^n \rightarrow 0 \quad (53)$$

as $L \rightarrow -\infty$ uniformly for u, v, w, u_k, v_k, w_k , where $h^L = \max(h, -L)$. Before we get into the estimates we observe that there exists $\psi \in \mathcal{W}_{2p+1}^+$ such that $\chi(l)/\psi(l)$ decreases to 0 as $l \rightarrow -\infty$ and $u \in \mathcal{E}_\psi(X, \omega)$. This implies that $u, v, w, u_k, v_k, w_k \in \mathcal{E}_\psi(X, \omega)$, $k \in \mathbb{N}$. As $\{v_k \leq -L\}, \{w_k \leq -L\} \subseteq \{u_k \leq -L\}$, since the Monge-Ampère measure is local in

the plurifine topology we can start writing:

$$\begin{aligned}
& \left| \int_X \chi\left(\frac{u_k - v_k}{N}\right) \omega_{w_k}^n - \int_X \chi\left(\frac{u_k^L - v_k^L}{N}\right) \omega_{w_k^L}^n \right| = \\
& = \left| \int_{\{u_k \leq -L\}} \chi\left(\frac{u_k - v_k}{N}\right) \omega_{w_k}^n - \int_{\{u_k \leq -L\}} \chi\left(\frac{u_k^L - v_k^L}{N}\right) \omega_{w_k^L}^n \right| \\
& \leq \int_{\{u_k \leq -L\}} \chi\left(\frac{u_k}{N}\right) \omega_{w_k}^n + \int_{\{u_k \leq -L\}} \chi\left(\frac{u_k^L}{N}\right) \omega_{w_k^L}^n \\
& \leq \frac{\chi(L/N)}{\psi(L/N)} \left(\int_{\{u_k \leq -L\}} \psi\left(\frac{u_k}{N}\right) \omega_{w_k}^n + \int_{\{u_k \leq -L\}} \psi\left(\frac{u_k^L}{N}\right) \omega_{w_k^L}^n \right) \\
& \leq \frac{\chi(L/N)}{\psi(L/N)} \left(\int_X \psi\left(\frac{u_k}{N}\right) \omega_{w_k}^n + \int_X \psi\left(\frac{u_k}{N}\right) \omega_{w_k^L}^n \right) \\
& \leq \frac{C(p, N)\chi(L/N)}{\psi(L/N)} \left(E_{\tilde{\psi}}(u_k) + E_{\tilde{\psi}}(w_k) + E_{\tilde{\psi}}(w_k^L) \right) \\
& \leq \frac{C(p, N)\chi(L/N)}{\psi(L/N)} E_{\tilde{\psi}}(u),
\end{aligned}$$

where in the penultimate line we have used [GZ1, Proposition 3.6] and in the last line we have used [GZ1, Lemma 3.5]. This justifies (52) and (53) is established the same way. As we explained above, all this implies that (51) holds for arbitrary u, u_k, v, v_k, w, w_k . \square

The following preliminary result will follow from (47), and is perhaps the crucial ingredient in proving Theorem 3.

Lemma 5.3. *Suppose $\chi \in \mathcal{W}_p^+$ and $u_0, u_1 \in \mathcal{E}_{\tilde{\chi}}(X, \omega)$. There exists $C(p) > 1$ such that*

$$d_\chi\left(u_0, \frac{u_0 + u_1}{2}\right) \leq C d_\chi(u_0, u_1).$$

Proof. Using (47) multiple times we can start writing:

$$\begin{aligned}
d_\chi\left(u_0, \frac{u_0 + u_1}{2}\right) & \leq C \left[d_\chi\left(u_0, P\left(u_0, \frac{u_0 + u_1}{2}\right)\right) + d_\chi\left(\frac{u_0 + u_1}{2}, P\left(u_0, \frac{u_0 + u_1}{2}\right)\right) \right] \\
& \leq C \left[d_\chi(u_0, P(u_0, u_1)) + d_\chi\left(\frac{u_0 + u_1}{2}, P(u_0, u_1)\right) \right] \\
& \leq C \left[\|u_0 - P(u_0, u_1)\|_{\chi, P(u_0, u_1)} + \left\| \frac{u_0 + u_1}{2} - P(u_0, u_1) \right\|_{\chi, P(u_0, u_1)} \right] \\
& \leq C \left[\frac{3}{2} \|u_0 - P(u_0, u_1)\|_{\chi, P(u_0, u_1)} + \frac{1}{2} \|u_1 - P(u_0, u_1)\|_{\chi, P(u_0, u_1)} \right] \\
& \leq C \left[d_\chi(u_0, P(u_0, u_1)) + d_\chi(u_1, P(u_0, u_1)) \right] \\
& \leq C d_\chi(u_0, u_1),
\end{aligned}$$

