K-POLYSTABILITY OF Q-FANO VARIETIES ADMITTING KÄHLER-EINSTEIN METRICS

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ABSTRACT. It is shown that any, possibly singular, Fano variety X admitting a Kähler-Einstein metric is K-polystable, thus confirming one direction of the Yau-Tian-Donaldson conjecture in the setting of $\mathbb{Q}$-Fano varieties equipped with their anti-canonical polarization. The proof is based on a new formula expressing the Donaldson-Futaki invariants in terms of the slope of the Ding functional along a geodesic ray in the space of all bounded positively curved metrics on the anti-canonical line bundle of X. One consequence is that a toric Fano variety X is K-polystable iff it is K-polystable along toric degenerations iff 0 is the barycenter of the canonical weight polytope P associated to X. The results also extend to the logarithmic setting and in particular to the setting of Kähler-Einstein metrics with edge-cone singularities. Applications to geodesic stability, bounds on the Ricci potential and Perelman's $\lambda-$entropy functional on $K-$unstable Fano manifolds are also given.

CONTENTS

1. INTRODUCTION

Let (X, L) be a polarized projective algebraic manifold. i.e. L is an ample line bundle over X. According to the fundamental Yau-Tian-Donaldson conjecture in Kähler geometry (see the recent survey [63]) the first Chern class $c_1(L)$ contains a Kähler metric ω with *constant scalar curvature* if and only if (X, L) is *K-polystable.* This notion of stability is of an algebro-geometric nature and has its origin in Geometric Invariant Theory (GIT). It was introduced by Tian [69] and in its most general form, due to Donaldson [23] it is formulated in terms of polarized $\mathbb{C}^*-$equivariant deformations $\mathcal{L} \to \mathcal{X} \to \mathbb{C}$ of (X, L) called *test configurations* for the polarized variety (X, L), where $\mathcal{X}_1 = X$. Briefly, to any test configuration $(\mathcal{X}, \mathcal{L})$ one associates a numerical invariant $DF(\mathcal{X}, \mathcal{L})$, called the *Donaldson-Futaki invariant* defined in terms of the polarized scheme $(\mathcal{X}_0, \mathcal{L}_{|\mathcal{X}_0})$ and X is said to be K-polystable if $DF(\mathcal{X}, \mathcal{L}) \geq 0$ with equality if and only if $(\mathcal{X}, \mathcal{L})$ is isomorphic to a product test configuration (the precise definitions are recalled in section 2.2). The

test configuration $(\mathcal{X}, \mathcal{L})$ thus plays the role of a one-parameter subgroup in GIT and the Donaldson-Futaki invariant corresponds to the Hilbert-Mumford weight in GIT. Accordingly, the Yau-Tian-Donaldson conjecture is sometimes also referred to as the non-linear version of the celebrated Kobayashi-Hitchin correspondence between Hermitian Yang-Mills metrics and polystable vector bundles.

In the case when the connected component $\mathrm{Aut}(X)_0$ containing the identity of the the automorphism group is trivial, i.e. X admits no non-trivial holomorphic vector fields, it was shown by Stoppa [66] that the existence of a constant scalar curvature metric in $c_1(L)$ indeed implies that (X, L) is K-polystable. The case when $\mathrm{Aut}(X)_0$ is non-trivial leads to highly non-trivial complications, related to the case when $DF = 0$ and was treated by Mabuchi in a series of papers [44, 45] (see also [68] where it is shown that if $c_1(L)$ contains an extremal Kähler metric, then (X, L) is K-polystable with respect to all test configurations whose $\mathbb{C}^*-$action commutes with a maximal torus of automorphisms). In this note we will be concerned with the special case when ω is a *Kähler-Einstein metric* of positive scalar curvature. Equivalently this means that the Ricci curvature of ω is positive and constant:

$$\mathrm{Ric}\ \omega = \omega,$$

i.e. L is the anti-canonical line bundle $-K_X$ and X is a Fano manifold. In the seminal paper of Tian [69] it was shown, in the case when $\mathrm{Aut}(X)_0$ is trivial, that X is K-stable along all test configurations $\mathcal{X}$ with normal central fiber $\mathcal{X}_0$. Here we will show that the assumption on $\mathrm{Aut}(X)_0$ can be removed, as well as the normality assumption on the central fiber $\mathcal{X}_0$. In fact, we will allow X to be a general, possibly singular, Fano variety and prove the following

Theorem 1.1. *Let X be a Fano variety admitting a Kähler-Einstein metric. Then X is K-polystable.*

It should be pointed out that, following Li-Xu [42], we assume that the total space $\mathcal{X}$ of the test configuration is normal to exclude some pathological test-configurations that had previously been overlooked in the literature (as explained in [42]). As follows from the results of Ross-Thomas [64] this does not affect the notion of K semi-stability. Moreover, by a remark of Stoppa [67] K-polystability for all normal test configuration is equivalent to having $DF(\mathcal{X}, \mathcal{L}) \geq 0$ for all test configurations with equality iff $(\mathcal{X}, \mathcal{L})$ is isomorphic to a product test configuration away from a subvariety of codimension at least two.

We recall that, by definition, X is a Fano variety if it is normal and the anti-canonical divisor $-K_X$ is defined as an ample $\mathbb{Q}-$line bundle (such a variety is also called a $\mathbb{Q}-$Fano variety in the literature) and, following [4], ω is said to be a *Kähler-Einstein metric* on X if ω is a bona fide Kähler-Einstein metric on the regular locus X_{reg} of X and the volume of ω on X_{reg} coincides with the top-intersection number $c_1(-K_X)^n[X]$. The existence of such a metric actually implies that the singularities are rather mild in the sense of the Minimal Model Program (MMP) in birational geometry [4], more precisely the singularities of X are *(Kawamata) log terminal (klt*, for short). In fact, even if X itself is smooth we will show that the singularity structure of the central fiber $\mathcal{X}_0$ of a given test configuration for X (or more precisely the log canonical threshold of $\mathcal{X}_0$) plays an important role in the metric analysis of the Donaldson-Futaki invariant $DF(\mathcal{X}, \mathcal{L})$, through the Lelong number l_0 at 0 of the L^2- type metric on a certain adjoint direct image sheaf over the base of the test configuration. Interestingly, this will single out test configurations $\mathcal{X}$ whose central

fibers have log canonical singularities as the "minimal" ones, in the sense that $l_0 = 0$. This appears to give a new metric incarnation of the appearance of log canonical singularities in the log MMP (which is reminiscent of the algebro-geometric results in [42]; compare Remark 3.1).

One motivation for considering singular Kähler-Einstein varieties X is that they naturally appear when taking Gromov-Hausdorff limits of smooth Kähler-Einstein varieties [27]. This is related to the expectation that one may be able to form *compact* moduli spaces of K-polystable Fano varieties if singular ones are included, or more precisely those with log terminal singularities; compare the discussions in [54] and [53] (where the surface case is considered).

Another motivation for allowing X to be singular comes from the toric setting considered in [2], where it was shown that the existence of a Kähler-Einstein metric on a toric Fano variety X is equivalent to X being $K-$polystable with respect to *toric* test configuration. In turn, this latter property is equivalent to the canonical rational weight polytope P associated to X having zero as its barycenter. However, the question whether the existence of a Kähler-Einstein metric on the toric variety X implies that X is K-polystable for *general* test configurations was left open in [2]. Combining the previous theorem with the results in [2] we thus deduce the following

Corollary 1.2. *A toric Fano variety is K-polystable iff it is K-polystable with respect to toric test configurations. In particular, if P is a reflexive lattice polytope, then the toric Fano variety X_P associated to P is $K-$polystable if and only if 0 is the barycenter of P.*

We recall that *reflexive* lattice polytopes P (i.e. those for which the dual P^* is also a lattice polytope) correspond to toric Fano varieties whose singularities are Gorenstein, i.e $-K_X$ is an ample line bundle (and not only a $\mathbb{Q}-$line bundle). This huge class of lattice polytopes plays an important role in string theory, as they give rise to many examples of mirror symmetric Calabi-Yau manifolds [1]. Already in dimension three there are 4319 isomorphism classes of such polytopes [40] and hence including *singular* Fano varieties leads to many new examples of K-polystable and K-unstable Fano threefolds (recall that there are, all in all, only 105 families of *smooth* Fano threefolds).

As explained in section 4.3 the theorem above extends to the logarithmic setting of Kähler-Einstein metrics on *log Fano varieties* (X, D), as considered in [4]. In particular, this shows that if D is an effective $\mathbb{Q}-$divisor with simple normal crossings, and coefficients < 1, on a projective manifold X such that the logarithmic first Chern class of (X, D) contains a Kähler-Einstein metric ω with *edge-cone singularities* along D in the sense of [26, 13, 34], then the pair (X, D) is log K-polystable in the sense of [26, 41, 52].

The starting point of the proof of Theorem 1.1 is the following result of independent interest, which expresses the Donaldson-Futaki invariant in terms of the Ding functional $\mathcal{D}$ (see formula 3.1):

Theorem 1.3. *Let X be a Fano variety with log terminal singularities, $(\mathcal{X}, \mathcal{L})$ a test configuration for $(X, -K_X)$ (assumed to have normal total space) and ϕ a locally bounded metric on $\mathcal{L}$ with positive curvature current. Setting $t := -\log |\tau|^2$ and denoting by $\phi^t = \rho(\tau)^* \phi_\tau$ the corresponding ray of locally bounded metrics on*

$-K_X$ the following formula holds:

$$(1.1) \qquad DF(\mathcal{X}, \mathcal{L}) = \lim_{t \to \infty} \frac{d}{dt} \mathcal{D}(\phi^t) + q,$$

where q is a non-negative rational number determined by the polarized scheme $(\mathcal{X}_0, \mathcal{L}_{|\mathcal{X}_0})$ with the following properties, if X is smooth:

- *$q = 0$ iff $\mathcal{X}$ is $\mathbb{Q}-$Gorenstein with $\mathcal{L}$ isomorphic to $-K_{\mathcal{X}/\mathbb{C}}$ and $\mathcal{X}_0$ is reduced and its normalization has log canonical singularities.*
- *In particular, if $\mathcal{X}_0$ is normal then $q = 0$ iff $\mathcal{X}_0$ is reduced and has log canonical singularities (and in particular $q = 0$ if it has log terminal singularities).*

More precisely, we will give an explicit expression for the number q in terms of a given log resolution $(\mathcal{X}, \mathcal{X}_0)$ (see Theorem 3.11). In order to prove Theorem 1.1 we apply Theorem 1.3 to a weak geodesic ray ϕ^t, emanating from the Kähler-Einstein metric on $-K_X$ (which is a critical point of the Ding functional). We can then exploit a result of Berndtsson [10] (and its generalization to singular Fano varieties in [4]) concerning the convexity of the Ding functional $\mathcal{D}$, also using a new triviality result for certain flat direct image sheaves, of independent interest (Proposition 3.3).

As for the proof of Theorem 1.3 it is based on the observation that $\mathcal{D}(\phi^t)$ extends to define a singular positively curved metric on a certain line bundle over the base $\mathbb{C}$ of the given test configuration, that we will accordingly call the *Ding line bundle*. To make the connection to $DF(\mathcal{X}, \mathcal{L})$ we use a result of Phong-Ross-Sturm [59] which expresses $DF(\mathcal{X}, \mathcal{L})$ in terms of the weight over 0 of another line bundle η over the base $\mathbb{C}$, involving certain Deligne pairings. This is also closely related to the intersection theoretic formulation of the Donaldson-Futaki invariant due to Wang [77] and Odaka [51], independently. The error term q in formula 1.1 can then be decomposed into two pieces where the first piece is the Lelong number l_0 at 0 of Ding metric referred to above, which is shown to coincide with the Lelong number at 0 of an L^2-type metric on a certain direct image sheaf. The non-negativity of l_0 then follows from the positivity results of Berndtsson-Paun for the L^2-metrics on direct images of adjoint line bundles [8, 11]. We show how to express l_0 explicitly in terms of a certain log canonical threshold of the central fiber $\mathcal{X}_0$ (Proposition 3.8). Finally, the vanishing properties of q are obtained using inversion of adjunction [38] (which can be seen as an algebro-geometric incarnation of the Ohsawa-Takegoshi extension theorem in complex analysis [19, 39]).

It should be pointed out that the information about the vanishing properties of q in Theorem 1.3 are not used in the proof of Theorem 1.1, but they appear to give a new link between differential geometry and the MMP (see Remark 3.1). Moreover, as discussed in section 5, the second point in Theorem 1.3 fits naturally into Tian's program [71] for establishing the existence part of the Yau-Tian-Donaldson conjecture - in particular when generalized to the setting of singular Fano varieties (compare Corollary 5.1).

We also give some applications of Theorem 1.3 to bounds on the Ricci potential and Perelman's $\lambda-$entropy functional [55] (see section 4.2), which can be seen as analogs of Donaldson's lower bound on the Calabi functional [24]. In particular, we obtain the following

Theorem 1.4. *Let X be an $n-$dimensional Fano manifold and set $V := c_1(X)^n$. If X is $K-$unstable, then Perelman's $\lambda-$entropy functional satisfies*

$$\sup_{\omega \in \mathcal{K}(X)} \lambda(\omega) < nV,$$

where $\mathcal{K}(X)$ denotes the space of all Kähler metrics in $c_1(X)$.

As is well-known $\lambda(\omega) \leq nV$ on the space $\mathcal{K}(X)$ and, as recently shown by Tian-Zhang [72] in their study of the Kähler-Ricci flow, if a Fano manifold X admits a Kähler-Einstein metric ω_{KE} then $\lambda(\omega_{KE}) = nV$, or more generally: if Mabuchi's K-energy is bounded from below on $\mathcal{K}(X)$, then supremum of λ is equal to nV. In the light of the Yau-Tian-Donaldson conjecture it seems thus natural to conjecture that X is $K-$semistable if and only if the supremum of λ is equal to nV (the "if direction" is the content of the previous theorem). In fact, a more precise version of Theorem 1.4 will be obtained, where the supremum of λ is explicitly bounded in terms of minus the supremum of the Donaldson-Futaki invariants over all (normalized) destabilizing test configurations for (X, L) (see Cor 4.5).

During the revision of the present paper the existence of Kähler-Einstein metrics on K-polystable smooth Fano varieties was finally settled by Chen-Donaldson-Sun [16] based on a modification of Tian's original program introduced by Donaldson [26], which uses metrics with conical singularities. In fact the proofs in [16] show that a Kähler-Einstein metric exists as soon as X is K-polystable with respect to *special* test configurations and hence combining the results in [16] with the main result of the present paper yields a new proof - not involving MMP - of the recent result of Li-Xu [42], saying that to test the $K-$polystability of a Fano manifolds it is enough to test it on special test configurations. Moreover, one also obtains a proof of an analog of a conjecture of Donaldson concerning "geodesic stability" saying that either a Fano manifold X admits a Kähler-Einstein metrics, or there exists a geodesic ray along which the Ding functional eventually becomes strictly decreasing (Theorem 4.1).

In section 5 we have included an outlook on the existence problem on singular Fano varieties, which is thus the missing piece in the Yau-Tian-Donaldson conjecture for projective varieties X polarized by $-K_X$. The opposite case of varieties polarized by K_X was very recently established in [6], building on [51].

1.1. **Further relations to previous work.** In the case when X is a smooth Fano manifold Theorem 1.3 (and its more precise version Theorem 3.11) should be compared with previous results of Ding-Tian [21] who considered the case when ϕ^t is a Bergman geodesic, induced by a fixed embedding in $\mathbb{P}^N$ by $-kK_X$ (and a $\mathbb{C}^*-$action on $\mathbb{P}^N$). In the case when the central fiber $\mathcal{X}_0$ is normal the results of Ding-Tian say that $DF(\mathcal{X}, \mathcal{L})$ is equal to the asymptotic slope of the Mabuchi functional (without any further restrictions on the nature of the singularities of $\mathcal{X}_0$). We also recall that in another direction Paul-Tian [56, I, Cor 1.2] and Phong-Ross-Sturm [59] considered the case of a general smooth and positively curved metric ϕ on $\mathcal{L} \to \mathcal{X}$, for a given test configuration $(\mathcal{X}, \mathcal{L})$ for a polarized manifold (X, L), but assumed that the total space $\mathcal{X}$ be smooth and then obtained a formula for $DF(\mathcal{X}, \mathcal{L})$ as the slope of the Mabuchi functional plus a correction term which vanishes if $\mathcal{X}_0$ is reduced.

It may also be illuminating to compare our proof of Theorem 1.1 with the original approach of Tian [69] in the case of a non-singular Fano variety. As shown by Tian

[69] assuming that the central fiber of test configuration is normal, and using the slope formula of Ding-Tian [21] referred to above, the Donaldson-Futaki invariant DF is expressed in terms of the asymptotics of *Mabuchi's K-energy* functional along a one-parameter family ϕ_k^t of Bergman metrics, i.e. a Bergman geodesic. The positivity properties of DF are then determined using that, in the presence of a Kähler-Einstein metric, the Mabuchi's K-energy functional is *proper* (if there are no non-trivial holomorphic vector fields on X), which is the content of deep result of Tian [69]. Here we thus show that the Mabuchi functional and the Bergman geodesic may be replaced by the Ding functional and a weak (bounded) geodesic, respectively, and the properness result with Berndtsson's convexity result. One technical advantage of the Ding functional is that, unlike the Mabuchi functional, it is indeed well-defined along a weak geodesic, as previously exploited in [10, 4] in the context of the uniqueness problem for Kähler-Einstein metrics. Thus the approach in this paper is in line with the programs of Phong-Sturm [60] and Chen-Tang [15] for calculating Donaldson-Futaki invariants by using (weak) geodesic rays associated to test configurations.

In the case when X is a smooth Kähler-Einstein Fano variety with $\mathrm{Aut}(X)_0$ trivial the properness of the Ding functional was shown by Tian [69] as a consequence of his properness result for the Mabuchi functional. It was later shown in [61] that if center of the group $\mathrm{Aut}(X)_0$ is finite then the Ding functional is still proper (in an appropriate sense), but the properness in the case of general Kähler-Einstein manifold is still open. The generalization of the properness result (even when $\mathrm{Aut}(X)_0$ is trivial) to singular Fano varieties and more generally log Fano varieties also appears to be a challenging open problem. Anyway, these subtle issues are bypassed in the present approach.

Organization. After having recalled some preliminary material in Section 2 the formula relating the Donaldson-Futaki invariant (Theorem 1.3 above) to the Ding function is established in Section 3 and then used, by exploiting positivity results for L^2-type metrics on direct images, to prove Theorem 1.1 concerning K-polystability. Section 3.3 also contains a detailed study of the singularities of the L^2-type metrics which is of independent interest, but not needed for the proof of Theorem 1.1. In section 4.2 various ramifications and applications are given to (i) an analog of conjecture of Donaldson (ii) bounds on the Ricci potential and Perelman's entropy functional and (iii) the log Fano setting. The paper is concluded with an outlook in Section 5 on the existence problem for Kähler-Einstein metrics on singular Fano varieties is given.

2. Preliminaries

In this section we will setup the notation and recall the basic tools to be used in the proofs of the main results.

