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Regularity of weak minimizers of the K-energy and applications to properness and K-stability

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Abstract

Let (X, ω) be a compact Kähler manifold and $\mathcal{H}$ the space of Kähler metrics cohomologous to ω. If a cscK metric exists in $\mathcal{H}$, we show that all finite energy minimizers of the extended K-energy are smooth cscK metrics, partially confirming a conjecture of Y.A. Rubinstein and the second author. As an immediate application, we obtain that existence of a cscK metric in $\mathcal{H}$ implies J-properness of the K-energy, thus confirming one direction of a conjecture of Tian. Exploiting this properness result we prove that an ample line bundle (L, X) admitting a cscK metric in $c_1(L)$ is K-polystable.

1 Introduction and main results

We borrow notations and terminology from [BDL, Da2, DR] throughout this note. Let (X, J, ω) be a compact connected Kähler manifold and let $\mathcal{H}_\omega = \{v \in C^\infty(X),\ \omega_v := \omega + i\partial\bar{\partial}v > 0\}$ denote the space of Kähler potentials. By Hodge theory, the level set

$$\mathcal{H} := \mathcal{H}_\omega \cap \mathrm{AM}^{-1}(0)$$

is isomorphic to the space of Kähler metrics cohomologous to ω, and we always work on the level of potentials unless specified otherwise (see (2) for the definition of the Aubin–Mabuchi energy AM). The connected Lie group $G := \mathrm{Aut}_0(X, J)$ has a natural action on $\mathcal{H}$ given by pullback of metrics (see [DR, Section 5.2] for a precise description on the level of potentials). We denote by $(\mathcal{E}^1, d_1)$ the metric completion of $\mathcal{H}_\omega$ with respect to the L^1-type Mabuchi path length metric d_1, realized in the space $\mathcal{E}^1$ of finite energy potentials [Da2] (see [Da1] for the corresponding picture for the L^2-Mabuchi metric, originally conjectured by Guedj [G]). The d_1-distance to ω in $\mathcal{H}$ is comparable to Aubin's J-functional J_ω.

Our first main result partially confirms a regularity conjecture of Y. Rubinstein and the second author [DR, Conjecture 2.9] and also a less general conjecture of X.X. Chen [Ch2, Conjecture 6.3]:

Theorem 1.1. *Suppose (X, ω) is a cscK manifold. If $u \in \mathcal{E}^1$ minimizes the extended K-energy $\mathcal{K} : \mathcal{E}^1 \to (-\infty, +\infty]$, then u is a smooth cscK potential. In particular there exists $g \in G$ such that $g^*\omega_u = \omega$.*

The last claim follows from the uniqueness result of [BB]. Using the above result and [DR, Theorem 2.10] we immediately obtain one direction of a conjecture of Tian from the 90's, relating properness of the K-energy to existence of cscK metrics [Ti1, Remark 5.2], [Ti3, Conjecture 7.12].

Theorem 1.2. *Suppose (X, ω) is a Kähler manifold. If there exists a cscK metric cohomologous to ω, then for some $C, D > 0$ we have*

$$\mathcal{K}(u) \geq C \inf_{g \in G} J_\omega(g.u) - D, \quad u \in \mathcal{H}. \tag{1}$$

For the precise definition of the K-energy $\mathcal{K}$, J_ω and other related functionals we refer to the next section. The proof of our theorems rely on the L^1-Mabuchi geometry of $\mathcal{H}$ explored in [Da1, Da2], the finite energy pluripotential theory of [BBEGZ, GZ1] and the methods of [BB]. Realizing that the metric geometry of $\mathcal{H}$ and energy properness should be related seems to have first appeared in [Ch2, Conjecture 6.1], but this work rather proposed the use of the L^2-Mabuchi metric on $\mathcal{H}$.

As a consequence of Theorem 1.2 we obtain a result on K-polystability, originally proved by Mabuchi [Ma2, Ma3] using a completely different argument. Slightly less general or different flavor results were obtained by Stoppa, Stoppa-Székelyhidi, Székelyhidi [Sto, StSz, Sz1] and others. We recall the relevant terminology in the last section of the paper.

Theorem 1.3. *Suppose $L \to X$ is a positive line bundle. If there exists a cscK metric in the class $c_1(L)$, then (X, L) is K-polystable.*

In case the group G is trivial it also follows directly from the properness result above that (X, L) is uniformly K-stable (see [BBJ, BHJ] and references therein).

We end the introduction with a brief (but by no means complete) discussion about further relations to previous results. Much work has been done on Tian's properness conjectures in the case when the Kähler class is anti-canonical ([Ti2, TZ, ZZ, PSSW, BBEGZ, DR] to name only a few). To our knowledge, in the case of cscK metrics, excluding perhaps the particular case of toric Kähler manifolds, very few partial results exist.

Proving the reverse direction of Tian's conjecture (that properness of the K-energy implies existence of cscK metrics), seems to require further progress on the weak non-linear theory of fourth order partial differential equations and seems to be out of reach for the moment (see [DR, Theorem 2.10] for more details on what needs to be overcome).

Following the techniques of [BB, DR], it is likely that different versions of the above properness theorem can be obtained assuming existence of extremal or soliton/edge type cscK metrics, but also for different spaces of potentials, mimicking [DR, Theorem 2.1, Theorem 2.11, Theorem 2.12]. We leave the discussion of these to the interested reader.

The K-stability results fit into a circle of ideas surrounding the seminal Yau–Tian–Donaldson conjecture on a polarized manifold (X,L), saying that the first Chern class of L contains a Kähler metric with constant scalar curvature if and only if (X, L) is stable in an appropriate sense, inspired by Geometric Invariant Theory. In the formulation introduced by Donaldson [Do1] the stability in question was formulated as K-polystability, but in view of an example in [ACGT] there is now a widespread belief that the notion of K-(poly)stability has to be strengthened (unless X is Fano and L is the anti-canonical polarization). In case G is trivial uniform K-stability was introduced in the thesis of Székelyhidy (see also [Sz1]) to provide such a stronger notion. In the light of the recent variational approach to the Yau–Tian–Donaldson conjecture introduced in [BBJ] it seems that the main analytical hurdle in proving that uniform K-stability conversely implies the existence of a constant scalar curvature metric is the general form of the regularity conjecture, alluded to above. It would be interesting to investigate if the techniques of this paper apply to other related notions of algebro-geometric stability as well, like CM stability, recently investigated in [Ti4]. Finally, it seems likely that our proof of K-polystability can be extended to the transcendental setting considered very recently in [SD], but we will not go further into this here.

