Hyperbolic algebraic varieties and holomorphic differential equations

Jean-Pierre Demailly

Université de Grenoble I, Institut Fourier

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§0. Introduction

The goal of these notes is to explain recent results in the theory of complex varieties, mainly projective algebraic ones, through a few geometric questions pertaining to hyperbolicity in the sense of Kobayashi. A complex space X is said to be hyperbolic if analytic disks $f : \mathbb{D} \to X$ through a given point form a normal family. If X is not hyperbolic, a basic question is to analyze entire holomorphic curves $f : \mathbb{C} \to X$, and especially to understand the Zariski closure $Y \subset X$ of the union $\bigcup f(\mathbb{C})$ of all those curves. A tantalizing conjecture by Green-Griffiths and Lang says that Y is a proper algebraic subvariety of X whenever X is a projective variety of general type. It is also expected that very generic algebraic hypersurfaces X of high degree in complex projective space $\mathbb{P}^{n+1}$ are Kobayashi hyperbolic, i.e. without any entire holomorphic curves $f : \mathbb{C} \to X$. A convenient framework for this study is the category of "directed manifolds", that is, the category of pairs (X, V) where X is a complex manifold and V a holomorphic subbundle of T_X, possibly with singularities – this includes for instance the case of holomorphic foliations. If X is compact, the pair (X, V) is hyperbolic if and only if there are no nonconstant entire holomorphic curves $f : \mathbb{C} \to X$ tangent to V, as a consequence of the Brody criterion. We describe here the construction of certain jet bundles J_kX, $J_k(X, V)$, and corresponding projectivized k-jet bundles P_kV. These bundles, which were introduced in various contexts (Semple in 1954, Green-Griffiths in 1978) allow to analyze hyperbolicity in terms of certain negativity properties of the curvature. For instance, $\pi_k : P_kV \to X$ is a tower of projective bundles over X and carries a canonical line bundle $\mathcal{O}_{P_kV}(1)$; the hyperbolicity of X is then conjecturally equivalent to the existence of suitable singular hermitian metrics of negative curvature on $\mathcal{O}_{P_kV}(-1)$ for k large enough. The direct images $(\pi_k)_*\mathcal{O}_{P_kV}(m)$ can be viewed as bundles of algebraic differential operators of order k and degree m, acting on germs of curves and invariant under reparametrization.

Following an approach initiated by Green and Griffiths, one can use the Ahlfors-Schwarz lemma in the situation where the jet bundle carries a (possibly singular) metric of negative curvature, to infer that every nonconstant entire curve $f : \mathbb{C} \to V$ tangent to V must be contained in the base locus of the metric. A related result is the fundamental vanishing theorem asserting that entire curves must be solutions of the algebraic differential equations provided by global sections of jet bundles, whenever their coefficients vanish on a given ample divisor; this result was obtained in the mid 1990's as the conclusion of contributions by Bloch, Green-Griffiths, Siu-Yeung and the author. It can in its turn be used to prove various important geometric statements. One of them is the Bloch theorem, which was confirmed at the end of the 1970's by Ochiai and Kawamata, asserting that the Zariski closure of an entire curve in a complex torus is a translate of a subtorus.

Since then many developments occurred, for a large part via the technique of constructing jet differentials – either by direct calculations or by various indirect methods: Riemann-Roch calculations, vanishing theorems ... In 1997, McQuillan introduced his "diophantine approximation" method, which was soon recognized to be an important tool in the study of holomorphic foliations, in parallel with Nevanlinna theory and the construction of Ahlfors currents. Around 2000, Siu showed that generic hyperbolicity results in the direction of the Kobayashi conjecture could be investigated by combining the algebraic techniques of Clemens, Ein and Voisin with the existence of certain "vertical" meromorphic vector fields on the jet space of the universal hypersurface of high degree; these vector fields are actually used to differentiate the global sections of the jet bundles involved, so as to produce new sections with a better control on the base locus. Also, in 2007, Demailly pioneered the use of holomorphic Morse inequalities to construct jet differentials; in 2010, Diverio, Merker and Rousseau were able in that way to prove the Green-Griffiths conjecture for generic hypersurfaces of high degree in projective space – their proof also makes an essential use of Siu's differentiation technique via meromorphic vector fields, as improved by Păun and Merker in 2008. The last sections of the notes are devoted to explaining the holomorphic Morse inequality technique; as an application, one obtains a partial answer to the Green-Griffiths conjecture in a very wide context : in particular, for every projective variety of general type X, there exists a global algebraic differential operator P on X (in fact many such operators P_j) such that every entire curve $f : \mathbb{C} \to X$ must satisfy the differential equations $P_j(f; f', \ldots, f^{(k)}) = 0$. We also recover from there the result of Diverio-Merker-Rousseau on the generic Green-Griffiths conjecture (with an even better bound asymptotically as the dimension tends to infinity), as well as a recent recent of Diverio-Trapani (2010) on the hyperbolicity of generic 3-dimensional hypersurfaces in $\mathbb{P}^4$.

§1. Basic hyperbolicity concepts

§1.A. Kobayashi hyperbolicity

We first recall a few basic facts concerning the concept of hyperbolicity, according to S. Kobayashi [Kob70, Kob76]. Let X be a complex space. An *analytic disk* in X a holomorphic map from the unit disk $\Delta = D(0,1)$ to X. Given two points $p, q \in X$, consider a *chain of analytic disks* from p to q, that is a chain of points $p = p_0, p_1, \ldots, p_k = q$ of X, pairs of points $a_1, b_1, \ldots, a_k, b_k$ of Δ and holomorphic maps $f_1, \ldots, f_k : \Delta \to X$ such that

