Space of Kähler Metrics (V)
– Kähler Quantization

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Dedicated to Jeff Cheeger for his 65th birthday

Abstract. Given a polarized Kähler manifold (X, L). The space $\mathcal{H}$ of Kähler metrics in $2\pi c_1(L)$ is an infinite-dimensional Riemannian symmetric space. As a metric space, it has non-positive curvature. There is associated to $\mathcal{H}$ a sequence of finite-dimensional symmetric spaces $\mathcal{B}_k (k \in \mathbb{N})$ of non-compact type. We prove that $\mathcal{H}$ is the limit of $\mathcal{B}_k$ as metric spaces in certain sense. As applications, this provides more geometric proofs of certain known geometric properties of the space $\mathcal{H}$.

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1. Introduction

Let (X, ω, J) be an n-dimensional compact Kähler manifold. By [9] this gives rise to two infinite-dimensional symmetric spaces: the Hamiltonian diffeomorphism group $\mathrm{Ham}(X, \omega)$ and the space $\mathcal{H}$ of smooth Kähler potentials under the natural (Weil-Petersson type) L^2 metric. The relation between these two spaces is analogous to that between a finite-dimensional compact group G and its noncompact dual $G^{\mathbb{C}}/G$, where $G^{\mathbb{C}}$ is a complexification of G. As in the finite-dimensional case, at least formally, $\mathrm{Ham}(X, \omega)$ has non-negative sectional curvature, while $\mathcal{H}$ has non-positive sectional curvature. It is proved by E. Calabi and the first author in [2] that $\mathcal{H}$ is non-positively curved in the sense of Alexandrov. On the other hand, it is well known in the literature of geometric quantization that $\mathcal{H}$ is the limit of a sequence of finite-dimensional symmetric spaces $\mathcal{B}_k (k \in \mathbb{N})$ when X is (Kähler) polarized. Indeed, the polarization is given by an ample line bundle L over X, and we consider the space $H^0(X, L^k)$ of holomorphic sections of L^k for large enough k.

Here k^{-1} plays the role of Planck constant $\hbar$, while $k \to \infty$ should correspond to the process of taking the classical limit. Denote by N_k the dimension of $H^0(X, L^k)$. By the Riemann-Roch theorem this is a polynomial in k of degree n for sufficiently large k. Then $\mathcal{B}_k$ is by definition the space of positive definite Hermitian forms on $H^0(X, L^k)$. It could also be viewed as the symmetric space $GL(N_k; \mathbb{C})/U(N_k)$. The spaces $\mathcal{B}_k$ are related to $\mathcal{H}$ through two naturally defined maps: $\mathrm{Hilb}_k : \mathcal{H} \to \mathcal{B}_k$ and $FS_k : \mathcal{B}_k \to \mathcal{H}$. (The precise definition of these maps will be given in the next section.) In [24], G. Tian proved that given any $\phi \in \mathcal{H}$, $FS_k \circ \mathrm{Hilb}_k(\phi) \to \phi$ in C^4 topology. Using Tian's peak section method and canonical coordinates, W.-D. Ruan proved the C^∞ convergence in [20]. S. Zelditch ([26]) beautifully generalized Tian's theorem and derived C^∞ convergence from the asymptotic expansion of the Bergman kernel. If we denote by $\mathcal{H}_k$ the image of $FS_k \circ \mathrm{Hilb}_k$, then these results tell us that

$$\mathcal{H} = \overline{\bigcup_{k \in \mathbb{N}} \mathcal{H}_k},$$

where the closure is taken in C^∞ topology.

In [10], S.K. Donaldson suggests that the geometry of $\mathcal{B}_k$ should also converge to that of $\mathcal{H}$. In particular, the geodesics in $\mathcal{H}$ should be approximated by geodesics in $\mathcal{H}_k$. In [3], the first author proved the existence of $C^{1,1}$ geodesics and it subsequently lead to many interesting applications in Kähler geometry (see [7], [4] for further references). The limitation of $C^{1,1}$ regularities is purely technical. In a very interesting paper([18]), Phong-Sturm proved that the $C^{1,1}$ geodesics in $\mathcal{H}$ are the weak C^0 limits of Bergman geodesics, assuming the existence of $C^{1,1}$ geodesics. It would provide a canonical smooth approximation of $C^{1,1}$ geodesics if one could show the convergence is in $C^{1,1}$ topology. More evidence comes from a series of beautiful works by S. Zeltdich and his collaborators, see Song-Zelditch ([23]), Rubinstein-Zelditch ([22]). They proved that on toric varieties both geodesics and harmonic maps in $\mathcal{H}$ (automatically smooth) are C^2 limits of corresponding objects in $\mathcal{H}_k$. Recently, J. Fine ([14]) proved a remarkable result that the Calabi flow in $\mathcal{H}$ could be approximated by balancing flows in $\mathcal{B}_k$. In this paper, we shall prove the following convergence of geodesic distance:

Theorem 1.1. *Given any $\phi_0, \phi_1 \in H$, we have*

$$\lim_{k \to \infty} k^{-\frac{n+2}{2}} d_{\mathcal{B}_k}(\mathrm{Hilb}_k(\phi_0), \mathrm{Hilb}_k(\phi_1)) = d_{\mathcal{H}}(\phi_0, \phi_1).$$

Remark 1.2. It is easy to identify the scaling factor by simply taking $\phi_1 = \phi_0 + c$, then $d(\phi_0, \phi_1) = c$, while $d(\mathrm{Hilb}_k(\phi_0), \mathrm{Hilb}_k(\phi_1)) = c \cdot k\sqrt{N_k}$. This scaling also indicates that the limit can not have bounded curvature, which can also be speculated from the expression of the infinitesimal curvature of $\mathcal{H}$:

$$R_\phi(\phi_1, \phi_2) = -\frac{1}{4} \frac{\|\{\phi_1, \phi_2\}_\phi\|_{L^2}^2}{\|\phi_1\|_{L^2}^2 \|\phi_2\|_{L^2}^2}.$$

Remark 1.3. This theorem indicates that the L^2 metric defined in [9] is in a sense canonical. We notice that there are also natural affine structures on both $\mathcal{H}$ and $\mathcal{B}_k$,

given by embedding them as convex subsets of the affine spaces $C^\infty(X)$ and the space of all Hermitian forms on $H^0(X, L^k)$ respectively. It would be interesting to see whether the affine structures on $\mathcal{H}$ and $\mathcal{B}_k$ have nice relations with each other as the above metric geometric structures do.

From the proof it follows that the convergence is uniform if both potentials vary in a C^l compact neighborhood for large l. So an easy corollary is the convergence of angles:

Corollary 1.4. *Given three points ϕ_i $(i = 1, 2, 3)$ in $\mathcal{H}$, let $H_{k,i} = \mathrm{Hilb}_k(\phi_1)$. Then*

$$\lim_{k \to \infty} \angle H_{k,1} H_{k,2} H_{k,3} = \angle \phi_1 \phi_2 \phi_3.$$

This corollary leads to that $\mathcal{H}$ is non-positively curved in the sense of Alexandrov, as was originally proved in [2].

Theorem 1.1 together with Theorem 5 in [14] imply the following corollary, because the downward gradient flow of a geodesically convex function on a finite-dimensional manifold is distance decreasing:

Corollary 1.5 (Calabi-Chen [2], [4]). *Calabi flow in $\mathcal{H}$ decreases geodesic distance.*

The "quantization" approach, namely using the approximation by $\mathcal{B}_k$ to handle problems about the Kähler geometry of (X, ω, J), turns out to be quite powerful and intriguing. It was shown in [9] that the existence and uniqueness problems in Kähler geometry are related to the geometry of $\mathcal{H}$. More precisely, the uniqueness of extremal metrics is implied by the existence of smooth geodesics connecting any two points in $\mathcal{H}$, and the non-existence is conjectured to be equivalent to the existence of a geodesic ray in $\mathcal{H}$ where the K-energy is strictly decreasing near infinity. The technical problem comes from the lack of regularity of geodesics in $\mathcal{H}$. There are two ways at present to circumvent this problem. The first way is to study the geodesic equation directly. One can use the continuity method, as in [3]. In [3], the first author constructed a continuous family of ϵ-approximate geodesics converging in weak $C^{1,1}$ topology to a $C^{1,1}$ geodesic. This gives the proof of the uniqueness of extremal metrics when $c_1(X) \leq 0$ and the other conjecture of Donaldson that $\mathcal{H}$ is a metric space. In [7] a new partial regularity was derived from studying the complex Monge-Ampère equation associated to the geodesic equation. Solutions with such regularity already have fruitful applications. The second way is to exploit the approximation of $\mathcal{H}$ by $\mathcal{B}_k$. This requires a polarization. The essential thing is to quantize the whole problem. Many problems have been settled using either of the two approaches: the uniqueness of cscK metrics ([7], [9]), the lower bound of Mabuchi energy assuming the existence of cscK metrics([7], [12]), the lower bound of Calabi energy ([5], [13]), etc.

