ALGEBRAIC REPRESENTATION OF SMOOTH ALMOST COMPLEX STRUCTURES

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1. Introduction and result

The goal of this work is to prove an embedding theorem for compact almost complex manifolds into complex algebraic varieties. As usual, an almost complex manifold of dimension n is a pair (X, J), where X is a real manifold of dimension $2n$ and J_X a smooth section of $\mathrm{End}(TX)$ such that $J_X^2 = -\mathrm{Id}$; we will assume here that all data are $\mathcal{C}^\infty$.

Let Z be a complex (holomorphic) manifold of complex dimension N. Such a manifold carries a natural integrable almost complex structure J_Z (conversely, by the Newlander-Nirenberg thorem any integrable almost complex structure can be viewed as a holomorphic structure). Now, assume that we are given a holomorphic distribution $\mathcal{D}$ in TZ, namely a holomorphic subbundle $\mathcal{D} \subset TZ$. Every fiber $\mathcal{D}_x$ of the distribution is then invariant under J_Z, i.e. $J_Z\mathcal{D}_x \subset \mathcal{D}_x$ for every $x \in Z$. Here, the distribution $\mathcal{D}$ is not assumed to be integrable. We recall that $\mathcal{D}$ is integrable in the sense of Frobenius (i.e. stable under the Lie bracket operation) if and only if the fibers $\mathcal{D}_x$ are the tangent spaces to leaves of a holomorphic a foliation. More precisely, $\mathcal{D}$ is integrable if and only if the torsion operator θ of $\mathcal{D}$, defined by

$$\begin{array}{rcccc} \theta & : & \mathcal{O}(\mathcal{D}) \times \mathcal{O}(\mathcal{D}) & \longrightarrow & \mathcal{O}(TZ/\mathcal{D}) \\ & & (\zeta, \eta) & \longmapsto & [\zeta, \eta] \mod \mathcal{D} \end{array}$$

vanishes identically. As is well known, θ is skew symmetric in (ζ, η) and can be viewed as a holomorphic section of the bundle $\Lambda^2\mathcal{D}^* \otimes (TZ/\mathcal{D})$.

Let M be a real submanifold of Z of class $\mathcal{C}^\infty$ and of real dimension $2n$ with $n < N$. We say that M is *transverse to* $\mathcal{D}$ if

$$\forall x \in M, \quad T_xM \oplus \mathcal{D}_x = T_xZ.$$

We could in fact assume more generally that the distribution $\mathcal{D}$ is singular, i.e. given by a certain satured subsheaf $\mathcal{O}(\mathcal{D})$ of $\mathcal{O}(TZ)$ (this means that the quotient sheaf $\mathcal{O}(TZ)/\mathcal{O}(\mathcal{D})$ has no torsion). Then $\mathcal{O}(\mathcal{D})$ is actually a subbundle of TZ outside an analytic subset $\mathcal{D}_{\mathrm{sing}} \subset Z$ of codimension ≥ 2, and we further assume in this case that $M \cap \mathcal{D}_{\mathrm{sing}} = \emptyset$.

When M is transverse to $\mathcal{D}$, one gets a natural $\mathbb{R}$-linear isomorphism

$$T_xM \simeq T_xZ/\mathcal{D}_x$$

at every point $x \in M$. Since $TZ/\mathcal{D}$ carries a structure of holomorphic vector bundle (at least over $Z \smallsetminus \mathcal{D}_{\mathrm{sing}}$), the complex structure J_Z induces a complex structure on the quotient, and therefore, through the above isomorphism, an almost complex structure $J_M^{Z,\mathcal{D}}$ on M.

Moreover, when $\mathcal{D}$ is a foliation (i.e. $\mathcal{O}(\mathcal{D})$ is an integrable subsheaf), then $J_M^{Z,\mathcal{D}}$ is an integrable almost complex structure on M. In that case, we may indeed realize the foliation associated to $\mathcal{D}$ near any regular point x as the fibers of a certain holomorphic submersion

$Z \supset \Omega \to \Omega' \subset \mathbb{C}^{N-n}$ where Ω is coordinate open set in Z containing x, which we also view as an open subset $\Omega \subset \mathbb{C}^N$. In suitable coordinates, this submersion is the first projection $\Omega \simeq \Omega' \times \Omega'' \to \Omega'$ with respect to $\Omega' \subset \mathbb{C}^n$, $\Omega'' \subset \mathbb{C}^{N-n}$, and then $\mathcal{D}$, $TZ/\mathcal{D}$ are identified with the trivial bundles $\Omega \times (\{0\} \times \mathbb{C}^{N-n})$ and $\Omega \times \mathbb{C}^n$. Then the composition

$$M \cap \Omega \subset \Omega \to \Omega'$$

provides M with holomorphic coordinates on $M \cap \Omega$, and it is clear that any other local trivialization of the foliation on a different chart $\widetilde{\Omega} = \widetilde{\Omega}' \times \widetilde{\Omega}''$ would give coordinates that are changed by local biholomorphisms $\Omega' \to \widetilde{\Omega}'$ in the intersection $\Omega \cap \widetilde{\Omega}$, thanks to the holomorphic character of $\mathcal{D}$. Thus we directly see in that case that $J_M^{Z,\mathcal{D}}$ comes from a holomorphic structure on M.

More generally, we say that $f : X \hookrightarrow Z$ is a transverse embedding of a smooth real manifold X in $(Z, \mathcal{D})$ if $M = f(X)$ is a transverse submanifold of Z, namely if $f_* T_x X \oplus \mathcal{D}_{f(x)} = T_{f(x)} Z$ for every point $x \in X$ (and $f(X)$ does not meet $\mathcal{D}_{\rm sing}$ in case there are singularities). One then gets a real isomorphism $TX \simeq f^*(TZ/\mathcal{D})$ and therefore an almost complex structure on X (for this it would be enough to assume that f is an immersion, but we will actually suppose that f is an embedding here).

A very interesting problem investigated about 20 years ago by F. Bogomolov is whether a given integrable complex structure on a compact manifold X can be realized by a transverse embedding $f : X \hookrightarrow Z$ into a projective manifold $(Z, \mathcal{D})$ equipped with an algebraic foliation $\mathcal{D}$, in such a way that $f(X) \cap \mathcal{D}_{{\rm sing}\,D} = \emptyset$, as described above. There are many examples of non Kähler compact complex manifolds which can be embedded in that way (the case of projective ones being of course trivial), and strong indications exist that every compact complex manifold should in fact be embeddable as a smooth submanifold transverse to an algebraic foliation on a complex projective variety. We prove here the corresponding statement in the almost complex category actually holds, when dealing of course with non integrable distributions rather than foliations.

Theorem 1.1. *Let (X, J) be a smooth compact almost complex manifold. There exists a complex affine algebraic manifold Z, a smooth embedding $f : X \hookrightarrow Z$ and an algebraic distribution $\mathcal{D}$ transverse to $f(X)$ such that the almost complex structures J and $f^* J_{f(X)}^{Z,\mathcal{D}}$ coincide on X.*

Since Z and $\mathcal{D}$ are algebraic and Z is affine, one can always compactify Z into a complex projective manifold $\overline{Z}$ (using Hironaka's desingularization theorem), and extend $\mathcal{D}$ into a saturated subsheaf of $T\overline{Z}$. In general such distributions will acquire singularities at infinity; getting $\mathcal{D}$ to be nonsingular on Z could possibly be achieved by removing some algebraic hypersurface in Z, but doing so on $\overline{Z}$ seems to be out of reach (if at all possible).

2. Transverse embeddings of almost complex manifolds

We consider the situation described above, where Z is a complex N-dimensional manifold equipped with a holomorphic distribution $\mathcal{D}$. More precisely, let X be a compact real manifold of class $\mathcal{C}^\infty$ and of real dimension $2n$ with $n < N$. We assume that there is an embedding $f : X \hookrightarrow Z$ that is transversal to $\mathcal{D}$, namely that

$$f_* T_x X \oplus \mathcal{D}_{f(x)} = T_{f(x)} Z$$

at every point $x \in X$. Here $\mathcal{D}_{f(x)}$ denotes the fiber at $f(x)$ of the distribution $\mathcal{D}$. As explained in section 1, this induces a $\mathbb{R}$-linear isomorphism $f_* : TX \to f^*(TZ/\mathcal{D})$, and from the complex structures of TZ and $\mathcal{D}$ we get an almost complex structure $f^*J^{Z,\mathcal{D}}_{f(X)}$ on TX which we will simply denote by J_f here.