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2 The finite energy continuity method

Let us first recall some terminology. Given a positive closed (1,1)-form χ, the functional $J_\chi : \mathcal{E}^1 \to \mathbb{R}$ is defined as follows:

$$J_\chi(u) = \mathrm{AM}_\chi(u) - \frac{1}{V}\Big(\int_X \chi \wedge \omega^{n-1}\Big)\mathrm{AM}(u),$$

where $V = \int_X \omega^n$, AM and AM_χ is the Aubin-Mabuchi energy and its "χ-contracted" version:

$$\mathrm{AM}(u) = \frac{1}{(n+1)V}\sum_{j=0}^{n}\int_X u\omega_u^j \wedge \omega^{n-j}, \quad \mathrm{AM}_\gamma(u) = \frac{1}{nV}\sum_{j=0}^{n-1}\int_X u\gamma \wedge \omega_u^j \wedge \omega^{n-1-j}. \quad (2)$$

Recall the definition of the $\mathcal{I}$ functional from [BBEGZ]:

$$\mathcal{I}(u_0, u_1) = \int_X (u_0 - u_1)(\omega_{u_1}^n - \omega_{u_0}^n), \quad u_0, u_1 \in \mathcal{E}^1_\omega.$$

This functional is non-negative and invariant under adding constants. An elementary calculation gives the following useful estimates:

$$\frac{1}{n(n+1)}\mathcal{I}(u_0, u_1) \leq J_{\omega_{u_0}}(u_1) - J_{\omega_{u_0}}(u_0) \leq \mathcal{I}(u_0, u_1), \quad u_0, u_1 \in \mathcal{E}^1 \cap \mathrm{AM}^{-1}(0). \quad (3)$$

The (extended) K-energy and its χ-twisted version $\mathcal{K}, \mathcal{K}_\chi : \mathcal{E}^1 \to (-\infty, \infty]$ are defined as follows:

$$\mathcal{K}(u) := \mathrm{Ent}(\omega^n, \omega_u^n) + \bar{S}\mathrm{AM}(u) - n\mathrm{AM}_{\mathrm{Ric}\,\omega}(u), \qquad \mathcal{K}_\chi(u) = \mathcal{K}(u) + nJ_\chi(u), \quad (4)$$

where $\mathrm{Ent}(\omega^n, \omega_u^n) = V^{-1}\int_X \log(\omega_u^n/\omega^n)\omega_u^n$ is the entropy of the measure ω_u^n with respect to ω^n. The above formula for the K-energy was originally introduced by Chen–Tian [Ch1]. The authors showed in [BDL] that this functional naturally extends to all L^p-Mabuchi completions of $\mathcal{H}_\omega$ and the extension is d_p-lsc (see [BBEGZ] for a different approach, in the particular case of $p = 1$). For more information on the metric spaces $(\mathcal{H}_\omega, d_p)$ we refer to [Da1, Da2]. In this note we will only focus on the case $p = 1$.

Let us introduce the $\mathcal{E}^1$-minimizer set of $\mathcal{K}$:

$$\mathcal{M}^1 = \{u \in \mathcal{E}^1 \cap \mathrm{AM}^{-1}(0) |\ \mathcal{K}(u) = \inf_{v \in \mathcal{E}^1} \mathcal{K}(v)\}.$$

Given that $\mathcal{K}$ is convex, it is standard to check that $\mathcal{M}^1$ is geodesically complete with respect to the finite energy geodesics of $\mathcal{E}^1$. Finally, as follows from [BDL, Theorem 4.12], there exits a unique $v_\chi \in \mathcal{M}^1$ such that $J_\chi(v_\chi) = \inf_{v \in \mathcal{M}^1} J_\chi(v)$. This fact will be used below.

The following two propositions represent the main analytic ingredient of this note and revolve around the finite energy continuity method for cscK metrics, whose smooth version was recently explored in [CPZ].

Proposition 2.1. *Assume that $\mathcal{M}^1$ is nonempty and $u \in \mathcal{H}_\omega$. Then for any $\lambda > 0$, there exists a unique minimizer $v^\lambda \in \mathcal{E}^1 \cap \mathrm{AM}^{-1}(0)$ of $\mathcal{K}_{\lambda\omega_u}$ and the curve $[0, \infty) \ni \lambda \to v^\lambda \in \mathcal{E}^1 \cap \mathrm{AM}^{-1}(0)$ is d_1-continuous and d_1-bounded with $v^0 = \lim_{\lambda \to 0} v^\lambda$ being the unique minimizer of J_{ω_u} on $\mathcal{M}^1$. Additionally, for any $w \in \mathcal{M}^1,\ \lambda \geq 0$ we have*

$$\mathcal{I}(v^\lambda, u) \leq n(n+1)\mathcal{I}(w, u). \tag{5}$$

Proof. First we show that the curve $\lambda \to v_\lambda$ described in the statement exists. Fixing $\lambda > 0$, observe that $\mathcal{K}_{\lambda\omega_u} = \mathcal{K} + \frac{\lambda}{n} J_{\omega_u} \geq \mathcal{K}$. Let $u_j \in \mathcal{E}^1 \cap \mathrm{AM}^{-1}(0)$ be a minimizing sequence of $\mathcal{K}_{\lambda\omega_u}$. As $\mathcal{M}^1$ is nonempty it follows that $\mathcal{K}$ is bounded from below, giving that $J_{\omega_u}(u_j) \geq 0$ is uniformly bounded from above. As $[\omega_u]_{dR} = [\omega]_{dR}$, [DR, Proposition 5.5] applies to give that $d_1(0, u_j)$ is also uniformly bounded. Again using that $\mathcal{K}(u_j)$ is uniformly bounded we get that $\mathrm{Ent}(\omega^n, \omega^n_{u_j}) \geq 0$ is also uniformly bounded from above, hence we can apply [BBEGZ, Theorem 2.17] (see [DR, Theorem 5.6] for an equivalent formulation that fits our context most). By this last result, from u_j we can extract a d_1-convergent subsequence, converging to $v^\lambda \in \mathcal{E}^1$. By the d_1-lower semi-continuity of $\mathcal{K}_{\lambda\omega_u}$ ([BDL, Theorem 1.2])we get that v^λ is a $\mathcal{E}^1$-minimizer of $\mathcal{K}_{\lambda\omega_u}$ and by [BDL, Theorem 4.13] this minimizer is unique.

Now we argue (5). Let $w \in \mathcal{M}^1$ and $\lambda > 0$. As both v^λ and w minimize a functional, we can write the following:

$$\mathcal{K}(w) + \frac{\lambda}{n} J_{\omega_u}(v^\lambda) \leq \mathcal{K}(v^\lambda) + \frac{\lambda}{n} J_{\omega_u}(v^\lambda) = \mathcal{K}_{\lambda\omega_u}(v^\lambda) \leq \mathcal{K}_{\lambda\omega_u}(w) = \mathcal{K}(w) + \frac{\lambda}{n} J_{\omega_u}(w),$$

hence $J_{\omega_u}(v^\lambda) \leq J_{\omega_u}(w)$. Subtracting $J_{\omega_u}(u)$ from both sides and using (3) we ultimately get (5) for $\lambda > 0$.