Due to [12], the Mabuchi functional (K-energy) E has a nice quantization defined on $\mathcal{B}_k$, which we denote by Z_k. More precisely, up to a constant, given any $\phi \in \mathcal{H}$, we have $Z_k(\mathrm{Hilb}_k(\phi)) \to E(\phi)$. It is well known that E is convex along smooth geodesics in $\mathcal{H}$, but it is not known that E is convex along $C^{1,1}$ geodesics,

although it is well defined (cf. [6]). One good thing is that Z_k is convex on a finite-dimensional space $\mathcal{B}_k$, which is geodesically complete. This makes it possible to derive some weak convexity of E, such as an alternative proof of two inequalities of the first author. Namely

Corollary 1.6 ([4]). *For any $\phi_0, \phi_1 \in \mathcal{H}$, let $\phi(t)(t \in [0,1])$ be the $C^{1,1}$ geodesic connecting them. Then*
$$dE_{\phi_0}(\dot{\phi}(0)) \leq dE_{\phi_1}(\dot{\phi}(1)).$$

Corollary 1.7 ([4]). *For any $\phi_0, \phi_1 \in \mathcal{H}$, we have*
$$E(\phi_1) - E(\phi_0) \leq d(\phi_0, \phi_1) \cdot \sqrt{Ca(\phi_1)}.$$

Both corollaries are useful in the study of Kähler geometry by the space of Kähler metrics. By Theorem 3.1 in [2], Corollary 1.6 immediately gives Corollary 1.5. Using these corollaries, the first author proved the sharp lower bound of the Calabi energy in any given Kähler class. The original proof of these two corollaries depends heavily on the delicate regularity results of Chen-Tian [7]. Our proof in the algebraic case is more geometric and substantially simpler.

Organization

In Section 2 we briefly recall the geometry of the space of Kähler metrics and the associated finite-dimensional spaces $\mathcal{B}_k$. In Section 3 we prove the convergence of geodesic distance, using the existence of $C^{1,1}$ geodesics in $\mathcal{H}$, the non-positivity of curvature of $\mathcal{B}_k$, and the convergence of infinitesimal geometry. In Section 4 the weak convexity of K-energy is discussed and we prove Corollary 1.6 and Corollary 1.7. The idea is that the K-energy could be approximated by convex functions Z_k on $\mathcal{B}_k$, while the norm of the gradient of Z_k approximates the norm of gradient of K-energy, which is the Calabi functional. So Corollary 1.7 follows. To prove Corollary 1.6, we need to estimate the difference between the initial direction of an "almost" geodesic γ (i.e., $|\ddot{\gamma}|$ is small) in $\mathcal{B}_k$ and that of the genuine geodesic connecting the two end points $\gamma(0)$ and $\gamma(1)$. This is done in Lemma 4.5. In the last section, we discuss some further problems.

2. Preliminaries

Let (X, ω, J) be a Kähler manifold. Here we always assume it is polarized, i.e., there is a holomorphic line bundle L whose first Chern class is $[\omega]/2\pi$. Fix the base metric ω and define the space of Kähler potentials as
$$\mathcal{H} = \{\phi \in C^\infty(M) | \omega + \sqrt{-1}\partial\bar{\partial}\phi > 0\}.$$

$\mathcal{H}$ can also be identified with the space of Hermitian metrics on L with positive curvature form. In the following we will not distinguish between these two meanings. We write ω_h to denote $\sqrt{-1}$ times the curvature of h for a Hermitian metric h, and $\omega = \omega_{h_0}$. Thus,
$$\omega_\phi := \omega_{e^{-\phi}h_0} = \omega + \sqrt{-1}\partial\bar{\partial}\phi.$$

Denote
$$d\mu_h = d\mu_\phi = \frac{\omega_h^n}{(2\pi)^n n!}.$$

There is a Weil-Petersson type metric defined on $\mathcal{H}$ by
$$(\delta_1\phi, \delta_2\phi)_\phi = \int_X \delta_1\phi\delta_2\phi d\mu_\phi,$$

for any $\delta_1\phi, \delta_2\phi \in T_\phi\mathcal{H} = C^\infty(X)$. Due to [9], $\mathcal{H}$ is formally an infinite-dimensional symmetric space, and the sectional curvature of the Levi-Cività connection is given by:
$$R_\phi(\delta_1\phi, \delta_2\phi) = -\frac{1}{4}\|\{\delta_1\phi, \delta_2\phi\}_\phi\|^2.$$

It was confirmed in [2] that $\mathcal{H}$ satisfies the triangle comparison theorem of a non-positive Alexandrov space, by using so-called ϵ-approximate geodesics. The geodesic equation in $\mathcal{H}$ is
$$\ddot{\phi} - |\nabla_\phi\dot{\phi}|_\phi^2 = 0, \tag{1}$$

where we have used the notation of complex gradient. The following theorem is proved in [3] by the first author:

Lemma 2.1 (Geodesic Approximation Lemma). *Given any $\phi_0, \phi_1 \in \mathcal{H}$, there is a positive number ϵ_0, and a one-parameter smooth family of smooth curves $\phi_\epsilon(\cdot)$: $[0,1] \to \mathcal{H}$ ($\epsilon \in (0, \epsilon_0]$), such that the following holds:*

(1) *For any $\epsilon \in (0, \epsilon_0]$, ϕ_ϵ is an ϵ-approximate geodesic, i.e., it solves the following equation:*
$$(\ddot{\phi}_\epsilon - |\nabla\dot{\phi}_\epsilon|^2)\omega_{\phi_\epsilon}^n = \epsilon \cdot \omega^n,$$
and $\phi_\epsilon(0) = \phi_0$, $\phi_\epsilon(1) = \phi_1$.

(2) *There exists a $C > 0$, such that for all $\epsilon \in (0, 1]$ and $t \in [0, 1]$, we have*
$$|\phi_\epsilon(t)| + |\dot{\phi}_\epsilon(t)| \leq C,$$
and
$$|\ddot{\phi}_\epsilon(t)| \leq C.$$

(3) *$\phi_\epsilon(\cdot)$ converges to the unique $C^{1,1}$ geodesic connecting ϕ_0 and ϕ_1 in the weak $C^{1,1}$ topology when $\epsilon \to 0$.*

(4) *We have uniform estimates when ϕ_0 and ϕ_1 varies in a C^k compact set for some large k.*

The space $\mathcal{H}$ has a global flat direction given by addition of a constant. The de Rham decomposition theorem turns out to be true in this case. There is a functional I giving the isometric decomposition
$$\mathcal{H} = \mathcal{H}_0 \oplus \mathbb{R}.$$

I is only defined up to an addition of a constant, and its derivative is given by:
$$\delta I = \int_X \delta\phi d\mu_\phi.$$

So I is linear along $C^{1,1}$ geodesics in $\mathcal{H}$. $\mathcal{H}_0$ is then an arbitrary level set of I, and it can also be regarded as the space of Kähler metrics cohomologous to ω. An interesting thing to notice is that $\mathcal{H}$ is embedded as a convex subset into an affine linear space $C^\infty(M)$. It is easy to show that I is indeed convex along any linear path in $\mathcal{H}$.

There is a well-known K-energy functional E on $\mathcal{H}$, defined up to an additive constant, with its variation given by:

$$\delta E = -\int_X (S - \underline{S})\delta\phi d\mu_\phi. \tag{2}$$

Here $\delta\phi$ is the infinitesimal variation of ϕ, and S is the scalar curvature of ω_ϕ. From the point of view of [8], this is a natural convex functional associated to a moment map of a Hamiltonian action (for infinite-dimensional manifold). Indeed, by a straightforward calculation, if $\phi(t)$ is a smooth geodesic in $\mathcal{H}$, then

$$\frac{d^2}{dt^2}E(\phi(t)) = \int_X |\mathcal{D}\dot\phi|^2 d\mu_\phi \geq 0,$$

where $\mathcal{D}$ is the Lichnerowicz-Laplacian operator. By studying the explicit expression of E, it is shown in [6] that E is well defined for $C^{1,1}$ Kähler potentials, but it is not obvious from the definition that E is convex along $C^{1,1}$ geodesics.

There is a sequence of finite-dimensional symmetric spaces $\mathcal{B}_k$ consisting of all positive definite Hermitian forms on $H^0(X, L^k)$. This can be identified with $GL(N_k; \mathbb{C})/U(N_k)$ by choosing a base point in $\mathcal{B}_k$, where $N_k = \dim H^0(X, L^k)$. We shall review some basic facts about these symmetric spaces. $U(N)$ is a compact Lie group and admits a natural bi-invariant Riemannian metric given by

$$(A_1, A_2)_A = -Tr(A_1 A^{-1} \cdot A_2 A^{-1}),$$

for any $A_1, A_2 \in T_A U(N)$. The sectional curvature is given by:

$$R(A_1, A_2) = \frac{1}{4}\|[A_1, A_2]_A\|^2.$$

The geodesic equation is

$$\ddot{A} = \dot{A}A^{-1}\dot{A}.$$

It is then clear that the geodesics all come from one parameter subgroups given by the usual exponential map. Now the non-compact dual of $U(N)$ is $GL(N; \mathbb{C})/U(N)$ – the space of positive definite $N \times N$ Hermitian matrices. We can explicitly write down the metric on it. For any $H_1, H_2 \in T_H(GL(N; \mathbb{C})/U(N))$, define

$$(H_1, H_2)_H = Tr(H_1 H^{-1} \cdot H_2 H^{-1}).$$

Then the sectional curvature becomes non-positive:

$$R(H_1, H_2) = -\frac{1}{4}\|[H_1, H_2]_H\|^2.$$

A path $H(t)$ is a geodesic if it satisfies the following equation:

$$\ddot{H} = \dot{H}H^{-1}\dot{H}. \tag{3}$$

This looks very similar to equation (1), except that it is a nonlinear ODE instead of a nonlinear PDE. In our setting, we get a sequence of equations (3) depending on k. For this reason, we may say that this sequence of equations "quantizing" equation (1). It is easy to see that all the geodesics in this space are of the form

$$H(t) = H(0)^{\frac{1}{2}} \exp(tA) H(0)^{\frac{1}{2}},$$

for some initial point $H(0)$ and initial tangent vector A.