$$f_i(a_i) = p_{i-1}, \quad f_i(b_i) = p_i, \qquad i = 1, \ldots, k.$$

Denoting this chain by α, define its length $\ell(\alpha)$ by

$$(1.1') \qquad \ell(\alpha) = d_P(a_1, b_1) + \cdots + d_P(a_k, b_k)$$

and a pseudodistance d^K_X on X by

$$(1.1'') \qquad d^K_X(p, q) = \inf_\alpha \ell(\alpha).$$

This is by definition the *Kobayashi pseudodistance* of X. In the terminology of Kobayashi [Kob75], a *Finsler metric* (resp. *pseudometric*) on a vector bundle E is a homogeneous positive (resp. nonnegative) positive function N on the total space E, that is,

$$N(\lambda\xi) = |\lambda|\, N(\xi) \qquad \text{for all } \lambda \in \mathbb{C} \text{ and } \xi \in E,$$

but in general N is not assumed to be subbadditive (i.e. convex) on the fibers of E. A Finsler (pseudo-)metric on E is thus nothing but a hermitian (semi-)norm on the tautological line bundle $\mathcal{O}_{P(E)}(-1)$ of lines of E over the projectivized bundle $Y = P(E)$. The *Kobayashi-Royden infinitesimal pseudometric* on X is the Finsler pseudometric on the tangent bundle T_X defined by

$$(1.2) \qquad \mathbf{k}_X(\xi) = \inf\big\{\lambda > 0\,;\, \exists f : \Delta \to X,\ f(0) = x,\ \lambda f'(0) = \xi\big\}, \qquad x \in X,\ \xi \in T_{X,x}.$$

Here, if X is not smooth at x, we take $T_{X,x} = (\mathfrak{m}_{X,x}/\mathfrak{m}^2_{X,x})^*$ to be the Zariski tangent space, i.e. the tangent space of a minimal smooth ambient vector space containing the germ (X, x); all tangent vectors may not be reached by analytic disks and in those cases we put $\mathbf{k}_X(\xi) = +\infty$. When X is a smooth manifold, it follows from the work of H.L. Royden ([Roy71], [Roy74]) that d^K_X is the integrated pseudodistance associated with the pseudometric, i.e.

$$d^K_X(p, q) = \inf_\gamma \int_\gamma \mathbf{k}_X(\gamma'(t))\, dt,$$

where the infimum is taken over all piecewise smooth curves joining p to q; in the case of complex spaces, a similar formula holds, involving jets of analytic curves of arbitrary order, cf. S. Venturini [Ven96].

1.3. Definition. *A complex space X is said to be hyperbolic* (*in the sense of Kobayashi*) *if d^K_X is actually a distance, namely if $d^K_X(p, q) > 0$ for all pairs of distinct points (p, q) in X.*

When X is hyperbolic, it is interesting to investigate when the Kobayashi metric is complete: one then says that X is a *complete hyperbolic* space. However, we will be mostly concerned with compact spaces here, so completeness is irrelevant in that case.

Another important property is the *monotonicity* of the Kobayashi metric with respect to holomorphic mappings. In fact, if $\Phi : X \to Y$ is a holomorphic map, it is easy to see from the definition that

$$(1.4) \qquad d^K_Y(\Phi(p), \Phi(q)) \leqslant d^K_X(p, q), \qquad \text{for all } p, q \in X.$$

The proof merely consists of taking the composition $\Phi \circ f_i$ for all clains of analytic disks connecting p and q in X. Clearly the Kobayashi pseudodistance $d_{\mathbb{C}}^K$ on $X = \mathbb{C}$ is identically zero, as one can see by looking at arbitrarily large analytic disks $\Delta \to \mathbb{C}$, $t \mapsto \lambda t$. Therefore, if there is any (non constant) *entire curve* $\Phi : \mathbb{C} \to X$, namely a non constant holomorphic map defined on the whole complex plane $\mathbb{C}$, then by monotonicity d_X^K is identically zero on the image $\Phi(\mathbb{C})$ of the curve, and therefore X cannot be hyperbolic. When X is hyperbolic, it follows that X cannot contain rational curves $C \simeq \mathbb{P}^1$, or elliptic curves $\mathbb{C}/\Lambda$, or more generally any non trivial image $\Phi : W = \mathbb{C}^p/\Lambda \to X$ of a p-dimensional complex torus (quotient of $\mathbb{C}^p$ by a lattice).

§1.B. The case of complex curves (i.e. Riemann surfaces)

The only case where hyperbolicity is easy to assess is the case of curves ($\dim_{\mathbb{C}} X = 1$). In fact, as the disk is simply connected, every holomorphic map $f : \Delta \to X$ lifts to the universal cover $\widehat{f} : \Delta \to \widehat{X}$, so that $f = \rho \circ \widehat{f}$ where $\rho : \widehat{X} \to X$ is the projection map.

Now, by the Poincaré-Koebe uniformization theorem, every simply connected Riemann surface is biholomorphic to $\mathbb{C}$, the unit disk Δ or the complex projective line $\mathbb{P}^1$. The complex projective line $\mathbb{P}^1$ has no smooth étale quotient since every automorphism of $\mathbb{P}^1$ has a fixed point; therefore the only case where $\widehat{X} \simeq \mathbb{P}^1$ is when $X \simeq \mathbb{P}^1$ already. Assume now that $\widehat{X} \simeq \mathbb{C}$. Then $\pi_1(X)$ operates by translation on $\mathbb{C}$ (all other automorphisms are affine nad have fixed points), and the discrete subgroups of $(\mathbb{C}, +)$ are isomorphic to $\mathbb{Z}^r$, $r = 0, 1, 2$. We then obtain respectively $X \simeq \mathbb{C}$, $X \simeq \mathbb{C}/2\pi i\mathbb{Z} \simeq \mathbb{C}^* = \mathbb{C} \smallsetminus \{0\}$ and $X \simeq \mathbb{C}/\Lambda$ where Λ is a lattice, i.e. X is an elliptic curve. In all those cases, any entire function $\widehat{f} : \mathbb{C} \to \mathbb{C}$ gives rise to an entire curve $f : \mathbb{C} \to X$, and the same is true when $X \simeq \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$.

