# A priori $L^\infty$-estimates for degenerate complex Monge-Ampère equations

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**Abstract :** We study families of complex Monge-Ampère equations, focusing on the case where the cohomology classes degenerate to a non big class. We establish uniform a priori $L^\infty$-estimates for the normalized solutions, generalizing the recent work of S. Kolodziej and G. Tian. This has interesting consequences in the study of the Kähler-Ricci flow.

## 1 Introduction

Let $\pi : X \longrightarrow Y$ be a non degenerate holomorphic mapping between compact Kähler manifolds such that $n := dim_{\mathbb{C}} X \geq m := dim_{\mathbb{C}} Y$. Let $\omega_X$, $\omega_Y$ Kähler forms on $X$ and $Y$ respectively. Let $F : X \longrightarrow \mathbb{R}^+$ be a non negative function such that $F \in L^p(X)$ for some $p > 1$.

Set $\omega_t := \pi^*(\omega_Y) + t\omega_X$, $t > 0$. We consider the following family of complex Monge-Ampère equations

$$(\star)_t \qquad \begin{cases} (\omega_t + dd^c\varphi_t)^n & = \ c_t t^{n-m} F \omega_X^n \\ \max_X \varphi_t = 0 & = \ 0 \end{cases}$$

where $\varphi_t$ is $\omega_t-$plurisubharmonic on $X$ and $c_t > 0$ is a constant given by

$$c_t t^{n-m} \int_X F \omega_X^n = \int_X \omega_t^n.$$

It follows from the seminal work of S.T. Yau [Y] and S. Kolodziej [K 1], [K 2] that the equation $(\star)_t$ admits a unique continuous solution. (Observe that for $t \in ]0,1]$, $\omega_t$ is a Kähler form).

Our aim here is to understand what happens when $t \to 0^+$, motivated by recent geometrical developpments [ST], [KT]. When $n = m$, the cohomology class $\omega_0$ is big and semi-ample and this problem has been adressed by several authors recently (see [CN], [EGZ], [TZ], [To]).

We focus here on the case $m < n$. This situation is motivated by the study of the Kähler-Ricci flow on manifolds $X$ of intermediate Kodaira dimension $1 \leq kod(X) \leq n-1$. When $n = 2$ this has been studied by J.Song and G.Tian [ST].

In a very recent and interesting paper [KT], S. Kolodziej and G. Tian were able to show, under a technical geometric assumption on the fibration $\pi$, that the solutions $(\varphi_t)$ are uniformly bounded on $X$ when $t \searrow 0^+$.

The purpose of this note is to (re)prove this result without any technical assumption and with a different method: we actually follow the strategy introduced by S. Kolodziej in [K] and further developped in [EGZ], [BGZ].

**THEOREM.** *There exists a uniform constant $M = M(\pi, \|F\|_p) > 0$ such that the solutions to the Monge-Ampère equations $(\star)_t$ satisfy*

$$\|\varphi_t\|_{L^\infty(X)} \ \leq \ M, \ \forall t \in ]0,1].$$

It follows from our result that Theorems 1 and 2 in [KT] hold without any technical assumption on the fibration (see condition 0.2 in [KT]).

This result has been announced by J-P. Demailly and N. Pali [DP].

# 2 Proof of the theorem

## 2.1 Preliminary remarks

**Uniform control of $c_t$.** Observe that $\omega_0^k = 0$ for $m < k \leq n$, hence for all $t \in ]0,1]$,

$$\omega_t^n = \sum_{k=1}^{m} \binom{n}{k} t^{n-k} {\omega_0}^k \wedge {\omega_X}^{n-k}.$$

Note that $]0,1] \ni t \longmapsto t^{m-n}\omega_t^n$ is increasing (hence decreases as $t \searrow 0^+$) and satisfies for $t \in ]0,1]$

$$(1) \quad \binom{n}{m} \qquad \frac{\omega_0^m \wedge \omega_X^{n-m}}{\int_X \omega_0^m \wedge \omega_X^{n-m}} \leq \frac{\omega_t^n}{t^{n-m}\int_X \omega_0^m \wedge \omega_X^{n-m}} \leq \frac{\omega_1^n}{\int_X \omega_0^m \wedge \omega_X^{n-m}}.$$