Similar to the infinite-dimensional case, $\mathcal{B}_k$ also admits an isometric splitting given by the function $I_k = \log \det$, defined up to an additive constant. It is easy to see that I_k is linear along geodesics in $\mathcal{B}_k$ and convex along a linear path when we regard $\mathcal{B}_k$ as a convex subset of the affine linear space of all $N_k \times N_k$ complex matrices. The splitting corresponds to the following isometry:

$$GL(N; \mathbb{C})/U(N) \simeq SL(N; \mathbb{C})/SU(N) \times \mathbb{R}.$$

We will see in Section 4 that I_k indeed "quantizes" the I functional.

To "quantize" functionals on $\mathcal{H}$, there are two natural maps relating $\mathcal{H}$ and $\mathcal{B}_k$, where we have adopted the notation of [12]:

$$\text{Hilb}_k : \mathcal{H} \to \mathcal{B}_k;$$
$$FS_k : \mathcal{B}_k \to \mathcal{H}.$$

Explicitly, given $h \in \mathcal{H}$, and $s \in H^0(X, L^k)$, we define

$$\|s\|^2_{\text{Hilb}_k(h)} = \int_X |s|^2_h d\mu_h.$$

For $H \in \mathcal{B}_k$, pick an orthonormal basis $\{s_\alpha\}$ of $H^0(X, L^k)$ with respect to H. Define

$$FS_k(H) = \frac{1}{k} \log \sum_\alpha |s_\alpha|^2_{h_0^k},$$

where h_0 is the base metric in $\mathcal{H}$. In particular, we have

$$\sum_\alpha |s_\alpha|^2_{e^{-k \cdot FS_k(H)} h_0} \equiv 1.$$

Notice that $\omega_{FS_k(H)}$ coincides with the pull-back of the Fubini-Study metric on $\mathbb{CP}^{N_k-1}$ under the projective embedding induced by $\{s_\alpha\}$, while $FS_k(H)$ coincides with the induced metric on the pull-back of $\mathcal{O}(1)$. Let $\mathcal{H}_k$ be the image of FS_k, then the well-known Tian-Yau-Zelditch theorem says that

$$\overline{\bigcup_k \mathcal{H}_k} = \mathcal{H}.$$

To be more precise, for any $\phi \in \mathcal{H}$, and $k > 0$, we choose an orthonormal basis $\{s_\alpha\}$ of $H^0(X, L^k)$ with respect to $\text{Hilb}_k(\phi)$, and define the density of state function:

$$\rho_k(\phi) = \sum_\alpha |s_\alpha|^2_{h^k}.$$

Then we have the following C^∞ expansion:

Lemma 2.2 ([26], [15]).

$$\rho_k(\phi) = k^n + A_1(\phi)k^{n-1} + A_2(\phi)k^{n-2} + \cdots , \tag{4}$$

with $A_1(\phi) = \frac{1}{2}S(\phi)$, where S denotes the scalar curvature. Moreover, the expansion is uniform in that for any l and $R \in \mathbb{N}$,

$$\left\| \rho_k(\phi) - \sum_{j \leq R} A_j k^{n-j} \right\|_{C^l} \leq C_{R,l} k^{n-R},$$

where $C_{R,l}$ only depends on R and l.

Remark 2.3. Later we will need a generalization of the previous expansion theorem. We want to differentiate the expansion of the density of state function several times along a smooth path $\phi(t)$ in $\mathcal{H}$, and still to get a uniform expansion. This could be done following the arguments of Z. Lu and G. Tian, see [16].

Now if we let $\phi_k = FS_k \circ \mathrm{Hilb}_k(\phi)$, then

$$\phi_k - \phi = \frac{1}{k} \log \rho_k(\phi) \to 0,$$

as $k \to \infty$.

To quantize the K-energy, following [12], we define

$$\mathcal{L}_k = I_k \circ \mathrm{Hilb}_k + \frac{kd_k}{V}I,$$

and

$$Z_k = \frac{kd_k}{V}I \circ FS_k + I_k + d_k(\log V - \log d_k),$$

where $d_k = \dim H^0(X, L^k)$, and $V = \mathrm{Vol}_{[\omega]}(X)$. Notice that the definition of both functionals requires choices of base points in both $\mathcal{H}$ and $\mathcal{B}_k$. From now on, we always fix the same base points for defining these two functionals. By a straightforward calculation,

$$\delta\mathcal{L}_k = \int [\Delta\rho_k - k\rho_k]_\phi \delta\phi d\mu_\phi,$$

where we define for any function f, $[f]_\phi := f - \frac{1}{V}\int f d\mu_\phi = f - \underline{f}$. From (4), we see that

$$[\Delta\rho_k - k\rho_k]_\phi \to -\frac{1}{2}k^n(S - \underline{S}).$$

Thus, there are constants c_k, such that

$$\frac{2}{k^n}\mathcal{L}_k + c_k \to E,$$

where the convergence is uniform on $C^l(l \gg 1)$ bounded subsets in $\mathcal{H}$. From now on, we will denote the left-hand side by $\mathcal{L}_k$, and $Z_k + c_k$ by Z_k.

In [12] the following relation between $\mathcal{L}_k$ and Z_k was shown:

$$\mathcal{L}_k(FS_k(H)) - Z_k(H) = \log \det(\mathrm{Hilb}_k \circ FS_k(H)) - \log \det H - d_k \log \frac{V}{d_k} \leq 0;$$

and

$$Z_k(\mathrm{Hilb}_k(\phi)) - \mathcal{L}_k(\phi) = \frac{kd_k}{V}(I(FS_k \circ \mathrm{Hilb}_k(\phi)) - I(\phi)) + d_k \log \frac{V}{d_k} \leq 0.$$

Thus, given any $\phi \in \mathcal{H}$, we have

$$\mathcal{L}_k(\phi_k) = \mathcal{L}_k(FS_k \circ \mathrm{Hilb}_k(\phi)) \leq Z_k(\mathrm{Hilb}_k(\phi)) \leq \mathcal{L}_k(\phi).$$

Since ϕ_k converges to ϕ smoothly, and $\mathcal{L}_k$ converges to E uniformly on $C^l(l \gg 1)$ bounded subsets in $\mathcal{H}$, we immediately obtain:

Lemma 2.4. Z_k *quantizes* E *in the sense that given any* $\phi \in \mathcal{H}$, *we have as* $k \to \infty$,

$$Z_k(\mathrm{Hilb}_k(\phi)) \to E,$$

and the convergence is uniform in $C^l(l \gg 1)$ *bounded subsets of* $\mathcal{H}$.

In Section 3 we will investigate more about the relation between Z_k and E.

The following is proved essentially in [12]:

Lemma 2.5 ([12]). $\mathcal{L}_k$ *is convex on* $\mathcal{H}$, *and* Z_k *is convex on* $\mathcal{B}_k$.

3. Convergence of geodesic distance

In this section we shall prove Theorem 1.1. The following lemma about the derivative of Hilb_k and FS_k follows from a simple calculation:

Lemma 3.1. *For* $\phi \in \mathcal{H}$, *and* $\delta\phi \in T_\phi\mathcal{H}$, *we have for any* $s_1, s_2 \in H^0(X, L^k)$, *that*

$$d_\phi \ \mathrm{Hilb}_k(\delta\phi)(s_1, s_2) = \int (s_1, s_2)_\phi(-k\delta\phi + \Delta\delta\phi)d\mu_\phi. \tag{5}$$

For $H \in \mathcal{B}_k$, *and* $\delta H \in T_H\mathcal{B}_k$, *we have*

$$d_H \ FS_k(\delta H) = -\frac{1}{k} \sum_{i,j} \delta H(s_i, s_j) \cdot (s_j, s_i)_{FS_k(H)}, \tag{6}$$

where $\{s_i\}$ *is an orthonormal basis of* H.

The following theorem proves the convergence of infinitesimal geometry.

Theorem 3.2. *Given a smooth path* $\phi(t) \in \mathcal{H}(t \in [0,1])$, *after normalization, the length of the induced path* $H_k(t) = \mathrm{Hilb}_k(\phi(t))$ *converges to that of* $\phi(t)$ *as* $k \to \infty$, *where the corresponding metrics on* $\mathcal{H}$ *and* $\mathcal{B}_k$ *are used. More precisely,*

$$\lim_{k \to \infty} k^{-n-2}\|\dot{H}_k\|^2 = \|\dot{\phi}\|^2 = \int_X \dot{\phi}^2 d\mu_\phi,$$

for any $t \in [0,1]$.