2.1. Kähler-Einstein metrics on Fano varieties and log pairs.

2.1.1. *Fano varieties and log pairs.* Let X be an $n-$dimensional normal compact projective algbraic variety. In analytic terms normality just means that any holomorphic function f on $U \cap X_{reg}$, where X_{reg} denotes the regular locus of X and $U \subset X$ is open, extends holomorphically to U in the strong sense, i.e. f is the restriction to U of holomorphic function in $\mathbb{C}^m$ under some local holomorphic embedding $F : U \hookrightarrow \mathbb{C}^m$ (after perhaps shrinking U). In particular, $\mathrm{codim}(X - X_{reg}) \geq 2$. More generally, the corresponding strong extension property holds for any plurisubharmonic (psh, for short) function ϕ on U (see [6] and references therein for a more detailed discussion of pluripotential theory on singular analytic spaces). By definition, X is said to be a *Fano variety* if the anti-canonical line bundle $-K_X := \det(TX)$ defined on the regular locus X_{reg} of X extends to an ample $\mathbb{Q}-$line bundle on X, i.e. there exists a positive integer m such that the mth tensor power $-mK_{X_{reg}}$ extends to an ample line bundle over X. Since X is normal this equivalently means that the anti-canonical divisor $-K_X$ of X defines an ample $\mathbb{Q}-$line bundle. More generally we will (in particular in Section 4.3) consider *log pairs* (X, D) in the sense of birational geometry [39]: i.e. X is normal and D is a $\mathbb{Q}-$divisor on X such that $K_X + D$ is a $\mathbb{Q}-$Cartier, i.e. defines a $\mathbb{Q}-$line bundle (called the log canonical bundle of (X, D). By definition, a *log resolution* of a log pair (X, D) is a proper birational morphism $X' \to X$ such that $p^*D + E$ has simple normal crossings, where E is the exceptional divisor of p. Then

$$(2.1) \qquad\qquad p^*(K + D) = K_{X'} + D',$$

for a $\mathbb{Q}-$divisor D' on X' (by Hironaka's theorem we may and will assume that p is an isomorphism away from $p^{-1}(X_{sing} \cup \mathrm{Supp} D_{sing})$. A log pair (X, D) is said to be *log canonical, or lc* for short, if the coefficients c_i of D' (along the corresponding prime divisors) satisfy $c_i \leq 1$. Similarly, (X, D) is said to be *(Kawamata) log terminal,* or *klt for short, if $c_i < 1$*. Setting $D = 0$ these notations also apply to the normal variety X, which is thus said to have *log canonical (log terminal) singularities* if $(X, 0)$ is log canonical (log terminal). In practice we will in what follows only consider Fano varieties X with log terminal singularities (the corresponding analytical characterization will be recalled below), but even if X is smooth the notion of log canonical singularities will be important in the study of test configurations $\mathcal{X}$ for X.

2.1.2. *Singular metrics on line bundles and (multi-) sections.* Throughout the paper we will use additive notation for line bundles, as well as metrics. This means that a metric $\|\cdot\|$ on a line bundle $L \to X$ is represented by a collection of local functions $\phi(:= \{\phi_U\})$ defined as follows: given a locally trivializing section of L, i.e. a local generator s_U of the invertible sheaf $\mathcal{O}(L)$ on an open subset $U \subset X$ we set $\phi_U := -\log\|s_U\|^2$, where ϕ_U is upper semi-continuous. It will be convenient to identify the additive object ϕ with the metric it represents. Of course, ϕ_U depends on s_U but the curvature current

$$dd^c\phi := \frac{i}{2\pi}\partial\bar{\partial}\phi_U$$

is globally well-defined on X and represents the first Chern class $c_1(L)$, which with our normalizations lies in the integer lattice of $H^2(X, \mathbb{R})$. We will say that a (singular) metric ϕ is *psh* (or have *positive curvature current*), $\phi \in PSH(X, L)$, if

ϕ_U is always psh (and in particular $dd^c\phi \geq 0$ holds in the sense of currents). By the normality of X the injection $X_{reg} \hookrightarrow X$ alows us to identity $PSH(X,L) = PSH(X_{reg}, L)$ and hence, given a smooth resolution $\pi : X' \to X$, we can also identify $PSH(X,L)$ with $PSH(X', \pi^*L)$, using the pull-back π^*. We will denote by $\mathcal{H}_b(X,L)$ the subspace of $PSH(X,L)$ consisting of all *locally bounded* metrics. Fixing $\phi_0 \in \mathcal{H}_b(X,L)$ and setting $\omega_0 := dd^c\phi_0$ the map $\phi \mapsto v := \phi - \phi_0$ thus gives an isomorphism between the space $\mathcal{H}_b(X,L)$ and the space $PSH(X,\omega_0) \cap L^\infty(X)$ of all bounded ω_0-psh functions, i.e. the space of all bounded usc functions v on X such that $dd^cv + \omega_0 \geq 0$. Similarly, a metric ϕ will be said to be *smooth* if ϕ_U is the restriction to U of a smooth function under a local embedding as above. A special class of smooth metrics with strictly positive curvature is given by *Bergman metrics*, i.e. metrics of the form ϕ_k/k, where ϕ_k is obtained by restricting the Fubini-Study metric ϕ_{FS} on $\mathbb{P}^N$ under a given Kodaira embedding of X in $\mathbb{P}^N$ induced by kL, for some k sufficiently large.

In particular, if s is a holomorphic section of $L \to X$ then $\phi_s := \log|s|^2$ defines a singular metric on L with positive curvature current $[D]$, i.e. integration along the zero divisor of s (taking multiplicities into account). More generally, it will often be convenient to use the terminiology of holomorphic *multisections* of L, which by definition consists of a pair (r, s_r), where r is a positive integer r and $s_r \in H^0(X, rL)$ and where two pairs (r, s_r) and $(r', s_{r'})$, where $r' \geq r$, are identified if there exists a positive integer p such that $r' = pr$ and $s_{r'} = s_r^{\otimes p}$. Denoting by s such an equivalence class $\phi_s := \frac{1}{r}\log|s_r|^2$ defines a singular metric on L with curvature current $[D]$, where D is the $\mathbb{Q}-$divisor defined by the zero-divisor of s (i.e. D is, by definition, equal to $1/r$ times the zero divisor of s_r). Accordingly we will occasionally also write $\phi_D := \phi_s$. More generally, abusing notation slightly, the statement "let s be a holomorphic multisection of L" will in the follopwing mean that we tacitly fix a pair (r, s_r) defining s and work with the bona fide section s_r and then make the appropriate scalings by r.

2.1.3. *Canonical measures.* In the special case when $L = -K_X$ any given metric on $\phi \in \mathcal{H}_b(X,L)$ induces a measure μ_ϕ on X, which may be concretely defined as follows: if U is a coordinate chart in X_{reg} with local holomorphic coordinates $z_1, ..., z_n$ we let ϕ_U be the representation of ϕ with respect to the local trivialization of $-K_X$ which is dual to $dz := dz_1 \wedge \cdots \wedge dz_n$. Then we define the restriction of μ_ϕ to $U \subset X_{reg}$ as

$$\mu_\phi = e^{-\phi_U} i^{n^2} dz \wedge d\bar{z}$$

This expression is readily verified to be independent of the local coordinates z and hence defines a measure μ_ϕ on X_{reg} which we then extend by zero to all of X. Note that since $-K_X$ is assumed $\mathbb{Q}-$Cartier we may cover X with a finite number of open sets V (not necessarily contained in X_{reg}) such that the restriction to V of μ_ϕ is given by $1_{X_{reg}} i^{n^2} \alpha_U \wedge \overline{\alpha_U} e^{-\phi_U}$, where α_U is a trivializing section of $K_{X|U}$ (whose restriction to $U \cap X_{reg}$ may thus be identified with a holomorphic $(n,0)-$form) and where $\phi_U = -\log\|s\|^2$ for s the dual of α. The Fano variety X has log terminal singularities (as defined above) precisely when the total mass of μ_ϕ is finite for some and hence any $\phi \in \mathcal{H}_b(X,L)$ (see Lemma 3.7). Abusing notation slightly we will often use the suggestive notation $e^{-\phi}$ for the measure μ_ϕ. This notation is compatible with the usual notation used in the context of adjoint bundles: if s is a holomorphic section of $L + K_X \to X$ and ϕ is a metric on L then $|s|^2 e^{-\phi}$

(sometimes written as $i^{n^2} s \wedge \bar{s} e^{-\phi}$) may be naturally identified with a measure on X. In particular, letting $L = -K_X$ and taking s to be the canonical section 1 in the trivial line bundle $L + K_X$ gives us back the measure μ_ϕ. More generally, if (X, D) is a log pair (see section 4.3 below) and ϕ is a locally bounded metric on $-(K_X + D)$ then one obtains a measure $\mu_{(X,D,\phi)}$ on X by using the natural identification between $-(K_X + D)$ and $-K_X$ on the complement of the support of D in X and extending by zero to all of X (compare [4]). Abusing notation, we will sometimes write $\mu_{(X,D;\phi)} = e^{-(\phi + \log |s_D|^2)}$, where s_D is the (multi-) section cutting out D. These constructions are compatible with taking resolutions p, as in 2.1: if ϕ is a metric on $-(K_X + D)$ then $p^*\phi$ is a metric on $-(K_{X'} + D')$ and $p_*(\mu_{(X',D';p^*\phi)}) = \mu_{(X,D;\phi)}$.

In the relative setting of a morphism $\pi : \mathcal{X} \to \mathbb{C}$ from a normal $\mathbb{Q}$−Gorenstein variety such that π is smooth over $\mathbb{C}^*$ (or more generally, $\mathcal{X}_\tau$ is reduced and defines a variety X_τ with log terminal singularities for $\tau \neq 0$) we denote by $K_{\mathcal{X}/\mathbb{C}} := K_{\mathcal{X}} - \pi^* K_{\mathbb{C}}$ the relative canonical line bundle (viewed as a $\mathbb{Q}$−line bundle). Denoting by τ the standard affine coordinate on $\mathbb{C}$ we will use $\pi^* d\tau$ to trivialize $\pi^* K_{\mathbb{C}}$ over $\mathcal{X}$. Accordingly, we will identify an element $s \in \mathcal{O}(U, K_{\mathcal{X}/\mathbb{C}})$ with a holomorphic $(n+1)$−form α on $U \cap \mathcal{X}_{reg}$ (i.e. $s = \alpha \otimes \pi^* \frac{\partial}{\partial \tau}$). Moreover, if $U = \pi^{-1}(V)$ for $V \subset \mathbb{C}^*$, then the natural isomorphism $K_{\mathcal{X}/\mathbb{C}|X_\tau} \approx K_{X_\tau}$ allows us to identify the restrictions s_τ of $s \in \mathcal{O}(U, K_{\mathcal{X}/\mathbb{C}})$ to X_τ with a family of holomorphic n−forms on X_τ (i.e. $\alpha = s_\tau \wedge \pi^* d\tau$). Similarly, replacing $K_{\mathcal{X}/\mathbb{C}}$ with $\mathcal{L} + K_{\mathcal{X}/\mathbb{C}}$ for a given line bundle $\mathcal{L} \to \mathcal{X}$ equipped with a metric ϕ we will use the symbolic notation $|s|^2 e^{-\phi}$ for the corresponding measures on $U \subset \mathcal{X}$ and $|s_\tau|^2 e^{-\phi_\tau}$ for the corresponding family of measures over $V \subset \mathbb{C}^*$.

2.1.4. *Kähler-Einstein metrics.*
Following [4] ω is said to be a *Kähler-Einstein metric* on X if it is Kähler metric on X_{reg} with constant Ricci curvature, i.e. Ric $\omega = \omega$ on X_{reg} and $\int_{X_{reg}} \omega^n = c_1(-K_X)^n$. By [4, Lemma 3.6 and Proposition 3.8] this equivalently means that the Fano variety X in fact has log terminal singularities and ω extends to a Kähler current defined on the whole Fano variety X, such that ω is the curvature current of a locally bounded (and in fact continuous) metric ϕ_{KE} on the $\mathbb{Q}$−line bundle $-K_X$ such that

$$(2.2) \qquad (dd^c \phi_{KE})^n = V e^{-\phi_{KE}} / \int_X e^{-\phi_{KE}}.$$

The measure appearing the left hand side above is the *Monge-Ampère measure* of ϕ_{KE} defined in sense of pluripotential theory, i.e. using the Bedford-Taylor product between positive closed currents with locally bounded potentials (see [4] and references therein for the general singular setting).

2.2. **K-polystability and test configurations.**
Let us start by recalling Donaldson's general definition [23] of K-stability of a polarized variety (X, L) generalizing the original definition of Tian [69]. First, a general *test configuration* $(\mathcal{X}, \mathcal{L}, \pi, \rho)$ for (X, L) consists of a scheme $\mathcal{X}$ with a $\mathbb{C}^*$−equivariant flat surjective morphism $\pi : \mathcal{X} \to \mathbb{C}$ (where the base $\mathbb{C}$ is equipped with its standard $\mathbb{C}^*$−action) and a relatively ample line bundle $\mathcal{L} \to \mathcal{X}$ with a $\mathbb{C}^*$−action ρ on $\mathcal{L}$ and such that $(X_1, \mathcal{L}_{|X_1}) = (X, rL)$ for some integer r. In fact, by allowing $\mathcal{L}$ to be a $\mathbb{Q}$−line bundle we may as well assume that $r = 1$. More specifically, following [42] we will

assume that the total space $\mathcal{X}$ is normal. Then the morphism π is automatically flat [35, Prop 9.7]).

To simplify the notation we will usually surpress the dependence on π and ρ denote a test configuration by $(\mathcal{X}, \mathcal{L})$. Occasionally we will use the notation X_0 for the reduction of the central fiber $\mathcal{X}_0$, i.e. the projective variety X_0 underlying the scheme theoretic central fiber $\mathcal{X}_0$. For a *semi-test configuration* we only require that $\mathcal{L}$ be relatively semi-ample. We recall that the total space $\mathcal{X}$ of a test configuration may, using the relative linear systems defined by $r\mathcal{L}$ for r sufficiently large, be equivariantly embedded as a subvariety of $\mathbb{P}^N \times \mathbb{C}$ so that $r\mathcal{L}$ becomes the pull-back of the relative $\mathcal{O}(1)-$hyperplane line bundle over $\mathbb{P}^N \times \mathbb{C}$. We will denote by ϕ_{FS} the metric on $\mathcal{L}$ obtained by restriction of the fiberwise Fubini-Study metrics on $\mathbb{P}^N \times \{\tau\}$ (see [64, Proposition 3.7] and the beginning of Section 5 [24]).

The *Donaldson-Futaki invariant* $DF(\mathcal{X}, \mathcal{L})$ of a test configuration is defined as follows: consider the N_k-dimensional space $H^0(\mathcal{X}_0, k\mathcal{L}_{|\mathcal{X}_0})$ over the central fiber $\mathcal{X}_0$ and let w_k be the weight of the $\mathbb{C}^*-$action on the complex line $H^0(\mathcal{X}_0, k\mathcal{L}_{|\mathcal{X}_0})$. Then the Donaldson-Futaki invariant of $DF(\mathcal{X}, \mathcal{L})$ is defined as minus the sub-leading coefficient in the expansion of w_k/kN_k in powers of $1/k$ (up to normalization):

$$\frac{w_k(\det H^0(\mathcal{X}_0, k\mathcal{L}_{|\mathcal{X}_0}))}{kN_k} = c_0 - \frac{1}{k}\frac{1}{2}DF(\mathcal{X}, \mathcal{L}) + O(\frac{1}{k^2}),$$

where $N_k := \dim(H^0(X_0, kL_0)$. The polarized variety (X, L) is said to be *K-semistable* if, for any test configuration, $DF(\mathcal{X}, \mathcal{L}) \geq 0$ and *K-(poly)stable* if moreover equality holds iff $(\mathcal{X}, \mathcal{L})$ is (equivariantly) isomorphic to $(X \times \mathbb{C}, p_1^*L)$. We also recall that (X, L) is said to be $K-unstable$ if it is not K-semistable, i.e. there exists a *destabilizing test configuration* in the sense that $DF(\mathcal{X}, \mathcal{L}) < 0$.

Example 2.1. Let V be a holomorphic vector field on X of type $(1, 0)$ with a fixed lift to $L \to X$. We will say that V *generates a* $\mathbb{C}^*-$*action on X*, denoted by $\rho_{(X,V)}$, if $\frac{d}{dt}\rho_{(X,V)}(e^{-t/2}) = e^{t\mathrm{Re}V}$, for $t \in \mathbb{R}$ (in other words the standard additive group $(\mathbb{C}, +)$ action on X determined by the complex flow of V descends to a multiplicative action of $\mathbb{C}^*$ on (X, L) under the homomorphism $\mathbb{C} \to \mathbb{C}^*, t \mapsto e^{-t/2}$). Such a vector field V determines a product test configuration $(\mathcal{X}, \mathcal{L}, \pi, \rho)$ by setting $(\mathcal{X}, \mathcal{L}, \pi) = (X \times \mathbb{C}, \mathcal{L} = p_1^*L, p_2)$ and defining the action $\rho : \mathbb{C}^* \times \mathcal{X} \to \mathcal{X}$ by $(\lambda, (x, \tau)) \mapsto (\rho_V(x), \lambda\tau)$. Note that the original action of $\rho_{(X.V)}$ on X may be identified with the restricted action of ρ on $\mathcal{X}_0$ and $DF(\mathcal{X}, \mathcal{L}, \rho)$ coincides with Futaki's invariant $F(X, V)$. Since, $F(X, V) = -F(X, V)$ a necessary condition for the K-polystability of (X, L) is that $DF(\mathcal{X}, \mathcal{L}, \rho) = 0$ for any V as above and a necassary condition for K-stability is that X admits no holomorphic vector fields as above (since, $(\mathcal{X}, \mathcal{L}, \rho)$ is equivariantly isomorphic to $(\mathcal{X}, \mathcal{L}, \rho_{triv})$, where $\rho_{triv}(\lambda, (x, \tau) = (x, \lambda\tau)$, iff $V = 0$).

In this paper we will be concerned with test configurations $(\mathcal{X}, \mathcal{L})$ for a Fano variety with its anti-canonical polarization, i.e. X is a Fano variety and $L = -K_X$ so that the restriction of $\mathcal{L}$ to the complement $\mathcal{X}^*$ of the central fiber coincides with the $\mathbb{Q}-$line bundle defined by the dual of the relative canonical divisor $K_{\mathcal{X}^*/\mathbb{C}} := K_{\mathcal{X}^*} - \pi^*K_{\mathbb{C}}$.

2.2.1. $\mathbb{Q}-Gorenstein$ *and special test configurations.* In general, $K_{\mathcal{X}/\mathbb{C}}$ does not extend as a $\mathbb{Q}-$line bundle over the central fiber, but following [42] we say that a normal variety $\mathcal{X}$ with a $\mathbb{C}^*-$equivariant surjective morphism π to $\mathbb{C}$ is a *special test configuration for the Fano variety X* if $\mathcal{X}_1 = X$, the total space $\mathcal{X}$ is $\mathbb{Q}-$Gorenstein

and the central fiber is reduced and irreducible and defines a Fano variety X_0 with log terminal singularities.

Lemma 2.2. *Let $(\mathcal{X}, \mathcal{L})$ be a general test configuration (with a priori non-normal total space) for $(X, -K_X)$, where X is a Fano variety. Assume that the central fiber $\mathcal{X}_0$ is normal. Then $\mathcal{X}$ and $\mathcal{X}_0$ are both normal $\mathbb{Q}-$Gorenstein varieties and $\mathcal{L}_{|\mathcal{X}_0}$ is isomorphic to $-K_{\mathcal{X}_0}$, i.e. $\mathcal{L}$ is isomorphic to $-K_{\mathcal{X}/\mathbb{C}}$. Moreover, if X_0 has log terminal singularities, then so has $\mathcal{X}$. In other words, a test configuration is special iff the central fiber is reduced and the underlying variety X_0 has log terminal singularities.*

Proof. This is essentially well-known, but for completeness we provide a proof (thanks to Yuji Odaka for his help in this matter). It follows from general commutative algebra that if $\pi : \mathcal{X} \to \mathbb{C}$ is a morphism projective and flat over $\mathbb{C}$, with normal fibers, then $\mathcal{X}$ is also normal [29, Theorem 1.101]. In particular, the canonical divisor $K_{\mathcal{X}}$ is a well-defined Weil divisor. By assumption $-K_{\mathcal{X}}$ and $\mathcal{L}$ are $\mathbb{Q}-$Cartier and linearly equivalent on $\mathcal{X}^*$ and hence $K_{\mathcal{X}} + \mathcal{L}$ is linearly equivalent to a Weil $\mathbb{Q}-$divisor D supported in the central fiber. But the central fiber is Cartier (since it is cut out by the function $\pi^* \tau$) and hence, since it is assumed irreducible $-m(K_{\mathcal{X}} + \mathcal{L})$ is linearly equivalent to a multiple of $\mathcal{X}_0$, which means that $-mK_{\mathcal{X}}$ is a sum of Cartier divisors, hence Cartier, i.e. $\mathcal{X}$ is $\mathbb{Q}-$Gorenstein. More precisely, $-mK_{\mathcal{X}}$ is linearly equivalent to $\mathcal{L}$ modulo a pull back from the base and thus it follows from adjunction that the restriction of $\mathcal{L}$ to $\mathcal{X}_0$ is linearly equivalent to $-mK_{\mathcal{X}_0}$, which concludes the proof of the first statement. Finally, if $\mathcal{X}_0$ has log terminal singularities it follows from inversion of adjunction that $\mathcal{X}$ also has log terminal singularities [39, Theorem 7.5] (see also the beginning of Section 3). $\square$

Since the Donaldson-Futaki is independent of the lift of the $\mathbb{C}^*-$action on $\mathcal{X}$ we may and will in the case when $\mathcal{L} := -K_{\mathcal{X}/\mathbb{C}}$ assume that the $\mathbb{C}^*-$action on $\mathcal{L} := -K_{\mathcal{X}/\mathbb{C}}$ is the canonical lift of the $\mathbb{C}^*-$action on $\mathcal{X}$ to $-K_{\mathcal{X}/\mathbb{C}}$.

