ALGEBRAIC EMBEDDINGS OF SMOOTH ALMOST COMPLEX STRUCTURES

JEAN-PIERRE DEMAILLY AND HERVÉ GAUSSIER

1. Introduction and main results

The goal of this work is to prove an embedding theorem for compact almost complex manifolds into complex algebraic varieties. As usual, an almost complex manifold of dimension n is a pair (X, J_X), where X is a real manifold of dimension $2n$ and J_X a smooth section of $\mathrm{End}(TX)$ such that $J_X^2 = -\mathrm{Id}$; we will assume here that all data are $\mathcal{C}^\infty$.

Let Z be a complex (holomorphic) manifold of complex dimension N. Such a manifold carries a natural integrable almost complex structure J_Z (conversely, by the Newlander-Nirenberg thorem any integrable almost complex structure can be viewed as a holomorphic structure). Now, assume that we are given a holomorphic distribution $\mathcal{D}$ in TZ, namely a holomorphic subbundle $\mathcal{D} \subset TZ$. Every fiber $\mathcal{D}_x$ of the distribution is then invariant under J_Z, i.e. $J_Z\mathcal{D}_x \subset \mathcal{D}_x$ for every $x \in Z$. Here, the distribution $\mathcal{D}$ is not assumed to be integrable. We recall that $\mathcal{D}$ is integrable in the sense of Frobenius (i.e. stable under the Lie bracket operation) if and only if the fibers $\mathcal{D}_x$ are the tangent spaces to leaves of a holomorphic foliation. More precisely, $\mathcal{D}$ is integrable if and only if the torsion operator θ of $\mathcal{D}$, defined by

$$\theta \;:\; \begin{array}{ccc} \mathcal{O}(\mathcal{D}) \times \mathcal{O}(\mathcal{D}) & \longrightarrow & \mathcal{O}(TZ/\mathcal{D}) \\ (\zeta, \eta) & \longmapsto & [\zeta, \eta] \mod \mathcal{D} \end{array} \tag{1.1}$$

vanishes identically. As is well known, θ is skew symmetric in (ζ, η) and can be viewed as a holomorphic section of the bundle $\Lambda^2\mathcal{D}^* \otimes (TZ/\mathcal{D})$.

Let M be a real submanifold of Z of class $\mathcal{C}^\infty$ and of real dimension $2n$ with $n < N$. We say that M is *transverse to* $\mathcal{D}$ if for every $x \in M$ we have

$$T_xM \oplus \mathcal{D}_x = T_xZ. \tag{1.2}$$

We could in fact assume more generally that the distribution $\mathcal{D}$ is singular, i.e. given by a certain saturated subsheaf $\mathcal{O}(\mathcal{D})$ of $\mathcal{O}(TZ)$ ("saturated" means that the quotient sheaf $\mathcal{O}(TZ)/\mathcal{O}(\mathcal{D})$ has no torsion). Then $\mathcal{O}(\mathcal{D})$ is actually a subbundle of TZ outside an analytic subset $\mathcal{D}_{\rm sing} \subset Z$ of codimension ≥ 2, and we further assume in this case that $M \cap \mathcal{D}_{\rm sing} = \emptyset$.

When M is transverse to $\mathcal{D}$, one gets a natural $\mathbb{R}$-linear isomorphism

$$T_xM \simeq T_xZ/\mathcal{D}_x \tag{1.3}$$

at every point $x \in M$. Since $TZ/\mathcal{D}$ carries a structure of holomorphic vector bundle (at least over $Z \smallsetminus \mathcal{D}_{\rm sing}$), the complex structure J_Z induces a complex structure on the quotient and therefore, through the above isomorphism (1.3), an almost complex structure $J_M^{Z,\mathcal{D}}$ on M.

Moreover, when $\mathcal{D}$ is a foliation (i.e. $\mathcal{O}(\mathcal{D})$ is an integrable subsheaf of $\mathcal{O}(TZ)$), then $J_M^{Z,\mathcal{D}}$ is an *integrable* almost complex structure on M. Indeed, such a foliation is realized near any

Date: July 22, 2014.

regular point x as the set of fibers of a certain submersion: there exists an open neighborhood Ω of x in Z and a holomorphic submersion $\sigma : \Omega \to \Omega'$ to an open subset $\Omega' \subset \mathbb{C}^{N-n}$ such that the fibers of σ are the leaves of $\mathcal{D}$ in Ω. We can take Ω to be a coordinate open set in Z centered at point x and select coordinates such that the submersion is expressed as the first projection $\Omega \simeq \Omega' \times \Omega'' \to \Omega'$ with respect to $\Omega' \subset \mathbb{C}^n$, $\Omega'' \subset \mathbb{C}^{N-n}$, and then $\mathcal{D}$, $TZ/\mathcal{D}$ are identified with the trivial bundles $\Omega \times (\{0\} \times \mathbb{C}^{N-n})$ and $\Omega \times \mathbb{C}^n$. The restriction

$$\sigma_{M\cap\Omega} : M \cap \Omega \subset \Omega \xrightarrow{\sigma} \Omega'$$

provides M with holomorphic coordinates on $M \cap \Omega$, and it is clear that any other local trivialization of the foliation on a different chart $\widetilde{\Omega} = \widetilde{\Omega}' \times \widetilde{\Omega}''$ would give coordinates that are changed by local biholomorphisms $\Omega' \to \widetilde{\Omega}'$ in the intersection $\Omega \cap \widetilde{\Omega}$, thanks to the holomorphic character of $\mathcal{D}$. Thus we directly see in that case that $J_M^{Z,\mathcal{D}}$ comes from a holomorphic structure on M.

More generally, we say that $f : X \hookrightarrow Z$ is a transverse embedding of a smooth real manifold X in $(Z, \mathcal{D})$ if f is an embedding and $M = f(X)$ is a transverse submanifold of Z, namely if $f_* T_x X \oplus \mathcal{D}_{f(x)} = T_{f(x)} Z$ for every point $x \in X$ (and $f(X)$ does not meet $\mathcal{D}_{\rm sing}$ in case there are singularities). One then gets a real isomorphism $TX \simeq f^*(TZ/\mathcal{D})$ and therefore an almost complex structure on X (for this it would be enough to assume that f is an immersion, but we will actually suppose that f is an embedding here). We denote by J_f the almost complex structure $J_f := f^*(J_{f(X)}^{Z,\mathcal{D}})$.

The following very interesting question was investigated about 20 years ago by F. Bogomolov [Bog96].

Basic question 1.1. *Given an integrable complex structure J on a compact manifold X, can one realize J, as described above, by a transverse embedding $f : X \hookrightarrow Z$ into a projective manifold $(Z, \mathcal{D})$ equipped with an algebraic foliation $\mathcal{D}$, in such a way that $f(X) \cap \mathcal{D}_{\rm sing} = \emptyset$ and $J = J_f$?*

There are indeed many examples of Kähler and non Kähler compact complex manifolds which can be embedded in that way (the case of projective ones being of course trivial): tori, Hopf and Calabi-Eckman manifolds, and more generally all manifolds given by the LVMB construction (see Section 2). Strong indications exist that every compact complex manifold should be embeddable as a smooth submanifold transverse to an algebraic foliation on a complex projective variety, see section 5 . We prove here that the corresponding statement in the almost complex category actually holds – provided that non integrable distributions are considered rather than foliations. In fact, there are even "universal solutions" to this problem.

Theorem 1.2. *For all integers $n \geq 1$ and $k \geq 4n$, there exists a complex affine algebraic manifold $Z_{n,k}$ of dimension $N = 2k + 2(k^2 + n(k - n))$ possessing a real structure (i.e. an anti-holomorphic algebraic involution) and an algebraic distribution $\mathcal{D}_{n,k} \subset TZ_{n,k}$ of codimension n, for which every compact n-dimensional almost complex manifold (X, J) admits an embedding $f : X \hookrightarrow Z_{n,k}^{\mathbb{R}}$ transverse to $\mathcal{D}_{n,k}$ and contained in the real part of $Z_{n,k}$, such that $J = J_f$. Moreover, f can be chosen to depend in a simple algebraic way on the almost complex structure J selected on X.*

The choice $k = 4n$ yields the explicit embedding dimension $N = 38n^2 + 8n$ (we will see that a quadratic bound $N = O(n^2)$ is optimal, but the above explicit value could perhaps be improved). Since $Z = Z_{n,k}$ and $\mathcal{D} = \mathcal{D}_{n,k}$ are algebraic and Z is affine, one can further compactify Z into a complex projective manifold $\overline{Z}$, and extend $\mathcal{D}$ into a saturated subsheaf $\overline{\mathcal{D}}$ of $T\overline{Z}$. In general such distributions $\overline{\mathcal{D}}$ will acquire singularities at infinity, and it is unclear whether one can achieve such embeddings with $\overline{\mathcal{D}}$ non singular on $\overline{Z}$, if at all possible.

Next, we consider the case of a compact almost complex symplectic manifold (X, J, ω) where the symplectic form ω is assumed to be J-compatible, i.e. $J^*\omega = \omega$ and $\omega(\xi, J\xi) > 0$. By a theorem of Tischler [Tis77], at least under the assumption that the De Rham cohomology class $\{\omega\}$ is integral, we know that there exists a smooth embedding $g : X \hookrightarrow \mathbb{CP}^s$ such that $\omega = g^*\omega_{\mathrm{FS}}$ is the pull-back of the standard Fubini-Study metric on ω_{FS} on $\mathbb{CP}^s$. A natural problem is whether the symplectic structure can be accommodated simultaneously with the almost complex structure by a transverse embedding. Let us introduce the following definition.

Definition 1.3. *Let $(Z, \mathcal{D})$ be a complex manifold equipped with a holomorphic distribution. We say that a closed semipositive $(1,1)$-form β on Z is a transverse Kähler structure if the kernel of β is contained in $\mathcal{D}$, in other terms, if β induces a Kähler form on any germ of complex submanifold transverse to $\mathcal{D}$.*

Using an effective version of Tischler's theorem stated by Gromov [Gro86], we prove :

Theorem 1.4. *For all integers $n \geq 1$, $b \geq 1$ and $k \geq 2n+1$, there exists a complex projective algebraic manifold $Z_{n,b,k}$ of dimension $N = 2bk(2bk+1) + 2n(2bk - n))$, equipped with a real structure and an algebraic distribution $\mathcal{D}_{n,b,k} \subset TZ_{n,b,k}$ of codimension n, for which every compact n-dimensional almost complex symplectic manifold (X, J, ω) with second Betti number $b_2 \leq b$ and a J-compatible symplectic form ω admits an embedding $f : X \hookrightarrow Z^{\mathbb{R}}_{n,b,k}$ transverse to $\mathcal{D}_{n,b,k}$ and contained in the real part of $Z_{n,k}$, such that $J = J_f$ and $\omega = f^*\beta$ for some transverse Kähler structure β on $(Z_{n,b,k}, \mathcal{D}_{n,b,k})$.*

In section 5, we discuss Bogomolov's conjecture for the integrable case. We first prove the following weakened version, which can be seen as a form of "algebraic embedding" for arbitrary compact complex manifolds.

Theorem 1.5. *For all integers $n \geq 1$ and $k \geq 4n$, let $(Z_{n,k}, \mathcal{D}_{n,k})$ be the affine algebraic manifold equipped with the algebraic distribution $\mathcal{D}_{n,k} \subset TZ_{n,k}$ introduced in Theorem 1.2. Then, for every compact n-dimensional (integrable) complex manifold (X, J), there exists an embedding $f : X \hookrightarrow Z^{\mathbb{R}}_{n,k}$ transverse to $\mathcal{D}_{n,k}$, contained in the real part of $Z_{n,k}$, such that*

(i) *$J = J_f$ and $\overline{\partial}_J f$ is injective;*

(ii) *$\mathrm{Im}(\overline{\partial}_J f)$ is contained in the isotropic locus $I_{\mathcal{D}_{n,k}}$ of the torsion operator θ of $\mathcal{D}_{n,k}$, the intrinsically defined algebraic locus in the Grassmannian bundle $\mathrm{Gr}(\mathcal{D}_{n,k}, n) \to Z_{n,k}$ of complex n-dimensional subspaces in $\mathcal{D}_{n,k}$, consisting of those subspaces S such that $\theta_{|S\times S} = 0$.*

The inclusion condition (ii) $\mathrm{Im}(\overline{\partial}_J f) \subset I_{\mathcal{D}_{n,k}}$ is in fact necessary and sufficient for the integrability of J_f.

In the last section we investigate the original Bogomolov conjecture and "reduce it" to a statement concerning approximations of holomorphic foliations. The flavor of the statement is that holomorphic objects (functions, sections of algebraic bundles, etc) defined on a polynomially convex open set of $\mathbb{C}^n$ can always be approximated by polynomials or algebraic sections. Our hope is that this might be true also for the approximation of holomorphic foliations by Nash algebraic ones. Recall that a Nash algebraic map $g : U \to V$ between open sets U, V of algebraic manifolds Y, Z is a map whose graph is a connected component of the intersection of $U \times V$ with an algebraic subset of $Y \times Z$. We say that a holomorphic foliation $\mathcal{F}$ of codimension n on U is Nash algebraic if the associated distribution $Y \supset U \to \mathrm{Gr}(TY, n)$ is Nash algebraic. In section 6, we show

Proposition 1.6. *Assume that holomorphic foliations can be approximated by Nash algebraic foliations uniformly on compact subsets of any polynomially convex open subset of $\mathbb{C}^N$. Then every compact complex manifold can be approximated by compact complex manifolds that are embeddable in the sense of Bogomolov in foliated projective manifolds.*

2. Transverse embeddings to foliations

We consider the situation described above, where Z is a complex N-dimensional manifold equipped with a holomorphic distribution $\mathcal{D}$. More precisely, let X be a compact real manifold of class $\mathcal{C}^\infty$ and of real dimension $2n$ with $n < N$. We assume that there is an embedding $f : X \hookrightarrow Z$ that is transverse to $\mathcal{D}$, namely that $f(X) \cap \mathcal{D}_{\mathrm{sing}} = \emptyset$ and

$$f_* T_x X \oplus \mathcal{D}_{f(x)} = T_{f(x)} Z \tag{2.1}$$

at every point $x \in X$. Here $\mathcal{D}_{f(x)}$ denotes the fiber at $f(x)$ of the distribution $\mathcal{D}$. As explained in section 1, this induces a $\mathbb{R}$-linear isomorphism $f_* : TX \to f^*(TZ/\mathcal{D})$, and from the complex structures of TZ and $\mathcal{D}$ we get an almost complex structure $f^* J^{Z,\mathcal{D}}_{f(X)}$ on TX which we will simply denote by J_f here. Next, we briefly investigate the effect of isotopies.