Proof. W.L.O.G, assume $t = 0$. Let $\{s_i(t)\}$ be an orthonormal basis of $H_k(t) = \text{Hilb}_k(\phi(t))$. Then by Lemma 3.1,

$$\dot{H}_{ij} = \dot{H}(s_i(t), s_j(t)) = \int_X (s_i, s_j)(-k\dot{\phi} + \Delta\dot{\phi})d\mu_\phi,$$

where for convenience we drop the subscript k of H_k. Diagonalize it at $t = 0$, we get

$$\dot{H}_{ij} = \delta_{ij} \cdot \int_X |s_i|_0^2 (-k\dot{\phi} + \Delta\dot{\phi})d\mu_\phi.$$

Now let $\phi_k(t) = FS_k(H_k(t))$, then

$$\dot{\phi}_k = -\frac{1}{k}\sum_i \dot{H}_{ii}|s_i|_{\phi_k}^2.$$

Therefore,

$$\|\dot{H}\|^2 = \sum_i |\dot{H}_{ii}H_{ii}^{-1}|^2$$

$$= \int_X \sum_i \dot{H}_{ii}|s_i|^2(-k\dot{\phi} + \Delta\dot{\phi})d\mu_\phi$$

$$= -k\int_X \dot{\phi}_k(-k\dot{\phi} + \Delta\dot{\phi})\rho_k(\phi)d\mu_\phi$$

By Lemma 2.2, we know

$$\rho_k(\phi) = k^n + \frac{1}{2}S \cdot k^{n-1} + O(k^{n-2}),$$

and

$$\dot{\phi}_k - \dot{\phi} = \frac{1}{k}\cdot\frac{\dot{\rho}_k(\phi)}{\rho_k(\phi)} = \frac{\dot{S}}{2k^2} + O\left(\frac{1}{k^3}\right).$$

Hence,

$$\|\dot{H}\|^2 = k^{n+2}\left(\int_X \dot{\phi}^2 d\mu_\phi + O\left(\frac{1}{k}\right)\right). \qquad \square$$

Remark 3.3. Actually we have proved that for any smooth ψ, and $\{s_i\}$ an orthonormal basis of H_k,

$$\lim_{k\to\infty} k^{-n-2}\sum_{i,j}\left|\int_X (s_i, s_j)_\phi(-k\psi + \Delta\psi)d\mu_\phi\right|^2 = \int_X \psi^2 d\mu_\phi.$$

Indeed, the convergence is uniform for ψ varying in a C^l compact set. So the following holds:

$$\lim_{k\to\infty} k^{-n}\sum_{i,j}\left|\int_X (s_i, s_j)_\phi\psi d\mu_\phi\right|^2 = \int_X \psi^2 d\mu_\phi. \qquad (7)$$

Now we can prove one side inequality of Theorem 1.1.

Corollary 3.4. *Given two metrics $\phi_1, \phi_2 \in \mathcal{H}$, and denote*

$$H_{k,i} = \text{Hilb}_k(\phi_i), \quad i = 1, 2,$$

then we have

$$\limsup_{k \to \infty} k^{-\frac{n}{2}-1} d_{\mathcal{B}_k}(H_{k,1}, H_{k,2}) \le d_{\mathcal{H}}(\phi_1, \phi_2).$$

Proof. From Lemma 2.1, we know that for any $\epsilon > 0$, there exists a smooth ϵ-approximate geodesic $\phi(t)(t \in [0,1])$ in $\mathcal{H}$ connecting ϕ_1 and ϕ_2 such that the length

$$L_{\mathcal{H}}(\phi(t)) \le d_{\mathcal{H}}(\phi_1, \phi_2) + \frac{\epsilon}{2}.$$

For this path and k sufficiently large, by Theorem 3.2,

$$k^{-\frac{n}{2}-1} L_{\mathcal{B}_k}(\text{Hilb}_k(\phi(t))) \le L_{\mathcal{H}}(\phi(t)) + \frac{\epsilon}{2}.$$

Then

$$k^{-\frac{n}{2}-1} d_{\mathcal{B}_k}(H_{k,1}, H_{k,2}) \le k^{-\frac{n}{2}-1} L_{\mathcal{B}_k}(\text{Hilb}_k(\phi(t))) \le d_{\mathcal{H}}(\phi_1, \phi_2) + \epsilon. \qquad \square$$

To prove the reversed inequality, we need the following lemma.

Lemma 3.5. *Given a smooth path $\phi(t) \in \mathcal{H}(t \in [0,1])$, then*

$$\lim_{k \to \infty} k^{-n-2} \|\nabla_{\dot{H}_k} \dot{H}_k\|^2 = \|\nabla_{\dot{\phi}} \dot{\phi}\|^2 = \int_X |\nabla_{\dot{\phi}} \dot{\phi}|^2 d\mu_\phi,$$

for any $t \in [0,1]$.

Proof. First notice that

$$\nabla_{\dot{\phi}} \dot{\phi} = \ddot{\phi} - |\nabla \dot{\phi}|^2,$$

and

$$\nabla_{\dot{H}} \dot{H} = \frac{d}{dt}(H^{-1}\dot{H})H = \ddot{H} - \dot{H}H^{-1}\dot{H},$$

where again we drop the subscript k for convenience. As before, we only need to prove the theorem at $t = 0$. We can pick an orthonormal basis $\{s_i(t)\}$ with respect to $H(t)$ such that $\dot{H}(0)$ is diagonalized, i.e.,

$$\int (s_i(t), s_j(t))_\phi d\mu_\phi = \delta_{ij},$$

and

$$\dot{H}_{ij} = \int (s_i, s_j)(-k\dot{\phi} + \Delta\dot{\phi})d\mu = \dot{H}_i \cdot \delta_{ij}, \tag{8}$$

holds at $t = 0$. Taking more derivatives, we obtain:

$$\ddot{H}_{ij} := \ddot{H}(s_i, s_j) = \int (s_i, s_j)[(-k\dot{\phi} + \Delta\dot{\phi})^2 - k\ddot{\phi} + \Delta\ddot{\phi} + \Psi(\dot{\phi})]d\mu, \tag{9}$$

where

$$\Psi(f) = -\sum_{i,j} f_{i\bar{j}} f_{j\bar{i}}.$$

$$\dddot{H}_{ij} := \dddot{H}(s_i, s_j)$$

$$= \int (s_i, s_j) \Big[(-k\dot{\phi} + \Delta\dot{\phi})^3 - k\ddot{\phi}(-k\dot{\phi} + \Delta\dot{\phi}) + (\Delta\ddot{\phi} + \Psi(\dot{\phi}))(-k\dot{\phi} + \Delta\dot{\phi})$$

$$+ 2k^2\dot{\phi}\ddot{\phi} - 2k\ddot{\phi}\Delta\dot{\phi} - 2k\dot{\phi}(\Delta\ddot{\phi} + \Psi(\dot{\phi})) + \frac{d}{dt}(\Delta\dot{\phi})^2 - k\dddot{\phi} + \frac{d^2}{dt^2}(\Delta\dot{\phi}) \Big] d\mu$$

$$= \int (s_i, s_j) \Big[-k^3\dot{\phi}^3 + 3k^2(\dot{\phi}^2\Delta\dot{\phi} + \ddot{\phi}\dot{\phi}) - 3k\Big(\dot{\phi}(\Delta\dot{\phi})^2 + \ddot{\phi}\Delta\dot{\phi} + \dot{\phi}\Delta\ddot{\phi}$$

$$+ \dot{\phi}\Psi(\dot{\phi}) + \frac{1}{3}\dddot{\phi} \Big) + 3\Delta\dot{\phi}\frac{d}{dt}(\Delta\dot{\phi}) + \frac{d^2}{dt^2}(\Delta\dot{\phi}) \Big] d\mu. \tag{10}$$

Let $\dot{s}_i = \sum_j a_{ij} s_j$, then we can further assume that (a_{ij}) is Hermitian, i.e., $a_{ji} = \overline{a_{ij}}$. Then it is easy to see that

$$a_{ij} = -\frac{1}{2}\dot{H}_{ij}.$$

Let $\phi_k(t) = FS_k(H_k(t))$, i.e.,

$$\sum_i |s_i|^2_{h_0 e^{-k\phi_k}} = 1,$$

and

$$\dot{\phi}_k(t) = -\frac{1}{k} \sum_{i,j} \dot{H}_{ij}(s_j, s_i)_{\phi_k} = -\frac{1}{k\rho_k(\phi)} \sum_{i,j} \dot{H}_{ij}(s_j, s_i)_\phi. \tag{11}$$

Taking time derivative, we get

$$\ddot{\phi}_k(t) = \frac{\dot{\rho}_k}{k\rho_k^2} \sum_{i,j} \dot{H}_{ij}(s_j, s_i)_\phi - \frac{1}{k\rho_k} \Big[\sum_{i,j} \ddot{H}_{ij}(s_j, s_i)_\phi - 2 \sum_{i,j,l} \dot{H}_{il}\dot{H}_{lj}(s_j, s_i)_\phi \Big] - k\dot{\phi}\dot{\phi}_k, \tag{12}$$

i.e.,

$$\sum_{i,j} \ddot{H}_{ij}(s_j, s_i)_\phi = -k\rho_k\ddot{\phi}_k - k\dot{\rho}_k\dot{\phi}_k - k^2\rho_k\dot{\phi}\dot{\phi}_k + 2 \sum_{i,j,l} \dot{H}_{il}\dot{H}_{lj}(s_j, s_i)_\phi. \tag{13}$$

