Bergman bundles and the plurigenera conjecture for polarized families of Kähler manifolds

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1. Exponential map and tubular neighborhoods

Let X be a compact n-dimensional complex manifold and $Y \subset X$ a smooth totally real submanifold, i.e. such that $T_Y \cap JT_Y = \{0\}$ for the complex structure J on X. By a well known result of Grauert, such a Y always admits a fundamental system of Stein tubular neighborhoods $U \subset X$ (this would be even true when X is noncompact, but we only need the compact case here). In fact, if (Ω_α) is a finite covering of X such that $Y \cap \Omega_\alpha$ is a smooth complete intersection $\{z \in \Omega_\alpha \,;\, x_{\alpha,j}(z) = 0\}$, $1 \leq j \leq q$ (where $q = \operatorname{codim}_{\mathbb{R}} Y \geq n$), then one can take $U = U_\varepsilon = \{\varphi(z) < \varepsilon\}$ where

$$(1.1) \qquad \varphi(z) = \sum_\alpha \theta_\alpha(z) \sum_{1 \leq j \leq q} (x_{\alpha,j}(z))^2 \geq 0$$

where (θ_α) is a partition of unity subordinate to (Ω_α). The reason is that φ is strictly plurisubharmonic near Y, as

$$i\partial\overline{\partial}\varphi_{|Y} = 2i \sum_\alpha \theta_\alpha(z) \sum_{1 \leq j \leq q} \partial x_{\alpha,j} \wedge \overline{\partial}\, x_{\alpha,j}$$

and $(\partial x_{\alpha,j})_j$ has rank n at every point of Y, by the assumption that Y is totally real.

Now, let $\overline{X}$ be the complex conjugate manifold associated with the integrable almost complex structure $(X, -J)$ (in other words, $\mathcal{O}_{\overline{X}} = \overline{\mathcal{O}_X}$); we denote by $x \mapsto \overline{x}$ the identity

map Id : $X \to \overline{X}$ to stress that it is conjugate holomorphic. The underlying real analytic manifold $X^{\mathbb{R}}$ can be embedded diagonally in $X \times \overline{X}$ by the diagonal map $\delta : x \mapsto (x, \overline{x})$, and the image $\delta(X^{\mathbb{R}})$ is a totally real submanifold of $X \times \overline{X}$. In fact, if $(z_{\alpha,j})_{1 \le j \le n}$ is a holomorphic coordinate system relative to a finite open covering (Ω_α) of X, then the $\overline{z}_{\alpha,j}$ define holomorphic coordinates on $\overline{X}$ relative to $\overline{\Omega}_\alpha$, and the "diagonal" $\delta(X^{\mathbb{R}})$ is the totally real submanifold of pairs (z, w) such that $w_{\alpha,j} = \overline{z}_{\alpha,j}$ for all α, j. In that case, we can take Stein tubular neighborhoods of the form $U_\varepsilon = \{\varphi < \varepsilon\}$ where

$$(1.2) \qquad \varphi(z, w) = \sum_\alpha \theta_\alpha(z)\theta_\alpha(w) \sum_{1 \le j \le q} |\overline{w}_{\alpha,j} - z_{\alpha,j}|^2.$$

Here, the strict plurisubharmonicity of φ near $\delta(X^{\mathbb{R}})$ is obvious from the fact that

$$|w_{\alpha,j} - \overline{z}_{\alpha,j}|^2 = |z_{\alpha,j}|^2 + |w_{\alpha,j}|^2 - 2\operatorname{Re}(z_{\alpha,j}w_{\alpha,j}).$$

For $\varepsilon > 0$ small, the first projection $\operatorname{pr}_1 : U_\varepsilon \to X$ gives a small bundle whose fibers are C^∞-diffeomorphic to balls, but they will not be biholomorphic to complex balls in general. In order to achieve this property, we can proceed in the following way. Pick a real analytic hermitian metric γ on X (take e.g. the $(1,1)$-part $\gamma = g^{(1,1)} = \frac{1}{2}(g + J^* g)$ of the Riemannian pull-back $g = \delta^*(\sum_j idf_j \wedge d\overline{f}_j)$ where the $(f_j)_{1 \le j \le N}$ provide a holomorphic immersion of the Stein neighborhood U_ε into $\mathbb{C}^N$). Let $\exp : T_X \to X$, $(z, \xi) \mapsto \exp_z(\xi)$ be the exponential map associated with the metric γ, in such a way that $\mathbb{R} \ni t \mapsto \exp_z(t\xi)$ are geodesics for the Levi-Civita connection (the Chern connection of T_X can be used alternatively, see [Dem??]). Then exp is real analytic, and we have Taylor expansions

$$\exp_z(\xi) = \sum_{\alpha,\beta \in \mathbb{N}^n} a_{\alpha\beta}(z)\xi^\alpha \overline{\xi}^\beta, \qquad \xi \in T_{X,z}$$

with real analytic coefficients $a_{\alpha\beta}$, where $\exp_z(\xi) = z + \xi + O(|\xi|^2)$ in local coordinates. The real analyticity means that these expansions are convergent on a neighborhood $|\xi|_\gamma < \varepsilon_0$ of the zero section of T_X. We define the fiber-holomorphic part of the exponential map to be

$$(1.3) \qquad \operatorname{exph} : T_X \to X, \qquad (z, \xi) \mapsto \operatorname{exph}_z(\xi) = \sum_{\alpha \in \mathbb{N}^n} a_{\alpha\,0}(z)\xi^\alpha.$$

It is uniquely defined, is convergent on the same tubular neighborhood $\{|\xi|_\gamma < \varepsilon_0\}$, has the property that $\xi \mapsto \operatorname{exph}_z(\xi)$ is holomorphic for $z \in X$ fixed, and satisfies again $\operatorname{exph}_z(\xi) = z + \xi + O(\xi^2)$ in coordinates. By the implicit function, theorem, the map $(z, \xi) \mapsto (z, \operatorname{exph}_z(\xi))$ is a biholomorphism from a neighborhood of the zero section of T_X onto a neighborhood V of the diagonal in $X \times X$. Therefore, we get an inverse real analytic map $X \times X \supset V \to T_X$, which we denote by $(z, w) \mapsto (z, \xi)$, $\xi = \operatorname{logh}_z(w)$, such that $w \mapsto \operatorname{logh}_z(w)$ is holomorphic on $V \cap (\{z\} \times X)$, and $\operatorname{logh}_z(w) = w - z + O((w - z)^2)$ in coordinates. The tubular neighborhood

$$U_{\gamma,\varepsilon} = \{(z, w) \in X \times \overline{X}\,;\, |\operatorname{logh}_z(\overline{w})|_\gamma < \varepsilon\}$$

is Stein for $\varepsilon > 0$ small; in fact, if $p \in X$ and $(z_1, \ldots, z_n)$ is a holomorphic coordinate system centered at p such that $\gamma_p = i\sum dz_j \wedge d\overline{z}_j$, then $|\operatorname{logh}_z(\overline{w})|_\gamma^2 = |\overline{w} - z|^2 + O(|\overline{w} - z|^3)$, hence $i\partial\overline{\partial}|\operatorname{logh}_z(\overline{w})|_\gamma^2 > 0$ at $(p, \overline{p}) \in X \times \overline{X}$. By construction, the fiber $\operatorname{pr}_1^{-1}(z)$ of $\operatorname{pr}_1 : U_{\gamma,\varepsilon} \to X$ is biholomorphic to the ε-ball of the complex vector space $T_{X,z}$ equipped with the hermitian metric γ_z. In this way, we get a locally trivial real analytic bundle $\operatorname{pr}_1 : U_{\gamma,\varepsilon}$ whose fibers are complex balls; it is important to notice, however, that this ball bundle need not – and in fact, will never – be holomorphically locally trivial.