Finally, assume that $\widehat{X} \simeq \Delta$; by what we have just seen, this must occur as soon as $X \not\simeq \mathbb{P}^1, \mathbb{C}, \mathbb{C}^*, \mathbb{C}/\Lambda$. Let us take on X the infinitesimal metric ω_P which is the quotient of the Poincaré metric on Δ. The Schwarz-Pick lemma shows that $d_\Delta^K = d_P$ coincides with the Poincaré metric on Δ, and it follows easily by the lifting argument that we have $\mathbf{k}_X = \omega_P$. In particular, d_X^K is non degenerate and is just the quotient of the Poincaré metric on Δ, i.e.

$$d_X^K(p, q) = \inf_{p' \in \rho^{-1}(p),\, q' \in \rho^{-1}(q)} d_P(p', q').$$

We can summarize this discussion as follows.

1.5. Theorem. *Up to bihomorphism, any smooth Riemann surface X belongs to one* (*and only one*) *of the following three types.*

(a) (*rational curve*) $X \simeq \mathbb{P}^1$.

(b) (*parabolic type*) $\widehat{X} \simeq \mathbb{C}$, $X \simeq \mathbb{C}$, $\mathbb{C}^*$ *or* $X \simeq \mathbb{C}/\Lambda$ (*elliptic curve*)

(c) (*hyperbolic type*) $\widehat{X} \simeq \Delta$. *All compact curves X of genus $g \geqslant 2$ enter in this category, as well as $X = \mathbb{P}^1 \smallsetminus \{a, b, c\} \simeq \mathbb{C} \smallsetminus \{0, 1\}$, or $X = \mathbb{C}/\Lambda \smallsetminus \{a\}$* (*elliptic curve minus one point*).

In some rare cases, the one-dimensional case can be used to study the case of higher dimensions. For instance, it is easy to see by looking at projections that the Kobayashi pseudodistance on a product $X \times Y$ of complex spaces is given by

$$d_{X \times Y}^K((x, y), (x', y')) = \max\big(d_X^K(x, x'), d_Y^K(y, y')\big), \tag{1.6}$$

$$\mathbf{k}_{X \times Y}(\xi, \xi') = \max\big(\mathbf{k}_X(\xi), \mathbf{k}_Y(\xi')\big), \tag{1.6$'$}$$

and from there it follows that a product of hyperbolic spaces is hyperbolic. As a consequence $(\mathbb{C} \smallsetminus \{0, 1\})^2$, which is also a complement of five lines in $\mathbb{P}^2$, is hyperbolic.

§1.C. Brody criterion for hyperbolicity

Throughout this subsection, we assume that X is a complex manifold. In this context, we have the following well-known result of Brody [Bro78]. Its main interest is to relate hyperbolicity to the non existence of entire curves.

1.7. Brody reparametrization lemma. *Let ω be a hermitian metric on X and let $f:\Delta\to X$ be a holomorphic map. For every $\varepsilon>0$, there exists a radius $R\geqslant(1-\varepsilon)\|f'(0)\|_\omega$ and a homographic transformation ψ of the disk $D(0,R)$ onto $(1-\varepsilon)\Delta$ such that*

$$\|(f\circ\psi)'(0)\|_\omega=1,\qquad \|(f\circ\psi)'(t)\|_\omega\leqslant\frac{1}{1-|t|^2/R^2}\quad\text{for every } t\in D(0,R).$$

Proof. Select $t_0\in\Delta$ such that $(1-|t|^2)\|f'((1-\varepsilon)t)\|_\omega$ reaches its maximum for $t=t_0$. The reason for this choice is that $(1-|t|^2)\|f'((1-\varepsilon)t)\|_\omega$ is the norm of the differential $f'((1-\varepsilon)t):T_\Delta\to T_X$ with respect to the Poincaré metric $|dt|^2/(1-|t|^2)^2$ on T_Δ, which is conformally invariant under $\mathrm{Aut}(\Delta)$. One then adjusts R and ψ so that $\psi(0)=(1-\varepsilon)t_0$ and $|\psi'(0)|\,\|f'(\psi(0))\|_\omega=1$. As $|\psi'(0)|=\frac{1-\varepsilon}{R}(1-|t_0|^2)$, the only possible choice for R is

$$R=(1-\varepsilon)(1-|t_0|^2)\|f'(\psi(0))\|_\omega\geqslant(1-\varepsilon)\|f'(0)\|_\omega.$$

The inequality for $(f\circ\psi)'$ follows from the fact that the Poincaré norm is maximum at the origin, where it is equal to 1 by the choice of R. Using the Ascoli-Arzelà theorem we obtain immediately:

1.8. Corollary (Brody)**.** *Let (X,ω) be a compact complex hermitian manifold. Given a sequence of holomorphic mappings $f_\nu:\Delta\to X$ such that $\lim\|f'_\nu(0)\|_\omega=+\infty$, one can find a sequence of homographic transformations $\psi_\nu:D(0,R_\nu)\to(1-1/\nu)\Delta$ with $\lim R_\nu=+\infty$, such that, after passing possibly to a subsequence, $(f_\nu\circ\psi_\nu)$ converges uniformly on every compact subset of $\mathbb{C}$ towards a non constant holomorphic map $g:\mathbb{C}\to X$ with $\|g'(0)\|_\omega=1$ and $\sup_{t\in\mathbb{C}}\|g'(t)\|_\omega\leqslant 1$.*

An entire curve $g:\mathbb{C}\to X$ such that $\sup_{\mathbb{C}}\|g'\|_\omega=M<+\infty$ is called a *Brody curve*; this concept does not depend on the choice of ω when X is compact, and one can always assume $M=1$ by rescaling the parameter t.

1.9. Brody criterion. *Let X be a compact complex manifold. The following properties are equivalent.*

(a) *X is hyperbolic.*

(b) *X does not possess any entire curve $f:\mathbb{C}\to X$.*

(c) *X does not possess any Brody curve $g:\mathbb{C}\to X$.*

(d) *The Kobayashi infinitesimal metric $\mathbf{k}_X$ is uniformly bouded below, namely*

$$\mathbf{k}_X(\xi)\geqslant c\|\xi\|_\omega,\qquad c>0,$$

for any hermitian metric ω on X.