Near a point $0 \in Z$ we can pick holomorphic coordinates $z = (z_1, \ldots, z_n)$ such that $\mathcal{D}_0 = \mathrm{Span}(\partial/\partial z_j)_{n+1\le j\le N}$ and thus we have

$$\mathcal{D}_z = \mathrm{Span}\left(\frac{\partial}{\partial z_j} + \sum_{1\le i\le n} a_{ij}(z)\frac{\partial}{\partial z_i}\right)_{n+1\le j\le N},\ a_{ij}(0) = 0.$$

In other words $\mathcal{D}_z$ is the set of vectors of the form $(a(z)\eta, \eta) \in \mathbb{C}^n \times \mathbb{C}^{N-n}$, where $a(z) = (a_{ij}(z))$ is a holomorphic map into the space of $n\times(N-n)$ matrices. A change of coordinates $\tilde z_i = z_i + \sum_{j,k} b_{ijk} z_j z_k$ with $b_{ijk} = b_{ikj}$ yields

$$\frac{\partial}{\partial z_j} = \sum_i \frac{\partial \tilde z_i}{\partial z_j}\frac{\partial}{\partial \tilde z_i} = \frac{\partial}{\partial \tilde z_j} + 2\sum_{i,k} b_{ijk} z_k \frac{\partial}{\partial \tilde z_i}.$$

If $a_{ij}(z) = \sum_k a_{ijk} z_k + O(z^2)$ is the first order expansion of the coefficients a_{ij}, changing the coordinates (z_i) by the $(\tilde z_i)$'s replaces a_{ijk} by $\tilde a_{ijk} = a_{ijk} + 2b_{ijk}$. By taking the $(\tilde z_i)$ such that $b_{ijk} = -\frac14(a_{ijk} + a_{ikj})$ whenever $1 \le i \le n$ and $n+1 \le j,k \le N$ and $b_{ijk} = 0$ otherwise, we find $\tilde a_{ijk} = \frac12(a_{ijk} - a_{ikj})$, hence $\tilde a_{ikj} = -\tilde a_{ijk}$. After changing coordinates, we can therefore assume without loss of generality that $a_{ijk} = -a_{ikj}$. A simple calculation shows that the vector fields $\zeta_j = \frac{\partial}{\partial z_j} + \sum_i a_{ij}(z)\frac{\partial}{\partial z_i}$ have brackets equal to

$$[\zeta_j, \zeta_k] = \sum_{1\le i\le n}\left(\frac{\partial a_{ik}}{\partial z_j}(0) - \frac{\partial a_{ij}}{\partial z_k}(0)\right)\frac{\partial}{\partial z_i} = -2\sum_{1\le i\le n} a_{ijk}\frac{\partial}{\partial z_i} \text{ at } 0,\ n+1 \le j,k \le N,$$

hence

$$\theta(0) = -2\sum_{1\le i\le n,\ n+1\le j,k\le N} a_{ijk}\, dz_j \wedge dz_k \otimes \frac{\partial}{\partial z_i}. \tag{2.1}$$

If Z is Stein, there exists a bilohomorphism $\Phi : TZ \to Z \times Z$ from a neighborhood of the section section of TZ to a neighborhood of the diagonal in $Z \times Z$, such that $\Phi(z, 0) = z$ and $d_\zeta\Phi(z,\zeta)_{|\zeta=0} = \mathrm{Id}$ on T_zZ. When Z is embedded in $\mathbb{C}^{N'}$ for some N', such a map can be obtained by taking $\Phi(z,\zeta) = \rho(z+\zeta)$, where ρ is a local holomorphic retraction $\mathbb{C}^{N'} \to Z$ and T_zZ is identified to a vector subspace of $\mathbb{C}^{N'}$. In general (i.e. when Z is not necessarily Stein), one can still find a $\mathcal{C}^\infty$ map Φ satisfying the same conditions, by taking e.g. $\Phi(z,\zeta) = (z, \exp_z(\zeta))$, where exp is the Riemannian exponential map of a smooth hermitian metric on Z; actually, we will not need Φ to be holomorphic in what follows. For any $r \in [1, +\infty]$, let us consider the group $\mathrm{Diff}^r_0(X)$ of diffeomorphisms of class C^r isotopic to identity in X, and let $\Gamma^r(X, Z, \mathcal{D})$ be the space of $\mathcal{C}^r$ embeddings of X into Z that are transverse to $\mathcal{D}$ (an open condition for $r \ge 1$, so this is an open subset in the space $\mathcal{C}^r(X, Z)$ of all $\mathcal{C}^r$ maps $X \to Z$ with $\mathcal{C}^r$ convergence topology).

Lemma 2.1. *For $r < +\infty$, $\mathrm{Diff}^r_0(X)$ is a Banach Lie group with Lie algebra $\mathcal{C}^r(X, TX)$, and $\Gamma^r(X, Z, \mathcal{D})$ is a Banach manifold whose tangent space at f is $\mathcal{C}^r(X, f^*TZ)$.*

Proof. A use of the map Φ allows us to parametrize small deformations of the embedding f as $\widetilde f(x) = \Phi(f(x), u(x))$ [or equivalently $u(x) = \Phi^{-1}(f(x), \widetilde f(x))$], where u is a smooth sufficiently small section of f^*TZ. This parametrization is one-to-one, and $\widetilde f$ is $\mathcal{C}^r$ if and only if u is $\mathcal{C}^r$ (provided f is). The argument is similar, and very well known indeed, for $\mathrm{Diff}_0^r(X)$. □

Now, $\mathrm{Diff}_0^r(X)$ acts on $\Gamma^r(X, Z, \mathcal{D})$ through the natural right action

$$\Gamma^r(X, Z, \mathcal{D}) \times \mathrm{Diff}_0^r(X) \longrightarrow \Gamma^r(X, Z, \mathcal{D}), \qquad (f, \psi) \mapsto f \circ \psi,$$

and the differential of this action at $\psi = \mathrm{Id}_X$ consists of moving an embedding f as $\Phi(f, f_*v)$ where $v \in \mathcal{C}^r(X, TX)$ and $f_*v \in \mathcal{C}^r(X, f^*TZ)$. By the transversality condition, we can uniquely decompose a section $u \in \mathcal{C}^r(X, f^*TZ)$ as $u = f_*v + w$ where $v \in \mathcal{C}^r(X, TX)$ and $w \in \mathcal{C}^r(X, f^*\mathcal{D})$. Therefore, modulo composition by elements of $\mathrm{Diff}_0^r(X)$ close to identity (i.e. in the quotient space $\Gamma^r(X, Z, \mathcal{D})/\,\mathrm{Diff}_0^r(X)$), small deformations of f are parametrized by $\Phi(f, u)$ where u is a small section of $\mathcal{C}^r(X, f^*\mathcal{D})$. The first variation of f depends only on the differential of Φ along the zero section of TZ, so it is actually independent of the choice of our map Φ. We can think of small variations of f as $f + u$, at least if we are working in local coordinates $(z_1, \ldots, z_N) \in \mathbb{C}^N$ on Z, and consider that $\mathcal{D}_z \subset T_zZ = \mathbb{C}^N$; the use of a map Φ like those already considered is however needed to make the arguments global. Let us summarize these observations as follows.