Definition 2.1. *An isotopy of smooth transverse embeddings of X into $(Z, \mathcal{D})$ is by definition a family $f_t : X \to Z$ of embeddings for $t \in [0, 1]$, such that the map $F(x, t) = f_t(x)$ is smooth on $X \times [0, 1]$ and f_t is transverse to $\mathcal{D}$ for every $t \in [0, 1]$.*

We then get a smooth variation J_{f_t} of almost complex structures on X. When $\mathcal{D}$ is integrable (i.e. a holomorphic foliation), these structures are integrable and we have the following simple but remarkable fact.

Proposition 2.2. *Let Z be a compact complex manifold equipped with a holomorphic foliation $\mathcal{D}$ and let $f_t : X \to Z$, $t \in [0, 1]$, be an isotopy of transverse embeddings of a compact smooth real manifold. Then all complex structures (X, J_{f_t}) are biholomorphic to (X, J_{f_0}) through a smooth variation of diffeomorphisms in $\mathrm{Diff}_0(X)$, the identity component of the group $\mathrm{Diff}(X)$ of diffeomorphisms of X.*

Proof. By an easy connectedness argument, it is enough to produce a smooth variation of biholomorphisms $\psi_{t,t_0} : (X, J_{f_{t_0}})$ to (X, J_{f_t}) when t is close to t_0, and then extend these to all $t, t_0 \in [0, 1]$ by the chain rule. Let $x \in X$. Thanks to the local triviality of the foliation at $z_0 = f_{t_0}(x) \in Z \smallsetminus \mathcal{D}_{\mathrm{sing}}$, $\mathcal{D}$ is locally near x the family of fibers of a holomorphic submersion $\sigma : Z \supset \Omega \to \Omega' \subset \mathbb{C}^n$ defined on a neighborhood Ω of z_0. Then $\sigma \circ f_t : X \supset f_t^{-1}(\Omega) \to \Omega'$ is by definition a local biholomorphism from (X, J_{f_t}) to Ω' (endowed with the standard complex structure of $\mathbb{C}^n$). Now, $\psi_{t,t_0} = (\sigma \circ f_t)^{-1} \circ (\sigma \circ f_{t_0})$ defines a local biholomorphism from $(X, J_{f_{t_0}})$ to (X, J_{f_t}) on a small neighborhood of x, and these local biholomorphisms glue together into a global one when x and Ω vary (this biholomorphism consists of "following the leaf of $\mathcal{D}$" from the position $f_{t_0}(X)$ to the position $f_t(X)$ of the embedding). Clearly ψ_{t,t_0} depends smoothly on t and satisfies the chain rule $\psi_{t,t_0} \circ \psi_{t_0,t_1} = \psi_{t,t_1}$. $\square$

Therefore when $\mathcal{D}$ is a foliation, to any triple $(Z,\mathcal{D},\alpha)$ where α is an isotopy class of transverse embeddings $X\to Z$, one can attach a point in the Teichmüller space $\mathcal{J}^{\rm int}(X)/\operatorname{Diff}_0(X)$ of integrable almost complex structures modulo biholomorphisms diffeotopic to identity. The question raised by Bogomolov can then be stated more precisely:

Question 2.3. *For any compact complex manifold (X,J), does there exist a triple $(Z,\mathcal{D},\alpha)$ formed by a smooth complex projective variety Z, an algebraic foliation $\mathcal{D}$ on Z and an isotopy class α of transverse embeddings $X\to Z$, such that $J=J_f$ for some $f\in\alpha$?*

The isotopy class of embeddings $X\to Z$ in a triple $(Z,\mathcal{D},\alpha)$ provides some sort of "algebraicization" of a compact complex manifold, in the sense that there is an atlas of X such that the transition functions are solutions of algebraic linear equations (rather than plain algebraic functions, as would be the case for usual algebraic varieties). In this setting, it should be observed that the isotopy classes α are just "topological classes" belonging to a discrete countable set – this set can be infinite as one already sees for real linear embeddings of a real even dimensional torus $X=(\mathbb{R}/\mathbb{Z})^{2n}$ into a complex torus $Z=\mathbb{C}^N/\Lambda$ equipped with a linear foliation $\mathcal{D}$. To see this, we first cover $Z\smallsetminus\mathcal{D}_{\rm sing}$ by a countable family of coordinate open sets $\Omega_\nu\simeq\Omega'_\nu\times\Omega''_\nu$ such that the first projections $\sigma_\nu:\Omega_\nu\to\Omega'_\nu\subset\mathbb{C}^n$ define the foliation. We assume here Ω'_ν and Ω''_ν to be balls of sufficiently small radius, so that all fibers $z'\times\Omega''_\nu$ are geodesically convex with respect to a given hermitian metric on the ambient manifold Z, and the geodesic segment joining any two points in those fibers is unique (of course, we mean here geodesics relative to the fibers – standard results of differential geometry guarantee that sufficiently small coordinate balls will satisfy this property). Then any nonempty intersection $\bigcap z'_j\times\Omega''_{\nu_j}$ of the fibers from various coordinate sets is still connected and geodesically convex. We further enlarge the family with all smaller balls whose centers have coordinates in $\mathbb{Q}[i]$ and radii in $\mathbb{Q}_+$, so that arbitrarily fine coverings can be extracted from the family. A transverse embedding $f:X\to Z$ is characterized by its image $M=f(X)$ up to right composition with an element $\psi\in\operatorname{Diff}(X)$, and thus, modulo isotopy, up to an element in the countable mapping class group $\operatorname{Diff}(X)/\operatorname{Diff}_0(X)$. The image $M=f(X)$ is itself given by a finite collection of graphs of maps $g_\nu:\Omega'_\nu\to\Omega''_\nu$ that glue together, for a certain finite subfamily of coordinate sets $(\Omega_\nu)_{\nu\in I}$ extracted from the initial countable family. However, any two such transverse submanifolds $(M_k)_{k=0,1}$ and associated collections of graphs $(g_{k,\nu})$ defined on the same finite subset I are isotopic: to see this, assume e.g. $I=\{1,2,\ldots,s\}$ and fix even smaller products of balls $\widetilde\Omega_\nu\simeq\widetilde\Omega'_\nu\times\widetilde\Omega''_\nu\Subset\Omega_\nu$ still covering M_0 and M_1, and a cut-off function $\theta_\nu(z')$ equal to 1 on $\widetilde\Omega'_\nu$ and with support in Ω'_ν. Then we construct isotopies $(F_{t,k})_{t\in[0,1]}:M_0\to M_{t,k}$ step by step, for $k=1,2,\ldots,s$, by taking inductively graphs of maps $(G_{t,k,\nu})_{t\in[0,1],\,k=1,\ldots,s,\,\nu\in I}$ such that

$$M_{t,1}\quad\text{given by}\quad\begin{cases}G_{t,1,1}(z')=\gamma\big(t\theta_1(z')\,;\,g_{0,1}(z'),g_{1,1}(z')\big)\quad\text{on }\Omega'_1,\\ G_{t,1,\nu}(z')=g_{0,\nu}(z')\quad\text{on }\sigma_\nu\big(\Omega_\nu\smallsetminus(\operatorname{Supp}(\theta_1)\times\widetilde\Omega''_1)\big),\ \nu\neq1,\end{cases}$$

$$M_{t,k}\quad\text{given by}\quad\begin{cases}G_{t,k,k}(z')=\gamma\big(t\theta_k(z')\,;\,G_{t,k-1,k}(z'),G_{t,k-1,k}(z')\big)\quad\text{on }\Omega'_k,\\ G_{t,k,\nu}(z')=G_{t,k-1,\nu}(z')\quad\text{on }\sigma_\nu\big(\Omega_\nu\smallsetminus(\operatorname{Supp}(\theta_k)\times\widetilde\Omega''_k)\big),\ \nu\neq k,\end{cases}$$

where $\gamma(t\,;\,a'',b'')$ denotes the geodesic segment between a'' and b'' in each fiber $z'\times\Omega''_\nu$. By construction, we have $M_{0,k}=M_0$ and $M_{1,k}\cap U_k=M_1\cap U_k$ on $U_k=\widetilde\Omega_1\cup\ldots\cup\widetilde\Omega_k$, thus $f_t:=F_{t,s}:M_0\to M_t$ is a transverse isotopy between M_0 and M_1. Therefore, we have at most as many isotopy classes as the cardinal of the mapping class group, times the cardinal of the set of finite subsets of a countable set, which is still countable. □

Of course, when $\mathcal{D}$ is non integrable, the almost complex structure J_{f_t} will in general vary under isotopies. One of the goals of the next sections is to investigate this phenomenon, but in this section we study further some integrable examples.

Example 2.4 (Complex tori)**.** Let $X = \mathbb{R}^{2n}/\mathbb{Z}^{2n}$ be an even dimensional real torus and $Z = \mathbb{C}^N/\Lambda$ a complex torus where $\Lambda \simeq \mathbb{Z}^{2N}$ is a lattice of $\mathbb{C}^N$, $N > n$. Any complex vector subspace $\mathcal{D} \subset \mathbb{C}^N$ of codimension n defines a linear foliation on Z (which may or may not have closed leaves, but for $\mathcal{D}$ generic, the leaves are everywhere dense). Let $f : X \to Z$ be a linear embedding transverse to $\mathcal{D}$. Here, there are countably many distinct isotopy classes of such linear embeddings, in fact up to a translation, f is induced by a $\mathbb{R}$-linear map $u : \mathbb{R}^{2n} \to \mathbb{C}^N$ that sends the standard basis $(e_1, \ldots, e_{2n})$ of $\mathbb{Z}^{2n}$ to a unimodular system of $2n$ $\mathbb{Z}$-linearly independent vectors $(\varepsilon_1, \ldots, \varepsilon_{2n})$ of Λ. Such $(\varepsilon_1, \ldots, \varepsilon_{2n})$ can be chosen to generate any $2n$-dimensional $\mathbb{Q}$-vector subspace V_ε of $\Lambda \otimes \mathbb{Q} \simeq \mathbb{Q}^{2N}$, thus the permitted directions for V_ε are dense, and for most of them f is indeed transverse to $\mathcal{D}$. For a transverse linear embedding, we get a $\mathbb{R}$-linear isomorphism $\tilde{u} : \mathbb{R}^{2n} \to \mathbb{C}^N/\mathcal{D}$, and the complex structure J_f on X is precisely the one induced by that isomorphism by pulling-back the standard complex structure on the quotient. For $N \ge 2n$, we claim that all possible translation invariant complex structures on X are obtained. In fact, we can then choose the lattice vector images $\varepsilon_1, \ldots, \varepsilon_{2n}$ to be $\mathbb{C}$-linearly independent, so that the map $u : \mathbb{Z}^{2n} \to \Lambda$, $e_j \mapsto \varepsilon_j$ extends into an injection $v : \mathbb{C}^{2n} \to \mathbb{C}^N$. Once this is done, the isotopy class of embedding is determined, and a translation invariant complex structure J on X is given by a direct sum decomposition $\mathbb{C}^{2n} = S \oplus \overline{S}$ with $\dim_{\mathbb{C}} S = n$ (and $\overline{S}$ the complex congugate of S). What we need is that the composition $\tilde{v} : \mathbb{C}^{2n} \to \mathbb{C}^N \to \mathbb{C}^N/\mathcal{D}$ defines a $\mathbb{C}$-linear isomorphism of S onto $\tilde{v}(S) \subset \mathbb{C}^N/\mathcal{D}$ and $\tilde{v}(\overline{S}) = \{0\}$, i.e. $\mathcal{D} \supset v(\overline{S})$ and $\mathcal{D} \cap v(S) = \{0\}$. The solutions are obtained by taking $\mathcal{D} = v(\overline{S}) \oplus H$, where H is any supplementary subspace of $v(S \oplus \overline{S})$ in $\mathbb{C}^N$ (thus the choice of $\mathcal{D}$ is unique if $N = 2n$, and parametrized by an affine chart of a Grassmannian $G(N-n, N-2n)$ if $N > 2n$). Of course, we can take here Z to be an Abelian variety – even a simple Abelian variety if we wish.