Proof. (a)⇒(b) If X possesses an entire curve $f:\mathbb{C}\to X$, then by looking at arbitrary large disks $D(0,R)\subset\mathbb{C}$, it is easy to see that the Kobayashi distance of any two points in $f(\mathbb{C})$ is zero, so X is not hyperbolic.

(b)⇒(c) is trivial.

(c)⇒(d) If (d) does not hold, there exists a sequence of tangent vectors $\xi_\nu \in T_{X,x_\nu}$ with $\|\xi_\nu\|_\omega = 1$ and $\mathbf{k}_X(\xi_\nu) \to 0$. By definition, this means that there exists an analytic curve $f_\nu : \Delta \to X$ with $f(0) = x_\nu$ and $\|f'_\nu(0)\|_\omega \geqslant (1 - \frac{1}{\nu})/\mathbf{k}_X(\xi_\nu) \to +\infty$. One can then produce a Brody curve $g = \mathbb{C} \to X$ by Corollary 1.8, contradicting (c).

(d)⇒(a). In fact (d) implies after integrating that $d_X^K(p,q) \geqslant c\, d_\omega(p,q)$ where d_ω is the geodesic distance associated with ω, so d_X^K must be non degenerate. □

Notice also that if $f : \mathbb{C} \to X$ is an entire curve such that $\|f'\|_\omega$ is unbounded, one can apply the Corollary 1.8 to $f_\nu(t) := f(t + a_\nu)$ where the sequence (a_ν) is chosen such that $\|f'_\nu(0)\|_\omega = \|f(a_\nu)\|_\omega \to +\infty$. Brody's result then produces repametrizations $\psi_\nu : D(0,R_\nu) \to D(a_\nu, 1-1/\nu)$ and a Brody curve $g = \lim f \circ \psi_\nu : \mathbb{C} \to X$ such that $\sup \|g'\|_\omega = 1$ and $g(\mathbb{C}) \subset \overline{f(\mathbb{C})}$. It may happen that the image $g(\mathbb{C})$ of such a limiting curve is disjoint from $f(\mathbb{C})$. In fact Winkelmann [Win07] has given a striking example, actually a projective 3-fold X obtained by blowing-up a 3-dimensional abelian variety Y, such that every Brody curve $g : \mathbb{C} \to X$ lies in the exceptional divisor $E \subset X$; however, entire curves $f : \mathbb{C} \to X$ can be dense, as one can see by taking f to be the lifting of a generic complex line embedded in the abelian variety Y. For further precise information on the localization of Brody curves, we refer the reader to the remarkable results of [Duv08].

The absence of entire holomorphic curves in a given complex manifold is often referred to as *Brody hyperbolicity.* Thus, in the compact case, Brody hyperbolicity and Kobayashi hyperbolicity coincide (but Brody hyeperbolicity is in general a strictly weaker property when X is non compact).

§1.D. Geometric applications

We give here two immediate consequences of the Brody criterion: the openness property of hyperbolicity and a hyperbolicity criterion for subvarieties of complex tori. By definition, a *holomorphic family* of compact complex manifolds is a holomorphic proper submersion $\mathcal{X} \to S$ between two complex manifolds.

1.10. Proposition. *Let $\pi : \mathcal{X} \to S$ be a holomorphic family of compact complex manifolds. Then the set of $s \in S$ such that the fiber $X_s = \pi^{-1}(s)$ is hyperbolic is open in the Euclidean topology.*

Proof. Let ω be an arbitrary hermitian metric on $\mathcal{X}$, $(X_{s_\nu})_{s_\nu \in S}$ a sequence of non hyperbolic fibers, and $s = \lim s_\nu$. By the Brody criterion, one obtains a sequence of entire maps $f_\nu : \mathbb{C} \to X_{s_\nu}$ such that $\|f'_\nu(0)\|_\omega = 1$ and $\|f'_\nu\|_\omega \leqslant 1$. Ascoli's theorem shows that there is a subsequence of f_ν converging uniformly to a limit $f : \mathbb{C} \to X_s$, with $\|f'(0)\|_\omega = 1$. Hence X_s is not hyperbolic and the collection of non hyperbolic fibers is closed in S. □

Consider now an n-dimensional complex torus W, i.e. an additive quotient $W = \mathbb{C}^n/\Lambda$, where $\Lambda \subset \mathbb{C}^n$ is a (cocompact) lattice. By taking a composition of entire curves $\mathbb{C} \to \mathbb{C}^n$ with the projection $\mathbb{C}^n \to W$ we obtain an infinite dimensional space of entire curves in W.

1.11. Theorem. *Let $X \subset W$ be a compact complex submanifold of a complex torus. Then X is hyperbolic if and only if it does not contain any translate of a subtorus.*

Proof. If X contains some translate of a subtorus, then it contains lots of entire curves and so X is not hyperbolic.

Conversely, suppose that X is not hyperbolic. Then by the Brody criterion there exists an entire curve $f : \mathbb{C} \to X$ such that $\|f'\|_\omega \leqslant \|f'(0)\|_\omega = 1$, where ω is the flat metric on W

inherited from $\mathbb{C}^n$. This means that any lifting $\widetilde{f} = (\widetilde{f}, \ldots, \widetilde{f}_\nu) : \mathbb{C} \to \mathbb{C}^n$ is such that

$$\sum_{j=1}^{n} |f_j'|^2 \leqslant 1.$$

Then, by Liouville's theorem, $\widetilde{f}'$ is constant and therefore $\widetilde{f}$ is affine. But then the closure of the image of f is a translate $a + H$ of a connected (possibly real) subgroup H of W. We conclude that X contains the analytic Zariski closure of $a + H$, namely $a + H^{\mathbb{C}}$ where $H^{\mathbb{C}} \subset W$ is the smallest closed complex subgroup of W containing H. □

§2. Directed manifolds

§2.A. Basic definitions concerning directed manifolds

Let us consider a pair (X, V) consisting of a n-dimensional complex manifold X equipped with a *linear subspace* $V \subset T_X$: assuming X connected, this is by definition an irreducible closed analytic subspace of the total space of T_X such that each fiber $V_x = V \cap T_{X,x}$ is a vector subspace of $T_{X,x}$; the rank $x \mapsto \dim_{\mathbb{C}} V_x$ is Zariski lower semicontinuous, and it may a priori jump. We will refer to such a pair as being a (complex) *directed manifold.* A morphism $\Phi : (X, V) \to (Y, W)$ in the category of (complex) directed manifolds is a holomorphic map such that $\Phi_*(V) \subset W$.