We shift focus to the d_1-continuity of the curve $\lambda \to v^\lambda$. By $J_{\omega_u}(v^\lambda) \leq J_{\omega_u}(w)$ and [DR, Proposition 5.5] notice that $d_1(0, v^\lambda)$ is uniformly bounded for any $\lambda > 0$. We know that $\{\mathcal{K}_{\lambda\omega_u}(v^\lambda)\}_{\lambda > 0}$ is bounded from above. Since all terms except the first in the expression of $\mathcal{K}_{\lambda\omega_u}(v^\lambda)$ from (4) are uniformly bounded by $d_1(0, v^\lambda)$ it follows that $\mathrm{Ent}(\omega^n, \omega^n_{v^\lambda}) \geq 0$ is also uniformly bounded from above, ultimately giving that $\{v^\lambda\}_{\lambda > 0}$ is relatively d_1-compact (again by [BBEGZ, Theorem 2.17]).

We claim now that $\lambda \to v^\lambda$ is d_1-continuous for $\lambda > 0$. Indeed, assume that $\{\lambda_j\}_j$ converges to $\lambda > 0$. As shown above, the sequence v^{λ_j} is relatively d_1-compact, hence it suffices to prove that any limit of this sequence coincides with v^λ. So, we can assume that $v^{\lambda_j} \to v$ and we will show that $v = v^\lambda$. For any $w \in \mathcal{E}^1$ we have

$$\mathcal{K}_{\lambda_j\omega_u}(w) \geq \mathcal{K}_{\lambda_j\omega_u}(v^{\lambda_j}) = \mathcal{K}(v^{\lambda_j}) + \frac{\lambda_j}{n} J_{\omega_u}(v^{\lambda_j}).$$

Letting $j \to +\infty$, we can use that $\mathcal{K}$ is d_1-lsc and J_{ω_u} is d_1-continuous, to obtain that $\mathcal{K}_{\lambda\omega_u}(w) \geq \mathcal{K}_{\lambda\omega_u}(v)$. Uniqueness of minimizers of $\mathcal{K}_{\lambda\omega_u}$ [BDL, Theorem 4.13] now gives that $v^\lambda = v$, what we wanted to prove.

Finally, we focus on continuity at $\lambda = 0$. Using compactness of $\{v^\lambda\}_{\lambda>0}$, we can find $\lambda_j \to 0$ and $v^0 \in \mathcal{E}^1$ such that $d_1(v^{\lambda_j}, v^0) \to 0$. We will show that v_0 is independent of the choice of λ_j. By the joint lower semi-continuity of $(h, \lambda) \to \mathcal{K}_{\lambda\omega_u}(h)$ it follows that $v^0 \in \mathcal{M}^1$. Let $q \in \mathcal{M}^1$ be arbitrary. Then we have that

$$\mathcal{K}(q) \leq \mathcal{K}(v^{\lambda_j}) \quad \text{and} \quad \mathcal{K}_{\lambda\omega_u}(v^{\lambda_j}) \leq \mathcal{K}_{\lambda\omega_u}(q),$$

implying that $\frac{\lambda}{n} J_{\omega_u}(v^{\lambda_j}) \leq \frac{\lambda}{n} J_{\omega_u}(q)$, hence $J_{\omega_u}(v^{\lambda_j}) \leq J_{\omega_u}(q)$. After taking $\lambda_j \to \infty$, it follows that $J_{\omega_u}(v^0) \leq J_{\omega_u}(q)$, hence v^0 is the unique minimizer of J_{ω_u} on $\mathcal{M}^1$ (see the comments preceding the theorem), finishing the proof. □

Recall that for $g \in G$ and $\varphi \in \mathcal{H}_\omega$ we denote $g.\varphi$ the unique function in $\mathcal{H}_\omega \cap \mathrm{AM}^{-1}(0)$ such that $\omega_{g.\varphi} = g.\omega_\varphi$ (see [DR] on how this action extends to $\mathcal{E}^1 \cap \mathrm{AM}^{-1}(0)$). As detailed in the next proposition, whose proof builds on the arguments of [BB], if a smooth cscK metric exists then the minimizer of J_{ω_u} on $\mathcal{M}^1$ can be given more specifically:

Proposition 2.2. *Assume that $v \in \mathcal{H}_\omega \cap \mathrm{AM}^{-1}(0)$ is a cscK potential and $u \in \mathcal{H}_\omega$. Then for some $g \in G$ we have:*

$$\inf_{w \in \mathcal{M}^1} J_{\omega_u}(w) = J_{\omega_u}(g.v).$$

Proof. By changing the reference metric ω to $\o_u$, we can assume that $u = 0$. As a cscK metric exists, the group G is reductive, hence there exists $g \in G$ such that $J_\omega(g.v) = \min\{J_\omega(f.v) : f \in G\}$. This is indeed well known and can be seen from the fact that J_ω is equivalent with the growth of the d_1-metric [DR, Proposition 5.5], and the Lie algebra of G has a very specific decomposition (for details see for example Section 6 of [DR], especially [DR, Proposition 6.2 and Proposition 6.9]).

We denote $v^1 = g.v \in \mathcal{H}_\omega \cap \mathrm{AM}^{-1}(0)$ and let $v^0 \in \mathcal{M}^1$ be the unique minimizer of J_ω on $\mathcal{M}^1$ (see comments before Propostion 2.1). We are done if we can show that $v^0 = v^1$. For each $\lambda \in (0, 1/2]$ let v^λ be the unique $\mathcal{E}^1$-minimizer of $\mathcal{K}_{\lambda\omega}$. Then by Proposition 2.1 above, $d_1(v^\lambda, v^0) \to 0$ as $\lambda \to 0$. Let F_λ and G denote the differential of $\mathcal{K}_{\lambda\omega}|_{\mathcal{H}_\omega}$ and $nJ_\omega|_{\mathcal{H}_\omega}$ respectively.