Example 2.5 (LVMB manifolds)**.** We refer to López de Medrano-Verjovsky [LoV97], Meersseman [[Mer00] and Bosio [Bos01] for the original constructions, and sketch here the more general definition given in [Bos01] (or rather an equivalent one, with very minor changes of notation). Let $m \ge 1$ and $N \ge 2m$ be integers, and let $\mathcal{E} = \mathcal{E}_{m,N+1}$ be a non empty set of subsets of $\{0, 1, \ldots, N\}$ of cardinal $2m+1$. For $J \in \mathcal{E}$, define U_J to be the open set of points $[z_0 : \ldots : z_N] \in \mathbb{C}P^N$ such that $z_j \neq 0$ for $j \in J$ and $U_{\mathcal{E}} = \bigcup_{J \in \mathcal{E}} U_J$. Then, consider the action of $\mathbb{C}^m$ on $U_{\mathcal{E}}$ given by

$$w \cdot [z_0 : \ldots : z_N] = \left[e^{\ell_0(w)} z_0 : \ldots : e^{\ell_N(w)} z_N\right]$$

where $\ell_j \in (\mathbb{C}^m)^*$ are complex linear forms $\mathbb{C}^m \to \mathbb{C}$, $0 \le j \le N$. Then Bosio proves ([Bos01], Théorème 1.4), that the space of orbits $X = U_{\mathcal{E}}/\mathbb{C}^m$ is a compact complex manifold if and only if the following two combinatorial conditions are met:

(i) for any two sets J_1, $J_2 \in \mathcal{E}$, the convex envelopes in $(\mathbb{C}^m)^*$ of $\{\ell_j\}_{j \in J_1}$ and $\{\ell_j\}_{j \in J_2}$ overlap on some open set.

(ii) for all $J \in \mathcal{E}$ and $k \in \{0, \ldots, N\}$, there exists $k' \in J$ such that $(J \smallsetminus \{k'\}) \cup \{k\} \in \mathcal{E}$.

The above action can be described in terms of m pairwise commuting Killing vector fields of the action of $\mathrm{PGL}(N+1,\mathbb{C})$ on $\mathbb{CP}^N$, given by

$$\zeta_j=\sum_{k=0}^{N}\lambda_{jk}z_k\frac{\partial}{\partial z_k},\qquad \lambda_{jk}=\frac{\partial\ell_k}{\partial w_j},\quad 1\le j\le m.$$

These vector fields generate a foliation $\mathcal{F}$ of dimension m on $\mathbb{CP}^N$ that is non singular over $U_{\mathcal{E}}$. Under the more restrictive condition defining LVM manifolds, it follows from [[Mer00] that X can be embedded as a smooth compact real analytic submanifold in $U_{\mathcal{E}}$ that is transverse to $\mathcal{F}$; such a submanifold is realized as the transverse intersection of hermitian quadrics $\sum_{0\le k\le N}\lambda_{j,k}|z_k|^2=0$, $1\le j\le m$ (this actually yields $2m$ real conditions by taking real and imaginary parts). In the more general case of LVMB manifolds, Bosio has observed that X can be also embedded smoothly in $U_{\mathcal{E}}\subset\mathbb{CP}^N$ (see [Bos01], Prop. 2.3 and discussion thereafter; and also [BoM06], Part III, section 12).

3. Deformation of transverse embeddings

Let $f:X\to(Z,\mathcal{D})$ be a transverse embedding. Then $J_f:=f^*(J^{Z,\mathcal{D}}_{f(X)})$ defines an almost complex structure on X. We give in this section sufficient conditions on the embedding f that ensure that small deformations of J_f, in a suitable space of almost complex structures on X, are given as $J_{\tilde f}$ where $\tilde f$ are small deformations of f in a suitable space of transverse embeddings of X into $(Z,\mathcal{D})$. Since the implicit function theorem will be needed, we have to introduce various spaces $\mathcal{C}^r$ of mappings. For any $r\in[1,\infty]$, we consider the group $\mathrm{Diff}^r(X)$ of diffeomorphisms of X of class $\mathcal{C}^r$, and $\mathrm{Diff}^r_0(X)\subset\mathrm{Diff}^r(X)$ the subgroup of diffeomorphisms diffeotopic to identity (when $r=s+\gamma$ is not an integer, $s=\lfloor r\rfloor$, we mean that all derivatives up to order s are γ-Hölder continuous). Similarly, we consider the space $\mathcal{C}^r(X,Z)$ of $\mathcal{C}^r$ mappings $X\to Z$ equipped with $\mathcal{C}^r$ convergence topology (of course, in $\mathrm{Diff}^r(X)$, the topology also requires convergence of sequences f_ν^{-1}). If Z is Stein, there exists a biholomorphism $\Phi:TZ\to Z\times Z$ from a neighborhood of the zero section of TZ to a neighborhood of the diagonal in $Z\times Z$, such that $\Phi(z,0)=z$ and $d_\zeta\Phi(z,\zeta)_{|\zeta=0}=\mathrm{Id}$ on T_zZ. When Z is embedded in $\mathbb{C}^{N'}$ for some N', such a map can be obtained by taking $\Phi(z,\zeta)=\rho(z+\zeta)$, where ρ is a local holomorphic retraction $\mathbb{C}^{N'}\to Z$ and T_zZ is identified to a vector subspace of $\mathbb{C}^{N'}$. In general (i.e. when Z is not necessarily Stein), one can still find a $\mathcal{C}^\infty$ map Φ satisfying the same conditions, by taking e.g. $\Phi(z,\zeta)=(z,\exp_z(\zeta))$, where exp is the Riemannian exponential map of a smooth hermitian metric on Z; actually, we will not need Φ to be holomorphic in what follows.

Lemma 3.1. *For $r\in[1,\infty[$, $\mathrm{Diff}^r_0(X)$ is a Banach Lie group with Lie algebra $\mathcal{C}^r(X,TX)$, and $\mathcal{C}^r(X,Z)$ is a Banach manifold whose tangent space at $f:X\to Z$ is $\mathcal{C}^r(X,f^*TZ)$.*

Proof. A use of the map Φ allows us to parametrize small deformations of the embedding f as $\widetilde f(x)=\Phi(f(x),u(x))$ [or equivalently $u(x)=\Phi^{-1}(f(x),\widetilde f(x))$], where u is a smooth sufficiently small section of f^*TZ. This parametrization is one-to-one, and $\widetilde f$ is $\mathcal{C}^r$ if and only if u is $\mathcal{C}^r$ (provided f is). The argument is similar, and very well known indeed, for $\mathrm{Diff}^r_0(X)$. □

Now, let $\mathcal{J}^r(X)$ denote the space of almost complex structures of class $\mathcal{C}^r$ on X. For $r<+\infty$, this is a Banach manifold whose tangent space at a point J is the space of sections

$h \in \mathcal{C}^r(X, \mathrm{End}_{\overline{\mathbb{C}}}(TX))$ satisfying $J\circ h + h\circ J = 0$ (namely conjugate $\mathbb{C}$ linear endomorphisms of TX). There is a natural right action of $\mathrm{Diff}^r_0(X)$ on $\mathcal{J}^{r-1}(X)$ defined by

$$(J,\psi) \mapsto \psi^* J, \qquad \psi^* J(x) = d\psi(x)^{-1} \circ J(\psi(x)) \circ d\psi(x).$$

As is well-known and as a standard calculation shows, the differential of $\psi \mapsto \psi^* J$ is precisely the $\overline{\partial}_J$ operator

$$\overline{\partial}_J : \mathcal{C}^r(X, TX) \longrightarrow \mathcal{C}^{r-1}(X, \Lambda^{0,1}TX^* \otimes TX^{1,0}) = \mathcal{C}^{r-1}(X, \mathrm{End}_{\overline{\mathbb{C}}}(TX)).$$

Let $\Gamma^r(X, Z, \mathcal{D})$ be the space of $\mathcal{C}^r$ embeddings of X into Z that are transverse to $\mathcal{D}$. Transversality is an open condition, so $\Gamma^r(X, Z, \mathcal{D})$ is an open subset in $\mathcal{C}^r(X, Z)$. Now, $\mathrm{Diff}^r_0(X)$ acts on $\Gamma^r(X, Z, \mathcal{D})$ through the natural right action

$$\Gamma^r(X, Z, \mathcal{D}) \times \mathrm{Diff}^r_0(X) \longrightarrow \Gamma^r(X, Z, \mathcal{D}), \qquad (f, \psi) \mapsto f \circ \psi.$$

With respect to the tangent space isomorphisms of Lemma 3.1, the differential of this action at point (f, ψ), $\psi = \mathrm{Id}_X$, is just the addition law in the bundle f^*TZ:

$$\mathcal{C}^r(X, f^*TZ) \times \mathcal{C}^r(X, TX) \to \mathcal{C}^r(X, f^*TZ), \qquad (u, v) \mapsto u + f_*v.$$

If we restrict u to be in $\mathcal{C}^r(X, f^*\mathcal{D})$, we actually get an isomorphism of Banach spaces

$$\mathcal{C}^r(X, f^*\mathcal{D}) \times \mathcal{C}^r(X, TX) \to \mathcal{C}^r(X, f^*TZ), \qquad (u, v) \mapsto u + f_*v \tag{3.1}$$

by the transversality condition. In fact, we can (non canonically) define on $\Gamma^r(X, Z, \mathcal{D})$ a "lifting"

$$\Phi(f, \bullet) : \mathcal{C}^r(X, f^*\mathcal{D}) \to \Gamma^r(X, Z, \mathcal{D}), \qquad u \mapsto \Phi(f, u)$$

on a small neighborhood of the zero section, and the differential of $\Phi(f, \bullet)$ at 0 is given by the inclusion $\mathcal{C}^r(X, f^*\mathcal{D}) \hookrightarrow \mathcal{C}^r(X, f^*TZ)$. Modulo composition by elements of $\mathrm{Diff}^r_0(X)$ close to identity (i.e. in the quotient space $\Gamma^r(X, Z, \mathcal{D})/\mathrm{Diff}^r_0(X)$), small deformations of f are parametrized by $\Phi(f, u)$ where u is a small section of $\mathcal{C}^r(X, f^*\mathcal{D})$. The first variation of f depends only on the differential of Φ along the zero section of TZ, so it is actually independent of the choice of our map Φ. We can think of small variations of f as $f + u$, at least if we are working in local coordinates $(z_1, \ldots, z_N) \in \mathbb{C}^N$ on Z, and consider that $\mathcal{D}_z \subset T_zZ = \mathbb{C}^N$; the use of a map Φ like those already considered is however needed to make the arguments global. Let us summarize these observations as follows.

Lemma 3.2. *For $r < +\infty$, the quotient space $\Gamma^r(X, Z, \mathcal{D})/\mathrm{Diff}^r_0(X)$ is a Banach manifold whose tangent space at f can be identified with $\mathcal{C}^r(X, f^*\mathcal{D})$ via the differential of the composition*

$$\mathcal{C}^r(X, f^*\mathcal{D}) \xrightarrow{\Phi(f,\bullet)} \Gamma^r(X, Z, \mathcal{D}) \longrightarrow \Gamma^r(X, Z, \mathcal{D})/\mathrm{Diff}^r_0(X)$$

at 0, where the first arrow is given by $u \mapsto \Phi(f, u)$ and the second arrow is the natural map to the quotient. □

Our next goal is to compute J_f and the differential dJ_f of $f \mapsto J_f$ when f varies in the above Banach manifold $\Gamma^r(X, Z, \mathcal{D})$. Near a point $z_0 \in Z$ we can pick holomorphic coordinates $z = (z_1, \ldots, z_N)$ centered at z_0, such that $\mathcal{D}_{z_0} = \mathrm{Span}(\partial/\partial z_j)_{n+1\le j\le N}$. Then we have

$$\mathcal{D}_z = \mathrm{Span}\left(\frac{\partial}{\partial z_j} + \sum_{1\le i\le n} a_{ij}(z)\frac{\partial}{\partial z_i}\right)_{n+1\le j\le N}, \ a_{ij}(z_0) = 0. \tag{3.2}$$

In other words $\mathcal{D}_z$ is the set of vectors of the form $(a(z)\eta,\eta)\in\mathbb{C}^n\times\mathbb{C}^{N-n}$, where $a(z)=(a_{ij}(z))$ is a holomorphic map into the space $\mathcal{L}(\mathbb{C}^{N-n},\mathbb{C}^n)$ of $n\times(N-n)$ matrices. A trivial calculation shows that the vector fields $e_j(z)=\frac{\partial}{\partial z_j}+\sum_i a_{ij}(z)\frac{\partial}{\partial z_i}$ have brackets equal to

$$[e_j,e_k]=\sum_{1\le i\le n}\left(\frac{\partial a_{ik}}{\partial z_j}(z_0)-\frac{\partial a_{ij}}{\partial z_k}(z_0)\right)\frac{\partial}{\partial z_i}\ \text{at } z_0,\ \ n+1\le j,k\le N,$$

in other words the torsion tensor θ is given by

$$\theta(z_0)=\sum_{1\le i\le n,\ n+1\le j,k\le N}\theta_{ijk}(z_0)\,dz_j\wedge dz_k\otimes\frac{\partial}{\partial z_i},\quad \theta_{ijk}(z_0)=\frac{1}{2}\left(\frac{\partial a_{ik}}{\partial z_j}(z_0)-\frac{\partial a_{ij}}{\partial z_k}(z_0)\right). \tag{3.3}$$

We now take a point $x_0\in X$ and apply this to $z_0=f(x_0)\in M=f(X)\subset Z$. With respect to coordinates $z=(z_1,\ldots,z_N)$ chosen as above, we have $T_{z_0}M\oplus\mathrm{Span}(\partial/\partial z_j)_{n+1\le j\le N}=T_{z_0}Z$, thus we can represent M in the coordinates $z=(z',z'')\in\mathbb{C}^n\times\mathbb{C}^{N-n}$ locally as a graph $z''=g(z')$ in a small polydisc $\Omega'\times\Omega''$ centered at z_0, and use $z'=(z_1,\ldots,z_n)\in\Omega'$ as local (non holomorphic!) coordinates on M. Here $g:\Omega'\to\Omega''$ is $\mathcal{C}^r$ differentiable and $g(z'_0)=z''_0$. The embedding $f:X\to Z$ is itself obtained as the composition with a certain local diffeomorphism $\varphi:X\supset V\to\Omega'\subset\mathbb{C}^n$, i.e.