2.3. **Deligne pairings, the energy functional $\mathcal{E}$ and the line bundle η.** The Donaldson-Futaki invariant may also be expressed in terms of Deligne pairings [59] (also called intersection bundles [48]). First recall that if $\pi : \mathcal{X} \to B$ is a proper flat projective morphism of relative dimension n (between normal schemes) and $L_0, ..., L_n$ are line bundles over $\mathcal{X}$ then the Deligne pairing $\langle L_0, ..., L_n \rangle$ is a line bundle over B, which depends in a multilinear fashion on L_i [75, **?**] and satisfies

$$c_1 \langle L_0, ..., L_n \rangle = \pi_* \left(c_1(L_0) \wedge \cdots \wedge c_1(L_n) \right)$$

In particular, if B is a non-singular projective curve then

(2.3)
$$\deg \langle L_0, ..., L_n \rangle = L_0 \cdots L_n$$

In our case $\mathcal{X}$ will be a normal variety (defined over $\mathbb{C}$) and $B = \mathbb{C}$. Given Hermitian metrics $\phi_0, ..., \phi_n$ on $L_0, ..., L_n$ there is a natural Hermitian metric $\langle \phi_0, ..., \phi_n \rangle$ on $\langle L_0, ..., L_n \rangle$ [75] which has the following fundamental properties[1]:

(1) Its curvature is given by

(2.4)
$$dd^c \langle \phi_0, ..., \phi_n \rangle = \pi_*(dd^c \phi_1 \wedge \cdots \wedge dd^c \phi_n)$$

---

[1]Following [32] the construction in [75] seems to require that π be Cohen-Macaulay in order to define the metric on $\langle L_0, ..., L_n \rangle$ by induction over the dimension n. Anyway all our arguments will be carried out on a non-singular resolution of $\mathcal{X}$ where the constructions in [75, 32] apply.

(2) If ϕ and ψ are metrics in $\mathcal{H}(L)$ with $\langle\phi\rangle$ and $\langle\psi\rangle$ denoting the induced metrics on the top Deligne pairing $\langle L, ..., L\rangle$ in the absolute case when B is a point, then we have the following "change of metric formula":

$$\langle\phi\rangle - \langle\psi\rangle = \sum_{j=0}^{n} \int_X (\phi - \psi)(dd^c\phi)^{n-j} \wedge (dd^c\psi)^j$$

In order to define $\langle\phi\rangle$ for ϕ merely locally bounded, i.e. in $\mathcal{H}_b(L)$, we fix a reference metric ψ_0 in $\mathcal{H}(L)$ and first set

$$(2.5) \qquad \mathcal{E}(\phi) := \mathcal{E}(\phi, \psi_0) := \frac{1}{(n+1)} \sum_{j=0}^{n} \int_X (\phi - \psi)(dd^c\phi)^{n-j} \wedge (dd^c\psi)^j$$

The functional $\mathcal{E}(\phi, \psi)$ is well-defined and finite for any $\phi, \psi \in \mathcal{H}(L)_b$, using the Bedford-Taylor product between the corresponding currents, and the functional $\mathcal{E}(\phi)$ on $\mathcal{H}_b(L)$ coincides with the restriction to $\mathcal{H}_b(L)$ of the functional E in [3, 4, Section 1.4] defined on the whole space of singular metrics on L with positive curvature current. In particular, the functional $\mathcal{E}(\phi)$ is a primitive for the one-form on $\mathcal{H}_b(L)$ defined by the Monge-Ampère measure, i.e.

$$(2.6) \qquad \frac{d}{dt}_{|t=0} \mathcal{E}(\phi_0(1-t) + \phi_1 t) = \int (\phi_1 - \phi_0)(dd^c\phi_0)^n$$

Now, for any $\phi \in \mathcal{H}_b(L)$ we can simply define the corresponding metric $\langle\phi\rangle$ on the Deligne pairing by

$$\langle\phi\rangle := \langle\psi\rangle + (n+1)\mathcal{E}(\phi, \psi)$$

for any fixed $\psi \in \mathcal{H}(L)$. It follows immediately from the cocycle formula for $\mathcal{E}(\phi, \psi)$ (which in turn follows from the variational property 2.6) that $\langle\phi\rangle$ is independent of the choice of ψ and still satisfies the change of metric formula above, i.e.

$$(2.7) \qquad\qquad\qquad \langle\phi\rangle - \langle\psi\rangle = (n+1)\mathcal{E}(\phi, \psi)$$

Similarly, the first property 1 above also holds in the singular setting of locally bounded metrics, by approximation, since the Bedford-Taylor product is local and continuous under local decreasing limits.

Remark 2.3. More generally, by the results in [3] the metric ϕ_D can be defined as long as $\phi_0, ..., \phi_n$ are in the finite energy space $\mathcal{E}^1(X, L)$, but the locally bounded setting above will be adequate for our purposes.

Let us now come back to the general setting of a test configuration $\mathcal{L} \to \mathcal{X} \to \mathbb{C}$ for a polarized variety (X, L). Under appropriate regularity assumptions it was shown in [59] that the Donaldson-Futaki invariant of a test configuration $(\mathcal{X}, \mathcal{L})$ is the weight over 0 :

$$(2.8) \qquad\qquad\qquad DF(\mathcal{X}, \mathcal{L}) = w_0(\eta)$$

of the following $\mathbb{Q}$−line bundle over $\mathbb{C}$:
(2.9)
$$\eta := \frac{1}{(n+1)L^n} \left(\mu \langle\mathcal{L}, ..., \mathcal{L}\rangle - (n+1) \langle -K_{\mathcal{X}/\mathbb{C}}, \mathcal{L}..., \mathcal{L}\rangle \right) \quad, \mu := n(-K_X) \cdot L^{n-1}/L^n$$

where in the case when $L = -K_X$ we have $\mu = n$. More precisely, it was shown in [59] that, up to natural isomorphisms, the Knudson-Mumford expansion of the determinant line bundle $\det(\pi_*(kL)) \to \mathbb{C}$ (with fibers $\det H^0(X_\tau, kL_\tau)$) satisfies

$$\det(\pi_*(kL))/kN_k = \frac{1}{(n+1)L^n} \langle \mathcal{L}, ..., \mathcal{L} \rangle - \frac{1}{k}\frac{1}{2}\eta + O(\frac{1}{k^2})$$

and η is thus naturally isomorphic to Tian's CM-line bundle [56]. The proofs in [59] were carried out under the assumption that the total space $\mathcal{X}$ and the central fiber X_0 be non-singular (in particular, there are no multiple fibers), but as pointed out in [59] the regularity assumptions can be relaxed. Here we observe that the formula 2.8 holds as long as $\mathcal{X}$ is $\mathbb{Q}-$Gorenstein as a consequence of the intersection theoretic expression for the Donaldson-Futaki invariant recalled in the following proposition. The expression involves the "$\mathbb{C}^*-$equivariant compactification" $\bar{\mathcal{L}} \to \bar{\mathcal{X}} \to \mathbb{P}^1$ of $\mathcal{L} \to \mathcal{X} \to \mathbb{C}$ obtained by using the given $\mathbb{C}^*-$action ρ on $(\mathcal{X}, \mathcal{L})$ to identify $(\mathcal{X}^*, \mathcal{L}_{|\mathcal{X}^*})$ with $(X \times \mathbb{C}, p_1^*L)$, which trivially extends to $(X \times \mathbb{P}^1 - \{0\}, p_1^*L)$ so that $\bar{\rho}$ acts trivially over $\infty \in \mathbb{P}^1$.

Proposition 2.4. *Let $(\mathcal{X}, \mathcal{L}, \rho)$ be a test configuration (in particular, $\mathcal{X}$ is normal) and denote by $\bar{\mathcal{L}} \to \bar{\mathcal{X}} \to \mathbb{P}^1$ the corresponding $\mathbb{C}^*-$equivariant compactification over $\mathbb{P}^1$. Then*

(2.10) $$(n+1)L^n(DF(\mathcal{X}, \mathcal{L})) = \mu\bar{\mathcal{L}} \cdot \bar{\mathcal{L}} \cdots \bar{\mathcal{L}} + (n+1)K_{\bar{\mathcal{X}}/\mathbb{P}^1} \cdot \bar{\mathcal{L}} \cdots \bar{\mathcal{L}},$$

where $K_{\bar{\mathcal{X}}/\mathbb{P}^1}$ is the relative canonical divisor (viewed as a Weil divisor).

The previous formula is shown in [77, Proposition 17] under the assumption that $\mathcal{X}$ is $\mathbb{Q}-$Gorenstein, so that $K_{\mathcal{X}/\mathbb{C}}$ is well-defined as a $\mathbb{Q}-$line bundle and in [51, Theorem 3.2] for $\mathcal{X}$ normal (or more generally, Gorenstein in codimension one. See also [42, Proposition 6] for a simple direct proof.

Proposition 2.5. *Let $(\mathcal{X}, \mathcal{L})$ be a test configuration such that $\mathcal{X}$ is $\mathbb{Q}-$Gorenstein. Then $DF(\mathcal{X}, \mathcal{L}) = w_0(\eta)$, where η is the $\mathbb{Q}-$line bundle defined by formula 2.9.*

Proof. If $K_{\bar{\mathcal{X}}}$ is well-defined as a $\mathbb{Q}-$Cartier divisor (i.e. as a $\mathbb{Q}-$line bundle) then it follows from the previous proposition and the standard push-forward formula that

$$(n+1)L^n(DF(\mathcal{X}, \mathcal{L})) = \int_{\mathbb{P}^1} \mu\pi_*(\mu(c_1(\bar{\mathcal{L}})^{n+1}) + (n+1)c_1(K_{\bar{\mathcal{X}}/\mathbb{P}^1}) \wedge c_1(\bar{\mathcal{L}})^n),$$

which, according to 2.3 coincides with the degree of the corresponding sum $\bar{\eta} \to \mathbb{P}^1$ of extended Deligne pairings. But, this is nothing but the weight over 0 of the $\mathbb{C}^*-$action on $\eta_{|\tau=0}$. Indeed, if F is a line bundle over $\mathbb{P}^1$ equipped with a $\mathbb{C}^*-$action covering the standard action on $\mathbb{P}^1$ (viewed as a compactification of $\mathbb{C}^*$) acting trivially on the line $F_{|\infty}$, then

(2.11) $$w_0(F) = \deg(F),$$

$\square$

2.3.1. *A singular Kempf-Ness formula for weights involving the Lelong number.* Next we give a generalization of the Kempf-Ness type formula for the weight appearing in Geometric Invariant Theory [49], which is essentially equivalent to Lemma 6 in [59]. Its formulation involves the classical notion of a *Lelong number* $l_0(\Phi)$ at zero of a subharmonic function Φ on the unit-disc in $\mathbb{C}$, which may be defined as

the sup over all numbers λ such that $\Phi(\tau) \leq \lambda \log |\tau|^2$ close to $\tau = 0$ (equivalently, $l_0(\Phi)$ is the mass of curvature current of Φ at the origin:

$$(2.12) \qquad l_0(\Phi) = \int_{\{0\}} (dd^c \Phi).$$

Lemma 2.6. *Let F be a line bundle over the unit-disc Δ in $\mathbb{C}$ equipped with a $\mathbb{C}^*-$action ρ compatible with the standard one on Δ and fix an S^1-invariant metric Φ on F with positive curvature current. Then the weight w_0 of the $\mathbb{C}^*$ action on the complex line F_0 is given by the following formula involving right derivatives:*

$$(2.13) \qquad w_0 = -\lim_{t\to\infty} \frac{d}{dt} \log \|\rho(\tau)s_1\|_\Phi^2 + l_0(\Phi)$$

for $t = -\log|\tau|^2$ and s_1 a fixed element in the complex line F_1.

Proof. This can be proved exactly as in Lemma 6 in [59], using that $l_0(\phi) = -\lim_{t\to\infty} \frac{d}{dt}\phi(e^{-t/2})$ if ϕ is subharmonic on Δ and S^1-invariant. Alternatively, a highbrow proof can be given as follows, using the equivariant compactification $\bar{F} \to \mathbb{P}^1$, as in the discussion preceeding Proposition 2.4 (and by extending Φ to a metric on $\bar{F} \to \mathbb{P}$, smooth close to $\infty \in \mathbb{P}^1$) : the section $s_\tau := \rho(\tau)s_1$ defines a trivializing holomorphic section of $\bar{F} \to \mathbb{C}$ and hence, setting $v(\tau) := -\log\|\rho(\tau)s_1\|_\Phi^2$ on $\mathbb{C}^*$ we can decompose $\deg \bar{F} = \int_{\mathbb{P}^1} dd^c\Phi = \int_{\mathbb{C}^*} dd^c v + \int_{\{0\}} dd^c\Phi$. But $\int_{\mathbb{C}} dd^c v(\tau) = \int_{-\infty}^{\infty} d(\frac{dv(e^{-t/2})}{dt}) = \lim_{t\to\infty} dv(t)/dt - 0$, which concludes the proof using formulae 2.12 and 2.11. $\qquad \square$

2.4. The Monge-Ampère equation on $\mathcal{X}$ and geodesic rays.

Next we explain how to attach a canonical metric on the line bundle $\mathcal{L} \to \mathcal{X}$ over a test configuration to a given metric ϕ on $L \to X$ and the relation to weak geodesic rays. This builds on ideas introduced in the work of Phong-Sturm [60, 62] and Chen-Tang [15].

Let (X, L) be a polarized normal variety and $(\mathcal{X}, \mathcal{L}, \rho)$ a test configuration for X (recall that $\mathcal{X}$ is assumed normal). Denote by M the variety with boundary obtained by restricting $\mathcal{X}$ to the unit-disc $\Delta \subset \mathbb{C}$. Given a locally bounded metric ϕ_1 with positive curvature on L we let ϕ be the metric on $\mathcal{L} \to M$ defined as the following envelope:

$$(2.14) \qquad \phi := \sup\{\psi : \ \psi \leq \phi_1 \text{ on } \partial M\}$$

where ψ ranges over all locally bounded metrics with positive curvature form on $\mathcal{L} \to M$ and ϕ_1 is identified with the S^1-invariant metric on ∂M induced by the given metric (since we are not a priori assuming that ψ is continuous the boundary condition above means that, locally, $\limsup_{z_i \to z} \psi(z_i) \leq \phi_1(z)$ for any sequence z_i approaching a boundary point z). Occasionally, we will use the logarithmic real coordinate $t = -\log|\tau|^2$ on the punctured disc Δ^*. We note that since X is identified with the fiber X_1 of $\mathcal{X}$ we can use the action ρ to identify the metrics ϕ_τ on X_τ with a curve of metric

$$(2.15) \qquad \phi^t := \rho(\tau)^*\phi_\tau, \quad t := -\log|\tau|^2$$

on L. Next we will show that the metric ϕ above can be seen as a solution to a Dirichlet problem for the Monge-Ampère operator on M. In fact, it will be convenient to formulate the result for any test configuration:

Proposition 2.7. *Let $(\mathcal{X}, \mathcal{L})$ be a test configuration for the polarized variety (X, L) with normal total space $\mathcal{X}$. Then the following holds:*

- ϕ is $S^1-invariant$
- ϕ is locally bounded with positive curvature current and upper semi-continuous in M
- $\phi_\tau \to \phi_1$ uniformly as $|\tau| \to 1$ (with respect to any fixed trivializing of $\mathcal{L}$ close to a given boundary point).
- In the interior of M we have that $(dd^c\phi)^{n+1} = 0$ in the sense of pluripotential theory.

Proof. The first point follows immediately from the extremal defining of ϕ. It will be convenient to identify the metric ϕ_1 with a $\mathbb{C}^*-$invariant metric on $\mathcal{L}$ over the punctured unit-disc Δ^* using the action ρ. We will also, abusing notation slightly, identify the coordinate τ with the psh function on $\pi^*\tau$ on $\mathcal{X}$. Let us first construct a *barrier,* i.e. a continuous metric $\tilde{\phi}$ on $\mathcal{L}$ with positive curvature current such that $\tilde{\phi} = \phi_1$ on ∂M and $\tilde{\phi}_\tau \to \phi_1$ as $|\tau| \to 1$. To this end first observe that for $\epsilon > 0$ sufficiently small there exist a continuous metric ϕ_U with positive curvature on $\mathcal{L} \to U$ over the open set $U := \{|\tau| \leq \epsilon\} \subset \mathcal{X}$. Indeed, we can set $\phi_U = \phi_{FS}$ for the Fubini-Study metric induced by a fixed embedding of $\mathcal{X}$ (see the end of section 2.2). Finally, we set $\tilde{\phi} := \max\{\phi_1 + \log|\tau|, \phi_U - C\}$ for C sufficiently large so that $\tilde{\phi} = \phi_U - C$ for $|\tau|$ sufficiently small and $\tilde{\phi} = \phi_1 + \log|\tau|$ for $|\tau| > \epsilon/2$. Since $\tilde{\phi}$ is a candidate for the sup defining ϕ we conclude that

$$(2.16) \qquad \phi \geq \tilde{\phi} \geq \phi_1 + \log|\tau|$$

Next, let us show that ϕ is locally bounded from above or equivalently that there exists a constant C' such that

$$(2.17) \qquad \phi \leq \phi_{FS} + C'$$

Accepting this for the moment we deduce that the envelope ϕ is finite with positive curvature current. Moreover, the upper bound also implies that the upper semi-continuous regularization ϕ^* of ϕ is a candidate for the sup defining ϕ, forcing $\phi = \phi^*$ in the interior of M, i.e. ϕ is upper semi-continuous there. To prove the previous upper bound we note that since any candidate ψ for the sup defining ϕ satisfies $\psi \leq \phi_{FS} + C$ on the set $E := \partial M$ it follows from general compactness properties of positively curved metrics (or more generally, $\omega-$psh functions) that there is a constant C' such that $\psi \leq \phi_{FS} + C'$ on all of M. Indeed, by a simple extension argument we may as well assume that $u := \psi - \phi_{FS}$ extends as an $\omega-$psh function to some compactification $\hat{\mathcal{X}}$ of $\mathcal{X}$ for some semi-positive form current ω with continuous potentials. But since $u \leq C$ on the non-pluripolar set E it then follows from Cor 5.3 in [30] that $u \leq C'$ on all of $\hat{\mathcal{X}}$ (strictly speaking the variety $\hat{\mathcal{X}}$ is assumed non-singular in [30], but we may as well deduce the result by pulling back u to a smooth resolution of $\hat{\mathcal{X}}$). Alternatively, u can be shown to be bounded from above by using the maximum principle to bound it by a solution to a Dirichlet type problem for the Laplace operator with respect to a fixed Kähler metric on a resolution of M (compare the argument for the upper bound in [62]).

Let us next consider the behavior of ϕ on Δ^* by identifying ϕ_τ with ϕ^t as above for $t \in [0, \infty[$. Since ϕ is positively curved and S^1-invariant it follows that ϕ^t is convex in t on $]0, \infty[$ and in particular the right derivative $\dot{\phi}(t)$ with respect to t exist and define an increasing function on $]0, \infty[$. Hence, $\dot{\phi}(t) \leq C_1 := \dot{\phi}(t_1)$ as $t \to 0$. Combined with the lower bound 2.16 this means that there exists a constant C_T such that $\left|\dot{\phi}\right| \leq C_T$ for any $t \in [0, T]$ and thus $|\phi^t - \phi^{t'}| \leq C_T|t - t'|$ for t and t'

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sufficiently small positive numbers. But then it follows that ϕ^t converges uniformly to the correct boundary values as $t \to 0$ (again using the lower bound 2.16).