In particular $t \longmapsto c_t$ is increasing in $t \in ]0,1]$ and

$$0 < \binom{n}{m} \frac{\int_X \omega_0^m \wedge \omega_X^{n-m}}{\int_X F\omega_X^n} =: c_0 \quad \leq \quad c_t \leq \quad c_1.$$

**Uniform control of densities.** Let $J_\pi$ denote the (modulus square) of the Jacobian of the mapping $\pi$, defined through

$$\omega_0^m \wedge \omega_X^{n-m} = J_\pi \omega_X^n.$$

Let us rewrite the equation $(\star)_t$ as follows

$$(\omega_t + dd^c\varphi_t)^n = f_t\omega_t^n,$$

where for $t \in ]0,1]$

$$0 \leq f_t := c_t t^{n-m} F \frac{\omega_X^n}{\omega_t^n} \leq c_1 \frac{F}{J_\pi}.$$

Observe that

$$\int_X f_t\omega_t^n = c_t t^{n-m} \int_X F\omega_t^n = \int_X \omega_t^n =: Vol_{\omega_t}(X),$$

hence $(f_t)$ is uniformly bounded in $L^1(\omega_t/V_t)$, $V_t := Vol_{\omega_t}(X)$. We actually need a slightly stronger information.

**Lemma 2.1** *There exists $p' > 1$ and a constant $C = C(\pi, \|F\|_{L^p(X)}) > 0$ such that for all $t \in ]0,1]$*

$$\int_X f_t^{p'}\omega_t^n \leq C\, Vol_{\omega_t}(X).$$

**Proof of the lemma.** Set $V_t := Vol_{\omega_t} = \int_X \omega_t^n$ and observe that

$$0 \leq f_t \frac{\omega_t^n}{V_t} \leq c_1 F \frac{\omega_X^n}{\int_X \omega_0^m \wedge \omega_X^{n-m}} = C_2 F \omega_X^n,$$

where $C_2 := c_1 \int_X J_\pi \omega_X^n$.

This shows that the densities $f_t$ are uniformly in $L^1$ w.r.t. the normalized volume fomrs $\omega_t^n/V_t$.

Since $J_\pi$ is locally given as the square of the modulus of a holomorphic function which does not vanish identically, there exists $\alpha \in ]0,1[$ such that $J_\pi^{-\alpha} \in L^1(X)$. Fix $\beta \in ]0,\alpha[$ satisfying the condition $\beta/p + \beta/\alpha = 1$. It follows from Hölder's inequality that

$$\int_X f_t^\beta \omega_X^n \leq \Big(\int_X F^p \omega_X^n\Big)^{\beta/p} \Big(\int_X J_\pi^{-\alpha}\omega_X^n\Big)^{\beta/\alpha}.$$

Setting $\varepsilon := \beta/q$ and using Hölder's inequality again , we obtain

$$\int_X f_t^{1+\varepsilon} \frac{\omega_t^n}{V_t} \leq C_2 \int_X f_t^\varepsilon F \omega_X^n.$$

Now applying again Hölder inequality we get

$$\int_X f_t^{1+\varepsilon} \frac{\omega_t^n}{V_t} \leq C_2 \Big(\int_X f_t^\beta \omega_X\Big)^{1/q} \|F\|_{L^p(X)}.$$

Therefore denoting by $p' := 1+\varepsilon$, we have the following uniform estimate

$$\int_X f_t^{p'} \frac{\omega_t^n}{V_t} \leq C(\pi, \|F\|_{L^p(X)}), \forall t \in ]0,1],$$

where

$$C(\pi, \|F\|_{L^p(X)}) := C_2 \Big( \int_X J_\pi^{-\alpha} \omega_X^n \Big)^{\beta/\alpha q} \|F\|_{L^p(X)}^{1+\beta/q}.$$

▶

## 2.2 Uniform domination by capacity

We now show that the measure $\mu_t := f_t \omega_t^n / Vol_{\omega_t}$ are uniformly strongly dominated by the normalized capacity $\mathrm{Cap}_{\omega_t}/Vol_{\omega_t}(X)$. It actually follows from a carefull reading of the no parameter proof given in [EGZ], [BGZ].