where in the second line we have used Lemma 4.2 and the fact that $P(u_0, u_1) \leq P(u_0, (u_0 + u_1)/2)$, in the third and fifth line Lemma 5.1, in the fourth line the sub-additivity of the norm, and in the sixth line we have used (47) again. \square

The max operator interacts well with the I_χ energy. This is showcased in the next auxiliary lemma which is essentially a result of Guedj [G, Proposition 2.16] adapted to our more general setting:

Lemma 5.4. *Given $\chi \in \mathcal{W}_p^+$ we have*

$$\begin{aligned} \frac{1}{2}(I_\chi(u_0, \max(u_0, u_1)) + I_\chi(\max(u_0, u_1), u_1)) &\leq I_\chi(u_0, u_1) \\ &\leq 2(I_\chi(u_0, \max(u_0, u_1)) + I_\chi(\max(u_0, u_1), u_1)) \end{aligned}$$

for $u_0, u_1 \in \mathcal{E}_{\bar{\chi}}(X, \omega)$.

Proof. Suppose $\alpha = \|u_1 - u_0\|_{\chi, u_0}$ and $\beta = \|u_1 - u_0\|_{\chi, u_1}$. As it will be clear at the end of the proof, we can assume that $\alpha, \beta \neq 0$. Then we have that

$$\int_X \chi\left(\frac{u_1 - u_0}{\alpha}\right) \omega_{u_0}^n = \int_X \chi\left(\frac{u_1 - u_0}{\beta}\right) \omega_{u_1}^n = \chi(1).$$

As the Monge-Ampère operator is local in the plurifine topology we also have that

$$\begin{aligned} \int_X \chi\left(\frac{\max(u_1, u_0) - u_0}{\beta}\right) \omega_{\max(u_0, u_1)}^n &= \int_{\{u_1 > u_0\}} \chi\left(\frac{u_1 - u_0}{\beta}\right) \omega_{u_1}^n \leq \chi(1) \\ \int_X \chi\left(\frac{\max(u_1, u_0) - u_0}{\alpha}\right) \omega_{u_0}^n &= \int_{\{u_1 > u_0\}} \chi\left(\frac{u_1 - u_0}{\alpha}\right) \omega_{u_0}^n \leq \chi(1) \end{aligned}$$

This implies that $I_\chi(u_0, \max(u_0, u_1)) \leq \alpha + \beta = I_\chi(u_0, u_1)$ and by symmetry we also have $I_\chi(u_1, \max(u_0, u_1)) \leq \alpha + \beta = I_\chi(u_0, u_1)$. Adding these two estimates yields the first inequality of the lemma. For the second inequality we first notice that

$$\int_X \chi\left(\frac{\max(u_1, u_0) - u_0}{\alpha}\right) \omega_{u_0}^n + \int_X \chi\left(\frac{\max(u_1, u_0) - u_1}{\alpha}\right) \omega_{\max(u_0, u_1)}^n = \chi(1)$$

Hence one of terms in the above sum is bigger than $\chi(1)/2$. We can assume that this is the first term, hence by convexity of χ we have

$$\int_X \chi\left(\frac{2(\max(u_1, u_0) - u_0)}{\alpha}\right) \omega_{u_0}^n \geq \int_X 2\chi\left(\frac{\max(u_1, u_0) - u_0}{\alpha}\right) \omega_{u_0}^n \geq \chi(1)$$

This implies that $I_\chi(\max(u_1, u_0), u_0) \geq \|\max(u_1, u_0) - u_0\|_{\chi, u_0} \geq \alpha/2$. One can establish a similar estimate for β and this finishes the proof. \square

We can now establish the main theorem of this section:

Theorem 5.5. *Given $\chi \in \mathcal{W}_p^+$, there exists $C(p) > 1$ such that*

$$\frac{1}{C} I_\chi(u_0 - u_1) \leq d_\chi(u_0, u_1) \leq C I_\chi(u_0 - u_1), \quad u_0, u_1 \in \mathcal{E}_{\bar{\chi}}(X, \omega).$$

Proof. The second estimate follows easily:

$$\begin{aligned} d_\chi(u_0, u_1) &\leq d_\chi(u_0, \max(u_0, u_1)) + d_\chi(\max(u_0, u_1), u_1) \\ &\leq I_\chi(u_0, \max(u_0, u_1)) + I_\chi(\max(u_0, u_1), u_1) \\ &\leq C I_\chi(u_0, u_1), \end{aligned}$$

where in the second line we have used Lemma 5.1 and in the third Lemma 5.4. Now we deal with the first estimate. By the previous result, (47) and Lemma 5.1 we can write

$$\begin{aligned} d_\chi(u_0, u_1) &\geq C d_\chi\left(u_0, \frac{u_0 + u_1}{2}\right) \geq C d_\chi\left(u_0, P\left(u_0, \frac{u_0 + u_1}{2}\right)\right) \\ &\geq C \left\| u_0 - P\left(u_0, \frac{u_0 + u_1}{2}\right) \right\|_{\chi, u_0}. \end{aligned}$$

By the same reasoning as above and the fact that $2^n \omega_{(u_0+u_1)/2}^n \geq \omega_{u_0}^n$ we can write:

$$\begin{aligned} d_\chi(u_0, u_1) &\geq C d_\chi\left(u_0, \frac{u_0 + u_1}{2}\right) \geq C d_\chi\left(\frac{u_0 + u_1}{2}, P\left(u_0, \frac{u_0 + u_1}{2}\right)\right) \\ &\geq C \left\| \frac{u_0 + u_1}{2} - P\left(u_0, \frac{u_0 + u_1}{2}\right) \right\|_{\chi, (u_0+u_1)/2} \\ &\geq C \left\| \frac{u_0 + u_1}{2} - P\left(u_0, \frac{u_0 + u_1}{2}\right) \right\|_{\chi, u_0}. \end{aligned}$$

Using the triangle inequality and the last two estimates, we obtain:

$$\begin{aligned} d_\chi(u_0, u_1) &\geq C \left[\left\| u_0 - P\left(u_0, \frac{u_0 + u_1}{2}\right) \right\|_{\chi, u_0} + \left\| \frac{u_0 + u_1}{2} - P\left(u_0, \frac{u_0 + u_1}{2}\right) \right\|_{\chi, u_0} \right] \\ &\geq C \|u_0 - u_1\|_{\chi, u_0}. \end{aligned}$$

By symmetry we also obtain $d_\chi(u_0, u_1) \geq C \|u_0 - u_1\|_{\chi, u_1}$ and adding these last two estimates together the result follows. \square

Remark 5.6. *From the previous two results it follows that there exists $C(p) > 1$ for which*

$$\begin{aligned} \frac{1}{C} (d_\chi(u_0, \max(u_0, u_1)) + d_\chi(u_1, \max(u_0, u_1))) &\leq d_\chi(u_0, u_1) \\ &\leq C (d_\chi(u_0, \max(u_0, u_1)) + d_\chi(u_1, \max(u_0, u_1))), \end{aligned}$$

for any $u_0, u_1 \in \mathcal{E}_{\bar{\chi}}(X, \omega)$. These estimates are the "max" analog of the ones in (47).

For $u_0, u_1 \in \mathcal{E}^1(X, \omega)$, as proposed in [BBGZ], one can introduce another notion of energy:

$$\mathcal{I}(u_0, u_1) = \int_X (u_0 - u_1)(\omega_{u_1}^n - \omega_{u_0}^n). \quad (54)$$

This functional will be useful for us in the proof of the next theorem.

Corollary 5.7. *Suppose $\chi \in \mathcal{W}_p^+$ and $d_\chi(u_k, u) \rightarrow 0$ with $u_k, u \in \mathcal{E}_{\bar{\chi}}(X, \omega)$. Then $u_k \rightarrow u$ in capacity in the sense of [K]. In particular, $\omega_{u_k}^n \rightarrow \omega_u^n$ weakly.*

Also, for any $v \in \mathcal{E}_1(X, \omega)$ and $R > 0$ there exists a continuous increasing function $f_R : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ with $f(0) = 0$ such that

$$\int_X |u_0 - u_1| \omega_v^n \leq f(d_1(u_0, u_1)), \quad (55)$$

for $u_0, u_1 \in \mathcal{E}_1(X, \omega)$ with $d_1(0, u_0), d_1(0, u_1) \leq R$.