Lemma 2.2. *For $r < +\infty$, the quotient space $\Gamma^r(X, Z, \mathcal{D})/\,\mathrm{Diff}_0^r(X)$ is a Banach manifold whose tangent space at f can be identified with $\mathcal{C}^r(X, f^*\mathcal{D})$.*

Our next goal is to compute J_f and the differential dJ_f of $f \mapsto J_f$ on the above Banach manifold. Fix a point $z_0 = f(x_0) \in Z$. We use local holomorphic coordinates $(z_1, \ldots, z_N)$ on Z centered at z_0, so that $z_0 = 0$ and $\mathcal{D}_0 = \mathrm{Span}(\partial/\partial z_j)_{n+1 \le j \le N}$. Let us write f as a graph in the coordinates $z = (x, y) \in \mathbb{C}^n \times \mathbb{C}^{N-n}$ and let us use $x = (z_1, \ldots, z_n)$ as the local coordinates on X. We can then write

$$f(x) = (x, g(x))$$

near $x_0 = 0$, where g is $\mathcal{C}^r$ differentiable and $g(0) = 0$. With respect to the (x, y) coordinates, we get a $\mathbb{R}$-linear isomorphism given by the differential df

$$\begin{array}{rcccl} df(x) & : & T_xX & \longrightarrow & f_*T_xX \subset T_xZ \\ & & \zeta & \longmapsto & (\zeta, dg(x).\zeta) = (\zeta, \partial g(x) \cdot \zeta + \overline{\partial} g(x) \cdot \zeta). \end{array}$$

Here $\overline{\partial} g$ is defined with respect to the standard structure of $\mathbb{C}^n \ni 0$ and has a priori no intrinsic meaning. The almost complex structure J_f can be explicitely defined by

$$J_f(x) = df(x)^{-1} \circ p_{f,x,Z,\mathcal{D}} \circ J_Z(f(x)) \circ df(x),$$

where $p_{f,x,Z,\mathcal{D}} : T_{f(x)}Z \to f_*T_xX$ denotes the $\mathbb{R}$-linear projection to f_*T_xX along $\mathcal{D}_{f(x)}$, and J_Z the complex structure on Z. Since this formula depends on the first derivatives of f, we see that J_f is of class $\mathcal{C}^{r-1}$ on X.

Using the identifications $T_xX \simeq \mathbb{C}^n$, $T_zZ \simeq \mathbb{C}^N$ given by the above choice of coordinates, we simply have $J_Z\eta = i\eta$ on TZ since the (z_j) are holomorphic, and we get therefore

$$\begin{aligned} J_Z(f(x)) \circ df(x) \cdot \zeta &= idf(x) \cdot \zeta = i(\zeta, dg(x) \cdot \zeta) = (i\zeta, \partial g(x) \cdot i\zeta - \overline{\partial} g(x) \cdot i\zeta) \\ &= (i\zeta, dg(x) \cdot i\zeta) - 2(0, \overline{\partial} g(x) \cdot i\zeta). \end{aligned}$$

The first term is already in f_*T_xX and the second term satisfies

$$(0, \overline{\partial} g(x) \cdot i\zeta) = -(a(f(x))\overline{\partial} g(x) \cdot i\zeta, 0) = (ia(f(x))\overline{\partial} g(x) \cdot \zeta, 0) \bmod \mathcal{D}_{f(x)}.$$

We find in this way

$$J_f(x)\cdot\zeta = i\zeta - 2df(x)^{-1}\circ p_{f,x,Z,\mathcal{D}}(ia(f(x))\overline{\partial}g(x)\cdot\zeta,0). \tag{2.2}$$

In particular, since $a(0)=0$, we have $J_f(x_0)\cdot\zeta = i\zeta$.

We want to evaluate the variation of the almost complex structure J_f when the embedding f varies. In local coordinates a nearby embedding can be expressed as $f+tu$ for some section of $f^*\mathcal{D}$ with $t\in\mathbb{C}$ small. By replacing f with $f+tu$ in (2.2) and taking the derivative in t, we find

$$dJ_f(u)(x_0)\cdot\zeta = -2df(x_0)^{-1}\circ p_{f,x,Z,\mathcal{D}}(ida(z_0)(u(x_0))\overline{\partial}g(x_0)\cdot\zeta,0)$$

since $a(0)=0$. Now, if we put $\theta = ida(z_0)(u(x_0))\overline{\partial}g(x_0)\cdot\zeta$, as $\mathcal{D}_0=\{0\}\times\mathbb{C}^{N-n}$ in our coordinates, we immediately get

$$p_{f_*T_{X,f(x)},\mathcal{D}_{f(x)}}(\theta,0) = (\theta, dg(0)\cdot\theta),\ \ df(x_0)^{-1}\circ p_{f,x,Z,\mathcal{D}}(\theta,0)=\theta.$$

Therefore we obtain

$$dJ_f(u)(x_0)\cdot\zeta = -2ida(0)(u(x_0))\overline{\partial}g(x_0)\cdot\zeta,$$

i.e. $dJ_f(u)(x_0) = -2ida(z_0)(u(x_0))\circ\overline{\partial}g(x_0)$. In fact, we compute $\overline{\partial}f : TX\to f^*TZ$ with respect to J_f, we see almost by definition that $\overline{\partial}f(TX)\subset f^*(\mathcal{D})$ and in particular that $\overline{\partial}f(x_0)$ can be identified with $\overline{\partial}g(x_0)$. Also, in our coordinates

$$da(0)_{|\{0\}\times\mathbb{C}^{N-n}} = \sum_{1\le i\le n,\ n+1\le j,k\le N} a_{ijk}dz_k\otimes dz_j\otimes\frac{\partial}{\partial z_i}$$

coincides with $\frac{1}{2}\theta(z_0)$ by (2.1). We finally get the formula

$$dJ_f(u)(x_0)\cdot\zeta = -i\theta(f(x_0))(u(x_0),\overline{\partial}f(x_0)\cdot\zeta),$$

i.e.

$$dJ_f(u) = -if^*\theta(u,\overline{\partial}f) \tag{2.3}$$

where $\overline{\partial}$ is computed with respect to the almost complex structures (X,J_f) and (Z,J_Z).

Let $\mathcal{J}^r(X)$ the space of almost complex structures of class $\mathcal{C}^r$ on X. For $r<+\infty$, this is a Banach manifold whose tangent space at a point J is the space of sections $h\in\mathcal{C}^r(X,\mathrm{End}_{\overline{\mathbb{C}}}(TX))$ satisfying $J\circ h+h\circ J=0$ (namely conjugate $\mathbb{C}$ linear endomorphisms of TX). The above discussion yields

Proposition 2.3. *The map*

$$\Gamma^r(X,Z,\mathcal{D})\longrightarrow\mathcal{J}^{r-1}(X),\qquad f\longmapsto J_f$$

obtained by varying infinitesimally f along a tangent vector $u\in\mathcal{C}^r(X,f^*\mathcal{D})$ has a differential equal to

$$\mathcal{C}^r(X,f^*\mathcal{D})\longrightarrow\mathcal{C}^{r-1}(X,\mathrm{End}_{\overline{\mathbb{C}}}(TX))\qquad u\longmapsto dJ_f(u) = -if^*\theta(u,\overline{\partial}f),$$

where the complex structure on TX is precisely given by J_f, and is also used to compute $\overline{\partial}f$.

The following result can now be obtained as a consequence of the Implicit Function Theorem.

Proposition 2.4. *Fix $r>1$ where r is not an integer. Let $(Z,\mathcal{D})$ be a complex manifold equipped with a holomorphic distribution, and let $f : X\hookrightarrow Z$ be a transverse embedding of class $\mathcal{C}^r$ with respect to $\mathcal{D}$. Assume that f and the torsion tensor θ of $\mathcal{D}$ satisfy the following additional conditions:*

(i) $\overline{\partial} f(x) \in \mathrm{End}_{\overline{\mathbb{C}}}(T_xX, T_{f(x)}Z)$ *is injective at every point* $x \in X$;

(ii) *For every* $y = f(x) \in f(X)$ *and every pair of vectors* $\xi \in \overline{\partial} f(T_xX) \subset T_yZ$, $\xi \neq 0$, *and* $\eta \in (TX/\mathcal{D})_y$, *there exists a vector* $\lambda \in \mathcal{D}_y$ *such that* $\theta(\lambda, \xi) = \eta$.

Then there is a neighborhood $\mathcal{V}$ *of* J_f *in* $\mathcal{J}^{r-1}(X)$ *and a neighborhood* $\mathcal{U}$ *of* f *in* $\mathcal{C}^r(X, Z)$ *such that:*

$$\forall J \in \mathcal{V},\ \exists g \in \mathcal{U} \text{ such that } J = J_g.$$

Proof. **Details to be completed** □

Remark 2.5. (i) Choosing $\mathcal{U}$ sufficiently small, every $g \in \mathcal{U}$ is an embedding and J_g is well defined.