As for the final point, the vanishing of the Monge-Ampère measure $(dd^c\phi)^{n+1}$ on the regular part of the interior of M is a standard local argument, which follows from comparison with the solution of the homogenuous Monge-Ampère equation on small balls. Since, the Monge-Ampère measure on a locally bounded metric does not charge pluripolar sets and in particular not the singular locus of M this concludes the proof. $\qquad\square$

According to the previous proposition the envelope ϕ thus induces a *weak geodesic ray* ϕ^t (formula 2.15) in the space $\mathcal{H}_b(X, L)$ of all bounded positively curved metrics, starting at a given metric (compare [60]). For much more precise regularity results (given suitably smooth data on ∂M) expressed on a smooth resolution of $\mathcal{X}$ we refer to the paper [62] and to [15]. However, the point here is that the modest regularity results above will be adequate for our purposes and that are valid for any given locally bounded positively curved metric ϕ_1.

Example 2.8. Given (X, L) and a metric ϕ_L in $\mathcal{H}_b(L)$ we set $(\mathcal{X}, \mathcal{L}, \pi) = (X \times \mathbb{C}, p_1^*L, p_2)$ and equip $\mathcal{L}$ with the metric $\phi := p_1^*\phi_L \in \mathcal{H}_b(\mathcal{L})$. Then ϕ is the unique solution to the Dirichlet problem for the complex Monge-Ampère equation on $\pi^{-1}(\Delta)$ with boundary data $p_1^*\phi_L$. This example fits into the setting above if one equips $(\mathcal{X}, \mathcal{L})$ with the "trivial" action ρ_{triv} covering the standard $\mathbb{C}^*-$action on $\mathbb{C}$ and then the ray ϕ_t, as defined by formula 2.15, is constant in $t : \phi^t = \phi_L$, since ρ_{triv} preserves ϕ. On the other hand, if we are given a non-trivial holomorphic vector field V generating a $\mathbb{C}^*-$action on (X, L), such that the corresponding S^1-action preserves ϕ_L, then we can endow $(\mathcal{X}, \mathcal{L})$ with the corresponding non-trivial action ρ (in the sense of Example 2.1), which still preserves the boundary data. The corresponding ray ϕ_t determined by ϕ and ρ is then given by $\phi_t = (e^{tV})^*\phi_L$. Note that $\rho_{triv} = \rho \circ \rho_{(X,-V)}$ in the notation of Example 2.1 and hence ϕ is invariant under $\rho \circ \rho_{(X,-V)}$ in the sense that $(\rho \circ \rho_{(X,-V)})^*\phi = \phi$.

3. Proofs of the main results

Recall that the Ding functional introduced in [20], in the setting of smooth Fano manifolds X, is the functional on the space of all smooth positively curved metrics on $-K_X$ defined, in our notation, by

$$(3.1) \qquad \mathcal{D}(\phi) := -\frac{1}{(-K_X)^n}\mathcal{E}(\phi) + \log \int_X e^{-\phi}.$$

It follows immediately from the variational property 2.6 of $\mathcal{E}$ that the critical points of $\mathcal{D}$ are Kähler-Einstein metrics. More generally, the functional $\mathcal{D}(\phi)$ is well-defined and finite on the space $\mathcal{H}_b(-K_X)$ of bounded metrics on $-K_X$ as long as X has log terminal singularities (see Lemma 3.7 for the finiteness of the integral piece). Moreover, for any curve ϕ_t in $\mathcal{H}_b(-K_X)$ such that ϕ_0 is a (singular) Kähler-Einstein metric and the right derivative $\frac{d\phi_t}{dt}_{|t=0^+}$ exists we have

$$(3.2) \qquad \frac{d\mathcal{D}(\phi_t)}{dt}_{|t=0^+} \geq 0$$

as follows from the affine concavity of $\mathcal{E}$ and the Kähler-Einstein equation (see [3, Formula 6.5]).

3.0.1. *Sketch of the proofs of Theorem 3.11 and Theorem 1.1 in the* $\mathbb{Q}-$*Gorenstein case.* First note that in the case when the total space $\mathcal{X}$ of the test configuration is $\mathbb{Q}-$Gorenstein and $\mathcal{L} = -K_{\mathcal{X}/\mathbb{C}}$ (as is, for example, the case for a special test configuration) the line bundle $\eta \to \mathbb{C}$, whose weight $w_0(\eta)$ over zero is equal to $DF(\mathcal{X}, \mathcal{L})$ (Section 2.3), is simply given by the top Deligne pairing $-\frac{1}{(-K_X)^n(n+1)} \left\langle -K_{\mathcal{X}/\mathbb{C}}, ..., -K_{\mathcal{X}/\mathbb{C}} \right\rangle$. [2] Given a locally bounded metric ϕ on $\mathcal{L} \to \mathbb{C}$ we define the corresponding *Ding metric* Φ on η as the induced Deligne metric $\langle \phi \rangle$ plus the function

$$(3.3) \qquad v_\phi(\tau) := -\log \int_{X_\tau} e^{-\phi_\tau}$$

(using, as before, additive notation for metrics) on $\mathbb{C}^*$, which, as will be explained in the next section extends to a unique subharmonic function on $\mathbb{C}$. By the change of metrics formula 2.7 the Ding metric $\Phi(\tau)$ may, as the name suggest, be identified with the Ding functional $\mathcal{D}(\phi^t)$ (formula 3.1) along the corresponding ray of metric ϕ^t on $-K_X$. More generally, when $\mathcal{X}$ is $\mathbb{Q}-$Gorenstein and $\mathcal{L}$ is a general polarization for $\mathcal{X} \to \mathbb{C}$ the idea is to replace η with another line bundle δ, that we shall call the *Ding line bundle,* defined by

$$\delta := -\frac{1}{L^n(n+1)} \langle \mathcal{L}, ..., \mathcal{L} \rangle + \pi_*(\mathcal{L} + K_{\mathcal{X}/\mathbb{C}}) \to \mathbb{C},$$

where $\pi_*(\mathcal{L} + K_{\mathcal{X}/\mathbb{C}})$ is the corresponding adjoint direct image sheaf. In this more general case we define the corresponding *Ding metric* Φ as the Deligne metric $\langle \phi \rangle$ plus the natural L^2-type metric on $\pi_*(\mathcal{L} + K_{\mathcal{X}/\mathbb{C}})$ induced by ϕ. Even if δ is, in general not isomorphic to η we will show that

$$(DF(\mathcal{X}, \mathcal{L}) =) \, w_0(\eta) = w_0(\delta) + p, \quad p \geq 0$$

where $p = 0$ iff $\mathcal{L}$ is isomorphic to $-K_{\mathcal{X}/\mathbb{C}}$. Applying the Kempf-Ness type Lemma 2.6 to $w_0(\delta)$ then gives

$$(3.4) \qquad DF(\mathcal{X}, \mathcal{L}) = \lim_{t \to \infty} \frac{d}{dt} \mathcal{D}(\phi^t) + q, \quad q = p + l_0,$$

where l_0 is the Lelong number l_0 at zero of the Ding metric, which in turn is shown to coincides with the non-negative Lelong number of the L^2-type metric on $\pi_*(\mathcal{L} + K_{\mathcal{X}/\mathbb{C}})$. As a consequence the error term q is non-negative. Finally, taking ϕ^t to be the geodesic ray associated to $(\mathcal{X}, \mathcal{L})$ emanating from a given Kähler-Einstein metric and using convexity properties of $\mathcal{D}(\phi^t)$ then proves Theorem 1.1. In the case of a general test configuration $(\mathcal{X}, \mathcal{L})$ the idea is to apply the previous argument to a log resolution of $\mathcal{X}$.

Remark 3.1. It should be pointed out that, by a recent result of Li-Xu [42] which uses recent advances in MMP, a Fano variety X is K-polystable iff it is K-polystable for all *special* test configurations and hence, if one invokes [42], it is enough to consider special test configurations in the proof of Theorem 1.1. Anyway, as indicated above the proof in the general case is not much more involved and combined with the existence results in [16] it yields an analytical/differential geometric proof of the result of Li-Xu referred to above, at least for X smooth (since it is shown in [16] that if a Fano manifold X is K-polystable for all special test configuration,

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[2] When making the identifications between $\mathcal{L} = -K_{\mathcal{X}/\mathbb{C}}$ we have to take the lift ρ of the $\mathbb{C}^*-$action on $\mathcal{X}$ to be the canonical lift to $-K_{\mathcal{X}/\mathbb{C}}$ (compare the beginning of Section 2.2.1).

then X admits a Kähler-Einstein metrics). We recall that an important ingredient in the proof in [42] is to show that $DF(\mathcal{X}, \mathcal{L})$ decreases along a relative MMP (first without and then with scaling) which, after an initial base change, modifies $(\mathcal{X}_0, \mathcal{L}_{|\mathcal{X}_0})$ so that (i) $(\mathcal{X}, \mathcal{X}_0)$ has log canonical singularities and (ii) $\mathcal{L}$ is isomorphic to $-K_{\mathcal{X}/\mathbb{C}}$. Interestingly, as we will show (independently of [42]) that (i) holds iff $l_0 = 0$ and (ii) holds iff $p = 0$ and thus the error term q in formula 3.4 achieves it minimal value 0 iff $(\mathcal{X}, \mathcal{L})$ is a test configuration of the form produced by the MMP procedure in [42].

3.1. The Ding line bundle and the Ding metric.
Let $(\mathcal{X}, \mathcal{L})$ be a test configuration for a Fano variety $(X, -K_X)$ and fix an equivariant log resolution $p : \mathcal{X}' \to \mathcal{X}$ of $(\mathcal{X}, \mathcal{X}_0)$ and write $\mathcal{L}' := p^*\mathcal{L}$. Then $(\mathcal{X}', \mathcal{L}')$ is a semi-test configuration for $(X, -K_X)$. First assume, to fix ideas, that the original Fano variety X is smooth with $\mathcal{L}$ a line bundle over $\mathcal{X}$ and define a the *Ding line bundle* $\delta' \to \mathbb{C}$ by

$$(3.5) \qquad \delta' := -\frac{1}{L^n(n+1)} \langle \mathcal{L}', ..., \mathcal{L}' \rangle + \pi'_*(\mathcal{L}' + K_{\mathcal{X}'/\mathbb{C}}) \to \mathbb{C},$$

(when X is smooth the direct image sheaf $\pi'_*(\mathcal{L}' + K_{\mathcal{X}'/\mathbb{C}})$ is indeed a line bundle, as explained below). Given a metric ϕ on $\mathcal{L} \to \mathcal{X}$ we denote by Φ' the *generalized Ding metric* on δ', defined as the Deligne metric on the top Deligne pairing of $\mathcal{L}$ twisted by the L^2-metric on $\pi'_*(\mathcal{L}' + K_{\mathcal{X}'/\mathbb{C}})$, induced by $\phi' := p^*\phi$. Note that in general $\mathcal{L}$ is only assumed to be a $\mathbb{Q}-$line bundle, i.e. $r\mathcal{L}$ is a line bundle for some positive integer r and then we may simply define $\pi'_*(\mathcal{L}' + K_{\mathcal{X}'/\mathbb{C}}) := \pi'_*(r(\mathcal{L}' + K_{\mathcal{X}'/\mathbb{C}})/r$ as a $\mathbb{Q}-$line bundle (which is easily seen to be independent of r) and let Φ' be the metric defined by the corresponding $L^{2/r}-$norm

$$\|s_\tau\|_{L^{2/r}_\phi} := \left(\int_{\mathcal{X}_\tau} |s_\tau|^{2/r} e^{-\phi'_\tau} \right)^{r/2},$$

where we have identified the restriction s_τ of $s \in H^0(X', \mathcal{L}' + K_{\mathcal{X}'/\mathbb{C}})$ with a holomorphic $(n, 0)-$form on X_τ with values in $\mathcal{L}'_{|X_\tau}$ (compare the notation in Section 2.1.3).

Turning to the case of a general Fano variety X with log terminal singularities first recall that, since the variety $\mathcal{X}^*(:= \mathcal{X} - \mathcal{X}_0)$ has log terminal singularities, we have $p^*K_{\mathcal{X}} = K_{\mathcal{X}'} + D^*$ on $\mathcal{X}' - \mathcal{X}'_0$ for a (sub) klt $\mathbb{Q}-$divisor D^*, whose closure in $\mathcal{X}'$ we will denote by D. We can decompose $D = D' - E'$ as a difference of effective $\mathbb{Q}-$divisors where E' has integral coefficients (but we are not claiming that the D' and E' have no common components) . We may and will assume that the log resolution is such that the support of D has simple normal crossings and is transversal to $\mathcal{X}'_0$. We then define

$$\delta' := -\frac{1}{L^n(n+1)} \langle \mathcal{L}', ..., \mathcal{L}' \rangle + \pi'_*(\mathcal{L}' + D' + K_{\mathcal{X}'/\mathbb{C}}) \to \mathbb{C},$$

and denote by Φ' the corresponding metric on δ', which is defined using the log adjoint $L^{2/r}-$metric on $\pi'_*(\mathcal{L}' + D' + K_{\mathcal{X}'/\mathbb{C}})$

$$\|s_\tau\|_{L^{2/r}_\phi} := \left(\int_{\mathcal{X}_\tau} |s'_\tau|^{2/r} e^{-(\phi'_\tau + \phi_{D'})} \right)^{r/2}$$

To see that $\pi'_*(\mathcal{L}' + D' + K_{\mathcal{X}'/\mathbb{C}})$ is indeed a line bundle over $\mathbb{C}$ first note that over $\mathbb{C}^*$, where the sheaf is globally free, any fiber may be identified with $H^0(X', E')$, where

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$X' = p^*X$ and E' is $p-$exceptional, so that $\dim H^0(X', E') = 1$. The extension property to all of $\mathbb{C}$ then follows from general principles. Indeed, the direct image sheaf is clearly torsion-free and since the base is a curve any torsion-free sheaf is automatically locally free (indeed, to get a local generator for the sheaf close to $0 \in \mathbb{C}$ one simply takes an element with minimal vanishing order at 0).

3.2. Positivity/continuity properties of the Ding metric.
In this section we assume given a Fano variety X (with log terminal singularities) and a locally bounded metric ϕ_1 on $-K_X$ with positive curvature current. Fixing a test configuration $(\mathcal{X}, \mathcal{L})$ for $(X, -K_X)$ and a resolution $(\mathcal{X}', \mathcal{L}')$, as in the previous section, we denote by ϕ the induced S^1-invariant locally bounded metric on $\mathcal{L} \to M(\subset \mathcal{X})$ (compare Section 2.4), by ϕ^t the corresponding weak geodesic in $\mathcal{H}_b(-K_X)$ and by Φ the corresponding Ding type metric on the Ding line bundle $\delta' \to \Delta$.

The study of the positivity properties of the Ding metric relies on the following fundamental positivity result of Berndtsson-Paun for direct image vector bundles (applied the rank one case):

Lemma 3.2. *Let $\mathcal{X}$ be a non-singular projective variety with a morphism $\pi : \mathcal{X} \to \mathbb{C}$ which is smooth (i.e. a submersion) over $\mathbb{C}^*$ and a $\mathbb{Q}-$line bundle $\mathcal{L} \to \mathcal{X}$ equipped with with a singular metric ϕ with positive curvature and such that the restriction ϕ_τ to each fiber X_τ, for $\tau \in \mathbb{C}^*$, satisfies $e^{-\phi_\tau} \in L^1_{loc}$. If $\pi_*(\mathcal{L} + K_{\mathcal{X}/\mathbb{C}}) \to \mathbb{C}$ is defined as a $\mathbb{Q}-$line bundle (i.e. if $\dim H^0(\mathcal{L}_{|X_\tau} + K_{\mathcal{X}_\tau}) = 1$ for $\tau \in \mathbb{C}^*$) then the corresponding $L^{2/r}-$metric on the line bundle $\pi_*(\mathcal{L} + K_{\mathcal{X}/\mathbb{C}}) \to \mathbb{C}$ has positive curvature in the sense of currents (where r is a positive integer such that $r\mathcal{L}$ is a line bundle).*

Proof. The positivity over $\mathbb{C}^*$ is a special case of the main results in [8, 11] (note that by assumption the $L^{2/r}-$metric is finite over $\mathbb{C}^*$). The positivity over a neighborhood of 0 also follows from the arguments in [11]. But as the latter positivity was not stated explicitly in [11] we provide a detailed proof[3]. Fix a local trivializing section of $\pi_*(r(\mathcal{L} + K_{\mathcal{X}/\mathbb{C}})) \to \mathbb{C}$ over a small neighborhood V of $0 \in \mathbb{C}$. It may be identified with a global holomorphic section s of $r(\mathcal{L} + K_{\mathcal{X}/\mathbb{C}}) \to \mathcal{X}_{|V}$ with the property that τ does not divide s. Fix a local coordinate τ on $\mathbb{C}$ and let

$$(3.6) \qquad v(\tau) := -\frac{1}{r} \log \|s\|^2_{L^{2/r}_\phi} \left(= -\log \|s\|^{2/r}_{L^{2/r}_\phi} \right)$$

be the corresponding local weight of the $L^{2/r}-$metric on the $\mathbb{Q}-$line bundle $\frac{1}{r}\pi_*(r(\mathcal{L} + K_{\mathcal{X}/\mathbb{C}}))$. By basic properties of subharmonic functions the positivity in question is equivalent to an upper bound on $v(\tau)$ on V or equivalently a lower bound

$$(3.7) \qquad \|s\|^{2/r}_{L^{2/r}_\phi} := \int_{\mathcal{X}_\tau} |s_\tau|^{2/r} e^{-\phi_\tau} \geq \epsilon > 0, \ \tau \in V$$

where we have identified $|s|^{2/r} e^{-\phi}$ with a family of measures over $\mathcal{X}^* := \pi^{-1}(\mathbb{C}^*)$ (as in Section 2.1.3). A subtle point is that the assumptions of the lemma do not exclude that the holomorphic section s vanishes identically on the reduction of $\mathcal{X}_0$ (i.e. on the underlying variety); for example, this can happen if $\mathcal{X}_0$ has components with different multiplicities. On the other hand it follows from a local application

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[3]The positivity in question is also a special case of the very general positivity results in [58, Theorem 1.1], which appeared during the revision of the present paper, whose proof uses among other things, semi-stable reduction.

of the generalized Ohsawa-Takegoshi extension theorem in [11, Lemma 1.1] that there exists a uniform constant C (i.e. independent of τ) such that

$$\int_V |s|^{2/r} e^{-\phi} \leq C \int_{\mathcal{X}_\tau} |s_\tau|^{2/r} e^{-\phi_\tau}, \ \ \tau \in V$$

But since s is non-vanishing over $V - \{0\}$ this implies the desired lower bound 3.7. □

In particular, by the previous proposition the function $v_\phi(\tau)$ is subharmonic, i.e $v_\phi(e^{-t/2})$ is convex, as long as ϕ is psh (as already observed in [3, 10] for X non-singular and in [4] in general). Next, we will show that if the curvature of the line bundle $\pi_*(\mathcal{L} + K_{\mathcal{X}/\mathbb{C}})$ vanishes identically, then - in the presence of a $\mathbb{C}^*-$action as in the definition of a test configuration - $\mathcal{X}$ has to be a product. This will be crucial when considering the case $DF(\mathcal{X}, \mathcal{L}) = 0$ in the proof of Theorem 1.1.