**Lemma 2.2** *There exists a constant* $C_0 = C_0(\pi, \|F\|_{L^p(\omega_X^n)}) > 0$ *such that for any compact set* $K \subset X$ *and* $t \in ]0,1]$,

$$\mu_t(K) \leq C_0^n \left( \frac{\mathrm{Cap}_{\omega_t}(K)}{Vol_{\omega_t}(X)} \right)^2 .$$

Proof: Fix a compact set $K \subset X$. Set $V_t := Vol_{\omega_t}(X)$. Hölder's inequality yields

$$\mu_t(K) \leq \left( \int_X f_t^{p'} \frac{\omega_t^n}{V_t} \right)^{1/p'} \left( \int_K \frac{\omega_t^n}{V_t} \right)^{1/q'} .$$

It remains to dominate uniformly the normalized volume forms $\omega_t^n / V_t$ by the normalized capacities $\mathrm{Cap}_{\omega_t}/V_t$. Fix $\sigma > 0$ and observe that for any $t \in ]0,1]$,

$$\int_K \frac{\omega_t^n}{V_t} \leq \int_X e^{-\sigma(V_{K,\omega_t} - \max_X V_{K,\omega_t})} \frac{\omega_t^n}{V_t} T_{\omega_t}(K)^\sigma,$$

where

$$V_{K,\omega_t} := \sup\{\psi \in PSH(X,\omega_t); \psi \leq 0, \text{ on } K\}$$

is the $\omega_t-$extremal function of $K$ and $T_{\omega_t}(K) := \exp(-\sup_X V_{K,\omega_t})$ is the associated $\omega_t-$capacity of $K$ (see [GZ 1] for their properties).

Observe that $\omega_t^n / V_t \leq c_1 \omega_1^n$ and $\omega_t \leq \omega_1$, hence the family of functions $V_{K,\omega_t} - \max_X V_{K,\omega_t}$ is a normalized family of $\omega_1-$psh functions. Thus there exists $\sigma > 0$ which depends only on $(X, \omega_1)$ and a constant $B = B(\sigma, X, \omega_1)$ such that ([Z])

$$\int_X e^{-\sigma(V_{K,\omega_t} - \max_X V_{K,\omega_t})} \frac{\omega_t^n}{V_t} \leq B, \forall t \in ]0,1].$$

The Alexander-Taylor comparison theorem (see Theorem 7.1 in [GZ 1]) now yields for a constant $C_3 = C_3(\pi, \|F\|_{L^p(X)})$

$$\mu_t(K) \leq C_3 \exp\left[-\sigma\left(\frac{V_t}{\mathrm{Cap}_{\omega_t}(K)}\right)^{1/n}\right], \forall t \in ]0,1].$$

We infer that there is a constant $C_4 = C_4(\pi, \|F\|_{L^p(X)})$ such that

$$\mu_t(K) \leq C_4 \left(\frac{\mathrm{Cap}_{\omega_t}(K)}{V_t}\right)^2, \forall t \in ]0,1]. \tag{2}$$

### 2.3 Uniform normalization

The comparison principle (see [K] ,[EGZ]) yields for any $s > 0$ and $\tau \in [0,1]$

$$\tau^n \frac{\mathrm{Cap}_{\omega_t}(\{\varphi_t \leq -s-\tau\})}{V_t} \leq \int_{\{\varphi_t \leq -s\}} \frac{(\omega_t + dd^c\varphi_t)^n}{V_t}.$$