2. Curvature of Bergman bundles

2.A. Bergman version of the Dolbeault complex

Let X be a n-dimensional compact complex manifold equipped with a real analytic hermitian metric γ, $U_\varepsilon = U_{\gamma,\varepsilon} \subset X \times \overline{X}$ the ball bundle considered in §1 and

$$p = (\mathrm{pr}_1)_{|U_\varepsilon} : U_\varepsilon \to X, \qquad \overline{p} = (\mathrm{pr}_2)_{|U_\varepsilon} : U_\varepsilon \to \overline{X}$$

the natural projections. Our goal is to compute the curvature of what we call the "Bergman direct image sheaf"

$$(2.1) \qquad \qquad \mathcal{B}_\varepsilon = p_*^{L^2}(\overline{p}^* \mathcal{O}(K_{\overline{X}})),$$

whose space of sections over an open subset $V \subset X$ is defined to be $\mathcal{B}_\varepsilon(V) = $ holomorphic sections f of $\overline{p}^* \mathcal{O}(K_{\overline{X}})$ on $p^{-1}(V)$ that are in $L^2(p^{-1}(K))$ for all compact subsets $K \Subset V$, i.e.

$$(2.2) \qquad \qquad \int_{p^{-1}(K)} i^{n^2} f \wedge \overline{f} \wedge \gamma^n < +\infty, \quad \forall K \Subset V.$$

Then $\mathcal{B}_\varepsilon$ is an $\mathcal{O}_X$-module, and by the Ohsawa-Takegoshi extension theorem applied to the subvariety $p^{-1}(z) \subset U_\varepsilon$, its fiber $B_{\varepsilon,z} = \mathcal{B}_{\varepsilon,z}/\mathfrak{m}_z \mathcal{B}_{\varepsilon,z}$ is isomorphic to the Hilbert space $\mathcal{H}^2(\mathbb{B}_n)$ of L^2 holomorphic n-forms on $p^{-1}(z) \simeq \mathbb{B}_n$. In fact, if we use orthonormal coordinates $(w_1, \ldots, w_n)$ provided by exph acting on the hermitian space $(T_{X,z}, \gamma_z)$ and centered at $\overline{z}$, we get a biholomorphism $\mathbb{B}_n \to p^{-1}(z)$ given by the homothety $\eta_\varepsilon : w \mapsto \varepsilon w$, and a corresponding isomorphism

$$(2.3) \qquad B_{\varepsilon,z} \longrightarrow \mathcal{H}^2(\mathbb{B}_n), \qquad f \longmapsto g = \eta_\varepsilon^* f, \quad \text{i.e. with } I = \{1, \ldots, n\},$$

$$(2.3') \qquad f_I(w)\, dw_1 \wedge \ldots \wedge dw_n \longmapsto \varepsilon^n f_I(\varepsilon w)\, dw_1 \wedge \ldots \wedge dw_n, \quad w \in \mathbb{B}_n,$$

$$(2.3'') \qquad \|g\|^2 = \int_{\mathbb{B}_n} 2^{-n} i^{n^2} g \wedge \overline{g}, \quad g = g(w)\, dw_1 \wedge \ldots \wedge dw_n \in \mathcal{H}^2(\mathbb{B}_n).$$

As a consequence, $B_\varepsilon \to X$ can be seen as a locally trivial (infinite dimensional) real analytic bundle of typical fiber $\mathcal{H}^2(\mathbb{B}_n)$, and we can equip $B_{\varepsilon,z}$ with the natural L^2 metric obtained by declaring (2.3) to be an isometry. In this way, we get a real analytic hermitian metric on the Bergman bundle $B_\varepsilon \to X$.

Similarly, we can consider a "Bergman version" of the Dolbeault complex, by introducing a sheaf $\mathcal{F}_\varepsilon^q$ over X of (n,q)-forms which can be written locally over small open sets $V \subset X$ as

$$(2.4) \qquad f(z,w) = \sum_{|J|=q} f_J(z,w)\, dw_1 \wedge \ldots \wedge dw_n \wedge d\overline{z}_J, \quad (z,w) \in U_\varepsilon \cap (V \times \overline{X}),$$

where the $f_J(z,w)$ are C^∞ smooth functions on $U_\varepsilon \cap (V \times \overline{X})$ such that $f_J(z,w)$ is holomorphic in w and both f and $\overline{\partial} f = \overline{\partial}_z f$ are in $L^2(p^{-1}(K))$ for all compact subsets $K \Subset V$. By construction, the usual $\overline{\partial}$ operator yields a complex of sheaves $(\mathcal{F}^\bullet, \overline{\partial})$ and the kernel $\mathrm{Ker}\, \overline{\partial} : \mathcal{F}^0 \to \mathcal{F}^1$ coincides with $\mathcal{B}_\varepsilon$. We are going to see that B_ε can somehow be seen as an infinite dimensional very ample vector bundle. This is already illustrated by the following result.

2.5. Proposition. *Assume here that $\varepsilon > 0$ is taken so small that $\psi(z,w) := |\log h_z(w)|^2$ is strictly plurisubharmonic up to the boundary on the compact set $\overline{U}_\varepsilon \subset X \times \overline{X}$. Then the complex of sheaves $(\mathcal{F}^\bullet, \overline{\partial})$ is a resolution of $\mathcal{B}_\varepsilon$ by soft sheaves over X (actually, by $\mathcal{C}_X^\infty$-modules), and for every holomorphic vector bundle $E \to X$ and every $q \geq 1$ we have*

$$H^q(X, \mathcal{B}_\varepsilon \otimes \mathcal{O}(E)) = H^q\big(\Gamma(X, \mathcal{F}_\varepsilon^q \otimes \mathcal{O}(E)), \overline{\partial}\big) = 0.$$

Moreover the fibers $B_{\varepsilon,z} \otimes E_z$ are always generated by global sections of $H^0(X, \mathcal{B}_\varepsilon \otimes \mathcal{O}(E))$.

Proof. By construction, we can equip U_ε with the the associated Kähler metric $\omega = i\partial\overline{\partial}\psi$ which is smooth and strictly positive on $\overline{U}_\varepsilon$. We can then take an arbitrary smooth hermitian metric h_E on E and multiply it by $e^{-C\psi_\varepsilon}$ to obtain a bundle with positive arbitrarily large curvature tensor. The exactness of $\mathcal{F}^\bullet$ and cohomology vanishing then follow from the standard Hörmander L^2 estimates applied either locally on $p^{-1}(V)$ for small Stein open sets $V \subset X$, or globally on U_ε. The global generation of fibers is an immediate consequence of the Ohsawa-Takegoshi L^2 extension theorem. $\square$

It would not be very hard to show that the same result holds for an arbitrary coherent sheaf $\mathcal{E}$ instead of a locally free sheaf $\mathcal{O}(E)$, the reason being that $p^*\mathcal{E}$ admits a resolution by (finite dimensional) locally free sheaves $\mathcal{O}_{U_{\varepsilon'}}^{\oplus N}$ on a Stein neighborhood $U_{\varepsilon'}$ of $\overline{U}_\varepsilon$.

2.B. Curvature tensor of the Bergman bundle on $(\mathbb{C}^n, \mathrm{std})$

In the model situation $X = \mathbb{C}^n$ with its standard hermitian metric, we consider the tubular neighborhood

$$(2.6) \qquad\qquad U_\varepsilon := \{(z,w) \in \mathbb{C}^n \times \mathbb{C}^n \,;\, |\overline{w} - z| < \varepsilon\}$$

and the projections

$$p = (\mathrm{pr}_1)_{|U_\varepsilon} : U_\varepsilon \to X = \mathbb{C}^n, \ \ (z,w) \mapsto z, \quad \overline{p} = (\mathrm{pr}_2)_{|U_\varepsilon} : U_\varepsilon \to X = \mathbb{C}^n, \ \ (z,w) \mapsto w$$

If one insists on working on a compact complex manifold, the geometry is locally identical to that of a complex torus $X = \mathbb{C}^n/\Lambda$ equipped with a constant hermitian metric γ.

2.7. Remark. Again, we have to insist that the Bergman bundle B_ε is not holomorphically locally trivial, even in the above situation where we have invariance by translation. In the category of real analytic bundles, we get a trivialization from $\mathcal{B}_\varepsilon(V)$ onto the topological tensor product of $\mathcal{O}_X(V)$ by the Hilbert space $\mathcal{H}^2(\mathbb{B}_n)$, namely

$$\tau : \mathcal{B}_\varepsilon(V) \xrightarrow{\simeq} \mathcal{O}_X(V) \widehat{\otimes} \mathcal{H}^2(\mathbb{B}_n), \quad f \longmapsto \tau(f) = g, \quad g(z,w) := f(z, \varepsilon w + \overline{z}), \ \ w \in \mathbb{B}_n.$$