(ii) If *codim* $Ker\ \Lambda_f < \infty$ then Condition (ii) of Proposition 2.4 is satisfied.

(iii) If f is a holomorphic embedding meaning that $X$is a complex manifold with complex structure J_X, then $p_{f_*T_{X,x}}$ is holomorphic and $J_f = J_X$.

(iv) If $\mathcal{D}$ is a foliation then $J_f(x) = df(x)^{-1} \circ J_Z(f(x)) \circ df(x)$.

(v) Under the conditions (iii) or (iv) we have $dJ_f \equiv 0$ and there is no variation of the structure when f varies.

3. Embedding in a twistor space

3.1. **Twistor space.** For $K > 0$ let $\mathcal{T}_{2K}$ denote the twistor space

$$\mathcal{T}_{2K} := \{(x, J) \in \mathbb{R}^{2K} \times End(\mathbb{R}^{2K}) /\ J^2 = -I\}.$$

Then $\mathcal{T}_{2K}$ is a trivial bundle over $\mathbb{R}^{2K}$ and $\mathcal{T}_{2K}$ is endowed with the almost complex structure $\mathcal{J}^{2K}$ defined by:

$$\forall (x, J) \in \mathcal{T}_{2K},\ \forall (\zeta, h) \in T_{(x,J)}\mathcal{T}_{2K},\ \mathcal{J}^{2K}_{(x,J)}(\zeta, h) := (J\zeta, J \circ h).$$

We denote by $\mathcal{T}^{\mathbb{C}}_{2K} := \mathbb{C} \otimes \mathcal{T}_{2K}$ the complexification of $\mathcal{T}_{2K}$. Then $\mathcal{T}^{\mathbb{C}}_{2K} = \{(z, J) \in \mathbb{C}^{2K} \times End(\mathbb{C}^{2K}) /\ J^2 = -I\}$ and $\mathcal{T}^{\mathbb{C}}_{2K}$ inherits two structures:

- The complexification $(\mathcal{J}^{2K})^{\mathbb{C}}$ of $\mathcal{J}^{2K}$,
- The standard complex structure $J_{\mathcal{T}^{\mathbb{C}}_{8n+2}}$.

Denote by $\pi : \mathcal{T}_{2K} \to \mathbb{R}^{2K}$ the projection defined by $\pi(x, J) = x$. Then we have the following decomposition of $T\mathcal{T}_{2K}$:

$$T\mathcal{T}_{2K} = \pi^* T\mathbb{R}^{2K} \oplus Ker(d\pi)$$

where $Ker(d\pi)$ denotes the vertical distribution whose restriction to each fiber of π corresponds to the tangent bundle of that fiber.

If N is a submanifold of $\mathbb{R}^{2K}$ and $J_N : N \to End(\mathbb{R}^{2K})$ is a section of $End(\mathbb{R}^{2K})$ satisfying $J_N^2 = -I$, we denote by Ψ_{2K,J_N} the embedding of N into $\mathcal{T}_{2K}$ defined by

$$\forall x \in N,\ \Psi_{2K,J_N}(x) = (x, J_N(x)). \tag{3.1}$$

3.2. **Smooth embedding of an almost complex manifold.** Let (X, J) be a smooth compact almost complex manifold. By this we mean that X is a compact smooth $\mathcal{C}^\infty$ real manifold and J is a smooth $\mathcal{C}^\infty$ almost complex structure on X: $J \in End(TX)$, $J^2 = -I$.

Let $\overline{X}$ denote the almost complex manifold $\overline{X} := (X, -J)$ and let $J_{X \times \overline{X}} := J \oplus (-J)$. Then $J_{X \times \overline{X}}$ defines an almost complex structure on $X \times \overline{X}$.

We denote by Δ the diagonal in $X \times \overline{X}$:

$$\Delta := \{(x,x) \in X \times \overline{X}\}.$$

Lemma 3.1. *Δ is a totally real submanifold of $(X \times \overline{X}, J_{X\times\overline{X}})$.*

Proof of Lemma 3.1. Let $(v,v) \in T_{(x,x)}\Delta \cap JT_{(x,x)}\Delta$. Then $(v,v) = J_{X\times\overline{X}}(w,w)$ for some $w \in T_xX$ implying that $v = Jw = -Jw$: $v = 0$. □

We denote by ι the embedding of X into $X \times \overline{X}$ defined by

$$\forall x \in X,\ \iota(x) = (x,x).$$

Then $\iota(X) = \Delta$ and $i_*(J)$ defines an almost complex structure on Δ. The almost complex structure $J_{X\times\overline{X}}$ on $X \times \overline{X}$ and the distribution given by $Ker(dpr_1)$ endow Δ with an almost complex structure. Here $pr_1 : X \times \overline{X} \to X$ denotes the projection defined by $pr_1(x,y) = x$ fo every $(x,y) \in X \times \overline{X}$.

Lemma 3.2. *The almost complex structure on Δ given by $J_{X\times\overline{X}}$ and by the distribution $Ker\, d(pr_1)$ coincides with $\iota_*(J)$.*

Proof of Lemma 3.2. For $(v,v) \in T\Delta$, $J_{X\times\overline{X}}(v,v) = (Jv,-Jv) = (Jv,Jv) + (0,-2Jv)$ with $(Jv,Jv) = \iota_*(J)(v) \in T\Delta$ and $(0,-2Jv) \in Ker\, d(pr_1)$. □

Moreover the projection pr_1 is $(J_{X\times\overline{X}}, J)$ holomorphic:

$$d(pr_1) \circ J_{X\times\overline{X}} = J \circ d(pr_1).$$

Since X is a smooth compact manifold, it follows from the Whitney embedding Theorem that X can be embedded into $\mathbb{R}^{4n+1}$. Hence we may embed $X \times \overline{X}$ into $\mathbb{R}^{8n+2}$. Notice that the Whitney embedding Theorem asserts that every compact real n-manifold may be embedded in $\mathbb{R}^{2n+1}$. Hence we may in fact embed $X \times \overline{X}$ into $\mathbb{R}^{8n+1}$. However for our purpose, dealing with almost complex structures, it is more convenient to consider spaces of even dimension. Denote by ϕ_X the smooth $\mathcal{C}^\infty$ embedding $\phi_X : X \hookrightarrow \mathbb{R}^{4n+1}$ and let $\phi := (\phi_X, \phi_X)$. Then $(\phi(X \times \overline{X}), \phi_*(J_{X\times\overline{X}}))$ is an almost complex manifold.

Let NX denote the normal bundle of $\phi_X(X)$ in $\mathbb{R}^{4n+1}$. This is the subbundle of $T_{|\phi_X(X)}\mathbb{R}^{4n+1}$ orthogonal to $T(\phi_X(X))$ for the Euclidean metric $g_{\mathbb{R}^{4n+1}}$ on $\mathbb{R}^{4n+1}$. Alternatively NX may be given by the short exact sequence:

$$0 \to T(\phi_X(X)) \to T_{|\phi_X(X)}\mathbb{R}^{4n+1} \to NX \to 0.$$

We have the Whitney sum decomposition:

$$T_{|\phi(X\times\overline{X})}(\mathbb{R}^{8n+2}) = T(\phi(X \times \overline{X})) \oplus NX \oplus NX.$$

Consider the bundle isomorphism $v \in NX \mapsto (g_{\mathbb{R}^{4n+1}} : NX \to \mathbb{R})$ from NX to its dual bundle N^*X and the associated bundle isomorphism from $NX \oplus NX$ to $NX \oplus N^*X$. We may define an almost complex structure on $NX \oplus NX$ by considering the non-degenerate skew symmetric form $\omega : ((x_1,v_1),(x_2,v_2)) \in (NX \times N^*X)^2 \mapsto v_2(x_1) - v_1(x_2) \in \mathbb{R}$. It is sufficient then to consider an almost complex structure tamed by ω. Here the closedness of the ω is not needed.