Proof. As d_1 is dominated by all d_χ norms, we will only establish the result for d_1 . We start with (55). From [BBGZ, Lemma 5.8] and its proof it follows that for any $R > 0$ there exists an increasing continuous function $f_R : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ with $f(0) = 0$ such that

$$\int_X (u_0 - u_1)(\omega_{v_1}^n - \omega_{v_2}^n) \leq f(\mathcal{I}(u_0, u_1)) \quad (56)$$

for any u_0, u_1, v_1, v_2 having supremum less than R and Aubin-Mabuchi energy greater than $-R$. By and Corollary 4 and Lemma 4.15, $d_1(0, u_0)$ dominates both $\sup_X u_0$ and $AM(u_0)$. Also by Theorem 5.5, $d_1(u_0, u_1)$ dominates $\mathcal{I}(u_0, u_1)$ and $\int_X (u_0 - u_1)\omega_{u_0}^n$. Hence, choosing $v_1 = v$ and $v_2 = u_0$, we can rewrite (56) in the following way

$$\int_X (u_0 - u_1)\omega_v^n \leq \tilde{f}(d_1(u_0, u_1)), \quad (57)$$

for any $u_0, u_1 \in \mathcal{E}_1(X, \omega)$ satisfying $d_1(0, u_0), d_1(0, u_1) \leq R$. Using this, the identity

$$\int_X |u_0 - u_1|\omega_v^n = 2 \int_X (\max(u_0, u_1) - u_1)\omega_v^n - \int_X (u_0 - u_1)\omega_v^n,$$

and Remark 5.6, estimate (55) follows.

Now we prove that $d_\chi(u_k, u) \rightarrow 0$ implies $u_k \rightarrow u$ in capacity. This again follows from the results of [BBGZ]. Indeed, by (55), it follows that

$$\int_X u_k \omega^n \rightarrow \int_X u \omega^n.$$

Now using [BBGZ, Theorem 5.7] it follows that $u_k \rightarrow u$ in capacity. Finally, $u_k \rightarrow u$ in capacity (for $u_k, u \in \mathcal{E}(X, \omega)$) is known to imply the weak convergence of measures $\omega_{u_k}^n \rightarrow \omega_u^n$ (see [DP], compare [BBGZ, Proposition 5.6]). \square

Building on this last result, we prove another continuity theorem:

Theorem 5.8. *Suppose $\chi \in \mathcal{W}_p^+$ and $u_k, u, v \in \mathcal{E}_\chi(X, \omega)$ with $d_\chi(u_k, u) \rightarrow 0$. Then $u_k, u \in L^\chi(\omega_v^n)$ and*

$$\|u_k - u\|_{\chi, v} \rightarrow 0. \quad (58)$$

Proof. The fact that $u_k, u \in L^\chi(\omega_v^n)$ follows from [GZ1, Proposition 3.6]. By Proposition 2.2 to prove (58), it suffices to argue that

$$\int_X \chi(u_k - u)\omega_v^n \rightarrow 0.$$

Given an arbitrary subsequence of u_k , there exists a sub-subsequence, again denoted by u_k , satisfying the sparsity condition:

$$d_\chi(u_k, u_{k+1}) \leq \frac{1}{2^k}, \quad k \in \mathbb{N}. \quad (59)$$

Using the sparsity condition and Proposition 4.12 we can write

$$\begin{aligned} d_\chi(P(u, u_0, \dots, u_k), P(u, u_0, \dots, u_{k+1})) &= \\ &= d_\chi(P(P(u, u_0, \dots, u_k), u_k), P(P(u, u_0, \dots, u_k), u_{k+1})) \\ &\leq d_\chi(u_k, u_{k+1}) \leq \frac{1}{2^k}. \end{aligned}$$

Hence, the decreasing sequence $h_k = P(u, u_0, u_1, \dots, u_k)$, $k \geq 1$ is bounded and Lemma 4.16 implies now that the decreasing limit $\lim_k h_k = h = P(u, u_0, u_1, u_2, \dots) \in \mathcal{E}_\chi(X, \omega)$.