$$f=F\circ\varphi \text{ on } V,\quad \varphi:V\ni x\mapsto z'=\varphi(x)\in\Omega'\subset\mathbb{C}^n,\quad F:\Omega'\ni z'\mapsto(z',g(z'))\in Z.$$

With respect to the (z',z'') coordinates, we get a $\mathbb{R}$-linear isomorphism

$$\begin{array}{rcccl} dF(z') & : & \mathbb{C}^n & \longrightarrow & T_{F(z')}M\subset T_{F(z')}Z\simeq\mathbb{C}^n\times\mathbb{C}^{N-n}\\ & & \zeta & \longmapsto & (\zeta,dg(z')\cdot\zeta)=(\zeta,\partial g(z')\cdot\zeta+\overline{\partial}g(z')\cdot\zeta).\end{array}$$

Here $\overline{\partial}g$ is defined with respect to the standard complex structure of $\mathbb{C}^n\ni z'$ and has a priori no intrinsic meaning. The almost complex structure J_f can be explicitly defined by

$$J_f(x)=d\varphi(x)^{-1}\circ J_F(\varphi(x))\circ d\varphi(x), \tag{3.4}$$

where J_F is the almost complex structure on M defined by the embedding $F:M\subset Z$, expressed in coordinates as $z'\mapsto(z',g(z'))$. We get by construction

$$J_F(z')=dF(z')^{-1}\circ\pi_{Z,\mathcal{D},M}(F(z'))\circ J_Z(F(z'))\circ dF(z') \tag{3.5}$$

where J_Z is the complex structure on Z and $\pi_{Z,\mathcal{D},M}(z):T_zZ\to T_zM$ is the $\mathbb{R}$-linear projection to T_zM along $\mathcal{D}_z$ at a point $z\in M$. Since these formulas depend on the first derivatives of F, we see that J_f is of class $\mathcal{C}^{r-1}$ on X and J_F is of class $\mathcal{C}^{r-1}$ on M. Using the identifications $T_{F(z')}M\simeq\mathbb{C}^n$, $T_zZ\simeq\mathbb{C}^N$ given by the above choice of coordinates, we simply have $J_Z\eta=i\eta$ on TZ since the (z_j) are holomorphic, and we get therefore

$$\begin{aligned} J_Z(F(z'))\circ dF(z')\cdot\zeta &= idF(z')\cdot\zeta = i(\zeta,dg(z')\cdot\zeta)=(i\zeta,\partial g(z')\cdot i\zeta-\overline{\partial}g(z')\cdot i\zeta)\\ &= (i\zeta,dg(z')\cdot i\zeta)-2(0,\overline{\partial}g(z')\cdot i\zeta).\end{aligned}$$

By definition of $z\mapsto a(z)$, we have $(a(z)\eta,\eta)\in\mathcal{D}_z$ for every $\eta\in\mathbb{C}^{N-n}$, and so

$$\pi_{Z,\mathcal{D},M}(z)(0,\eta)=\pi_{Z,\mathcal{D},M}(z)\big((0,\eta)-(a(z)\eta,\eta)\big)=-\pi_{Z,\mathcal{D},M}(z)(a(z)\eta,0).$$

We take here $\eta=\overline{\partial}g(z')\cdot i\zeta$. As $(i\zeta,dg(z')\cdot i\zeta)\in T_{F(z')}M$ already, we find

$$\pi_{Z,\mathcal{D},M}(F(z'))\circ J_Z(F(z'))\circ dF(z')\cdot\zeta=(i\zeta,dg(z')\cdot i\zeta)+2\pi_{Z,\mathcal{D},M}(F(z'))\big(a(F(z'))\overline{\partial}g(z')\cdot i\zeta,0\big).$$

From (3.5), we get in this way

$$J_F(z')\cdot\zeta = i\zeta - 2dF(z')^{-1}\circ\pi_{Z,\mathcal{D},M}(F(z'))\big(ia(F(z'))\overline\partial g(z')\cdot\zeta,0\big). \tag{3.6}$$

In particular, since $a(z_0)=0$, we simply have $J_F(z'_0)\cdot\zeta = i\zeta$.

We want to evaluate the variation of the almost complex structure J_f when the embedding $f_t = F_t\circ\varphi_t$ varies with respect to some parameter $t\in[0,1]$. Let $w\in\mathcal{C}^r(X,f^*TZ)$ be a given infinitesimal variation of f_t and $w = u + f_*v$, $u\in\mathcal{C}^r(X,f^*\mathcal{D})$, $v\in\mathcal{C}^r(X,TX)$ its direct sum decomposition. With respect to the trivialization of $\mathcal{D}$ given by our local holomorphic frame $(e_j(z))$, we can write in local coordinates

$$u(\varphi^{-1}(z')) = \big(a(F(z'))\cdot\eta(z'),\eta(z')\big)\in\mathcal{D}_{F(z')}$$

for some section $z'\mapsto\eta(z')\in\mathbb{C}^{N-n}$. Therefore

$$u(\varphi^{-1}(z')) = \big(0,\eta(z') - dg(z')\cdot a(F(z'))\cdot\eta(z')\big) + F_*\big(a(F(z'))\cdot\eta(z')\big)$$

where the first term is "vertical" and the second one belongs to $T_{F(z')}M$. We then get a slightly different decomposition $\widetilde w := w\circ\varphi^{-1} = \widetilde u + F_*\widetilde v\in\mathcal{C}^r(\Omega',F^*TZ)$ where

$$\widetilde u(z') = \big(0,\eta(z') - dg(z')\cdot a(F(z'))\cdot\eta(z')\big)\in\{0\}\times\mathbb{C}^{N-n},$$
$$\widetilde v(z') = \varphi_*v(z') + a(F(z'))\cdot\eta(z')\in\mathbb{C}^n.$$

This allows us to perturb $f = F\circ\varphi$ as $f_t = F_t\circ\varphi_t$ with

$$\begin{cases} X\ni x \longmapsto z' = \varphi_t(x) = \varphi(x) + t\widetilde v(\varphi(x))\in\mathbb{C}^n,\\ \mathbb{C}^n\ni z'\longmapsto F_t(z') = (z',g_t(z'))\in Z,\\ g_t(z') = g(z') + t\widetilde u(z') = g(z') + t\big(\eta(z') - dg(z')\cdot a(F(z'))\cdot\eta(z')\big),\end{cases} \tag{3.7}$$

in such a way that $\dot f_t = \frac{d}{dt}(f_t)_{|t=0} = w$; in the sequel, all derivatives $\frac{d}{dt}{}_{|t=0}$ will be indicated by a dot. We replace f, g, F, M by f_t, g_t, F_t, M_t in (3.6) and compute the derivative for $t=0$ and $z'=z'_0$. Since $a(z_0)=0$, the only non zero term is the one involving the derivative of the map $t\mapsto a(F_t(z'))$. We have $\dot F_t(z'_0) = (0,\eta(z'_0)) = u(x_0)$ where $\eta(z'_0)\in\mathbb{C}^{N-n}$, thus $\dot J_{F_t}$ can be expressed at z'_0 as

$$\dot J_{F_t}(z'_0)\cdot\zeta := \frac{d}{dt}\big(J_{F_t}(z'_0)\cdot\zeta\big)_{|t=0} = -2dF(z'_0)^{-1}\circ\pi_{Z,\mathcal{D},M}(z'_0)\big(ida(z_0)(u(x_0))\cdot\overline\partial g(z'_0)\cdot\zeta,0\big).$$

Now, if we put $\lambda = ida(z_0)(u(x_0))\overline\partial g(z'_0)\cdot\zeta$, as $\mathcal{D}_{z_0} = \{0\}\times\mathbb{C}^{N-n}$ in our coordinates, we immediately get

$$\pi_{Z,\mathcal{D},M}(z'_0)(\lambda,0) = (\lambda,dg(z'_0)\cdot\lambda) = dF(z'_0)\cdot\lambda \implies dF(z'_0)^{-1}\circ\pi_{Z,\mathcal{D},M}(z'_0)(\lambda,0) = \lambda.$$

Therefore, we obtain the very simple expression

$$\dot J_{F_t}(z'_0) = -2i\,da(z_0)(u(x_0))\cdot\overline\partial g(z'_0)\in\mathrm{End}_{\overline{\mathbb{C}}}(\mathbb{C}^n) \tag{3.8}$$

where $da(z_0)(\xi)\in\mathcal{L}(\mathbb{C}^{N-n},\mathbb{C}^n)$ is the derivative of the matrix function $z\mapsto a(z)$ at point $z=z_0$ in the direction $\xi\in\mathbb{C}^N$, and $\overline\partial g(z'_0)$ is viewed as an element of $\mathcal{L}_{\overline{\mathbb{C}}}(\mathbb{C}^n,\mathbb{C}^{N-n})$. What we want is the derivative of $J_{f_t} = d\varphi_t^{-1}\circ J_{F_t}(\varphi_t)\circ d\varphi_t$ at x_0 for $t=0$. Writing φ_* as an abbreviation for $d\varphi$, we find for $t=0$

$$\begin{aligned}\dot J_{f_t} &= -\varphi_*^{-1}\circ d\dot\varphi_t\circ\varphi_*^{-1}\circ J_F(\varphi)\circ\varphi_* + \varphi_*^{-1}\circ J_F(\varphi)\circ d\dot\varphi_t + \varphi_*^{-1}\circ\dot J_{F_t}(\varphi)\circ\varphi_*\\ &= 2J_f\,\overline\partial_{J_f}(\varphi_*^{-1}\dot\varphi_t) + \varphi_*^{-1}\circ\dot J_{F_t}(\varphi)\circ\varphi_*,\end{aligned} \tag{3.9}$$

where the first term in the right hand side comes from the identity $-ds\circ J_f+J_f\circ ds=2J_f\,\overline\partial_{J_f}s$ with $s=\varphi_*^{-1}\dot\varphi_t\in\mathcal{C}^r(X,TX)$ and $ds=\varphi_*^{-1}d\dot\varphi_t$. Our choices $\widetilde v=\varphi_*v+a\circ F\cdot\eta$ and $\varphi_t=\varphi+t\widetilde v\circ\varphi$ yield

$$\dot\varphi_t=\widetilde v\circ\varphi=\varphi_*v+a\circ f\cdot\eta\circ\varphi\implies\varphi_*^{-1}\dot\varphi_t=v+\varphi_*^{-1}(a\circ f\cdot\eta\circ\varphi).$$

If we recall that $a(z_0)=0$ and $\eta(\varphi(x_0))=\eta(z_0')=\mathrm{pr}_2\,u(x_0)$, we get at $x=x_0$

$$\overline\partial_{J_f}(\varphi_*^{-1}\dot\varphi_t)(x_0)=\overline\partial_{J_f}v(x_0)+\varphi_*^{-1}\big(da(z_0)(\overline\partial_{J_f}f(x_0))\cdot\mathrm{pr}_2\,u(x_0)\big).\tag{3.10}$$

By construction, $\varphi_*=d\varphi$ is compatible with the respective almost complex structures (X,J_f) and $(\mathbb{C}^n,J_F)$. A combination of (3.8), (3.9) and (3.10) yields

$$\dot J_{f_t}(x_0)=2J_f\,\overline\partial_{J_f}v(x_0)+\varphi_*^{-1}\Big(2i\,da(z_0)(\overline\partial_{J_f}f(x_0))\cdot\mathrm{pr}_2\,u(x_0)-2i\,da(z_0)(u(x_0))\cdot\overline\partial g(z_0')\circ\varphi_*\Big).$$

As $\overline\partial_{J_f}f(x_0)=(\overline\partial_{J_F}F)(z_0')\circ d\varphi(x_0)=(0,\overline\partial g(z_0'))\circ\varphi_*$ and $f_*=F_*\circ\varphi_*$, we get

$$\dot J_{f_t}(x_0)=f_*^{-1}F_*\Big(2i\,da(z_0)(\overline\partial_{J_f}f(x_0))\cdot\mathrm{pr}_2\,u(x_0)-2i\,da(z_0)(u(x_0))\cdot\mathrm{pr}_2\,\overline\partial_{J_f}f(x_0)\Big)+2J_f\,\overline\partial_{J_f}v(x_0).$$

By (3.3), the torsion tensor $\theta(z_0):\mathcal{D}_{z_0}\times\mathcal{D}_{z_0}\to T_{z_0}Z/\mathcal{D}_{z_0}\simeq F_*T_{z_0}M=f_*T_{x_0}X$ is given by

$$\theta(\eta,\lambda)=\sum_{1\le i\le n,\,n+1\le j,k\le N}\left(\frac{\partial a_{ik}}{\partial z_j}(z_0)-\frac{\partial a_{ij}}{\partial z_k}(z_0)\right)\eta_j\lambda_k\frac{\partial}{\partial z_i}=da(z_0)(\eta)\cdot\lambda-da(z_0)(\lambda)\cdot\eta.$$

Since our point $x_0\in X$ was arbitrary and $\dot J_{f_t}(x_0)$ is the value of the differential $dJ_f(w)$ at x_0, we finally get the global formula

$$dJ_f(w)=2J_f\big(f_*^{-1}\theta(\overline\partial_{J_f}f,u)+\overline\partial_{J_f}v\big)$$

(observe that $\overline\partial_{J_f}f\in\mathcal{L}_{\overline{\mathbb{C}}}(TX,f^*TZ)$ actually takes values in $f^*\mathcal{D}$, so taking a projection to $f^*\mathcal{D}$ is not needed). We conclude:

Proposition 3.3. *The differential of the natural map*

$$\Gamma^r(X,Z,\mathcal{D})\to\mathcal{J}^{r-1}(X),\qquad f\mapsto J_f$$

along every infinitesimal variation $w=u+f_*v:X\to f^*TZ=f^*\mathcal{D}\oplus f_*TX$ of f is given by

$$dJ_f(w)=2J_f\big(f_*^{-1}\theta(\overline\partial_{J_f}f,u)+\overline\partial_{J_f}v\big)$$

where $\theta:\mathcal{D}\times\mathcal{D}\to TZ/\mathcal{D}$ is the torsion tensor of the holomorphic distribution $\mathcal{D}$, and $\overline\partial f=\overline\partial_{J_f}f$, $\overline\partial v=\overline\partial_{J_f}v$ are computed with respect to the almost complex structure (X,J_f).