Complex structures of these bundles are defined by the $(0,1)$-connections $\overline{\partial}_z$ of the associated Dolbeault complexes, but obviously $\overline{\partial}_z f$ and $\overline{\partial}_z g$ do not match. In fact, if we write $g(z,w) = u(z,w)\, dw_1 \wedge \ldots \wedge dw_n \in C^\infty(V) \widehat{\otimes} \mathcal{H}^2(\mathbb{B}_n)$ (i.e. $u(z,w)$ is a smooth function that is holomorphic in w), then we get

$$f(z,w) = g(z, (w-\overline{z})/\varepsilon) = \varepsilon^{-n} u(z, (w-\overline{z})/\varepsilon)\, dw_1 \wedge \ldots \wedge dw_n,$$

$$\overline{\partial}_z f(z,w) = \varepsilon^{-n}\Big(\overline{\partial}_z u(z, (w-\overline{z})/\varepsilon) - \varepsilon^{-1} \sum_{1 \leq j \leq n} \frac{\partial u}{\partial w_j}(z, (w-\overline{z})/\varepsilon)\, d\overline{z}_j\Big) \wedge dw_1 \wedge \ldots \wedge dw_n.$$

Therefore the trivialization $\tau_* : f \mapsto u$ yields at the level of $\overline{\partial}$-connections an identification

$$\tau_* : \overline{\partial}_z \, f \stackrel{\simeq}{\longmapsto} \overline{\partial}_z \, u + Au$$

where the "connection matrix" $A \in \Gamma(V, \Lambda^{0,1} T_X^* \otimes_{\mathbb{C}} \mathrm{End}(\mathcal{O}_X(V) \,\widehat{\otimes}\, \mathcal{H}^2(\mathbb{B}_n)))$ is the linear operator $A = \mathrm{Id}_{\mathcal{O}_X(V)} \,\widehat{\otimes}\, \Psi$ induced by the unbounded Hilbert space operator

$$\Psi : \mathcal{H}^2(\mathbb{B}_n) \to \mathcal{H}^2(\mathbb{B}_n) \otimes_{\mathbb{C}} \Lambda^{0,1} T_X^*, \quad u \mapsto \Psi u = -\varepsilon^{-1} \sum_{1 \leq j \leq n} \frac{\partial u}{\partial w_j} \, d\overline{z}_j.$$

We see that the holomorphic structure of B_ε is given by a $(0,1)$-connection that differs by the matrix A from the trivial $(0,1)$-connection, and as A is unbounded, there is no way we can make it trivial by a real analytic gauge change with values in Lie algebra of continuous endomorphisms of $\mathcal{H}^2(\mathbb{B}_n)$. $\qquad\square$

We are now going to compute the curvature tensor of the Bergman bundle B_ε. For the sake of simplicity, we identify here $\mathcal{H}^2(\mathbb{B}_n)$ to the Hardy space of L^2 holomorphic functions via $u \mapsto g = u(w) \, dw_1 \wedge \ldots \wedge dw_n$. After rescaling, we can also assume $\varepsilon = 1$, and at least in a first step, we perform our calculations on B_1 rather than B_ε. Let us write $w^\alpha = \prod_{1 \leq j \leq n} w_j^{\alpha_j}$ for a multiindex $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, and denote by λ the Lebesgue measure on $\mathbb{C}^n$. A straightforward calculation gives

$$\int_{\mathbb{B}_n} |w^\alpha|^2 d\lambda(w) = \pi^n \frac{\alpha_1! \ldots \alpha_n!}{(|\alpha| + n)!}, \quad |\alpha| = \alpha_1 + \cdots + \alpha_n.$$

In fact, by using polar coordinates $w_j = r_j e^{i\theta_j}$ and writing $t_j = r_j^2$, we get

$$\int_{\mathbb{B}_n} |w^\alpha|^2 d\lambda(w) = (2\pi)^n \int_{r_1^2 + \cdots + r_n^2 < 1} r^{2\alpha} \, r_1 dr_1 \ldots r_n dr_n = \pi^n I(\alpha)$$

with

$$I(\alpha) = \pi^n \int_{t_1 + \cdots + t_n < 1} t^\alpha \, dt_1 \ldots dt_n.$$

Now, an induction on n together with the Fubini formula gives

$$I(\alpha) = \int_0^1 t_n^{\alpha_n} dt_n \int_{t_1 + \cdots + t_{n-1} < 1 - t_n} (t')^{\alpha'} dt_1 \ldots dt_{n-1}$$

$$= I(\alpha') \int_0^1 (1 - t_n)^{\alpha_1 + \cdots + \alpha_{n-1} + n - 1} t_n^{\alpha_n} dt_n$$

where $t' = (t_1, \ldots, t_{n-1})$ and $\alpha' = (\alpha_1, \ldots, \alpha_{n-1})$. As $\int_0^1 x^a (1-x)^b dt = \frac{a! b!}{(a+b+1)!}$, we get inductively

$$I(\alpha) = \frac{(|\alpha'| + n - 1)! \alpha_n!}{(|\alpha| + n)!} I(\alpha') \quad \Rightarrow \quad I(\alpha) = \frac{\alpha_1! \ldots \alpha_n!}{(|\alpha| + n)!}.$$

This implies that a Hilbert (orthonormal) basis of $\mathcal{O} \cap L^2(\mathbb{B}_n)$ is

$$(2.8) \qquad\qquad e_\alpha(w) = \pi^{-n/2} \sqrt{\frac{(|\alpha| + n)!}{\alpha_1! \ldots \alpha_n!}} \, w^\alpha.$$

As a consequence, and quite classically, the Bergman kernel of the unit ball $\mathbb{B}_n \subset \mathbb{C}^n$ is

$$(2.9) \qquad K_n(w) = \sum_{\alpha \in \mathbb{N}^n} |e_\alpha(w)|^2 = \pi^{-n} \sum_{\alpha \in \mathbb{N}^n} \frac{(|\alpha| + n)!}{\alpha_1! \dots \alpha_n!} |w^\alpha|^2 = n! \pi^{-n} (1 - |w|^2)^{-n-1}.$$

If we come back to U_ε for $\varepsilon > 0$ not necessarily equal to 1 (and do not omit any more the trivial n-form $dw_1 \wedge \dots \wedge dw_n$), we have to use a rescaling $(z, w) \mapsto (\varepsilon^{-1} z, \varepsilon^{-1} w)$. This gives for the Hilbert bundle B_ε a real analytic orthonormal frame

$$(2.10) \qquad e_\alpha(z, w) = \pi^{-n/2} \varepsilon^{-|\alpha|-n} \sqrt{\frac{(|\alpha| + n)!}{\alpha_1! \dots \alpha_n!}} \, (w - \overline{z})^\alpha \, dw_1 \wedge \dots \wedge dw_n$$

A germ of holomorphic section $\sigma \in \mathcal{O}(B_\varepsilon)$ near $z = 0$ (say) is thus given by a convergent power series

$$\sigma(z, w) = \sum_{\alpha \in \mathbb{N}^n} \xi_\alpha(z) \, e_\alpha(z, w)$$

such that the functions ξ_α are real analytic on a neighborhood of 0 and satisfy the following two conditions:

$$(2.11) \qquad \sum_{\alpha \in \mathbb{N}^n} |\xi_\alpha(z)|^2 \text{ is uniformly convergent,}$$

$$(2.12) \qquad \overline{\partial}_{z_k} \sigma(z, w) = \sum_{\alpha \in \mathbb{N}^n} \overline{\partial}_{z_k} \xi_\alpha(z) \, e_\alpha(z, w) + \xi_\alpha(z) \, \overline{\partial}_{z_k} e_\alpha(z, w) \equiv 0.$$

Let $c_k = (0, \dots, 1, \dots, 0)$ be the canonical basis of the $\mathbb{Z}$-module $\mathbb{Z}^n$. A straightforward calculation from (2.10) yields

$$\overline{\partial}_{z_k} e_\alpha(z, w) = -\varepsilon^{-1} \sqrt{\alpha_k(|\alpha| + n)} \, e_{\alpha - c_k}(z, w).$$