The rank $r \in \{0, 1, \ldots, n\}$ of V is by definition the dimension of V_x at a generic point. The dimension may be larger at non generic points; this happens e.g. on $X = \mathbb{C}^n$ for the rank 1 linear space V generated by the Euler vector field: $V_z = \mathbb{C} \sum_{1 \leqslant j \leqslant n} z_j \frac{\partial}{\partial z_j}$ for $z \neq 0$, and $V_0 = \mathbb{C}^n$. Our philosophy is that directed manifolds are also useful to study the "absolute case", i.e. the case $V = T_X$, because there are certain fonctorial constructions which are quite natural in the category of directed manifolds (see e.g. § 5, 6, 7). We think of directed manifolds as a kind of "relative situation", covering e.g. the case when V is the relative tangent space to a holomorphic map $X \to S$. In general, we can associate to V a sheaf $\mathcal{V} = \mathcal{O}(V) \subset \mathcal{O}(T_X)$ of holomorphic sections. These sections need not generate the fibers of V at singular points, as one sees already in the case of the Euler vector field when $n \geqslant 2$. However, $\mathcal{V}$ is a *saturated* subsheaf of $\mathcal{O}(T_X)$, i.e. $\mathcal{O}(T_X)/\mathcal{V}$ has no torsion: in fact, if the components of a section have a common divisorial component, one can always simplify this divisor and produce a new section without any such common divisorial component. Instead of defining directed manifolds by picking a linear space V, one could equivalently define them by considering saturated coherent subsheaves $\mathcal{V} \subset \mathcal{O}(T_X)$. One could also take the dual viewpoint, looking at arbitrary quotient morphisms $\Omega_X^1 \to \mathcal{W} = \mathcal{V}^*$ (and recovering $\mathcal{V} = \mathcal{W}^* = \mathrm{Hom}_{\mathcal{O}}(\mathcal{W}, \mathcal{O})$, as $\mathcal{V} = \mathcal{V}^{**}$ is reflexive). We want to stress here that no assumption need be made on the Lie bracket tensor $[\ ,\ ] : \mathcal{V} \times \mathcal{V} \to \mathcal{O}(T_X)/\mathcal{V}$, i.e. we do not assume any kind of integrability for $\mathcal{V}$ or $\mathcal{W}$.

The singular set $\mathrm{Sing}(V)$ is by definition the set of points where $\mathcal{V}$ is not locally free, it can also be defined as the indeterminacy set of the (meromorphic) classifying map $\alpha : X \dashrightarrow G_r(T_X)$, $z \mapsto V_z$ to the Grasmannian of r dimensional subspaces of T_X. We thus have $V_{|X \smallsetminus \mathrm{Sing}(V)} = \alpha^* S$ where $S \to G_r(T_X)$ is the tautological subbundle of $G_r(T_X)$. The singular set $\mathrm{Sing}(V)$ is an analytic subset of X of codim $\geqslant 2$, hence V is always a holomorphic subbundle outside of codimension 2. Thanks to this remark, one can most often treat linear spaces as vector bundles (possibly modulo passing to the Zariski closure along $\mathrm{Sing}(V)$).

§2.B. Hyperbolicity properties of directed manifolds

Most of what we have done in §1 can be extended to the category of directed manifolds.

2.1. Definition. *Let (X,V) be a complex directed manifold.*

i) *The Kobayashi-Royden infinitesimal metric of (X,V) is the Finsler metric on V defined for any $x \in X$ and $\xi \in V_x$ by*

$$\mathbf{k}_{(X,V)}(\xi) = \inf\big\{\lambda > 0\,;\, \exists f : \Delta \to X,\ f(0) = x,\ \lambda f'(0) = \xi,\ f'(\Delta) \subset V\big\}.$$

Here $\Delta \subset \mathbb{C}$ is the unit disk and the map f is an arbitrary holomorphic map which is tangent to V, i.e., such that $f'(t) \in V_{f(t)}$ for all $t \in \Delta$. We say that (X,V) is infinitesimally hyperbolic if $\mathbf{k}_{(X,V)}$ is positive definite on every fiber V_x and satisfies a uniform lower bound $\mathbf{k}_{(X,V)}(\xi) \geqslant \varepsilon\|\xi\|_\omega$ in terms of any smooth hermitian metric ω on X, when x describes a compact subset of X.

ii) *More generally, the Kobayashi-Eisenman infinitesimal pseudometric of (X,V) is the pseudometric defined on all decomposable p-vectors $\xi = \xi_1 \wedge \cdots \wedge \xi_p \in \Lambda^p V_x$, $1 \leqslant p \leqslant r = \operatorname{rank} V$, by*

$$\mathbf{e}^p_{(X,V)}(\xi) = \inf\big\{\lambda > 0\,;\, \exists f : \mathbb{B}_p \to X,\ f(0) = x,\ \lambda f_*(\tau_0) = \xi,\ f_*(T_{\mathbb{B}_p}) \subset V\big\}$$

where $\mathbb{B}_p$ is the unit ball in $\mathbb{C}^p$ and $\tau_0 = \partial/\partial t_1 \wedge \cdots \wedge \partial/\partial t_p$ is the unit p-vector of $\mathbb{C}^p$ at the origin. We say that (X,V) is infinitesimally p-measure hyperbolic if $\mathbf{e}^p_{(X,V)}$ is positive definite on every fiber $\Lambda^p V_x$ and satisfies a locally uniform lower bound in terms of any smooth metric.