A difficulty occurring here is the loss of regularity from $\mathcal{C}^r$ to $\mathcal{C}^{r-1}$ coming from the differentiations of f and v. To overcome this difficulty, we have to introduce a slightly smaller space of transverse embeddings that will make possible to apply the implicit function theorem without trouble.

Definition 3.4. *We consider the space*

$$\widetilde\Gamma^r(X,Z,\mathcal{D})\subset\Gamma^r(X,Z,\mathcal{D})\subset\mathcal{C}^r(X,Z)$$

of transverse embeddings $f:X\to Z$ such that f is of class $\mathcal{C}^r$ as well as all "transverse" derivatives $h\cdot df$, where h runs over conormal holomorphic 1-form with values in $(TZ/\mathcal{D})^*$. When $r=\infty$ or $r=\omega$ (real analytic case), we put $\widetilde\Gamma^r(X,Z,\mathcal{D})=\Gamma^r(X,Z,\mathcal{D})$.

Proposition 3.5. *For $r < \infty$, the space $\widetilde\Gamma^r(X, Z, \mathcal{D})$ is a Banach manifold whose tangent space at a point $f : X \to Z$ is*

$$\mathcal{C}^r(X, f^*\mathcal{D}) \oplus \mathcal{C}^{r+1}(X, TX).$$

Moreover:

(i) *The group $\mathrm{Diff}_0^{r+1}(X)$ acts on the right on $\widetilde\Gamma^r(X, Z, \mathcal{D})$;*

(ii) *The natural map $f \mapsto J_f$ sends $\widetilde\Gamma^r(X, Z, \mathcal{D})$ in $\mathcal{J}^r(X)$;*

(iii) *The differential dJ_f of $f \mapsto J_f$ on $\widetilde\Gamma^r(X, Z, \mathcal{D})$ is a continuous morphism*

$$\mathcal{C}^r(X, f^*\mathcal{D}) \oplus \mathcal{C}^{r+1}(X, TX) \longrightarrow \mathcal{C}^r(X, \mathrm{End}_{\overline{\mathbb{C}}}(TX)), \quad (u, v) \longmapsto 2i\big(\theta(\overline\partial f, u) + \overline\partial v\big).$$

Proof. Parts (i), (ii) are clear, as it can be easily seen that $\overline\partial f$ depends only on the transversal part of df by the very definition of J_f and of $\overline\partial f = \frac{1}{2}(df + J_Z \circ df \circ J_f)$. Now, to check the Banach manifold statement, pick $f \in \widetilde\Gamma^r(X, Z, \mathcal{D})$, $u \in \mathcal{C}^r(X, f^*\mathcal{D})$ and $v \in \mathcal{C}^{r+1}(X, TX)$. The flow of v yields a family of diffeomorphisms $\psi_t \in \mathrm{Diff}_0^{r+1}(X)$ with $\psi_0 = \mathrm{Id}_X$ and $\dot\psi_{t|t=0} = v$. Now, fix $\tilde u \in \mathcal{C}^r(Z, \mathcal{D})$ such that $u = \tilde u \circ f$, by extending the $\mathcal{C}^r$ vector field f_*u from $f(X)$ to Z. The extension mapping $u \mapsto \tilde u$ can be chosen to be a continuous linear map of Banach spaces, using e.g. a retraction from a tubular neighborhood of the $\mathcal{C}^r$ submanifold $f(X) \subset Z$. Let f_t be the flow of $\tilde u$ starting at $f_0 = f$ i.e. such that $\frac{d}{dt} f_t = \tilde u(f_t)$. Let $(e_j)_{1 \le j \le N}$ be a local holomorphic frame of TZ such that $(e_j)_{n+1 \le j \le N}$ is a holomorphic frame of $\mathcal{D}$, (e_j^*) its dual frame and ∇ the unique local holomorphic connection of TZ such that $\nabla e_j = 0$. For $j = 1, \ldots, n$, we find

$$\frac{d}{dt}(e_j^* \circ df_t) = e_j^*(f_t) \circ \nabla \frac{df_t}{dt} = e_j^*(f_t) \circ \nabla(\tilde u(f_t)) = e_j^*(f_t) \circ (\nabla \tilde u)(f_t) \cdot df_t.$$

However, if we write $\tilde u = \sum_{n+1 \le k \le N} \tilde u_k e_k$ we see that the composition vanishes since $e_j^* e_k = 0$. Therefore $\frac{d}{dt}(e_j^* \circ df_t) = 0$ and $e_j^* \circ df_t = e_j^*(f) \circ df \in \mathcal{C}^r(X)$. This shows that $f_t \in \widetilde\Gamma^r(X, Z, \mathcal{D})$ for all t, and by definition we have $\dot f_t = \tilde u \circ f = u$. Now, if we define $g_t = f_t \circ \psi_t$, we find $g_t \in \widetilde\Gamma^r(X, Z, \mathcal{D})$ by (i), and $\dot g_t = u + f_* v$ since $\dot\psi_t = v$. The mapping $(u, v) \mapsto g_1 = (f_t \circ \psi_t)_{|t=1}$ defines a local "linearization" of $\widetilde\Gamma^r(X, Z, \mathcal{D})$ near f, and our statement follows, since (iii) is a trivial consequence of the general variation formula. □

Our goal, now, is to understand under which conditions $f \mapsto J_f$ can be a local submersion from $\widetilde\Gamma^r(X, Z, \mathcal{D})$ to $\mathcal{J}^r(X)$. If we do not take into account the quotient by the action of Diff_0^{r+1} on $\mathcal{J}^r(X)$, we obtain a more demanding condition. For that stronger requirement, we see that a sufficient condition is that the continuous linear map

$$\mathcal{C}^r(X, f^*\mathcal{D}) \longrightarrow \mathcal{C}^r(X, \mathrm{End}_{\overline{\mathbb{C}}}(TX)), \quad u \longmapsto 2i\,\theta(\overline\partial f, u) \tag{3.11}$$

be surjective.

Theorem 3.6. *Fix $r \in [1, \infty] \cup \{\omega\}$ (again, ω means real analyticity here). Let $(Z, \mathcal{D})$ be a complex manifold equipped with a holomorphic distribution, and let $f \in \widetilde\Gamma^r(X, Z, \mathcal{D})$ be a transverse embedding with respect to $\mathcal{D}$. Assume that f and the torsion tensor θ of $\mathcal{D}$ satisfy the following additional conditions:*

(i) *f is a totally real embedding, i.e. $\overline\partial f(x) \in \mathrm{End}_{\overline{\mathbb{C}}}(T_xX, T_{f(x)}Z)$ is injective at every point $x \in X$;*

(ii) *for every* $x \in X$ *and every* $\eta \in \mathrm{End}_{\overline{\mathbb{C}}}(TX)$, *there exists a vector* $\lambda \in \mathcal{D}_{f(x)}$ *such that* $\theta(\overline{\partial} f(x) \cdot \xi, \lambda) = \eta(\xi)$ *for all* $\xi \in TX$.

Then there is a neighborhood $\mathcal{U}$ *of* f *in* $\widetilde{\Gamma}^r(X, Z, \mathcal{D})$ *and a neighborhood* $\mathcal{V}$ *of* J_f *in* $\mathcal{J}^r(X)$ *such that* $\mathcal{U} \to \mathcal{V}$, $f \mapsto J_f$ *is a submersion.*

Proof. This is an immediate consequence of the implicit function theorem in the Banach space situation $r < +\infty$. In fact the differential $u \mapsto dJ_f(u)$ in (3.11) is simply given by $u \mapsto L(u)$ where $L \in \mathcal{C}^r(X, \mathrm{Hom}(f^*\mathcal{D}, \mathrm{End}_{\overline{\mathbb{C}}}(TX)))$ is a surjective morphism of bundles of finite rank. Such a morphism always has a smooth splitting $\eta \mapsto \sigma(\eta)$ (also of class $\mathcal{C}^r$), which induces a splitting of our differential on the level of Banach spaces of $\mathcal{C}^r$ sections. In the cases $r = \infty$ or $r = \omega$ where we have a projective (resp. inductive) limit of Banach spaces, the argument is similar. □

Remark 3.7. (a) When $\mathcal{D}$ is a foliation, i.e. $\theta \equiv 0$ identically, or when f is holomorphic or pseudo-holomorphic, i.e. $\overline{\partial} f = 0$, we have $dJ_f \equiv 0$ up to the action of $\mathrm{Diff}_0^{r+1}(X)$. Therefore, when $n > 1$, one can never attain the submersion property by means of a foliation $\mathcal{D}$ or when starting from a (pseudo-)holomorphic map f.

(b) Condition (ii) of Theorem. 3.6 is easily seen to be equivalent to (3.11). When one of these is satisfied, condition (i) on the injectivity of $\overline{\partial} f$ is in fact automatically implied: otherwise a vector $\xi \in \mathrm{Ker}\,\overline{\partial} f(x)$ could never be mapped to a nonzero element $\eta(\xi)$ assigned by η.

(c) For condition (ii) or (3.11) to be satisfied, a necessary condition is that the rank $N - n$ of $\mathcal{D}$ be such that

$$N - n \geq \mathrm{rank}(\mathrm{End}_{\overline{\mathbb{C}}}(TX)) = n^2,$$

i.e. $N \geq n^2 + n$, so the dimension of Z must be rather large compared to $n = \dim_{\mathbb{C}} X$.

We will see in the next section that is indeed possible to find a quasi-projective algebraic variety Z whose dimension is quadratic in n, for which any n-dimensional almost complex manifold (X, J) admits a transverse embedding $f : X \hookrightarrow Z$ satisfying (i), (ii) and $J = J_f$. The present remark shows that one cannot improve the quadratic character $N = O(n^2)$ of the embedding dimension under condition (ii).

4. Universal embedding spaces

We prove here the existence of the universal embedding spaces $(Z_{n,k}, \mathcal{D}_{n,k})$ claimed in Theorem 1.2. They will be constructed as some sort of combination of Grassmannians and twistor bundles. For $k > n$, we let $W \subset \mathbb{R}^{2k} \times G_{\mathbb{R}}(2k, 2n) \times \mathrm{End}_{\mathbb{R}}(\mathbb{R}^{2k})$ be the set of triples (w, S, J) where $w \in \mathbb{R}^{2k}$, S lies in the real Grassmannian of $2n$-codimensional subspaces of $\mathbb{R}^{2k}$, $J \in \mathrm{End}_{\mathbb{R}}(\mathbb{R}^{2k})$ satisfies $J^2 = -I$ and $J(S) \subset S$. Clearly, W is a quasi-projective real algebraic variety, and it has a complexification $W^{\mathbb{C}}$ which can be described as a component of the set of triples

$$(z, S, J) \in \mathbb{C}^{2k} \times G_{\mathbb{C}}(2k, 2n) \times \mathrm{End}_{\mathbb{C}}(\mathbb{C}^{2k})$$

such that $J^2 = -I$ and $J(S) \subset S$. Such an endomorphism J actually induces almost complex structures on $\mathbb{C}^{2k}$ and on S, and thus yields direct sum decompositions $\mathbb{C}^{2k} = \Sigma' + \Sigma''$ and $S = S' \oplus S''$ where $S' \subset \Sigma'$, $S'' \subset \Sigma''$ correspond respectively to the $+i$ and $-i$ eigenspaces. If J is the complexification of some $J^{\mathbb{R}} \in \mathrm{End}_{\mathbb{R}}(\mathbb{R}^{2k})$ and S is the complexification of some $S^{\mathbb{R}} \subset \mathbb{R}^{2k}$, we have

$$\dim S' = \dim S'' = \frac{1}{2}\dim S = k-n \quad\text{and}\quad \dim\Sigma' = \dim\Sigma'' = \frac{1}{2}\dim\mathbb{C}^{2k} = k. \tag{4.1}$$

We let Z be the irreducible nonsingular quasi-projective algebraic variety consisting of triples (z,S,J) as above where J has such "balanced" eigenspaces S', S'', Σ', Σ''. Alternatively, we could view Z as the set of 5-tuples $(z,S',S'',\Sigma',\Sigma'')$ with $S'\subset\Sigma'$, $S''\subset\Sigma''$ and $\mathbb{C}^{2k}=\Sigma'\oplus\Sigma''$, and with dimensions given as above (the decomposition $\mathbb{C}^{2k}=\Sigma'\oplus\Sigma''$ then defines J uniquely). Therefore we have by (4.1)

$$N := \dim_{\mathbb{C}} Z = 2k + 2(k^2+n(k-n))$$

since k^2 is the dimension of the Grassmannian of subspaces $\Sigma'\subset\mathbb{C}^{2k}$ (or $\Sigma''\subset\mathbb{C}^{2k}$), and $n(k-n)$ the dimension of the Grassmannian of subspaces $S'\subset\Sigma'$ (or $S''\subset\Sigma''$). The real part $W=Z^{\mathbb{R}}\subset Z$ can also be seen as the set of 5-tuples $p=(w,S',S'',\Sigma',\Sigma'')$ for which $w=z=\overline{z}\in\mathbb{R}^{2k}$, $S''=\overline{S'}$ and $\Sigma''=\overline{\Sigma'}$.

