Bergman bundles and the plurigenera conjecture for polarized families of Kähler manifolds

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Abstract. We introduce the concept of Bergman bundle attached to a hermitian manifold X, assuming the manifold X to be compact – although the results are local for a large part. The Bergman bundle is some sort of infinite dimensional very ample Hilbert bundle whose fibers are isomorphic to the standard Hardy space on the complex unit ball; however the bundle is locally trivial only in the real analytic category, and its complex structure is strongly twisted. We compute the Chern curvature of the Bergman bundle, and show that it is strictly positive. As an application, we investigate the conjecture on the invariance of plurigenera in the general situation of polarized families of compact Kähler manifolds – a long standing and still unsolved conjecture of Yum-Tong Siu.

0. Introduction

0.1. Conjecture. *Let $\pi : \mathcal{X} \to S$ be a proper holomorphic map defining a family of smooth compact Kähler manifolds over an irreducible base S. Assume that π admits local polarizations, i.e. every point $s_0 \in S$ has a neighborhood V such that $\pi^{-1}(V)$ carries a closed smooth $(1,1)$-form ω such that $\omega_{|X_t}$ is positive definite on $X_t := \pi^{-1}(t)$. Then the plurigenera $p_m(X_t) = h^0(X_t, mK_{X_t})$ of fibers are independent of t for all $m \geq 0$.*

1. Exponential map and tubular neighborhoods

Let X be a compact n-dimensional complex manifold and $Y \subset X$ a smooth totally real submanifold, i.e. such that $T_Y \cap JT_Y = \{0\}$ for the complex structure J on X. By a well known result of Grauert, such a Y always admits a fundamental system of Stein tubular neighborhoods $U \subset X$ (this would be even true when X is noncompact, but we only need the compact case here). In fact, if (Ω_α) is a finite covering of X such that $Y \cap \Omega_\alpha$ is a

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smooth complete intersection $\{z \in \Omega_\alpha\,;\ x_{\alpha,j}(z)=0\}$, $1\le j\le q$ (where $q=\operatorname{codim}_{\mathbb R} Y\ge n$), then one can take $U=U_\varepsilon=\{\varphi(z)<\varepsilon\}$ where

$$\varphi(z)=\sum_\alpha \theta_\alpha(z)\sum_{1\le j\le q}(x_{\alpha,j}(z))^2\ge 0 \tag{1.1}$$

where (θ_α) is a partition of unity subordinate to (Ω_α). The reason is that φ is strictly plurisubharmonic near Y, as

$$i\partial\overline\partial\varphi_{|Y}=2i\sum_\alpha\theta_\alpha(z)\sum_{1\le j\le q}\partial x_{\alpha,j}\wedge\overline\partial x_{\alpha,j}$$

and $(\partial x_{\alpha,j})_j$ has rank n at every point of Y, by the assumption that Y is totally real.

Now, let $\overline X$ be the complex conjugate manifold associated with the integrable almost complex structure $(X,-J)$ (in other words, $\mathcal O_{\overline X}=\overline{\mathcal O_X}$); we denote by $x\mapsto\overline x$ the identity map $\mathrm{Id}:X\to\overline X$ to stress that it is conjugate holomorphic. The underlying real analytic manifold $X^{\mathbb R}$ can be embedded diagonally in $X\times\overline X$ by the diagonal map $\delta : x\mapsto(x,\overline x)$, and the image $\delta(X^{\mathbb R})$ is a totally real submanifold of $X\times\overline X$. In fact, if $(z_{\alpha,j})_{1\le j\le n}$ is a holomorphic coordinate system relative to a finite open covering (Ω_α) of X, then the $\overline z_{\alpha,j}$ define holomorphic coordinates on $\overline X$ relative to $\overline\Omega_\alpha$, and the "diagonal" $\delta(X^{\mathbb R})$ is the totally real submanifold of pairs (z,w) such that $w_{\alpha,j}=\overline z_{\alpha,j}$ for all α,j. In that case, we can take Stein tubular neighborhoods of the form $U_\varepsilon=\{\varphi<\varepsilon\}$ where

$$\varphi(z,w)=\sum_\alpha\theta_\alpha(z)\theta_\alpha(w)\sum_{1\le j\le q}|\overline w_{\alpha,j}-z_{\alpha,j}|^2. \tag{1.2}$$

Here, the strict plurisubharmonicity of φ near $\delta(X^{\mathbb R})$ is obvious from the fact that

$$|w_{\alpha,j}-\overline z_{\alpha,j}|^2=|z_{\alpha,j}|^2+|w_{\alpha,j}|^2-2\operatorname{Re}(z_{\alpha,j}w_{\alpha,j}).$$

For $\varepsilon>0$ small, the first projection $\mathrm{pr}_1 : U_\varepsilon\to X$ gives a complex fibration whose fibers are C^∞-diffeomorphic to balls, but they need not be biholomorphic to complex balls in general. In order to achieve this property, we proceed in the following way. Pick a real analytic hermitian metric γ on X ; take e.g. the $(1,1)$-part $\gamma=g^{(1,1)}=\frac12(g+J^*g)$ of the Riemannian metric obtained as the pull-back $g=\delta^*(\sum_j idf_j\wedge d\overline f_j)$, where the $(f_j)_{1\le j\le N}$ provide a holomorphic immersion of the Stein neighborhood U_ε into $\mathbb C^N$. Let $\exp : T_X\to X$, $(z,\xi)\mapsto\exp_z(\xi)$ be the exponential map associated with the metric γ, in such a way that $\mathbb R\ni t\mapsto\exp_z(t\xi)$ are geodesics $\frac{D}{dt}(\frac{du}{dt})=0$ for the the Chern connection D on T_X (see e.g. [Dem??], [??]). Then exp is real analytic, and we have Taylor expansions

$$\exp_z(\xi)=\sum_{\alpha,\beta\in\mathbb N^n}a_{\alpha\beta}(z)\xi^\alpha\overline\xi^\beta,\qquad \xi\in T_{X,z}$$

with real analytic coefficients $a_{\alpha\beta}$, where $\exp_z(\xi)=z+\xi+O(|\xi|^2)$ in local coordinates. The real analyticity means that these expansions are convergent on a neighborhood $|\xi|_\gamma<\varepsilon_0$ of the zero section of T_X. We define the fiber-holomorphic part of the exponential map to be

$$\mathrm{exph} : T_X\to X,\qquad (z,\xi)\mapsto\mathrm{exph}_z(\xi)=\sum_{\alpha\in\mathbb N^n}a_{\alpha0}(z)\xi^\alpha. \tag{1.3}$$

It is uniquely defined, is convergent on the same tubular neighborhood $\{|\xi|_\gamma < \varepsilon_0\}$, has the property that $\xi \mapsto \mathrm{exph}_z(\xi)$ is holomorphic for $z \in X$ fixed, and satisfies again $\mathrm{exph}_z(\xi) = z+\xi+O(\xi^2)$ in coordinates. By the implicit function, theorem, the map $(z,\xi) \mapsto (z, \mathrm{exph}_z(\xi))$ is a real analytic diffeomorphism from a neighborhood of the zero section of T_X onto a neighborhood V of the diagonal in $X \times X$. Therefore, we get an inverse real analytic mapping $X \times X \supset V \to T_X$, which we denote by $(z,w) \mapsto (z,\xi)$, $\xi = \mathrm{logh}_z(w)$, such that $w \mapsto \mathrm{logh}_z(w)$ is holomorphic on $V \cap (\{z\} \times X)$, and $\mathrm{logh}_z(w) = w - z + O((w-z)^2)$ in coordinates. The tubular neighborhood

$$U_{\gamma,\varepsilon} = \{(z,w) \in X \times \overline{X}\,;\ |\mathrm{logh}_z(\overline{w})|_\gamma < \varepsilon\}$$

is Stein for $\varepsilon > 0$ small; in fact, if $p \in X$ and $(z_1, \ldots, z_n)$ is a holomorphic coordinate system centered at p such that $\gamma_p = i\sum dz_j \wedge d\overline{z}_j$, then $|\mathrm{logh}_z(\overline{w})|^2_\gamma = |\overline{w}-z|^2 + O(|\overline{w}-z|^3)$, hence $i\partial\overline{\partial}|\mathrm{logh}_z(\overline{w})|^2_\gamma > 0$ at $(p,\overline{p}) \in X \times \overline{X}$. By construction, the fiber $\mathrm{pr}_1^{-1}(z)$ of $\mathrm{pr}_1 : U_{\gamma,\varepsilon} \to X$ is biholomorphic to the ε-ball of the complex vector space $T_{X,z}$ equipped with the hermitian metric γ_z. In this way, we get a locally trivial real analytic bundle $\mathrm{pr}_1 : U_{\gamma,\varepsilon}$ whose fibers are complex balls; it is important to notice, however, that this ball bundle need not – and in fact, will never – be holomorphically locally trivial.

2. Curvature of Bergman bundles

2.A. Bergman version of the Dolbeault complex

Let X be a n-dimensional compact complex manifold equipped with a real analytic hermitian metric γ, $U_\varepsilon = U_{\gamma,\varepsilon} \subset X \times \overline{X}$ the ball bundle considered in §1 and

$$p = (\mathrm{pr}_1)_{|U_\varepsilon} : U_\varepsilon \to X, \qquad \overline{p} = (\mathrm{pr}_2)_{|U_\varepsilon} : U_\varepsilon \to \overline{X}$$

the natural projections. Our goal is to compute the curvature of what we call the "Bergman direct image sheaf"

$$\mathcal{B}_\varepsilon = p_*^{L^2}(\overline{p}^*\mathcal{O}(K_{\overline{X}})), \tag{2.1}$$

whose space of sections over an open subset $V \subset X$ is defined to be $\mathcal{B}_\varepsilon(V)$ = holomorphic sections f of $\overline{p}^*\mathcal{O}(K_{\overline{X}})$ on $p^{-1}(V)$ that are in $L^2(p^{-1}(K))$ for all compact subsets $K \Subset V$, i.e.

$$\int_{p^{-1}(K)} i^{n^2} f \wedge \overline{f} \wedge \gamma^n < +\infty, \quad \forall K \Subset V. \tag{2.2}$$

Then $\mathcal{B}_\varepsilon$ is an $\mathcal{O}_X$-module, and by the Ohsawa-Takegoshi extension theorem applied to the subvariety $p^{-1}(z) \subset U_\varepsilon$, its fiber $B_{\varepsilon,z} = \mathcal{B}_{\varepsilon,z}/\mathfrak{m}_z\mathcal{B}_{\varepsilon,z}$ is isomorphic to the Hilbert space $\mathcal{H}^2(\mathbb{B}_n)$ of L^2 holomorphic n-forms on $p^{-1}(z) \simeq \mathbb{B}_n$. In fact, if we use orthonormal coordinates $(w_1, \ldots, w_n)$ provided by exph acting on the hermitian space $(T_{X,z}, \gamma_z)$ and centered at $\overline{z}$, we get a biholomorphism $\mathbb{B}_n \to p^{-1}(z)$ given by the homothety $\eta_\varepsilon : w \mapsto \varepsilon w$, and a corresponding isomorphism

$$B_{\varepsilon,z} \longrightarrow \mathcal{H}^2(\mathbb{B}_n), \qquad f \longmapsto g = \eta_\varepsilon^* f, \quad \text{i.e. with } I = \{1, \ldots, n\}, \tag{2.3}$$

$$f_I(w)\, dw_1 \wedge \ldots \wedge dw_n \longmapsto \varepsilon^n\, f_I(\varepsilon w)\, dw_1 \wedge \ldots \wedge dw_n, \quad w \in \mathbb{B}_n, \tag{2.3'}$$

$$\|g\|^2 = \int_{\mathbb{B}_n} 2^{-n} i^{n^2} g \wedge \overline{g}, \quad g = g(w)\, dw_1 \wedge \ldots \wedge dw_n \in \mathcal{H}^2(\mathbb{B}_n). \tag{2.3''}$$

As a consequence, $B_\varepsilon \to X$ can be seen as a locally trivial (infinite dimensional) real analytic bundle of typical fiber $\mathcal{H}^2(\mathbb{B}_n)$, and we can equip $B_{\varepsilon,z}$ with the natural L^2 metric obtained by declaring (2.3) to be an isometry. In this way, we get a real analytic hermitian metric on the Bergman bundle $B_\varepsilon \to X$.

Similarly, we can consider a "Bergman version" of the Dolbeault complex, by introducing a sheaf $\mathcal{F}^q_\varepsilon$ over X of (n,q)-forms which can be written locally over small open sets $V \subset X$ as

$$f(z,w) = \sum_{|J|=q} f_J(z,w)\,dw_1 \wedge \ldots \wedge dw_n \wedge d\overline{z}_J, \quad (z,w) \in U_\varepsilon \cap (V \times \overline{X}), \tag{2.4}$$

where the $f_J(z,w)$ are C^∞ smooth functions on $U_\varepsilon \cap (V \times \overline{X})$ such that $f_J(z,w)$ is holomorphic in w and both f and $\overline{\partial} f = \overline{\partial}_z f$ are in $L^2(p^{-1}(K))$ for all compact subsets $K \Subset V$. By construction, the usual $\overline{\partial}$ operator yields a complex of sheaves $(\mathcal{F}^\bullet, \overline{\partial}\,)$ and the kernel $\operatorname{Ker} \overline{\partial} : \mathcal{F}^0 \to \mathcal{F}^1$ coincides with $\mathcal{B}_\varepsilon$. We are going to see that B_ε can somehow be seen as an infinite dimensional very ample vector bundle. This is already illustrated by the following result.

2.5. Proposition. *Assume here that $\varepsilon > 0$ is taken so small that $\psi(z,w) := |\operatorname{logh}_z(w)|^2$ is strictly plurisubharmonic up to the boundary on the compact set $\overline{U}_\varepsilon \subset X \times \overline{X}$. Then the complex of sheaves $(\mathcal{F}^\bullet, \overline{\partial})$ is a resolution of $\mathcal{B}_\varepsilon$ by soft sheaves over X (actually, by $\mathcal{C}^\infty_X$-modules), and for every holomorphic vector bundle $E \to X$ and every $q \geq 1$ we have*

$$H^q(X, \mathcal{B}_\varepsilon \otimes \mathcal{O}(E)) = H^q\big(\Gamma(X, \mathcal{F}^q_\varepsilon \otimes \mathcal{O}(E)), \overline{\partial}\big) = 0.$$

Moreover the fibers $B_{\varepsilon,z} \otimes E_z$ are always generated by global sections of $H^0(X, \mathcal{B}_\varepsilon \otimes \mathcal{O}(E))$.

Proof. By construction, we can equip U_ε with the the associated Kähler metric $\omega = i\partial\overline{\partial}\psi$ which is smooth and strictly positive on $\overline{U}_\varepsilon$. We can then take an arbitrary smooth hermitian metric h_E on E and multiply it by $e^{-C\psi_\varepsilon}$ to obtain a bundle with positive arbitrarily large curvature tensor. The exactness of $\mathcal{F}^\bullet$ and cohomology vanishing then follow from the standard Hörmander L^2 estimates applied either locally on $p^{-1}(V)$ for small Stein open sets $V \subset X$, or globally on U_ε. The global generation of fibers is an immediate consequence of the Ohsawa-Takegoshi L^2 extension theorem. □

It would not be very hard to show that the same result holds for an arbitrary coherent sheaf $\mathcal{E}$ instead of a locally free sheaf $\mathcal{O}(E)$, the reason being that $p^*\mathcal{E}$ admits a resolution by (finite dimensional) locally free sheaves $\mathcal{O}^{\oplus N}_{U_{\varepsilon'}}$ on a Stein neighborhood $U_{\varepsilon'}$ of $\overline{U}_\varepsilon$.

2.B. Curvature tensor of the Bergman bundle on $(\mathbb{C}^n$, std$)$

In the model situation $X = \mathbb{C}^n$ with its standard hermitian metric, we consider the tubular neighborhood

$$U_\varepsilon := \{(z,w) \in \mathbb{C}^n \times \mathbb{C}^n \,;\; |\overline{w} - z| < \varepsilon\} \tag{2.6}$$

and the projections

$$p = (\mathrm{pr}_1)_{|U_\varepsilon} : U_\varepsilon \to X = \mathbb{C}^n, \quad (z,w) \mapsto z, \quad \overline{p} = (\mathrm{pr}_2)_{|U_\varepsilon} : U_\varepsilon \to X = \mathbb{C}^n, \quad (z,w) \mapsto w$$

If one insists on working on a compact complex manifold, the geometry is locally identical to that of a complex torus $X = \mathbb{C}^n/\Lambda$ equipped with a constant hermitian metric γ.