We have the slight problem that the coefficients are unbounded as $|\alpha| \to +\infty$, and therefore the two terms occurring in (2.12) need not form convergent series when taken separately. However if we take $\sigma \in \mathcal{O}(B_{\rho\varepsilon})$ in a slightly bigger tubular neighborhood ($\rho > 1$), the L^2 condition implies that $\sum_\alpha (\rho')^{2|\alpha|} |\xi_\alpha|^2$ is uniformly convergent for every $\rho' \in \,]1, \rho[$, and this is more than enough to ensure convergence, since the growth of $\alpha \mapsto \sqrt{\alpha_k(|\alpha| + n)}$ is at most linear; we can even iterate as many derivatives as we want. For a smooth section $\sigma \in C^\infty(B_{\rho\varepsilon})$, the coefficients ξ_α are smooth, with $\sum \rho^{2|\alpha|} |\partial_z^\beta \overline{\partial}_z^\gamma \xi_\alpha|^2$ convergent for all β, γ, and we get

$$\overline{\partial}_{z_k} \sigma(z, w) = \sum_{\alpha \in \mathbb{N}^n} \overline{\partial}_{z_k} \xi_\alpha(z) \, e_\alpha(z, w) + \xi_\alpha(z) \, \overline{\partial}_{z_k} e_\alpha(z, w)$$

$$= \sum_{\alpha \in \mathbb{N}^n} \overline{\partial}_{z_k} \xi_\alpha(z) \, e_\alpha(z, w) - \varepsilon^{-1} \sqrt{\alpha_k(|\alpha| + n)} \, \xi_\alpha(z) \, e_{\alpha - c_k}(z, w)$$

$$= \sum_{\alpha \in \mathbb{N}^n} \left(\overline{\partial}_{z_k} \xi_\alpha(z) - \varepsilon^{-1} \sqrt{(\alpha_k + 1)(|\alpha| + n + 1)} \, \xi_{\alpha + c_k}(z) \right) e_\alpha(z, w),$$

after replacing α by $\alpha + c_k$ in the terms containing ε^{-1}. The $(0, 1)$-part $\nabla_h^{0,1}$ of the Chern connection ∇_h of (B_ε, h) with respect to the orthonormal frame (e_α) is thus given by

$$(2.13) \qquad \nabla_h^{0,1} \sigma = \sum_{\alpha \in \mathbb{N}^n} \left(\overline{\partial} \xi_\alpha - \sum_k \varepsilon^{-1} \sqrt{(\alpha_k + 1)(|\alpha| + n + 1)} \, \xi_{\alpha + c_k} \, d\overline{z}_k \right) \otimes e_\alpha.$$

Now, we replace the natural L^2 norm by

$$|\sigma|_h^2 = \sum_{\alpha \in \mathbb{N}^n} \lambda_{|\alpha|} |\xi_\alpha|^2, \quad \lambda_{|\alpha|} > 0.$$

The $(1,0)$-part can be derived from the identity $\partial |\sigma|_h^2 = \langle \nabla_h^{1,0} \sigma, \sigma \rangle_h + \langle \sigma, \nabla_h^{0,1} \sigma \rangle_h$. However

$$\partial_{z_j} |\sigma|_h^2 = \partial_{z_j} \sum_{\alpha \in \mathbb{N}^n} \lambda_{|\alpha|} \xi_\alpha \overline{\xi}_\alpha = \sum_{\alpha \in \mathbb{N}^n} \lambda_{|\alpha|} (\partial_{z_j} \xi_\alpha) \, \overline{\xi}_\alpha + \lambda_{|\alpha|} \xi_\alpha \, (\overline{\partial_{\overline{z}_j} \xi_\alpha})$$

$$= \sum_{\alpha \in \mathbb{N}^n} \lambda_{|\alpha|} \Big(\partial_{z_j} \xi_\alpha + \varepsilon^{-1} \frac{\lambda_{|\alpha|-1}}{\lambda_{|\alpha|}} \sqrt{\alpha_j (|\alpha| + n)} \, \xi_{\alpha - c_j} \Big) \overline{\xi}_\alpha$$

$$+ \sum_{\alpha \in \mathbb{N}^n} \lambda_{|\alpha|} \xi_\alpha \Big(\overline{\partial_{\overline{z}_j} \xi_\alpha - \varepsilon^{-1} \sqrt{(\alpha_j + 1)(|\alpha| + n + 1)} \, \xi_{\alpha + c_j}} \Big).$$

For $\sigma \in C^\infty(B_{\rho\varepsilon})$, it follows from there that

$$(2.14) \qquad \nabla_h^{1,0} \sigma = \sum_{\alpha \in \mathbb{N}^n} \Big(\partial \xi_\alpha + \varepsilon^{-1} \sum_j \frac{\lambda_{|\alpha|-1}}{\lambda_{|\alpha|}} \sqrt{\alpha_j(|\alpha| + n)} \, \xi_{\alpha - c_j} dz_j \Big) \otimes e_\alpha.$$

Finally, to find the curvature tensor of (B_ε, h), we only have to compute the $(1,1)$-form $(\nabla_h^{1,0} \nabla_h^{0,1} + \nabla_h^{0,1} \nabla_h^{1,0}) \sigma$ and take the terms that contain no differentiation at all, especially in view of the usual identity $\partial \overline{\partial} + \overline{\partial} \partial = 0$ and the fact that we also have here $(\nabla_h^{1,0})^2 = 0$, $(\nabla_h^{0,1})^2 = 0$. As $(\alpha - c_j)_k = \alpha_k - \delta_{jk}$ and $(\alpha + c_k)_j = \alpha_j + \delta_{jk}$, we are left with

$$\big(\nabla_h^{1,0} \nabla_h^{0,1} + \nabla_h^{0,1} \nabla_h^{1,0} \big) \sigma$$

$$= -\varepsilon^{-2} \sum_{\alpha \in \mathbb{N}^n} \sum_{j,k} \frac{\lambda_{|\alpha|-1}}{\lambda_{|\alpha|}} \sqrt{\alpha_j (\alpha_k - \delta_{jk} + 1)} \, (|\alpha| + n) \, \xi_{\alpha - c_j + c_k} \, dz_j \wedge d\overline{z}_k \otimes e_\alpha$$

$$+ \varepsilon^{-2} \sum_{\alpha \in \mathbb{N}^n} \sum_{j,k} \frac{\lambda_{|\alpha|}}{\lambda_{|\alpha|+1}} \sqrt{(\alpha_j + \delta_{jk})(\alpha_k + 1)} \, (|\alpha| + n + 1) \, \xi_{\alpha - c_j + c_k} \, dz_j \wedge d\overline{z}_k \otimes e_\alpha.$$

$$= -\varepsilon^{-2} \sum_{\alpha \in \mathbb{N}^n} \sum_{j,k} \frac{\lambda_{|\alpha|-2}}{\lambda_{|\alpha|-1}} \sqrt{(\alpha_j - \delta_{jk})(\alpha_k - \delta_{jk})} \, (|\alpha| + n - 1) \, \xi_{\alpha - c_j} \, dz_j \wedge d\overline{z}_k \otimes e_{\alpha - c_k}$$

$$+ \varepsilon^{-2} \sum_{\alpha \in \mathbb{N}^n} \sum_{j,k} \frac{\lambda_{|\alpha|-1}}{\lambda_{|\alpha|}} \sqrt{\alpha_j \alpha_k} \, (|\alpha| + n) \, \xi_{\alpha - c_j} \, dz_j \wedge d\overline{z}_k \otimes e_{\alpha - c_k}.$$

$$= \varepsilon^{-2} \sum_{\alpha \in \mathbb{N}^n} \sum_{j,k} \sqrt{\alpha_j \alpha_k} \Big(\frac{\lambda_{|\alpha|-1}}{\lambda_{|\alpha|}} (|\alpha| + n) - \frac{\lambda_{|\alpha|-2}}{\lambda_{|\alpha|-1}} (|\alpha| + n - 1) \Big) \xi_{\alpha - c_j} \, dz_j \wedge d\overline{z}_k \otimes e_{\alpha - c_k}$$

$$+ \varepsilon^{-2} \sum_{\alpha \in \mathbb{N}^n} \sum_j \frac{\lambda_{|\alpha|-2}}{\lambda_{|\alpha|-1}} (|\alpha| + n - 1) \, \xi_{\alpha - c_j} \, dz_j \wedge d\overline{z}_j \otimes e_{\alpha - c_j},$$

where the last summation comes from the subtraction of the diagonal terms $j = k$. By changing α into $\alpha + c_j$ in that summation, we obtain the following expression of the curvature tensor of (B_ε, h).

2.15. Theorem. *The curvature tensor of the Bergman bundle (B_ε, h) is given by*

$$\langle \Theta_{B_\varepsilon, h} \sigma(v, Jv), \sigma \rangle_h = \varepsilon^{-2} \sum_{\alpha \in \mathbb{N}^n} \Big(\Big| \sum_j \sqrt{\alpha_j} \, \xi_{\alpha - c_j} v_j \Big|^2 + \sum_j (|\alpha| + n) |\xi_\alpha|^2 |v_j|^2 \Big)$$

for every $\sigma = \sum_\alpha \xi_\alpha e_\alpha \in B_{\rho\varepsilon}$, $\rho > 1$, and every tangent vector $v = \sum v_j\, \partial/\partial z_j$.