It is now an exercise to derive from this inequality an a priori $L^\infty-$estimate,

$$\|\varphi_t\|_{L^\infty(X)} \leq C_5 + \ s_0(\omega_t),$$

where $s_0(\omega_t)$ (see [EGZ],[BGZ]) is the smallest number $s > 0$ satisfying the condition $e^n C_0^n \mathrm{Cap}_{\omega_t}(\{\psi \leq -s\})/V_t \leq 1$ for all $\psi \in PSH(X, \omega_t)$ such that $\sup_X \psi = 0$. Recall from ([GZ 1], Prop. 3.6) that

$$\frac{\mathrm{Cap}_{\omega_t}(\{\psi \leq -s-\tau\})}{V_t} \leq \frac{1}{s}\left(\int_X (-\psi)\frac{\omega_t^n}{V_t} + n\right).$$

Since $\frac{\omega_t^n}{V_t} \leq C_1\omega_1^n$, it follows that

$$\frac{\mathrm{Cap}_{\omega_t}(\{\psi \leq -s-\tau\})}{V_t} \leq \frac{1}{s}\left(C_1\int_X (-\psi)\omega_1^n + n\right).$$

Since $\psi$ is $\omega_1-$psh and normalized, we know that there is a constant $A = A(X, \omega_1) > 0$ such that $C_1 \int_X (-\psi)\omega_1^n \leq A$ for any such $\psi$. Therefore $s_0(\omega_t) \leq s_0 := e^n C_0^n (A+n)$ for any $t \in ]0,1]$. Finally we obtain the required uniform estimate for all $t \in ]0,1]$.

## References


[BT1] E. BEDFORD & B. A. TAYLOR: The Dirichlet problem for a complex Monge-Ampère equation. Invent. Math. **37** (1976), no. 1, 1–44.

[BGZ] S.BENELKOURCHI & V.GUEDJ & A.ZERIAHI: A priori estimates for weak solutions of complex Monge-Ampère equations. Ann. Sc. N. Sup. Pisa, to appear.

[CN] P. CASCINI & G. LA NAVE: Kähler-Ricci Flow and the Minimal Model Program for Projective Varieties. Preprint, arXiv:math/06.03.064.

[DP] J-P. DEMAILLY & N.PALI: Degenerate complex Monge-Ampère equations over compact Kähler manifolds. Preprint, arXiv:math/0710.5109.

[EGZ] P.EYSSIDIEUX & V.GUEDJ & A.ZERIAHI: Singular Kähler-Einstein metrics. Preprint arXiv:math/06.03.431.

[GZ 1] V.GUEDJ & A.ZERIAHI: Intrinsic capacities on compact Kähler manifolds, J. Geom. Anal. 15-4 (2005), 607-639.

[GZ 2] V.GUEDJ & A.ZERIAHI: The weighted Monge-Ampère energy of quasiplurisubharmonic functions, J. Funct. Anal. **250** (2007), 442-482.

[K] S.KOLODZIEJ: The complex Monge-Ampre equation. Acta Math. **180** (1998), no. 1, 69–117.

[KT] S. KOLODZIEJ & G.TIAN: A uniform $L^\infty-$estimate for complex Monge-Ampère equations. Preprint arXiv:math/07.10.1144.

[ST] J. SONG & G.TIAN: The Kähler-Ricci flow on surfaces of positive Kodaira dimension,. Preprint, arXiv:math/06.02.150.

[KT] G. TIAN & Z.ZHANG: On the Kähler-Ricci flow on projective manifolds of general type, Chinese Ann. Math. B **27 (2)** (2006), 179-192.

[To] V. TOSATTI: Limits of Calabi-Yau metrics when the Kähler class degenerates. Preprint arXiv:math/07.10.4579.

[Y] S.T.YAU: On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation. I. Comm. Pure Appl. Math. **31** (1978), no. 3, 339–411.

[Z] A.ZERIAHI: Volume and capacity of sublevel sets of a Lelong class of plurisubharmonic functions. Indiana Univ. Math. J. **50** (2001), no. 1, 671-703.