2.7. Remark. Again, we have to insist that the Bergman bundle B_ε is not holomorphically locally trivial, even in the above situation where we have invariance by translation. In the category of real analytic bundles, we get a trivialization from $\mathcal{B}_\varepsilon(V)$ onto the topological tensor product of $\mathcal{O}_X(V)$ by the Hilbert space $\mathcal{H}^2(\mathbb{B}_n)$, namely

$$\tau : \mathcal{B}_\varepsilon(V) \xrightarrow{\simeq} \mathcal{O}_X(V)\,\widehat{\otimes}\,\mathcal{H}^2(\mathbb{B}_n), \quad f \longmapsto \tau(f) = g, \quad g(z,w) := f(z, \varepsilon w + \overline{z}), \quad w \in \mathbb{B}_n.$$

Complex structures of these bundles are defined by the $(0,1)$-connections $\overline{\partial}_z$ of the associated Dolbeault complexes, but obviously $\overline{\partial}_z f$ and $\overline{\partial}_z g$ do not match. In fact, if we write $g(z,w) = u(z,w)\,dw_1 \wedge \ldots \wedge dw_n \in C^\infty(V)\,\widehat{\otimes}\,\mathcal{H}^2(\mathbb{B}_n)$ (i.e. $u(z,w)$ is a smooth function that is holomorphic in w), then we get

$$\begin{aligned} &f(z,w) = g(z,(w-\overline{z})/\varepsilon) = \varepsilon^{-n} u(z,(w-\overline{z})/\varepsilon)\,dw_1 \wedge \ldots \wedge dw_n, \\ &\overline{\partial}_z f(z,w) = \varepsilon^{-n}\Big(\overline{\partial}_z u(z,(w-\overline{z})/\varepsilon) - \varepsilon^{-1} \sum_{1\le j\le n} \frac{\partial u}{\partial w_j}(z,(w-\overline{z})/\varepsilon)\,d\overline{z}_j\Big) \wedge dw_1 \wedge \ldots \wedge dw_n. \end{aligned}$$

Therefore the trivialization $\tau_* : f \mapsto u$ yields at the level of $\overline{\partial}$-connections an identification

$$\tau_* : \overline{\partial}_z f \xmapsto{\simeq} \overline{\partial}_z u + Au$$

where the "connection matrix" $A \in \Gamma(V, \Lambda^{0,1}T^*_X \otimes_{\mathbb{C}} \mathrm{End}(\mathcal{O}_X(V)\,\widehat{\otimes}\,\mathcal{H}^2(\mathbb{B}_n))$ is the linear operator $A = \mathrm{Id}_{\mathcal{O}_X(V)}\,\widehat{\otimes}\,\Lambda$ induced by the unbounded Hilbert space operator

$$\Lambda : \mathcal{H}^2(\mathbb{B}_n) \to \mathcal{H}^2(\mathbb{B}_n) \otimes_{\mathbb{C}} \Lambda^{0,1}T^*_X, \quad u \mapsto \Lambda u = -\varepsilon^{-1} \sum_{1\le j\le n} \frac{\partial u}{\partial w_j}\,d\overline{z}_j.$$

We see that the holomorphic structure of B_ε is given by a $(0,1)$-connection that differs by the matrix A from the trivial $(0,1)$-connection, and as A is unbounded, there is no way we can make it trivial by a real analytic gauge change with values in Lie algebra of continuous endomorphisms of $\mathcal{H}^2(\mathbb{B}_n)$. □

We are now going to compute the curvature tensor of the Bergman bundle B_ε. For the sake of simplicity, we identify here $\mathcal{H}^2(\mathbb{B}_n)$ to the Hardy space of L^2 holomorphic functions via $u \mapsto g = u(w)\,dw_1 \wedge \ldots \wedge dw_n$. After rescaling, we can also assume $\varepsilon = 1$, and at least in a first step, we perform our calculations on B_1 rather than B_ε. Let us write $w^\alpha = \prod_{1\le j\le n} w_j^{\alpha_j}$ for a multiindex $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, and denote by λ the Lebesgue measure on $\mathbb{C}^n$. A straightforward calculation gives

$$\int_{\mathbb{B}_n} |w^\alpha|^2 d\lambda(w) = \pi^n \frac{\alpha_1! \ldots \alpha_n!}{(|\alpha|+n)!}, \quad |\alpha| = \alpha_1 + \cdots + \alpha_n.$$

In fact, by using polar coordinates $w_j = r_j e^{i\theta_j}$ and writing $t_j = r_j^2$, we get

$$\int_{\mathbb{B}_n} |w^\alpha|^2 d\lambda(w) = (2\pi)^n \int_{r_1^2+\cdots+r_n^2<1} r^{2\alpha}\, r_1 dr_1 \ldots r_n dr_n = \pi^n I(\alpha)$$

with

$$I(\alpha) = \pi^n \int_{t_1+\cdots+t_n<1} t^\alpha\, dt_1 \ldots dt_n.$$

Now, an induction on n together with the Fubini formula gives

$$\begin{aligned} I(\alpha) &= \int_0^1 t_n^{\alpha_n}\,dt_n \int_{t_1+\cdots+t_{n-1}<1-t_n} (t')^{\alpha'}\,dt_1\ldots dt_{n-1} \\ &= I(\alpha')\int_0^1 (1-t_n)^{\alpha_1+\cdots+\alpha_{n-1}+n-1}t_n^{\alpha_n}\,dt_n \end{aligned}$$

where $t' = (t_1, \ldots, t_{n-1})$ and $\alpha' = (\alpha_1, \ldots, \alpha_{n-1})$. As $\int_0^1 x^a(1-x)^b dt = \frac{a!b!}{(a+b+1)!}$, we get inductively

$$I(\alpha) = \frac{(|\alpha'|+n-1)!\alpha_n!}{(|\alpha|+n)!}I(\alpha') \quad\Rightarrow\quad I(\alpha) = \frac{\alpha_1!\ldots\alpha_n!}{(|\alpha|+n)!}.$$

This implies that a Hilbert (orthonormal) basis of $\mathcal{O}\cap L^2(\mathbb{B}_n)$ is

(2.8)
$$e_\alpha(w) = \pi^{-n/2}\sqrt{\frac{(|\alpha|+n)!}{\alpha_1!\ldots\alpha_n!}}\,w^\alpha.$$

As a consequence, and quite classically, the Bergman kernel of the unit ball $\mathbb{B}_n \subset \mathbb{C}^n$ is

(2.9)
$$K_n(w) = \sum_{\alpha\in\mathbb{N}^n}|e_\alpha(w)|^2 = \pi^{-n}\sum_{\alpha\in\mathbb{N}^n}\frac{(|\alpha|+n)!}{\alpha_1!\ldots\alpha_n!}\,|w^\alpha|^2 = n!\pi^{-n}(1-|w|^2)^{-n-1}.$$

If we come back to U_ε for $\varepsilon > 0$ not necessarily equal to 1 (and do not omit any more the trivial n-form $dw_1\wedge\ldots\wedge dw_n$), we have to use a rescaling $(z,w)\mapsto(\varepsilon^{-1}z,\varepsilon^{-1}w)$. This gives for the Hilbert bundle B_ε a real analytic orthonormal frame

(2.10)
$$\pi^{-n/2}\varepsilon^{-|\alpha|-n}\sqrt{\frac{(|\alpha|+n)!}{\alpha_1!\ldots\alpha_n!}}\,(w-\overline{z})^\alpha\,dw_1\wedge\ldots\wedge dw_n.$$

We modify it by taking cofficients $\chi(\ell) > 0$, $\ell\in\mathbb{N}$, and putting

(2.10′)
$$e_\alpha(z,w) = \sqrt{c_0\,\frac{\prod_{0\le\ell\le|\alpha|}\chi(\ell)}{\alpha_1!\ldots\alpha_n!}}\,(w-\overline{z})^\alpha\,dw_1\wedge\ldots\wedge dw_n,$$

e.g. with $c_0 = \pi^{-n}\,(n-1)!\,\varepsilon^{-2n}$, $\chi = \chi_\varepsilon$, $\chi_\varepsilon(\ell) = \varepsilon^{-2}(\ell+n)$, to recover exactly (2.10). A germ of holomorphic section σ near $z = 0$ (say) is then given by a convergent power series

$$\sigma(z,w) = \sum_{\alpha\in\mathbb{N}^n}\xi_\alpha(z)\,e_\alpha(z,w).$$

We define $\mathcal{O}(B_\chi)$ to be the (real analytic) Hilbert bundle of sections σ such that the ξ_α are real analytic on a neighborhood of 0 and satisfy the following two conditions:

(2.11)
$$|\sigma(z)|_h^2 := \sum_{\alpha\in\mathbb{N}^n}|\xi_\alpha(z)|^2 \ \text{ is uniformly convergent,}$$

(2.12)
$$\overline{\partial}_{z_k}\sigma(z,w) = \sum_{\alpha\in\mathbb{N}^n}\overline{\partial}_{z_k}\xi_\alpha(z)\,e_\alpha(z,w) + \xi_\alpha(z)\,\overline{\partial}_{z_k}e_\alpha(z,w) \equiv 0.$$

Then $\mathcal{O}(B_\varepsilon)=\mathcal{O}(B_{\chi_\varepsilon})$ by definition. Let $c_k=(0,\ldots,1,\ldots,0)$ be the canonical basis of the $\mathbb{Z}$-module $\mathbb{Z}^n$. A straightforward calculation from (2.10′) yields

$$\overline{\partial}_{z_k}e_\alpha(z,w)=-\sqrt{\alpha_k\,\chi(|\alpha|)}\;e_{\alpha-c_k}(z,w).$$

We have the slight problem that the coefficients maye be unbounded as $|\alpha|\to+\infty$, and therefore the two terms occurring in (2.12) need not form convergent series when taken separately. However if we take σ in a suitable dense Hilbert subbundle $\mathcal{O}(B_{\tilde\chi})\subset\mathcal{O}(B_\chi)$, we may eventually be able to ensure convergence. This is the case if we take $\chi=\chi_\varepsilon$ and $\sigma\in\mathcal{O}(B_{\rho\varepsilon})$ for a slightly bigger tubular neighborhood ($\rho>1$), as the L^2 condition implies that $\sum_\alpha(\rho')^{2|\alpha|}|\xi_\alpha|^2$ is uniformly convergent for every $\rho'\in\,]1,\rho[\,$, and this is more than enough to ensure convergence, since the growth of $\alpha\mapsto\sqrt{\alpha_k\,\chi_\varepsilon(|\alpha|)}$ is at most linear; we can even iterate as many derivatives as we want. In case of convergence, we get

$$\begin{aligned}\overline{\partial}_{z_k}\sigma(z,w)&=\sum_{\alpha\in\mathbb{N}^n}\overline{\partial}_{z_k}\xi_\alpha(z)\,e_\alpha(z,w)+\xi_\alpha(z)\,\overline{\partial}_{z_k}e_\alpha(z,w)\\&=\sum_{\alpha\in\mathbb{N}^n}\overline{\partial}_{z_k}\xi_\alpha(z)\,e_\alpha(z,w)-\sqrt{\alpha_k\,\chi(|\alpha|)}\;\xi_\alpha(z)\;e_{\alpha-c_k}(z,w)\\&=\sum_{\alpha\in\mathbb{N}^n}\big(\overline{\partial}_{z_k}\xi_\alpha(z)-\sqrt{(\alpha_k+1)\,\chi(|\alpha|+1)}\;\xi_{\alpha+c_k}(z)\big)\;e_\alpha(z,w),\end{aligned}$$

after replacing α by $\alpha+c_k$ in the terms containing χ. The $(0,1)$-part $\nabla_h^{0,1}$ of the Chern connection ∇_h of (B_χ,h) with respect to the orthonormal frame (e_α) is thus given by

$$\nabla_h^{0,1}\sigma=\sum_{\alpha\in\mathbb{N}^n}\Big(\overline{\partial}\xi_\alpha-\sum_k\sqrt{(\alpha_k+1)\,\chi(|\alpha|+1)}\;\xi_{\alpha+c_k}\;d\overline{z}_k\Big)\otimes e_\alpha. \tag{2.13}$$

The $(1,0)$-part can be derived from the identity $\partial|\sigma|_h^2=\langle\nabla_h^{1,0}\sigma,\sigma\rangle_h+\langle\sigma,\nabla_h^{0,1}\sigma\rangle_h$. However

$$\begin{aligned}\partial_{z_j}|\sigma|_h^2=\partial_{z_j}\sum_{\alpha\in\mathbb{N}^n}\xi_\alpha\overline{\xi}_\alpha&=\sum_{\alpha\in\mathbb{N}^n}(\partial_{z_j}\xi_\alpha)\;\overline{\xi}_\alpha+\xi_\alpha\;(\overline{\overline{\partial}_{z_j}\xi_\alpha}\,)\\&=\sum_{\alpha\in\mathbb{N}^n}\Big(\;\partial_{z_j}\xi_\alpha+\sqrt{\alpha_j\,\chi(|\alpha|)}\;\xi_{\alpha-c_j}\Big)\;\overline{\xi}_\alpha\\&\quad+\sum_{\alpha\in\mathbb{N}^n}\xi_\alpha\Big(\;\overline{\overline{\partial}_{z_j}\xi_\alpha-\sqrt{(\alpha_j+1)\,\chi(|\alpha|+1)}\;\xi_{\alpha+c_j}}\;\Big).\end{aligned}$$

For $\sigma\in C^\infty(B_{\rho\varepsilon})$, it follows from there that

$$\nabla_h^{1,0}\sigma=\sum_{\alpha\in\mathbb{N}^n}\Big(\partial\xi_\alpha+\sum_j\sqrt{\alpha_j\,\chi(|\alpha|)}\;\xi_{\alpha-c_j}dz_j\Big)\otimes e_\alpha. \tag{2.14}$$

Finally, to find the curvature tensor of (B_χ,h), we only have to compute the $(1,1)$-form $(\nabla_h^{1,0}\nabla_h^{0,1}+\nabla_h^{0,1}\nabla_h^{1,0})\sigma$ and take the terms that contain no differentiation at all, especially in view of the usual identity $\partial\overline{\partial}+\overline{\partial}\partial=0$ and the fact that we also have here $(\nabla_h^{1,0})^2=0$,

$(\nabla_h^{0,1})^2 = 0$. As $(\alpha - c_j)_k = \alpha_k - \delta_{jk}$ and $(\alpha + c_k)_j = \alpha_j + \delta_{jk}$, we are left with

$$\begin{aligned}
&\big(\nabla_h^{1,0}\nabla_h^{0,1} + \nabla_h^{0,1}\nabla_h^{1,0}\big)\sigma \\
&= -\sum_{\alpha\in\mathbb{N}^n}\sum_{j,k}\sqrt{\alpha_j(\alpha_k-\delta_{jk}+1)}\,\chi(|\alpha|)\,\xi_{\alpha-c_j+c_k}\,dz_j\wedge d\overline{z}_k\otimes e_\alpha \\
&\quad+\sum_{\alpha\in\mathbb{N}^n}\sum_{j,k}\sqrt{(\alpha_j+\delta_{jk})(\alpha_k+1)}\,\chi(|\alpha|+1)\,\xi_{\alpha-c_j+c_k}\,dz_j\wedge d\overline{z}_k\otimes e_\alpha \\
&= -\sum_{\alpha\in\mathbb{N}^n}\sum_{j,k}\sqrt{(\alpha_j-\delta_{jk})(\alpha_k-\delta_{jk})}\,\chi(|\alpha|-1)\,\xi_{\alpha-c_j}\,dz_j\wedge d\overline{z}_k\otimes e_{\alpha-c_k} \\
&\quad+\sum_{\alpha\in\mathbb{N}^n}\sum_{j,k}\sqrt{\alpha_j\alpha_k}\,\chi(|\alpha|)\,\xi_{\alpha-c_j}\,dz_j\wedge d\overline{z}_k\otimes e_{\alpha-c_k}. \\
&= \sum_{\alpha\in\mathbb{N}^n}\sum_{j,k}\sqrt{\alpha_j\alpha_k}\,\big(\chi(|\alpha|)-\chi(|\alpha|-1)\big)\,\xi_{\alpha-c_j}\,dz_j\wedge d\overline{z}_k\otimes e_{\alpha-c_k} \\
&\quad+\sum_{\alpha\in\mathbb{N}^n}\sum_{j}\chi(|\alpha|-1)\,\xi_{\alpha-c_j}\,dz_j\wedge d\overline{z}_j\otimes e_{\alpha-c_j},
\end{aligned}$$

where the last summation comes from the subtraction of the diagonal terms $j = k$. By changing α into $\alpha+c_j$ in that summation, we obtain the following expression of the curvature tensor of (B_χ, h).

2.15. Theorem. *The curvature tensor of the Bergman bundle (B_χ, h) is given by*

$$\langle\Theta_{B_\chi,h}\sigma(v,Jv),\sigma\rangle_h = \sum_{\alpha\in\mathbb{N}^n}\left(\big(\chi(|\alpha|)-\chi(|\alpha|-1)\big)\Big|\sum_j\sqrt{\alpha_j}\,\xi_{\alpha-c_j}v_j\Big|^2+\sum_j\chi(|\alpha|)\,|\xi_\alpha|^2|v_j|^2\right)$$

for every $\sigma = \sum_\alpha \xi_\alpha e_\alpha \in B_{\rho\varepsilon}$, $\rho > 1$, and every tangent vector $v = \sum v_j\,\partial/\partial z_j$. In particular, the curvature is ≥ 0 if the weight function χ is nondecreasing. In the standard case $\chi = \chi_\varepsilon$, $\chi(\ell) = \ell + n$, we get

$$\langle\Theta_{B_\varepsilon,h}\sigma(v,Jv),\sigma\rangle_h = \varepsilon^{-2}\sum_{\alpha\in\mathbb{N}^n}\left(\Big|\sum_j\sqrt{\alpha_j}\,\xi_{\alpha-c_j}v_j\Big|^2+\sum_j(|\alpha|+n)\,|\xi_\alpha|^2|v_j|^2\right).$$

In the standard case $\chi = \chi_\varepsilon$, the above curvature hermitian tensor is positive definite, and even positive definite unbounded if we view it as a hermitian form on $T_X \otimes B_\varepsilon$ rather than on $T_X \otimes B_{\rho\varepsilon}$. This is not so surprising since the connection matrix was already an unbounded operator. Philosophically, the very ampleness of B_ε was also a strong indication that the curvature should have been positive. Observe that we have in fact

$$\begin{aligned}
&\varepsilon^{-2}\sum_{\alpha\in\mathbb{N}^n}\sum_j(|\alpha|+n)\,|\xi_\alpha|^2|v_j|^2 \\
&\qquad\leq\langle\Theta_{B_\varepsilon,h}\sigma(v,Jv),\sigma\rangle_h\leq 2\varepsilon^{-2}\sum_{\alpha\in\mathbb{N}^n}\sum_j(|\alpha|+n)\,|\xi_\alpha|^2|v_j|^2,
\end{aligned}\tag{2.16}$$

thanks to the Cauchy-Schwarz inequality

$$\sum_{\alpha\in\mathbb{N}^n}\Big|\sum_j\sqrt{\alpha_j}\;\xi_{\alpha-c_j}v_j\Big|^2\le\sum_\ell|v_\ell|^2\sum_{\alpha\in\mathbb{N}^n}\sum_j\alpha_j|\xi_{\alpha-c_j}|^2=\sum_\ell|v_\ell|^2\sum_j\sum_{\alpha\in\mathbb{N}^n}\alpha_j|\xi_{\alpha-c_j}|^2$$
$$=\sum_\ell|v_\ell|^2\sum_j\sum_{\alpha\in\mathbb{N}^n}(\alpha_j+1)|\xi_\alpha|^2=\sum_\ell|v_\ell|^2\sum_{\alpha\in\mathbb{N}^n}(|\alpha|+n)|\xi_\alpha|^2.$$

2.C. Curvature of Bergman bundles on compact hermitian manifolds

We consider here the general situation of a compact hermitian manifold (X,γ) described in §1, where γ is real analytic and exph is the associated partially holomorphic exponential map. Fix a point $x_0\in X$, and use a holomorphic system of coordinates $(z_1,\ldots,z_n)$ centered at x_0, provided by $\mathrm{exph}_{x_0}: T_{X,x_0}\supset V\to X$. If we take γ_{x_0} orthonormal coordinates on T_{X,x_0}, then by construction the fiber of $p:U_\varepsilon\to X$ over x_0 is the standard ε-ball in the coordinates $(w_j)=(\overline{z}_j)$. Let $T_X\to V\times\mathbb{C}^n$ be the trivialization of T_X in the coordinates (z_j), and

$$X\times X\to T_X,\quad (z,w)\mapsto\xi=\mathrm{logh}_z(w)$$

the expression of logh near $(x_0,\overline{x}_0)$, that is, near $(z,w)=(0,0)$. By our choice of coordinates, we have $\mathrm{logh}_0(w)=w$ and of course $\mathrm{logh}_z(z)=0$, hence we get a real analytic expansion of the form

$$\begin{aligned}\mathrm{logh}_z(w)=w-z&+\sum z_ja_j(w-z)+\sum\overline{z}_ja'_j(w-z)\\&+\sum z_jz_kb_{jk}(w-z)+\sum\overline{z}_j\overline{z}_kb'_{jk}(w-z)+\sum z_j\overline{z}_kc_{jk}(w-z)+O(|z|^3)\end{aligned}$$

with holomorphic coefficients a_j, a'_j, b_{jk}, b'_{jk}, c_{jk} vanishing at 0. In fact by [Dem??], we always have $da'_j(0)=0$, and if γ is Kähler, the equality $da_j(0)=0$ also holds; we will not use these properties here. In coordinates, we then have locally near $(0,0)\in\mathbb{C}^n\times\mathbb{C}^n$

$$U_{\varepsilon,z}=\big\{(z,w)\in\mathbb{C}^n\times\mathbb{C}^n\,;\ |\Psi_z(w)|<\varepsilon\big\}$$

where $\Psi_z(w)=\overline{\mathrm{logh}_z(\overline{w})}$ has a similar expansion

$$\begin{aligned}\Psi_z(w)=w-\overline{z}&+\sum z_ja_j(w-\overline{z})+\sum\overline{z}_ja'_j(w-\overline{z})\\&+\sum z_jz_kb_{jk}(w-\overline{z})+\sum\overline{z}_j\overline{z}_kb'_{jk}(w-\overline{z})+\sum z_j\overline{z}_kc_{jk}(w-\overline{z})+O(|z|^3)\end{aligned}\tag{2.17}$$