Proposition 3.3. *Let X be a Fano variety with log terminal singularities and $(\mathcal{X}, \mathcal{L})$ a test configuration for $(X, -K_X)$ such that $\mathcal{X}$ is $\mathbb{Q}-$Gorenstein and $\mathcal{L} = -K_{\mathcal{X}/\mathbb{C}}$. Assume that $\mathcal{L}$ is equipped with an S^1-invariant locally bounded metric ϕ with positive curvature current such that the induced curvature current of the direct image sheaf $\pi_*(\mathcal{L} + K_{\mathcal{X}/\mathbb{C}})$ vanishes identically on $\mathbb{C}$ (or more generally, over some neighbourhood of $0 \in \mathbb{C}$). Then $\mathcal{X}$ is isomorphic to $X \times \mathbb{C}$.*

Proof. First recall that it was shown in [10, Theorem 6.1] in the case of X smooth and [4, Theorem 5.1] in the general case, that if $v_\phi(\tau)$ is harmonic, for $\tau \in \Delta^*$ (i.e. $v_\phi(e^{-t})$ is affine in t), then there is a family of biholomorphic maps F^t indexed by $t \in \mathbb{R}$ such that

$$(3.8) \qquad\qquad (F^t)^* dd^c \phi^t = dd^c \phi^0,$$

where ϕ^t denotes the ray of metric on $-K_X$ corresponding to ϕ, i.e. $\phi^t = \rho(\tau)^* \phi_\tau$, for $t = -\log|\tau|^2$ (compare Section 2.4). Moreover, as shown in [10, Section 4.1] $F^t = \exp(t\mathrm{Re}V)$for a holomorphic $(1,0)-$vector field V on X such that the flow of its imaginary part $\mathrm{Im}V$ preserves $dd^c \phi^0$. In fact, in the present setting we even have

$$(3.9) \qquad\qquad (F^t)^* \phi^t = \phi^0,$$

where we have used the same notation F^t for the canonical lift of F^t to $-K_X$. To see this first note that the relation 3.8 implies that $(F^t)^* \phi^t = \phi^0 + a(t)$ for some function $a(t)$ and since $v_\phi(e^{-t}) = -\log \int_X e^{-\phi_0} + a(t)$ is assumed affine it follows that $a(t)$ is also affine, i.e. $a(t) = at + c$ for some real numbers a and c. But then it follows that $v_\phi(\tau) = -\log \int_X e^{-\phi_0} + a\log|\tau|^2 + c$ and hence a is equal to the Lelong number l_0 of $v_\phi(\tau)$ at 0. Now, l_0 coincides with the mass at 0 of the curvature current on $\pi_*(\mathcal{L} + K_{\mathcal{X}/\mathbb{C}})$ (see formula 2.12) which is assumed to vanish and hence $a = 0$ and since F^0 is the identity this means that c also vanishes, which proves 3.9. Next, we make the following

$$(3.10) \qquad\qquad \text{Claim:} V \text{ generates a } \mathbb{C}^*-\text{action } \rho_{(X,V)} \text{on } X$$

(i.e. the flow of $\mathrm{Im}V$ has period 2π). This is obvious if X admits no non-trivial holomorphic vector fields (since $V = 0$ then) and in the general case the claim follows from Lemma 3.4 below, by observing that $\mathcal{X}_0$ is reduced. Indeed, as observed above, the Lelong number $l_0 = 0$ and hence it follows from Proposition 3.8

below that $\mathcal{X}_0$ is indeed reduced.[4] We will write $F_\tau := \rho_{(X,V)}(\tau)$ for the family of biholomorphic maps on X, indexed by $\tau \in \mathbb{C}^*$, generated by V (so that $F_\tau = F^t$ for $\tau = e^{-t/2}$). Next, we set

$$G_\tau := \rho_\tau \circ F_\tau^{-1}; \quad \mathcal{X}_0 \to \mathcal{X}_\tau$$

so that

(3.11) $$G_\tau^* \phi_\tau = \phi_1$$

Using an equivariant embedding into $\mathbb{P}^N$ (see section 2.2) we can identify G_τ with a family of holomorphic embeddings

$$G_\tau : X \to \mathbb{P}^N, \quad X_\tau := G_\tau(X), \quad G_\tau^* \mathcal{O}(1) = -K_X, \quad \tau \in \mathbb{C}^*$$

Since ϕ is a locally bounded metric on $\mathcal{L} \to \mathcal{X}$ we have that $|\phi - p_1^* \phi_{FS}| \leq C$ over Δ, where ϕ_{FS} denotes the Fubini-Study metric on $\mathcal{O}(1) \to \mathbb{P}^N$ and p_1 the natural projection $\mathbb{P}^N \times \Delta \to \mathbb{P}^N$. Hence, 3.11 gives

(3.12) $$\sup_X |G_\tau^* \phi_{FS} - \phi_1| \leq C$$

for $\tau \in \Delta^*$. We claim that the corresponding holomorphic map G from $X \times \Delta^*$ to $\mathbb{P}^N \times \Delta$ extends to a holomorphic map $X \times \Delta \to \mathbb{P}^N \times \Delta$. To see this introduce local coordinates on an open set $U \subset X$ centered at a given point x_0 in X and fix an affine piece $\mathbb{C}^N$ of $\mathbb{P}^N$ such that $G_1(U) \subset \mathbb{C}^N$. Then there exists a bounded subset B of $\mathbb{C}^{N+1}$ such that $G_\tau(U) \subset B$ for any $\tau \in \mathbb{C}^*$. Indeed, by the very definition of the Fubini-Study metric ϕ_{FS} the bound 3.12 gives that there exists a constant C' such that $|G_\tau(z)| \leq C'$ for any $z \in U$ and $\tau \in \mathbb{C}^*$ and hence B can be taken as as ball of radius C'. Hence, applying Harthog's extension theorem (coordinatewise) thus gives that $G_{|U \times \Delta^*}$ extends to a unique holomorphic map of $U \times \Delta$ into $\mathbb{C}^N \times \Delta$ and since the point x_0 was arbitrary this proves the claim. Moreover, since $\mathcal{X}$ is normal and in particular closed and irreducible, it follows that G maps $X \times \Delta$ surjectively onto $\mathcal{X} \subset \mathbb{P}^N \times \Delta$. We claim that G is a finite map. Since G is, by construction, injective on $X \times \Delta^*$ it will be enough to prove that the restriction of G to $X \times \{0\}$, that we denote by G_0, defines a finite map from X to $\mathbb{P}^N$. To this end we note that that G_0 pulls back the cohomology class $c_1(\mathcal{O}(1))$ on $\mathbb{P}^N$ to $c_1(L)(= c_1(-K_X))$. Indeed, by construction

(3.13) $$G_\tau^* c_1(\mathcal{O}(1)) = c_1(L)$$

for any $\tau \in \mathbb{C}^*$ and since $G_\tau \to G_0$ as $\tau \to 0$ (in the sense established above) it follows that 3.13 also holds for $\tau = 0$ and hence G_0 pulls back a Kähler class on $\mathbb{P}^n$ to a Kähler class on X. But then G_0 has to be finite, since otherwise G_0 would contract some p- dimensional subvariety V_p of X. This would mean that the p th intersection number of $c_1(L)$ with V_p vanishes, contradicting the fact that $c_1(L)$ is a Kähler class. All in all this means that G defines a finite birational morphism from $X \times \mathbb{C}$ to $\mathcal{X}$ and since $\mathcal{X}$ is assumed normal it then follows from Zariski's Main Theorem that G is a biholomorphism, as desired. The same argument also applies if $\tau = 1$ is replaced with any $\tau_0 \neq 0$ such that $v_\phi(\tau)$ is harmonic for $|\tau| \leq |\tau_0|$. $\square$

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[4]In fact, one does not have to use the fact that $\mathcal{X}_0$ is reduced if one instead performs a base change followed by a normalization to get a new normal test configuration $\mathcal{X}'$ with reduced central fiber to which Lemma 3.4 can be applied; this corresponds to replacing V with mV for some positive integer m.

The previous proposition can be seen as a partial generalization to singular fibrations of a result in [9] (see Theorem 1.2 and the discussion in Section 4.1 in [9] where the fibration is assumed to be a submersion, but without any assumptions on $\mathbb{C}^*-$equivariance). In the proof we used the following lemma of independent interest:

Lemma 3.4. *Assume given a non-trivial holomorphic vector field V on a normal variety X of type $(1,0)$ with a fixed lift to $L \to X$ and a metric $\phi^0 \in \mathcal{H}_b(L)$, which is invariant under the flow $\mathrm{Im}V$. Then V generates a $\mathbb{C}^*-$action on X iff there exists a (normal) test configuration $(\mathcal{X}, \mathcal{L}, \pi, \rho)$ with reduced central fiber $\mathcal{X}_0$ such that $(e^{t\mathrm{Re}V})^* \phi^0 = \rho(\tau)^* \phi_\tau$, where $\phi_\tau = \phi_{|\mathcal{X}_\tau}$ for some $\phi \in \mathcal{H}_b(\pi^{-1}(\Delta), \mathcal{L})$.*

Proof. If V generates a $\mathbb{C}^*-$action then the test configuration can be taken as a product, as explained in Example 2.8. But in order to prove the converse, which is what was used in the proof of Proposition 3.3, we will have to show that the total space $\mathcal{X}$ of the given test configuration is necessarily a product. We will continue with the notation from the Proposition 3.3. First observe that the map G_τ above is well-defined for $\tau = e^{-t/2} \in [0, 1[$ and set $G^t := G_{e^{-t}}$. By the argument above the map $G_0 := \lim_{t_j \to \infty} G^t$ still exists and defines a holomorphic map from X to $\mathcal{X} \subset \mathbb{P}^N$, if one uses "normal families" instead of Harthog's extension theorem, for some subsequence $t_j \to \infty$ (but we are not claiming that G_0 is independent of the subsequence at this point).

Step 1: $\mathcal{X}_0$ is reduced and irreducible, i.e. defined by a variety X_0 and G_0 is finite and generically one-to-one, mapping X_1 onto X_0 and

$$(3.14) \qquad G_0^* \phi_0 = \phi_1$$

where ϕ_1 is restricted metric on $\mathcal{L}_{|X_0}$.

To prove the first point we decompose the central fiber $\mathcal{X}_0$, viewed as a divisor on the normal variety $\mathcal{X}$, in its irreduicble components: $\mathcal{X}_0 = \sum_{i=1}^p m_i E_i$, where E_i are distinct prime divisors on $\mathcal{X}$ (i.e. reduced and irreducible). Since X is assumed irreducible it follows that G_0 maps X_1 onto one of the components of $\mathcal{X}_0$ that we may take to be the one labeled by $i = 1$. By the definition of G_0 we have the following convergence in the sense of currents on $\mathbb{P}^N$

$$\lim_{t_j}[X_{t_j}] = (G_0)_*[X] = d[E_1],$$

where d is the degree of the surjective finite map $G_0 : X \to E_1$. But the lhs above is also equal to $[\mathcal{X}_0]$ (by basic convergence properties of currents, or using the Chow variety) and hence $d[E_1] = \sum_{i=1}^p m_i[E_i]$, which forces $d = m_1$ and $p = 1$. Now, $\mathcal{X}_0$ was assumed reduced and hence $d = m_1 = 1$, which implies (by basic properties of the degree) that G_0 is a finite generally one-to-one map from X_1 onto the irreducible variety $X_0(= E_1)$. Next, to prove 3.14 we fix a point $x_0 \in X_0 \cap \mathcal{X}_{reg} \cap (X_0)_{reg}$ (i.e. $x_0 \in X_0 - Z$ where Z has codimension one in X_0, since $\mathcal{X}$ is normal). Then there exists a neighbourhood U of x_0 in $\mathcal{X}$ with holomorphic coordinates centered at x_0 of the form (z, τ). The relation 3.14 then follows from 3.11 and a simple continuity argument at x_0. This means that the two psh metrics $G_0^* \phi_0$ and ϕ_1 on $L \to X$ coincide on the Zariski open subset $G_0^{-1}(X_0 - Z)$ of X and hence everywhere, by the local identity principle for psh functions.

Step 2: The "pull-back" ρ' to X of the restricted action of ρ to X_0 coincides with the flow of V

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Denote by Y^ν the normalization of an irreducible variety Y and by ν the normalization map $\nu : Y^\nu \to Y$, which is a finite generically injective morphism. By the universal property of the normalization G_0 lifts to X_0^ν and hence the lifted map G_0^ν satisfies $G_0^{\nu*}\nu^*\phi_0 = \phi_1$. Moreover, since X is normal it follows from Zariski's main theorem that G_0 is an isomorphism and hence X isomorphic to the normalization of X_0 and G_0 may be identified with ν. Using the universal property of the normalization again this means that the the pull-back $G_0^*\rho$ which is a priori only well-defined on a Zariski open subset of X where G_0 is holomorphically invertible, extends to give a well-defined holomorphic $\mathbb{C}^*-$action ρ' on X. To prove that ρ' is generated by V we will use a (singular) Hamiltonian formalism. To a given pair (ψ, W) consisting of a locally bounded psh metric ψ on a line bundle $L \to Y$ over a complex variety Y and a holomorphic vector field W on Y with a fixed lift to L, preserving ψ, we associate a function $h_{(\psi, W)}$ on Y, that we will call the *Hamiltonian:*

$$h_{(\psi, W)} = \frac{d}{ds}_{|s=0}(e^{sW})^*\psi$$

in the sense of right derivatives ($h_{(\psi, W)}$ exists and is finite since ψ is locally psh and hence $(e^{sW})^*\psi$ is convex wrt s). In the particular case when W is the generator of a $\mathbb{C}^*-$action ρ we set $h_{(\psi, \rho)} := h_{(\psi, W)}$. We let h, h_0 and h_1 be the Hamiltonian functions on $\mathcal{X}, X_0$ and X_1 corresponding to $(\rho, \phi), (\rho_{|X_0}, \phi_0)$ and (V, ϕ_1), respectively. Note that it follows directly from the definition that $h_{|X_0} = h_0$. Next we will show that

(3.15) $$G_0^*h_0 = h_1$$

To this end first observe that

(3.16) $$h_1(x) = h(G^t(x)).$$

Indeed, under the isomorphism $X_1 \times \mathbb{C}^* \to \mathcal{X}^*$, $(x, \tau) \mapsto x_\tau := \rho(\tau)x$ determined by ρ the action ρ on $\mathcal{X}^*$ may be identified with the "trivial" action on $X_1 \times \mathbb{C}^*$ generated by the vector field $\tau\frac{\partial}{\partial \tau}$ and the metric ϕ on $\mathcal{L}$ may be identified with the metric on p_1^*L suggestively written as $\phi(x, \tau) := \phi^t(x)$, where $t = -\log|\tau|^2$. In the present setting we have, by assumption, that $\phi_t = \exp(tV)^*\phi_1$ where V also determines the map G^t from X_1 to $X_{e^{-t/2}}$ defined above, which may be identified with the map $(x, 1) \mapsto (\exp(-tV)x, e^{-t/2}) \in X_1 \times \mathbb{C}^*$. Using these identifications we may write

$$h(G_t(x)) = \frac{d}{ds}_{|s=0}\phi(\exp(-tV)x, e^{-(t+s)/2}) = \frac{d}{ds}_{|s=0}\phi^{t+s}(\exp(-tV)x) =$$

$$= \frac{d}{ds}\phi_1(\exp(t + sV)\exp(-tV)x) = \frac{d}{ds}_{|s=0}\phi_1(\exp(sV)x) =: h_1(x),$$

which proves 3.16. Finally, setting $t = t_i$ and letting $t_t \to \infty$ gives, since $G_0(x) := \lim_{t_j \to \infty} G^{t_j}(x)$ that $h_1(x) = h(G_0(x))$ and hence $h_1(x) = h_0(G_0(x))$, proving 3.15. All in all, combining the pull-back relations 3.14 and 3.15 reveals that $h_{(\phi, V)} = h_{(\phi, G_0^*\rho)}$. But for a fixed metric $\phi \in \mathcal{H}_b(X)$ we have that $W \mapsto h_{(\phi, W)}$ is injective and hence V is the generator of the $\mathbb{C}^*-$action $G_0^*\rho$ which proves the claim 3.10. The injectivity used above is standard under the regularity assumption that there exists a point $x \in X$ such that $dd^c\phi$ is smooth and strictly positive close to x, since $dd^c\phi(\operatorname{Im}V, \cdot) = dh_{(\phi, V)}$, which may be inverted to determine $\operatorname{Im}V$ and hence V. In

the general case, the injectivity follows from the general formalism in [7] (or from Proposition 8.2 in [10]). Anyway, in the application to the proof of Theorem 1.1 ϕ_1 will be a Kähler-Einstien metric and in particular the regularity assumption above holds. □

Proposition 3.5. *Let $(\mathcal{X}, \mathcal{L})$ be a test configuration for a Fano variety $(X, -K_X)$ with log terminal singularities. Then the Ding metric associated to a weak geodesic ray ϕ^t as above has the following positivity properties:*

- *Its curvature defines a positive current on Δ (and in particular the function $\mathcal{D}(\phi^t)$ is convex in t)*
- *If $\mathcal{X}$ is $\mathbb{Q}-$Gorenstein, $\mathcal{L} = -K_{\mathcal{X}/\mathbb{C}}$ and the curvature current of the Ding metric vanishes on some disc centered at 0 (i.e. $\mathcal{D}(\phi^t)$ is affine on $]T, \infty[$ and the Lelong number l_0 of the Ding metric vanishes) then $\mathcal{X}$ is a product test configuration.*

Proof. First we note that the curvature of the Deligne metric $\langle \phi \rangle$ on $\langle \mathcal{L}, ..., \mathcal{L} \rangle$ is non-negative if ϕ is psh and vanishes if the corresponding ray ϕ^t is a weak geodesic, as follows from the push-forward formula 2.4 Alternatively, since $\langle \phi \rangle$ is locally bounded from above (by the continuity result in Prop 3.6) it is enough to consider the holomorphically trivial case over Δ^* where the result amounts to a well-known property of the functional $\mathcal{E}$ (see [4]). Combined with the positivity in the previous lemma this shows that the Ding metric has positive curvature current. More precisely, in the case when X is singular we apply the previous lemma to the line bundle $p^*\mathcal{L} + D' \to \mathcal{X}$ equipped with the metric $p^*\phi + \phi_{D'}$, where $\phi_{D'}$ is the singular psh metric on the line bundle $\mathcal{O}(D')$ induced by D' which satisfies $e^{-\phi_{D'}} \in L^1_{loc}$, since D' is klt. The last point follows immediately from the previous proposition. □

Proposition 3.6. *The Ding metric associated to a weak geodesic ray is continuous on Δ^* up to the boundary circle.*

Proof. Let us first verify that if ϕ is a locally bounded positively curved metric on $\mathcal{L} \to \mathcal{X}$ then the Deligne metric $\langle \phi \rangle$ on $\langle \mathcal{L}, ..., \mathcal{L} \rangle \to \mathbb{C}$ is locally bounded on Δ and continuous at the boundary of Δ. To this end we first recall that if ψ is a smooth metric on $\mathcal{L}$(i.e. the restriction to $\mathcal{X}$ of a smooth metric) then it was shown by Moriwaki [46, Theorem A] that the corresponding Deligne metric $\langle \psi \rangle$ on the top Deligne product on $\langle \mathcal{L}, ..., \mathcal{L} \rangle \to \mathbb{C}$ is continuous. But since ϕ is a locally bounded metric on $\mathcal{L}$ we have that $u := \phi - \psi$ is a bounded function on $\mathcal{X}$ and hence it follows from the change of metric formula 2.7 that

$$|\langle \phi \rangle - \langle \psi \rangle| \leq c^1(L)^n \sup_{\mathcal{X}} |u|$$

is bounded (where L denotes the restriction of $\mathcal{L}$ to a generic fiber). Hence $\langle \phi \rangle$ is locally bounded, as desired. Alternatively, the local boundedness of $\langle \phi \rangle$ can be verified directly by induction over the relative dimension, using the recursive definition of $\langle \phi \rangle$ [75]. Similarly, the continuity at $\tau = 1$ follows from continuity properties at $\tau = 1$ of ϕ_τ. Indeed, by Prop 2.7 we have that $\phi_\tau \to \phi_1$ uniformly as $\tau \to 1$, i.e. $\phi^t \to \phi^0$ and hence it follows from the change of metrics formula that, in a fixed local trivialization close to $\tau = 1$, we have

$$|\langle \rho(\tau)\phi_\tau \rangle - \langle \phi_1 \rangle| \leq c(L)^n \sup_X |(\rho(\tau)^*\phi_\tau) - \phi_1| \to 0,$$

as $\tau \to 1$ and moreover $v_\phi(\tau) \to v_\phi(1)$. This shows in particular that, over Δ^*, the Ding metric $\Phi(= \langle\phi\rangle + v_\phi)$ may be identified with a locally bounded S^1-invariant convex function (by the previous proposition) which is continuous up to $\partial\Delta$. $\quad\square$

3.3. **Singularity structure of the Ding metric.** We continue with the setup and notation in the previous section. We will give a detailed description of the singularity of the Ding metric at $\tau = 0$ (which however is not used in the proof of Theorem 1.1). The key point is the observation that the Lelong number l_0 at 0 of $L^{2/r}$- metrics as above can be expressed in terms the log canonical thresholds of the corresponding central fiber. First recall that the *complex singularity exponent* $c_x(v)$ at a point x in a complex manifold X of a local psh function v (i.e. defined on some neighborhood U_x of x) is defined by

$$c_x(v) := \sup_{c\in\mathbb{R}} \left\{c : \exists U_x \ e^{-cv} \in L^1(U_x, dV)\right\},$$

where dV is a local volume form. When

(3.17) $$v = v_D := \sum_i a_i \log|f_i|^2 \quad D := \sum_i a_i D_i$$

where f_i is a local holomorphic function determining a zero prime divisor D_i the number $c_x(v)$ coincides with the l*og canonical threshold at x, denoted by*$c_x(D)$, of the $\mathbb{Q}-$divisor D :

(3.18) $$c_x(v) = c_x(D)$$

(see [39, Proposition 8.2]). The latter number admits a purely algebraic definition valid for any log pair (X, D), i.e. without assuming X non-singular:

$$c_x(D) := \sup_{c\in\mathbb{R}} \left\{c : cD \text{ is log canonical close to } x\right\},$$

(compare Section 2.1.1). More generally, given a log pair (X, Δ) and an effective $\mathbb{Q}-$Cartier divisor D on X the *log canonical threshold of* (X, Δ, D), *along* Z, may be defined by

$$c_Z(X, \Delta, D) := \sup_{c\in\mathbb{R}} \left\{c : \Delta + cD \text{ is lc close to } Z\right\}$$

[39, Definition 8.1]. In particular, by definition, (X, D) is lc iff $c_X(X, 0, D) \geq 1$. It will be convenient to introduce the following analytic counterpart of $c_Z(X, \Delta, D)$ obtained by replacing D with a psh function v defined in a neighborhood of Z :

$$c_Z(X, \Delta, v) := \sup_{c\in\mathbb{R}} \left\{c : \exists U_Z \ e^{-cv} \in L^1_{loc}(U_Z, \mu_{(X,\Delta;\phi_0)})\right\},$$

where ϕ_0 is a fixed locally bounded metric on $-(K_X + \Delta)$ and $\mu_{(X,\Delta;\phi_0)}$ denotes the corresponding measure on X (see Section 2.1.3). By the boundedness assumption on ϕ_0 the definition above is independent of the choice of ϕ_0. More generally, v can be taken as a metric on a $\mathbb{Q}-$line bundle $L \to X$. The following generalization of the identity 3.18 holds:

Lemma 3.7. *Let (X, Δ) be a log pair. For $v = v_D$ as in formula 3.17 we have that*

$$c_Z(X, \Delta, v) = c_Z(X, \Delta, D)$$

In particular, (X, Δ) has log terminal singularities iff $\mu_{(X,\Delta;\phi_0)}$ gives finite volume to X.