In our first interpretation, the tangent space TZ at a point $p=(z,S,J)$ consists of triples (ζ,u,h) where $\zeta\in\mathbb{C}^{2k}$, $u\in\mathrm{Hom}(S,\mathbb{C}^{2k}/S)$ and $v\in\mathrm{End}(\mathbb{C}^{2k})$ is such that $v\circ J+J\circ v=0$ and $v(S)\subset S$. In the second interpretation, T_pZ is given by 5-tuples (ζ,u',u'',v',v'') with $\zeta\in\mathbb{C}^{2k}$, $u'\in\mathrm{Hom}(S',\Sigma'/S')$, $u''\in\mathrm{Hom}(S'',\Sigma''/S'')$, $v'\in\mathrm{Hom}(\Sigma',\mathbb{C}^{2k}/\Sigma')$ and $v''\in\mathrm{Hom}(\Sigma'',\mathbb{C}^{2k}/\Sigma'')$. We let $\mathcal{D}_p\subset T_pZ$ be the set of 5-tuples (ζ,u',u'',v',v'') for which $\zeta\in S'\oplus\Sigma''\subset\mathbb{C}^{2k}$ (with no conditions on the other components (u',u'',v',v'')). Therefore we have a canonical isomorphism $T_pZ/\mathcal{D}_p\simeq\Sigma'/S'$, and we see that $\mathcal{D}$ is an algebraic subbundle of corank n, i.e. $\mathrm{rank}(\mathcal{D})=N-n$, and $\mathcal{D}$ is isomorphic to the tautological bundle Σ'/S' arising from the flag manifold structure of pairs (S',Σ') with $S'\subset\Sigma'\subset\mathbb{C}^{2k}$.

Proof of Theorem 1.2. Let (X,J_X) be an arbitrary compact n-dimensional almost complex manifold, where J_X is of class $\mathcal{C}^{r+1}$, $r\in[0,\infty]\cup\{\omega\}$ (we may always assume here that the differential structure of X itself is $\mathcal{C}^\omega$, by well-known results, one could even take X to be given by a real algebraic variety). Since $\dim_{\mathbb{R}}X=2n$, the strong Whitney embedding theorem [Whi44] shows that there exists a $\mathcal{C}^\omega$ embedding $g:X\to\mathbb{R}^k$ where $k=2(2n)=4n$. Let NX be the normal bundle of $g(X)$ in $\mathbb{R}^k$ (with a slight abuse of notation consisting of identifying X and $g(X)$). We use here the Euclidean structure of $\mathbb{R}^k$ to view NX as a subbundle of the trivial tangent bundle $T\mathbb{R}^k$. Next, we embed X in $\mathbb{R}^{2k}$ by the diagonal embedding $x\mapsto G(x)=(g(x),g(x))$, whose normal bundle is $TX\oplus NX\oplus NX$. We have

$$T\mathbb{R}^{2k}_{|G(X)} = TX\oplus TX\oplus NX\oplus NX.$$

On $NX\oplus NX$ (or, for that purpose, on the double of any real vector bundle), there is a tautological almost complex structure $J_{NX\oplus NX}$ given by $(u,v)\mapsto(-v,u)$. For every $x\in X$, we consider the complex structure $\widetilde{J}(x)$ on $T\mathbb{R}^{2k}_{|G(x)}=\mathbb{R}^{2k}$ given by

$$\widetilde{J}(x) := J_X(x)\oplus(-J_X(x))\oplus J_{NX\oplus NX}(x).$$

Notice that $(X,-J_X)$ is the complex conjugate almost complex manifold $\overline{X}$. In some sense, we have embedded X diagonally into $X\times\overline{X}$ (this embedding is totally real and has normal bundle $T\overline{X}$), and composed that diagonal embedding with the product embedding

$$g\times g : X\times\overline{X}\to\mathbb{R}^k\times\mathbb{R}^k=\mathbb{R}^{2k}$$

which has normal bundle $\mathrm{pr}_1^*NX\oplus\mathrm{pr}_2^*NX$. Let $\widetilde{J}^{\mathbb{C}}(x)\in\mathrm{End}(\mathbb{C}^{2k})$ be the complexification of $\widetilde{J}(x)$, and let $\Sigma'_x\subset\mathbb{C}^{2k}$, $\Sigma''_x\subset\mathbb{C}^{2k}$ be the $+i$ and $-i$ eigenspaces of $\widetilde{J}(x)$ respectively (both are k-dimensional). By construction, the bundle Σ' consists of vectors of the form $(\xi^{1,0},\eta^{0,1},u,-iu)$, $\xi^{1,0}\in T^{1,0}X$, $\eta^{0,1}\in T^{0,1}X$, $u\in N^{\mathbb{C}}X$, and similarly Σ'' consists of vectors of the form $(\xi^{0,1},\eta^{1,0},u,iu)$. We further define $S^{\mathbb{R}}\subset T\mathbb{R}^{2k}$ and its fiberwise complexification $S_x=S_x^{\mathbb{R}}\otimes\mathbb{C}\subset\mathbb{C}^{2k}$ by

$$S^{\mathbb{R}}=\{0\}\oplus T\overline{X}\oplus NX\oplus NX,\qquad S=\{0\}\oplus T^{\mathbb{C}}\overline{X}\oplus N^{\mathbb{C}}X\oplus N^{\mathbb{C}}X.$$

Clearly $S_x^{\mathbb{R}}$ is stable by $\widetilde{J}(x)$ and

$$\begin{aligned}S' &:= \Sigma'\cap S=\{0\}\oplus T^{0,1}X\oplus\{(u,-iu),\ u\in N^{\mathbb{C}}X\},\\ S'' &:= \Sigma''\cap S=\{0\}\oplus T^{1,0}X\oplus\{(u,iu),\ u\in N^{\mathbb{C}}X\}\end{aligned}$$

are the $+i$ and $-i$ eigenspaces of $\widetilde{J}^{\mathbb{C}}_{|S}$, respectively. We finally get an embedding of class $\mathcal{C}^{r+1}$

$$f:X\hookrightarrow Z,\qquad x\mapsto\big(G(x),S'_x,S''_x,\Sigma'_x,\Sigma''_x\big),$$

and since $(TZ/\mathcal{D})_{f(x)}\simeq\Sigma'_x/S'_x\simeq T_x^{1,0}X$, we see that the almost complex structure J_f induced by the natural complex structure of $TZ/\mathcal{D}$ coincides with J_X. As this point, Z is quasi-projective but not affine. However $f(X)$ is contained in the real part $W=Z^{\mathbb{R}}$, especially the corresponding subspaces $S=S'\oplus S''$ lie in the real part $G_{\mathbb{R}}(2k,2n)\subset G(2k,2n)$ of the complex Grassmannian. In this situation, we can find an ample divisor Δ of $G(2k,2n)$ that is disjoint from $G_{\mathbb{R}}(2k,2n)$ and invariant by complex conjugation (to see this, we embed the Grassmannian into a complex projective space $\mathbb{CP}^s$ by the Plücker embedding, and observe that the real hyperquadric $Q=\{\sum_{0\le j\le s}z_j^2=0\}$ is disjoint from $\mathbb{RP}^s$; we can thus take Δ to be the inverse image of Q by the Plücker embedding). By restricting the situation to the complement $G(2k,2n)\smallsetminus\Delta$, we obtain an affine algebraic open set $Z'\subset Z$ that is invariant by complex conjugation, so that $f(X)\subset Z'^{\mathbb{R}}$. Theorem 1.2 is proved with $Z_{n,k}=Z'$ and $\mathcal{D}_{n,k}=\mathcal{D}_{|Z'}$. □

Remark 4.1. A computation in coordinates shows that conditions (i) and (ii) of Theorem 3.6 are satisfied in this construction. Actually (i) is already implied by the fact that the image $M=f(X)\subset Z^{\mathbb{R}}_{n,k}$ is totally real.

Remark 4.2. It is easy to find a non singular model for a projective compactification $\overline{Z}_{n,k}$ of $Z_{n,k}$: just consider the set of 5-tuples $p=(z,S',S'',\Sigma',\Sigma'')$ where $z\in\mathbb{CP}^{2k}$, $S'\subset\Sigma'\subset T\mathbb{CP}^{2k}$, $S''\subset\Sigma''\subset T\mathbb{CP}^{2k}$, so that $\pi:\overline{Z}_{n,k}\to\mathbb{CP}^{2k}$ is a fiber bundle whose fibers are products of flag manifolds constructed from the tangent bundle of the base. The associated distribution $\overline{\mathcal{D}}_{n,k}=(\pi_*)^{-1}(S'+\Sigma'')$, however, does possess singularities at all points where the sum $S'+\Sigma''$ is not direct.

Symplectic case: Proof of Theorem 1.4. Let (X,J,ω) be a compact n-dimensional almost complex symplectic manifold with second Betti number $b_2\le b$ and a J-compatible symplectic form ω. We choose b_2 rational cohomology classes on X, denoted $[\omega_1],\ldots,[\omega_{b_2}]$, that form a basis of the De Rham cohomology space $H^2(X,\mathbb{R})$. For this, we take classes $[\omega_j]\in H^2(X,\mathbb{Q})$ very close to $[\omega]$, such that $[\omega]$ lies in the interior of the simplex of vertices $[0]$, $[\omega_1]$, … , $[\omega_{b_2}]$. The 2-form ω_j can be taken to be very close to ω in uniform norm over X. This ensures that the ω_j's are symplectic and that $[\omega]$ is a convex combination $[\omega]=\sum\lambda_j[\omega_j]$ for some $\lambda_1,\ldots,\lambda_{b_2}>0$ with $0<\sum\lambda_j<1$ and $\sum\lambda_j\simeq1$. Since $u=\omega-\sum\lambda_j\omega_j$ is a very small exact 2-form u, we can in fact achieve $u=0$ after replacing one of the ω_j's by $\omega_j+\lambda_j^{-1}u$. Also, after replacing each ω_j by an integer multiple, we obtain $\omega=\sum\lambda_j\omega_j$ where ω_j is a system of integral symplectic forms and $\lambda_j > 0$, $\sum \lambda_j < 1$. After replacing ω_{b_2} by $b - b_2 + 1$ identical copies $\omega_j = \omega_{b_2}$, we can assume that $\omega = \sum_{1\le j\le b} \lambda_j\omega_j$, $\lambda_j > 0$.

According to the effective version of Tischler's theorem stated by Gromov [Gro86] (page 335), for every $j = 1, \ldots, b$, there exists a symplectic embedding $g_j : (X, \omega_j) \to (\mathbb{CP}^k, \gamma_{\rm FS})$ with $k = 2n + 1$, where $\mathbb{CP}^k$ denotes the complex projective space of (complex) dimension k and $\gamma_{\rm FS}$ denotes the Fubini-Study form on $\mathbb{CP}^k$. Then $g := (g_1, \ldots, g_b)$ is a symplectic embedding of (X, ω) into the Kähler complex projective manifold

$$(Y, \gamma_\lambda) := \bigg(\prod_{j=1}^{b} \mathbb{CP}^k\ ,\ \sum_{j=1}^{b} \lambda_j\,{\rm pr}_j^*\,\gamma_{\rm FS}\bigg).$$

Here ${\rm pr}_j : \prod_{j=1}^{b} \mathbb{CP}^k \to \mathbb{CP}^k$ denotes the j-th projection. By construction, we have

$$\omega = \sum_{j=1}^{b} \lambda_j\omega_j = g^*\gamma_\lambda.$$

Let NX be the normal bundle of $g(X)$ in Y. Here, we identify the normal bundle with a subbundle of $TY_{|g(X)}$ by using the symplectic structure, namely we define

$$NX = \big\{\eta \in TY\,;\,\forall\xi \in TX,\ \gamma_\lambda(g_*\xi, \eta) = 0\big\};$$

the positivity condition $\gamma_\lambda(g_*\xi, g_*J_X\xi) = \omega(\xi, J_X\xi) > 0$ for $\xi \ne 0$ implies that we indeed have $g_*TX \cap NX = \{0\}$, and thus $TY_{|g(X)} = g_*TX \oplus NX$. Although we will not make use of this, one can see that the Riemannian and symplectic normal bundles are linked by the relation $NX^{\rm riem} = J_{\rm st}NX^{\rm symp}$ where $J_{\rm st}$ is the standard complex structure of Y, the latter being unrelated to J_X. We embed X in $Y \times \overline{Y}$ by the "diagonal" embedding $x \mapsto G(x) = (g(x), \overline{g(x)})$ (set theoretically $\overline{Y}$ coincides with Y, but we take the conjugate complex structure $J_{\overline{Y}} = -J_Y$). We have a decomposition of the tangent bundle given by

$$T(Y \times \overline{Y})_{G(x)} = TX_x \oplus \overline{TX}_x \oplus NX_x \oplus \overline{NX}_x$$

where the first factor TX_x consists of diagonal vectors $(g_*\xi, \overline{g_*\xi})$ (with the slight abuse of notation consisting in identifying X and $g(X) \subset Y$), the second consists of "antidiagonal" vectors $(g_*\xi, -\overline{g_*\xi})$, and the two normal bundle copies are ${\rm pr}_1^*\,NX$ and ${\rm pr}_2^*\,\overline{NX}$. With respect to this decomposition, we define a complex structure $\tilde{J}(x)$ on $T(Y \times \overline{Y})_{G(x)}$ by

$$\tilde{J}(x) = J_X(x) \oplus (-J_X(x)) \oplus J_{NX\oplus\overline{NX}}(x)$$

where $J_{NX\oplus\overline{NX}}$ is the tautological almost complex structure $(u, v) \mapsto (-\overline{v}, \overline{u})$ on $NX \oplus \overline{NX}$. Clearly, we have $G^*\tilde{J} = J_X$ [in fact $Y \times \overline{Y}$ is just the complexification of the underlying real algebraic structure $Y^{\mathbb{R}}$ on Y, under the anti-holomorphic involution $(x, y) \mapsto (y, x)$]. Let us consider the Kähler structure

$$\tilde{\gamma} = \frac{1}{2}\big(\,{\rm pr}_1^*\,\gamma_\lambda - {\rm pr}_2^*\,\gamma_\lambda\big) \quad \text{on } Y \times \overline{Y}.$$