(when going from logh to Ψ, the coefficients a_j, a'_j and b_j, b'_j get twisted, but we do not care and keep the same notation for Ψ, as we will not refer to logh any more). In this situation, the Hilbert bundle B_ε has a real analytic normal frame given by $\widetilde{e}_\alpha=\Psi^*e_\alpha$ where

$$e_\alpha(w)=\pi^{-n/2}\varepsilon^{-|\alpha|-n}\sqrt{\frac{(|\alpha|+n)!}{\alpha_1!\ldots\alpha_n!}}\;w^\alpha\,dw_1\wedge\ldots\wedge dw_n\tag{2.18}$$

and the pull-back Ψ^*e_α is taken with respect to $w\mapsto\Psi_z(w)$ (z being considered as a parameter). For a local section $\sigma=\sum_\alpha\xi_\alpha\widetilde{e}_\alpha\in C^\infty(B_{\rho\varepsilon})$ we can write

$$\overline{\partial}_{z_k}\sigma(z,w)=\sum_{\alpha\in\mathbb{N}^n}\overline{\partial}_{z_k}\xi_\alpha(z)\,\widetilde{e}_\alpha(z,w)+\xi_\alpha(z)\,\overline{\partial}_{z_k}\widetilde{e}_\alpha(z,w).$$

Near $z=0$, by taking the derivative of $\Psi^*e_\alpha(z,w)$, we find

$$\begin{aligned}\overline\partial_{z_k}\widetilde e_\alpha(z,w) = {}&-\varepsilon^{-1}\sqrt{\alpha_k(|\alpha|+n)}\,\widetilde e_{\alpha-c_k}(z,w)\\ &+\varepsilon^{-1}\sum_m\sqrt{\alpha_m(|\alpha|+n)}\,\Big(a'_{k,m}(w-\overline z)+\sum_j z_jc_{jk,m}(w-\overline z)\Big)\,\widetilde e_{\alpha-c_m}(z,w)\\ &+\sum_m\Big(\frac{\partial a'_{k,m}}{\partial w_m}(w-\overline z)+\sum_j z_j\frac{\partial c_{jk,m}}{\partial w_m}(w-\overline z)\Big)\,\widetilde e_\alpha(z,w)+O(\overline z,|z|^2),\end{aligned}$$

where the last sum comes from the expansion of $dw_1\wedge\ldots\wedge dw_n$, and $a'_{k,m}$, $c_{jk,m}$ are the m-th components of a'_k and c_{jk}. This gives two additional terms in comparison to the translation invariant case, but these terms are "small" in the sense that the first one vanishes at $(z,w)=(0,0)$ and the second one does not involve ε^{-1}. If $\nabla^{0,1}_{h,0}$ is the $\overline\partial$-connection associated with the standard tubular neighborhood $|\overline w-z|<\varepsilon$, we thus find in terms of the local trivialization $\sigma\simeq\xi=\sum\xi_\alpha\widetilde\varepsilon_\alpha$ an expression of the form

$$\nabla^{0,1}_h\sigma\simeq\nabla^{0,1}_{h,0}\xi+A^{0,1}\xi,$$

where

$$\begin{aligned}A^{0,1}\Big(\sum_\alpha\xi_\alpha\widetilde e_\alpha\Big)={}&\sum_{\alpha\in\mathbb N^n}\sum_k\\ &\xi_\alpha\Bigg(\varepsilon^{-1}\sum_m\sqrt{\alpha_m(|\alpha|+n)}\,\Big(a'_{k,m}(w)+\sum_j z_jc_{jk,m}(w)\Big)\,d\overline z_k\otimes\widetilde e_{\alpha-c_m}\\ &+\sum_m\Big(\frac{\partial a'_{k,m}}{\partial w_m}(w)+\sum_j z_j\frac{\partial c_{jk,m}}{\partial w_m}(w)\Big)\,d\overline z_k\otimes\widetilde e_\alpha\Bigg)+O(\overline z,|z|^2).\end{aligned}$$

The corresponding $(1,0)$-parts satisfy

$$\nabla^{1,0}_h\sigma\simeq\nabla^{1,0}_{h,0}\xi+A^{1,0}\xi,\qquad A^{1,0}=-(A^{0,1})^*,$$

and the corresponding curvature tensors are related by

$$\Theta_{\beta_\varepsilon,h}=\Theta_{\beta_\varepsilon,h,0}+\partial A^{0,1}+\overline\partial A^{1,0}+A^{1,0}\wedge A^{0,1}+A^{0,1}\wedge A^{1,0}. \tag{2.19}$$

At $z=0$ we have

$$\begin{aligned}A^{0,1}\xi={}&\sum_{\alpha\in\mathbb N^n}\sum_k\xi_\alpha\Bigg(\varepsilon^{-1}\sum_m\sqrt{\alpha_m(|\alpha|+n)}\,a'_{k,m}(w)\,d\overline z_k\otimes\widetilde e_{\alpha-c_m}\\ &+\sum_m\frac{\partial a'_{k,m}}{\partial w_m}(w)\,d\overline z_k\otimes\widetilde e_\alpha\Bigg),\\ \partial A^{0,1}\xi={}&\sum_{\alpha\in\mathbb N^n}\sum_k\xi_\alpha\Bigg(\varepsilon^{-1}\sum_{j,m}\sqrt{\alpha_m(|\alpha|+n)}\,c_{jk,m}(w)\,dz_j\wedge d\overline z_k\otimes\widetilde e_{\alpha-c_m}\\ &+\sum_{j,m}\frac{\partial c_{jk,m}}{\partial w_m}(w)\,dz_j\wedge d\overline z_k\otimes\widetilde e_\alpha\Bigg),\end{aligned}$$

and $A^{1,0}$, $\overline{\partial}A^{1,0}$ are, up to the sign, the adjoint endomorphisms of $A^{0,1}$ and $\partial A^{0,1}$. The unboundedness comes from the fact that we have unbounded factors $\sqrt{(\alpha_m+1)(|\alpha|+n+1)}$; it is worth noticing that multiplication by a holomorphic factor $u(w)$ is a continuous operator on the fibers $B_{\varepsilon,z}$, whose norm remains bounded as $\varepsilon\to 0$. In this setting, it can be seen that the only term in (2.19) that is (a priori) not small with respect to the main term $\Theta_{B_\varepsilon,h,0}$ is the term involving ε^{-2} in $A^{1,0}\wedge A^{0,1}+A^{0,1}\wedge A^{1,0}$, and that the other terms appearing in the quadratic form $\langle\Theta_{B_\varepsilon,h}\xi,\xi\rangle$ are $O(\varepsilon^{-1}\sum(|\alpha|+n)|\xi_\alpha|^2)$ or smaller. In order to check this, we expand $c_{jk,m}(w)$ into a power series $\sum_\mu c_{jk,m,\mu}\,g_\mu(w)$ where

$$g_\mu(w)=s_\mu^{-1}w^\mu,\quad \text{with } s_\mu=\sup_{|w|\le 1}|w^\mu|=\prod_{1\le j\le n}\left(\frac{\mu_j}{|\mu|}\right)^{\mu_j/2}=\frac{\prod\mu_j^{\mu_j/2}}{|\mu|^{|\mu|/2}},\tag{2.20}$$

so that $\sup_{|w|\le\varepsilon}|g_\mu(w)|=\varepsilon^{|\mu|}$. We get from the term $\langle\partial A^{0,1}\xi,\xi\rangle$ a summation

$$\Sigma(\xi)=\varepsilon^{-1}\sum_{j,k,m}\sum_{\alpha\in\mathbb{N}^n}\sqrt{\alpha_m(|\alpha|+n)}\sum_{\mu\in\mathbb{N}^n}c_{jk,m,\mu}\,dz_j\wedge d\overline{z}_k\otimes\langle\xi_\alpha\,g_\mu\widetilde{e}_\alpha,\xi\rangle.$$

At $z=0$, $g_\mu\widetilde{e}_\alpha=g_\mu e_\alpha$ is proportional to $e_{\alpha+\mu}$, and by (2.20) and the definition of the L^2 norm, we have $\|g_\mu\widetilde{e}_\alpha\|\le\varepsilon^{|\mu|}$ and $|\langle\xi_\alpha\,g_\mu\widetilde{e}_\alpha,\xi\rangle|\le\varepsilon^{|\mu|}\,|\xi_\alpha||\xi_{\alpha+\mu}|$. We infer

$$\big|\Sigma(\xi)\big|\le\varepsilon^{-1}\sum_{j,k,m}\sum_{\alpha\in\mathbb{N}^n}\sqrt{\alpha_m(|\alpha|+n)}\sum_{\mu\in\mathbb{N}^n}|c_{jk,m,\mu}|\,\varepsilon^{|\mu|}\,|\xi_\alpha||\xi_{\alpha+\mu}|.$$

Let r be the infimum of the radius of convergence of $w\mapsto\Psi_z(w)$ over all $z\in X$. Then for $\varepsilon<r$ and $r'\in\,]\varepsilon,r[$, we have a uniform bound $|c_{jk,m,\mu}|\le C(1/r')^{|\mu|}$, hence

$$\big|\Sigma(\xi)\big|\le C'\varepsilon^{-1}\sum_{\alpha\in\mathbb{N}^n}\sum_{\mu\in\mathbb{N}^n}\left(\frac{\varepsilon}{r'}\right)^{|\mu|}\sqrt{\alpha_m(|\alpha|+n)}\,|\xi_\alpha||\xi_{\alpha+\mu}|.$$

If we write

$$\begin{aligned}\sqrt{\alpha_m(|\alpha|+n)}\,|\xi_\alpha||\xi_{\alpha+\mu}|&\le\frac{1}{2}(|\alpha|+n)\big(|\xi_\alpha|^2+|\xi_{\alpha+\mu}|^2\big)\\&\le\frac{1}{2}\big((|\alpha|+n)|\xi_\alpha|^2+(|\alpha+\mu|+n)|\xi_{\alpha+\mu}|^2\big),\end{aligned}$$

the above bound implies

$$\big|\Sigma(\xi)\big|\le C'\varepsilon^{-1}(1-\varepsilon/r')^{-n}\sum_{\alpha\in\mathbb{N}^n}(|\alpha|+n)|\xi_\alpha|^2=O\bigg(\varepsilon^{-1}\sum_{\alpha\in\mathbb{N}^n}(|\alpha|+n)|\xi_\alpha|^2\bigg).$$

We now come to the more annoying term $A^{1,0}\wedge A^{0,1}+A^{0,1}\wedge A^{1,0}$, and especially to the part containing ε^{-2} (the other parts can be treated as above or are smaller). We compute explicitly that term by expanding $a'_{k,m}(w)$ into a power series $\sum_\mu a'_{k,m,\mu}\,g_\mu(w)$ as above. Let us write $g_m(w)=s_\mu^{-1}w^\mu$. As $a'_{k,m}(0)=0$, the relevant term in $A^{0,1}$ is

$$\varepsilon^{-1}\sum_{k,m}\sum_{\mu\in\mathbb{N}^n\smallsetminus\{0\}}a'_{k,m,\mu}s_\mu^{-1}\,d\overline{z}_k\otimes W^\mu D_m$$

where D_m and $W^\mu = W_1^{\mu_1}\ldots W_n^{\mu_n}$ are operators on the Hilbert space $\mathcal{H}^2(B_{\varepsilon,0})$, defined by

$$D_m\widetilde{e}_\alpha = \sqrt{\alpha_m(|\alpha|+n)}\,\widetilde{e}_{\alpha-c_m},\quad W_m(f)=w_mf.$$

The corresponding term in $A^{1,0}$ is the opposite of the adjoint, namely

$$-\varepsilon^{-1}\sum_{j,\ell}\sum_{\lambda\in\mathbb{N}^n\smallsetminus\{0\}} a'_{k,\ell,\lambda}s_\lambda^{-1}dz_j\otimes D_\ell^*W^{*\lambda}$$

and the annoying term in $A^{1,0}\wedge A^{0,1}+A^{0,1}\wedge A^{1,0}$ is

$$Q=-\,\varepsilon^{-2}\sum_{j,k,\ell,m}\sum_{\lambda,\mu\in\mathbb{N}^n\smallsetminus\{0\}} a'_{k,\ell,\lambda}s_\lambda^{-1}\,a'_{k,m,\mu}s_\mu^{-1}\,dz_j\wedge d\overline{z}_k\otimes \Big(D_\ell^*W^{*\lambda}W^\mu D_m - W^\mu D_m D_\ell^* W^{*\lambda}\Big). \tag{2.21}$$

We have here $\|W^\mu\|\le s_\mu\,\varepsilon^{|\mu|}$ (as W^μ is the multiplication by $w^\mu = s_\mu\,g_\mu(w)$, and $|g_\mu|\le\varepsilon^{|\mu|}$ on $B_{\varepsilon,0}$). The operators D_ℓ^* and D_m are unbounded, but the important point is that their commutators have substantially better continuity than what could be expected a priori. We have for instance

$$\begin{aligned}
&D_m\widetilde{e}_\alpha=\sqrt{\alpha_m(|\alpha|+n)}\,\widetilde{e}_{\alpha-c_m},\quad D_\ell^*(\widetilde{e}_\alpha)=\sqrt{(\alpha_\ell+1)(|\alpha|+n+1)}\,\widetilde{e}_{\alpha+c_\ell},\\
&[D_\ell^*,D_m](\widetilde{e}_\alpha)=\Big(\sqrt{(\alpha_\ell+1-\delta_{\ell m})\alpha_m}\,(|\alpha|+n)\\
&\qquad\qquad -\sqrt{(\alpha_\ell+1)(\alpha_m+\delta_{\ell m})}\,(|\alpha|+n+1)\Big)\,\widetilde{e}_{\alpha+c_\ell-c_m}
\end{aligned}$$

and the coefficient between braces is controlled by $2(|\alpha|+n)$, as one sees by considering separately the two cases $\ell\ne m$, where we get $-\sqrt{(\alpha_\ell+1)\alpha_m}$, and $\ell=m$, where we get $\alpha_\ell(|\alpha|+n)-(\alpha_\ell+1)(|\alpha|+n+1)$. Therefore $\|[D_\ell^*,D_m](\widetilde{e}_\alpha)\|\le 2(|\alpha|+n)$. We obtain similarly

$$\begin{aligned}
&W_m(\widetilde{e}_\alpha)=\varepsilon\sqrt{\frac{\alpha_m+1}{|\alpha|+n+1}}\,\widetilde{e}_{\alpha+c_m},\quad W_\ell^*(\widetilde{e}_\alpha)=\varepsilon\sqrt{\frac{\alpha_\ell}{|\alpha|+n}}\,\widetilde{e}_{\alpha-c_\ell},\\
&[W_\ell^*,W_m](\widetilde{e}_\alpha)=\varepsilon^2\bigg(\frac{\sqrt{(\alpha_\ell+\delta_{\ell m})(\alpha_m+1)}}{|\alpha|+n+1}-\frac{\sqrt{\alpha_\ell(\alpha_m+1-\delta_{\ell m})}}{|\alpha|+n}\bigg)\,\widetilde{e}_{\alpha-c_\ell+c_m},
\end{aligned}$$

and it is easy to see that the coefficient between large braces is bounded for $\ell\ne m$ by $\sqrt{\alpha_\ell(\alpha_m+1)}/((|\alpha|+n)(|\alpha|+n+1))\le(|\alpha|+n)^{-1}$, and for $\ell=m$ we have as well

$$\left|\frac{(\alpha_\ell+1)(|\alpha|+n)-\alpha_\ell(|\alpha|+n+1)}{(|\alpha|+n)(|\alpha|+n+1)}\right|\le(|\alpha|+n)^{-1}.$$

Therefore $\|[W_\ell^*,W_m](\widetilde{e}_\alpha)\|\le\varepsilon^2(|\alpha|+n)^{-1}$. Finally

$$[W_\ell^*,D_m](\widetilde{e}_\alpha)=\varepsilon\bigg(\sqrt{\frac{(\alpha_\ell-\delta_{\ell m})\alpha_m(|\alpha|+n)}{|\alpha|+n-1}}-\sqrt{\frac{\alpha_\ell(\alpha_m-\delta_{\ell m})(|\alpha|+n-1)}{|\alpha|+n}}\bigg)\,\widetilde{e}_{\alpha-c_\ell-c_m}$$

with a coefficient between braces less than 1, thus $\|[W_\ell^*,D_m](\widetilde e_\alpha)\|\le\varepsilon$. By adjunction, the same is true for $[D_\ell^*,W_m]$, and we can summarize our estimates as follows:

$$\tag{2.22}\begin{cases}\|W^\mu\|\le s_\mu\varepsilon^{|\mu|},\quad \|W^{*\lambda}\|\le s_\lambda\varepsilon^{|\lambda|},\quad \|D_\ell^*(\widetilde e_\alpha)\|\le|\alpha|+n+1,\quad \|D_m(\widetilde e_\alpha)\|\le|\alpha|+n,\\ \|[D_\ell^*,D_m](\widetilde e_\alpha)\|\le 2(|\alpha|+n),\quad \|[W_\ell^*,W_m](\widetilde e_\alpha)\|\le(|\alpha|+n)^{-1},\\ \|[W_\ell^*,D_m](\widetilde e_\alpha)\|\le\varepsilon,\quad \|[D_\ell^*,W_m](\widetilde e_\alpha)\|\le\varepsilon\end{cases}$$

Now, we observe that both $D_\ell^*W^{*\lambda}W^\mu D_m(\widetilde e_\alpha)$ and $W^\mu D_m D_\ell^* W^{*\lambda}(\widetilde e_\alpha)$ are multiples of $\widetilde e_{\alpha+c_\ell-c_m-\lambda+\mu}$. By considering the second product $W^\mu D_m D_\ell^* W^{*\lambda}$ and permuting successively its factors $D_mD_\ell^*$, $D_mW^{*\lambda}$, $W^\mu D_\ell^*$, $W^\mu W^{*\lambda}$, the difference with $D_\ell^*W^{*\lambda}W^\mu D_m$ is expressed as a sum of $1+|\lambda|+|\mu|+|\lambda||\mu|$ terms involving commutators. We derive from our estimates (2.22) precise bounds for the image of $\widetilde e_\alpha$ by the commutators. For instance, when we arrive at $D_\ell^*W^\mu W^{*\lambda}D_m$ and permute $W^\mu W^{*\lambda}$, we go through intermediate steps $D_\ell W^{*\lambda'}W^{\mu'}W_kW_j^*W^{\mu''}W^{*\lambda''}D_m$ with $\lambda=\lambda'+\lambda''+c_j$, $\mu=\mu'+\mu''+c_k$, $|\lambda|=|\lambda'|+|\lambda''|+1$, $|\mu|=|\mu'|+|\mu''|+1$, and have to evaluate the commutators

$$D_\ell W^{*\lambda'}W^{\mu'}[W_j^*,W_k]W^{\mu''}W^{*\lambda''}D_m(\widetilde e_\alpha).$$