Alternatively, since $NX \oplus N^*X$ is isomorphic to T^*NX, where T^*NX denotes the cotangent bundle of NX, we obtain a bundle isomorphism ψ from $NX \oplus NX$ to T^*NX. If g_{NX} denotes the restriction of $g_{\mathbb{R}^{4n+1}}$ to the subbbundle NX of $\mathbb{R}^{4n+1}$ then $\psi_*(g_{NX} \times g_{NX})$ defines a Riemannian metric on T^*NX.

Let ω_{T^*NX} be the canonical symplectic form on T^*NX. We may associate to $\psi_*(g_{NX} \times g_{NX})$ a ω_{T^*NX}-calibrated almost complex structure J_{T^*NX} defined uniquely by

$$\forall (u,v) \in T^*NX \times T^*NX,\ \psi_*(g_{NX} \times g_{NX}) = \omega_{T^*\Phi_*(X\times NX}(u, J_{T^*NX}v).$$

Hence we endow $NX \oplus NX$ with the almost complex structure $\psi^* J_{T^*NX}$. The Whitney sum

$$J_{8n+2,X} := \phi_*(J_{X\times\overline{X}}) \oplus \psi^* J_{T^*NX}$$

defines an almost complex structure on $T_{|\phi(X\times\overline{X})}(\mathbb{R}^{8n+2})$ that extends $\phi_*(J_{X\times\overline{X}})$:

$$(3.2)\qquad J_{8n+2,X} \in End(T_{|\phi(X\times\overline{X})}(\mathbb{R}^{8n+2})),\ J^2_{8n+2,X} = -I,\ (J_{8n+2,X})_{|T\phi(X\times\overline{X})} = \phi_*(J_{X\times\overline{X}}).$$

Let $\Phi := \phi \circ \iota : X \hookrightarrow \mathbb{R}^{8n+2}$ where ϕ and ι are defined just above. By construction the distribution $\mathcal{D}_{\mathbb{R}^{8n+2}} := \Phi_* Ker(dpr_1) \oplus NX \oplus NX$ is transverse to $\Phi(X)$ in $\mathbb{R}^{8n+2}$. Moreover, it follows from Lemma 3.2 and from (3.2) that the almost complex structure given on $\Phi_*(TX)$ by $J_{8n+2,X}$ and by the distribution $\mathcal{D}_{\mathbb{R}^{8n+2}}$ coincides with $\Phi_*(J)$.

According to Subsection 3.2 we may consider the embedding $\Psi_{8n+2,\Phi_*(J)} : \Phi(X) \hookrightarrow \mathcal{T}_{8n+2}$ defined in (3.1):

$$\forall x \in X, \Psi_{8n+2,\Phi_*(J)}(\Phi(x)) = (\Phi(x), J_{8n+2,X}(x)).$$

Then $\Psi := \Psi_{8n+2,\Phi(X)} \circ \Phi$ is a smooth embedding of X into $\mathcal{T}_{8n+2}$. Finally the distribution $\mathcal{D}_{\mathcal{T}_{8n+2}} := \pi^*[\Phi_* Ker(dpr_1) \oplus (NX \oplus NX)] \oplus Ker(d\pi) \subset T\mathcal{T}_{8n+2}$ is transverse to $\Psi(X)$.

We denote by $\iota_{\mathbb{C}}$ the embedding of $\mathcal{T}_{8n+2}$ into $\mathcal{T}^{\mathbb{C}}_{8n+2}$ and by f the embedding

$$f := \iota_{\mathbb{C}} \circ \Psi$$

of X into $\mathcal{T}^{\mathbb{C}}_{8n+2}$.

Finally we denote by $T^{(0,1)}_{\mathcal{J}^{8n+2}}\mathcal{T}_{8n+2}$ the fiber bundle

$$T^{(0,1)}_{\mathcal{J}^{8n+2}}\mathcal{T}_{8n+2} := \{\zeta \in T\mathcal{T}^{\mathbb{C}}_{8n+2} /\ (\mathcal{J}^{8n+2})^{\mathbb{C}}\zeta = -i\zeta\}.$$

Then the distribution

$$\mathcal{D}_{\mathcal{T}^{\mathbb{C}}_{8n+2}} := (\iota_{\mathbb{C}})_* (\mathcal{D}_{\mathbb{R}^{8n+2}}) + T^{(0,1)}_{\mathcal{J}^{8n+2}}\mathcal{T}_{8n+2}$$

is transverse to $f(X)$ in $\mathcal{T}^{\mathbb{C}}_{8n+2}$ and is of constant maximal rank.

For simplicity and if no confusion is possible, we will denote by $\mathcal{D}$ the distribution $\mathcal{D}_{\mathcal{T}^{\mathbb{C}}_{8n+2}}$. Following the notations of Section 3, for $y \in \mathcal{T}^{\mathbb{C}}_{8n+2}$ we will denote by $\mathcal{D}_y$ the fiber at y of the distribution $\mathcal{D}$.

By construction the almost complex structure induced on $f(X)$ by $J_{\mathcal{T}^{\mathbb{C}}_{8n+2}}$ and by $\mathcal{D}$ coincides with $f_*(J)$. Hence we have proved the following

Lemma 3.3. *The almost complex structures J and J_f coincide on X.*

We recall that J_f is defined for every $x \in X$ by

$$J_f(x) = df(x)^{-1} \circ p_{f_*T_{X,f(x)},\mathcal{D}_{f(x)}} \circ J_Z(f(x)) \circ df(x),$$

where $p_{f_*T_{X,f(x)},\mathcal{D}_{f(x)}} : T_{Z,f(x)} \to f_*T_{X,x}$ denotes the $\mathbb{R}$-linear projection along $\mathcal{D}_{f(x)}$.

Given two almost complex manifolds (X_1, J_1) and (X_2, J_2) and a smooth map $g : X_1 \to X_2$ we denote by

$$\overline{\partial}_{J_1,J_2} g := \frac{1}{2}(dg + J_2(g) \circ dg \circ J_1)$$

the d-bar operator with respect to the structures J_1 and J_2.

In particular $\overline{\partial}_{J_f,J_{\mathcal{T}^{\mathbb{C}}_{8n+2}}}$ denotes the d-bar operator with respect to the almost complex structure J_f on X and to the complex structure $J_{\mathcal{T}^{\mathbb{C}}_{8n+2}}$ on $\mathcal{T}^{\mathbb{C}}_{8n+2}$. Then by definition of J_f

$$\forall x \in X,\ \overline{\partial}_{J_f,J_{\mathcal{T}^{\mathbb{C}}_{8n+2}}} f(x) = (p_{f_*T_{X,f(x)}} - I) \circ J_{\mathcal{T}^{\mathbb{C}}_{8n+2}}(f(x)) \circ df(x). \tag{3.3}$$

We have:

Lemma 3.4. *(i)* $\overline{\partial}_{J_f,J_{\mathcal{T}^{\mathbb{C}}_{8n+2}}} f \in \Lambda^{(0,1)}T^*_X \otimes \mathcal{D}$ *and* $\overline{\partial}_{J_f,J_{\mathcal{T}^{\mathbb{C}}_{8n+2}}} f$ *is injective,*

(ii) $Im(\overline{\partial}_{J_f,(\mathcal{J}_{8n+2})^{\mathbb{C}}} f) \subset \mathcal{D}$ *and* $\overline{\partial}_{J_f,\mathcal{J}_{8n+2}} \Psi$ *is injective.*

Proof of Lemma 3.4. The first statement of Point (i) is direct from (3.3) since $p_{f_*T_{X,f(x)}}$ is the projection on f_*T_X along $\mathcal{D}$. For the second statement we assume to get a contradiction that there is $v_X \in TX$, $v_X \neq 0$, such that $df(v_X) = -J_{\mathcal{T}^{\mathbb{C}}_{8n+2}}(df(Jv_X))$. Since $f = \iota_{\mathbb{C}} \circ \Psi$ then

$$J_{\mathcal{T}^{\mathbb{C}}_{8n+2}}((\iota_{\mathbb{C}})_*(d\Psi(v_X))) = (\iota_{\mathbb{C}})_*(d\Psi(Jv)). \tag{3.4}$$

But $(\iota_{\mathbb{C}})_*(d\Psi(v_X)) \in (\iota_{\mathbb{C}})_*(T\mathcal{T}_{8n+2})$ and $J_{\mathcal{T}^{\mathbb{C}}_{8n+2}}((\iota_{\mathbb{C}})_*(d\Psi(v_X))) \in (\iota_{\mathbb{C}})_*(T\mathcal{T}_{8n+2})$ according to (3.4). This is contradiction since by definition $\iota_{\mathbb{C}}(\mathcal{T}_{8n+2})$ is a totally real submanifold of $(\mathcal{T}^{\mathbb{C}}_{8n+2}, J_{\mathcal{T}^{\mathbb{C}}_{8n+2}})$.