By the same argument as in the proof of [BB, Theorem 4.4,Theorem 4.7] we can find $h \in C^\infty(X)$ such that

$$D_h F_0|_{v^1} = -G(v^1).$$

Again going back to the arguments in [BB, Theorem 4.4, Theorem 4.7], for small enough λ this identity implies

$$|F_\lambda(v^1 + \lambda h).w| \leq C\lambda^2 \sup |w|, \ \forall w \in C(X).$$

Given this last estimate and the explicit formula for F_λ going back to Mabuchi, we can write

$$F_\lambda(v^1 + \lambda h).w = \int_X w f_\lambda \omega^n, \tag{6}$$

where the density $f_\lambda = (\overline{S}_\omega - S_{\omega_{v^1+\lambda h}} + \lambda \mathrm{Tr}^{\omega_{v^1+\lambda h}}\omega)\omega^n_{v^1+\lambda h}/\omega^n \in C^\infty(X)$ satisfies $f_\lambda = O(\lambda^2)$. Let $[0,1] \ni t \to u^\lambda_t \in \mathcal{E}^1$ be the finite energy geodesic connecting $u^\lambda_0 := v^\lambda$ with $u^\lambda_1 := v^1 + \lambda h$, $\lambda \in [0,1/2]$. By Lemma 2.3 proved below, we can write:

$$\frac{d}{dt}\Big|_{t=1^-} \mathcal{K}_{\lambda\omega}(u^\lambda_t) \leq \int_X \dot{u}^\lambda_1 f_\lambda \omega^n.$$

Since $u^\lambda_1 = v^1 + \lambda h$ is smooth and $u^\lambda_0 = v^\lambda$ is uniformly d_1-bounded (Proposition 2.1) we have $|\dot{u}^\lambda_1| \leq |v^\lambda - C|$ for some $C > 1$ independent of $\lambda \in [0,1/2]$ [Da2], ultimately giving the following sequence of estimates:

$$\int_X |\dot{u}^\lambda_1|\omega^n \leq \int_X |v^\lambda - C|\omega^n \leq d_1(v^\lambda, 0) + C,$$

where in the last inequality we have used [Da2, Theorem 3]. Again, since $\{v^\lambda\}_{\lambda\in[0,1/2]}$ is d_1-bounded (Proposition 2.1), it follows that the right most quantity above is uniformly bounded, and since $f_\lambda = O(\lambda^2)$ we can write

$$\frac{d}{dt}\Big|_{t=1^-} \mathcal{K}_{\lambda\omega}(u^\lambda_t) \leq O(\lambda^2).$$

Recall that v^λ is the unique $\mathcal{E}^1$-minimizer of $\mathcal{K}_{\lambda\omega}$, thus $\frac{d}{dt}\big|_{t=1^-}\mathcal{K}_{\lambda\omega}(u^\lambda_t) \geq \frac{d}{dt}\big|_{t=0^+}\mathcal{K}_{\lambda\omega}(u^\lambda_t) \geq 0$. Consequently, as both $t \to J_\omega(u^\lambda_t) = \mathrm{AM}_\omega(u^\lambda_t)$ and $t \to \mathcal{K}(u^\lambda_t)$ are convex, we obtain the following sequence of estimates

$$0 \leq n\lambda\Big(\frac{d}{dt}\Big|_{t=1^-} - \frac{d}{dt}\Big|_{t=0^+}\Big)\mathrm{AM}_\omega(u^\lambda_t) \leq \Big(\frac{d}{dt}\Big|_{t=1^-} - \frac{d}{dt}\Big|_{t=0^+}\Big)\mathcal{K}_{\lambda\omega}(u^\lambda_t) \leq O(\lambda^2).$$

Using convexity of $t \to \mathrm{AM}_\omega(u^\lambda_t)$ again, this last estimate gives

$$0 \leq t\mathrm{AM}_\omega(u^\lambda_1) + (1-t)\mathrm{AM}_\omega(u^\lambda_0) - \mathrm{AM}_\omega(u^\lambda_t) \leq t(1-t)O(\lambda), \ t \in (0,1).$$

Letting $\lambda \to 0$, using the endpoint stability of finite energy geodesic segments [BDL, Proposition 4.3] and the d_1-continuity of AM [Da2, Lemma 4.15], we obtain that $t \to \mathrm{AM}_\omega(u_t)$ is linear along the finite energy geodesic $[0,1] \ni t \to u_t \in \mathcal{E}^1$ connecting v^0 to v^1. By [BDL, Theorem 4.12] this implies that $v^0 = v^1$, what we desired to prove. □

As promised in the above argument, we provide the following lemma, which generalizes [BB, Lemma 3.5] to finite energy geodesics with one smooth endpoint:

Lemma 2.3. *Given $u_1 \in \mathcal{E}^1$ and $u_0 \in \mathcal{H}_\omega$ let $[0,1] \ni t \to u_t \in \mathcal{E}^1$ be the finite energy geodesic connecting u_0, u_1 and χ is a smooth closed and positive (1,1)-form. Then*

$$\lim_{t\to 0^+} \frac{\mathcal{K}_\chi(u_t) - \mathcal{K}_\chi(u_0)}{t} \geq \int_X (\bar{S}_\chi - S_{\omega_{u_0}} + \mathrm{Tr}^{\omega_{u_0}}\chi)\dot{u}_0\omega^n_{u_0},$$

where $\bar{S}_\chi = nV^{-1}\int_X (\mathrm{Ric}\,\omega - \chi)\wedge\omega^{n-1}$.

Proof. For fixed $t \in [0,1]$ we need to show that

$$\frac{\mathcal{K}_\chi(u_t) - \mathcal{K}_\chi(u_0)}{t} \geq \int_X (\bar{S}_\chi - S_{\omega_{u_0}} + \mathrm{Tr}^{\omega_{u_0}}\chi)\dot{u}_0\omega^n_{u_0}. \tag{7}$$

By [BDL, Theorem 1.2] there exists $u_t^k \in \mathcal{H}_\omega$ such that $d_1(u_t^k, u_t) \to 0$ and $\mathcal{K}_\chi(u_t^k) \to \mathcal{K}_\chi(u_t)$. Let $[0,t] \ni l \to v_l^k \in \mathcal{E}^1$ be the weak $C^{1\bar{1}}$ geodesic connecting $u_0^k := u_0$ with u_t^k. By [BB, Lemma 3.5] we can write:

$$\frac{\mathcal{K}_\chi(u_t^k) - \mathcal{K}_\chi(u_0)}{t} \geq \int_X (\bar{S}_\chi - S_{\omega_{u_0}} + \mathrm{Tr}^{\omega_{u_0}}\chi)\dot{v}_0^k \omega_{u_0}^n.$$

By the next lemma, after perhaps passing to a subsequence, we can apply the dominated convergence theorem on the right hand side and obtain (7). □

Lemma 2.4. *Suppose $u_1^j, u_1 \in \mathcal{E}^1$ satisfies $d_1(u_1^j, u_1) \to 0$ and $u_0 \in \mathcal{H}_\omega$. Let $[0,1] \ni t \to u_t, u_t^j \in \mathcal{E}^1$ be the finite energy geodesics connecting u_0, u_1^j and u_0, u_1 respectively. Then there exists $f \in L^1(\omega_{u_0}^n)$ and a $j_k \to \infty$ such that $|\dot{u}_0^{j_k}| \leq f$ and $\dot{u}_0^{j_k} \to \dot{u}_0$ a.e. as $k \to \infty$.*