If $\Phi : (X,V) \to (Y,W)$ is a morphism of directed manifolds, it is immediate to check that we have the monotonicity property

$$\mathbf{k}_{(Y,W)}(\Phi_*\xi) \leqslant \mathbf{k}_{(X,V)}(\xi), \qquad \forall \xi \in V, \tag{2.2}$$

$$\mathbf{e}^p_{(Y,W)}(\Phi_*\xi) \leqslant \mathbf{e}^p_{(X,V)}(\xi), \qquad \forall \xi = \xi_1 \wedge \cdots \wedge \xi_p \in \Lambda^p V. \tag{2.2^p}$$

The following proposition shows that virtually all reasonable definitions of the hyperbolicity property are equivalent if X is compact (in particular, the additional assumption that there is locally uniform lower bound for $\mathbf{k}_{(X,V)}$ is not needed). We merely say in that case that (X,V) is *hyperbolic.*

2.3. Proposition. *For an arbitrary directed manifold (X,V), the Kobayashi-Royden infinitesimal metric $\mathbf{k}_{(X,V)}$ is upper semicontinuous on the total space of V. If X is compact, (X,V) is infinitesimally hyperbolic if and only if there are no non constant entire curves $g : \mathbb{C} \to X$ tangent to V. In that case, $\mathbf{k}_{(X,V)}$ is a continuous* (*and positive definite*) *Finsler metric on V.*

Proof. The proof is almost identical to the standard proof for $\mathbf{k}_X$, for which we refer to Royden [Roy71, Roy74]. One of the main ingredients is that one can find a Stein neighborhood of the graph of any analytic disk (thanks to a result of [Siu76], cf. also [Dem90a] for more general results). This allows to obtain "free" small deformations of any given analytic disk, as there are many holomorphic vector fields on a Stein manifold. □

Another easy observation is that the concept of p-measure hyperbolicity gets weaker and weaker as p increases (we leave it as an exercise to the reader, this is mostly just linear algebra).

2.4. Proposition. *If (X,V) is p-measure hyperbolic, then it is $(p+1)$-measure hyperbolic for all $p\in\{1,\ldots,r-1\}$.*

Again, an argument extremely similar to the proof of 1.10 shows that relative hyperbolicity is an open property.

2.5. Proposition. *Let $(\mathcal{X},\mathcal{V})\to S$ be a holomorphic family of compact directed manifolds (by this, we mean a proper holomorphic map $\mathcal{X}\to S$ together with an analytic linear subspace $\mathcal{V}\subset T_{\mathcal{X}/S}\subset T_{\mathcal{X}}$ of the relative tangent bundle, defining a deformation $(X_s,V_s)_{s\in S}$ of the fibers). Then the set of $s\in S$ such that the fiber (X_s,V_s) is hyperbolic is open in S with respect to the Euclidean topology.*

Let us mention here an impressive result proved by Marco Brunella [Bru03, Bru05, Bru06] concerning the behavior of the Kobayashi metric on foliated varieties.

2.6. Theorem (Brunella)**.** *Let X be a compact Kähler manifold equipped with a (possibly singular) rank 1 holomorphic foliation which is not a foliation by rational curves. Then the canonical bundle $K_{\mathcal{F}}=\mathcal{F}^*$ of the foliation is pseudoeffective (i.e. the curvature of $K_{\mathcal{F}}$ is $\geqslant 0$ in the sense of currents).*

The proof is obtained by putting on $K_{\mathcal{F}}$ precisely the metric induced by the Kobayashi metric on the leaves whenever they are generically hyperbolic (i.e. covered by the unit disk). The case of parabolic leaves (covered by $\mathbb{C}$) has to be treated separately.

§3. Algebraic hyperbolicity

In the case of projective algebraic varieties, hyperbolicity is expected to be related to other properties of a more algebraic nature. Theorem 3.1 below is a first step in this direction.

3.1. Theorem. *Let (X,V) be a compact complex directed manifold and let $\sum\omega_{jk}dz_j\otimes d\overline{z}_k$ be a hermitian metric on X, with associated positive $(1,1)$-form $\omega=\frac{i}{2}\sum\omega_{jk}dz_j\wedge d\overline{z}_k$. Consider the following three properties, which may or not be satisfied by (X,V) :*

i) *(X,V) is hyperbolic.*

ii) *There exists $\varepsilon>0$ such that every compact irreducible curve $C\subset X$ tangent to V satisfies*

$$-\chi(\overline{C})=2g(\overline{C})-2\geqslant\varepsilon\,\deg_\omega(C)$$

where $g(\overline{C})$ is the genus of the normalization $\overline{C}$ of C, $\chi(\overline{C})$ its Euler characteristic and $\deg_\omega(C)=\int_C\omega$. (This property is of course independent of ω.)

iii) *There does not exist any non constant holomorphic map $\Phi:Z\to X$ from an abelian variety Z to X such that $\Phi_*(T_Z)\subset V$.*

Then i) $\Rightarrow$ ii) $\Rightarrow$ iii)*.*

Proof. i) $\Rightarrow$ ii). If (X,V) is hyperbolic, there is a constant $\varepsilon_0>0$ such that $\mathbf{k}_{(X,V)}(\xi)\geqslant\varepsilon_0\|\xi\|_\omega$ for all $\xi\in V$. Now, let $C\subset X$ be a compact irreducible curve tangent to V and let $\nu:\overline{C}\to C$ be its normalization. As (X,V) is hyperbolic, $\overline{C}$ cannot be a rational or elliptic curve, hence $\overline{C}$ admits the disk as its universal covering $\rho:\Delta\to\overline{C}$.