Define:

$$g_k = k^{-1}(-k+\Delta)^{-1} \Big[(-k\dot{\phi} + \Delta\dot{\phi})^2 - k\ddot{\phi} + \Delta\ddot{\phi} + \Psi(\dot{\phi}) \Big]$$

$$= -\dot{\phi}^2 - \frac{1}{k}(2|\nabla\dot{\phi}|^2 - \ddot{\phi}) - \frac{1}{k^2} \Big[(\Delta\dot{\phi})^2 + \Psi(\dot{\phi}) - 2\Delta(\dot{\phi}\Delta\dot{\phi}) + \Delta^2(\dot{\phi}^2) \Big] + O\Big(\frac{1}{k^3}\Big).$$

Now let $\psi_k = d\,FS_k \circ d\,\mathrm{Hilb}_k(g_k)$, then

$$-\frac{1}{k^2\rho_k} \sum_{i,j} \ddot{H}_{ij}(s_j, s_i) = \psi_k = g_k - \frac{\Phi(\dot{\phi}^2)}{k^2} + O(k^{-3}),$$

where

$$\Phi(f) = \frac{1}{2}\delta_f S.$$

From (12) we then see that

$$2 \sum_{i,j,l} \dot{H}_{il} \dot{H}_{lj} (s_j, s_i)_\phi$$

$$= -k^2 \rho_k g_k + k \rho_k \ddot{\phi}_k + k \dot{\rho}_k \dot{\phi}_k + k^2 \rho_k \dot{\phi} \dot{\phi}_k + \Phi(\dot{\phi}^2) k^n + O(k^{n-1})$$

$$= 2k^2 \rho_k \dot{\phi}^2 + 2k \rho_k |\nabla \dot{\phi}|^2 + \Big[(\Delta \dot{\phi})^2 + \Psi(\dot{\phi}) - 2\Delta(\dot{\phi} \Delta \dot{\phi}) + 2\dot{\phi} \Phi(\dot{\phi})$$

$$+ \Phi(\dot{\phi}^2) + \Delta^2(\dot{\phi}^2) \Big] k^n + O(k^{n-1}).$$

Thus,

$$\sum_{i,j} (\ddot{H}_{ij} - \dot{H}_i \dot{H}_{ij})(s_j, s_i)$$

$$= k \rho_k (|\nabla \dot{\phi}|^2 - \ddot{\phi}) + \frac{\rho_k}{2} \Big[\Phi(\dot{\phi}^2) + \Delta^2(\dot{\phi}^2) - 2\dot{\phi} \Phi(\dot{\phi}) + (\Delta \dot{\phi})^2$$

$$+ \Psi(\dot{\phi}) - 2\Delta(\dot{\phi} \Delta \dot{\phi}) \Big] + O(k^{n-1}). \tag{14}$$

By (10),

$$\ddot{H} = k^2 d \ \mathrm{Hilb}_k(f_k),$$

where

$$f_k = k^{-2} (-k + \Delta)^{-1} \Big[-k^3 \dot{\phi}^3 + 3k^2 (\dot{\phi}^2 \Delta \dot{\phi} + \ddot{\phi} \dot{\phi}) - 3k \Big(\dot{\phi} (\Delta \dot{\phi})^2 + \ddot{\phi} \Delta \dot{\phi}$$

$$+ \dot{\phi} \Delta \ddot{\phi} + \dot{\phi} \Psi(\dot{\phi}) + \frac{1}{3} \dddot{\phi} \Big) + 3\Delta \dot{\phi} \frac{d}{dt} (\Delta \dot{\phi}) + \frac{d^2}{dt^2} (\Delta \dot{\phi}) \Big]$$

$$= -\frac{1}{k^3} \Big[-k^3 \dot{\phi}^3 + 3k^2 (\dot{\phi}^2 \Delta \dot{\phi} + \ddot{\phi} \dot{\phi}) - 3k \Big[\dot{\phi} (\Delta \dot{\phi})^2 + \ddot{\phi} \Delta \dot{\phi} + \dot{\phi} \Delta \ddot{\phi}$$

$$+ \dot{\phi} \Psi(\dot{\phi}) + \frac{1}{3} \dddot{\phi} \Big] - (3k^2 \dot{\phi}^2 \Delta \dot{\phi} + 6k^2 \dot{\phi} |\nabla \dot{\phi}|^2) + 3k \Big[\Delta(\dot{\phi}^2 \Delta \dot{\phi})$$

$$+ \Delta(\ddot{\phi} \dot{\phi}) - \Delta^2(\dot{\phi}^3) \Big] + O(1) \Big]$$

$$= \dot{\phi}^3 - \frac{3}{k} \dot{\phi} (\ddot{\phi} - 2|\nabla \dot{\phi}|^2) + \frac{3}{k^2} \Big[\dot{\phi} (\Delta \dot{\phi})^2 + \ddot{\phi} \Delta \dot{\phi} + \dot{\phi} \Delta \ddot{\phi} + \dot{\phi} \Psi(\dot{\phi})$$

$$+ \frac{1}{3} \dddot{\phi} - \Delta(\dot{\phi}^2 \Delta \dot{\phi}) - \Delta(\ddot{\phi} \dot{\phi}) - \Delta^2(\dot{\phi}^3) \Big] + O\Big(\frac{1}{k^3} \Big).$$

Therefore,

$$\sum_{i,j} \ddot{H}_{ij} (s_j, s_i) = -k^3 \rho_k f_k - k^{n+1} \Phi(\dot{\phi}^3) + O(k^n)$$

$$= -k^3 \rho_k \dot{\phi}^3 + 3k^2 \rho_k \dot{\phi} (\ddot{\phi} - 2|\nabla \dot{\phi}|^2) - k \rho_k \Big[\Phi(\dot{\phi}^3) + 3\dot{\phi} (\Delta \dot{\phi})^2$$

$$+ 3\ddot{\phi} \Delta \dot{\phi} + 3\dot{\phi} \Delta \ddot{\phi} + 3\dot{\phi} \Psi(\dot{\phi}) + \dddot{\phi} - 3\Delta(\dot{\phi}^2 \Delta \dot{\phi})$$

$$- 3\Delta(\ddot{\phi} \dot{\phi}) - \Delta^2(\dot{\phi}^3) \Big] + O(k^n). \tag{15}$$

Now differentiating (13), we obtain

$$\sum_{i,j} \dddot{H}_{ij}(s_j, s_i) - \sum_{i,j} \ddot{H}_{i,j}(\dot{H}_i + \dot{H}_j)(s_j, s_i) - k\dot{\phi} \sum_{i,j} \ddot{H}_{ij}(s_j, s_i)$$

$$= -2k\dot{\rho}_k\ddot{\phi}_k - k\rho_k\dddot{\phi}_k - k\ddot{\rho}_k\dot{\phi}_k - k^2\dot{\rho}_k\dot{\phi}\dot{\phi}_k - k^2\rho_k\ddot{\phi}\dot{\phi}_k - k^2\rho_k\dot{\phi}\ddot{\phi}_k$$

$$+ 2\sum_{i,j} \ddot{H}_{i,j}(\dot{H}_i + \dot{H}_j)(s_j, s_i) - 6\sum_i \dot{H}_i^3|s_i|^2 - 2k\dot{\phi}\sum_i \dot{H}_i^2|s_i|^2. \qquad (16)$$

Thus,

$$3\sum_{i,j} \ddot{H}_{i,j}(\dot{H}_i + \dot{H}_j)(s_j, s_i) - 6\sum_i \dot{H}_i^3|s_i|^2$$

$$= \sum_{i,j} \dddot{H}_{i,j}(s_j, s_i) + k\dot{\phi}(k\rho_k\ddot{\phi}_k + k\dot{\rho}_k\dot{\phi}_k + k^2\rho_k\dot{\phi}\dot{\phi}_k)$$

$$+ 2k\dot{\rho}_k\ddot{\phi}_k + k\rho_k\dddot{\phi}_k + k\ddot{\rho}_k\dot{\phi}_k + k^2\rho_K\dot{\phi}\dot{\phi}_k + k^2\rho_k\ddot{\phi}\dot{\phi}_k + k^2\rho_k\dot{\phi}\ddot{\phi}_k$$

$$= \sum_{i,j} \dddot{H}_{i,j}(s_j, s_i) + k^3\rho_k\dot{\phi}^2\dot{\phi}_k + 3k^2\rho_k\dot{\phi}\ddot{\phi} + k^{n+1}(2\Phi(\dot{\phi})\dot{\phi}^2 + \dddot{\phi}) + O(k^n)$$

$$= 6k^2\rho_k\dot{\phi}(\ddot{\phi} - |\nabla\dot{\phi}|^2) + k^{n+1}\Big[3\Phi(\dot{\phi})\dot{\phi}^2 + \dddot{\phi} - \Phi(\dot{\phi}^3) - 3\dot{\phi}(\Delta\dot{\phi})^2 - 3\dot{\phi}\Delta\ddot{\phi} - 3\ddot{\phi}\Delta\dot{\phi}$$

$$- 3\dot{\phi}\Psi(\dot{\phi}) - \dddot{\phi} + 3\Delta(\dot{\phi}^2\Delta\dot{\phi}) + 3\Delta(\ddot{\phi}\dot{\phi}) - \Delta^2(\dot{\phi}^3)\Big] + O(k^n)$$

$$= 6k^2\rho_k\dot{\phi}(\ddot{\phi} - |\nabla\dot{\phi}|^2) + k^{n+1}\Big[3\Phi(\dot{\phi})\dot{\phi}^2 - \Phi(\dot{\phi}^3) - \Delta^2(\dot{\phi}^3) - 3\dot{\phi}(\Delta\dot{\phi})^2 - 6\nabla\dot{\phi}\cdot\nabla\ddot{\phi}$$

$$- 3\dot{\phi}\Psi(\dot{\phi}) + 3\Delta(\dot{\phi}^2\Delta\dot{\phi})\Big] + O(k^n). \qquad (17)$$