The above curvature hermitian tensor is positive definite, and even positive definite unbounded if we view it as a hermitian form on $T_X \otimes B_\varepsilon$ rather than on $T_X \otimes B_{\rho\varepsilon}$. This is not so surprising since the connection matrix was already an unbounded operator. Philosophically, the very ampleness of B_ε was also a strong indication that the curvature should have been positive. Observe that we have in fact

$$\varepsilon^{-2} \sum_{\alpha \in \mathbb{N}^n} \sum_j (|\alpha| + n)\, |\xi_\alpha|^2 |v_j|^2$$

$$(2.16) \qquad\qquad \leq \langle \Theta_{B_\varepsilon,h}\sigma(v, Jv), \sigma \rangle_h \leq 2\varepsilon^{-2} \sum_{\alpha \in \mathbb{N}^n} \sum_j (|\alpha| + n)\, |\xi_\alpha|^2 |v_j|^2,$$

thanks to the Cauchy-Schwarz inequality

$$\sum_{\alpha \in \mathbb{N}^n} \left| \sum_j \sqrt{\alpha_j}\, \xi_{\alpha - c_j} v_j \right|^2 \leq \sum_\ell |v_\ell|^2 \sum_{\alpha \in \mathbb{N}^n} \sum_j \alpha_j |\xi_{\alpha - c_j}|^2 = \sum_\ell |v_\ell|^2 \sum_j \sum_{\alpha \in \mathbb{N}^n} \alpha_j |\xi_{\alpha - c_j}|^2$$

$$= \sum_\ell |v_\ell|^2 \sum_j \sum_{\alpha \in \mathbb{N}^n} (\alpha_j + 1)|\xi_\alpha|^2 = \sum_\ell |v_\ell|^2 \sum_{\alpha \in \mathbb{N}^n} (|\alpha| + n)|\xi_\alpha|^2.$$

2.C. Curvature of Bergman bundles on compact hermitian manifolds

3. Invariance of plurigenera for polarized Kähler families

The goal of this section is to prove that for every polarized family $\mathfrak{X} \to S$ of compact Kähler manifolds, the plurigenera $p_m(X_t) = h^0(X_t, mK_{X_t})$ of fibers are independent of t for all $m \geq 0$. This result has first been proved by Y.T. Siu [Siu98] in the case of projective varieties of general type (in which case the proof has been translated in a purely algebraic form by Y. Kawamata [Kaw99]), and then by [Siu00] and Păun [Pau04] in the case of arbitrary projective varieties; in the second case, no algebraic proof of the result is known. We extend here the result to the Kähler context. This requires substantial modifications of the proof, since the technique of Siu and Păun involved in a crucial manner the use of an auxiliary ample line bundle. We replace it here by a use of the Hilbert bundles studied in the previous section.

3.1. Conjecture. *Let $\pi : \mathfrak{X} \to S$ be a proper holomorphic map defining a family of smooth compact Kähler manifolds over an irreducible base S. Assume that π admits local polarizations, i.e. every point $s_0 \in S$ has a neighborhood U such that $\pi^{-1}(U)$ carries a closed smooth $(1,1)$-form ω such that $\omega_{|X_t}$ is positive definite on $X_t := \pi^{-1}(t)$. Then the plurigenera $p_m(X_t) = h^0(X_t, mK_{X_t})$ of fibers are independent of t for all $m \geq 0$.*

The above statement would follow directly from the following more technical result.

3.2. Conjecture (generalized version of the Claudon-Păun theorem). *Let $\pi : \mathfrak{X} \to \Delta$ be a polarized family of compact Kähler manifolds over a disc $\Delta \subset \mathbb{C}$, and let $(\mathcal{L}_j, h_j)_{0 \leq j \leq N-1}$ be (singular) hermitian line bundles with semi-positive curvature currents $i\Theta_{\mathcal{L}_j, h_j} \geq 0$ on $\mathfrak{X}$. Assume that*

(a) *the restriction of h_j to the central fiber X_0 is well defined (i.e. not identically $+\infty$).*

(b) *the multiplier ideal sheaf $\mathcal{I}(h_{j|X_0})$ is trivial for $1 \leq j \leq N-1$.*

Then any section σ of $\mathcal{O}(mK_{\mathcal{X}} + \sum \mathcal{L}_j)_{|X_0} \otimes \mathcal{I}(h_{0|X_0})$ over the central fiber X_0 extends into a section $\widetilde{\sigma}$ of $\mathcal{O}(mK_{\mathcal{X}} + \sum \mathcal{L}_j)$ over a certain neighborhood $\mathcal{X}' = \pi^{-1}(\Delta')$ of X_0, where $\Delta' \subset \Delta$ is a sufficienty small disc centered at 0.

3.3. Remark. A standard cohomological argument shows that we can in fact take $\mathcal{X}' = \mathcal{X}$ in the conclusion of Theorem 3.2, because the direct image sheaf $\mathcal{E} = \pi_* \mathcal{O}(mK_{\mathcal{X}} + \sum \mathcal{L}_j)$ is coherent, and the restriction $\mathcal{E} \to \mathcal{E} \otimes (\mathcal{O}_\Delta / \mathfrak{m}_0 \mathcal{O}_\Delta)$ induces a surjective map at the H^0 level on the Stein space Δ, so we can extend $\widetilde{\sigma}$ mod $\pi^* \mathfrak{m}_0$ to $\mathcal{X}$.

Conjecture 3.2 *implies Conjecture* 3.1. The invariance of plurigenera is in fact just obtained as the special case of Conjecture 3.2 when all line bundles $\mathcal{L}_j$ and their metrics h_j are trivial. Since the dimension $t \mapsto h^0(X_t, mK_{X_t})$ is always upper semicontinuous and since Conjecture 3.2 implies the lower semicontinuity, we conclude that the dimension must be constant along analytic discs, hence along the irreducible base S, by joining any two points through a chain of analytic discs. □

3.4. Lemma. *For $\varepsilon \leq \varepsilon_0$ small enough, one can find a Stein open subset $\mathcal{U}'_\varepsilon \subset \mathcal{X}' \times \overline{X}_0$, where $\mathcal{X}' = \pi^{-1}(\Delta') \to \Delta'$ is the restriction of $\pi : \mathcal{X} \to \Delta$ to a disc $\Delta' \Subset \Delta$ centered at 0, of sufficiently small radius $r' \leq r_0(\varepsilon)$, such that the projection $\mathrm{pr}_1 : \mathcal{U}'_\varepsilon \to \mathcal{X}'$ is a complex ball bundle over $\mathcal{X}'$ that is locally trivial real analytically.*

Proof. The argument is very similar to what we did in §1, applied to the central fiber X_0 only. Let γ_0 be a real analytic hermitian metric on X_0 and $\exp h : T_{X_0} \to X_0$ be the corresponding real analytic and fiber-holomorphic exponential map associated with γ_0, as in §1. By construction, $Y = \mathcal{X} \times \overline{X}_0$ has a tangent bundle T_Y that admits a direct sum decomposition $T_Y = \mathrm{pr}_1^* T_{\mathcal{X}} \oplus \mathrm{pr}_2^* T_{\overline{X}_0}$ into holomorphic subbundles. Let $\Phi_t : X_0 \to X_t$, $t \in \Delta$, be a real analytic family of diffeomorphisms such that $\Phi_0 = \mathrm{Id}$ (by the Ehresmann lemma, such a family can be obtained by taking the flow of a real analytic vector field τ on $\mathcal{X}$ that lifts $d/dt \in T_\Delta$ on a neighborhood of $\overline{\Delta'}$). We consider the graph Γ' of (Φ_t), namely the real analytic set $\Gamma' \subset \mathcal{X}' \times \overline{X}_0$ of points of the form $(\Phi_t(x), \overline{x})$, $x \in X_0$, $t \in \Delta'$, and take

$$\mathcal{U}'_\varepsilon = \big\{ (\Phi_t(x), \overline{\exp h_x(\xi)}) \, ; \, (\Phi_t(x), \overline{x}) \in \Gamma', \, \xi \in T_{X_0, x}, \, |\xi|^2_\gamma < \varepsilon^2 - |t|^2) \big\}.$$

Let $\log h : X_0 \times X_0 \to T_{X_0}$ be the inverse of $\exp h$ near the diagonal. Then $\mathcal{U}'_\varepsilon$ is defined as the set of points $(w, \overline{z}) \in \mathcal{X}' \times \overline{X}_0$ such that

$$t = \pi(w) \text{ satisfies } |t| < r' = \mathrm{radius} \, \Delta', \quad |\log h(\Phi_t^{-1}(w), \overline{z})|^2_{\gamma_0} + |t|^2 < \varepsilon^2.$$

Now, the function

$$(3.5) \qquad \psi(w, z) = |\log h(\Phi_t^{-1}(w), \overline{z})|^2_{\gamma_0} + |t|^2 = |\log h(\Phi_{\pi(w)}^{-1}(w), \overline{z})|^2_{\gamma_0} + |\pi(w)|^2$$

is strictly plurisubharmonic along its zero variety $t = 0$, $w = \overline{z}$ (equal to the diagonal of $X_0 \times \overline{X}_0$, therefore it must be strictly plurisubharmonic on the domain defined by conditions (3.5) (let us recall that $\pi : \mathcal{X} \to \Delta$ is holomorphic). As a consequence $\mathcal{U}'_\varepsilon$ is stein for $\varepsilon \leq \varepsilon_0$ small enough, and the inequality $r' \leq r_0(\varepsilon) := \varepsilon/2$ guarantees that the fibers of $p = \mathrm{pr}_1 : \mathcal{U}'_\varepsilon \to \mathcal{X}'$ are nonempty balls. The other stated properties can be checked exactly as in §1. □

In order to study Conjecture 3.2, we first state a technical extension theorem needed for the proof, which is a special case of the well-known and extremely powerful Ohsawa-Takegoshi theorem [OT87], see also [Oh??], [Dem??].