By (2.22), the norm of these $|\lambda||\mu|$ terms is bounded by

$$\begin{aligned}&\big((|\alpha|-|\lambda|+|\mu|-1)_++n+1\big)s_{\lambda'}\varepsilon^{|\lambda'|}s_{\mu'}\varepsilon^{|\mu'|}\big((|\alpha|-|\lambda''|+|\mu''|)_++n\big)^{-1}s_{\mu''}\varepsilon^{|\mu''|}s_{\lambda''}\varepsilon^{|\lambda''|}(|\alpha|+n)\\ &\qquad\le s_{\lambda'}s_{\lambda''}s_{\mu'}s_{\mu''}\,\varepsilon^{|\lambda|+|\mu|}\,\frac{((|\alpha|-|\lambda|+|\mu|-1)_++n+1)(|\alpha|+n)}{(|\alpha|-|\lambda|)_++n}.\end{aligned}\tag{2.23}$$

The remaining commutators are easier, they lead to bounds

$$\tag{2.24}\begin{cases}s_\lambda s_\mu\ \varepsilon^{|\lambda|+|\mu|}\ 2((|\alpha|-|\lambda|)_++n) & \text{(once)},\\ s_{\lambda'}s_{\lambda''}s_\mu\ \varepsilon^{|\lambda|+|\mu|}\ (|\alpha|-|\lambda|-1)_++n+1 & (|\lambda|\ \text{times}),\\ s_\lambda s_{\mu'}s_{\mu''}\ \varepsilon^{|\lambda|+|\mu|}\ (|\alpha|+n) & (|\mu|\ \text{times}).\end{cases}$$

In the final estimates, we will have to bound some combinatorial factors of the form

$$\tag{2.25}\frac{s_{\lambda'}s_{\lambda''}}{s_\lambda}\,\frac{s_{\mu'}s_{\mu''}}{s_\mu}\quad\text{(worst case)},$$

and we want the ratios $s_{\lambda'}s_{\lambda''}/s_\lambda$ to be as small as possible (clearly they are at least equal to 1). For this, we try to keep the proportions $\lambda'_j/|\lambda'|$, $\lambda''_j/|\lambda''|$ as close as possible to $\lambda_j/|\lambda|$ by selecting carefully which factor W_ℓ^* (and W_m) we exchange at each step. After a permutation of the coordinates, we may assume than $\lambda_n\ge\max_{j<n}\lambda_j$, hence $\lambda_n\ge\frac1n|\lambda|$. If $t'=|\lambda'|/|\lambda|$ and $t''=|\lambda''|/|\lambda|=1-t'-1/|\lambda|$, we take $\lambda'_j=\lfloor t'\lambda_j\rfloor$, $\lambda''_j=\lfloor t''\lambda_j\rfloor$ for $j\le n-1$ and compensate by taking ad hoc values of λ'_n, λ''_n and $c_\ell=\lambda-(\lambda'+\lambda'')$. Then $t'\lambda_j-1<\lambda'_j\le t'\lambda_j$ for $j<n$ and

$$\lambda'_n=|\lambda'|-\sum_{j<n}\lambda'_j\ \begin{cases}<t'|\lambda|-\sum_{j<n}t'\lambda_j+n-1=t'\lambda_n+n-1,\\ \ge t'|\lambda|-\sum_{j<n}t'\lambda_j=t'\lambda_n.\end{cases}$$

Therefore

$$\frac{\lambda'_j}{|\lambda'|}\le\frac{\lambda_j}{|\lambda|}\quad\text{for } j<n,\qquad \frac{\lambda'_n}{|\lambda'|}\le\frac{t'\lambda_n+n-1}{t'|\lambda|}=\frac{\lambda_n}{|\lambda|}\Big(1+\frac{n-1}{t'\lambda_n}\Big)\quad\text{if }\lambda_n>0.$$

These inequalities imply respectively

$$\Big(\frac{\lambda'_j}{|\lambda'|}\Big)^{\lambda'_j/2}\le\Big(\frac{\lambda_j}{|\lambda|}\Big)^{(t'\lambda_j-1)/2},\quad \Big(\frac{\lambda'_n}{|\lambda'|}\Big)^{\lambda'_n/2}\le\Big(\frac{\lambda_n}{|\lambda|}\Big)^{t'\lambda_n/2}\Big(1+\frac{n-1}{t'\lambda_n}\Big)^{(t'\lambda_n+n-1)/2}.$$

In the last inequality we have $t'\lambda_n\ge\frac{1}{|\lambda|}\lambda_n\ge\frac{1}{n}$ unless $\lambda'=0$. Thus, if $\lambda'\ne 0$, we get

$$\Big(1+\frac{n-1}{t'\lambda_n}\Big)^{(t'\lambda_n+n-1)/2}\le\exp\Big(\frac12\frac{n-1}{t'\lambda_n}(t'\lambda_n+n-1)\Big)\le\exp\Big(\frac12\big(n-1+n(n-1)^2\big)\Big),$$

and by taking the product over all $j\in\{1,\ldots,n\}$ we find

$$s_{\lambda'}\le e^{n^3/2}\prod_{j\le n}\Big(\frac{\lambda_j}{|\lambda|}\Big)^{t'\lambda_j/2}\prod_{j<n}\Big(\frac{|\lambda|}{\lambda_j}\Big)^{1/2}\le e^{n^3/2}\,(\sigma_\lambda)^{t'}\;|\lambda|^{(n-1)/2}$$

(notice that for $\lambda_j=0$ we also have $\lambda'_j=0$, and the corresponding factors are then equal to 1). Notice also that

$$(s_\lambda)^{-1/|\lambda|}=\prod_{j\le n}\Big(\frac{|\lambda|}{\lambda_j}\Big)^{\lambda_j/2|\lambda|}\le\prod_{j\le n}|\lambda|^{\lambda_j/2|\lambda|}=|\lambda|^{1/2}.$$

For $\lambda',\lambda''\ne 0$, this implies

$$s_{\lambda'}s_{\lambda''}\le e^{n^3}(s_\lambda)^{t'+t''}\;|\lambda|^{n-1}=e^{n^3}(s_\lambda)^{1-1/|\lambda|}\;|\lambda|^{n-1}\le e^{n^3}s_\lambda\;|\lambda|^n,\tag{2.26}$$

and our combinatorial factor (2.25) is less than $e^{2n^3}|\lambda|^n|\mu|^n$. When $\lambda'=0$ or $\lambda''=0$ (say $\lambda''=0$), we have $\lambda'=\lambda-c_j$ for some j and $s_{\lambda''}=1$, thus

$$s_{\lambda'}s_{\lambda''}=s_{\lambda'}=s_\lambda\;\Big(\frac{|\lambda|}{|\lambda|-1}\Big)^{(|\lambda|-1)/2}|\lambda|^{1/2}\;\frac{(\lambda_j-1)^{(\lambda_j-1)/2}}{\lambda_j^{\lambda_j/2}}\le e^{1/2}\;s_\lambda\;|\lambda|^{1/2}$$

and inequality (2.26) still holds. We now put all our bounds together. For all $r'<r=$ radius of convergence of $w\mapsto\Psi_z(w)$, the coefficients $a'_{k,\ell,\lambda}$ satisfy $|a'_{k,\ell,\lambda}|\le C_0(1/r')^{|\lambda|}$ with $C_0=C_0(r')>0$, and for every $\xi=\sum_\alpha\xi_\alpha\widetilde e_\alpha$, (2.21–2.26) imply a bound of the form

$$\big|\langle Q(\xi),\xi\rangle\big|\le\varepsilon^{-2}\sum_{\alpha\in\mathbb{N}^n}\;\sum_{\lambda,\mu\in\mathbb{N}^n\smallsetminus\{0\}}C_1\;\Big(\frac{\varepsilon}{r'}\Big)^{|\lambda|+|\mu|}(2+|\lambda|+|\mu|+|\lambda||\mu|)\;|\lambda|^n|\mu|^n\times$$
$$\frac{((|\alpha|-|\lambda|+|\mu|-1)_++n+1)(|\alpha|+n)}{(|\alpha|-|\lambda|)_++n}|\xi_\alpha||\xi_{\alpha-\lambda+\mu}|.$$

Here $|\lambda|+|\mu|\ge 2$, and for $\delta>0$ arbitrary, there exists $C_2=C_2(\delta)$ such that

$$(2+|\lambda|+|\mu|+|\lambda||\mu|)\;|\lambda|^n|\mu|^n\le C_2\;(1+\delta)^{|\lambda|+|\mu|-2},$$

thus

$$\big|\langle Q(\xi),\xi\rangle\big|\le\frac{C_1C_2}{r'^2}\sum_{\alpha\in\mathbb{N}^n}\;\sum_{\lambda,\mu\in\mathbb{N}^n\smallsetminus\{0\}}\Big(\frac{(1+\delta)\varepsilon}{r'}\Big)^{|\lambda|+|\mu|-2}\times$$
$$\frac{((|\alpha|-|\lambda|+|\mu|)_++n)(|\alpha|+n)}{(|\alpha|-|\lambda|)_++n}|\xi_\alpha||\xi_{\alpha-\lambda+\mu}|.$$

Now, we split the summation with respect to (λ,μ) between the two subsets $|\lambda|+|\mu|\le(|\alpha|+n)/2$ and $|\lambda|+|\mu|>(|\alpha|+n)/2$. We find respectively

$$\frac{((|\alpha|-|\lambda|+|\mu|)_++n)(|\alpha|+n)}{(|\alpha|-|\lambda|)_++n}\le\begin{cases}\sqrt{6}\sqrt{|\alpha|+n)((|\alpha|-|\lambda|+|\mu|)_++n)} & \text{(first case)}\\ \frac{6}{n}(|\lambda|+|\mu|)^2 & \text{(second case).}\end{cases}$$

In the first case, we use the inequality

$$\begin{aligned}2\sqrt{|\alpha|+n)((|\alpha|-|\lambda|+|\mu|)_++n)}\,|\xi_\alpha||\xi_{\alpha-\lambda+\mu}|\\ \le(|\alpha|+n)|\xi_\alpha|^2+(|\alpha|-|\lambda|+|\mu|)_++n)|\xi_{\alpha-\lambda+\mu}|^2,\end{aligned}$$

and in the second case we content ourselves with the simpler bound

$$2|\xi_\alpha||\xi_{\alpha-\lambda+\mu}|\le|\xi_\alpha|^2+|\xi_{\alpha-\lambda+\mu}|^2.$$

For $\varepsilon\in\,]0,r[$, the series

$$\sum_{\lambda,\mu\in\mathbb{N}^n\smallsetminus\{0\}}\Big(\frac{(1+\delta)\varepsilon}{r'}\Big)^{|\lambda|+|\mu|-2}\quad\text{and}\quad\sum_{\lambda,\mu\in\mathbb{N}^n\smallsetminus\{0\}}\Big(\frac{(1+\delta)\varepsilon}{r'}\Big)^{|\lambda|+|\mu|-2}(|\lambda|+|\mu|)^2$$

can be made convergent by choosing $r'=(r+\varepsilon)/2\in\,]\varepsilon,r[$ and $1+\delta=\sqrt{r'/\varepsilon}$, thus there exists a positive continuous and increasing function $\varepsilon\mapsto C(\varepsilon)$ on $]0,r[$ such that

$$\big|\langle Q(\xi),\xi\rangle\big|\le C(\varepsilon)\sum_{\alpha\in\mathbb{N}^n}(|\alpha|+n)|\xi_\alpha|^2\quad\text{for all }\xi\in B_\varepsilon,$$

which is what we wanted. This bound, together with Theorem 2.15 and the estimates from the preliminary discussion yield the following result.

2.27. Theorem. *Let (X,γ) be a compact hermitian manifold equipped with a real analytic metric, and let r we the supremum of the radii r' of the ball bundles $\{\|\zeta\|_\gamma<r'\}$ on which the related exponential map* $\mathrm{exph}=\mathrm{exph}_\gamma:\{\|\zeta\|_\gamma<r'\}\subset T_X\to X\times X$ *defines a real analytic diffeomorphism $(z,\zeta)\mapsto(z,\mathrm{exph}_z(\zeta))$. Then, for all $\varepsilon<r$, the curvature tensor of the Bergman bundle (B_ε,h) satisfies an estimate*

$$\langle\Theta_{B_\varepsilon,h}\sigma(v,Jv),\sigma\rangle_h=\varepsilon^{-2}\sum_{\alpha\in\mathbb{N}^n}\left(\Big|\sum_j\sqrt{\alpha_j}\,\xi_{\alpha-c_j}v_j\Big|^2+(1+O(\varepsilon))\sum_j(|\alpha|+n)\,|\xi_\alpha|^2|v_j|^2\right)$$

for every $\sigma=\sum_\alpha\xi_\alpha e_\alpha\in B_{\rho\varepsilon}$, $\rho>1$, and every tangent vector $v=\sum v_j\;\partial/\partial z_j$, where $O(\varepsilon)=\varepsilon\,C(\varepsilon)$ for a continuous increasing function $\varepsilon\mapsto C(\varepsilon)$ on $]0,r[$. In particular $\Theta_{B_\varepsilon,h}$ is positive definite (and even coercive unbounded) for $\varepsilon<\varepsilon_0$ small enough.

3. Invariance of plurigenera for polarized Kähler families

The goal of this section is to prove that for every polarized family $\mathcal{X}\to S$ of compact Kähler manifolds, the plurigenera $p_m(X_t)=h^0(X_t,mK_{X_t})$ of fibers are independent of t for all $m\ge0$. This result has first been proved by Y.T. Siu [Siu98] in the case of projective varieties of general type (in which case the proof has been translated in a purely algebraic form by Y. Kawamata [Kaw99]), and then by [Siu00] and Păun [Pau04] in the case of arbitrary projective varieties; in the nonrestricted projective case, no algebraic proof of the result is known. We extend here the result to the Kähler context. This requires substantial modifications of the proof, since the technique of Siu and Păun involved in a crucial manner the use of an auxiliary ample line bundle. We replace it here by a use of the Hilbert bundles studied in the previous section.

3.1. Theorem. *Let $\pi : \mathcal{X} \to S$ be a proper holomorphic map defining a family of smooth compact Kähler manifolds over an irreducible base S. Assume that π admits local polarizations, i.e. every point $s_0 \in S$ has a neighborhood V such that $\pi^{-1}(V)$ carries a closed smooth $(1,1)$-form ω for which $\omega_{|X_t}$ is positive definite on $X_t := \pi^{-1}(t)$, $t \in V$. Then the plurigenera $p_m(X_t) = h^0(X_t, mK_{X_t})$ of fibers are independent of t for all $m \geq 0$.*

The above statement follows directly from the following more technical result.

3.2. Theorem (generalized version of the Claudon-Păun theorem)**.** *Let $\pi : \mathcal{X} \to \Delta$ be a polarized family of compact Kähler manifolds over a disc $\Delta \subset \mathbb{C}$, and let $(\mathcal{L}_j, h_j)_{0\leq j\leq N-1}$ be (singular) hermitian line bundles with semi-positive curvature currents $i\Theta_{\mathcal{L}_j,h_j} \geq 0$ on $\mathcal{X}$. Assume that*

(a) *the restriction of h_j to the central fiber X_0 is well defined (i.e. not identically $+\infty$).*

(b) *the multiplier ideal sheaf $\mathcal{I}(h_{j|X_0})$ is trivial for $1 \leq j \leq N-1$.*

Then any section σ of $\mathcal{O}(mK_{\mathcal{X}} + \sum \mathcal{L}_j)_{|X_0} \otimes \mathcal{I}(h_{0|X_0})$ over the central fiber X_0 extends into a section $\widetilde{\sigma}$ of $\mathcal{O}(mK_{\mathcal{X}} + \sum \mathcal{L}_j)$ over a certain neighborhood $\mathcal{X}' = \pi^{-1}(\Delta')$ of X_0, where $\Delta' \subset \Delta$ is a sufficienty small disc centered at 0.

3.3. Remark. A standard cohomological argument shows that we can in fact take $\mathcal{X}' = \mathcal{X}$ in the conclusion of Theorem 3.2, because the direct image sheaf $\mathcal{E} = \pi_*\mathcal{O}(mK_{\mathcal{X}} + \sum \mathcal{L}_j)$ is coherent, and the restriction $\mathcal{E} \to \mathcal{E} \otimes (\mathcal{O}_\Delta/\mathfrak{m}_0\mathcal{O}_\Delta)$ induces a surjective map at the H^0 level on the Stein space Δ, so we can extend $\widetilde{\sigma}$ mod $\pi^*\mathfrak{m}_0$ to $\mathcal{X}$.

Proof of Theorem 3.1. The invariance of plurigenera is in fact just obtained as the special case of Theorem 3.2 when all line bundles $\mathcal{L}_j$ and their metrics h_j are trivial. Since the dimension $t \mapsto h^0(X_t, mK_{X_t})$ is always upper semicontinuous and since Theorem 3.2 implies the lower semicontinuity, we conclude that the dimension must be constant along analytic discs, hence along the irreducible base S, by joining any two points through a chain of analytic discs. □

3.4. Lemma. *Let $\mathcal{X}' = \pi^{-1}(\Delta') \to \Delta'$ be the restriction of $\pi : \mathcal{X} \to \Delta$ to a disc $\Delta' \Subset \Delta$ centered at 0, of radius $R' < R$. For $\varepsilon \leq \varepsilon_0 = \varepsilon_0(R')$ small enough, one can find a Stein open subset $\mathcal{U}'_\varepsilon \subset \mathcal{X}' \times \overline{\mathcal{X}}$, where such that the projection $\mathrm{pr}_1 : \mathcal{U}'_\varepsilon \to \mathcal{X}'$ is a complex ball bundle over $\mathcal{X}'$ that is locally trivial real analytically.*

Proof. The arguments are very similar to those of §1, except for the fact that $\mathcal{X}$ is no longer compact, but this is not a problem since $\mathcal{X} \to \Delta$ is proper, and since we can always shrink Δ a little bit to achieve uniform bounds (would they be needed). Let γ be a real analytic hermitian metric on $\mathcal{X}$, and $\mathrm{exph} : T_{\mathcal{X}} \to \mathcal{X}$ be the corresponding real analytic and fiber-holomorphic exponential map associated with γ, as in §1. The map exph is no longer everywhere defined, but if we restrict it to the ε-tubular neighborhood of the zero section in $T_{\mathcal{X}'}$, we get for $\varepsilon > 0$ small enough a real analytic diffeomorphism $(z,\xi) \mapsto (z, \mathrm{exph}_z(\xi))$ onto a tubular neighborhood of the diagonal of $\mathcal{X}' \times \mathcal{X}'$. The rest of the proof is identical to what we did in §1, taking

$$\mathcal{U}'_\varepsilon = \big\{(z,w) \in \mathcal{X}' \times \overline{\mathcal{X}}\,;\ |\,\mathrm{logh}_z(\overline{w})|_\gamma < \varepsilon\big\}. \tag{3.5}$$ □

In order to study Theorem 3.2, we first state a technical extension theorem needed for the proof, which is a special case of the well-known and extremely powerful Ohsawa-Takegoshi theorem [OT87], see also [Oh??], [Dem??].

