REVIEW OF "BERGMAN BUNDLES AND APPLICATIONS TO THE GEOMETRY OF COMPACT COMPLEX MANIFOLDS"
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This articles defines and studies a "Bergman bundle" $p : B_\epsilon \to X$ over a compact complex Hermitian manifold X. It is an infinite dimensional Hilbert bundle, whose fibers are H^2 Hardy spaces of balls, defined by embedding X as a totally real submanifold of its complexification $X \times \bar{X}$. A "Chern" connection ∇ is defined. The main result is that its curvature tensor $\Theta_{B_\epsilon,h}$ is strictly positive. In the model case, it is explicitly calculated and in general it is calculated asymptotically as the tube radius $\epsilon \to 0$. The last Section 4 sketches potential applications of the Bergman bundle and the curvature calculation to a Conjecture 4.1 on invariance of plurigenera for polarized Kähler families.

I find the Bergman bundle interesting in and of itself and recommend publication of this article in the PAMQ volume devoted to B. Shiffman.

I have a number of questions about the definition and properties of the Bergman bundle; the article could use some pedagogical background for readers unfamiliar with infinite dimensional bundles, to clarify which kinds of holomorphic notions exist and which are used. I also think the author should compare the set-up and results to those of Berndtsson [1] and Wang [2]. I won't comment on the possible application in the last section.

0.1. Questions.

- Is the total space of B_ϵ a complex manifold? What is meant by a complex structure on an infinite dimensional 'Hilbert bundle'. Is $p : B_\epsilon \to X$ a holomorphic map?

- Several times it is written that the Bergman bundle is locally trivial only in the real analytic category, but is not a holomorphic vector bundle. A proof of this statement should be given to indicate where holomorphic triviality goes wrong. First, as above, there is the question in what sense $p : B_\epsilon \to X$ is holomorphic map. Even if it is, it might not be locally holomorphically trivial.

  On page 5, it is said, "the desired holomorphic bundle B_ϵ will not even be locally trivial in the complex analytic sense". See again Remark 3.2. "Again, we have to insist that the Bergman bundle is not holomorphically locally trivial..." Since this is so important, a proof should be given.

  Given that it is not a holomorphic vector bundle, how much of the theory of connections on a holomorphic vector bundle is well-defined and valid in this context? The same statement is made in [1, Page 2] for the bundle A^2_t with the explanation, "the spaces A^2_t will not be identical as vector spaces, so the bundle is not locally trivial..." This statement pinpoints the problem in the variation of L^2 structure. But the Bergman bundles in the article under submission are more complicated.

- My impression is that, even if $p : B_\epsilon \to X$ is not a holomorphic vector bundle, one does have a definition of 'holomorphic section of B_ϵ'. One has smooth functions $\sigma(z, w), z \in X$ where $w \to \sigma(z, w)$ is L^2 holomorphic on the fibres of $pr_1 : U_{\gamma,\epsilon} \to X$. There is a $\bar{\partial}_z$ operator and it makes sense to ask if $\bar{\partial}_z \sigma(z, w) = 0$. This is a definition of a holomorphic section, even if there is no local holomorhpic trivialization of the bundle.

  It seems that one has to explain whether B_ϵ is a complex manifold since one wishes to use local holomorphic maps $s : X \to B_\epsilon$ with $p \circ s = 0$. They seem to work as well as sections, e.g. in defining projective embeddings.
- Above (3.8), the connection ∇_h on (B_ϵ, h) is called the Chern connection. The term is generally used for a Hermitian metric connection on a holomorphic vector bundle which has type $(1, 0)$ in a holomorphic frame. What properties does a Chern connection have on the Bergman bundle? In some sense, there is only one relevant Hermitian metric, so it is not a general notion. There does not

exist a local holomorphic frame, but one might ask if the formula $H^{-1}\partial H$ for the Chern connection in a holomorphic frame has a generalization to this setting?
- In the model case, one considers the fibration of a Grauert tube by the unit balls of its quadratic pluri-subharmonic defining function. These unit balls are bi-holomophic to the upper-half space, whose boundaries are Heisenberg groups. This is reflected in the construction of the orthonormal frame $e_\alpha(z, w)$, which are constructed using creation/annihilation operators. That is, $\bar{\partial}_{z_k}$ acts as an annihilation operator lowering the weight α by $c_k = (0, \ldots, 1, \ldots, 0)$. The curvature is rather like a harmonic oscillator on the fibers.

0.2. **Relation to prior works.** Perhaps the most relevant prior articles on Bergman bundles are Berndtsson's article [1] on the bundles A_t^2 and Wang's article [2]. Berndtsson's is not cited or mentioned, as far as I saw. It seems that the motivation of this article and those of [1, 2] is the same: to construct a connection with strictly positive curvature.

There seems to be another approach to 'Bergman bundles' that might deserve comparison to the one of this article. Implicit in [3] is a 'Bargmann-Fock bundle' $\mathcal{BF} \to X$ over a Kähler manifold. The model Bargmann-Fock $\mathcal{H}_J(\mathbb{C}^m)$ spaces are the Gaussian Hilbert spaces of entire holomorphic functions on $\mathbb{C}^m$ determined by a choice of complex structure J and a constant Hermitian metric h. On a Kähler manifold (M, J, g, ω), at each point $z \in X$, one defines the 'osculating Bargmann-Fock space' $\mathcal{BF}_z$ of entire holomorphic functions on $T_z^{*(1,0)}X$ in the Gaussian Hilbert space with respect to its Hermitian metric and complex structure.

This bundle may be a kind of linearization, in the $\epsilon \to 0$ limit, of the one in Demailly's paper. It is naturally a Hermitian vector bundle, in some infinite dimensional sense. The family of Bergman kernels $\Pi^{T_z^{*(1,0)}}(w, w')$ of the BF spaces as z varies gives a kind of natural frame for this bundle. I don't know if this Hermitian bundle admits a Chern connection in the sense of Demailly, although many of the same types of formulae are valid. I also don't know it if has the positive curvature properties of the Bergman bundle of the article, nor how it compares to the Bergman bundles of the paper. The calculations seem similar to those in the set-up of the submitted article but also seem to connect to the geometry of X more.

0.3. **Typos.**
- Page 2, line 5 below (c): 'fivers' should be 'fibers'.

References

[1] Bo Berndtsson, Curvature of vector bundles associated to holomorphic fibrations. Ann. of Math. (2) 169 (2009), no. 2, 531-560.

[2] X. Wang, A curvature formula associated to a family of pseudoconvex domains. Ann. Inst. Fourier (Grenoble) 67 (2017), no. 1, 269-313.

[3] Zelditch, Steve; Zhou, Peng Pointwise Weyl law for partial Bergman kernels. Algebraic and analytic microlocal analysis, 589-634, Springer Proc. Math. Stat., 269, Springer, Cham, 2018.