3.6. Proposition. *Let $\pi : \mathcal{Z} \to \Delta$ be a smooth and proper morphism from a (non compact) Kähler manifold $\mathcal{Z}$ to a disc $\Delta \subset \mathbb{C}$ and let $(\mathcal{L}, h)$ be a (singular) hermitian line bundle with semi-positive curvature current $i\Theta_{\mathcal{L},h} \geq 0$ on $\mathcal{Z}$. Let ω be a global Kähler metric on $\mathcal{Z}$, and let $dV_{\mathcal{Z}}$, dV_{Z_0} the respective induced volume elements on $\mathcal{Z}$ and $Z_0 = \pi^{-1}(0)$. Assume that h_{Z_0} is well defined (i.e. almost everywhere finite). Then any holomorphic section s of $\mathcal{O}(K_{\mathcal{Z}} + \mathcal{L}) \otimes \mathcal{I}(h_{|Z_0})$ extends into a section $\widetilde{s}$ over $\mathcal{Z}$ satisfying an L^2 estimate*

$$\int_{\mathcal{Z}} \|\widetilde{s}\|^2_{\omega \otimes h} dV_{\mathcal{Z}} \leq C_0 \int_{Z_0} \|s\|^2_{\omega \otimes h} dV_{Z_0},$$

where $C_0 \geq 0$ is some universal constant (depending on $\dim \mathcal{Z}$ and $\operatorname{diam} \Delta$, but otherwise independent of $\mathcal{Z}$, $\mathcal{L}$, ...).

3.7. Remark. The assumptions of Proposition 3.6 imply that $\mathcal{Z}$ is holomorphically convex and complete Kähler, thus the technique of [Dem??] does apply to yield the result.

Attempt of proof of Conjecture 3.2. Let $p = \operatorname{pr}_1 : \mathcal{U}'_\varepsilon \to \mathcal{X}'$ be as in Lemma 3.4, and $q = \operatorname{pr}_2 : \mathcal{U}'_\varepsilon \to \overline{X}_0$. We take $\varepsilon < \varepsilon_0$ and use on $\mathcal{Z} := \mathcal{U}'_\varepsilon$ a Kähler metric ω_0 defined on the Stein manifold $\mathcal{U}'_{\varepsilon_0}$. On can define e.g. ω_0 as the $i\partial\bar{\partial}$ of a strictly plurisubharmonic exhaustion function on $\mathcal{U}'_{\varepsilon_0}$, but we can also take the restriction of $\operatorname{pr}_1^* \omega + \operatorname{pr}_2^* \overline{\omega}_{|\overline{X}_0}$ where ω is the Kähler metric on the total space $\mathcal{X}$, and $\overline{\omega} = -\omega$ the corresponding Kähler metric on the conjugate space $\overline{\mathcal{X}}$.

First step: construction of a sequence of extensions on $\mathcal{Z} = \mathcal{U}'_\varepsilon$ via the Ohsawa-Takegoshi extension theorem.

The strategy is to apply iteratively the special case 3.6 of the Ohsawa-Takegoshi extension theorem on the total space of the fibration

$$\pi' = \pi \circ p : \mathcal{Z} = \mathcal{U}'_\varepsilon \to \mathcal{X}' \to \Delta',$$

and to extend sections of ad hoc pull-backs $p^* \mathcal{G}$ from the zero fiber $Z_0 = \mathcal{U}'_\varepsilon \cap (X_0 \times \overline{X}_0)$ to the whole of $\mathcal{Z} = \mathcal{U}'_\varepsilon$. We write $h_j = e^{-\varphi_j}$ in terms of local plurisubharmonic weights, and define inductively a sequence of line bundles $\mathcal{G}_m$ by putting $\mathcal{G}_0 = \mathcal{O}_X$ and

$$\mathcal{G}_m = \mathcal{G}_{m-1} + K_{\mathcal{X}} + \mathcal{L}_r \qquad \text{if } m = Nq + r, \ 0 \leq r \leq N - 1.$$

By construction we have

$$\mathcal{G}_m = mK_{\mathcal{X}} + \mathcal{L}_1 + \cdots + \mathcal{L}_m, \qquad \text{for } 1 \leq m \leq N - 1,$$
$$\mathcal{G}_{m+N} - \mathcal{G}_m = \mathcal{G}_N = NK_{\mathcal{X}} + \mathcal{L}_0 + \cdots + \mathcal{L}_{N-1}, \quad \text{for all } m \geq 0.$$

The game is to construct inductively families of sections, say $\{\widetilde{f}_j^{(m)}\}_{j=1,\dots,J(m)}$, of $p^* \mathcal{G}_m$ over $\mathcal{Z}$, together with ad hoc L^2 estimates, in such a way that

(3.8) for $m = 0, \dots, N-1$, $p^* \mathcal{G}_m$ is generated by L^2 sections $\{\widetilde{f}_j^{(m)}\}_{j=1,\dots,J(m)}$ on $\mathcal{U}'_{\varepsilon_0}$;

(3.9) we have the m-periodicity relations $J(m+N) = J(m)$ and $\widetilde{f}_j^{(m)}$ is an extension of
$$f_j^{(m)} := (p^*\sigma)^q f_j^{(r)} \text{ over } \mathcal{Z} \text{ for } m = Nq + r, \text{ where } f_j^{(r)} := \widetilde{f}_{j|Z_0}^{(r)}, 0 \leq r \leq N-1.$$

Property (3.8) can certainly be achieved since $\mathcal{U}'_{\varepsilon_0}$ is Stein, and for $m = 0$ we can take $J(0) = 1$ and $\widetilde{f}_1^{(0)} = 1$. Now, by induction, we equip $p^*\mathcal{G}_{m-1}$ with the tautological metric $|\xi|^2 / \sum |\widetilde{f}_j^{(m-1)}(x)|^2$, and

$$\widetilde{\mathcal{G}}_m := p^*\mathcal{G}_m - K_{\mathcal{Z}} = p^*\mathcal{G}_m - (p^*K_{\mathcal{X}} + q^*K_{\overline{X}_0}) = p^*(\mathcal{G}_{m-1} + \mathcal{L}_r) - q^*K_{\overline{X}_0}$$

with that metric multiplied by $p^*h_r = e^{-p^*\varphi_r}$ and a fixed smooth metric $e^{-\psi}$ of positive curvature on $(-q^*K_{\overline{X}_0})_{|\mathcal{U}'_{\varepsilon_0}}$ (remember that $\mathcal{U}'_{\varepsilon_0}$ is Stein!). It is clear that these metrics have semi-positive curvature currents on $\mathcal{Z}$ (by adjusting ψ, we could even take them to be strictly positive if we wanted). In this setting, we apply the Ohsawa-Takegoshi theorem to the line bundle $K_{\mathcal{Z}} + \widetilde{\mathcal{G}}_m = p^*\mathcal{G}_m$, and extend in this way $f_j^{(m)}$ into a section $\widetilde{f}_j^{(m)}$ over $\mathcal{Z}$. By construction the pointwise norm of that section in $p^*\mathcal{G}_{m|X_0}$ in a local trivialization of the bundles involved is the ratio

$$\frac{|f_j^{(m)}|^2}{\sum_\ell |f_\ell^{(m-1)}|^2} e^{-p^*\varphi_r - \psi},$$

up to some fixed smooth positive factor depending only on the metric induced by ω_0 on $K_{\mathcal{Z}}$. However, by the induction relations, we have

$$\frac{\sum_j |f_j^{(m)}|^2}{\sum_\ell |f_\ell^{(m-1)}|^2} e^{-p^*\varphi_r} = \begin{cases} \dfrac{\sum_j |f_j^{(r)}|^2}{\sum_\ell |f_\ell^{(r-1)}|^2} e^{-p^*\varphi_r} & \text{for } m = Nq + r, \, 0 < r \leq N-1, \\[2ex] \dfrac{\sum_j |f_j^{(0)}|^2}{\sum_\ell |f_\ell^{(N-1)}|^2} |p^*\sigma|^2 e^{-p^*\varphi_0} & \text{for } m \equiv 0 \bmod N, \, m > 0. \end{cases}$$