Proof. If $[0,1] \ni t \to v_t \in \mathcal{E}^1$ is an arbitrary finite energy geodesic, we observe that $t \to v_t + t\alpha + (1-t)\beta$ is the finite energy geodesic connecting $v_0 + \beta$ and $v_1 + \alpha$, $\alpha, \beta \in \mathbb{R}$. Hence after possibility adding a constant, we can assume that $u_0 - 1 \geq u_1, u_1^j$, without loss of generality. We first show that

$$d_1(u_0, u_1) = \int_X -\dot{u}_0 \omega_{u_0}^n. \tag{8}$$

Indeed, let $\tilde{u}_1^j \in \mathcal{H}_\omega$ be a sequence decreasing to u_1 with $\tilde{u}_1^j \geq u_0$. By [Da2, Theorem 1] we have $d_1(u_0, \tilde{u}_1^j) = \int_X -\dot{\tilde{u}}_0 \omega_{u_0}^n$, where $t \to \tilde{u}_t^j$ is the weak $C^{1\bar{1}}$ geodesic connecting $u_0, \tilde{u}_1^j$, which is decreasing in t. As $\tilde{u}_0 = u_0$ and $u_t^j \searrow u_t$, (8) follows from the monotone convergence theorem.

By [BDL, Proposition 2.6] there exists $j_k \to \infty$, $v_1^{j_k} \in \mathcal{E}^1$ increasing and $w_1^{j_k} \in \mathcal{E}^1$ decreasing such that $v_1^{j_k} \leq u_1^{j_k} \leq w_1^{j_k} \leq u_0$ and $d_1(u_1, v_1^{j_k}), d_1(u_1, w_1^{j_k}) \to 0$. Let $[0,1] \ni t \to v_t^{j_k}, w_t^{j_k} \in \mathcal{E}^1$ be the finite energy geodesics connecting $u_0, v_1^{j_k}$ and $u_0, w_1^{j_k}$ respectively. By the comparison principle for finite energy geodesic segments we ultimately get $\dot{v}_0^{j_k} \leq \dot{u}_0^{j_k} \leq \dot{w}_0^{j_k} \leq 0$. We claim that for the monotone limits $\dot{v}_0 := \lim_k \dot{v}_0^{j_k}$ and $\dot{w}_0 := \lim_k \dot{w}_0^{j_k}$ we have $\dot{v}_0 = \dot{w}_0 = \dot{u}_0$ a.e., with this showing that $\dot{u}_0^{j_k} \to \dot{u}_0$ a.e. as $k \to \infty$. Indeed, by (8) we have $d_1(u_0, w_1^{j_k}) = \int_X -\dot{w}_0^{j_k} \omega_{u_0}^n$ and $d_1(u_0, v_1^{j_k}) = \int_X -\dot{v}_0^{j_k} \omega_{u_0}^n$. Applying the monotone/dominated convergence theorems, we can write $d_1(u_0, u_1) = \int_X -\dot{v}_0 \omega_{u_0}^n = \int_X -\dot{w}_0 \omega_{u_0}^n$. Since $\dot{v}_0 \leq \dot{u}_0 \leq \dot{w}_0$, it follows that $\dot{v}_0 = \dot{w}_0 = \dot{u}_0$ a.e. with respect to $\omega_{u_o}^n$, as we claimed.

Finally, as $\dot{v}_0^{j_1} \leq \dot{u}_0^{j_k} \leq 0$, using (8) we conclude that the function $f = |\dot{v}_0^{j_1}|$ satisfies the requirements in the statement. □

Remark 2.5. *One could approximate u_t by a decreasing sequence along with convergence of the K-energy by using the Ricci flow technique [GZ2, DL]. But the approximation as above is enough for our purpose.*

Remark 2.6. *Propositions 2.1, 2.2 together imply that whenever a cscK potential $u \in \mathcal{H}_\omega \cap \mathrm{AM}^{-1}(0)$ exists, then every "finite energy continuity path" $[0,\infty) \ni \lambda \to v^\lambda \in \mathcal{E}^1 \cap \mathrm{AM}^{-1}(0)$ d_1-converges to $g.u$ for some $g \in G$, with the crucial uniform estimate (5). Though we will not need it in this work, it is worth noting that (using the implicit function theorem and additional estimates) in [CPZ, Theorem 1.1] it is shown that $v^\lambda \in \mathcal{H}_\omega$ for small enough λ, and in fact $v^\lambda \to_{C^\infty} g.u$.*

In the proof of Theorem 1.1 we will need one last auxiliary result:

Lemma 2.7. *Suppose $\mathcal{H}_\omega \cap \mathrm{AM}^{-1}(0)$ contains a cscK potential. If $u \in \mathcal{H}_\omega \cap \mathrm{AM}^{-1}(0)$ and $g_j \in G$ such that $d_1(g_j.u, h) \to 0$ for some $h \in \mathcal{E}^1$ as $j \to \infty$, then there exists $g \in G$ such that $g.u = h$.*

Proof. Let $v \in \mathcal{H}_\omega \cap \mathrm{AM}^{-1}(0)$ be a cscK potential . By [DR, Propositions 6.2 and 6.9] there exists $k_j \in \mathrm{Isom}_0(X, \omega_v)$ and a Hamiltonian vector field $X_j \in \mathrm{isom}(X, \omega_v)$ such that $g_j = k_j \exp_I JX_j$. It is clear from the definition of the action of G on the level of potentials that $k_j.v = v$. Thus we can write

$$\begin{aligned} d_1(v, g_j.u) &= d_1(v, k_j \exp_I(JX_j).u) = d_1(k_j^{-1}v, \exp_I(JX_j).u) = d_1(v, \exp_I(JX_j).u) \\ &= d_1(\exp_I(-JX_j)v, u) \geq d_1(\exp_I(-JX_j)v, v) - d_1(v, u), \end{aligned}$$

giving that $d_1(\exp_I(-JX_j)v, v)$ is bounded independently of j. As $G = \mathrm{Aut}_0(X, J)$ is reductive, the curve $[0, \infty) \ni t \to \exp_I(-tJX_j).v \in \mathcal{H}_\omega \cap \mathrm{AM}^{-1}(0)$ is a d_1-geodesic ray (see [DR, Section 7.1]), hence $\|X_j\|$ has to be uniformly bounded in $\mathrm{isom}(X, \omega_v)$. By compactness, after possibly relabeling the sequences, we can choose $X_\infty \in \mathrm{isom}(X, \omega_v)$ and $k \in \mathrm{Isom}_0(X, \omega_v)$ such that $k_j \to k$ and $X_j \to X_\infty$ smoothly, hence also $g_j = k_j \exp_I(JX_j) \to g := k \exp_I(JX_\infty)$ smoothly. In particular this implies $d_1(g_j.u, g.u) \to 0$, hence $g.u = h$ by the non-degeneracy of d_1. □

Proof of Theorem 1.1. Let $v \in \mathcal{M}^1$. We have to show that $v = g.u$ for some $g \in G$. By [Da2] there exists $v^j \in \mathcal{H}_\omega \cap \mathrm{AM}^{-1}(0)$ with $d_1(v_j, v) \to 0$.