The Kobayashi-Royden metric $\mathbf{k}_\Delta$ is the Finsler metric $|dz|/(1-|z|^2)$ associated with the Poincaré metric $|dz|^2/(1-|z|^2)^2$ on Δ, and $\mathbf{k}_{\overline{C}}$ is such that $\rho^*\mathbf{k}_{\overline{C}}=\mathbf{k}_\Delta$. In other words, the metric $\mathbf{k}_{\overline{C}}$ is induced by the unique hermitian metric on $\overline{C}$ of constant Gaussian curvature -4. If $\sigma_\Delta = \frac{i}{2} dz \wedge d\overline{z}/(1-|z|^2)^2$ and $\sigma_{\overline{C}}$ are the corresponding area measures, the Gauss-Bonnet formula (integral of the curvature $= 2\pi\,\chi(\overline{C})$) yields

$$\int_{\overline{C}} d\sigma_{\overline{C}} = -\frac{1}{4}\int_{\overline{C}} \mathrm{curv}(\mathbf{k}_{\overline{C}}) = -\frac{\pi}{2}\chi(\overline{C})$$

On the other hand, if $j : C \to X$ is the inclusion, the monotonicity property (2.2) applied to the holomorphic map $j \circ \nu : \overline{C} \to X$ shows that

$$\mathbf{k}_{\overline{C}}(t) \geqslant \mathbf{k}_{(X,V)}\big((j\circ\nu)_* t\big) \geqslant \varepsilon_0 \big\|(j\circ\nu)_* t\big\|_\omega, \qquad \forall t \in T_{\overline{C}}.$$

From this, we infer $d\sigma_{\overline{C}} \geqslant \varepsilon_0^2 (j\circ\nu)^*\omega$, thus

$$-\frac{\pi}{2}\chi(\overline{C}) = \int_{\overline{C}} d\sigma_{\overline{C}} \geqslant \varepsilon_0^2 \int_{\overline{C}} (j\circ\nu)^*\omega = \varepsilon_0^2 \int_C \omega.$$

Property ii) follows with $\varepsilon = 2\varepsilon_0^2/\pi$.

ii) $\Rightarrow$ iii). First observe that ii) excludes the existence of elliptic and rational curves tangent to V. Assume that there is a non constant holomorphic map $\Phi : Z \to X$ from an abelian variety Z to X such that $\Phi_*(T_Z) \subset V$. We must have $\dim \Phi(Z) \geqslant 2$, otherwise $\Phi(Z)$ would be a curve covered by images of holomorphic maps $\mathbb{C} \to \Phi(Z)$, and so $\Phi(Z)$ would be elliptic or rational, contradiction. Select a sufficiently general curve Γ in Z (e.g., a curve obtained as an intersection of very generic divisors in a given very ample linear system $|L|$ in Z). Then all isogenies $u_m : Z \to Z$, $s \mapsto ms$ map Γ in a $1:1$ way to curves $u_m(\Gamma) \subset Z$, except maybe for finitely many double points of $u_m(\Gamma)$ (if $\dim Z = 2$). It follows that the normalization of $u_m(\Gamma)$ is isomorphic to Γ. If Γ is general enough, similar arguments show that the images

$$C_m := \Phi(u_m(\Gamma)) \subset X$$

are also generically $1:1$ images of Γ, thus $\overline{C}_m \simeq \Gamma$ and $g(\overline{C}_m) = g(\Gamma)$. We would like to show that C_m has degree $\geqslant \mathrm{Const}\, m^2$. This is indeed rather easy to check if ω is Kähler, but the general case is slightly more involved. We write

$$\int_{C_m} \omega = \int_\Gamma (\Phi\circ u_m)^*\omega = \int_Z [\Gamma] \wedge u_m^*(\Phi^*\omega),$$

where Γ denotes the current of integration over Γ. Let us replace Γ by an arbitrary translate $\Gamma + s$, $s \in Z$, and accordingly, replace C_m by $C_{m,s} = \Phi \circ u_m(\Gamma + s)$. For $s \in Z$ in a Zariski open set, $C_{m,s}$ is again a generically $1:1$ image of $\Gamma + s$. Let us take the average of the last integral identity with respect to the unitary Haar measure $d\mu$ on Z. We find

$$\int_{s\in Z}\left(\int_{C_{m,s}} \omega\right) d\mu(s) = \int_Z \left(\int_{s\in Z} [\Gamma + s]\, d\mu(s)\right) \wedge u_m^*(\Phi^*\omega).$$

Now, $\gamma := \int_{s\in Z} [\Gamma + s]\, d\mu(s)$ is a translation invariant positive definite form of type $(p-1, p-1)$ on Z, where $p = \dim Z$, and γ represents the same cohomology class as $[\Gamma]$, i.e. $\gamma \equiv c_1(L)^{p-1}$. Because of the invariance by translation, γ has constant coefficients and so $(u_m)_*\gamma = m^2\gamma$. Therefore we get

$$\int_{s\in Z} d\mu(s) \int_{C_{m,s}} \omega = m^2 \int_Z \gamma \wedge \Phi^*\omega.$$

In the integral, we can exclude the algebraic set of values z such that $C_{m,s}$ is not a generically $1:1$ image of $\Gamma+s$, since this set has measure zero. For each m, our integral identity implies that there exists an element $s_m \in Z$ such that $g(\overline{C}_{m,s_m}) = g(\Gamma)$ and

$$\deg_\omega(C_{m,s_m}) = \int_{C_{m,s_m}} \omega \geqslant m^2 \int_Z \gamma \wedge \Phi^*\omega.$$

As $\int_Z \gamma \wedge \Phi^*\omega > 0$, the curves C_{m,s_m} have bounded genus and their degree is growing quadratically with m, contradiction to property ii). □

3.2. Definition. *We say that a projective directed manifold* (X, V) *is "algebraically hyperbolic" if it satisfies property* 3.1 ii), *namely, if there exists* $\varepsilon > 0$ *such that every algebraic curve* $C \subset X$ *tangent to* V *satisfies*

$$2g(\overline{C}) - 2 \geqslant \varepsilon \deg_\omega(C).$$

A nice feature of algebraic hyperbolicity is that it satisfies an algebraic analogue of the openness property.