By Corollary 4, there exists $M > 0$ such that $h \leq u_k, u \leq M$. Putting everything together we have

$$h - M \leq u_k - u \leq M - h.$$

This implies that $\int_X \chi(u_k - u) \omega_v^n \leq \int_X \chi(M - h) \omega_v^n < \infty$. As $d_\chi(u_k, u) \rightarrow 0$ implies $u_k \rightarrow u$ in capacity, which in turn implies pointwise convergence a.e., by the dominated convergence theorem it follows that $\int_X \chi(u_k - u) \omega_v^n \rightarrow 0$, finishing the proof. \square

As promised we argue that the strong convergence introduced in [BBEGZ, Section 2.1] is equivalent to χ_1 -convergence. This results is well known to experts, however there does not seem to exist an adequate reference for it in the literature:

Proposition 5.9. *Suppose $u_k, u \in \mathcal{E}^1(X, \omega)$. Then $\int_X |u_k - u| \omega^n \rightarrow 0$ and $AM(u_k) \rightarrow AM(u)$ if and only if $I_1(u_k, u) \rightarrow 0$.*

Proof. As follows from Lemma 4.15 and Corollary 5.7, $I_1(u_k, u) \rightarrow 0$ implies $\int_X |u_k - u| \omega^n \rightarrow 0$ and $AM(u_k) \rightarrow AM(u)$. Now we argue the other direction. By [BBEGZ, Propostion 2.3] it follows that $\mathcal{I}(u_k, u) \rightarrow 0$. As in the proof of Corollary 5.7, we can conclude that $\int_X (u_k - u) \omega_{u_k}^n \rightarrow 0$ and $\int_X (u_k - u) \omega^n \rightarrow 0$. Using the locality of the complex Monge-Ampère measure in the plurifine topology we observe

$$\mathcal{I}(\max(u_k, u), u) = \int_{\{u_k > u\}} (u_k - u) (\omega_u^n - \omega_{u_k}^n) \leq \mathcal{I}(u_k, u),$$

hence we also have $\mathcal{I}(\max(u_k, u), u) \rightarrow 0$. By similar considerations as above we get $\int_X (\max(u_k, u) - u) \omega_{u_k}^n \rightarrow 0$ and $\int_X (\max(u_k, u) - u) \omega^n \rightarrow 0$. This concludes the argument as we note that $|u_k - u| = 2(\max(u_k, u) - u) - (u_k - u)$. \square

6 Applications to Kähler-Einstein metrics

For this section we assume that (X, J, ω) is Fano, i.e. $c_1(X) = [\omega]$. We review a few facts about convergence of the Kähler-Ricci flow. Under the normalization $\int_X e^{-\dot{r}_t} \omega^n = \text{Vol}(X)$, equation (8) can be rewritten as the scalar equation

$$e^{\dot{r}_t - r_t - h - \log(\int_X e^{-r_t - h} \omega^n)} \omega^n = \omega_{r_t}^n.$$

We refer to [BEG, Section 6.2] for details. As it follows from a theorem of Perelman and work of Chen-Tian, Tian-Zhu and Phong-Song-Sturm-Weinkove, ω_{r_t} converges exponentially fast to some Kähler-Einstein metric $\omega_{u_{KE}}$, whenever such metric exists (see [CT], [TZ], [PSSW]). More precisely we have $\|d\omega_{r_t}/dt\|_\omega = \|i\partial\bar{\partial}\dot{r}_t\|_\omega \leq Ce^{-Ct}$ for some $C > 0$. Given our normalization assumption, this implies that $\|\dot{r}_t\|_{L^\infty(X)} \leq Ce^{-Ct}$, in particular $\|r_t - u_{KE}\|_{L^\infty(X)} \rightarrow 0$. This will be the starting point in establishing the equivalence between (i) and (ii) in Theorem 6:

Theorem 6.1. *There exists a Kähler-Einstein metric cohomologous to ω if and only if (X, J, ω) is χ -stable for any $\chi \in \mathcal{W}_p^+$, $p \geq 1$.*

Proof. Suppose that there exists Kähler-Einstein metric cohomologous to ω and $t \rightarrow \tilde{r}_t$ is a Ricci trajectory normalized by $\int_X e^{-d\tilde{r}_t/dt} \omega^n = \text{Vol}(X)$ with initial metric $v \in \mathcal{H}_{AM}$. By what we recalled above, \tilde{r}_t converges to some Kähler-Einstein potential $\tilde{u}_{KE} \in \mathcal{H}$ uniformly.