The first statement of Point (ii) comes from the inclusion $T^{(0,1)}_{\mathcal{J}^{8n+2}}(\mathcal{T}_{8n+2}) \subset \mathcal{D}$. For the second statement we use the fact that by construction $\Psi(X)$ is totally real in $(\mathcal{T}_{8n+2}, \mathcal{J}^{8n+2})$. In particular $\overline{\partial}_{J,\mathcal{J}^{8n+2}}\Psi$ is injective. Moreover since $(\mathcal{J}^{8n+2})^{\mathbb{C}}$ is the complexification of $\mathcal{J}^{8n+2}$ then $\iota_{\mathbb{C}}$ satisfies $d(\iota_{\mathbb{C}}) \circ \mathcal{J}^{8n+2} = (\mathcal{J}^{8n+2})^{\mathbb{C}} \circ d(\iota_{\mathbb{C}})$. Hence $\overline{\partial}_{J,(\mathcal{J}^{8n+2})^{\mathbb{C}}} f = d(\iota_{\mathbb{C}}) \circ \overline{\partial}_{J,\mathcal{J}^{8n+2}}\Psi$. This proves the injectivity of $\overline{\partial}_{J,(\mathcal{J}^{8n+2})^{\mathbb{C}}} f$. □

4. Surjectivity of the horizontal torsion operator

The aim of this section is to prove the following

Proposition 4.1. *There is a positive integer a, a smooth embedding Φ_a of X into $\mathbb{R}^{8n+2+2a}$, an almost complex structure $J_{\mathbb{R}^{8n+2+2a},X}$ on $T\mathbb{R}^{8n+2+2a}$ and a smooth distribution $\mathcal{D}_{\mathbb{R}^{8n+2+2a}}$ transverse to $\Phi_a(X)$ such that*

(i) The almost complex structure induced by $J_{\mathbb{R}^{8n+2+2a},X}$ and by $\mathcal{D}_{\mathbb{R}^{8n+2+2a}}$ on $T(\Phi_a(X))$ coincides with $(\Phi_a)_*(J)$,

(ii) For every $v \in Im\left(\overline{\partial}_{J,J_{8n+2+2a}}\Phi_a\right)$ the operator

$$\begin{array}{cccc} (\tau_a)^H_v & : & \mathcal{D}_{\mathbb{R}^{8n+2+2a}} & \rightarrow & T\mathbb{R}^{8n+2+2a}/\mathcal{D}_{\mathbb{R}^{8n+2+2a}} \\ & & w & \mapsto & [v,w]\, mod\ \mathcal{D}_{\mathbb{R}^{8n+2+2a}} \end{array}$$

is surjective.

We point out that the different objects given by Proposition 4.1 for $a = 0$ are the ones constructed in Section 3. In particular, we may consider the operator $(\tau_0)^H_v$ for every $v \in T\mathbb{R}^{8n+2}$. However it is not clear whether $(\tau_0)^H_v$ is surjective for every $v \in Im(\overline{\partial}_{J,J_{8n+2}}\Phi)$. The aim of Proposition 4.1 is just to increase the dimension of the Euclidean space in which we embed X in order to get the surjectivity of the operator $(\tau_a)^H_v$ defined in Proposition 4.1. For simplicity we write $\kappa := 8n + 2 + 2a$.

We first start with the following

Lemma 4.2. *For every $p \in \Phi(X)$ there is a neighborhhod $\mathcal{U}_p$ of p such that the operator $(\tau_0)^H_v$ is surjective for every $v \in Im(\overline{\partial}_{J,J_{8n+2,X}}\Phi)_{|\mathcal{U}_p}$.*

Proof of Lemma 4.2. Since by assumption $\mathcal{D}$ is a smooth distribution there is a neighborhood $\mathcal{U}$ of p in $\Phi(X)$ and a diffeomorphism $\lambda_{\mathcal{U}_p}$ from $\mathcal{U}$ to $\lambda_p(\mathcal{U}) \subset \mathbb{R}^{8n+2}$ such that $\lambda_{\mathcal{U}}(p) = 0$, $\mathcal{D}_0 = Span\left(\frac{\partial}{\partial x_{2n+1}}, \dots, \frac{\partial}{\partial x_{8n+2}}\right)$ and $\mathcal{D}_x = Span\left(\frac{\partial}{\partial x_j} + \sum_{1\leq i\leq 2n} \alpha_{ij}(x)\frac{\partial}{\partial x_i}\right)_{2n+1\leq j\leq 8n+2}$ for every $x \in \lambda_{\mathcal{U}}(\mathcal{U})$.

Since $T_0\mathbb{R}^{8n+2}/\mathcal{D}_0 \simeq T_0\Phi(X) = \mathrm{Span}(\partial/\partial x_1, \dots, \partial/\partial x_{2n})$ we have to prove:

$$\forall\, 2n+1 \leq j \leq 8n+2,\ \forall\, 1 \leq k \leq 2n,\ \exists\, (b_l)_{2n+1\leq l\leq 8n+2}/\ [v,w]_0 = \frac{\partial}{\partial x_k}\ mod\ \mathcal{D}_0$$

where $v = \frac{\partial}{\partial x_j} + \sum_{1\leq i\leq 2n} \alpha_{ij}\frac{\partial}{\partial x_i}$ and $w = \sum_{l=2n+1}^{8n+2} b_l\left(\frac{\partial}{\partial x_l} + \sum_{1\leq p\leq 2n}\alpha_{pl}\frac{\partial}{\partial x_p}\right)$.

By definition

$$[v,w]_0 = \sum_{i,l} b_l\left(\frac{\partial \alpha_{ij}}{\partial x_l}(0) - \frac{\partial \alpha_{il}}{\partial x_j}(0)\right)\frac{\partial}{\partial x_i}.$$

If we choose $(\alpha^k_{ij})_{ij}$ such that

$$\frac{\partial \alpha^k_{ij}}{\partial x_l}(0) - \frac{\partial \alpha^k_{il}}{\partial x_j}(0) \neq 0 \tag{4.1}$$

then we obtain the surjectivity at point 0. Then since α^k_{ij} is smooth and satisfies $\alpha^k_{ij}(0) = 0$ for every i, j, k it follows from (4.1) that τ^H_v is surjective on $\mathcal{U}$, shrinking $\mathcal{U}$ if necessary. □

Proof of Proposition 4.1. Since X is compact we may cover $\Phi(X)$ with open sets $\mathcal{V}_1, \dots, \mathcal{V}_a$ for some sufficiently large a. We choose a and $\mathcal{V}_1, \dots, \mathcal{V}_a$ such that Lemma 4.2 is satisfied on every $\mathcal{V}_k$ for $k = 1, \dots, a$. Let $\mu_1, \dots, \mu_a$ be a partition of unity associated with $\mathcal{V}_1, \dots, \mathcal{V}_a$. We denote by $(x_1, \dots, x_\kappa)$ the coordinates in $\mathbb{R}^\kappa$. Then the map

$$\Phi_a := (\phi_X, \mu_1, \dots, \mu_a, \phi_X, \mu_1, \dots, \mu_a)$$

is a smooth embedding of X into $\mathbb{R}^\kappa$.