Putting (8), (9), (14), (17) together, we get

$$|\nabla_{\dot{H}}\dot{H}|^2 = \sum_{i,j} (\ddot{H}_{ij} - \dot{H}_i\dot{H}_{ij}) \int (s_j, s_i)(k^2\dot{\phi}^2 - 2k\dot{\phi}\Delta\dot{\phi} - k\ddot{\phi} + O(1))d\mu$$

$$- \sum_{i,j} (\dddot{H}_{ij}\dot{H}_i - \dot{H}_i^3) \int (s_j, s_i)(-k\dot{\phi} + \Delta\dot{\phi})d\mu$$

$$= \int \Big\{k\rho_k(|\nabla\dot{\phi}|^2 - \ddot{\phi}) + \frac{\rho_k}{2}\Big[\Phi(\dot{\phi}^2) - 2\dot{\phi}\Phi(\dot{\phi}) + \Delta^2(\dot{\phi}^2) + (\Delta\dot{\phi})^2 + \Psi(\dot{\phi})$$

$$- 2\Delta(\dot{\phi}\Delta\dot{\phi})\Big] + O(k^{n-1})\Big\}(k^2\dot{\phi}^2 - 2k\dot{\phi}\Delta\dot{\phi} - k\ddot{\phi} + O(1))d\mu$$

$$+ \int \Big\{k^2\rho_k\dot{\phi}(\ddot{\phi} - |\nabla\dot{\phi}|^2) + \frac{k\rho_k}{6}\Big[3\Phi(\dot{\phi})\dot{\phi}^2 - \Phi(\dot{\phi}^3) - \Delta^2(\dot{\phi}^3)$$

$$+ 6\nabla\dot{\phi}\nabla\ddot{\phi} - 3\dot{\phi}\Psi(\dot{\phi}) - 3\dot{\phi}(\Delta\dot{\phi})^2 + 3\Delta(\dot{\phi}^2\Delta\dot{\phi}) + O(k^n)\Big]\Big\}(k\dot{\phi} - \Delta\dot{\phi})d\mu$$

$$= \int -k^2\rho_k(|\nabla\dot{\phi}|^2 - \ddot{\phi})(2\dot{\phi}\Delta\dot{\phi} + \ddot{\phi}) + \frac{k^2}{2}\rho_k\dot{\phi}^2\Big[\Phi(\dot{\phi}^2) - 2\dot{\phi}\Phi(\dot{\phi}) + \Delta^2(\dot{\phi}^2)$$

$$+ (\Delta\dot{\phi})^2 + \Psi(\dot{\phi}) - 2\Delta(\dot{\phi}\Delta\dot{\phi})\Big] - k^2\rho_k\dot{\phi}\Delta\dot{\phi}(\ddot{\phi} - |\nabla\dot{\phi}|^2)$$

$$+ \frac{k^2 \rho_k}{6} \dot{\phi} \Big[3\Phi(\dot{\phi})\dot{\phi}^2 - \Phi(\dot{\phi}^3) - \Delta^2(\dot{\phi}^3) + 6\nabla\dot{\phi}\nabla\ddot{\phi} - 3\dot{\phi}\Psi(\dot{\phi})$$

$$- 3\dot{\phi}(\Delta\dot{\phi})^2 + 3\Delta(\dot{\phi}^2\Delta\dot{\phi}) \Big] d\mu + O(k^{n+1})$$

Further manipulating the above expression, we get

$$|\nabla_{\dot{H}}\dot{H}|^2$$

$$= -\frac{k^2}{2} \int \rho_k \Delta(\dot{\phi}^2)|\nabla\dot{\phi}|^2 d\mu$$

$$+ \frac{k^2}{2} \int \rho_k \dot{\phi}^2 [-R_{i\bar{j}}\dot{\phi}_{\bar{i}}\dot{\phi}_j + \Delta^2(\dot{\phi}^2) + \dot{\phi}\Delta^2\dot{\phi} + (\Delta\dot{\phi})^2 - 2\Delta(\dot{\phi}\Delta\dot{\phi})] d\mu$$

$$+ \frac{k^{n+2}}{6} \int \dot{\phi} \Big[3R_{i\bar{j}}\dot{\phi}\dot{\phi}_{\bar{i}}\dot{\phi}_j - \frac{3}{2}\dot{\phi}^2\Delta^2\dot{\phi} + \frac{1}{2}\Delta^2(\dot{\phi}^3) - \Delta^2(\dot{\phi}^3)$$

$$- 3\dot{\phi}(\Delta\dot{\phi})^2 + 3\Delta(\dot{\phi}^2\Delta\dot{\phi}) \Big] d\mu + O(k^{n+1})$$

$$= -k^2 \int \rho_k(\dot{\phi}\Delta\dot{\phi} + |\nabla\dot{\phi}|^2)|\nabla\dot{\phi}|^2 d\mu$$

$$+ k^{n+2} \int \frac{1}{2}\dot{\phi}^3\Delta^2\dot{\phi} + \frac{1}{4}\dot{\phi}^2\Delta^2(\dot{\phi}^2) + \frac{1}{2}\dot{\phi}^2(\Delta\dot{\phi})^2 - 2\dot{\phi}^2(\Delta\dot{\phi})^2 - 2\dot{\phi}\Delta\dot{\phi}|\nabla\dot{\phi}|^2 d\mu$$

$$- \frac{k^{n+2}}{3} \int \dot{\phi}^3\Delta^2\dot{\phi} d\mu + O(k^{n+1})$$

$$= k^{n+2} \int (\ddot{\phi} - |\nabla\dot{\phi}|)^2 d\mu + O(k^{n+1}).$$

where we have used:

$$\Phi(\dot{\phi}) = -\frac{1}{2}(\dot{\phi}_{i\bar{j}}R_{j\bar{i}} + \Delta^2\dot{\phi}),$$

$$\Phi(\dot{\phi}^2) = -\dot{\phi}\dot{\phi}_{i\bar{j}}R_{j\bar{i}} - \dot{\phi}_i\dot{\phi}_{\bar{j}}R_{j\bar{i}} - \frac{1}{2}\Delta^2(\dot{\phi}^2),$$

$$\Phi(\dot{\phi}^3) = -\frac{3}{2}\dot{\phi}^2\dot{\phi}_{i\bar{j}}R_{j\bar{i}} - 3\dot{\phi}\dot{\phi}_i\dot{\phi}_{\bar{j}}R_{j\bar{i}} - \frac{1}{2}\Delta^2(\dot{\phi}^3).$$

$$\int \dot{\phi}^2\Delta^2(\dot{\phi}^2)d\mu = 4 \int \dot{\phi}^2(\Delta\dot{\phi}^2)d\mu + 8 \int \dot{\phi}\Delta\dot{\phi}|\nabla\dot{\phi}|^2 d\mu + 4 \int |\nabla\dot{\phi}|^4 d\mu.$$

$$\int \dot{\phi}^3\Delta^2(\dot{\phi})d\mu = 3 \int \dot{\phi}^2(\Delta\dot{\phi})^2 d\mu + 6 \int \dot{\phi}\Delta\dot{\phi}|\nabla\dot{\phi}|^2 d\mu. \qquad \square$$

Proof of Theorem 1.1. In view of Corollary 3.4, it suffices to show that

$$\liminf_{k\to\infty} k^{-\frac{n}{2}-1} d_{\mathcal{B}_k}(H_{k,0}, H_{k,1}) \geq d_{\mathcal{H}}(\phi_0, \phi_1).$$

By Lemma 2.1, for any $\epsilon > 0$ small enough, there exists a smooth family $\phi_\epsilon(\cdot) :$ $[0,1] \to \mathcal{H}$, such that $\phi_\epsilon(0) = \phi_0$, $\phi_\epsilon(1) = \phi_1$, and

$$(\ddot{\phi}_\epsilon - |\nabla\dot{\phi}_\epsilon|_t^2)\omega_{\epsilon,t}^n = \epsilon\omega^n.$$

Moreover, $\ddot{\phi}_\epsilon - |\nabla\dot{\phi}_\epsilon|^2_t \leq C$, where C is a constant independent of t and ϵ. Let $H_\epsilon(t) = \text{Hilb}_k(\phi_\epsilon(t))$, then by Lemma 3.5, for k large enough, we have

$$k^{-n-2}|\nabla_{\dot{H}_\epsilon}\dot{H}_\epsilon|^2 \leq C\epsilon.$$

By the following simple lemma, we know that

$$k^{-\frac{n}{2}-1}d_{\mathcal{B}_k}(H_{k,0}, H_{k,1}) \geq k^{-\frac{n}{2}-1}L(H_\epsilon) - \sqrt{C\epsilon} \to L(\phi_\epsilon) - \sqrt{C\epsilon},$$

as $k \to \infty$. Hence,

$$\liminf_{k\to\infty} k^{-\frac{n}{2}-1}d_{\mathcal{B}_k}(H_{k,0}, H_{k,1}) \geq d_{\mathcal{H}}(\phi_0, \phi_1) - \sqrt{C\epsilon}.$$

Let $\epsilon \to 0$, we obtain the desired result. $\square$

Now we prove a simple lemma from Riemannian geometry.