Proof. This is essentially well-known, but for completeness we provide a proof (see [19, 39] for the standard case when X is smooth and Δ is trivial and [28, Lemma 6.8][4, Lemma 3.2] for the last statement of the lemma). First note that it follows directly from the definitions that it will be enough to show that $e^{-v} \in L^1_{loc}(U_Z, \mu_{(X,\Delta;\phi_0)})$ iff $(X, \Delta + D)$ is klt (since we may then replace v by cv and D by cD and take the sup with respect to c). To this end take a log resolution of $(X, \Delta + D)$ and denote by π the corresponding morphism from X' to X. Denote by Δ' the divisor on X' such that (X', Δ') corresponds to (X, Δ) as in formula 2.1. Since D is $\mathbb{Q}-$Cartier this means that $(X', \Delta' + \pi^*(D))$ corresponds to $(X, \Delta + D)$. Next, using $\mu_{(X,\Delta;\phi_0)} = \mu_{(X',\Delta';\pi^*\phi_0)}$ gives that $v \in L^1_{loc}(U_Z, \mu_{(X,\Delta;\phi_0)}$ iff $I := \int_{\pi^{-1}(U_Z)} e^{-((v_{\Delta'+\pi^*(D)})-v_0)} dV < \infty$ (after perhaps shrinking U_Z) for some volume form dV on X, where v_0 is a fixed locally bounded metric on the $\mathbb{Q}-$line bundle $\mathcal{O}(\Delta' + \pi^*(D))$. But X' is smooth and $\Delta' + \pi^*(D)$ has simple normal crossings and hence it follows from the basic fact that $c_0(\log |z|^2) = 1$ in $\mathbb{C}$ and Fubini's theorem that the the integral I is finite iff the coefficients of $\Delta' + \pi^*(D)$ are < 1 (just as in [28, Lemma 6.8][4, Lemma 3.2]) which equivalently means that $(X, \Delta + D)$ is klt, as desired. $\qquad\square$

Proposition 3.8. *Assume that $\mathcal{X}$ is a normal $\mathbb{Q}-$Gorenstein variety and $\pi : \mathcal{X} \to \mathbb{C}$ a projective morphism over $\mathbb{C}$ which is smooth (i.e. a submersion) over $\mathbb{C}^*$. Let $\mathcal{L} \to \mathcal{X}$ be a semi-positive $\mathbb{Q}-$line coinciding with $-K_{\mathcal{X}/\mathbb{C}}$ over $\mathbb{C}^*$ and ϕ a locally bounded metric on $\mathcal{L}$ with positive curvature current. Denote by l_0 the Lelong number l_0 at $\tau = 0$ of the induced $L^{2/r}-$metric on the line bundle $\pi_*(\mathcal{L} + K_{\mathcal{X}/\mathbb{C}}) \to \mathbb{C}$. Then*

$$(3.19) \qquad l_0 = 1 - c_{\mathcal{X}_0}(\mathcal{X}, -\Delta, \mathcal{X}_0)$$

where Δ is the zero-divisor in $\mathcal{X}$ of any local trivialization section of $\pi_*(\mathcal{L} + K_{\mathcal{X}/\mathbb{C}}) \to \mathbb{C}$, identified with an element of $H^0(U, \mathcal{L} + K_{\mathcal{X}/\mathbb{C}})$. Moreover, denoting by E_i the reduced components of $\mathcal{X}_0$ we define the numbers m_i and c_i by

$$\mathcal{X}_0 = \sum_i m_i E_i, \;\; \Delta' = \sum_i c_i E_i,$$

the following holds:

- *If $\mathcal{X}$ is smooth and the central fiber $\mathcal{X}_0$ has simple normal crossings, then*

$$(3.20) \qquad l_0 = \max_i \frac{m_i - 1 - c_i}{m_i}$$

- *If $\mathcal{L} = -K_{\mathcal{X}/\mathbb{C}}$, then $l_0 = 0$ iff $(\mathcal{X}, \mathcal{X}_0)$ is log canonical near $\mathcal{X}_0$ iff $\mathcal{X}_0$ is reduced and the normalization of $\mathcal{X}_0$ has log canonical singularities.*

Proof. Fix a local trivializing section of $\pi_*(r(\mathcal{L} + K_{\mathcal{X}/\mathbb{C}})) \to \mathbb{C}$ over a neighborhood V of $0 \in \mathbb{C}$ identified with a global holomorphic section s of $r(\mathcal{L} + K_{\mathcal{X}/\mathbb{C}}) \to \mathcal{X}_{|V}$ as in the proof of Lemma and denote by $v(\tau)$ the corresponding weight on V (formula 3.6). By Lemma 3.2 $v(\tau)$ is subharmonic and we denote by l_0 the Lelong number of v at $\tau = 0$ (as in Lemma 2.6). It will be very useful to represent the Lelong number l_0 as follows

$$(3.21) \qquad l_0 = \inf \left\{ l : \int_V e^{-(v(\tau)+(1-l)\log|\tau|^2)} id\tau \wedge d\bar\tau < \infty \right\}$$

(the equivalence with the ordinary definition follows immediately from $c_0(\log |\tau|^2) = 1$). Next recall that $\nu_\phi := |s|^{2/r} e^{-\phi}$ defines a measure on $\mathcal{X}_{|V}$, naturally attached

to ϕ (see Section 2.1.3). The measure ν_ϕ has the property that for, any continuous function g on $\mathbb{C}$,

$$(3.22) \qquad \int_{\mathcal{X}_{|V}} \nu_\phi \pi^* g = \int_V e^{-v(\tau)} g(\tau) i d\tau \wedge d\bar{\tau}$$

The proof of formula 3.19 is simply a matter of unraveling definitions. First assume, to fix ideas, that $\mathcal{X}$ is smooth. Then there exists a locally bounded metric ψ on $-K_{\mathcal{X}}$ and $\|\cdot\|$ on $\mathcal{L}$ such that

$$(3.23) \qquad \nu_\phi = \|s\|^2 \mu_\psi$$

where μ_ψ is the measure on $\mathcal{X}$ corresponding to ψ (see Section 2.1.3). Indeed, fixing holomorphic coordinates $w = (w_0, ..., w_n)$ on $U \subset \mathcal{X}$ and a trivialization $s_{\mathcal{L}}$ of $\mathcal{L} \to U$ the section s of $r(\mathcal{L} + K_{\mathcal{X}/\mathbb{C}})$, restricted to U, may be written as $s = f_U s_{\mathcal{L}} \otimes dw \otimes \pi^* \frac{\partial}{\partial \tau}$, for a holomorphic function f_U on U and thus on U

$$\nu_{\phi U} = |f_U(w)|^{2/r} e^{-\phi_U(w)} i^{n^2} dw \wedge d\bar{w},$$

where, by assumption, ϕ is bounded on U. This proves formula 3.23 in case $\mathcal{X}$ is smooth. More generally, if $\mathcal{X}$ is $\mathbb{Q}-$Gorenstein, then ν_ϕ and μ_ψ are still well-defined and by the argument above above the relation 3.23 holds on the regular locus of of $\mathcal{X}$ and thus everywhere since the two measures do not charge the singular locus of $\mathcal{X}$. In particular, combining formula 3.22 (for $g(\tau) = e^{-(1-l)\log|\tau|^2)}$) and formula 3.23 gives

$$(3.24) \qquad l_0 = \inf \left\{ l : \int_{\mathcal{X}_{|V}} e^{-(1-l)\log|\tau|^2)} \|s\|^2 \mu_\psi < \infty \right\} = 1 - c_{\mathcal{X}_0}(\mathcal{X}, -\Delta, \mathcal{X}_0)$$

using in the last equality Lemma 3.7 (applied to the log pair $(\mathcal{X}, -\Delta)$ where Δ is the zero-divisor of s and with $v = \log|\tau|^2$ and $D = \mathcal{X}_0$) which concludes the proof of formula 3.19. Formula 3.20 then follows immediately from basic formula for log canonical thresholds of simple normal crossing divisors. For completeness we provide a proof: by assumption $\Delta + c\mathcal{X}_0$ has simple normal crossings and

$$-\Delta + c\mathcal{X}_0 = \sum_i (-c_i + cm_i) E_i$$

and since $c_{\mathcal{X}_0}(\mathcal{X}, \Delta, \mathcal{X}_0)$ is the sup over all c such that the coefficients above are ≤ 1 we get $c_{\mathcal{X}_0}(\mathcal{X}, \Delta, \mathcal{X}_0) = \min_i \frac{1+c_i}{m_i}$ which, by formula 3.19, proves the formula in the first point. To prove the second point we apply formula 3.24 to the case where $\Delta = 0$ and thus $l_0 = 0$ iff $1 - c_{\mathcal{X}_0}(\mathcal{X}, \mathcal{X}_0) = 0$, i.e. iff $(\mathcal{X}, \mathcal{X}_0)$ if log canonical. Now, if $(\mathcal{X}, \mathcal{X}_0)$ is log canonical then it follows that $\mathcal{X}_0$ is reduced and, by adjunction, that its normalization has log canonical singularities (see [?, 2.7]). Finally, the converse follows from "inversion of adjunction", i.e. from the main result of [38], previously conjectured by Shokurov (the special case when $\mathcal{X}_0$ has log terminal singularities follows from a previous result of Kollar et al [39, Theorem 7.5]). $\qquad \square$

Combining the last point in the previous proposition with Lemma 2.2 gives the following

Corollary 3.9. *Let $(\mathcal{X}, \mathcal{L})$ be a test configuration (with a priori non-normal total space $\mathcal{X}$) for a smooth Fano manifold $(X, -K_X)$ such that the central fiber $\mathcal{X}_0$ is normal. Then $\mathcal{X}$ is $\mathbb{Q}-$Gorenstein with $\mathcal{L} = -K_{\mathcal{X}/\mathbb{C}}$ and $l_0 = 0$ iff the variety X_0*

has log canonical singularities. In particular, if $\mathcal{X}$ is a special test configuration then $l_0 = 0$.

3.3.1. *Interlude on Calabi-Yau degenerations.* Before continuing we make a brief detour to point out that Proposition 3.8 also has some applicatiotions to the non-Fano case when $\mathcal{X}$ is Gorenstein and $K_{\mathcal{X}}$ is a trivial line bundle and hence the generic fiber X_τ is a Calabi-Yau manifold. Then $F := \pi_*(K_{\mathcal{X}/\mathbb{C}}) \to \mathbb{C}$ is the Hodge line bundle and its curvature $dd^c v(\tau)$ (wher $v(\tau)$ is given by formula 3.6) coincides with the Weil-Peterson metric ω_{WP} on the punctured base $\mathbb{C}^*$ i.e. the pull-back of the Weil-Peterson metric on the moduli space of Calabai-Yau manifolds (see [76] and references therein). In this case $v(\tau)$ admits an expansion of the form $v(\tau) = l_0 \log |\tau|^2 + \beta \log(|\log |\tau|^2|^{-1}) + O(1)$ as $\tau \to 0$, for some integer $\beta \in [0, n]$, where $O(1)$ denotes a term which is bounded in C^2_{loc} [76, Theorem 4.1]. Accordingly, Proposition 3.8 applied to this case says that $l_0 = c(\mathcal{X}, \mathcal{X}_0)$ and $l_0 = 0$ iff $v(\tau)$ has at worst log log singularities, i.e.

$$(3.25) \qquad v(\tau) = \beta \log(|\log |\tau|^2|^{-1}) + O(1)$$

iff $\mathcal{X}_0$ is reduced and its normalization has log canonical singularities. This observation can be used to simplify the proof of Theorem 1.2 in [74] (which answers in the affirmative a question of Wang) saying that *if the Weil-Peterson metric ω_{WP} on the base of the Calabi-Yau fibration $\pi : \mathcal{X}^* \to \mathbb{C}^*$ as above is not complete as $\tau \to 0$, then, after a base change, the central fiber $\mathcal{X}_0$ may be modified so that $\mathcal{X}_0$ is reduced and has canonical singularities.* The starting point is, following [74], the recent advances in the MMP which give that, after a base change, one can assume that $(\mathcal{X}, \mathcal{X}_0)$ is relatively minimal (i.e. divisorially log terminal, dlt) and in particular log canonical. Hence, by Prop 3.8 $l_0 = 0$ i.e. $v(\tau)$ has at worst a log log singularity as in formula 3.25. The incompleteness assumption on $\omega_{WP} = dd^c v = \beta\omega_P + O(1)$, where ω_P is the Poincaré form on $\mathbb{C}^*$ thus forces $\beta = 0$ and hence $v(\tau)$ is bounded as $\tau \to 0$. But then one concludes that X_0 is irreducible with log terminal singularities (and hence canonical singularities since K_X is assumed Cartier), as desired (the last claim is the content of the implication $(c) \implies (a)$ in Theorem 1.1 in [74], whose proof is due to Sebastien Boucksom: by Fatou's lemma $i^{n^2} \int_{X_0} \Omega \wedge \Omega < \infty$ for a non-trivial $\Omega \in H^0(X_0, K_{X_0})$ and adjunction, using the dlt assumption, then implies that X_0 is irreducible and normal and thus log terminal by Lemma 3.7).

3.4. **Expressing the Donaldson-Futaki invariant in terms of the Ding functional.** Consider the following $\mathbb{Q}-$line bundle over $\mathbb{C}$ defined in terms of the fixed log resolution:

$$\eta' := -\frac{1}{(n+1)L^n} \langle \mathcal{L}', ..., \mathcal{L}' \rangle + \frac{1}{L^n} \langle \mathcal{L}' + K_{\mathcal{X}'/\mathbb{C}} + D', \mathcal{L}'..., \mathcal{L}' \rangle,$$

(recall that $L = -K_X$ here so that $\mu = n$ in formula 2.9). Then

$$(3.26) \qquad DF(\mathcal{X}, \mathcal{L}) = w_0(\eta')$$

Indeed, combining Prop 2.4 with the push-forward formula for intersection numbers gives

$$(n+1)L^n(DF(\mathcal{X}, \mathcal{L})) = np^*(\bar{\mathcal{L}}) \cdot p^*(\bar{\mathcal{L}}) \cdots p^*(\bar{\mathcal{L}}) + (n+1)p^* K_{\overline{\mathcal{X}}/\mathbb{P}^1} \cdot p^*(\bar{\mathcal{L}}) \cdots p^*(\bar{\mathcal{L}}),$$

Now, $p^* K_{\overline{\mathcal{X}}/\mathbb{P}^1}$ is equal to $K_{\overline{\mathcal{X}'}/\mathbb{P}^1} + D'$ modulo the $p-$exceptional divisor E', which give no contribution to the intersection number above, since $p^* \mathcal{L}$ is trivial on E'. Formula 3.26 then follows precisely as in the proof of Prop 2.5.

Lemma 3.10. *We have that $DF(\mathcal{X}, \mathcal{L}) = w_0(\eta') \geq w_0(\delta')$*

Proof. Using $DF(\mathcal{X}, \mathcal{L}) = w_0(\eta')$ and decomposing

$$(3.27) \qquad \eta' = \delta' + \left(\frac{1}{L^n} \left\langle K_{\mathcal{X}'/\mathbb{C}} + D' + \mathcal{L}', \mathcal{L}', ..., \mathcal{L}' \right\rangle - \pi'_*(\mathcal{L}' + K_{\mathcal{X}'/\mathbb{C}} + D') \right)$$

reveals that it is enough to show that $w_0 \left(\frac{1}{L^n} \left\langle K_{\mathcal{X}'/\mathbb{C}} + D' + \mathcal{L}', \mathcal{L}', ..., \mathcal{L}' \right\rangle - \pi'_*(\mathcal{L}' + K_{\mathcal{X}'/\mathbb{C}}) + D' \right) =$

$$= \frac{1}{L^n}(K_{\overline{\mathcal{X}'}/\mathbb{P}^1} + D' + \bar{\mathcal{L}}') \cdot \bar{\mathcal{L}}' \cdots \bar{\mathcal{L}}' - \deg \pi'_*(\bar{\mathcal{L}}' + K_{\overline{\mathcal{X}'}/\mathbb{P}^1} + D') \geq 0,$$

where we have used the the compactification $\bar{\mathcal{X}}'$ of the resolution $\mathcal{X}'$ and the corresponding extension $\bar{\mathcal{L}}'$ of $\mathcal{L}'$ in the first equality (together with formula 2.11). To simplify the notation we consider the case when X is smooth so that $D' = 0$, but the general case is essentially the same. Note that the formula above involving the degrees is invariant under $\mathcal{L}' \to \mathcal{L}' \otimes \pi'^* \mathcal{O}_{\mathbb{P}^1}(m)$ and hence we may as well assume that $\deg \pi'_*(\bar{\mathcal{L}}' + K_{\overline{\mathcal{X}'}/\mathbb{P}^1}) = 0$ (this corresponds to a performing an overall twisting of the original action ρ on $\mathcal{L}$). But the latter vanishing means that the line bundle $\pi'_*(\bar{\mathcal{L}}' + K_{\overline{\mathcal{X}'}/\mathbb{P}^1}) \to \mathbb{P}^1$ admits a global trivializing holomorphic section s, unique up to scaling by a non-zero complex constant. In particular, s induces a global holomorphic section $\bar{\mathcal{L}}' + K_{\overline{\mathcal{X}'}/\mathbb{P}^1} \to \bar{\mathcal{X}}'$. This means that $\bar{\mathcal{L}}' + K_{\overline{\mathcal{X}'}/\mathbb{P}^1}$ is linearly equivalent to an effective divisor E (whose support is contained in the central fiber). But then it follows, since $\bar{\mathcal{L}}'$ is relatively semi-ample, that

$$(3.28) \qquad (K_{\overline{\mathcal{X}'}/\mathbb{P}^1} + \bar{\mathcal{L}}') \cdot \bar{\mathcal{L}}' \cdots \bar{\mathcal{L}}' = E \cdot \bar{\mathcal{L}}' \cdots \bar{\mathcal{L}}' \geq 0$$

which thus concludes the proof. $\qquad \square$

Now we are ready to prove the following more precise version of Theorem 1.3, stated in the introduction:

Theorem 3.11. *Let X be a Fano variety with log terminal singularities and $(\mathcal{X}, \mathcal{L})$ a test configuration (with normal total space) for $(X, -K_X)$ with ϕ denoting a locally bounded metric on $\mathcal{L} \to \mathcal{X} \to \Delta$ with positive curvature current. Then, setting $\phi^t := \rho(\tau)^* \phi_\tau$, identified with a ray of metrics on $-K_X$ we have*

$$(3.29) \qquad DF(\mathcal{X}, \mathcal{L}) = \lim_{t \to \infty} \frac{d}{dt} \mathcal{D}(\phi^t) + q,$$

where q is a non-negative rational number determined by the polarized central fiber $(\mathcal{X}_0, \mathcal{L}_{|\mathcal{X}_0})$ with the following properties, in the case that X is smooth:

- *If $(\mathcal{X}', \mathcal{X}'_0)$ is a given log resolution of $(\mathcal{X}, \mathcal{X}_0)$ with E_i denoting the reduced components of $\mathcal{X}'_0$, then the following formula holds*

$$(3.30) \qquad q = \max_i \frac{m_i - 1 - c_i}{m_i} + \frac{1}{L^n} \sum_i c_i \mathcal{L}'^n \cdot E_i,$$

  where m_i and c_i are the order of vanishing along E_i of $\mathcal{X}'_0$ of $\pi'^* \tau$ and any given non-trivial meromorphic (multi-)section s' of $\mathcal{L}' + \mathcal{K}_{\mathcal{X}'/\mathbb{C}} \to \mathcal{X}'$, respectively, i.e. if Δ' denotes the zero-divisor of s', then

$$\mathcal{X}'_0 = \sum_i m_i E_i, \quad \Delta' = \sum_i c_i E_i$$

- $q = 0$ iff $\mathcal{X}$ is $\mathbb{Q}-$Gorenstein with $\mathcal{L}$ isomorphic to $-K_{\mathcal{X}/\mathbb{C}}$ and $\mathcal{X}_0$ is reduced and its normalization has log canonical singularities.