Moreover if we denote by $\Phi_{a,X}$ the smooth embedding $\Phi_{a,X} = (\phi_X, \mu_1, \dots, \mu_a)$ from X into $\mathbb{R}^{4n+1+a}$ and by $N(X)$ the distribution orthogonal to $\Phi_{a,X}(X)$ in $\mathbb{R}^{4n+1+a}$ with respect to the Euclidean norm in $\mathbb{R}^{4n+1+a}$ then the distribution $N(X) \oplus N(X)$ is transverse to $\Phi_a(X)$ in $\mathbb{R}^\kappa$. Hence we may define the almost complex structure $J_{\kappa,X}$ on $(T\mathbb{R}^\kappa)_{|\Phi_a(X)}$ the same way we defined $J_{8n+2,X}$ on $(T\mathbb{R}^{8n+2})_{|\Phi(X)}$.

Then the distribution $\mathcal{D}_{\mathbb{R}^\kappa} := (\Phi_a)_* Ker(d\,pr_1) \oplus N(X) \oplus N(X)$ is transverse to $\Phi_a(X)$ in $\mathbb{R}^\kappa$. After a change of coordinates on $\mathcal{U}_k$ this is given on $(T\mathbb{R}^\kappa)_{|\mathcal{U}_k}$, for each $1 \leq k \leq a$, by

$$\mathcal{D}_{\mathbb{R}^\kappa} := Span\left(\frac{\partial}{\partial x_j} + \sum_{1\leq i\leq 2n}\alpha^k_{ij}\frac{\partial}{\partial x_i}\right)_{2n+1\leq j\leq\kappa}.$$

It follows now from Lemma 4.2 that the operator $(\tau_a)^H_v$ is surjective for every $v \in Im(\overline{\partial}_{J,J_\kappa}\Phi_a)$. □

5. Surjectivity on the complex twistor space

Following the construction of Section 3 we define:

• The embedding Ψ_a of X into the twistor space $\mathcal{T}_\kappa$, the almost complex structure $\mathcal{J}^\kappa$ on $(T\mathcal{T}_\kappa)_{|\Psi_a(X)}$, the distribution $\mathcal{D}_{\mathcal{T}_\kappa} \subset T\mathcal{T}_\kappa$,

• The embedding f_a of X into $\mathcal{T}_\kappa^{\mathbb{C}}$, the almost complex structure $(\mathcal{J}^\kappa)^{\mathbb{C}}$ on $(T\mathcal{T}_\kappa^{\mathbb{C}})_{|f_a(X)}$, the distribution $\mathcal{D}_{\mathcal{T}_\kappa^{\mathbb{C}}} \subset T(\mathcal{T}_\kappa^{\mathbb{C}})$ and the complex structure $J_{\mathcal{T}_\kappa^{\mathbb{C}}}$ on $\mathcal{T}_\kappa^{\mathbb{C}}$.

For simplicity and if no possible confusion we still write $\mathcal{D}$ instead of $\mathcal{D}_{\mathcal{T}_\kappa^{\mathbb{C}}}$.

Consider the following operators:

• For $v \in T\mathcal{T}_\kappa^{\mathbb{C}}$ the operator

$$\begin{array}{cccc} \tau_v^{\mathbb{C}} & : & \mathcal{D} & \rightarrow & T\mathcal{T}_\kappa^{\mathbb{C}}/\mathcal{D} \\ & & w & \mapsto & [v,w]\, mod\, \mathcal{D}, \end{array}$$

• For every $v \in T\mathcal{T}_\kappa$ the operator

$$\begin{array}{cccc} \tau_v & : & \mathcal{D}_{\mathcal{T}_\kappa} & \rightarrow & T\mathcal{T}_\kappa^{\mathbb{C}}/\mathcal{D} \\ & & w & \mapsto & [v,w]\, mod\, \mathcal{D}_{\mathcal{T}_\kappa}. \end{array}$$

The aim of this section is to prove the following

Proposition 5.1. *For every $v \in Im(\overline{\partial}_{J,J_{\mathcal{T}_\kappa^{\mathbb{C}}}} f_a)$ the operator $\tau_v^{\mathbb{C}}$ is surjective.*

We start with the following

Lemma 5.2. *If for every $v \in Im(\overline{\partial}_{J,J_\kappa}\Phi_a)$ the operator $(\tau_a)_v^H$ is surjective then for every $v \in Im(\overline{\partial}_{J,(\mathcal{J}^\kappa)}\Psi_a)$ the operator τ_v is surjective.*

Since $\Psi_a = (\Phi_a, J_\kappa(\Phi_a))$ then

$$d\Psi_a = (d\Phi_a, dJ_\kappa(\Phi_a) \circ d\Phi_a).$$

We recall that by construction

$$\mathcal{J}^\kappa(\Psi_a(x))(h,k) = (J_\kappa(\Psi_a(x)) \cdot k, J_\kappa(\Psi_a(x)) \circ k)$$

for every $x \in X$ and every $(h,k) \in T_{\Psi_a(x)}\mathcal{T}_\kappa$. This implies

$$\begin{aligned} \overline{\partial}_{J,\mathcal{J}^\kappa}\Psi_a &:= (d\Psi_a + \mathcal{J}^\kappa \circ d\Psi_a \circ J) \\ &= (\overline{\partial}_{J,J_\kappa}\Phi_a, dJ_\kappa(\Phi_a) \circ d\Phi_a \\ &\quad + J_\kappa(\Phi_a) \circ dJ_\kappa(\Phi_a) \circ d\Phi_a \circ J). \end{aligned}$$

Since $J_\kappa^2 = -I$ then $J_\kappa(\Phi_a) \circ dJ_\kappa(\Phi_a) \circ d\Phi_a = -dJ_\kappa(\Phi_a)(J_\kappa \circ d\Phi_a)$. We obtain

$$\begin{aligned} \mathcal{J}^\kappa \circ d\Psi_a \circ J &= dJ_\kappa(\Phi_a)(d\Phi_a - J_\kappa \circ d\Phi_a \circ J) \\ &= dJ_\kappa(\Phi_a)(\partial_{J,J_\kappa}\Phi_a) \end{aligned}$$

where $\partial_{J,J_\kappa} = d - \overline{\partial}_{J,J_\kappa}$. We finally have:

$$\overline{\partial}_{J,\mathcal{J}^\kappa}\Psi_a = (\overline{\partial}_{J,J_\kappa}\Phi_a, dJ_\kappa(\Phi_a) \circ \partial_{J,J_\kappa}\Phi_a). \tag{5.1}$$

Let $V \in Im(\overline{\partial}_{J,\mathcal{J}^\kappa}\Psi_a)$, $V \neq 0$. According to (5.1) there is $v_X \in TX$, $v_X \neq 0$, such that

$$V = (\overline{\partial}_{J,J_\kappa}\Phi_a(v_X), dJ_\kappa(\Phi_a) \circ \partial_{J,J_\kappa}\Phi_a(v_X)).$$

We recall that $\mathcal{T}_\kappa \xrightarrow{\pi} \mathbb{R}^\kappa$ is a trivial bundle. By definition $\mathcal{D}_{\mathcal{T}_\kappa} := \pi^*[\Phi_* Ker(dpr_1) \oplus (NX \oplus NX)] \oplus Ker(d\pi)$. Hence we have

$$U \in T\mathcal{T}_\kappa/\mathcal{D}_{\mathcal{T}_\kappa} \Leftrightarrow \exists\ u \in T\mathbb{R}^\kappa/\mathcal{D}_{\mathbb{R}^\kappa}\ /\ U = (u, 0).$$

Take $U = (u, 0) \in T\mathcal{T}_\kappa/\mathcal{D}_{\mathcal{T}_\kappa}$. By assumption the operator $(\tau_a)^H_{\overline{\partial}_{J,J_\kappa}\Phi_a(v)}$ is surjective. So there is $w \in \mathcal{D}_{\mathbb{R}^\kappa}$ such that

$$[\overline{\partial}_{J,J_\kappa}\Phi_a(v_X), w] = u \, mod\, \mathcal{D}_{\mathbb{R}^\kappa}.$$

Hence if we consider $W := (w, 0)$ then $W \in \mathcal{D}_{\mathcal{T}_\kappa}$ and we have:

$$[V, W] = U\ mod\ \mathcal{D}_{\mathcal{T}_\kappa}.$$

This proves Lemma 5.2. □

Lemma 5.3. *If for every $v \in Im(\overline{\partial}_{J,\mathcal{J}^\kappa}\Psi_a)$ the operator τ_v is surjective then for every $v \in Im(\overline{\partial}_{J,(\mathcal{J}^\kappa)^{\mathbb{C}}} f_a)$ the operator $\tau_v^{\mathbb{C}}$ is surjective.*