Since the sections $\{f_j^{(r)}\}_{0 \leq r < N}$ generate their line bundle on $\mathcal{U}_{\varepsilon_0} \supset \overline{\mathcal{U}'_\varepsilon}$, the ratios involved are positive functions without zeroes and poles, hence smooth and bounded [possibly after shrinking a little bit the base disc Δ', as is permitted]. On the other hand, assumption 3.2 (b) and the fact that σ has coefficients in the multiplier ideal sheaf $\mathcal{I}(h_{0|X_0})$ tell us that $e^{-p^*\varphi_r}$, $1 \leq r < m$ and $|p^*\sigma|^2 e^{-p^*\varphi_0}$ are locally integrable on Z_0. It follows that there is a constant $C_1 \geq 0$ such that

$$\int_{Z_0} \frac{\sum_j |f_j^{(m)}|^2}{\sum_\ell |f_\ell^{(m-1)}|^2} e^{-p^*\varphi_r - \psi} dV_{\omega_0} \leq C_1$$

for all $m \geq 1$ (of course, the integral certainly involves finitely many trivializations of the bundles involved, whereas the integrand expression is just local in each chart). Inductively, the L^2 extension theorem produces sections $\widetilde{f}_j^{(m)}$ of $p^*\mathcal{G}_m$ over $\mathcal{Z}$ such that

$$\int_{\mathcal{Z}} \frac{\sum_j |\widetilde{f}_j^{(m)}|^2}{\sum_\ell |\widetilde{f}_\ell^{(m-1)}|^2} e^{-p^*\varphi_r - \psi} dV_{\omega_0} \leq C_2 = C_0 C_1.$$

Second step: applying the Hölder inequality. Put $k = Nq(k) + r(k)$ with $0 \le r(k) < N$, and take $m = Nq(m)$ to be a multiple of N. The Hölder inequality $|\int \prod_{1 \le k \le m} u_k d\mu| \le \prod_{1 \le k \le m} (\int |u_k|^m d\mu)^{1/m}$ applied to the measure $\mu = dV_{\omega_0}$ and to the product of functions

$$\left(\frac{\sum_j |\widetilde{f}_j^{(m)}|^2}{\sum_\ell |\widetilde{f}_\ell^{(0)}|^2} \right)^{1/m} e^{-\frac{1}{N}p^*(\varphi_0 + \ldots + \varphi_{N-1}) - \psi} = \prod_{1 \le k \le m} \left(\frac{\sum_j |\widetilde{f}_j^{(k)}|^2}{\sum_\ell |\widetilde{f}_\ell^{(k-1)}|^2} e^{-p^* \varphi_{r(k)} - \psi} \right)^{1/m}$$

in which $\sum_\ell |\widetilde{f}_\ell^{(0)}|^2 = |\widetilde{f}_1^{(0)}|^2 = 1$ and $\sum_j |\widetilde{f}_j^{(m)}|^2 = |\widetilde{f}_1^{(m)}|^2$, implies that

$$(3.10) \qquad \int_{\mathcal{Z}} |\widetilde{f}_1^{(m)}|^{2/m} e^{-\frac{1}{N}p^*(\varphi_0 + \ldots + \varphi_{N-1}) - \psi} dV_{\omega_0} \le C_2.$$

As the functions $\varphi_{r(k)}$ and ψ are locally bounded from above, we infer from there the weaker inequality

$$(3.10') \qquad \int_{\mathcal{Z}} |\widetilde{f}_1^{(m)}|^{2/m} dV_{\omega_0} \le C_3.$$

The last inequality is to be understood as an inequality that holds in fact only locally over $\mathcal{X}'$, on sets of the form $p^{-1}(V)$, where $V \Subset \mathcal{X}'$ are small coordinate open sets where our line bundles are trivial, so that the section $\widetilde{f}_1^{(m)}$ of $q(m) \, p^*(NK_{\mathcal{X}} + \sum \mathcal{L}_j)$ can be viewed as a holomorphic function on $p^{-1}(V)$.

Third step: construction of a singular hermitian metric on $NK_{\mathcal{X}} + \sum \mathcal{L}_j$. The rough idea is to extract a weak limit of the m-th root occurring in (3.10), (3.10′), combined with an integration on the fibers of $p : \mathcal{Z} = \mathcal{U}'_\varepsilon \to \mathcal{X}'$, to get a singular hermitian metric on $NK_{\mathcal{X}} + \sum \mathcal{L}_j$. This is the crucial step in the proof, and the place where the Kähler setup requires new arguments; especially, the integration on fibers makes the weak limit argument much less obvious than in the projective setup, and requires at least the results of §2 on Bergman bundles.

3.11. Proposition. *Assume that the sections $\widetilde{f}_1^{(m)}$ have been constructed on a slightly bigger tubular neighborhood $\mathcal{U}''_{\rho''\varepsilon} \to \mathcal{X}'' := \pi^{-1}(\Delta'')$ with $\rho'' > 1$, and Δ'' of radius $r'' > r'$ [this condition can of course always be achieved, since we have some flexibility on the choice of the tubular neighborhoods]. Then there exists a subsequence $m \in M_0 \subset \mathbb{N}$ such that the limit*

$$\theta(z) = \lim_{\substack{m \in M_0 \\ m \to +\infty}} \frac{1}{m} \log \int_{w \in \mathcal{U}'_{\varepsilon,z}} |\widetilde{f}_1^{(m)}(z,w)|^2 dV_{\omega_0}(w), \quad z \in \mathcal{X}'$$

exists almost everywhere on $\mathcal{X}'$, and $H = e^{-N\theta}$ defines a singular hermitian metric on $p^*(NK_{\mathcal{X}} + \sum \mathcal{L}_j)$. i.e. $i\partial\overline{\partial}\theta \ge 0$. Moreover, it satisfies the estimates

(a) $|\sigma|_H^2 = |\sigma|^2 e^{-N\theta} = 1$ *on* $X_0 \subset \mathcal{X}'$;

(b) $\displaystyle\int_{\mathcal{X}'} e^{-\theta} e^{-\frac{1}{N}(\varphi_0 + \ldots + \varphi_{N-1})} dV_\omega < \infty$:

(c) *if $\rho \in \,]1, \rho''[$, there is a constant $C_4 = C_4(\rho) > 0$ such that for every $\beta > 0$*

$$i\partial\overline{\partial}e^{\beta\theta} \ge -\varepsilon^{-2}(\log\rho)^{-1} e^{C_4\beta}\omega.$$

Proof. First notice that the choice of the Kähler metric ω_0 on $\mathcal{X} \times \overline{X}_0$ is irrelevant in the definition of θ (the L^2 integrals may eventually change by bounded multiplicative factors, which get killed as $m \to +\infty$). The mean value inequality for plurisubharmonic functions applied on small balls of $\mathcal{U}''_{\rho''\varepsilon}$ centered at points $(z, w) \in \mathcal{U}'_{\rho'\varepsilon}$, $\rho' \in]\rho, \rho''[$, implies that we have a uniform upper bound $\frac{1}{m}\log|\widetilde{f}_1^{(m)}|^2 \leq C_5 = C_5(\rho')$ on $\mathcal{U}'_{\rho'\varepsilon}$ (that is, in fact, locally over trivializing open sets of the line bundles in $\mathcal{X}'$, but we can arrange that there are only finitely many of these). By construction $\widetilde{f}_1^{(m)} = (p^*\sigma)^{q(m)}$ on $Z_0 = p^{-1}(X_0)$, thus $\frac{1}{m}\log|\widetilde{f}_1^{(m)}|^2 = \frac{1}{N}p^*\log|\sigma|^2$ on Z_0 and $\theta = \frac{1}{N}\log|\sigma|^2$ on X_0. Therefore the limit exists at least on X_0 and estimate (a) is satisfied.