Proof. First observe that we may as well assume that ϕ^t is a weak geodesic ray. Indeed, if ψ^t is the ray corresponding to a locally bounded metric ψ on $\mathcal{L}$ then $\phi - \psi$ is uniformly bounded and hence $f(t) := f_1(t) - f_2(t) := v_\phi(e^{-t/2}) - v_\psi(e^{-t/2})$ (compare formula 3.3) is bounded as $t \to \infty$. But since $f_i(t)$ is convex (by Lemma 3.2) the limit of $df_i(t)/dt$ as $t \to \infty$ exists (a priori in $]0, \infty]$) and since $f(t)$ is bounded it follows that the limits of $df_i(t)/dt$ coincide. Similarly, $g(t) := \mathcal{E}(\phi_t) - \mathcal{E}(\psi_t)$ is a difference of convex functions (compare the proof of Prop 3.6) and hence the limits of $d\mathcal{E}(\phi_t)/dt$ and $d\mathcal{E}(\psi_t)/dt$ coincide and thus so do the limits of $d\mathcal{D}(\phi_t)/dt$ and $d\mathcal{D}(\psi_t)/dt$.

To simplify the notation we will in the rest of the proof assume that X is smooth so that $D' = 0$, but the proof in the general case is essentially the same. Fix a trivializing section s of $\pi'_*(\mathcal{L}' + K_{\mathcal{X}'/\mathbb{C}}) \to \mathbb{C}$. The section s induces an isomorphism between $\mathcal{L}$ and $-K_{\mathcal{X}^*/\mathbb{C}^*}$ over $\mathcal{X}^*$. In fact, since the formula for $DF(\mathcal{X}, \mathcal{L})$ is invariant under an overall twist of the action ρ on $\mathcal{L}$ we may as well assume that s is an invariant section and hence, using the notation in the previous lemma $\deg \pi'_*(\bar{\mathcal{L}}' + K_{\bar{\mathcal{X}}'/\mathbb{P}^1}) = 0$. We also fix a trivializing (mulit-)section σ_1 of the $\mathbb{Q}-$line $-\frac{1}{L^n(n+1)} \langle \mathcal{L}', ..., \mathcal{L}' \rangle_{|\tau=1}$. By Lemma 2.6

$$w(\delta') = -\lim_{t \to \infty} \frac{d}{dt} \log \|\rho(\tau) S_1\|^2_{\Phi'} + l_0,$$

where $S_1 = \sigma_1 \otimes s_1 \in \delta'_{|\tau=1}$ and l_0 is the Lelong number of the metric Φ' on δ'. Now, $\|\rho(\tau) S_1\|^2_{\Phi'} = \|S_1\|^2_{\rho(\tau)^* \Phi'_{|\mathcal{X}_\tau}}$ and hence setting $\phi^t = \rho(\tau)^* \phi_\tau$ and fixing a metric ψ on $-K_X$ we can write

$$-\log \|\rho(\tau) S_1\|^2_{\Phi'} + \log \|\sigma_1\|^2_{\psi_D} = -\frac{1}{L^n} \mathcal{E}(\phi_t, \psi) - \log \int_X e^{-\phi_t} := \mathcal{D}(\phi_t)$$

using the previous identifications and the change of metrics formula for the Deligne pairing 2.7. Now, using $DF(\mathcal{X}, \mathcal{L}) = w_0(\eta')$ and the decomposition formula in Lemma 3.27 together with formula 3.28 and Lemma 2.6 gives

$$(3.31) \qquad DF(\mathcal{X}, \mathcal{L}) = \lim_{t \to \infty} \frac{d}{dt} \mathcal{D}(\phi^t) + q, \quad q := l_0 + \frac{1}{L^n} \sum_i c_i \mathcal{L}'^n \cdot E_i,$$

where c_i is the order of vanishing of s along E_i, when s is viewed as a global holomorpic section of $\mathcal{L}' + K_{\mathcal{X}'/\mathbb{C}} \to \mathcal{X}'$. Moreover, by the trivializing assumption on s the numbers c_i above coincide with thouse appearing in the formula for l_0 in Prop 3.8 and hence formula 3.30 follows. Note that both terms appearing in the definition of q above are non-negative and hence $q \geq 0$. Indeed, by Prop 3.5 $l_0 \geq 0$ and the non-negativity of the second terms follows directly from the definitions giving that $c_i \geq 0$ and $\mathcal{L}'$ is semi-ample. Next note that a general meromorphic section of $\mathcal{L}' + K_{\mathcal{X}'/\mathbb{C}} \to \mathcal{X}'$ may be written as $f(\tau)s'$ for $f(\tau)$ a meromorphic function, whose vanishing (or pole) order at $\tau = 0$ we denote by m. Since the formula for q is invariant under $c_i \to c_i + m$ the case of a general section thus follows. Accordingly, in the rest of the proof we will take c_i to be the non-negative numbers determined by the globally trivializing section s'.

Now, by formula 3.31 $q = 0$ iff the follows condition holds: $l_0 = 0$ and $c_i = 0$ for all index i in the set I defined by the condition $\mathcal{L}'^n \cdot E_i > 0$. But by formula 3.30

the latter condition holds iff $m_i = 1$ and $c_i = 0$ for any $i \in I$, i.e. any i such that E_i is not $p-$exceptional for the log resolution p (since $\mathcal{L}$ is assumed relative ample). Since $\mathcal{X}$ is normal we may, by Hironaka's theorem, take p to be an isomorphism on $p^{-1}(\mathcal{X} - \mathcal{Z})$, where $\mathcal{Z}$ is a subvariety of codimension at least two (containing the singular locus of $\mathcal{X}$). Hence, if $q = 0$ then $\mathcal{X}_0$ is reduced at any point in $\mathcal{X} - \mathcal{Z}$ (using that, by the previous argument, $m_i = 1$ for any non $p-$exceptional E_i). In other words, $q = 0$ implies that the central fiber $\mathcal{X}_0$, viewed as a divisor on the normal variety $\mathcal{X}$, is reduced. But then, since $q = 0$ also implies that $c_i = 0$ for any $i \in I$ it also follows that $\mathcal{L}$ is isomorphic to $-K_{\mathcal{X}/\mathbb{C}}$ on $\mathcal{X} - \mathcal{Z}$ and since the codimension of $\mathcal{X} - \mathcal{Z}$ is at least two $\mathcal{L}$ is thus the unique extension of $-K_{\mathcal{X}/\mathbb{C}}$ from the regular locus of $\mathcal{X}$, which, by definition, means that $\mathcal{X}$ is $\mathbb{Q}-$Gorenstein. Conversely, if $\mathcal{X}_0$ is reduced and $\mathcal{X}$ is $\mathbb{Q}-$Gorenstein, it follows from Prop 3.8 that $l_0 = 0$ and hence $q = 0$. $\qquad \square$

3.5. **Conclusion of the proof of Theorem 1.1.** Given a test configuration $(\mathcal{X}, \mathcal{L})$ for $(X, -K_X)$ Theorem 3.11 gives that for any weak geodesic ϕ^t ray emanating from any given metric on $\mathcal{L}$ which is associated to $(\mathcal{X}, \mathcal{L})$ we have

$$DF(\mathcal{X}, \mathcal{L}) = \lim_{t \to \infty} \frac{d}{dt} \mathcal{D}(\phi^t) + q, \quad q \geq 0$$

Next, by the convexity of $\mathcal{D}(\phi^t)$ the limit in the right hand side above is bounded from below by the right derivative $\frac{d}{dt} \mathcal{D}(\phi_t)_{|t=0^+}$ which, by formula 3.2, is non-negative if ϕ^0 is taken as a Kähler-Einstein metric. Thus $DF(\mathcal{X}, \mathcal{L}) \geq 0$ and if $DF(\mathcal{X}, \mathcal{L}) = 0$ then it must, since $q \geq 0$, be that $\lim_{t \to \infty} \frac{d}{dt} \mathcal{D}(\phi_t) = 0$ and hence $\mathcal{D}(\phi_t)$ is affine so that the second point in Prop 3.5 implies that $\mathcal{X}$ is isomorphic to a product test configuration.

4. Ramifications and applications

4.1. **An analog of Donaldson's conjecture about geodesic stability.** Combining the results above with the very recent existence result in [16] one arrives at the following analog of a conjecture of Donaldson [22] (see [14] for partial results about Donaldson's original conjecture):

Theorem 4.1. *Let X be a Fano manifold. Then precisely one of the following two alternatives holds:*

(1) *X admits a Kähler-Einstein metric*
(2) *For any given $\phi^0 \in \mathcal{H}_b(-K_X)$ there exists a weak geodesic ray ϕ^t in $\mathcal{H}_b(-K_X)$ emenating from ϕ^0 such that the Ding functional $\mathcal{D}(\phi^t)$ is strictly descreasing for sufficently large times.*

Proof. If the first alternative holds, then it follows immediately, by the convexity of $\mathcal{D}(\phi^t)$ (just as in the proof of Theorem 1.1) that the second alternative cannot hold. Now assume that the first alternative does not hold. Then, by the results in [16] X is not K-polystable along special test configurations, i.e. there exists a special test configuration $\mathcal{X}$ such that one of the following alternatives hold (a) $DF(\mathcal{X}) < 0$ or (b) $DF(\mathcal{X}) = 0$, but $\mathcal{X}$ is a not a product test configuration. Now, by Theorem 3.11 $DF(\mathcal{X})$ is the large time limit of $d\mathcal{D}(\phi^t)/dt$ where ϕ^t is any weak geodesic ray attached to $\mathcal{X}$. Assuming, to get a contradiction, that alternative above 2 does not hold, there exists a sequence of $t_i \to \infty$ such that $d\mathcal{D}(\phi^t)/dt \geq 0$. By the convexity of $\mathcal{D}(\phi^t)$ this means that there exists a $T > 0$ such that $d\mathcal{D}(\phi^t)/dt \geq 0$ on $[T, \infty[$.

In particular, $DF(\mathcal{X}) \geq 0$ and hence it must be that alternative (b) holds, i.e. $DF(\mathcal{X}) = 0$ and thus by, convexity, $\mathcal{D}(\phi^t)$ is affine on $[T, \infty[$. But then it follows from Prop 3.5 that $\mathcal{X}$ is a product test configuration, which contradicts (b). $\qquad\square$

In the original conjecture of Donaldson $(X, -K_X)$ is replaced by a general polarized manifold (X, L) and the Ding functional with the Mabuchi functional. Moreover, originally Donaldson's conjecture asked for bona fide geodesic rays ϕ^t of smooth and stricly positively curved metrics, but in view of the recent theory about geodesics one would expect that the best regularity that one can hope for is that $\omega^t := dd^c\phi^t$ be locally bounded, if ω^0 is a Kähler form. By the regularity results in [62], this is indeed the case in Theorem 4.1 above. Finally it should be pointed out that a weaker version of Theorem 4.1 has independently been obtained in [18], where it is assumed that X admits no holomorphic vector fields and where the "destabilizing" weak geodesic ray ϕ^t appearing in item 2 is merely in a finite energy class. On the other hand the proof in [18] dos not rely on the results in [16] (but rather estimates along the Kähler-Ricci flow).

4.2. Bounds on the Ricci potential and Perelman's $\lambda-$entropy functional.
Let now X be a Fano manifold and denote by $\mathcal{K}(X)$ the space of all Kähler metrics ω in $c_1(X)$ (equivalently, $\omega = dd^c\phi$ for some strictly positively curved metric ϕ on $-K_X$). In this section we will use the normalization $V := c_1(X)^n := \int_X \omega^n$. Recall that the Ricci potential h_ω is the function on X defined by $dd^c h_\omega = \text{Ric } \omega - \omega$ together with the normalization condition $\int e^{h_\omega}\omega^n/V = 1$, which in terms of the previous notation means that $h_{dd^c\phi} := h_\phi := -\log(\frac{(dd^c\phi)^n/V}{e^{-\phi}/\int e^{-\phi}})$. Note in particular that

$$\left\|1 - e^{h_\omega}\right\|_{L^1(X,\omega)} = \left\|\frac{1}{V}(dd^c\phi)^n - \frac{e^{-\phi}}{\int e^{-\phi}}\right\|,$$

where the norm in the right hand side is the total variation norm on the space of absolutely continuous probability measures on X.

Next, let $(\mathcal{X}, \mathcal{L})$ be a test configuration of a polarized manifold (X, L) and define its "$L^\infty-$norm" by

$$\|(\mathcal{X}, \mathcal{L})\|_\infty := \left\|\frac{d\phi^t}{dt}_{|t=0}\right\|_{L^\infty(X)}, \tag{4.1}$$

where ϕ^t is the (weak) geodesic determined by $\mathcal{X}$, emanating from any fixed reference metric $\phi^0 \in \mathcal{H}(X, L)$. The point is that if $\|(\mathcal{X}, \mathcal{L})\|_\infty \neq 0$ then the *normalized Donaldson-Futaki invariant* $DF(\mathcal{X}, \mathcal{L})/\|(\mathcal{X}, \mathcal{L})\|_\infty$ is independent of base changes of $(\mathcal{X}, \mathcal{L})$, induced by $\tau \to \tau^m$ (which correspond to reparametrizations of ϕ^t, induced by $t \mapsto mt$). We will be relying on the following lemma which is a special case of a very recent result of Hisamoto [36, Theorem 1.1]:

Lemma 4.2. *The number $\|(\mathcal{X}, \mathcal{L})\|_\infty$ is well-defined, i.e. it is independent of ϕ_0.*

Now we can prove the following theorem using a slight variant of the proof of Theorem 1.3; the result can be seen as an analog of Donaldson's lower bound on the Calabi functional [24],

Theorem 4.3. *Let X be a Fano manifold. Then*

$$\inf_{\omega \in \mathcal{K}(X)} \left\|1 - e^{h_\omega}\right\|_{L^1(X,\omega)} \geq \sup_{(\mathcal{X}, \mathcal{L})} \frac{-DF(\mathcal{X}, \mathcal{L})}{\|(\mathcal{X}, \mathcal{L})\|_\infty},$$

where $(\mathcal{X}, \mathcal{L})$ ranges over all test configurations $(\mathcal{X}, \mathcal{L})$ such that $\|(\mathcal{X}, \mathcal{L})\|_\infty \neq 0$. Moreover, if equality holds and the infimum is attained at some ω and the supremum is attained at $(\mathcal{X}, \mathcal{L})$ (with $\mathcal{X}$ normal), then $(\mathcal{X}, \mathcal{L})$ is isomorphic to a product test configuration. In particular,

$$\inf_{\omega \in \mathcal{K}(X)} \int h_\omega e^{h_\omega} \frac{\omega^n}{V} \geq \frac{1}{2} \sup_{(\mathcal{X}, \mathcal{L})} \left(\frac{DF(\mathcal{X}, \mathcal{L})}{\|(\mathcal{X}, \mathcal{L})\|_\infty} \right)^2$$

where the sup ranges over all destabilizing $(\mathcal{X}, \mathcal{L})$ (i.e. $DF(\mathcal{X}, \mathcal{L}) > 0$) with the same same necessary conditions for equality as before. In particular, if X is K-unstable then both infimums above are strictly positive.

Proof. Fix $(\mathcal{X}, \mathcal{L})$ and $\phi^0 \in \mathcal{H}(X, -K_X)$ and denote by ϕ^t the corresponding (weak) geodesic. By convexity of the Ding functional, combined with Theorem 1.3 (using that $q \geq 0$), we have

(4.2)
$$\int_X \left(\frac{1}{V}(dd^c \phi_0)^n - \frac{e^{-\phi_0}}{\int e^{-\phi_0}} \right) \frac{d\phi^t}{dt} \geq -\frac{d}{dt} \mathcal{D}(\phi^t)_{t=0} \geq -\lim_{t \to \infty} \frac{d}{dt} \mathcal{D}(\phi^t) \geq -DF(\mathcal{X}, \mathcal{L}).$$

Applying Hölder's inequality with exponents $(q, p) = (1, \infty)$ thus gives

(4.3)
$$\left\| 1 - e^{h_\omega} \right\|_{L^1(X, \omega)} \left\| \frac{d\phi^t}{dt}_{|t=0} \right\|_{L^\infty(X)} \geq -DF(\mathcal{X}, \mathcal{L})$$

and using the independence in the previous lemma then concludes the proof of the first inequality of the Theorem. The second inequality then follows immediately from the classical Csiszar-Kullback-Pinsker inequality between the relative entropy and the total variation norm [17]. As for the equality case it follows, just as in the second proof of Theorem 1.1, from the equality cases in 4.2. Finally, if X is $K-$unstable then there exists, by definition, a test configuration such that $DF(\mathcal{X}, \mathcal{L}) > 0$ and for any such test configuration the inequality 4.3 forces $\|(\mathcal{X}, \mathcal{L})\|_\infty > 0$, which concludes the proof. □

Recall that in the definition of a test configuration $(\mathcal{X}, \mathcal{L})$ we have fixed an action ρ on $\mathcal{L}$ and thus the norm $\|(\mathcal{X}, \mathcal{L})\|_\infty$ certainly depends on ρ. Indeed, twisting ρ with a character of $\mathbb{C}^*$ shifts the tangent of ϕ^t with a constant. On the other hand, $DF(\mathcal{X}, \mathcal{L})$ is independent of such a twist and hence the previous theorem still holds if we replace $\|(\mathcal{X}, \mathcal{L})\|_\infty$ with its (smaller) normalized version obtained by replacing the $L^\infty(X)-$norm in the definition 4.1 with the quotient norm on the quotient space $L^\infty(X)/\mathbb{R}$.

Remark 4.4. As pointed out above Lemma 4.2 is a special case of a general result of Hisamoto [36], saying that the measure $(\frac{d\phi^t}{dt})_* MA(\phi^t)$ on $\mathbb{R}$ only depends on the test configuration $(\mathcal{X}, \mathcal{L})$ and moreover is equal to the limiting normalized weight measures for the $\mathbb{C}^*-$action, as conjectured by Witt-Nyström [50], who settled the case of product test configurations. In particular, by [36] all the L^p-norms $\|(\mathcal{X}, \mathcal{L})\|_p$ of $\frac{d\phi^t}{dt}$ (integrating against $MA(\phi^t)$) only depend on $(\mathcal{X}, \mathcal{L})$ and coincide with the limits of the corresponding l^p-norms of the weights $\{\lambda_i^{(k)}\}$. In particular, letting $p \to \infty$ gives Lemma 4.2. Using this the proof of the previous theorem shows that the theorem holds, more generally, when $\|(\mathcal{X}, \mathcal{L})\|_\infty$ is replaced by $\|(\mathcal{X}, \mathcal{L})\|_p$ for $p \in [1, \infty]$ and the L^1-norm with the corresponding L^q-norm, where q is the Young (Hölder) dual of p. In fact, as shown in [5] a similar argument can be used

to give a new proof and extend to general L^p-norms Donaldson's lower bound on the Calabi functional [24].