3.6. Proposition. *Let $\pi : \mathcal{Z} \to \Delta$ be a smooth and proper morphism from a (non compact) Kähler manifold $\mathcal{Z}$ to a disc $\Delta \subset \mathbb{C}$ and let $(\mathcal{L}, h)$ be a (singular) hermitian line bundle with semi-positive curvature current $\mathrm{i}\Theta_{\mathcal{L},h} \geq 0$ on $\mathcal{Z}$. Let ω be a global Kähler metric on $\mathcal{Z}$, and let $dV_{\mathcal{Z}}$, dV_{Z_0} the respective induced volume elements on $\mathcal{Z}$ and $Z_0 = \pi^{-1}(0)$. Assume that h_{Z_0} is well defined (i.e. almost everywhere finite). Then any holomorphic section s of $\mathcal{O}(K_{\mathcal{Z}} + \mathcal{L}) \otimes \mathcal{I}(h_{|Z_0})$ extends into a section $\widetilde{s}$ over $\mathcal{Z}$ satisfying an L^2 estimate*

$$\int_{\mathcal{Z}} \|\widetilde{s}\,\|^2_{\omega\otimes h} dV_{\mathcal{Z}} \leq C_0 \int_{Z_0} \|s\|^2_{\omega\otimes h} dV_{Z_0},$$

where $C_0 \geq 0$ is some universal constant (depending on $\dim \mathcal{Z}$ and $\mathrm{diam}\,\Delta$, but otherwise independent of $\mathcal{Z}$, $\mathcal{L}$, ...).

3.7. Remark. The assumptions of Proposition 3.6 imply that $\mathcal{Z}$ is holomorphically convex and complete Kähler, thus the technique of [Dem??] does apply to yield the result.

Proof of Theorem 3.2. Let $p = \mathrm{pr}_1 : \mathcal{U}'_\varepsilon \to \mathcal{X}'$ be as in Lemma 3.4, and $q = \mathrm{pr}_2 : \mathcal{U}'_\varepsilon \to \overline{\mathcal{X}}$. We take $\varepsilon < \varepsilon_0$ and use on $\mathcal{Z} := \mathcal{U}'_\varepsilon$ a Kähler metric ω_0 defined on the Stein manifold $\mathcal{U}'_{\varepsilon_0}$. On can define e.g. ω_0 as the $i\partial\overline{\partial}$ of a strictly plurisubharmonic exhaustion function on $\mathcal{U}'_{\varepsilon_0}$, but we can also take the restriction of $\mathrm{pr}_1^*\,\omega + \mathrm{pr}_2^*\,\overline{\omega}_{|\overline{\mathcal{X}}}$ where ω is the Kähler metric on the total space $\mathcal{X}$, and $\overline{\omega} = -\,\omega$ the corresponding Kähler metric on the conjugate space $\overline{\mathcal{X}}$.

First step: construction of a sequence of extensions on $\mathcal{Z} = \mathcal{U}'_\varepsilon$ via the Ohsawa-Takegoshi extension theorem.

The strategy is to apply iteratively the special case 3.6 of the Ohsawa-Takegoshi extension theorem on the total space of the fibration

$$\pi' = \pi \circ p : \mathcal{Z} = \mathcal{U}'_\varepsilon \to \mathcal{X}' \to \Delta',$$

and to extend sections of ad hoc pull-backs $p^*\mathcal{G}$ from the zero fiber $Z_0 = \pi'^{\,-1}(0) = p^{-1}(X_0)$ to the whole of $\mathcal{Z} = \mathcal{U}'_\varepsilon$. We write $h_j = e^{-\varphi_j}$ in terms of local plurisubharmonic weights, and define inductively a sequence of line bundles $\mathcal{G}_m$ by putting $\mathcal{G}_0 = \mathcal{O}_{\mathcal{X}'}$ and

$$\mathcal{G}_m = \mathcal{G}_{m-1} + K_{\mathcal{X}'} + \mathcal{L}_r \qquad \text{if } m = Nq + r,\ 0 \leq r \leq N-1.$$

By construction we have

$$\begin{aligned} \mathcal{G}_m &= mK_{\mathcal{X}'} + \mathcal{L}_1 + \cdots + \mathcal{L}_m, &&\text{for } 1 \leq m \leq N-1\,,\\ \mathcal{G}_{m+N} - \mathcal{G}_m = \mathcal{G}_N &= NK_{\mathcal{X}'} + \mathcal{L}_0 + \cdots + \mathcal{L}_{N-1}\,, &&\text{for all } m \geq 0. \end{aligned}$$

The game is to construct inductively families of sections, say $\{\widetilde f_j^{(m)}\}_{j=1,\ldots,J(m)}$, of $p^*\mathcal{G}_m$ over $\mathcal{Z}$, together with ad hoc L^2 estimates, in such a way that

(3.8) for $m=0,\ldots,N-1$, $p^*\mathcal{G}_m$ is generated by L^2 sections $\{\widetilde f_j^{(m)}\}_{j=1,\ldots,J(m)}$ on $\mathcal{U}'_{\varepsilon_0}$;

(3.9) we have the m-periodicity relations $J(m+N)=J(m)$ and $\widetilde f_j^{(m)}$ is an extension of $f_j^{(m)} := (p^*\sigma)^q f_j^{(r)}$ over $\mathcal{Z}$ for $m=Nq+r$, where $f_j^{(r)} := \widetilde f^{(r)}_{j|Z_0}$, $0\le r\le N-1$.

Property (3.8) can certainly be achieved since $\mathcal{U}'_{\varepsilon_0}$ is Stein, and for $m=0$ we can take $J(0)=1$ and $\widetilde f_1^{(0)}=1$. Now, by induction, we equip $p^*\mathcal{G}_{m-1}$ with the tautological metric $|\xi|^2/\sum_\ell|\widetilde f_\ell^{(m-1)}(x)|^2$, and

$$\widetilde{\mathcal{G}}_m := p^*\mathcal{G}_m - K_{\mathcal{Z}} = p^*\mathcal{G}_m - (p^*K_{\mathcal{X}'} + q^*K_{\overline{\mathcal{X}}}) = p^*(\mathcal{G}_{m-1}+\mathcal{L}_r) - q^*K_{\overline{\mathcal{X}}}$$

with that metric multiplied by $p^*h_r = e^{-p^*\varphi_r}$ and a fixed smooth metric $e^{-\psi}$ of positive curvature on $(-q^*K_{\overline{\mathcal{X}}})_{|\mathcal{U}'_{\varepsilon_0}}$ (remember that $\mathcal{U}'_{\varepsilon_0}$ is Stein!). It is clear that these metrics have semi-positive curvature currents on $\mathcal{Z}$ (by adjusting ψ, we could even take them to be strictly positive if we wanted). In this setting, we apply the Ohsawa-Takegoshi theorem to the line bundle $K_{\mathcal{Z}}+\widetilde{\mathcal{G}}_m = p^*\mathcal{G}_m$, and extend in this way $f_j^{(m)}$ into a section $\widetilde f_j^{(m)}$ over $\mathcal{Z}$. By construction the pointwise norm of that section in $p^*\mathcal{G}_{m|X_0}$ in a local trivialization of the bundles involved is the ratio

$$\frac{|f_j^{(m)}|^2}{\sum_\ell|f_\ell^{(m-1)}|^2}e^{-p^*\varphi_r-\psi},$$

up to some fixed smooth positive factor depending only on the metric induced by ω_0 on $K_{\mathcal{Z}}$. However, by the induction relations, we have

$$\frac{\sum_j|f_j^{(m)}|^2}{\sum_\ell|f_\ell^{(m-1)}|^2}e^{-p^*\varphi_r} = \begin{cases} \dfrac{\sum_j|f_j^{(r)}|^2}{\sum_\ell|f_\ell^{(r-1)}|^2}e^{-p^*\varphi_r} & \text{for } m=Nq+r,\ 0<r\le N-1,\\[2ex] \dfrac{\sum_j|f_j^{(0)}|^2}{\sum_\ell|f_\ell^{(N-1)}|^2}|p^*\sigma|^2e^{-p^*\varphi_0} & \text{for } m\equiv 0 \bmod N,\ m>0.\end{cases}$$

Since the sections $\{f_j^{(r)}\}_{0\le r<N}$ generate their line bundle on $\mathcal{U}_{\varepsilon_0}\supset\overline{\mathcal{U}'_\varepsilon}$, the ratios involved are positive functions without zeroes and poles, hence smooth and bounded [possibly after shrinking a little bit the base disc Δ', as is permitted]. On the other hand, assumption 3.2 (b) and the fact that σ has coefficients in the multiplier ideal sheaf $\mathcal{I}(h_{0|X_0})$ tell us that $e^{-p^*\varphi_r}$, $1\le r<m$ and $|p^*\sigma|^2e^{-p^*\varphi_0}$ are locally integrable on Z_0. It follows that there is a constant $C_1\ge 0$ such that

$$\int_{Z_0}\frac{\sum_j|f_j^{(m)}|^2}{\sum_\ell|f_\ell^{(m-1)}|^2}e^{-p^*\varphi_r-\psi}dV_{\omega_0}\le C_1$$

for all $m\ge 1$ (of course, the integral certainly involves finitely many trivializations of the bundles involved, whereas the integrand expression is just local in each chart). Inductively, the L^2 extension theorem produces sections $\widetilde f_j^{(m)}$ of $p^*\mathcal{G}_m$ over $\mathcal{Z}$ such that

$$\int_{\mathcal{Z}}\frac{\sum_j|\widetilde f_j^{(m)}|^2}{\sum_\ell|\widetilde f_\ell^{(m-1)}|^2}e^{-p^*\varphi_r-\psi}dV_{\omega_0}\le C_2=C_0C_1.$$

Second step: applying the Hölder inequality. Put $k = Nq(k) + r(k)$ with $0 \le r(k) < N$, and take $m = Nq(m)$ to be a multiple of N. The Hölder inequality $|\int \prod_{1\le k\le m} u_k d\mu| \le \prod_{1\le k\le m}(\int |u_k|^m d\mu)^{1/m}$ applied to the measure $\mu = dV_{\omega_0}$ and to the product of functions

$$\left(\frac{\sum_j |\widetilde{f}_j^{(m)}|^2}{\sum_\ell |\widetilde{f}_\ell^{(0)}|^2}\right)^{1/m} e^{-\frac{1}{N}p^*(\varphi_0+\ldots+\varphi_{N-1})-\psi} = \prod_{1\le k\le m}\left(\frac{\sum_j |\widetilde{f}_j^{(k)}|^2}{\sum_\ell |\widetilde{f}_\ell^{(k-1)}|^2} e^{-p^*\varphi_{r(k)}-\psi}\right)^{1/m}$$

in which $\sum_\ell |\widetilde{f}_\ell^{(0)}|^2 = |\widetilde{f}_1^{(0)}|^2 = 1$ and $\sum_j |\widetilde{f}_j^{(m)}|^2 = |\widetilde{f}_1^{(m)}|^2$, implies that

$$\int_{\mathcal{Z}} \big|\widetilde{f}_1^{(m)}\big|^{2/m} e^{-\frac{1}{N}p^*(\varphi_0+\ldots+\varphi_{N-1})-\psi} dV_{\omega_0} \le C_2. \tag{3.10}$$

As the functions $\varphi_{r(k)}$ and ψ are locally bounded from above, we infer from this the weaker inequality

$$\int_{\mathcal{Z}} \big|\widetilde{f}_1^{(m)}\big|^{2/m} dV_{\omega_0} \le C_3. \tag{3.10′}$$

The last inequality is to be understood as an inequality that holds in fact only locally over $\mathcal{X}'$, on sets of the form $p^{-1}(V)$, where $V \Subset \mathcal{X}'$ are small coordinate open sets where our line bundles are trivial, so that the section $\widetilde{f}_1^{(m)}$ of $q(m)\, p^*(NK_{\mathcal{X}'} + \sum \mathcal{L}_j)$ can be viewed as a holomorphic function on $p^{-1}(V)$.

Third step: construction of a singular hermitian metric on $NK_{\mathcal{X}'} + \sum \mathcal{L}_j$. The rough idea is to extract a weak limit of the m-th root occurring in (3.10), (3.10′), combined with an integration on the fibers of $p : \mathcal{Z} = \mathcal{U}'_\varepsilon \to \mathcal{X}'$, to get a singular hermitian metric on $NK_{\mathcal{X}'} + \sum \mathcal{L}_j$. This is the crucial step in the proof, and the place where the Kähler setup requires new arguments; especially, the integration on fibers makes the weak limit argument much less obvious than in the projective setup, and requires the results of §2 on Bergman bundles.

3.11. Proposition. *Assume that the sections $\widetilde{f}_1^{(m)}$ and vector fields τ_j have been constructed on a slightly bigger tubular neighborhood $\mathcal{U}'_{\rho_0\varepsilon} \to \mathcal{X}'$ with $\rho_0 > 1$, [this condition can of course always be achieved, since we have a lot of flexibility on the choice of the tubular neighborhoods, by taking e.g. $\varepsilon \le \frac{1}{2}\varepsilon_0(R')$]. Then there exists $c > 0$ and a subsequence $m \in M_0 \subset \mathbb{N}$ such that, with respect to local trivializations of the $\mathcal{L}_j$ and local holomorphic sections $dw = dw_1 \wedge \ldots \wedge dw_{n+1}$ of $K_{\overline{\mathcal{X}}}$), we have a well defined limit*

$$\theta(z) = \lim_{\substack{m\in M_0\\ m\to+\infty}} \frac{1}{m} \log \int_{w\in \mathcal{U}'_{\varepsilon,z}} \big|\widetilde{f}_1^{(m)}(z,w)\big|^2\, i^{(n+1)^2} dw \wedge d\overline{w}, \quad z \in \mathcal{X}'$$

exists almost everywhere on $\mathcal{X}'$, and $H = e^{-N\theta}$ defines a singular hermitian metric of positive curvature current on $p^*(NK_{\mathcal{X}'} + \sum \mathcal{L}_j)$, i.e. $i\partial\overline{\partial}\theta \ge 0$. Moreover, it satisfies the estimates

(a) $|\sigma|_H^2 = |\sigma|^2 e^{-N\theta} = 1$ *on* $X_0 \subset \mathcal{X}'$;

(b) $\displaystyle\int_{\mathcal{X}'} e^{-\theta} e^{-\frac{1}{N}(\varphi_0+\ldots+\varphi_{N-1})} dV_\omega < \infty$:

(c) *if* $\rho_0 \in]1, \rho_0''[$, *there is a constant* $C_4 = C_4(\rho_0) > 0$ *such that for every* $\beta > 0$

$$i\partial\overline{\partial} e^{\beta\theta} \ge -\varepsilon^{-2}(\log\rho_0)^{-1} e^{C_4\beta}\omega.$$

Proof. First notice that the choice of the w local coordinates on $\overline{\mathcal{X}}$ is irrelevant in the definition of θ (the L^2 integrals may eventually change by bounded multiplicative factors, which get killed as $m \to +\infty$). We take $\rho_1 = \frac{\rho_0+1}{2} \in]1, \rho_0[$ and use estimate (3.10′) on $\mathcal{Z} = \mathcal{U}'_{\rho_0\varepsilon}$, together with the mean value inequality for plurisubharmonic functions, applied on ω_0-geodesic balls of $\mathcal{Z}$ centered at points $(z,w) \in \mathcal{U}'_\varepsilon$ and of radius $d_{\omega_0}(\mathcal{U}'_{\rho_1\varepsilon}, \complement\,\mathcal{U}'_{\rho_0\varepsilon}) \sim (\rho_0-1)\varepsilon/2$. As $\dim\mathcal{Z} = 2(n+1)$, we obtain by (3.10′) a uniform upper bound

$$\begin{aligned}\sup_{\mathcal{U}'_{\rho_1\varepsilon,z}} |\widetilde{f}_1^{(m)}|^{2/m} &\le \frac{C_5}{((\rho_0-1)\varepsilon)^{4(n+1)}} \int_{\mathcal{U}'_{\rho_0\varepsilon}} |\widetilde{f}_1^{(m)}|^{2/m}\, i^{(n+1)^2} dw\wedge d\overline{w}\\ &\le \frac{C_6}{((\rho_0-1)\varepsilon)^{4(n+1)}}, \quad \forall z\in\mathcal{X}'.\end{aligned}\tag{3.12}$$

Here our sections can be seen as functions only locally over trivializing open sets of the line bundles in $\mathcal{X}'$, but we can arrange that there are only finitely many of these; hence the transition automorphisms only involve bounded constants, after raising to power $1/m$. At this point, let us consider the Bergman bundle $B_\varepsilon \to \mathcal{X}'$ and write locally over $\mathcal{X}'$

$$\widetilde{f}_1^{(m)}(z,w)\,dw = \sum_{\alpha\in\mathbb{N}^{n+1}} \xi_\alpha(z)\,\widetilde{e}_\alpha(z,w)\otimes g(z)^{q(m)}, \quad z\in\mathcal{X}',\ w\in\mathcal{U}'_{\varepsilon,z}$$

in terms of an orthonormal frame $(\widetilde{e}_\alpha)_{\alpha\in\mathbb{N}^{n+1}}$ of B_ε as defined in §2, where g is a local generator of $\mathcal{O}_\mathcal{X}(NK_{\mathcal{X}'}+\sum\mathcal{L}_j)$ and $dw = dw_1\wedge\ldots\wedge dw_{n+1}$ in local coordinates. Therefore, we have an equality

$$\begin{aligned}\theta_{m,\rho}(z) &:= \frac{1}{m}\log\int_{w\in\mathcal{U}'_{\rho\varepsilon,z}} \big|\widetilde{f}_1^{(m)}(z,w)\big|^2\, i^{(n+1)^2} dw\wedge d\overline{w}\\ &= \frac{1}{m}\log\left(\sum_{\alpha\in\mathbb{N}^{n+1}} \rho^{2(|\alpha|+n+1)}\,|\xi_\alpha(z)|^2\right),\end{aligned}\tag{3.13}$$

and by (3.12), we get an upper bound

$$\theta_{m,\rho}(z) \le \frac{1}{m}\log\frac{C_7\,(\rho\varepsilon)^{2(n+1)}\,C_6^m}{((\rho_0-1)\varepsilon)^{4m(n+1)}} \le C_8 + 4(n+1)\log\frac{1}{\varepsilon} =: C_{9,\varepsilon}, \quad \rho\le\rho_1.$$

The sum $\sum_{\alpha\in\mathbb{N}^{n+1}} \rho^{2(|\alpha|+n+1)}\,|\xi_\alpha(z)|^2 = e^{m\theta_{m,\rho}(z)}$ is nothing else than the square of the norm of the section $\widetilde{f}_1^{(m)}$, expressed with respect to the natural hermitian metric $\langle\bullet,\bullet\rangle_\rho$ of the Bergman bundle $B_{\rho\varepsilon}$, $\rho\in]0,\rho_1]$. The inequalities (3.13) show that the series converge uniformly over the whole of $\mathcal{X}'$. As $\nabla^{0,1}\xi = 0$, a standard calculation with respect to the Bergman connection $\nabla = \nabla^{1,0}+\nabla^{0,1}$ of $B_{\rho\varepsilon}$ implies

$$\begin{aligned}i\partial\overline{\partial}\theta_{m,\rho} &= \frac{1}{m}\,\frac{1}{\|\xi\|^2}\left(i\langle\nabla^{1,0}\xi,\nabla^{1,0}\xi\rangle_\rho - \langle i\Theta_{B_{\rho\varepsilon}}\xi,\xi\rangle_\rho - \frac{i\langle\nabla^{1,0}\xi,\xi\rangle_\rho\wedge\overline{\langle\nabla^{1,0}\xi,\xi\rangle_\rho}}{\|\xi\|_\rho^2}\right)\\ &\ge -\frac{1}{m}\,\frac{\langle i\Theta_{B_{\rho\varepsilon}}\xi,\xi\rangle_\rho}{\|\xi\|_\rho^2}\end{aligned}$$

by the Cauchy-Schwarz inequality. On the other hand, as the orthonormal coordinates expressed in $B_{\rho\varepsilon}$ are the $(\rho^{|\alpha|+n+1}\xi_\alpha)$, the curvature bound obtained in §2 yields

$$\langle i\Theta_{B_{\rho\varepsilon}}\xi,\xi\rangle_\rho \le (2+O(\rho\varepsilon))(\rho\varepsilon)^{-2}\sum_{\alpha\in\mathbb{N}^{n+1}}(|\alpha|+n+1)\,\rho^{2(|\alpha|+n+1)}\;|\xi_\alpha|^2\,\omega.$$