Notice that $-\gamma_\lambda$ is in fact a Kähler structure on $\overline{Y}$ and that $\omega = g^*\gamma_\lambda = \overline{g}^*(-\gamma_\lambda)$. We thus have $G^*\tilde{\gamma} = g^*\gamma_\lambda = \omega$, and further claim that $\tilde{J}$ is compatible with $\tilde{\gamma}$. In order to check this, let us take two tangent vectors $(\xi_1, \overline{\xi}_2)$, $(\eta_1, \overline{\eta}_2) \in TY \times T\overline{Y}$, and write $\xi = \xi' + \xi''$ for the decomposition of $\xi \in TY$ along $TX \oplus NX$. We find

$$\begin{aligned}
\tilde J(\xi_1,\overline{\xi}_2) &= \tilde J\left(\frac12(\xi_1'+\xi_2',\overline{\xi}_1'+\overline{\xi}_2')+\frac12(\xi_1'-\xi_2',\overline{\xi}_2'-\overline{\xi}_1')+(\xi_1'',0)+(0,\overline{\xi}_2'')\right)\\
&= \frac12\left(J_X(\xi_1'+\xi_2'),-\overline{J_X(\xi_1'+\xi_2')}\right)+\frac12\left(-J_X(\xi_1'-\xi_2'),\overline{J_X(\xi_2'-\xi_1')}\right)\\
&\quad+(-\xi_2'',0)+(0,\overline{\xi}_1'')\\
&= \left(J_X\xi_2'-\xi_2'',\overline{-J_X\xi_1'+\xi_1''}\right).
\end{aligned}$$

Since J_X and ω are compatible and $\omega = g^*\gamma_\lambda$, we have (with our abuse of notation $\xi_1' \simeq g_*\xi_1'$)

$$\gamma_\lambda(J_X\xi_1',J_X\eta_1') = \omega(J_X\xi_1',J_X\eta_1') = \omega(\xi_1',\eta_1') = \gamma_\lambda(\xi_1',\eta_1')$$

and a similar formula for (ξ_2',η_2'). We infer from this and from the γ_λ-orthogonality of the decomposition $TX \oplus NX$ that

$$\begin{aligned}
\tilde\gamma\left(\tilde J(\xi_1,\overline{\xi}_2),\tilde J(\eta_1,\overline{\eta}_2)\right) &= \frac12\gamma_\lambda\left(J_X\xi_2'-\xi_2'',J_X\eta_2'-\eta_2''\right)-\frac12\gamma_\lambda\left(\overline{-J_X\xi_1'+\xi_1''},\overline{-J_X\eta_1'+\eta_1''}\right)\\
&= \frac12\gamma_\lambda\left(J_X\xi_2'-\xi_2'',J_X\eta_2'-\eta_2''\right)+\frac12\gamma_\lambda\left(-J_X\xi_1'+\xi_1'',-J_X\eta_1'+\eta_1''\right)\\
&= \frac12\gamma_\lambda\left(\xi_2',\eta_2'\right)+\frac12\gamma_\lambda\left(\xi_2'',\eta_2''\right)+\frac12\gamma_\lambda\left(\xi_1',\eta_1'\right)+\frac12\gamma_\lambda\left(\xi_1'',\eta_1''\right)\\
&= \frac12\gamma_\lambda\left(\xi_1,\eta_1\right)-\frac12\gamma_\lambda\left(\overline{\xi}_2,\overline{\eta}_2\right)\\
&= \tilde\gamma\left((\xi_1,\overline{\xi}_2),(\eta_1,\overline{\eta}_2)\right).
\end{aligned}$$

This proves that $\tilde J$ is compatible with the restriction of the Kähler structure $\tilde\gamma$ to $G(X)$.

We now construct $Z_{n,b,k}$, following essentially the same lines as for the proof of Theorem 1.2. We view $\tilde Y = Y \times \overline{Y}$ as a real algebraic manifold equipped with a real algebraic symplectic form $\tilde\gamma$ (though $\tilde Y$ is in fact complex projective and $\tilde\gamma$ Kähler). We consider

$$W = \left\{(w,S,J) \in \tilde Y \times G_{\mathbb R}(T\tilde Y,2n) \times \mathrm{End}_{\mathbb R}(T\tilde Y)\,;\ J^2=-I,\ J^*\tilde\gamma=\tilde\gamma,\ J(S)\subset S\right\}$$

and its complexification $W^{\mathbb C}$ which is defined by the same algebraic equations over $\mathbb C$. We define $Z_{n,b,k}$ to be the component of $W^{\mathbb C}$ for which J and $J_{|S}$ have balanced $+i$ and $-i$ eigenspaces $\Sigma'\oplus\Sigma'' = T\tilde Y^{\mathbb C}$ and $S'\oplus S''=S$. There is a natural projection

$$\pi = \pi_{n,b,k} : Z_{n,b,k} \to \tilde Y^{\mathbb C} = Y^2 \times \overline{Y}^2,$$

and $\tilde\gamma$ can be complexified into a Kähler form $\tilde\gamma^{\mathbb C}$ on $\tilde Y^{\mathbb C}$ which restricts to $\tilde\gamma$ on the real part. Our construction produces a canonical lifting $f : X \to Z_{n,b,k}$ of $G : X \to \tilde Y = \tilde Y^{\mathbb R} \subset \tilde Y^{\mathbb C}$. Then, as above, the bundle $\mathcal D_{n,b,k}$ of tangent vectors $\zeta \in TZ_{n,b,k}$ such that $\pi_*\zeta \in S'\oplus\Sigma''$ defines an algebraic distribution transverse to $G(X)$, and additionally $\beta = \pi^*\tilde\gamma^{\mathbb C}$ is a transverse Kähler form that induces the given symplectic structure ω on X. A calculation of dimensions shows that $\dim_{\mathbb C}\tilde Y^{\mathbb C} = 4bk$ and $\dim_{\mathbb C}\tilde Z_{n,b,k} = 2bk(2bk+1)+2n(2bk-n)$, since the symplectic twistor space $\{J\}$ in dimension $2m=4bk$ has dimension $m(m-1)$, and we have additionally to select n-dimensional subspaces S', S'' in the given $2bk$ dimensional eigenspaces of J. The above variety $Z_{n,b,k}$ is merely quasi-projective, but we can of course replace it with a relative projective compactification over $\tilde Y^{\mathbb C}$, and extend $\mathcal D_{n,b,k}$ as a torsion free algebraic subsheaf of $TZ_{n,b,k}$. □

Remark 4.3. In the above result, we could even take β to be a genuine Kähler metric on $Z_{n,b,k}$. In fact $Z_{n,b,k}$ is (quasi-)projective, so it possesses a Kähler metric γ. One can easily conclude by a perturbation argument, after replacing β by $\beta+\varepsilon\gamma$ and letting ω vary in a neighborhood of the original symplectic form on X.

5. A weak version of Bogomolov's conjecture: proof of Theorem 1.5

We start with a formula computing the Nijenhuis tensor of the almost complex structure J_f given by an embedding $f : X \hookrightarrow Z$ transverse to a holomorphic distribution $\mathcal{D}$. Recall that for any smooth (real) vector fields ζ, η of TX, the Nijenhuis tensor N_J of an almost complex structure J is defined in terms of Lie brackets of $\zeta^{0,1}=\frac{1}{2}(\zeta+iJ\zeta)$, $\eta^{0,1}=\frac{1}{2}(\eta+iJ\eta)$ as

$$N_J(\zeta,\eta)=4\,\mathrm{Re}\,[\zeta^{0,1},\eta^{0,1}]^{1,0}=[\zeta,\eta]-[J\zeta,J\eta]+J[\zeta,J\eta]+J[J\zeta,\eta].$$

Proposition 5.1. *If θ denotes the torsion operator of the distribution $\mathcal{D}$ on Z, the Nijenhuis tensor of the almost complex structure J_f induced by a transverse embedding $f : X \hookrightarrow Z$ is given by*

$$\forall z\in X,\ \forall\zeta,\eta\in T_zX,\quad N_{J_f}(\zeta,\eta)=4\,\theta(\overline{\partial}_{J_f}f(z)\cdot\zeta,\overline{\partial}_{J_f}f(z)\cdot\eta). \tag{5.1}$$

Proof. We keep the same notation as in Section 3. Especially, we put $M=f(X)$, and near any point $x_0\in X$, we write $f=F\circ\varphi$ where φ is a local diffeomorphism defined in a neighborhood V of x_0 and $F:\varphi\ni z'\mapsto(z',g(z'))\in Z$. According to (3.6), the almost complex structure J_F on $\varphi(V)\subset M$ and the corresponding one J_f on $V\subset X$ are given by

$$\forall z'\in\varphi(V),\ \forall\zeta\in T_zM,\ J_F(z)\cdot\zeta=i\zeta-2dF(z')^{-1}\pi_{Z,\mathcal{D},M}(ia(F(z'))(\overline{\partial}g(z')\cdot\zeta),0)$$

and $J_f(x)=d\varphi(x)^{-1}\circ J_F(\varphi(x))\circ d\varphi(x)$ for every $x\in V$. Thus, by construction, we have $\overline{\partial}_Jf=\overline{\partial}_{J_F}F\circ d\varphi$ (i.e. $d\varphi$ is compatible with J_f and J_F), and (5.1) is equivalent to

$$\forall z\in M,\ \forall\zeta,\eta\in T_zM,\ N_{J_F}(\zeta,\eta)=4\,\theta(\overline{\partial}_{J_F}F(z)\cdot\zeta,\overline{\partial}_{J_F}F(z)\cdot\eta).$$

Let $\zeta=\sum_{j=1}^n\Big(\zeta_j\frac{\partial}{\partial z_j}+\overline{\zeta_j}\frac{\partial}{\partial\overline{\zeta}_j}\Big)$, $\eta=\sum_{j=1}^n\Big(\eta_j\frac{\partial}{\partial z_j}+\overline{\eta_j}\frac{\partial}{\partial\overline{\eta}_j}\Big)$ be real vector fields in $z'\mapsto T_{z'}M$. For the sake of clarity we denote by $J_{\mathrm{st}}\zeta$ the vector field $i\zeta$ associated with the "standard" almost complex structure of $\mathbb{C}^n$, so that

$$J_{\mathrm{st}}\zeta=\sum_{j=1}^n\left(i\zeta_j\frac{\partial}{\partial z_j}-i\overline{\zeta_j}\frac{\partial}{\partial\overline{\zeta}_j}\right).$$

Without loss of generality (and in order to simplify calculations), we assume ζ, η to have constant coefficients ζ_j, η_j. At the central point $z'_0\in\varphi(V)$ we have $a(F(z'_0))=0$ and $J_F=J_{\mathrm{st}}$, hence (omitting vanishing terms such as $[\zeta,\eta]$), the Nijenhuis tensor of J_F at z'_0 is given by

$$\begin{aligned}N_{J_F}(\zeta,\eta)_{|z'_0}=\ &-J_{\mathrm{st}}\zeta\cdot\big(-2dF(z')^{-1}\pi_{Z,\mathcal{D},M}(ia(F(z'))\cdot(\overline{\partial}g(z')\cdot\eta),0)\big)_{|z'=z'_0}\\&+J_{\mathrm{st}}\eta\cdot\big(-2dF(z')^{-1}\pi_{Z,\mathcal{D},M}(ia(F(z'))\cdot(\overline{\partial}g(z')\cdot\zeta),0)\big)_{|z'=z'_0}\\&+J_{\mathrm{st}}\big[\zeta\cdot\big(-2dF(z')^{-1}\pi_{Z,\mathcal{D},M}(ia(F(z'))\cdot(\overline{\partial}g(z')\cdot\eta),0)\big)\big]_{|z'=z'_0}\\&+J_{\mathrm{st}}\big[-\eta\cdot\big(-2dF(z')^{-1}\pi_{Z,\mathcal{D},M}(ia(F(z'))\cdot(\overline{\partial}g(z')\cdot\zeta),0)\big)\big]_{|z'=z'_0}.\end{aligned}$$

We recall that by our normalization $a(F(z_0'))=0$ and $dF(z_0')\circ\pi_{Z,\mathcal{D},M}(z_0')=\mathrm{id}$. Hence for all ζ, $\eta\in T_{z_0'}M$ we get at z_0'

$$\begin{aligned}N_{J_F}(\zeta,\eta) &= \big(2ida(z_0')(dF(z_0')\cdot J_{\rm st}\zeta)\cdot(\overline{\partial}g(z_0')\cdot\eta),0\big)\\ &\quad + J_{\rm st}\big(-2ida(z_0')(dF(z_0')\cdot\zeta)\cdot\overline{\partial}g(z_0')\cdot\eta,0\big)\\ &\quad - \big(2ida(z_0')(dF(z_0')\cdot J_{\rm st}\eta)\cdot\overline{\partial}g(z_0')\cdot\zeta,0\big)\\ &\quad - J_{\rm st}\big(-2ida(z_0')(dF(z_0')\cdot\eta)\cdot\overline{\partial}g(z_0')\cdot\zeta,0\big)\,.\end{aligned}$$