For the moment fix $\lambda > 0$. As u realizes the minimum of $\mathcal{K}$, by Proposition 2.1 $\mathcal{K}_{\lambda\omega_{v_j}}$ admits a unique minimum $v_j^\lambda \in \mathcal{E}^1 \cap \mathrm{AM}^{-1}(0)$ satisfying $\mathcal{I}(v_j^\lambda, v_j) \leq n(n+1)\mathcal{I}(v_j, v)$. As $\mathcal{I}$ satisfies the quasi-triangle inequality (see [BBEGZ, Theorem 1.8]), it follows that for some $C := C(X, \omega) > 0$ we have in fact

$$\mathcal{I}(v_j^\lambda, v) \leq C\mathcal{I}(v_j, v). \tag{9}$$

Now we use that $u \in \mathcal{H}_\omega \cap \mathrm{AM}^{-1}(0)$ is a cscK potential. Fixing j, Proposition 2.2 gives that the potentials v_j^λ d_1-converge to $v^0 = g_j.u$ for some $g_j \in G$. Hence, letting $\lambda \to 0$ in (9) we can conclude

$$\mathcal{I}(g_j.u, v) \leq C\mathcal{I}(v_j, v).$$

Letting $j \to \infty$ in the above estimate we obtain $\mathcal{I}(g_j.u, v) \to 0$, which by [BBEGZ, Proposition 2.3] is equivalent to $\|g_j.u - v\|_{L^1(X)} \to 0$ and $\mathrm{AM}(g_j.u) \to \mathrm{AM}(v)$. By [Da2, Proposition 5.9] this is further equivalent to $d_1(g_j.u, v) \to 0$. Finally, by Lemma 2.7 there exists $g \in G$ such that $g.u = v$, finishing the proof. □

3 K-polystability as a consequence of properness

Let us first fix some terminology. Let $L \to X$ be an ample line bundle over a Kähler manifold (X, ω) such that $c_1(L) = [\omega]$. A test configuration $(\mathcal{L}, \mathcal{X}, \pi, \rho)$ for (X, L) consists of a scheme $\mathcal{X}$ with a $\mathbb{C}^*$-equivariant flat surjective morphism $\pi : \mathcal{X} \to \mathbb{C}$ and a relatively ample line bundle $\mathcal{L} \to \mathcal{X}$ with a $\mathbb{C}^*$-action $\tau \to \rho_\tau$ on $\mathcal{L}$ such that $(X_1, \mathcal{L}|_{X_1}) = (X, kL)$ for some $k > 1$. Without loss of generality we can assume that $k = 1$, by treating $\mathcal{L}$ as a Q-line bundle. Following the findings of [LX] we will always assume that $\mathcal{X}$ is normal which automatically makes the projection π flat.

Given any test configuration $(\mathcal{L},\mathcal{X},\pi,\rho)$, after raising $\mathcal{L}$ to a sufficiently high power, it is possible to find an equivariant embedding into $(\mathbb{CP}^N,\mathbb{C})$, such that $\mathcal{L}$ becomes the pullback of the relative $\mathcal{O}(1)$-hyperplane bundle (see [Do1, Th, PS1]). This automatically allows to fix a a semi-positive smooth "background" metric h on $\mathcal{L}$, that is positive on every X_τ slice and is S^1-invariant. For the restrictions we introduce the notation $h_\tau = h|_{X_\tau}$, $\tau \in \mathbb{C}$.

Any other positive metric $\tilde{h}$ on $\mathcal{L}$ can be uniquely represented by a potential $u^{\tilde{h},\mathcal{X}} \in \mathrm{PSH}(\mathcal{X},\Theta(h))$ using the identification

$$\tilde{h} = he^{-u^{\tilde{h},\mathcal{X}}}.$$

Additionally, one can associate to $\tilde{h}$ another potential $u^{\tilde{h},\mathbb{C}^*} \in \mathrm{PSH}(\mathbb{C}^* \times X, \mathrm{pr}_2^*\omega)$ using the identification

$$\rho_\tau^*\tilde{h}\big|_{X_1} = h_1 e^{-u_\tau^{\tilde{h},\mathbb{C}^*}},\ \tau \in \mathbb{C}^*. \tag{10}$$

By analyzing the action of ρ restricted to global sections of $\mathcal{L}^r,\ r \geq 1$ on X_0, we can associate to $(\mathcal{X},\mathcal{L},\pi,\rho)$ the Donalson–Futaki invariant $DF(\mathcal{X},\mathcal{L})$. For details we refer to [Sz2, Th]. We say that (X,L) is K-polystable if for any test configuration $(\mathcal{X},\mathcal{L},\pi,\rho)$ we have $DF(\mathcal{X},\mathcal{L}) \geq 0$ with $DF(\mathcal{X},\mathcal{L}) = 0$ if and only if $\mathcal{X}$ is a product.

Let us fix $\phi \in \mathrm{PSH}(X,\Theta(h_1))$. According to Phong–Sturm [PS1, PS2] (see also [Be1, Section 2.4]), to $(\mathcal{X},\mathcal{L},\pi,\rho)$ one can also associate a bounded geodesic ray $[0,\infty) \ni t \to \phi_t \in \mathrm{PSH}(X,\Theta(h_1)) \cap L^\infty$ (with $\phi_0 = \phi$) by first constructing a metric $\tilde{h} := he^{-\phi^{\mathcal{X}}}$ on $\mathcal{L}$, using the following upper envelope:

$$\phi^{\mathcal{X}} = \sup\{v \in \mathrm{PSH}(\mathcal{X}|_{\overline{\Delta}},\Theta(h)), v_\tau \leq \rho^*_{\tau^{-1}}\phi,\ |\tau| = 1\}.$$

The envelope $\phi^{\mathcal{X}}$ is seen to be S^1-invariant, and one can introduce $\phi_t = \phi^{\tilde{h},\mathbb{C}^*}_{e^{-t/2}} \in \mathrm{PSH}(X,\omega) \cap L^\infty(X)$ for any $t \in [0,\infty)$. As argued in [PS1, PS2], this last curve $t \to \phi_t$ is indeed a weak $C^{1\bar{1}}$-geodesic ray. In general, $t \to \phi_t$ is not normalized, i.e., $\mathrm{AM}(\phi_t)$ is not identically zero (as this depends on the $\mathbb{C}^*$-action). As follows from the proof of [Be1, Proposition 2.7] (see specifically the argument that gives (2.16),(2.17)), there exists $C := C(\phi,\mathcal{L},\mathcal{X},h) > 0$ such that

$$he^{-C} \leq he^{-\phi^{\mathcal{X}}} \leq he^{C}. \tag{11}$$

We give now the proof of Theorem 1.3 that will be a combination of finite energy techniques of this paper with the more algebro-geometric ideas of [Be1]:

Proof of Theorem 1.3. Let $(\mathcal{X},\mathcal{L},\pi,\rho)$ be a test configuration equivariantly embedded into $\mathbb{CP}^N \times \mathbb{C}$ with a $\mathbb{C}^*$-action $\mathbb{C}^* \ni \tau \to \rho_\tau \in GL(N+1,\mathbb{C})$. By possibly composing ρ_τ with an inner automorphism, we can and assume that the S^1-invarant background metric is just the restriction of the relative Fubini-Study metric h^{FS} on $\mathcal{O}(1) \to (\mathbb{CP}^N,\mathbb{C})$. For the background Kähler metric on X we choose $\omega := \Theta(h_1^{FS})$.

We will prove that $DF(\mathcal{X},\mathcal{L}) \geq 0$ with $DF(\mathcal{X},\mathcal{L}) = 0$ if and only if $(\mathcal{X},\mathcal{L},\pi,\rho)$ is a product test configuration.

We will be relying on the following formula relating the Donaldson-Futaki invariant to the asymptotics of the K-energy [PT, PRS, Ti4, BHJ, SD]:

$$\mathcal{K}(u_\tau^{h^{FS},\mathbb{C}^*}) = -(DF(\mathcal{X},\mathcal{L}) - a(\mathcal{X},\mathcal{L}))\log|\tau|^2 + O(1),\ \tau \in \mathbb{C}^*, \tag{12}$$

where $a(\mathcal{X},\mathcal{L}) \geq 0$, and $a(\mathcal{X},\mathcal{L}) = 0$ precisely when the central fiber X_0 is reduced (recall the notation introduced in (10) above). From Theorem 1.2 it follows that $\mathcal{K}$ is bounded from below, hence $DF(\mathcal{X},\mathcal{L}) - a(\mathcal{X},\mathcal{L}) \geq 0$, giving that $DF(\mathcal{X},\mathcal{L}) \geq 0$.

Now assume that $DF(\mathcal{X},\mathcal{L}) = 0$. To finish the proof we will argue that $(\mathcal{X},\mathcal{L},\pi,\rho)$ is a product test configuration. Let $\phi \in \mathcal{H}_\omega \cap \mathrm{AM}^{-1}(0)$ be a cscK potential (recall that $\Theta(h_1) = \omega$ by choice) and let $[0,\infty) \ni t \to \phi_t \in \mathcal{E}^1$ be the associated $C^{1\bar{1}}$-geodesic ray with $\phi_0 = \phi$.

First notice that (1) gives $\inf_{g\in G} J(g.(u^{h^{FS},\mathbb{C}^*}_{e^{-t/2}} - \mathrm{AM}(u^{h^{FS},\mathbb{C}^*}_{e^{-t/2}}))) < C'$. Pulling back the estimates of (11) by ρ_τ, we immediately obtain $u^{h^{FS},\mathbb{C}^*}_{e^{-t/2}} - C \leq \phi_t \leq u^{h^{FS},\mathbb{C}^*}_{e^{-t/2}} + C$. Putting these last two facts together, and possibly increasing C', we arrive at:

$$\inf_{g\in G} J(g.(\phi_t - \mathrm{AM}(\phi_t))) < C'.$$

Given that ϕ_0 is a cscK potential, Lemma 3.1 below implies that the normalized ray $t \to \phi_t - \mathrm{AM}(\phi_t)$ is induced by $t \to \exp_I(tJV)$, where V is a real holomorphic Hamiltonian Killing field of (X,J,ω). By Lemma 3.2 it is possible to find a lift $\tilde{V}$ to $L \to X$ such that

$$\exp_I(tJ\tilde{V})^* h_1^{FS} e^{-u_0} = h_1^{FS} e^{-u_t}.$$

Since $DF(\mathcal{X},\mathcal{L}) = 0$, (12) gives that $a(\mathcal{X},\mathcal{L}) = 0$, hence X_0 is reduced. Consequently, we can apply Proposition 3.3 to conclude that $\mathcal{X}$ is isomorphic to $X \times \mathbb{C}$, finishing the proof. □

Lemma 3.1. *Suppose (X,ω) is a Kähler manifold. Let $u_0 \in \mathcal{H}_\omega \cap \mathrm{AM}^{-1}(0)$ be a cscK potential and a finite energy geodesic ray $[0,\infty) \ni t \to u_t \in \mathcal{E}^1 \cap \mathrm{AM}^{-1}(0)$ emanating from u_0. If there exists $C > 0$ such that*

$$\inf_{g\in G} J(g.u_t) < C, \ t \in [0,\infty),$$

then there exists a real holomorphic Hamiltonian vector field $V \in isom(X,\omega_{u_0})$ such that $u_t = \exp_I(tJV).u_0$, where $t \to \exp_I(tJV)$ is the flow of JV.

Proof. Let $g_k \in G$ such that $J(g_k.u_k) < C$. As u_0 is a cscK potential there exists $h_k \in \mathrm{Isom}_0(X,\omega_{u_0})$ and a Hamiltonian vector field $V_k \in \mathrm{isom}(X,\omega_{u_0})$ such that $g_k = h_k \exp_I(-JV_k)$ (see [DR, Propositions 6.2 and 6.9]). As the growth of the J functional is the same as that of the d_1 metric [DR, Proposition 5.5], and G acts by d_1-isometries on $\mathcal{E}^1 \cap \mathrm{AM}^{-1}(0)$ [DR, Lemma 5.9], by possibly increasing the constant C we can write:

$$C > d_1(u_0, g_k.u_k) = d_1(g_k^{-1}u_0, u_k) = d_1(\exp(JV_k).u_0, u_k). \tag{13}$$

We can assume without loss of generality that $t \to u_t$ has unit d_1-speed, i.e., $d_1(u_0,u_t) = t$. Using the above inequality, the triangle inequality gives the following double estimate:

$$k - C \leq d_1(u_0, \exp_I(JV_k).u_0) \leq k + C.$$

The analytic expression of $\exp_I(JV_k).u_0$ (see [DR, Lemma 5.8]) implies that in fact $1/D \leq \|JV_k/k\| \leq D$ for some $D > 1$. As the space of holomorphic Hamiltonian Killing fields of (X,ω_{u_0},J) is finite dimensional, it follows that there exists a nonzero Killing field V such that $V_{k_j}/k_j \to V$ for some $k_j \to \infty$.