The last two inequalities imply the fundamental estimate

$$\begin{aligned} i\partial\overline{\partial}\theta_{m,\rho} &\ge -\frac{(2+O(\rho\varepsilon))(\rho\varepsilon)^{-2}}{m}\;\frac{\displaystyle\sum_{\alpha\in\mathbb{N}^{n+1}}(|\alpha|+n+1)\,\rho^{2(|\alpha|+n+1)}\;|\xi_\alpha|^2}{\displaystyle\sum_{\alpha\in\mathbb{N}^{n+1}}\rho^{2(|\alpha|+n+1)}\;|\xi_\alpha|^2}\;\omega\\ &\ge -\frac{1+O(\rho\varepsilon)}{\varepsilon^2\rho}\Big(\frac{\partial}{\partial\rho}\theta_{m,\rho}\Big)\,\omega. \end{aligned}\tag{3.14}$$

From its definition, we see that $\theta_{m,\rho}$ is a convex function of $\log\rho$. Therefore, for $\rho\le 1$, we have

$$\rho\,\frac{\partial}{\partial\rho}\theta_{m,\rho}\le\frac{\theta_{m,\rho_1}-\theta_{m,\rho}}{\log\rho_1-\log\rho}\le\frac{C_{9,\varepsilon}-\theta_{m,\rho}}{\log\rho_1},$$

and by (3.14) we find

$$i\partial\overline{\partial}\theta_{m,\rho}\ge-\frac{C_{10}}{\varepsilon^2\rho^2}\Big(C_{11}+\log\frac{1}{\varepsilon}-\theta_{m,\rho}\Big)\,\omega.$$

The ρ-derivative of $\theta_{m,\rho}$ may be difficult to estimate, thus we replace (3.14) by an integrated form and consider

$$\widetilde{\theta}_{m,\rho,\rho'}(z)=\frac{2}{\rho^2-\rho'^2}\int_{\rho'}^{\rho}\theta_{m,u}(z)\;u\,du.$$

Then we obtain

$$i\partial\overline{\partial}\,\widetilde{\theta}_{m,\rho,\rho'}\ge-\frac{2+O(\rho\varepsilon)}{\varepsilon^2(\rho^2-\rho'^2)}(\theta_{m,\rho}-\theta_{m,\rho'})\,\omega.\tag{3.15}$$

However, formula (2.18) shows that

$$\partial_{w_k}\widetilde{e}_\alpha(z,w)=\varepsilon^{-1}\;\sqrt{\alpha_k(|\alpha|+n)}\;\widetilde{e}_{\alpha-c_k}(z,w),$$

thus

$$\sum_k\|\partial_{w_k}\xi_\beta\|^2=\varepsilon^{-2}\sum_k\sum_{\alpha\in\mathbb{N}^{n+1}}\alpha_k(|\alpha|+n)\;|\xi_{\alpha\beta}|^2=\varepsilon^{-2}\sum_{\alpha\in\mathbb{N}^{n+1}}|\alpha|(|\alpha|+n)\;|\xi_{\alpha\beta}|^2$$

and

$$\varepsilon^{-2}\sum_{\alpha\in\mathbb{N}^{n+1}}(|\alpha|+n)^2\;|\xi_{\alpha\beta}|^2\le n^2\,\|\xi_\beta\|^2+2\sum_k\|\partial_{w_k}\xi_\beta\|^2\le C_8\Big(\|\xi_\beta\|^2+\sum_k\|\tau_k\xi_\beta\|^2\Big).$$

Since $\tau_k\xi_\beta = \xi_{\beta+c_k}$, we have

$$\sum_{|\beta|=\ell}\sum_k \|\tau_k\xi_\beta\|^2 \le \nu \sum_{|\beta'|=\ell+1} \|\xi_{\beta'}\|^2,$$

therefore

$$\varepsilon^{-2}\sum_{\beta\in\mathbb{N}^\nu} c^{|\beta|} \sum_{\alpha\in\mathbb{N}^{n+1}} (|\alpha|+n)^2|\xi_{\alpha\beta}|^2 \le C_8\big(\|\xi\|^2+\nu c^{-1}\|\xi\|^2\big) \le \frac{C_9}{(\rho-1)^2\varepsilon^2}\,\|\xi\|^2,$$

if we recall the value of $c=(8C_6^2)^{-1}(\rho-1)\varepsilon^2$. By (3.16) we get

$$\big|\langle i\Theta_{B_\varepsilon\,\widehat{\otimes}\,\ell^2(\mathbb{N}^{n+1})}\xi,\xi\rangle\big|_\omega \le \frac{C_{10}}{(\rho-1)\varepsilon^2}\,\|\xi\|^2$$

and in fine, by (3.15), we have

$$\text{(3.17)}\qquad i\partial\overline{\partial}\theta_m \ge -\frac{C_{10}}{m(\rho-1)\varepsilon^2}\,\omega.$$

Here the constant C_{10} can be seen to depend only on the geometry of $(T_{\mathfrak{X}},\omega)$. On the other hand, estimate (3.14) is equivalent to a uniform upper bound

$$\int_{z\in\mathfrak{X}'}\theta_m\,dV_{\omega_0} \le C_{11}+4(n+1)\ \log\frac{1}{(\rho-1)\varepsilon}$$

The above lower bound is uniform with respect to m as $m\to+\infty$, and we see that $e^{\lambda\theta_m}$ is quasi plurisubharmonic. By well known facts of pluripotential theory, there exists an upper semicontinuous regularization

$$\theta = \Big(\limsup_{m\to+\infty}\theta_m\Big)^*$$

and a subsequence $m\in M_0\subset\mathbb{N}$ such that

$$\theta = \limsup_{\substack{m\in M_0\\ m\to+\infty}}\theta_m\ \ \text{almost everywhere on } \mathfrak{X}',$$

and satisfying in the limit the Hessian estimate

$$\text{(3.17)}\qquad i\partial\overline{\partial}e^{\lambda\theta} \ge -\varepsilon^{-2}(\log\rho)^{-1}e^{C_6\lambda-1/2}\omega$$

for every $\lambda>0$. Property (c) is proved. □

3.18. What to do ?. Our hope (possibly after modifying the sections $f_j^{(m)}$ in an adequate manner) is that one can gain a factor converging to zero in estimate (3.16), and thus in (3.17). This would prove that $e^{\lambda\theta}$ is plurisubharmonic for every $\lambda>0$, hence the weight $\theta=\lim_{\lambda\to0}\frac{1}{\lambda}(e^{\lambda\theta}-1)$ would also be plurisubharmonic.

Fourth step: applying Ohsawa-Takegoshi once again with the singular hermitian metric produced in the third step.

Assuming that 3.18 holds, we have proved that $NK_{\mathcal{X}'}+\sum\mathcal{L}_j$ possesses a hermitian metric $H=e^{-N\theta}$ such that $\|\sigma\|_H\le 1$ oon X_0 and $\Theta_H\ge 0$ on $\mathcal{X}'$. In order to conclude, we equip the bundle

$$\mathcal{E}=(N-1)K_{\mathcal{X}'}+\sum\mathcal{L}_j$$

with the metric $\eta=H^{1-1/N}\prod h_j^{1/N}$, and $NK_{\mathcal{X}'}+\sum\mathcal{L}_j=K_{\mathcal{X}'}+\mathcal{E}$ with the metric $\omega\otimes\eta$. It is important here that $\mathcal{X}$ possesses a global Kähler polarization ω, otherwise the required estimates would not be valid. Clearly η has a semi-positive curvature current on $\mathcal{X}'$ and in a local trivialization we have

$$\|\sigma\|^2_{\omega\otimes\eta}\le C|\sigma|^2\exp\Big(-(N-1)\theta-\frac{1}{N}\sum\varphi_j\Big)\le C\Big(|\sigma|^2\prod e^{-\varphi_j}\Big)^{1/N}$$

on X_0. Since $|\sigma|^2e^{-\varphi_0}$ and $e^{-\varphi_r}$, $r>0$ are all locally integrable, we see that $\|\sigma\|^2_{\omega\otimes\eta}$ is also locally integrable on X_0 by the Hölder inequality. A new (and final) application of the L^2 extension theorem to the hermitian line bundle $(\mathcal{E},\eta)$ implies that σ can be extended to $\mathcal{X}'$. Theorem 3.2 is proved. □

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and a uniform upper bound $\theta_m\le C_9$ on $\mathcal{X}'$. Here the double summation is real analytic and $\theta_m(z)\to\frac{1}{N}\log|\sigma(z)|^2$ uniformly on X_0. The next idea is to estimate the Hessian form of $z\mapsto e^{\lambda\theta_m(z)}$ for every $\lambda>0$ fixed. We have of course

$$e^{\lambda\theta_m(z)}=\Bigg(\sum_{\alpha\in\mathbb{N}^n}|\xi_\alpha(z)|^2\Bigg)^{\lambda/m}=\|\xi(z)\|^{2\lambda/m}=\langle\xi(z),\xi(z)\rangle^{\lambda/m}$$

where ξ is nothing else than the expression of the section $\widetilde{f}_1^{(m)}$ is the (real analytic) trivialization of the Bergman bundle B_ε, and $\langle\bullet,\bullet\rangle$ the natural hermitian metric on B_ε. Now, a standard calculation with respect to the Bergman connection $\nabla=\nabla^{1,0}+\nabla^{0,1}$ of B_ε yields $\nabla^{0,1}\xi=0$, hence

$$i\partial\overline{\partial}e^{\lambda\theta_m(z)}=\frac{\lambda}{m}\|\xi\|^{2\lambda/m-2}\left(i\langle\nabla^{1,0}\xi,\nabla^{1,0}\xi\rangle-\langle i\Theta_{B_\varepsilon}\xi,\xi\rangle-\Big(1-\frac{\lambda}{m}\Big)\frac{i\langle\nabla^{1,0}\xi,\xi\rangle\wedge\overline{\langle\nabla^{1,0}\xi,\xi\rangle}}{\|\xi\|^2}\right)$$

$$\geq-\frac{\lambda}{m}\|\xi\|^{2\lambda/m-2}\langle i\Theta_{B_\varepsilon}\xi,\xi\rangle \tag{3.13}$$

by the Cauchy-Schwarz inequality. On the other hand, the curvature bound obtained in §2 yields

$$\big|\langle i\Theta_{B_\varepsilon}\xi,\xi\rangle\big|_\omega\le 2\varepsilon^{-2}\sum_{\alpha\in\mathbb{N}^n}(|\alpha|+n)|\xi_\alpha|^2$$

$$\le 2\varepsilon^{-2}\Bigg(\sum_{\alpha\in\mathbb{N}^n}|\xi_\alpha|^2\Bigg)^{1-\lambda/m}\Bigg(\sum_{\alpha\in\mathbb{N}^n}(|\alpha|+n)^{m/\lambda}|\xi_\alpha|^2\Bigg)^{\lambda/m} \tag{3.14}$$

by the discrete Hölder inequality. We compare the last summation to (3.12) by taking the maximum of $t\mapsto(t+n)^{m/\lambda}\rho^{-2t}$ which is reached for $t_0+n=\frac{m}{2\lambda}(\log\rho)^{-1}$ by an elementary calculation. This gives

$$\begin{aligned}\Big(\sum_{\alpha\in\mathbb{N}^n}(|\alpha|+n)^{m/\lambda}|\xi_\alpha|^2\Big)^{\lambda/m}&\le(t_0+n)\rho^{-t_0\lambda/m}\Big(\sum_{\alpha\in\mathbb{N}^n}\rho^{2|\alpha|}|\xi_\alpha(z)|^2\Big)^{\lambda/m}\\ &\le\frac{m}{2\lambda}(\log\rho)^{-1}\rho^{n\lambda/m}e^{C_6\lambda-1/2}.\end{aligned}\tag{3.15}$$

A combination of the last three estimates (3.13–3.15) gives

$$i\partial\overline{\partial}e^{\lambda\theta_m(z)}\ge-\varepsilon^{-2}(\log\rho)^{-1}\rho^{n\lambda/m}e^{C_6\lambda-1/2}\omega.\tag{3.16}$$

The above lower bound is uniform with respect to m as $m\to+\infty$, and we see that $e^{\lambda\theta_m}$ is quasi plurisubharmonic. By well known facts of pluripotential theory, there exists an upper semicontinuous regularization

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and a subsequence $m\in M_0\subset\mathbb{N}$ such that

$$\theta=\limsup_{\substack{m\in M_0\\ m\to+\infty}}\theta_m\quad\text{almost everywhere on }\mathcal{X}',$$

and satisfying in the limit the Hessian estimate

$$i\partial\overline{\partial}e^{\lambda\theta}\ge-\varepsilon^{-2}(\log\rho)^{-1}e^{C_6\lambda-1/2}\omega\tag{3.17}$$

for every $\lambda>0$. Property (c) is proved. □

3.18. What to do ?. Our hope (possibly after modifying the sections $f_j^{(m)}$ in an adequate manner) is that one can gain a factor converging to zero in estimate (3.16), and thus in (3.17). This would prove that $e^{\lambda\theta}$ is plurisubharmonic for every $\lambda>0$, hence the weight $\theta=\lim_{\lambda\to0}\frac{1}{\lambda}(e^{\lambda\theta}-1)$ would also be plurisubharmonic.

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$$\|\sigma\|^2_{\omega\otimes\eta}\le C|\sigma|^2\exp\Big(-(N-1)\theta-\frac{1}{N}\sum\varphi_j\Big)\le C\Big(|\sigma|^2\prod e^{-\varphi_j}\Big)^{1/N}$$

on X_0. Since $|\sigma|^2e^{-\varphi_0}$ and $e^{-\varphi_r}$, $r>0$ are all locally integrable, we see that $\|\sigma\|^2_{\omega\otimes\eta}$ is also locally integrable on X_0 by the Hölder inequality. A new (and final) application of the L^2 extension theorem to the hermitian line bundle $(\mathcal{E},\eta)$ implies that σ can be extended to $\mathcal{X}'$. Theorem 3.2 is proved. □

4. Invariance of plurigenera for polarized Kähler families

The goal of this section is to prove that for every polarized family $\mathcal{X} \to S$ of compact Kähler manifolds, the plurigenera $p_m(X_t) = h^0(X_t, mK_{X_t})$ of fibers are independent of t for all $m \geq 0$. This result has first been proved by Y.T. Siu [Siu98] in the case of projective varieties of general type (in which case the proof has been translated in a purely algebraic form by Y. Kawamata [Kaw99]), and then by [Siu00] and Păun [Pau04] in the case of arbitrary projective varieties; in the nonrestricted projective case, no algebraic proof of the result is known. We extend here the result to the Kähler context. This requires substantial modifications of the proof, since the technique of Siu and Păun involved in a crucial manner the use of an auxiliary ample line bundle. We replace it here by a use of the Hilbert bundles studied in the previous section.

4.1. Theorem. *Let $\pi : \mathcal{X} \to S$ be a proper holomorphic map defining a family of smooth compact Kähler manifolds over an irreducible base S. Assume that π admits local polarizations, i.e. every point $s_0 \in S$ has a neighborhood V such that $\pi^{-1}(V)$ carries a closed smooth $(1,1)$-form ω for which $\omega_{|X_t}$ is positive definite on $X_t := \pi^{-1}(t)$, $t \in V$. Then the plurigenera $p_m(X_t) = h^0(X_t, mK_{X_t})$ of fibers are independent of t for all $m \geq 0$.*

The above statement follows directly from the following more technical result.

4.2. Theorem (generalized version of the Claudon-Păun theorem). *Let $\pi : \mathcal{X} \to \Delta$ be a polarized family of compact Kähler manifolds over a disc $\Delta \subset \mathbb{C}$, and let $(\mathcal{L}_j, h_j)_{0 \leq j \leq N-1}$ be (singular) hermitian line bundles with semi-positive curvature currents $i\Theta_{\mathcal{L}_j, h_j} \geq 0$ on $\mathcal{X}$. Assume that*

(a) *the restriction of h_j to the central fiber X_0 is well defined (i.e. not identically $+\infty$).*

(b) *the multiplier ideal sheaf $\mathcal{I}(h_{j|X_0})$ is trivial for $1 \leq j \leq N-1$.*

Then any section σ of $\mathcal{O}(mK_{\mathcal{X}} + \sum \mathcal{L}_j)_{|X_0} \otimes \mathcal{I}(h_{0|X_0})$ over the central fiber X_0 extends into a section $\widetilde{\sigma}$ of $\mathcal{O}(mK_{\mathcal{X}} + \sum \mathcal{L}_j)$ over a certain neighborhood $\mathcal{X}' = \pi^{-1}(\Delta')$ of X_0, where $\Delta' \subset \Delta$ is a sufficienty small disc centered at 0.

4.3. Remark. A standard cohomological argument shows that we can in fact take $\mathcal{X}' = \mathcal{X}$ in the conclusion of Theorem 4.2, because the direct image sheaf $\mathcal{E} = \pi_* \mathcal{O}(mK_{\mathcal{X}} + \sum \mathcal{L}_j)$ is coherent, and the restriction $\mathcal{E} \to \mathcal{E} \otimes (\mathcal{O}_\Delta / \mathfrak{m}_0 \mathcal{O}_\Delta)$ induces a surjective map at the H^0 level on the Stein space Δ, so we can extend $\widetilde{\sigma}$ mod $\pi^* \mathfrak{m}_0$ to $\mathcal{X}$.