Since $J_{\rm st}\circ da(F(z_0'))\circ J_{\rm st}=-da(F(z_0'))$, we infer

$$\begin{aligned}&\big(da(z_0')(dF(z_0')\cdot J_{\rm st}\zeta)\cdot(\overline{\partial}g(z_0')\cdot\eta)-J_{\rm st}(da(z_0')(dF(z_0')\cdot\zeta))\cdot\overline{\partial}g(z_0')\cdot\eta,0\big)\\ &\qquad = -\big(J_{\rm st}\,(da(F(z_0'))\,(dF(z_0')+J_{\rm st}\circ dF(z_0')\circ J_{\rm st})\cdot\zeta)\cdot(\overline{\partial}g(z_0')\cdot\eta),0\big)\,.\end{aligned}$$

Finally, since $dF(z_0')+J_{\rm st}\circ dF(z_0')\circ J_{\rm st}=2\overline{\partial}F(z_0')=2\overline{\partial}g(z_0')$ we obtain for every ζ, $\eta\in T_{z_0'}M$

$$\begin{aligned}N_{J_F}(\zeta,\eta) &= -4i\left(J_{\rm st}\left(da(F(z_0'))(\overline{\partial}_{J_F}F(z_0')\cdot\zeta)\cdot(\overline{\partial}_{J_F}F(z_0')\cdot\eta)\right),0\right)\\ &\quad +4i\left(J_{\rm st}\left(da(F(z_0'))(\overline{\partial}_{J_F}F(z_0')\cdot\eta)\cdot(\overline{\partial}_{J_F}F(z_0')\cdot\zeta)\right),0\right)\\ &= -4i\,J_{\rm st}\,\theta\left(\overline{\partial}_{J_F}F(z_0')\cdot\zeta,\overline{\partial}_{J_F}F(z_0')\cdot\eta\right),\end{aligned}$$

which yields the expected formula. □

Proof of Theorem 1.5. Let (X,J) be a complex manifold and let $f:X\hookrightarrow Z_{n,k}$ be an embedding into the universal space $(Z_{n,k},\mathcal{D}_{n,k})$, such that $J_f=J$. Then J_f is an integrable structure, hence N_{J_f} vanishes identically. It follows from (5.1) that $\mathrm{Im}(\overline{\partial}_Jf)$ is contained in the isotropic locus of θ, namely the set of n-dimensional subspaces S in the Grassmannian bundle $\mathrm{Gr}(\mathcal{D}_{n,k},n)\to Z_{n,k}$ such that $\theta_{z|S\times S}=0$ at any point $z\in Z_{n,k}$. □

6. Relation to Nash Algebraic approximations of holomorphic foliations

We prove in this section Proposition 1.6. Let X be a compact complex manifold. We denote by J_X the corresponding (almost) complex structure. We first embed X diagonally into $X\times\overline{X}$. This is a totally real embedding with normal bundle $T\overline{X}$. If we denote by φ the embedding then $\varphi(X)$ is totally real and compact in $X\times\overline{X}$. Moreover if $\mathrm{pr}_1:X\times\overline{X}\to X$ is the projection on the first factor then $\mathrm{Ker}(d\,\mathrm{pr}_1)$ is a holomorphic foliation of $T(X\times\overline{X})$ and, quite trivially, $\varphi(X)$ is transverse to $\mathrm{Ker}(d\,\mathrm{pr}_1)$.

Since $X\times\overline{X}$ is a complexification of the real analytic manifold $\varphi(X)$, a well known result of Grauert [Gra58]) shows that $\varphi(X)$ possesses a Stein tubular neighborhood U in $X\times\overline{X}$ (by Nirenberg and Wells [NiW69], every totally real submanifold of a complex manifold has in fact a fundamental system of Stein neighborhoods, and in the compact case they can be obtained as tubes $U_\varepsilon=\{d(z,w)<\varepsilon\}$ for the geodesic distance associated with any hermitian metric on X). According to a result of E.L. Stout [Sto84], the Stein neighborhood U can be shrunk to a Runge open subset $U'\Subset U$ that is biholomorphic to a bounded polynomial polyhedron Ω in an affine complex algebraic manifold $Z=\{P_j(z)=0\}\subset\mathbb{C}^N$, say $\Omega=\{z\in Z\,;\ |Q_j(z)|<1\}\Subset Z$, where P_j, $Q_j\in\mathbb{C}[z_1,\ldots,z_N]$. Let $\psi:U'\to\Omega$ be this biholomorphism, let $f=\psi\circ\varphi:X\hookrightarrow Z$ be the resulting real analytic embedding and let J_Z be the complex structure on Z. We denote by $M=f(X)=\psi(\varphi(X))$ the image of X by f and by $\mathcal{F}=f_*(\mathrm{Ker}(d\,\mathrm{pr}_1)_{|U'})$ the direct image of $\mathrm{Ker}(d\,\mathrm{pr}_1)$ restricted to U'. Then

$\mathcal{F} \subset TZ_{|\Omega}$ is a holomorphic foliation of codimension n on Ω, and M is transverse to $\mathcal{F}$. By construction, the complex structure $J_M^{Z,\mathcal{F}}$ on M induced by $(TZ/\mathcal{F}, J_Z)$ on M coincides with $f_*(J_X)$. Now, we invoke the following

Proposition 6.1. *There exists a Runge open subset $\tilde{\Omega} \subset \mathbb{C}^N$ such that $\Omega = \tilde{\Omega} \cap Z$, a holomorphic retraction $\rho : \tilde{\Omega} \to \Omega$, and a holomorphic foliation $\tilde{\mathcal{F}}$ of codimension n on $\tilde{\Omega}$ such that $M \subset \Omega$ is transverse to $\tilde{\mathcal{F}}$.*

Proof. The normal bundle sequence

$$0 \to TZ \to T\mathbb{C}^N_{|Z} \to NZ \to 0$$

admits an algebraic splitting $\sigma : NZ \to T\mathbb{C}^N_{|Z}$ since $H^1_{\mathrm{alg}}(Z, \mathrm{Hom}(NZ, TZ)) = 0$ (Z being affine). Then $h(z, \zeta) = z + \sigma(z) \cdot \zeta$, $\zeta \in NZ_z$, defines an algebraic biholomorphism h from a tubular neighborhood V of the zero section of NZ onto a neighborhood $h(V)$ of Z in $\mathbb{C}^N$. Clearly, if $\pi : NZ \to Z$ is the natural projection, $\rho = \pi \circ h^{-1}$ is a Nash algebraic retraction from $\tilde{V} := h(V)$ onto Z. We take

$$\tilde{\Omega} = \Big\{ z \in \mathbb{C}^N\,;\ |P(z)|^2 := \sum |P_j(z)|^2 < \varepsilon(1 + |z|^2)^{-A},\ |Q_j(z)|^2 + C|P(z)| < 1 \Big\}$$

whith $\varepsilon \ll 1$ and A, $C \gg 1$ chosen so large that $\tilde{\Omega} \Subset h(V)$. Then ρ maps $\tilde{\Omega}$ submersively onto Ω, and $\tilde{\Omega}$ is a Runge open subset in $\mathbb{C}^N$. We simply take $\tilde{\mathcal{F}} = (\rho_*)^{-1}\mathcal{F} \subset T\tilde{\Omega}$ to be the inverse image of $\mathcal{F}$ in $\tilde{\Omega}$. □

End of proof of Proposition 1.6. Thanks to Prop. 6.1 and our preliminary discussion, we may assume that $f : X \hookrightarrow Z \cap \tilde{\Omega} \subset \tilde{\Omega}$ is a real analytic embedding into a Runge open subset $\tilde{\Omega} \subset \mathbb{C}^N$, transversally to a holomorphic foliation $\tilde{\mathcal{F}}$ on $\tilde{\Omega}$, with $J_X = J_f$. Assume that such foliations can be approximated by Nash algebraic foliations $\tilde{\mathcal{F}}_\nu$ on $\tilde{\Omega}$, uniformly on compact subsets. This means that $\tilde{\mathcal{F}}_\nu$ is given by a Nash algebraic distribution $\delta_\nu : \tilde{\Omega} \to \mathrm{Gr}(\mathbb{C}^N, n)$ that is moreover integrable. It is worth observing that if the integrability assumption is dropped, then the existence of the Nash algebraic approximating sequence δ_ν is actually granted by [DLS93] (but is seems quite difficult to enforce the integrability condition in this context).

Now $M = f(X)$ is still transverse to $\mathcal{F}_\nu$ for $\nu \geq \nu_0$, and in this way we would obtain a sequence of integrable complex structures $J_\nu = f^* J_M^{Z,\mathcal{F}_\nu}$ on X that approximate $J_X = f^* J_M^{Z,\mathcal{F}}$ in the Kuranishi space of small deformations of X. □

Remark 6.2. Instead of embedding X in an affine algebraic manifold, we could instead embed the whole Kuranishi space $\mathcal{X} \to S$ of X into a product $Z \times S$, keeping S as a parameter space (it is enough to take a small Stein neighborhood $S' \subset S$ containing the base point 0 of the central fiber $\mathcal{X}_0 = X$). For this we simply observe that X embeds diagonally as a totally real submanifold in $\mathcal{X} \times \overline{X}$, hence it admits a Stein neighborhood as before, and we can embed the latter as a Runge open set in an affine algebraic manifold. It is then not unlikely that one could also embed the original complex structure J_X (without having to take an approximation), by using some sort of openness argument and the fact that we have embedded the whole Kuranishi space.

Remark 6.3. It should also be observed that there is probably no topological obstruction to the approximation problem. In fact, it is enough to consider the case where we have a real analytic embedding $f : X \hookrightarrow Z$, transversal to a holomorphic foliation given by a *trivial* subbundle $\mathcal{F} \subset TZ$ on some Runge open set $\Omega \Subset Z$ of an affine algebraic manifold. Otherwise, thanks to [DLS93], one can always assume that $\mathcal{F}$ is isomorphic to the restriction of an algebraic vector bundle $\mathcal{F}'$ on Z (possibly after shrinking Ω and replacing Z by some finite cover $Z' \to Z$). Then, since Z is affine, one can find a surjective algebraic morphism $\mu : \mathcal{O}_Z^{\oplus p} \to \mathcal{F}'$. Its kernel $\mathcal{G} = \operatorname{Ker} \mu$ satisfies $\mathcal{F}' \oplus \mathcal{G} \simeq \mathcal{O}_Z^{\oplus p}$, i.e. $\mathcal{F}' \oplus \mathcal{G}$ is algebraically trivial. We replace Z by $\tilde{Z} = \mathcal{G}$ (the total space of $\mathcal{G}$), and $\mathcal{F}$ by the inverse image $\tilde{\mathcal{F}} = (\pi_*)^{-1}(\mathcal{F}) \subset T\tilde{Z}$ via $\pi : \tilde{Z} \to Z$. Then $\tilde{\mathcal{F}} \simeq \pi^*(\mathcal{F}' \oplus \mathcal{G})$, hence $\tilde{\mathcal{F}}$ is holomorphically trivial. We can therefore always reduce ourselves to the case where $\mathcal{F}$ is holomorphically trivial. When this is the case, $\mathcal{F}$ admits a global holomorphic frame (ζ_j), and the Lie brackets satisfy $[\zeta_j, \zeta_k] = \sum_\ell u_{jk\ell}\zeta_\ell$ for some uniquely defined holomorphic functions $u_{jk\ell}$ on Ω. These functions can of course be approximated by a sequence of polynomials $p^\nu_{jk\ell} \in \mathbb{C}[Z]$, $\nu \in \mathbb{N}$, but it is unclear how to construct Nash algebraic foliations from these data . . .

References

[Bos01] Bosio, F., *Variétés complexes compactes : une généralisation de la construction de Meersseman et López de Medrano-Verjovsky*, Ann. Inst. Fourier, Grenoble **51** (2001) 1259–1297.

[BoM06] Bosio, F., Meersseman, L. *Real quadrics in $\mathbb{C}^n$, complex manifolds and convex polytopes*, Acta Math. **197** (2006) 53–127.

[Bog96] Bogomolov, F., *Complex Manifolds and Algebraic Foliation*, Publ. RIMS-1084, Kyoto, June 1996 (unpublished).

[DLS93] Demailly, J.-P., Lempert, L., Shiffman, B., *A*lgebraic approximation of holomorphic maps from Stein domains to projective manifolds, Duke Math. J. **76** (1994) 333–363.

[Gra58] Grauert, H., *On Levi's problem and the imbedding of real-analytic manifolds*, Ann. of Math. **68** (1958) 460-472.

[Gro86] Gromov, M., *Partial differential relations*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Vol. 9, Springer-Verlag, Berlin, Heidelberg, 1986, 363 pp.

[LoV97] López de Medrano, S., Verjovsky, A., *A new family of complex compact non symplectic manifolds*, Bol. Soc. Math. Bra. bf 28-2 (1997), 253–269.

[[Mer00] Meersseman, L., *A new geometric construction of compact complex manifolds in any dimension*, Math. Annalen **317** (2000), 79–115.

[NiW69] Nirenberg, R. and and Wells, R.O., *Approximation theorems on differentiable submanifolds of a complex manifold*, Trans. Amer. Math. Soc. **142** (1969), 15–35.

[Sto84] Stout, E.L., *Algebraic domains in Stein manifolds.* Proceedings of the conference on Banach algebras and several complex variables (New Haven, Conn., 1983), 259266, Contemp. Math., 32, Amer. Math. Soc., Providence, RI, 1984.

[Tis77] Tischler, D., *Closed 2-forms and an embedding theorem for symplectic manifolds*, J. Differential Geometry **12** (1977), 229–235.

[Whi44] Whitney, H., *The self-intersections of a smooth n-manifold in 2n-space*, Ann. of Math. **45** (1944), 220–246.

Jean-Pierre Demailly & Hervé Gaussier
Institut Fourier, Université de grenoble I
BP74, 100 rue des maths, 38402 Saint-Martin d'Hères, France
e-mail: jean-pierre.demailly@ujf-grenoble.fr, herve.gaussier@ujf-grenoble.fr