Let us consider now the Bergman bundle $B_\varepsilon \to \mathcal{X}'$, and let us write locally over $\mathcal{X}'$

$$\widetilde{f}_1^{(m)}(z, w) = \sum_{\alpha \in \mathbb{N}^n} \xi_\alpha(z)\, e_\alpha(z, w) \otimes g(z)^{q(m)}, \quad z \in \mathcal{X}', \ w \in \mathcal{U}'_{\varepsilon, z}$$

in terms of an orthonormal frame $(e_\alpha)_{\alpha \in \mathbb{N}^n}$ of $\mathcal{U}'_\varepsilon$ as defined in §2, where g is a local generator of $\mathcal{O}_{\mathcal{X}}(NK_{\mathcal{X}} + \sum \mathcal{L}_j)$. The fact that we have uniform convergence of the series on $\mathcal{U}'_{\rho'\varepsilon}$ and a uniform upper bound $e^{C_5 m}$ implies for the holomorphic power series in w a uniform bound of the form

$$(3.12) \qquad \sum_{\alpha \in \mathbb{N}^n} \rho^{2|\alpha|}|\xi_\alpha(z)|^2 \leq e^{C_6 m}, \quad \text{for some } C_6 = C_6(\rho).$$

For a suitable smooth (and essentially irrelevant) correcting factor $\widetilde{\psi}(z, w)$ on $\mathcal{X}''$, we have a uniform bound

$$\theta_m(z) := \frac{1}{m}\log \int_{w \in \mathcal{U}'_{\varepsilon, z}} |\widetilde{f}_1^{(m)}(z, w)|^2 e^{-\widetilde{\psi}(z, w)} dV_{\omega_0}(w) = \frac{1}{m}\log \sum_{\alpha \in \mathbb{N}^n} |\xi_\alpha(z)|^2 \leq C_5$$

where the summation $\sum_{\alpha \in \mathbb{N}^n} |\xi_\alpha(z)|^2$ is real analytic and $\theta_m(z) \to \frac{1}{N}\log|\sigma(z)|^2$ uniformly on X_0. The next idea is to estimate the Hessian form of $z \mapsto e^{\beta\theta_m(z)}$ for every $\beta > 0$ fixed. We have of course

$$e^{\beta\theta_m(z)} = \left(\sum_{\alpha \in \mathbb{N}^n} |\xi_\alpha(z)|^2\right)^{\beta/m} = \|\xi(z)\|^{2\beta/m} = \langle \xi(z), \xi(z)\rangle^{\beta/m}$$

where ξ is nothing else than the expression of the section $\widetilde{f}_1^{(m)}$ is the (real analytic) trivialization of the Bergman bundle B_ε, and $\langle \bullet, \bullet \rangle$ the natural hermitian metric on B_ε. Now, a standard calculation with respect to the Bergman connection $\nabla = \nabla^{1,0} + \nabla^{0,1}$ of B_ε yields $\nabla^{0,1}\xi = 0$, hence

$$i\partial\overline{\partial}e^{\beta\theta_m(z)} = \frac{\beta}{m}\|\xi\|^{2\beta/m-2}\left(i\langle\nabla^{1,0}\xi, \nabla^{1,0}\xi\rangle - \langle i\Theta_{B_\varepsilon}\xi, \xi\rangle - \left(1 - \frac{\beta}{m}\right)\frac{i\langle\nabla^{1,0}\xi, \xi\rangle \wedge \overline{\langle\nabla^{1,0}\xi, \xi\rangle}}{\|\xi\|^2}\right)$$

$$(3.13) \qquad \geq -\frac{\beta}{m}\|\xi\|^{2\beta/m-2}\langle i\Theta_{B_\varepsilon}\xi, \xi\rangle$$

by the Cauchy-Schwarz inequality. On the other hand, the curvature bound obtained in §2 yields

$$\left|\langle i\Theta_{B_\varepsilon}\xi, \xi\rangle\right|_\omega \leq 2\varepsilon^{-2}\sum_{\alpha \in \mathbb{N}^n}(|\alpha| + n)|\xi_\alpha|^2$$

$$(3.14) \qquad \leq 2\varepsilon^{-2}\left(\sum_{\alpha \in \mathbb{N}^n}|\xi_\alpha|^2\right)^{1-\beta/m}\left(\sum_{\alpha \in \mathbb{N}^n}(|\alpha| + n)^{m/\beta}|\xi_\alpha|^2\right)^{\beta/m}$$

by the discrete Hölder inequality. We compare the last summation to (3.12) by taking the maximum of $t \mapsto (t+n)^{m/\beta}\rho^{-2t}$ which is reached for $t_0 + n = \frac{m}{2\beta}(\log\rho)^{-1}$ by an elementary calculation. This gives

$$
\left(\sum_{\alpha \in \mathbb{N}^n}(|\alpha|+n)^{m/\beta}|\xi_\alpha|^2\right)^{\beta/m} \leq (t_0+n)\rho^{-t_0\beta/m}\left(\sum_{\alpha \in \mathbb{N}^n}\rho^{2|\alpha|}|\xi_\alpha(z)|^2\right)^{\beta/m}
$$

$$
(3.15) \qquad\qquad\qquad\qquad\qquad \leq \frac{m}{2\beta}(\log\rho)^{-1}\rho^{n\beta/m}e^{C_6\beta-1/2}.
$$

A combination of the last three estimates (3.13–3.15) gives

$$
(3.16) \qquad\qquad i\partial\overline{\partial}e^{\beta\theta_m(z)} \geq -\varepsilon^{-2}(\log\rho)^{-1}\rho^{n\beta/m}e^{C_6\beta-1/2}\omega.
$$

The above lower bound is uniform with respect to m as $m \to +\infty$, and we see that $e^{\beta\theta_m}$ is quasi plurisubharmonic. By well known facts of pluripotential theory, there exists an upper semicontinuous regularization

$$
\theta = \left(\limsup_{m\to+\infty}\theta_m\right)^*
$$

and a subsequence $m \in M_0 \subset \mathbb{N}$ such that

$$
\theta = \limsup_{\substack{m\in M_0 \\ m\to+\infty}}\theta_m \quad \text{almost everywhere on } \mathcal{X}',
$$

and satisfying in the limit the Hessian estimate

$$
(3.17) \qquad\qquad i\partial\overline{\partial}e^{\beta\theta} \geq -\varepsilon^{-2}(\log\rho)^{-1}e^{C_6\beta-1/2}\omega
$$

for every $\beta > 0$. Property (c) is proved. □

3.18. What to do ?. Our hope (possibly after modifying the sections $f_j^{(m)}$ in an adequate manner) is that one can gain a factor converging to zero in estimate (3.16), and thus in (3.17). This would prove that $e^{\beta\theta}$ is plurisubharmonic for every $\beta > 0$, hence the weight $\theta = \lim_{\beta\to 0}\frac{1}{\beta}(e^{\beta\theta}-1)$ would also be plurisubharmonic.

Fourth step: applying Ohsawa-Takegoshi once again with the singular hermitian metric produced in the third step.

Assuming that 3.18 holds, we have proved that $NK_\mathcal{X}+\sum\mathcal{L}_j$ possesses a hermitian metric $H = e^{-N\theta}$ such that $\|\sigma\|_H \leq 1$ oon X_0 and $\Theta_H \geq 0$ on $\mathcal{X}'$. In order to conclude, we equip the bundle

$$
\mathcal{E} = (N-1)K_\mathcal{X} + \sum\mathcal{L}_j
$$

with the metric $\eta = H^{1-1/N}\prod h_j^{1/N}$, and $NK_\mathcal{X} + \sum\mathcal{L}_j = K_\mathcal{X} + \mathcal{E}$ with the metric $\omega \otimes \eta$. It is important here that $\mathcal{X}$ possesses a global Kähler polarization ω, otherwise the required estimates would not be valid. Clearly η has a semi-positive curvature current on $\mathcal{X}'$ and in a local trivialization we have

$$
\|\sigma\|_{\omega\otimes\eta}^2 \leq C|\sigma|^2\exp\left(-(N-1)\theta - \frac{1}{N}\sum\varphi_j\right) \leq C\left(|\sigma|^2\prod e^{-\varphi_j}\right)^{1/N}
$$

on X_0. Since $|\sigma|^2 e^{-\varphi_0}$ and $e^{-\varphi_r}$, $r > 0$ are all locally integrable, we see that $\|\sigma\|^2_{\omega \otimes \eta}$ is also locally integrable on X_0 by the Hölder inequality. A new (and final) application of the L^2 extension theorem to the hermitian line bundle $(\mathcal{E}, \eta)$ implies that σ can be extended to $\mathcal{X}'$. Conjecture 3.2 would then be proved. $\qquad \square$

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