Proof of Theorem 4.1. The invariance of plurigenera is in fact just obtained as the special case of Theorem 4.2 when all line bundles $\mathcal{L}_j$ and their metrics h_j are trivial. Since the dimension $t \mapsto h^0(X_t, mK_{X_t})$ is always upper semicontinuous and since Theorem 4.2 implies the lower semicontinuity, we conclude that the dimension must be constant along analytic discs, hence along the irreducible base S, by joining any two points through a chain of analytic discs. □

4.4. Lemma. *Let $\mathcal{X}' = \pi^{-1}(\Delta') \to \Delta'$ be the restriction of $\pi : \mathcal{X} \to \Delta$ to a disc $\Delta' \Subset \Delta$ centered at 0, of radius $r' \leq r$. For $\varepsilon \leq \varepsilon_0 = \varepsilon_0(r')$ small enough, one can find a Stein open subset $\mathcal{U}'_\varepsilon \subset \mathcal{X}' \times \overline{\mathcal{X}}$, where such that the projection $\mathrm{pr}_1 : \mathcal{U}'_\varepsilon \to \mathcal{X}'$ is a complex ball bundle over $\mathcal{X}'$ that is locally trivial real analytically.*

Proof. The arguments are very similar to those of §1, except for the fact that $\mathcal{X}$ is no longer compact, but this is not a problem since $\mathcal{X} \to \Delta$ is proper, and since we can always shrink Δ a little bit to achieve uniform bounds (would they be needed). Let γ be a real analytic hermitian metric on $\mathcal{X}$, and $\mathrm{exph} : T_{\mathcal{X}} \to \mathcal{X}$ be the corresponding real analytic and fiber-holomorphic exponential map associated with γ, as in §1. The map exph is no longer everywhere defined, but if we restrict it to the ε-tubular neighborhood of the zero section in $T_{\mathcal{X}'}$, we get for $\varepsilon > 0$ small enough a real analytic diffeomorphism $(z, \xi) \mapsto (z, \mathrm{exph}_z(\xi))$ onto a tubular neighborhood of the diagonal of $\mathcal{X}' \times \mathcal{X}'$. The rest of the proof is identical to what we did in §1, taking

$$\mathcal{U}'_\varepsilon = \big\{(z, w) \in \mathcal{X}' \times \overline{\mathcal{X}}\,;\ |\,\mathrm{logh}_z(\overline{w})|_\gamma < \varepsilon\big\}. \tag{4.5}$$ □

In order to study Theorem 4.2, we first state a technical extension theorem needed for the proof, which is a special case of the well-known and extremely powerful Ohsawa-Takegoshi theorem [OT87], see also [Oh??], [Dem??].

4.6. Proposition. *Let $\pi : \mathcal{Z} \to \Delta$ be a smooth and proper morphism from a (non compact) Kähler manifold $\mathcal{Z}$ to a disc $\Delta \subset \mathbb{C}$ and let $(\mathcal{L}, h)$ be a (singular) hermitian line bundle with semi-positive curvature current $\mathrm{i}\Theta_{\mathcal{L},h} \geq 0$ on $\mathcal{Z}$. Let ω be a global Kähler metric on $\mathcal{Z}$, and let $dV_{\mathcal{Z}}$, dV_{Z_0} the respective induced volume elements on $\mathcal{Z}$ and $Z_0 = \pi^{-1}(0)$. Assume that h_{Z_0} is well defined (i.e. almost everywhere finite). Then any holomorphic section s of $\mathcal{O}(K_{\mathcal{Z}} + \mathcal{L}) \otimes \mathcal{I}(h_{|Z_0})$ extends into a section $\widetilde{s}$ over $\mathcal{Z}$ satisfying an L^2 estimate*

$$\int_{\mathcal{Z}} \|\widetilde{s}\|^2_{\omega\otimes h} dV_{\mathcal{Z}} \leq C_0 \int_{Z_0} \|s\|^2_{\omega\otimes h} dV_{Z_0},$$

where $C_0 \geq 0$ is some universal constant (depending on $\dim \mathcal{Z}$ and $\operatorname{diam} \Delta$, but otherwise independent of $\mathcal{Z}$, $\mathcal{L}$, ...).

4.7. Remark. The assumptions of Proposition 4.6 imply that $\mathcal{Z}$ is holomorphically convex and complete Kähler, thus the technique of [Dem??] does apply to yield the result.

Proof of Theorem 4.2. Let $p = \mathrm{pr}_1 : \mathcal{U}'_\varepsilon \to \mathcal{X}'$ be as in Lemma 4.4, and $q = \mathrm{pr}_2 : \mathcal{U}'_\varepsilon \to \overline{\mathcal{X}}$. We take $\varepsilon < \varepsilon_0$ and use on $\mathcal{Z} := \mathcal{U}'_\varepsilon$ a Kähler metric ω_0 defined on the Stein manifold $\mathcal{U}'_{\varepsilon_0}$. On can define e.g. ω_0 as the $i\partial\overline{\partial}$ of a strictly plurisubharmonic exhaustion function on $\mathcal{U}'_{\varepsilon_0}$, but we can also take the restriction of $\mathrm{pr}_1^*\,\omega + \mathrm{pr}_2^*\,\overline{\omega}_{|\overline{\mathcal{X}}}$ where ω is the Kähler metric on the total space $\mathcal{X}$, and $\overline{\omega} = -\,\omega$ the corresponding Kähler metric on the conjugate space $\overline{\mathcal{X}}$.

First step: construction of a sequence of extensions on $\mathcal{Z} = \mathcal{U}'_\varepsilon$ via the Ohsawa-Takegoshi extension theorem.

The strategy is to apply iteratively the special case 4.6 of the Ohsawa-Takegoshi extension theorem on the total space of the fibration

$$\pi' = \pi \circ p : \mathcal{Z} = \mathcal{U}'_\varepsilon \to \mathcal{X}' \to \Delta',$$

and to extend sections of ad hoc pull-backs $p^*\mathcal{G}$ from the zero fiber $Z_0 = \pi'^{\,-1}(0) = p^{-1}(X_0)$ to the whole of $\mathcal{Z} = \mathcal{U}'_\varepsilon$. We write $h_j = e^{-\varphi_j}$ in terms of local plurisubharmonic weights, and define inductively a sequence of line bundles $\mathcal{G}_m$ by putting $\mathcal{G}_0 = \mathcal{O}_{\mathcal{X}'}$ and

$$\mathcal{G}_m = \mathcal{G}_{m-1} + K_{\mathcal{X}'} + \mathcal{L}_r \qquad \text{if } m = Nq + r,\ \ 0 \leq r \leq N - 1.$$

By construction we have

$$\begin{aligned}\mathcal{G}_m &= mK_{\mathcal{X}'} + \mathcal{L}_1 + \cdots + \mathcal{L}_m, &&\text{for } 1 \le m \le N-1\,,\\ \mathcal{G}_{m+N} - \mathcal{G}_m = \mathcal{G}_N &= NK_{\mathcal{X}'} + \mathcal{L}_0 + \cdots + \mathcal{L}_{N-1}\,, &&\text{for all } m \ge 0.\end{aligned}$$

The game is to construct inductively families of sections, say $\{\widetilde{f}_j^{(m)}\}_{j=1,\ldots,J(m)}$, of $p^*\mathcal{G}_m$ over $\mathcal{Z}$, together with ad hoc L^2 estimates, in such a way that

(4.8) for $m = 0, \ldots, N-1$, $p^*\mathcal{G}_m$ is generated by L^2 sections $\{\widetilde{f}_j^{(m)}\}_{j=1,\ldots,J(m)}$ on $\mathcal{U}'_{\varepsilon_0}$;

(4.9) we have the m-periodicity relations $J(m+N) = J(m)$ and $\widetilde{f}_j^{(m)}$ is an extension of $f_j^{(m)} := (p^*\sigma)^q f_j^{(r)}$ over $\mathcal{Z}$ for $m = Nq + r$, where $f_j^{(r)} := \widetilde{f}^{(r)}_{j|Z_0}$, $0 \le r \le N-1$.

Property (4.8) can certainly be achieved since $\mathcal{U}'_{\varepsilon_0}$ is Stein, and for $m = 0$ we can take $J(0) = 1$ and $\widetilde{f}_1^{(0)} = 1$. Now, by induction, we equip $p^*\mathcal{G}_{m-1}$ with the tautological metric $|\xi|^2/\sum_\ell |\widetilde{f}_\ell^{(m-1)}(x)|^2$, and

$$\widetilde{\mathcal{G}}_m := p^*\mathcal{G}_m - K_{\mathcal{Z}} = p^*\mathcal{G}_m - (p^*K_{\mathcal{X}'} + q^*K_{\overline{\mathcal{X}}}) = p^*(\mathcal{G}_{m-1} + \mathcal{L}_r) - q^*K_{\overline{\mathcal{X}}}$$

with that metric multiplied by $p^*h_r = e^{-p^*\varphi_r}$ and a fixed smooth metric $e^{-\psi}$ of positive curvature on $(-q^*K_{\overline{\mathcal{X}}})_{|\mathcal{U}'_{\varepsilon_0}}$ (remember that $\mathcal{U}'_{\varepsilon_0}$ is Stein!). It is clear that these metrics have semi-positive curvature currents on $\mathcal{Z}$ (by adjusting ψ, we could even take them to be strictly positive if we wanted). In this setting, we apply the Ohsawa-Takegoshi theorem to the line bundle $K_{\mathcal{Z}} + \widetilde{\mathcal{G}}_m = p^*\mathcal{G}_m$, and extend in this way $f_j^{(m)}$ into a section $\widetilde{f}_j^{(m)}$ over $\mathcal{Z}$. By construction the pointwise norm of that section in $p^*\mathcal{G}_{m|X_0}$ in a local trivialization of the bundles involved is the ratio

$$\frac{|f_j^{(m)}|^2}{\sum_\ell |f_\ell^{(m-1)}|^2} e^{-p^*\varphi_r - \psi},$$

up to some fixed smooth positive factor depending only on the metric induced by ω_0 on $K_{\mathcal{Z}}$. However, by the induction relations, we have

$$\frac{\sum_j |f_j^{(m)}|^2}{\sum_\ell |f_\ell^{(m-1)}|^2} e^{-p^*\varphi_r} = \begin{cases} \dfrac{\sum_j |f_j^{(r)}|^2}{\sum_\ell |f_\ell^{(r-1)}|^2} e^{-p^*\varphi_r} & \text{for } m = Nq + r,\ 0 < r \le N-1,\\[2ex] \dfrac{\sum_j |f_j^{(0)}|^2}{\sum_\ell |f_\ell^{(N-1)}|^2} |p^*\sigma|^2 e^{-p^*\varphi_0} & \text{for } m \equiv 0 \bmod N,\ m > 0.\end{cases}$$

Since the sections $\{f_j^{(r)}\}_{0 \le r < N}$ generate their line bundle on $\mathcal{U}_{\varepsilon_0} \supset \overline{\mathcal{U}'_\varepsilon}$, the ratios involved are positive functions without zeroes and poles, hence smooth and bounded [possibly after shrinking a little bit the base disc Δ', as is permitted]. On the other hand, assumption 4.2 (b) and the fact that σ has coefficients in the multiplier ideal sheaf $\mathcal{I}(h_{0|X_0})$ tell us that $e^{-p^*\varphi_r}$, $1 \le r < m$ and $|p^*\sigma|^2 e^{-p^*\varphi_0}$ are locally integrable on Z_0. It follows that there is a constant $C_1 \ge 0$ such that

$$\int_{Z_0} \frac{\sum_j |f_j^{(m)}|^2}{\sum_\ell |f_\ell^{(m-1)}|^2} e^{-p^*\varphi_r - \psi} dV_{\omega_0} \le C_1$$

for all $m \ge 1$ (of course, the integral certainly involves finitely many trivializations of the bundles involved, whereas the integrand expression is just local in each chart). Inductively, the L^2 extension theorem produces sections $\widetilde f_j^{(m)}$ of $p^*\mathcal{G}_m$ over $\mathcal{Z}$ such that

$$\int_{\mathcal{Z}} \frac{\sum_j |\widetilde f_j^{(m)}|^2}{\sum_\ell |\widetilde f_\ell^{(m-1)}|^2} e^{-p^*\varphi_r - \psi} dV_{\omega_0} \le C_2 = C_0C_1.$$

Second step: applying the Hölder inequality. Put $k = Nq(k) + r(k)$ with $0 \le r(k) < N$, and take $m = Nq(m)$ to be a multiple of N. The Hölder inequality $|\int \prod_{1\le k\le m} u_k d\mu| \le \prod_{1\le k\le m}(\int |u_k|^m d\mu)^{1/m}$ applied to the measure $\mu = dV_{\omega_0}$ and to the product of functions

$$\left(\frac{\sum_j |\widetilde f_j^{(m)}|^2}{\sum_\ell |\widetilde f_\ell^{(0)}|^2}\right)^{1/m} e^{-\frac{1}{N}p^*(\varphi_0+\ldots+\varphi_{N-1})-\psi} = \prod_{1\le k\le m}\left(\frac{\sum_j |\widetilde f_j^{(k)}|^2}{\sum_\ell |\widetilde f_\ell^{(k-1)}|^2} e^{-p^*\varphi_{r(k)}-\psi}\right)^{1/m}$$

in which $\sum_\ell |\widetilde f_\ell^{(0)}|^2 = |\widetilde f_1^{(0)}|^2 = 1$ and $\sum_j |\widetilde f_j^{(m)}|^2 = |\widetilde f_1^{(m)}|^2$, implies that

$$\int_{\mathcal{Z}} \big|\widetilde f_1^{(m)}\big|^{2/m} e^{-\frac{1}{N}p^*(\varphi_0+\ldots+\varphi_{N-1})-\psi} dV_{\omega_0} \le C_2. \tag{4.10}$$

As the functions $\varphi_{r(k)}$ and ψ are locally bounded from above, we infer from this the weaker inequality

$$\int_{\mathcal{Z}} \big|\widetilde f_1^{(m)}\big|^{2/m} dV_{\omega_0} \le C_2. \tag{4.10'}$$

The last inequality is to be understood as an inequality that holds in fact only locally over $\mathcal{X}'$, on sets of the form $p^{-1}(V)$, where $V \Subset \mathcal{X}'$ are small coordinate open sets where our line bundles are trivial, so that the section $\widetilde f_1^{(m)}$ of $q(m)\, p^*(NK_{\mathcal{X}'} + \sum \mathcal{L}_j)$ can be viewed as a holomorphic function on $p^{-1}(V)$.

Third step: construction of "vertical" holomorphic vector fields. For technical reasons, we need to differentiate the sections $\widetilde f_1^{(m)}$ along the fibers of $p : \mathcal{Z} \to \mathcal{X}'$, i.e. along the extra factor $\overline{\mathcal{X}}$. Since our sections take values in pull-backs $q(m)\, p^*(NK_{\mathcal{X}'} + \sum \mathcal{L}_j)$ (which are trivial along the fibers of p, namely ε-balls in $\overline{\mathcal{X}}$), such "vertical" derivatives make sense and do not require the use of a connection. Since $\mathcal{Z}$ is Stein and $q^*T_{\overline{\mathcal{X}}}$ is a holomorphic subbundle of $T_{\mathcal{Z}}$, we can find global holomorphic vector fields $\tau_1, \ldots, \tau_\nu \in H^0(\mathcal{Z}, q^*T_{\overline{\mathcal{X}}})$ that generate the subbundle $q^*T_{\overline{\mathcal{X}}}$ over the whole of $\mathcal{Z}$; a standard Whitney type argument shows that one needs only $\nu := \dim_{\mathbb{C}} \mathcal{Z} + \operatorname{rank} q^*T_{\overline{\mathcal{X}}} = 3(n+1)$ such vector fields to achieve the global generation property, but we do not need the precise value of ν here. For any finite sequence $\beta = (\beta_1, \ldots, \beta_\ell) \in \{1, \ldots, \nu\}^\ell$ of length $|\beta| = \ell$ and any holomorphic section $f \in H^0(\mathcal{Z}, p^*\mathcal{G})$ of the pull-pack of a line or vector bundle $\mathcal{G} \to \mathcal{X}'$, we define

$$\tau^\beta f = \tau_{\beta_1} \ldots \tau_{\beta_\ell} f \tag{4.11}$$

to be the iterated derivative of f with respect to the vector fields τ_j. This is again a holomorphic section of $p^*\mathcal{G}$ on $\mathcal{Z}$.

Fourth step: construction of a singular hermitian metric on $NK_{\mathcal{X}'} + \sum \mathcal{L}_j$. The rough idea is to extract a weak limit of the m-th root occurring in (4.10), (4.10′), combined with an integration on the fibers of $p : \mathcal{Z} = \mathcal{U}'_\varepsilon \to \mathcal{X}'$, to get a singular hermitian metric on $NK_{\mathcal{X}'} + \sum \mathcal{L}_j$. This is the crucial step in the proof, and the place where the Kähler setup requires new arguments; especially, the integration on fibers makes the weak limit argument much less obvious than in the projective setup, and requires the results of §2 on Bergman bundles. The estimates do not work directly for the given sections $\widetilde{f}_1^{(m)}$, and we have to add a suitable combination of the derivatives to obtain good estimates.

4.12. Proposition. *Assume that the sections* $\widetilde{f}_1^{(m)}$ *and vector fields* τ_j *have been constructed on a slightly bigger tubular neighborhood* $\mathcal{U}'_{\rho\varepsilon} \to \mathcal{X}'$ *with* $\rho > 1$, [*this condition can of course always be achieved, since we have a lot of flexibility on the choice of the tubular neighborhoods, by taking e.g.* $\varepsilon \le \frac{1}{2}\varepsilon_0(r')$]. *Then there exists* $c > 0$ *and a subsequence* $m \in M_0 \subset \mathbb{N}$ *such that the limit*

$$\theta(z) = \lim_{\substack{m\in M_0\\ m\to+\infty}} \frac{1}{m} \log \int_{w\in\mathcal{U}'_{\varepsilon,z}} \sum_{\beta\in\mathbb{N}^\nu} \frac{c^{|\beta|}}{(|\beta|!)^2}\, \big|\tau^\beta \widetilde{f}_1^{(m)}(z,w)\big|^2 dV_{\omega_0}(w), \quad z \in \mathcal{X}'$$

exists almost everywhere on $\mathcal{X}'$, *and* $H = e^{-N\theta}$ *defines a singular hermitian metric of positive curvature current on* $p^*(NK_{\mathcal{X}'} + \sum \mathcal{L}_j)$, *i.e.* $i\partial\overline{\partial}\theta \ge 0$. *Moreover, it satisfies the estimates*

(a) $|\sigma|_H^2 = |\sigma|^2 e^{-N\theta} = 1$ *on* $X_0 \subset \mathcal{X}'$;

(b) $\displaystyle\int_{\mathcal{X}'} e^{-\theta} e^{-\frac{1}{N}(\varphi_0+\ldots+\varphi_{N-1})} dV_\omega < \infty$:

(c) *if* $\rho \in\,]1, \rho''[$, *there is a constant* $C_4 = C_4(\rho) > 0$ *such that for every* $\beta > 0$

$$i\partial\overline{\partial} e^{\beta\theta} \ge -\varepsilon^{-2}(\log\rho)^{-1} e^{C_4\beta}\omega.$$

Proof. First notice that the choice of the Kähler metric ω_0 on $\mathcal{X}' \times \overline{\mathcal{X}}$ is irrelevant in the definition of θ (the L^2 integrals may eventually change by bounded multiplicative factors, which get killed as $m \to +\infty$). We use estimate (4.10′) on $\mathcal{Z} = \mathcal{U}'_{\rho\varepsilon}$, together with the mean value inequality for plurisubharmonic functions, applied on balls of the fibers $\mathcal{U}'_{\rho\varepsilon,z} = p^{-1}(z)$ centered at points $(z, w) \in \mathcal{U}'_\varepsilon$ and of radius $(\rho - 1)\varepsilon/2$ in w. Let us take $\rho' = \frac{\rho+1}{2} \in\,]1, \rho[$. If we view the fibers as standard hermitian balls in the hermitian vector space $(T_{\overline{\mathcal{X}}}, \omega_0)$, we obtain a uniform upper bound

$$\sup_{\mathcal{U}'_{\rho'\varepsilon,z}} |\widetilde{f}_1^{(m)}|^{2/m} \le \frac{C_5}{((\rho-1)\varepsilon)^{2(n+1)}} \int_{\mathcal{U}'_{\rho\varepsilon,z}} |\widetilde{f}_1^{(m)}|^{2/m} dV_{\omega_0}, \quad \forall z \in \mathcal{X}'.$$