Proof of Lemma 5.3. Since $(\mathcal{J}^{8n+2})^{\mathbb{C}} \circ d(\iota_{\mathbb{C}}) = d(\iota_{\mathbb{C}}) \circ \mathcal{J}^{8n+2}$ we have

$$\forall X, Y, Z \in T\mathcal{T}_\kappa,\ [X, Y] = Z \Rightarrow [(\iota_{\mathbb{C}})_* X, (\iota_{\mathbb{C}})_* Y] = (\iota_{\mathbb{C}})_* Z. \tag{5.2}$$

Let now $V \in Im(\overline{\partial}_{J,(\mathcal{J}^\kappa)^{\mathbb{C}}})f_a$. Then there is $v_X \in X$ such that

$$\begin{aligned} V &= \left(df_a + (\mathcal{J}^\kappa)^{\mathbb{C}} \circ df_a \circ J\right)(v_X) \\ &= (d(\iota_{\mathbb{C}}) \circ d\Psi_a + (\mathcal{J}^\kappa)^{\mathbb{C}} \circ d(\iota_{\mathbb{C}}) \circ d\Psi_a \circ J)(v_X) \\ &= d(\iota_{\mathbb{C}}) \circ (d\Psi_a + \mathcal{J}^\kappa \circ d\Psi_a \circ J)(v_X) \end{aligned}$$

since $(\mathcal{J}^\kappa)^{\mathbb{C}}$ is the complexification of $\mathcal{J}^\kappa$. Hence $V = d(\iota_{\mathbb{C}})\left(\overline{\partial}_{J,\mathcal{J}^\kappa}\Psi_a(v_X)\right)$.

Let $U \in T\mathcal{T}_\kappa^{\mathbb{C}}/\mathcal{D}$. Then there is $u \in T\mathcal{T}_\kappa/\mathcal{D}_{\mathcal{T}_\kappa}$ such that $U = d(\iota_{\mathbb{C}})(u)$. Moreover by assumption there is $w \in \mathcal{D}_{\mathcal{T}_\kappa}$ such that $[\overline{\partial}_{J,\mathcal{J}^\kappa}\Psi_a(v_X), w] = u$. It follows now from (5.2) that $\left[d(\iota_{\mathbb{C}})\left(\overline{\partial}_{J,\mathcal{J}^\kappa}\Psi_a(v_X)\right), d(\iota_{\mathbb{C}})w\right] = d(\iota_{\mathbb{C}})u$ with $d(\iota_{\mathbb{C}})w \in \mathcal{D}$. This proves lemma 5.3. □

To conclude the proof of Proposition 5.1 it remains to prove the following

Lemma 5.4. *If for every $v \in Im(\overline{\partial}_{J,(\mathcal{J}^\kappa)^{\mathbb{C}}})f_a)$ the operator $\tau_v^{\mathbb{C}}$ is surjective then for every $v \in Im(\overline{\partial}_{J,J_{\mathcal{T}_\kappa^{\mathbb{C}}}} f_a)$ the operator $\tau_v^{\mathbb{C}}$ is surjective.*

Proof of Lemma 5.4. To be completed

6. Algebraic approximation and proof of Theorem 1.1

We first point out that by definition $T^{(0,1)}_{\mathcal{J}^{8n+2}}\mathcal{T}^{\mathbb{C}}_{8n+2}$ is an algebraic distribution. Moreover $\pi^*[(\Phi \circ \iota)_* Ker(dpr_1) \oplus (NX \oplus NX)] \oplus Ker(d\pi)$ is a smooth distribution contained in the totally real submanifold $\mathcal{T}_{8n+2}$ of the complex manifold $(\mathcal{T}^{\mathbb{C}}_{8n+2}, J_{\mathcal{T}^{\mathbb{C}}_{8n+2}})$.

We have the following generalization of the Weierstrass Approximation theorem [2]:

Theorem A: Let $1 \leq k \leq \infty$ and let M be a $\mathcal{C}^k$ submanifold totally real in a complex manifold X; then there exists a Stein open neighborhood U of M in X such that the set of *restrictions to M of all holomorphic functions on U is dense in the Fréchet space of all $\mathcal{C}^k$ functions on M.*

It follows from Theorem A that there is a neighborhood U of $\iota_{\mathbb{C}} \circ \Phi \circ \iota(X)$ in $\mathcal{T}^{\mathbb{C}}_{8n+2}$ and there is a family $(\mathcal{H}_\nu)$ of holomorphic distributions defined on a neighborhood of $\overline{U}$ and converging (in the $\mathcal{C}^\infty$ topology on $\overline{U}$) to $\iota_{\mathbb{C}*}(\pi^*[(\Phi \circ \iota)_* Ker(dpr_1) \oplus (NX \oplus NX)])$. Moreover there is a neighborhood V of $\iota_{\mathbb{C}} \circ \Phi \circ \iota(X)$ in $\mathcal{T}^{\mathbb{C}}_{8n+2}$, $V \subset\subset U$, and a family $(\mathcal{A}_\nu)$ of algebraic distributions, with possible singularities outside $\overline{V}$ approximating $(\mathcal{H}_\nu)$. In particular, we have

Lemma 6.1. *The family $(\mathcal{A}_\nu)$ converges, in the $\mathcal{C}^\infty$ topology on $\overline{V}$, to $\iota_{\mathbb{C}*}(\pi^*[(\Phi \circ \iota)_* Ker(dpr_1) \oplus (NX \oplus NX)])$.*

Consider now the holomorphic algebraic distribution $\mathcal{D}_\nu := \mathcal{A}_\nu \oplus T^{(0,1)}_{\mathcal{J}^{8n+2}} \mathcal{T}^{\mathbb{C}}_{8n+2}$. The complex structure $J_{\mathcal{T}^{\mathbb{C}}_{8n+2}}$ and the distribution $\mathcal{D}_\nu$ define an almost complex structure J_ν on $\iota_{\mathbb{C}} \circ \Phi\iota(X) \subset \mathcal{T}^{\mathbb{C}}_{8n+2}$. Moreover it follows from Lemma 6.1 that for every non negative integer l:

$$\lim_{\nu \to \infty} \sup_{x \in \iota_{\mathbb{C}} \circ \Phi \circ \iota(X)} \| D^l J_\nu(x) - D^l(\iota_{\mathbb{C}} \circ \Phi \circ \iota)_*(J_f)(x) \| = 0$$

where $\| \cdot \|$ denotes any norm on the space of $(2n \times 2n)$ matrices and D^l denotes the l-th derivative.

Hence we have proved the following:

Proposition 6.2. *There is a family $(\mathcal{D}_\nu)_\nu$ of compact algebraic distributions on $\mathcal{T}^{\mathbb{C}}_{8n+2}$ satisfying the following properties:*

(i) For every ν, $\mathcal{D}_\nu$ is transverse to $\iota_{\mathbb{C}} \circ \Phi \circ \iota(X)$,

(ii) If J_ν denotes the almost complex structure induced on $\iota_{\mathbb{C}} \circ \Phi\iota(X)$ by the complex structure $J_{\mathcal{T}^{\mathbb{C}}_{8n+2}}$ and by the distribution $\mathcal{D}_\nu$ then $\lim_{\nu \to \infty} \sup_{x \in \iota_{\mathbb{C}} \circ \Phi \circ \iota(X)} \| D^l J_\nu(x) - D^l(\iota_{\mathbb{C}} \circ \Phi \circ \iota)_*(J_f)(x) \| = 0$.

It follows from Proposition 6.2 that if $J_{f,\nu}$ is the almost complex structure defined on X by

$$J_{f,\nu}(x) := df(x)^{-1} \circ p_{f_* T_{X,f(x)}, \mathcal{D}_\nu f(x)} \circ J_Z(f(x)) \circ df(x),$$

then for every integer l

$$\lim_{\nu \to \infty} \sup_{x \in X} \| D^l J_{f,\nu}(x) - D^l(J_f)(x) \| = 0.$$

We can now prove Theorem 1.1.

To be continued

References

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