Let us introduce the smooth geodesic segments

$$[0,k] \ni t \to u_t^k = \exp_I\Big(t\frac{JV_k}{k}\Big).u_0 \in \mathcal{H}_\omega \cap \mathrm{AM}^{-1}(0).$$

By [BDL, Proposition 5.1] the function $t \to d_1(u_t^k, u_t)$ is convex, hence (13) gives that $d_1(u_t^k, u_t) \leq Ct/k,\ t \in [0,k]$. This implies that for fixed t we have $d_1(u_t^k, u_t) \to 0$. But examining convergence in the expressions defining $\exp_I(tJV_{k_j}/k_j).u_0$ we conclude that $u_t^{k_j} \to \exp_I(tJV).u_0$ smoothly, ultimately giving $u_t = \exp_I(tJV).u_0$. □

In case the Kähler class is integral, we have the following addendum to the previous lemma:

Lemma 3.2. *Suppose $(L,h) \to X$ is a hermitian line bundle with $\omega := \Theta(h) > 0$. Let $\phi_0 \in \mathcal{H}_\omega$, a real holomorphic Hamiltonian vector field $V \in \mathrm{isom}(X, J, \omega_{u_0})$, and $[0,\infty) \ni t \to \phi_t \in \mathcal{E}^1$ a geodesic ray. If the "normalization" of $t \to \phi_t$ is induced by V, i.e., $\phi_t - \mathrm{AM}(\phi_t) = \exp_I(tJV).(\phi_0 - \mathrm{AM}(\phi_0))$, then it is possible to find a lift $\tilde{V}$ of V to the line bundle $L \to X$ such that $\exp_I(tJ\tilde{V})^* he^{-\phi_0} = he^{-\phi_t}$.*

This is essentially well-known, but as we could not find an adequate reference we include a proof here.

Proof. It is shown in [Do1, Lemma 12] that it is possible to lift V to a vector field $\tilde{V}$ on $L \to X$. Below we recall the construction of $\tilde{V}$ and show that one of the lifts satisfies the required properties.

By computing the curvature of both sides and using the dd^c lemma, we see that for any lift there exists a smooth function $f : [0,\infty) \to \mathbb{R}$ such that

$$\exp_I(tJ\tilde{V})^* he^{-\phi_0} = he^{-\phi_t + f(t)}. \tag{14}$$

We will show that for the right choice of $\tilde{V}$ we have $f(t) \equiv 0$. In fact, as it will be clarified below, it all depends on how we choose the Hamiltonian potential of V.

Suppose $v \in C^\infty(X)$ such that $i_V \omega_{\phi_0} = dv$. After perhaps adjusting v by a constant, one can compute that (see [Ma1], [Sz2, Example 4.26])

$$\phi_t(x) - \phi_0(x) = 2\int_0^t v(\exp_I(lJV)x)dl.$$

Let us fix $x_0 \in \mathrm{Crit}(v)$, i.e., $dv(x_0) = 0$. This gives $V(x_0) = 0$, hence by the above

$$\phi_t(x_0) = \phi_0(x_0) + 2tv(x_0). \tag{15}$$

We now recall the main elements of [Be2, Lemma 13] and its proof. For this, it will be more convenient to use the complex notation for holomorphic vector fields. To avoid confusion, recall that $V^{\mathbb{C}} = V - iJV$ and $V = \mathrm{Re}\ V := (V^{\mathbb{C}} + \overline{V^{\mathbb{C}}})/2$.

Let $(z_1, \ldots, z_n)$ be coordinates on X in a neighborhood U of x_0. Let s be a non-vanishing section of L on U and we introduce $e^{-\phi(z)} := he^{-\phi_0}(s, \bar{s})(z)$. Let $W^{\mathbb{C}}$ be the generator of the natural $\mathbb{C}^*$-action along the fibres of L. In local holomorphic coordinates $(z_1, \ldots, z_n, w)$ of $L \to X$ on U we have $W^{\mathbb{C}} = w\frac{\partial}{\partial w}$ and $V^{\mathbb{C}} = V^j \frac{\partial}{\partial z_j} = -2i\phi^{j\bar{k}} v_{\bar{k}} \frac{\partial}{\partial z_j}$ (note the missing factor in the corresponding formula in the proof of [Be2, Lemma 13]), where we have used that V is Hamiltonian (in holomorphic coordinates $iV^j \phi_{j\bar{k}} = 2v_{\bar{k}}$).

If $V^{\mathbb{C}}_{hor}$ is the horizontal lift of $V^{\mathbb{C}}$ with respect to the connection of the metric $he^{-\phi_0}$ on L, then one can compute that $V^{\mathbb{C}}_{hor} = V^{\mathbb{C}} + \partial\phi(V^{\mathbb{C}})W^{\mathbb{C}}$. An elementary calculation gives that

$$\tilde{V}^{\mathbb{C}} = V^{\mathbb{C}}_{hor} + 2ivW^{\mathbb{C}} = V^{\mathbb{C}} + W^{\mathbb{C}}(\partial\phi(V^{\mathbb{C}}) + 2iv)$$

is a holomorphic lift of V to $L \to X$. By this last formula, at the critical point x_0 we actually have $J\tilde{V}^{\mathbb{C}}(x_0) = -2v(x_0)w\frac{\partial}{\partial w}$. This immediately gives that the flow of $J\tilde{V} = \text{Re } J\tilde{V}^{\mathbb{C}}$ satisfies $\exp_I(tJ\tilde{V})(x_0, w) = (x_0, e^{-v(x_0)t}w)$, ultimately implying

$$\exp_I(tJ\tilde{V})^* he^{-\phi_0(x_0)}(x_0) = he^{-\phi_0(x_0)-2tv(x_0)}(x_0). \tag{16}$$

A comparison of (14), (15) and (16) gives that $f(t) \equiv 0$, finishing the proof. □

The proof of Theorem 1.3 is now completed by invoking the following result from [Be1]:

Proposition 3.3. *Let X be a Kähler manifold with positive line bundle $L \to X$ and a normal test configuration $(\mathcal{X}, \mathcal{L}, \pi, \rho)$ with S^1-invariant smooth background metric h, and reduced central fiber X_0. Given $\phi \in \mathcal{H}_{\Theta(h_1)}$, suppose that the associated geodesic ray $t \to \phi_t$ is induced by a vector field $\tilde{V}$ of $L \to X$, i.e., $\exp_I(tJ\tilde{V})^* h_1 e^{-\phi_0} = h_1 e^{-\phi_t}$. Then $\mathcal{X}$ is isomorphic to $X \times \mathbb{C}$.*

Although formulated in a slightly different context of Fano varieties in [Be1], the proof of the previous proposition follows word for word from the proof of Proposition 3.3 in [Be1], thanks to the lifting property established in the previous lemma (which generalizes formula (3.9) in [Be1]).

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