Here our sections can be seen as functions only locally over trivializing open sets of the line bundles in $\mathcal{X}'$, but we can arrange that there are only finitely many of these. Now, Cauchy inequalities applied on the fibers give an upper bound

$$\sup_{\mathcal{U}'_{\varepsilon,z}} \frac{1}{|\beta|!} |\tau^\beta g| \le \left(\frac{C_6}{(\rho'-1)\varepsilon)}\right)^{|\beta|} \sup_{\mathcal{U}'_{\rho'\varepsilon}} |g|, \quad C_6 > 0 \tag{4.13}$$

for every holomorphic function g. By integrating over $z \in \mathcal{X}'$, a combination of these estimates implies

$$\int_{(z,w)\in\mathcal{U}'_\varepsilon} \sum_{\beta\in\mathbb{N}^\nu} \frac{c^{|\beta|}}{(|\beta|!)^2} \big|\tau^\beta \widetilde{f}_1^{(m)}\big|^2\, dV_{\omega_0} \le \sum_{\beta\in\mathbb{N}^\nu} \bigg(\frac{c\, C_6^2}{(\rho'-1)^2\varepsilon^2}\bigg)^{|\beta|} \frac{C_5^m}{((\rho-1)\varepsilon)^{2m(n+1)}}.$$

The choice $c = (8C_6^2)^{-1}(\rho-1)^2\varepsilon^2$ yields

$$\int_{(z,w)\in\mathcal{U}'_\varepsilon} \sum_{\beta\in\mathbb{N}^\nu} \frac{c^{|\beta|}}{(|\beta|!)^2} \big|\tau^\beta \widetilde{f}_1^{(m)}\big|^2\, dV_{\omega_0} \le \frac{2^\nu\ C_5^m}{((\rho-1)\varepsilon)^{2m(n+1)}},$$

and the Jensen inequality $\int g\,d\mu \le \log \int e^g d\mu$ for the probability measures $\mu_z = \frac{1}{\mathrm{Vol}(\mathcal{U}'_{\varepsilon,z})} dV_{\omega_0}$ on the fibers implies by Fubini

$$\int_{z\in\mathcal{X}'} \frac{1}{m} \log \int_{w\in\mathcal{U}'_{\varepsilon,z}} \sum_{\beta\in\mathbb{N}^\nu} \frac{c^{|\beta|}}{(|\beta|!)^2} \big|\tau^\beta \widetilde{f}_1^{(m)}\big|^2\, dV_{\omega_0} \le C_7 + 4(n+1)\ \log \frac{1}{(\rho-1)\varepsilon} \tag{4.14}$$

(the factor $4(n+1)$ could be replaced by $2(1+1/m)(n+1)$, but this is irrelevant). By construction $\widetilde{f}_1^{(m)} = (p^*\sigma)^{q(m)}$ on $Z_0 = p^{-1}(X_0)$ and all higher derivatives $\tau^\beta \widetilde{f}_1^{(m)}$ vanish on Z_0, thus we are left with the sum $\frac{1}{m}\log|\widetilde{f}_1^{(m)}|^2 = \frac{1}{N}p^*\log|\sigma|^2$ on Z_0 and $\theta = \frac{1}{N}\log|\sigma|^2$ on X_0. Therefore the limit exists at least on X_0, and estimate (a) is satisfied.

Now, let us consider now the Bergman bundle $B_\varepsilon \to \mathcal{X}'$ where $\dim \mathcal{X}' = n+1$, and write locally over $\mathcal{X}'$

$$\frac{1}{|\beta|!}\ \tau^\beta \widetilde{f}_1^{(m)}(z,w) = \sum_{\alpha\in\mathbb{N}^{n+1}} \xi_{\alpha\beta}(z)\, e_\alpha(z,w) \otimes g(z)^{q(m)}, \quad z\in\mathcal{X}',\ w\in\mathcal{U}'_{\varepsilon,z}$$

in terms of an orthonormal frame $(e_\alpha)_{\alpha\in\mathbb{N}^{n+1}}$ of B_ε as defined in §2, where g is a local generator of $\mathcal{O}_{\mathcal{X}}(NK_{\mathcal{X}'} + \sum \mathcal{L}_j)$. For a suitable smooth (and essentially irrelevant) correcting factor $\widetilde{\psi}(z,w)$ on $\mathcal{U}'_{\varepsilon_0}$ depending only on $|dw_1\wedge\ldots\wedge dw_n|^2_{\omega_0}$, we have an equality

$$\begin{aligned}\theta_m(z) :&= \frac{1}{m}\log \int_{w\in\mathcal{U}'_{\varepsilon,z}} \sum_{\beta\in\mathbb{N}^\nu} \frac{c^{|\beta|}}{(|\beta|!)^2}\ \big|\tau^\beta \widetilde{f}_1^{(m)}(z,w)\big|^2 e^{-\widetilde{\psi}(z,w)} dV_{\omega_0}(w)\\ &= \frac{1}{m}\log\bigg(\sum_{\beta\in\mathbb{N}^\nu} c^{|\beta|} \sum_{\alpha\in\mathbb{N}^{n+1}} |\xi_{\alpha\beta}(z)|^2\bigg) = \frac{1}{m}\log\bigg(\sum_{\beta\in\mathbb{N}^\nu} c^{|\beta|}\|\xi_\beta(z)\|^2\bigg)\end{aligned}$$

where $\xi_\beta = (\xi_{\alpha\beta})_{\alpha\in\mathbb{N}^{n+1}}$ is nothing else than the expression of the section $\tau^\beta\widetilde{f}_1^{(m)}$ is the (real analytic) trivialization of the Bergman bundle B_ε, equipped with its natural hermitian metric $\langle\bullet,\bullet\rangle$, and $\xi = (\xi_\beta)_{\beta\in\mathbb{N}^\nu} \in B_\varepsilon \mathbin{\widehat{\otimes}} \ell^2(\mathbb{N}^{n+1})$ equipped with the metric

$$\|\xi\|^2 = \sum_{\beta\in\mathbb{N}^\nu} c^{|\beta|}\|\xi_\beta(z)\|^2 = \sum_{\beta\in\mathbb{N}^\nu} c^{|\beta|} \sum_{\alpha\in\mathbb{N}^{n+1}} |\xi_{\alpha\beta}(z)|^2.$$

The inequalities (4.13) show that the series converge uniformly over the whole of $\mathcal{X}'$. A standard calculation with respect to the Bergman connection $\nabla = \nabla^{1,0} + \nabla^{0,1}$ of B_ε and the induced connection on $B_\varepsilon \mathbin{\widehat{\otimes}} \ell^2(\mathbb{N}^{n+1})$ yields $\nabla^{0,1}\xi = 0$, hence

$$\begin{aligned} i\partial\overline{\partial}\theta_m &= \frac{1}{m}\ \frac{1}{\|\xi\|^2}\bigg(i\langle\nabla^{1,0}\xi,\nabla^{1,0}\xi\rangle - \langle i\Theta_{B_\varepsilon}\xi,\xi\rangle - \frac{i\langle\nabla^{1,0}\xi,\xi\rangle\wedge\overline{\langle\nabla^{1,0}\xi,\xi\rangle}}{\|\xi\|^2}\bigg)\\ &\ge -\frac{1}{m}\ \frac{\langle i\Theta_{B_\varepsilon}\xi,\xi\rangle}{\|\xi\|^2} \end{aligned} \tag{4.15}$$

by the Cauchy-Schwarz inequality. On the other hand, the curvature bound obtained in §2 yields

$$\big|\langle i\Theta_{B_\varepsilon\,\widehat{\otimes}\,\ell^2(\mathbb{N}^{n+1})}\xi,\xi\rangle\big|_\omega \le (2+O(\varepsilon))\varepsilon^{-2}\sum_{\beta\in\mathbb{N}^\nu} c^{|\beta|}\sum_{\alpha\in\mathbb{N}^{n+1}}(|\alpha|+n)|\xi_{\alpha\beta}|^2$$

$$\le (2+O(\varepsilon))\varepsilon^{-2}\bigg(\sum_{\beta\in\mathbb{N}^\nu} c^{|\beta|}\sum_{\alpha\in\mathbb{N}^{n+1}}|\xi_{\alpha\beta}|^2\bigg)^{1/2}\bigg(\sum_{\beta\in\mathbb{N}^\nu} c^{|\beta|}\sum_{\alpha\in\mathbb{N}^{n+1}}(|\alpha|+n)^2|\xi_{\alpha\beta}|^2\bigg)^{1/2} \tag{4.16}$$

by the Cauchy-Schwarz inequality. However, formula (2.18) shows that

$$\partial_{w_k}\widetilde{e}_\alpha(z,w) = \varepsilon^{-1}\ \sqrt{\alpha_k(|\alpha|+n)}\ \widetilde{e}_{\alpha-c_k}(z,w),$$

thus

$$\sum_k\|\partial_{w_k}\xi_\beta\|^2 = \varepsilon^{-2}\sum_k\sum_{\alpha\in\mathbb{N}^{n+1}}\alpha_k(|\alpha|+n)\ |\xi_{\alpha\beta}|^2 = \varepsilon^{-2}\sum_{\alpha\in\mathbb{N}^{n+1}}|\alpha|(|\alpha|+n)\ |\xi_{\alpha\beta}|^2$$

and

$$\varepsilon^{-2}\sum_{\alpha\in\mathbb{N}^{n+1}}(|\alpha|+n)^2\ |\xi_{\alpha\beta}|^2 \le n^2\,\|\xi_\beta\|^2 + 2\sum_k\|\partial_{w_k}\xi_\beta\|^2 \le C_8\bigg(\|\xi_\beta\|^2+\sum_k\|\tau_k\xi_\beta\|^2\bigg).$$

Since $\tau_k\xi_\beta = \xi_{\beta+c_k}$, we have

$$\sum_{|\beta|=\ell}\sum_k\|\tau_k\xi_\beta\|^2 \le \nu\sum_{|\beta'|=\ell+1}\|\xi_{\beta'}\|^2,$$

therefore

$$\varepsilon^{-2}\sum_{\beta\in\mathbb{N}^\nu}c^{|\beta|}\sum_{\alpha\in\mathbb{N}^{n+1}}(|\alpha|+n)^2|\xi_{\alpha\beta}|^2 \le C_8\big(\|\xi\|^2+\nu c^{-1}\|\xi\|^2\big) \le \frac{C_9}{(\rho-1)^2\varepsilon^2}\ \|\xi\|^2,$$

if we recall the value of $c=(8C_6^2)^{-1}(\rho-1)\varepsilon^2$. By (4.16) we get

$$\big|\langle i\Theta_{B_\varepsilon\,\widehat{\otimes}\,\ell^2(\mathbb{N}^{n+1})}\xi,\xi\rangle\big|_\omega \le \frac{C_{10}}{(\rho-1)\varepsilon^2}\ \|\xi\|^2$$

and in fine, by (4.15), we have

$$i\partial\overline{\partial}\theta_m \ge -\frac{C_{10}}{m(\rho-1)\varepsilon^2}\ \omega. \tag{4.17}$$

Here the constant C_{10} can be seen to depend only on the geometry of $(T_{\mathcal{X}},\omega)$. On the other hand, estimate (4.14) is equivalent to a uniform upper bound

$$\int_{z\in\mathcal{X}'}\theta_m\ dV_{\omega_0} \le C_{11}+4(n+1)\ \log\frac{1}{(\rho-1)\varepsilon}$$

The above lower bound is uniform with respect to m as $m \to +\infty$, and we see that $e^{\lambda\theta_m}$ is quasi plurisubharmonic. By well known facts of pluripotential theory, there exists an upper semicontinuous regularization

$$\theta = \Big(\limsup_{m\to+\infty} \theta_m\Big)^*$$

and a subsequence $m \in M_0 \subset \mathbb{N}$ such that

$$\theta = \limsup_{\substack{m\in M_0\\ m\to+\infty}} \theta_m \quad \text{almost everywhere on } \mathcal{X}',$$

and satisfying in the limit the Hessian estimate

$$i\partial\overline{\partial} e^{\lambda\theta} \geq -\varepsilon^{-2}(\log\rho)^{-1} e^{C_6\lambda - 1/2}\omega \tag{4.17}$$

for every $\lambda > 0$. Property (c) is proved. □

4.18. What to do ?. Our hope (possibly after modifying the sections $f_j^{(m)}$ in an adequate manner) is that one can gain a factor converging to zero in estimate (4.16), and thus in (4.17). This would prove that $e^{\lambda\theta}$ is plurisubharmonic for every $\lambda > 0$, hence the weight $\theta = \lim_{\lambda\to 0} \frac{1}{\lambda}(e^{\lambda\theta} - 1)$ would also be plurisubharmonic.

Fourth step: applying Ohsawa-Takegoshi once again with the singular hermitian metric produced in the third step.

Assuming that 4.18 holds, we have proved that $NK_{\mathcal{X}'} + \sum \mathcal{L}_j$ possesses a hermitian metric $H = e^{-N\theta}$ such that $\|\sigma\|_H \leq 1$ oon X_0 and $\Theta_H \geq 0$ on $\mathcal{X}'$. In order to conclude, we equip the bundle

$$\mathcal{E} = (N-1)K_{\mathcal{X}'} + \sum \mathcal{L}_j$$

with the metric $\eta = H^{1-1/N} \prod h_j^{1/N}$, and $NK_{\mathcal{X}'} + \sum \mathcal{L}_j = K_{\mathcal{X}'} + \mathcal{E}$ with the metric $\omega \otimes \eta$. It is important here that $\mathcal{X}$ possesses a global Kähler polarization ω, otherwise the required estimates would not be valid. Clearly η has a semi-positive curvature current on $\mathcal{X}'$ and in a local trivialization we have

$$\|\sigma\|^2_{\omega\otimes\eta} \leq C|\sigma|^2 \exp\Big(-(N-1)\theta - \frac{1}{N}\sum \varphi_j\Big) \leq C\Big(|\sigma|^2 \prod e^{-\varphi_j}\Big)^{1/N}$$

on X_0. Since $|\sigma|^2 e^{-\varphi_0}$ and $e^{-\varphi_r}$, $r > 0$ are all locally integrable, we see that $\|\sigma\|^2_{\omega\otimes\eta}$ is also locally integrable on X_0 by the Hölder inequality. A new (and final) application of the L^2 extension theorem to the hermitian line bundle $(\mathcal{E}, \eta)$ implies that σ can be extended to $\mathcal{X}'$. Theorem 4.2 is proved. □

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and a uniform upper bound $\theta_m \leq C_9$ on $\mathcal{X}'$. Here the double summation is real analytic and $\theta_m(z) \to \frac{1}{N}\log|\sigma(z)|^2$ uniformly on X_0. The next idea is to estimate the Hessian form of $z \mapsto e^{\lambda\theta_m(z)}$ for every $\lambda > 0$ fixed. We have of course

$$e^{\lambda\theta_m(z)} = \bigg(\sum_{\alpha\in\mathbb{N}^n} |\xi_\alpha(z)|^2\bigg)^{\lambda/m} = \|\xi(z)\|^{2\lambda/m} = \langle \xi(z), \xi(z)\rangle^{\lambda/m}$$

where ξ is nothing else than the expression of the section $\widetilde{f}_1^{(m)}$ is the (real analytic) trivialization of the Bergman bundle B_ε, and $\langle \bullet, \bullet \rangle$ the natural hermitian metric on B_ε. Now, a standard calculation with respect to the Bergman connection $\nabla = \nabla^{1,0} + \nabla^{0,1}$ of B_ε yields $\nabla^{0,1}\xi = 0$, hence

$$i\partial\overline{\partial} e^{\lambda\theta_m(z)} = \frac{\lambda}{m}\|\xi\|^{2\lambda/m-2}\bigg(i\langle\nabla^{1,0}\xi,\nabla^{1,0}\xi\rangle - \langle i\Theta_{B_\varepsilon}\xi,\xi\rangle - \Big(1-\frac{\lambda}{m}\Big)\frac{i\langle\nabla^{1,0}\xi,\xi\rangle\wedge\overline{\langle\nabla^{1,0}\xi,\xi\rangle}}{\|\xi\|^2}\bigg)$$

$$\geq -\frac{\lambda}{m}\|\xi\|^{2\lambda/m-2}\langle i\Theta_{B_\varepsilon}\xi,\xi\rangle \tag{4.13}$$

by the Cauchy-Schwarz inequality. On the other hand, the curvature bound obtained in §2 yields

$$\begin{aligned}\big|\langle i\Theta_{B_\varepsilon}\xi,\xi\rangle\big|_\omega &\leq 2\varepsilon^{-2}\sum_{\alpha\in\mathbb{N}^n}(|\alpha|+n)|\xi_\alpha|^2\\ &\leq 2\varepsilon^{-2}\Big(\sum_{\alpha\in\mathbb{N}^n}|\xi_\alpha|^2\Big)^{1-\lambda/m}\Big(\sum_{\alpha\in\mathbb{N}^n}(|\alpha|+n)^{m/\lambda}|\xi_\alpha|^2\Big)^{\lambda/m}\end{aligned} \tag{4.14}$$

by the discrete Hölder inequality. We compare the last summation to (4.12) by taking the maximum of $t\mapsto(t+n)^{m/\lambda}\rho^{-2t}$ which is reached for $t_0+n=\frac{m}{2\lambda}(\log\rho)^{-1}$ by an elementary calculation. This gives

$$\begin{aligned}\Big(\sum_{\alpha\in\mathbb{N}^n}(|\alpha|+n)^{m/\lambda}|\xi_\alpha|^2\Big)^{\lambda/m} &\leq (t_0+n)\rho^{-t_0\lambda/m}\Big(\sum_{\alpha\in\mathbb{N}^n}\rho^{2|\alpha|}|\xi_\alpha(z)|^2\Big)^{\lambda/m}\\ &\leq \frac{m}{2\lambda}(\log\rho)^{-1}\rho^{n\lambda/m}e^{C_6\lambda-1/2}.\end{aligned} \tag{4.15}$$

A combination of the last three estimates (4.13–4.15) gives

$$i\partial\overline{\partial}e^{\lambda\theta_m(z)} \geq -\varepsilon^{-2}(\log\rho)^{-1}\rho^{n\lambda/m}e^{C_6\lambda-1/2}\omega. \tag{4.16}$$

The above lower bound is uniform with respect to m as $m\to+\infty$, and we see that $e^{\lambda\theta_m}$ is quasi plurisubharmonic. By well known facts of pluripotential theory, there exists an upper semicontinuous regularization

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$$\|\sigma\|^2_{\omega\otimes\eta} \le C|\sigma|^2 \exp\Big(-(N-1)\theta - \frac{1}{N}\sum \varphi_j\Big) \le C\Big(|\sigma|^2 \prod e^{-\varphi_j}\Big)^{1/N}$$

on X_0. Since $|\sigma|^2 e^{-\varphi_0}$ and $e^{-\varphi_r}$, $r > 0$ are all locally integrable, we see that $\|\sigma\|^2_{\omega\otimes\eta}$ is also locally integrable on X_0 by the Hölder inequality. A new (and final) application of the L^2 extension theorem to the hermitian line bundle $(\mathcal{E}, \eta)$ implies that σ can be extended to $\mathcal{X}'$. Theorem 4.2 is proved. □

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