As $AM(\cdot)$ is continuous with respect to uniform convergence, it follows that the "AM-normalized" Kähler-Ricci trajectory $t \rightarrow r_t = \tilde{r}_t - AM(\tilde{r}_t) \in \mathcal{H}_{AM}$ also converges to $u_{KE} = \tilde{u}_{KE} - AM(\tilde{u}_{KE})$ uniformly, hence also with respect to d_X , finishing one direction of the proof.

For the other direction, let us assume that there exists no Kähler-Einstein metric cohomologous to ω . By [CR, Lemma 6.5], along any Kähler-Ricci trajectory $[0, \infty) \ni t \rightarrow r_t \in \mathcal{H}_{AM}$, one can find a sequence $t_j \rightarrow \infty$ and an non-empty subvariety $S \subset X$ of positive dimension such that

$$\lim_{j \rightarrow \infty} \int_K \omega_{r_{t_j}}^n = 0,$$

for any $K \subset X \setminus S$ compact. This implies that there is no $u \in \mathcal{E}_{\tilde{\chi}}(X, \omega)$ satisfying $d_X(r_{t_j}, u) \rightarrow 0$, as by Corollary 5.7 this would imply that $\omega_{r_{t_j}}^n \rightarrow \omega_u^n$ weakly. By the above, this in turn implies that ω_u^n is concentrated on S , in particular $\int_S \omega_u^n = \text{Vol}(X) > 0$. But this is impossible as ω_u^n does not charge pluripolar sets, as follows from the definition of ω_u^n (see [GZ1, Theorem 1.3]). \square

Now we prove the equivalence between (i) and (iii) in Theorem 6:

Theorem 6.2. *Suppose that (X, J) does not have non-trivial holomorphic vector fields. Then there exists a Kähler-Einstein metric cohomologous to ω if and only if \mathcal{F} is d_1 -proper on $\mathcal{E}_{AM}^1(X, \omega)$.*

Proof. By [T1, Theorem 1.6] it is enough to prove that the concept of \mathcal{J} -properness is equivalent to d_1 -properness. Suppose $u \in \mathcal{E}_{AM}^1(X, \omega)$. By Theorem 5.5 there exists $C > 1$ such that

$$\int_X u \omega^n \leq C d_1(u, 0). \quad (60)$$

On the other hand, by (46) and the fact that $AM(u) = AM(0) = 0$ we can write:

$$d_1(u, 0) = AM(u) + AM(0) - 2AM(P(0, u)) = 2(AM(u) - AM(P(0, u))).$$

We have $\sup_X (u - P(0, u)) \leq \sup_X u$, hence (49) implies that $AM(u) - AM(P(0, u)) \leq \sup_X u$. It is well know that there exists $C > 0$ depending only on X and ω such that $\sup_X u - C \leq \int_X u \omega^n \leq \sup_X u$ for all $u \in \text{PSH}(X, \omega)$. Putting the above together we arrive at:

$$d_1(u, 0) \leq 2 \int_X u \omega^n + 2C. \quad (61)$$

By (60) and (61) we have

$$\mathcal{J}(u)/C \leq d_1(u, 0) \leq 2\mathcal{J}(u) + 2C, \quad (62)$$

hence \mathcal{J} -properness is equivalent to d_1 -properness. \square

From (62) the following estimate of independent interest follows:

Remark 6.3. *There exists $C > 1$ such that*

$$\frac{1}{C} \sup_X u - C \leq d_1(u, 0) \leq C \sup_X u + C, \quad u \in \mathcal{E}_{AM}^1(X, \omega).$$

As a consequence of this, there exists $C > 1$ such that $\mathcal{F}(u) \leq Cd_1(0, u) + C$ for any $u \in \mathcal{E}_{AM}^1(X, \omega)$, regardless whether (X, J, ω) admits a Kähler-Einstein metric or not.

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