Lemma 3.6. *Suppose (M, g) is a simply connected Riemmannian manifold with non-positive curvature. Let $\gamma : [0,1] \to M$ be a path in M such that $|\nabla_{\dot{\gamma}(t)}\dot{\gamma}(t)| \leq \epsilon$ for all $t \in [0,1]$. Then*

$$d(\gamma(0), \gamma(1)) \geq L(\gamma) - \epsilon.$$

Proof. Denote $p = \gamma(0)$ and $q = \gamma(1)$. By the theorem of Cartan-Hadamard, we know that the exponential map at p is a diffeomorphism. Now let $\gamma_t(s)(s \in [0,1])$ be the unique geodesic connecting p and $\gamma(t)$. By standard calculation of the second variation, we obtain:

$$
\begin{aligned}
\frac{d^2}{dt^2}L(\gamma_t) &= \frac{1}{L(\gamma_t)}\left\langle \nabla_{\dot{\gamma}(t)}\dot{\gamma}(t), \frac{\partial}{\partial s}\gamma_t(s)|_{s=1}\right\rangle + \frac{1}{L(\gamma_t)}\int_0^1 \left|\left(\frac{\partial^2}{\partial s\partial t}\gamma_t(s)\right)^\perp\right|^2 \\
&\quad - R\left(\frac{\partial}{\partial s}\gamma_t(s), \frac{\partial}{\partial t}\gamma_t(s), \frac{\partial}{\partial s}\gamma_t(s), \frac{\partial}{\partial t}\gamma_t(s)\right) ds \\
&\geq -\epsilon,
\end{aligned}
$$

where $\perp$ denotes projection to the orthogonal complement of $\frac{\partial}{\partial s}\gamma_t(s)$. It is also easy to see that

$$L(\gamma_0) = 0,$$

and

$$\frac{d}{dt}L(t)|_{t=0} = |\dot{\gamma}(0)|.$$

Therefore,

$$L(t) \geq |\dot{\gamma}(0)|t - \frac{\epsilon}{2}t^2.$$

In particular, $d(p, q) = L(\gamma(1)) \geq |\dot{\gamma}(0)| - \frac{\epsilon}{2}$. On the other hand,

$$L(\gamma) = \int_0^1 |\dot{\gamma}(t)|dt \leq \int_0^1 (|\dot{\gamma}(0)| + \epsilon t)dt = |\dot{\gamma}(0)| + \frac{\epsilon}{2}.$$

Hence, $d(p, q) \geq L(\gamma) - \epsilon$. $\square$

4. Weak convexity of K-energy

In this section, we shall prove Corollary 1.7 and 1.6. Before doing this, we want to show the functional I is quantized by I_k, which is an analogue of the fact that Z_k quantizes E.

Proposition 4.1. *There are constants c_k such that for $\phi \in \mathcal{H}$, and $H_k = \mathrm{Hilb}_k(\phi)$, we have*

$$\lim_{k \to \infty} k^{-n-1} I_k(H_k) + c_k = I(\phi).$$

The convergence is uniform when ϕ varies in a C^l bounded sets as before.

Proof. It suffices to show for any $\psi \in T_\phi \mathcal{H}$

$$d_\phi \ I(\psi) = d_{\mathrm{Hilb}_k(\phi)} \ I_k \circ d_\phi \ \mathrm{Hilb}_k(\psi). \tag{18}$$

By definition,

$$d_\phi \ I(\psi) = \int \psi d\mu_\phi.$$

On the other hand, by Lemma 3.1,

$$d_{\mathrm{Hilb}_k(\phi)} \ I_k \circ d_\phi \ \mathrm{Hilb}_k(\psi) = \sum_i \int |s_i|_\phi^2 (-k\psi + \Delta\psi) d\mu_\phi$$

$$= \int \rho_k(\phi)(-k\psi + \Delta\psi) d\mu_\phi$$

Then the result follows from Lemma 3.1. □

Proposition 4.2. *Let $\phi \in \mathcal{H}$, and $H_k = \mathrm{Hilb}_k(\phi)$, then*

$$\lim_{k \to \infty} k^{n+2} \|\nabla Z_k(H_k)\|^2 = \|\nabla E(\phi)\|^2.$$

Proof. Note

$$\|\nabla E(\phi)\|^2 = \int (S - \underline{S})^2 d\mu_\phi$$

is simply Calabi's functional. An inequality of this form is essentially proved in [13] for obtaining a lower bound of the Calabi functional. We can easily calculate the first variation of Z_k:

$$\delta Z_k = -\frac{d_k}{k^n \cdot V} \int_X \delta H_{ij}[(s_j, s_i)_{FS_k(H)}]_0 d\mu_{\omega_{FS_k(H)}},$$

where $[A]_0$ denote the trace-free part of a matrix A, and $\{s_i\}$ is an orthonormal basis with respect to H. Here

$$V = \int_X d\mu_\phi,$$

and

$$d_k = \dim H^0(X, L^k) = \int_X \rho_k(\phi) d\mu_\phi = k^n \cdot V + \frac{k^{n-1} \cdot V}{2} \underline{S} + O(k^{n-2}).$$

So
$$(\nabla Z_k)_{ij} = -\frac{d_k}{k^n \cdot V} \int_X [(s_i, s_j)_{FS_k(H)}]_0 d\mu_{\omega_{FS_k(H)}}.$$

Diagonalize the above matrix so that its diagonal entries are
$$\lambda_i = \frac{d_k}{k^n \cdot V} \int_X (|s_i|^2_{\psi_2} - \frac{1}{d_k}) d\mu_{\psi_2}.$$

Let $\phi_k = FS_k(H_k)$, then
$$\phi_k - \phi = \frac{1}{k} \log(\rho_k(\phi)) = \frac{1}{k} \log(k^n + \frac{S}{2} k^{n-1} + O(k^{n-2})).$$

So
$$|s_i|^2_{\phi_k} = k^{-n} |s_i|^2_\phi \left(1 - \frac{S}{2k} + O\left(\frac{1}{k^2}\right)\right),$$

and
$$\omega^n_{\phi_k} = \omega^n_\phi (1 + O(k^{-2})).$$

Since
$$\int_X |s_i|^2_\phi d\mu_\phi = 1,$$

we obtain
$$\lambda_i = -\frac{1}{k^{n+1}} \int_X |s_i|^2_\phi (S - \underline{S}) d\mu_\phi + O\left(\frac{1}{k^{n+2}}\right).$$

Hence by Remark 3.3,
$$\|\nabla Z_k\|^2 = \sum_i |\lambda_i|^2$$
$$= k^{-2n-2} \sum_i \left[\int_X |s_i|^2_\phi (S - \underline{S}) d\mu_\phi + O\left(\frac{1}{k^{n+2}}\right)\right]^2$$
$$= k^{-n-2} \left(Ca(\phi_2) + O\left(\frac{1}{k}\right)\right). \qquad \square$$

The proof of the following proposition is similar and we omit it.

Proposition 4.3. *Let $\phi(t)$ be a smooth path in $\mathcal{H}$, and $H_k(t) = \mathrm{Hilb}_k(\phi)$, then*
$$\lim_{k \to \infty} \frac{d}{dt} Z_k(H_k) = \frac{d}{dt} E(\phi).$$

Proof of Corollary 1.7. By Lemma 2.5, we already know that Z_k is a genuine convex function on $\mathcal{B}_k$. So it is easy to see that
$$Z_k(H_2) - Z_k(H_1) \leq d_{\mathcal{B}_k}(H_1, H_2) \cdot \|\nabla Z_k(H_2)\|.$$

Using Theorem 1.1, Lemma 2.4 and Proposition 4.2, and let $k \to \infty$, we get the desired inequality for E. $\qquad \square$