Next, we recall that Perelman's W-functional [55], when restricted to the space all pairs (ω, f) such that ω as in the space $\mathcal{K}(X)$ of all Kähler metrics ω in $c_1(X)$ and f is a smooth function such that $e^{-f}\omega^n$ has unit mass, is given by

$$ W(\omega, f) := \int_X (R_\omega + |\nabla f|^2 + f)e^{-f}\omega^n, $$

where R_ω is the scalar curvature of ω normalized so that $\int R_\omega \omega = n$ for any $\omega \in c_1(X)$ (as usual in the Kähler setting where the volume of the metrics is fixed we have set Perelman's parameter τ to be equal to $1/2$ and as in [55, 72, 73, 33] we have subtraced the universal constant $2n$ from Perelman's original definition). Then Perelman's $\lambda-$entropy functional on $\mathcal{K}(X)$ is defined as

$$ \lambda(\omega) = \inf_{f \in \mathcal{C}^\infty(X): \int e^{-f}\omega^n = 1} W(\omega, f) $$

[55, 72, 73, 33] and in particular $\lambda(\omega) \leq W(\omega, 0) = nV$.

Corollary 4.5. *Let X be an $n-$dimension Fano manifold. Then*

$$ \sup_{\omega \in \mathcal{K}(X)} \lambda(\omega) \leq nV - \frac{1}{2} \sup_{(\mathcal{X}, \mathcal{L})} \left(\frac{DF(\mathcal{X}, \mathcal{L})}{\|(\mathcal{X}, \mathcal{L})\|_\infty} \right)^2 $$

where $V = c_1(X)^n$ and $(\mathcal{X}, \mathcal{L})$ ranges of all destabilizing test configurations for $(X, -K_X)$. In particular, if X is K-unstable then $\lambda \leq nV - \epsilon$ for some positive number ϵ.

Proof. As explained in [33] $\lambda(\omega) + \int h_\omega e^{h_\omega}\omega^n \leq nV$ (using $W(\omega, f) \leq W(\omega, -h_\omega)$ and one integration by parts) and hence the corollary follows immediately from the previous theorem. $\square$

Remark 4.6. The previous inequality was inspired by the result in [73] and its extension to general non-invariant Kähler metrics in [33], saying that

$$ \sup_{\omega \in \mathcal{K}(X)} \lambda(\omega) \leq nV - \sup_{\xi \in \text{Lie}G} H(\xi), $$

with equality if X admits a Kähler-Ricci soliton, where $\text{Lie}G$ is the Lie algebra of a maximal compact subgroup in $\text{Aut}_0(X)$ and H is a certain concave functional on $\text{Lie}G$, defined in [73]. The proof in [33] was based on the convexity of the functional v_{ϕ^t}, while we here use the convexity of the whole Ding functional.

4.3. **The log Fano setting.** Let us briefly recall the more general setting of Kähler-Einstein metrics on log Fano varieties [4] and log K-stability [26, 41, 52]. In a nutshell, this setting is obtained from the previous one by replacing the canonical line bundle K_X with the *log canonical line bundle* $K_{(X,D)} := K_X + D$ of a given log pair (X, D). For example, (X, D) is said to be a *(weak) log Fano variety* if $-K_{(X,D)}$ is ample (nef and big). A *log Kähler-Einstein metric* ω associated to (X, D) is, by definition, a current ω in $c_1(-K_{(X,D)})$, defining a Kähler metric on $X_{reg} - D$, with locally bounded potentials on X and such that

$$ \text{Ric } \omega - [D] = \omega, $$

holds in the sense of currents, where $[D]$ denotes the current of integration defined by D. Equivalently [4], this means that ω is the curvature current of a locally bounded metric ϕ_{KE} on $-K_{(X,D)}$ satisfying

$$(dd^c\phi_{KE})^n = Ce^{-(\phi_{KE}+\log|s_D|^2)}$$

(for some constant C) in the sense of pluripotential theory, where we recall that $e^{-(\phi+\log|s_D|^2)}$ denotes the measure associated to a metric ϕ on $-K_{(X,D)}$; see section 2.1.3. The definitions are compatible with log resolutions (as in formula 2.1). Hence if (X,D) is a weak log Fano variety, then so is (X',D') and ϕ_{KE} is a log Kähler-Einstein metric for (X,D) iff $p^*\phi_{KE}$ is a log Kähler-Einstein metric for (X',D').

Example 4.7. If (X,D) is log smooth and klt, i.e. X is smooth and D has simple normal crossings with coefficents < 1, it follows the regularity results in [13, 31] give that any log Kähler-Einstein metric for (X,D) has edge-cone singularities along D. Moreover, by [34], if the log Mabuchi functional (or the log Ding functional) for (X,D) is proper the metric even admits a complete polyhomogenous expansion along D (this is shown in [34] when D has a singular component and the general case is anounced in [34]). However, one of the main points of the approach in the present paper is that it only relies on very weak regularity properties of the metric (the local boundedness of ϕ_{KE}) and that it is independent of any properness assumption.

The notion of K-stability has also been generalized to the log setting (see [26, 41, 52]). Briefly, a test configuration for a log Fano variety (X,D) consists of a test configuration $(\mathcal{X},\mathcal{L})$ for (X,L) where $L = -K_{(X,D)}$. The $\mathbb{C}^*-$action, applied to the support of D in $\mathcal{X}_1$, induces a $\mathbb{C}^*-$invariant divisor $\mathcal{D}^*$ in $\mathcal{X}^*$ and we denote by $\mathcal{D}$ its closure in $\mathcal{X}$. The corresponding *log Donaldson-Futaki invariant* $DF(\mathcal{X},\mathcal{L};\mathcal{D})$ was defined in [41] (by imposing linearity it is enough to consider the case when D is reduced and irreducible). A direct calculation reveals that, up to normalization, the definition in [41] is equivalent to replacing the relative canonical divisor K in the intersection theoretic formula 2.10 with the relative log canonical divisor $K+\mathcal{D}$, defined as a Weil divisor (compare [52]):

$$(4.4) \qquad DF(\mathcal{X},\mathcal{L};\mathcal{D}) = \mu\bar{\mathcal{L}}\cdot\bar{\mathcal{L}}\cdots\bar{\mathcal{L}} + (n+1)(K+\mathcal{D})\cdot\cdot\bar{\mathcal{L}}\cdots\bar{\mathcal{L}},,$$

where now $\mu = n(-(K_X+D))\cdot L^{n-1}/L^n$. We can hence take the latter formula as the definition of the invariant $DF(\mathcal{X},\mathcal{L};\mathcal{D})$. Finally, (X,D) is said to be *log K-polystable* if, for any test configuration, $DF(\mathcal{X},\mathcal{L};\mathcal{D}) \geq 0$ with equality iff the test configuration is equivariantly isomorphic to a product test configuration.

Theorem 4.8. *Let (X,D) be a log Fano variety admitting a log Kähler-Einstein metric, where D is an effective $\mathbb{Q}-$divisor on X. Then (X,D) is log K-polystable.*

Proof. The proof given for Theorem 1.1 actually proves Theorem 4.8 as well, since, in the case when X is singular, it uses a log resolution to replace $\mathcal{X}$ with a pair $(\mathcal{X}'^*, D^*)$ and then uses the closure of the divisor D^* in $\mathcal{X}'$. $\qquad\square$

The theorem thus confirms one direction of the log version of the Yau-Tian-Donaldson conjecture formulated in [41].

5. Outlook on the existence problem for Kähler-Einstein metrics on $\mathbb{Q}-$Fano varieties

Very recently the existence of a Kähler-Einstein metric on a K-polystable Fano manifold X was finally settled by Donaldson-Chen-Sun [16]. In this section we will briefly discuss how some of the results in the present paper may be useful when considering the corresponding existence problem on a singular Fano variety. We will follow Tian's original program, which is based on Aubin's continuity method (see the outline in [71] and references therein), but using the Ding functional as a replacement for the Mabuchi functional used in [71]. Howeer, it should be emphasized that the recent results in [16] are based on a modification of Tian's program introduced by Donaldson which involves Kähler-Einstein metrics with conical singularities (obtained by replacing the smooth form η in Aubin's equation 5.1 below, with the current defined by a suitable anti-canonical $\mathbb{Q}-$divisor D). One motivation for using Aubin's original method here is that Tian's conjecture on the partial C^0-estimate (see H1 below) has now been proved along Aubin's continuity method when X is smooth (see [65] which builds on[16]) and one can thus hope that it will eventually also be established on singular Fano varieties.

The main connection to the present paper stems from the following immediate consequence of Theorem 3.11 applied to a special test configuration, which, as will be explained below, together with two the general hypotheses H1 and H2 gives the existence of a Kähler-Einstein metric on a K-stable Fano manifold.

Corollary 5.1. *Let X be a Fano variety with log terminal singularities and $\mathcal{X}$ a special test configuration for X such that $DF(\mathcal{X}) > 0$. Fix a smooth and positively curved metric ϕ on $-K_{\mathcal{X}/\mathbb{C}}$ (more generally, local boundedness is enough) and set $\phi^t := \rho^* \phi_\tau$. Then the Ding functional $\mathcal{D}$ and the Mabuchi functional $\mathcal{M}$ both tend to infinity along ϕ^t, as $t \to \infty$.*

Proof. By Theorem 3.11 we have that $\lim_{t \to \infty} \frac{d}{dt} \mathcal{D}(\phi^t) > 0$ and hence $\mathcal{D}(\phi^t) \to \infty$ which, by the well-known inequality $\mathcal{M} \geq \mathcal{D}$, concludes the proof. $\square$

We recall that the Mabuchi functional $\mathcal{M}$ admits a natural extension to the space $\mathcal{H}_b(-K_X)$ taking values in $]-\infty, \infty]$ such that $\mathcal{M}(\phi)$ is finite precisely when the measure $MA(\phi)$ has finite pluricomplex energy and relative entropy [4]. In particular, by the regularity results in [62], $\mathcal{M}(\phi^t)$ is finite, for any fixed t, if the initial metric ϕ_0 is smooth and hence under the assumption in the previous corollary $\mathcal{M}(\phi^t) \to \infty$ tends to infinity as $t \to \infty$. See [60, 15] for related results in the case when the total space $\mathcal{X}$ is assumed smooth.

After recalling Tian's program in the smooth setting with an eye towards the singular case we will comment on further complications arising when considering the existence problem on general $\mathbb{Q}$- Fano varieties.

5.1. The case of a smooth Fano variety X.
The starting point of Tian's program is the continuity equation

$$(5.1) \qquad\qquad \operatorname{Ric} \omega_t = t\omega_t + (1-t)\eta,$$

where ω_0 is a given Kähler metric of positive Ricci curvature η and $t \in [0, 1]$ is a fixed parameter. Let I be the set of all t such that a solution ω_t exists. As shown by Aubin $I \cap [0, 1[$ is open and non-empty and hence to prove the existence of a Kähler-Einstein metrics, i.e. that $1 \in I$, it is enough to show that I is closed. More precisely,

denoting by T the boundary of I and taking $t_i \to T$ we can write $\omega_{t_i} = dd^c \phi_{t_i}$ for suitably normalized metrics ϕ_{t_i} on $-K_X$ (e.g. satisfying $\sup_X(\phi_{t_i} - \phi_0) = 0$) and to show that I is closed it is enough to establish the following C^0-estimate:

(5.2)
$$\sup_X |\phi_{t_i} - \phi_0| \leq C$$

(then the higher order estimates follow using the Aubin-Yau C^2-estimate, Evans-Krylov theory and elliptic boot strapping). Before continuing we recall that the following properties hold along the continuity path 5.1 for $t \geq t_0$ (for a fixed $t_0 \in I$) :

(5.3)
$$(i)\, \mathrm{Ric}\, \omega_t \geq t_0 \omega_t, \quad (ii)\, \mathcal{M}(\phi_t) \leq C_0,$$

where the first property follows immediately from the fact that $\eta \geq 0$ and the second one from the well-known fact that $\mathcal{M}(\phi_t)$ is decreasing in t.

In order to relate the desired C^0-estimate 5.2 to algebraic properties of X Tian proposed the following conjecture stated in terms of the Bergman function $\rho_\omega^{(k)}(x)$, at level k, associated to a Kähler metric ω on X :

$$\rho_\omega^{(k)}(x) = \sum_{i=0}^{N_k} |s_i^{(\phi)}|^2 e^{-k\phi},$$

where ϕ is any metric on $-K_X$ with curvature form ω and $\{s_i^{(\phi)}\}$ is any base in $H^0(X, -kK_X)$ which is orthonormal with respect to the L^2-norm $\|\cdot\|_{k\phi}$ on $H^0(X, -kK_X)$ determined by ϕ, i.e. $\|s\|_{k\phi}^2 = \int_X |s|^2 e^{-k\phi} \omega^n$.

- **(H1)** (Tian's partial C^0-estimate). Given $t_0 \in]0,1]$, let $\mathcal{K}(X, t_0)$ be the space of all Kähler metrics ω in $c_1(X)$ such that $\mathrm{Ric}\,\omega \geq t_0 \omega$. Then there exists a $k > 0$ and $\delta > 0$ such that kL is very ample and for any $\omega \in \mathcal{K}(X, t_0)$,
$$\inf_X \rho_\omega^{(k)}(x) \geq \delta$$

(more precisely, the conjecture says that k can be chosen arbitrarily large). If the previous conjecture holds then, as follows immediately from the definition of $\rho_\omega^{(k)}$, the desired C^0-estimate holds 5.2 iff

(5.4)
$$\sup_X \left| \phi_{t_i}^{(k)} - \phi_0 \right| \leq C$$

where now $\phi_{t_i}^{(k)}$ is the Bergman metric at level k determined by ϕ_{t_j}, i.e. $\phi_{t_j}^{(k)} = \frac{1}{k} \log \sum_{i=0}^{N_k} |s_i^{(\phi_{t_j})}|^2$. In other words: $\phi_{t_i}^{(k)}$ is the scaled pull-back of the Fubini-Study metric ϕ_{FS} on $\mathcal{O}(1) \to \mathbb{P}^{N_k}$ under the Kodaira map F_j determined by ϕ_{t_j} :

$$F_j : X \to \mathbb{P}^{N_k}, \quad \phi_{t_i}^{(k)} = F_j^* \phi_{FS}/k, \quad F_j(X) := V_j$$

i.e. $F_i(x) = [s_0(x) : s_1(x) : \cdots s_{N_k}(x)]$, where now (s_i) is a fixed base, which is orthonormal with respect to the L^2-norm determined by ϕ_{t_j} (strictly speaking, due to the choice of base V_i is only determined modulo action of the the unitary group $U(N_k + 1)$, but since this group is compact this fact will be immaterial in the following). After passing to a subsequence we may assume that the projective subvariety $V_j := F_j(X) \subset \mathbb{P}^{N_k}$, converges, in the sense of cycles, to an algebraic cycle V_∞ in $\mathbb{P}^{N_k}$. It was indicated by Tian [71] that the validity of the previous conjecture would imply that the cycle V_∞ is reduced, irreducible and even defines a *normal* variety. More precisely, we will make the following

- **(H2)** V_∞ is normal with log terminal singularities and there is a one parameter subgroup $\rho : \mathbb{C}^* \to GL(N_k + 1, \mathbb{C})$ such that

$$\sup_X \left| \phi_{t_j}^{(k)} - \rho(\tau_i)^* \phi_{FS} \right| \leq C$$

where $\rho(\tau_i) V_0$ also converges (in the corresponding Hilbert scheme) to the normal variety V_∞

Then, by standard properties of the Hilbert scheme ρ determines a (special) test configuration $(\mathcal{X}, \mathcal{L})$ with central fiber V_∞. In fact, as explained in [16] the existence of ρ in the case of Donaldson's contintuity method follows from the reductivity of the automorphism group of (V_∞, D_∞) established in [16], where D_∞ is a divisor on V_∞ induced by $\eta = [D]$. Now, assuming that X is K-stable (for simplicity we consider the case when X admits no non-trivial holomorphic vector fields, but the general argument is similar) we have that $DF(\mathcal{X}, \mathcal{L}) \geq 0$ with equality iff $(\mathcal{X}, \mathcal{L})$ is equivariantly isomorphic to a product test-configuration (recall that the total space $\mathcal{X}$ here is automatically normal and even $\mathbb{Q}$−Gorenstein, by Lemma 2.2). In the latter case, $\rho(\tau)^* \phi_{FS} - \phi$ is trivially bounded and hence the desired C^0−estimate 5.2 then holds, showing that X indeed admits a Kähler-Einstein metric. The main issue is thus to exclude the case of $DF(\mathcal{X}, \mathcal{L}) > 0$ and this is where Cor 5.1 enters into the picture. Thus, assuming the validity of H1 and H2 above we deduce from Cor 5.1 (with ϕ the restriction of the Fubini-Study metric) that if the second alternative $DF(\mathcal{X}, \mathcal{L}) > 0$ holds, then the Ding functional $\mathcal{D}$ tends to infinity along $\rho(\tau_i)^* \phi_{FS}$ and hence it is unbounded from above along $\phi_{t_j}^{(k)}$. But this implies that $\mathcal{D}$ is also unbounded along the original sequence ϕ_{t_j} (also using that if $|\psi - \psi'| \leq C$ then $|\mathcal{D}(\psi) - \mathcal{D}(\psi')| \leq 2C$, as follows immediately from the definition 3.1). But, since $\mathcal{D} \leq \mathcal{M}$, this contradicts the property (ii) in formula 5.3 hence it must be that the first alternative, $DF(\mathcal{X}, \mathcal{L}) = 0$, holds and thus X admits a Kähler-Einstein metric, as desired.

5.2. Towards the case of $\mathbb{Q}$−Fano varieties.

Let us finally discuss some of the new complications that arise when trying to generalize Tian's program to the case of singular K-polystable Fano varieties X (by [51] such a Fano variety X automatically has log terminal singularities). Taking η to be a smooth semi-positive form in $c_1(X)$ the continuity equations 5.1 are defined as before and, by the results in [4], the set I is still non-empty, i.e. $T > 0$ (using the positivity of the alpha invariant of X). The solutions ω_t define Kähler forms on X_{reg} with volume $c_1(X)^n / n!$. Next, we note that Tian's conjecture admits a natural generalization to general $\mathbb{Q}$−Fano varieties if one uses the notion of (singular) Ricci curvature appearing in [4] (and similarly for general log Fano varieties (X, D)). However, one new difficulty that arises is the *openness* of I. From the point of view of the implicit function theorem the problem is to find appropriate Banach spaces, encoding the singularities of X (the uniqueness of solutions to the formally linearized version of equation 5.1, for $t \in]0, 1[$, follows from the results in [4]). On the other hand, another approach could be to use the following lemma, where the properness refers to the exhaustion function defined by the J−functional (see [4] for the singular case).

Lemma 5.2. *The set I is open iff the twisted Ding (Mabuchi) functional $\mathcal{D}_t$ (associated to the twisting form $(1 - t)\eta$) is proper for any $t \in I$.*

Proof. If $\mathcal{D}_t$ is proper, then it follows from the results in [4] that a solution ω_t exists. Conversely, if a solution ω_t exists and I is open, i.e. solutions $\omega_{t+\delta}$ exist for δ sufficiently small, then it follows from the convexity of $\mathcal{D}_{t+\delta}$ along weak geodesics that $\mathcal{D}_{t+\delta} \geq C$. But since δ may be taken to be positive this implies that $\mathcal{D}_t$ is proper (and even coercive; compare [4]). $\square$

Note that in the case $n = 2$ it is a basic fact that a projective variety X has log terminal singularities iff it has quotient singularities (defining an orbifold structure on X) and hence the two-dimensional Fano varieties are precisely the orbifold Del Pezzo surfaces. In the general Fano orbifold case, if one takes η to be an orbifold Kähler metric, the usual implicit function theorem applies to give that I above is indeed open. For the case of K-polystable Del Pezzo surfaces with canonical singularities (i.e. ADE singularities) the existence of Kähler-Einstein metrics was established very recently in [53], using a different method, thus generalizing the case of smooth Del Pezzo surfaces settled by Tian [70].

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