Proof of Corollary 1.6. As in the proof of Theorem 1.1, for any $\epsilon > 0$ sufficiently small, we choose ϵ-approximate geodesic $\phi_\epsilon : [0,1] \to \mathcal{H}$, such that $|\nabla_{\dot{\phi}_\epsilon} \dot{\phi}_\epsilon| \le C\epsilon$. Denote $H_\epsilon(t) = \mathrm{Hilb}_k(\phi_\epsilon(t))$, and $H_i = \mathrm{Hilb}_k(\phi_i)$, $i = 0, 1$. Then by Lemma 3.5, we see that

$$k^{-\frac{n}{2}-1} |\nabla_{\dot{H}_\epsilon} \dot{H}_\epsilon| \le C\epsilon,$$

for k sufficiently large and a different constant C. Denote by $\tilde{H} : [0,1] \to \mathcal{B}_k$ the geodesic connecting H_0 and H_1. By Lemma 4.5 which will be proved later, we know that

$$k^{-\frac{n}{2}-1} |\dot{H}_\epsilon(i) - \dot{\tilde{H}}(i)| \le C\sqrt{\epsilon},$$

for $i = 0, 1$. Since $\tilde{H}(t)$ is a geodesic in $\mathcal{B}_k$, we have by Lemma 2.5 that

$$dZ_{k_{H_1}}(\dot{\tilde{H}}(1)) \ge dZ_{k_{H_0}}(\dot{\tilde{H}}(0)).$$

On the other hand, by Proposition 4.2,

$$|dZ_{k_{H_1}}(\dot{H}_\epsilon(1) - \dot{\tilde{H}}(1))| \le |\nabla Z_k|_{H_1} \cdot |\dot{H}_\epsilon(1) - \dot{\tilde{H}}(1)| \le C\sqrt{\epsilon}\left(\sqrt{Ca(\phi_1)} + O\left(\frac{1}{k}\right)\right).$$

Similarly,

$$|dZ_{k_{H_0}}(\dot{H}_\epsilon(0) - \dot{\tilde{H}}(0))| \le C\sqrt{\epsilon}(\sqrt{Ca(\phi_0)} + O(\frac{1}{k})).$$

So,

$$dZ_{k_{H_1}}(\dot{H}_\epsilon(1)) \ge dZ_{k_{H_0}}(\dot{H}_\epsilon(0)) - C\sqrt{\epsilon}.$$

By Proposition 4.3, that as $k \to \infty$

$$dZ_{k_{H_1}}(\dot{H}_\epsilon(1)) \to dE_{\phi_1}(\dot{\phi}_\epsilon(1)),$$

and

$$dZ_{k_{H_0}}(\dot{H}_\epsilon(0)) \to dE_{\phi_0}(\dot{\phi}_\epsilon(0)).$$

Thus,

$$dE_{\phi_1}(\dot{\phi}_\epsilon(1)) \ge dE_{\phi_0}(\dot{\phi}_\epsilon(0)) - C\sqrt{\epsilon}.$$

Letting $\epsilon \to 0$, this proves Corollary 1.6. □

Remark 4.4. If we denote $\phi_{k,\epsilon}(t) = FS_k(\mathrm{Hilb}_k \phi_\epsilon(t))$ and $\tilde{\phi}_k(t) = FS_k(H(t))$, then by a similar estimate as in the proof of Proposition 4.2 we have

$$|\dot{\tilde{\phi}}_k(0) - \dot{\phi}_{k,\epsilon}(0)|_{L^2} \le C\sqrt{\epsilon}.$$

Thus, the time derivative of $\tilde{\phi}_k$ at the end points converge in L^2 to the time derivative of the $C^{1,1}$ geodesic. This seems to be an interesting fact, compare [18].

Lemma 4.5. *Suppose (M, g) is a simply connected Riemmannian manifold with non-positive curvature. Let $\gamma : [0,1] \to M$ be a path in M such that $|\nabla_{\dot{\gamma}(t)} \dot{\gamma}(t)| \le \epsilon$ for all $t \in [0,1]$. Let $\tilde{\gamma} : [0,1]$ be the unique geodesic segment joining $\gamma(0)$ and $\gamma(1)$. Then*

$$|\dot{\gamma}(0) - \dot{\tilde{\gamma}}(0)|^2 \le \frac{9}{4}\epsilon^2 + 4\epsilon|\dot{\gamma}(1)|,$$

and

$$|\dot\gamma(1) - \dot{\tilde\gamma}(1)|^2 \le \frac{9}{4}\epsilon^2 + 4\epsilon|\dot\gamma(0)|,$$

Proof. It suffices to prove the second inequality. As before, let $\gamma_t(s) : [0,1] \to M$ be the smooth family of geodesic segments connecting $\gamma(0)$ and $\gamma(t)$. Then by the proof of Lemma 3.6 we see that

$$\frac{d}{dt}\left\langle \frac{d\gamma}{dt}, \frac{\dot\gamma_t(1)}{|\dot\gamma_t(1)|} \right\rangle \ge -\epsilon,$$

and so

$$\left\langle \frac{d\gamma}{dt}, \frac{\dot\gamma_t(1)}{|\dot\gamma_t(1)|} \right\rangle \ge \lim_{t\to 0}\left\langle \frac{d\gamma}{dt}, \frac{\dot\gamma_t(1)}{|\dot\gamma_t(1)|} \right\rangle - \epsilon t = \left|\frac{d\gamma}{dt}(0)\right| - \epsilon t.$$

Let

$$\frac{d\gamma}{dt} = A_t \cdot \frac{\dot\gamma_t(1)}{|\dot\gamma_t(1)|} + B_t$$

be the orthogonal decomposition. Then

$$A_t \ge \left|\frac{d\gamma}{dt}(0)\right| - \epsilon t.$$

It is clear that

$$\left|\frac{d}{dt}\left|\frac{d\gamma}{dt}\right|\right| \le \left|\frac{d^2\gamma}{dt^2}\right| \le \epsilon,$$

so

$$\left|\frac{d\gamma}{dt}(0)\right| - \epsilon t \le \left|\frac{d\gamma}{dt}\right| \le \left|\frac{d\gamma}{dt}(0)\right| + \epsilon t.$$

Hence,

$$|B_t|^2 \le \left(\left|\frac{d\gamma}{dt}(0)\right| + \epsilon t\right)^2 - \left(\left|\frac{d\gamma}{dt}(0)\right| - \epsilon t\right)^2 = 4\epsilon t\left|\frac{d\gamma}{dt}(0)\right|.$$

From the proof of Lemma 3.6, we know that

$$\left|\frac{d\gamma}{dt}(0)\right|t - \frac{\epsilon}{2}t^2 \le |\dot\gamma_t(1)| \le \left|\frac{d\gamma}{dt}(0)\right|t + \frac{\epsilon}{2}t^2.$$

Finally,

$$\left|\frac{d\gamma}{dt} - \frac{1}{t}\dot\gamma_t(1)\right|^2 = \left(A_t - \frac{1}{t}|\dot\gamma_t(1)|\right)^2 + B_t^2$$

$$\le \frac{9}{4}\epsilon^2 t^2 + 4\epsilon t\left|\frac{d\gamma}{dt}(0)\right|,$$

Let $t = 1$, we obtain the desired bound. $\square$

5. Open problems

In this section, we list some open problems and speculations on which the results of this paper may help in the future.

Problem 5.1. Theorem 1.1 essentially says that $\mathcal{H}$ is a weak Gromov-Hausdorff limit of a sequence of finite-dimensional symmetric spaces and $\mathcal{H}$ can be viewed as a generalized $Cat(0)$ space which has been extensively studied in the literature (cf. [1]). It would be interesting to investigate this more carefully and develop a suitable notion for this type of convergence. From this convergence it follows that the negativity of curvature is inherited by the limit. A natural question would be what else properties for the finite-dimensional symmetric spaces would survive on $\mathcal{H}$. For example can we describe the algebraic structure of $\mathcal{H}$. In particular does it admit a local involution? In other words, for any $\phi \in \mathcal{H}$, there exists a small constant $\delta(\phi)$ such that there is an local involution

$$\sigma : B_\delta(\varphi) \bigcap \mathcal{H} \to B_\delta(\varphi) \bigcap \mathcal{H}$$

with $\sigma^2 = id$. This is not trivial since the initial value problem for the geodesic equation in $\mathcal{H}$ is not well posed.

We know that $\mathcal{H}$ is not complete, then the following is very interesting:

Problem 5.2. What is the structure of $\partial \mathcal{H}$?

It would be extremely interesting to understand the structure of the boundary. Note that for every k, the Bergman metric space $\mathcal{B}_k$ is complete and $\mathcal{H}$ is a limit of these nice symmetric spaces after appropriate scaling down (by a factor of $k^{-\frac{n+2}{2}}$). It is natural to hope that $\mathcal{H}$ should look like the limit of the tangent cone of $\mathcal{B}_k$ at infinity.

By the work of J. Fine [14], we can approximate Calabi flow over a bounded interval by the Balancing flow in $\mathcal{B}_k$. The next intriguing question is can we approximate the Calabi flow over $[0, \infty)$ if we know it converges (the complex structure might jump). This is related to

Problem 5.3. In an algebraic manifold, if there is a cscK metric, does the Calabi flow exist for long time and converge to a cscK metric?

This is only proved for metrics near a cscK metric. The corresponding problems for Kähler-Ricci flow was proved first by G. Perelman and a written version is provided by Tian-Zhu([25]).

Problem 5.4. The existence of cscK metrics implies that the K-energy is proper.

The corresponding results for Kähler-Einstein metrics is proved by Tian in [24]. Note the definition of properness could vary. In [24], properness means bounded below by another positive functional J. Since the K-energy is convex along smooth geodesics and its Hessian at a cscK metric is strictly positive (if the automorphism group of the manifold is discrete), we would expect it bounds

the distance function. Note the corresponding statements in the finite-dimensional case is clear: a convex function with a non-degenerate critical point automatically bounds the distance function. Since both the distance function and the K-energy could be approximated by the corresponding quantities on $\mathcal{B}_k$, to prove the corresponding statement for $\mathcal{H}$, it is important to derive a uniform constant.

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