3.3. Proposition. *Let* $(\mathcal{X}, \mathcal{V}) \to S$ *be an algebraic family of projective algebraic directed manifolds* (*given by a projective morphism* $\mathcal{X} \to S$). *Then the set of* $t \in S$ *such that the fiber* (X_t, V_t) *is algebraically hyperbolic is open with respect to the "countable Zariski topology" of* S (*by definition, this is the topology for which closed sets are countable unions of algebraic sets*).

Proof. After replacing S by a Zariski open subset, we may assume that the total space $\mathcal{X}$ itself is quasi-projective. Let ω be the Kähler metric on $\mathcal{X}$ obtained by pulling back the Fubini-Study metric via an embedding in a projective space. If integers $d > 0$, $g \geqslant 0$ are fixed, the set $A_{d,g}$ of $t \in S$ such that X_t contains an algebraic 1-cycle $C = \sum m_j C_j$ tangent to V_t with $\deg_\omega(C) = d$ and $g(\overline{C}) = \sum m_j\, g(\overline{C}_j) \leqslant g$ is a closed algebraic subset of S (this follows from the existence of a relative cycle space of curves of given degree, and from the fact that the geometric genus is Zariski lower semicontinuous). Now, the set of non algebraically hyperbolic fibers is by definition

$$\bigcap_{k>0} \; \bigcup_{2g-2<d/k} A_{d,g}.$$

This concludes the proof (of course, one has to know that the countable Zariski topology is actually a topology, namely that the class of countable unions of algebraic sets is stable under arbitrary intersections; this can be easily checked by an induction on dimension). □

3.4. Remark. More explicit versions of the openness property have been dealt with in the literature. H. Clemens ([Cle86] and [CKL88]) has shown that on a very generic surface of degree $d \geqslant 5$ in $\mathbb{P}^3$, the curves of type (d,k) are of genus $g > kd(d-5)/2$ (recall that a very generic surface $X \subset \mathbb{P}^3$ of degree $\geqslant 4$ has Picard group generated by $\mathcal{O}_X(1)$ thanks to the Noether-Lefschetz theorem, thus any curve on the surface is a complete intersection with another hypersurface of degree k ; such a curve is said to be of type (d,k) ; genericity is taken here in the sense of the countable Zariski topology). Improving on this result of Clemens, Geng Xu [Xu94] has shown that every curve contained in a very generic surface of degree $d \geqslant 5$ satisfies the sharp bound $g \geqslant d(d-3)/2 - 2$. This actually shows that a very generic surface of degree $d \geqslant 6$ is algebraically hyperbolic. Although a very generic quintic surface has no rational or elliptic curves, it seems to be unknown whether a (very) generic quintic surface is algebraically hyperbolic in the sense of Definition 3.2.

In higher dimension, L. Ein ([Ein88], [Ein91]) proved that every subvariety of a very generic hypersurface $X \subset \mathbb{P}^{n+1}$ of degree $d \geqslant 2n+1$ ($n \geqslant 2$), is of general type. This was reproved by a simple efficient technique by C. Voisin in [Voi96].

3.5. Remark. It would be interesting to know whether algebraic hyperbolicity is open with respect to the Euclidean topology ; still more interesting would be to know whether Kobayashi hyperbolicity is open for the countable Zariski topology (of course, both properties would follow immediately if one knew that algebraic hyperbolicity and Kobayashi hyperbolicity coincide, but they seem otherwise highly non trivial to establish). The latter openness property has raised an important amount of work around the following more particular question: is a (very) generic hypersurface $X \subset \mathbb{P}^{n+1}$ of degree d large enough (say $d \geqslant 2n+1$) Kobayashi hyperbolic ? Again, "very generic" is to be taken here in the sense of the countable Zariski topology. Brody-Green [BrGr77] and Nadel [Nad89] produced examples of hyperbolic surfaces in $\mathbb{P}^3$ for all degrees $d \geqslant 50$, and Masuda-Noguchi [MaNo93] gave examples of such hypersurfaces in $\mathbb{P}^n$ for arbitrary $n \geqslant 2$, of degree $d \geqslant d_0(n)$ large enough. The question of studying the hyperbolicity of complements $\mathbb{P}^n \smallsetminus D$ of generic divisors is in principle closely related to this; in fact if $D = \{P(z_0, \ldots, z_n) = 0\}$ is a smooth generic divisor of degree d, one may look at the hypersurface

$$X = \left\{z_{n+1}^d = P(z_0, \ldots, z_n)\right\} \subset \mathbb{P}^{n+1}$$

which is a cyclic $d\,{:}\,1$ covering of $\mathbb{P}^n$. Since any holomorphic map $f : \mathbb{C} \to \mathbb{P}^n \smallsetminus D$ can be lifted to X, it is clear that the hyperbolicity of X would imply the hyperbolicity of $\mathbb{P}^n \smallsetminus D$. The hyperbolicity of complements of divisors in $\mathbb{P}^n$ has been investigated by many authors. □

In the "absolute case" $V = T_X$, it seems reasonable to expect that properties 3.1 i), ii) are equivalent, i.e. that Kobayashi and algebraic hyperbolicity coincide. However, it was observed by Serge Cantat [Can00] that property 3.1 (iii) is not sufficient to imply the hyperbolicity of X, at least when X is a general complex surface: a general (non algebraic) K3 surface is known to have no elliptic curves and does not admit either any surjective map from an abelian variety; however such a surface is not Kobayashi hyperbolic. We are uncertain about the sufficiency of 3.1 (iii) when X is assumed to be projective.

§4. The Ahlfors-Schwarz lemma for metrics of negative curvature

One of the most basic ideas is that hyperbolicity should somehow be related with suitable negativity properties of the curvature. For instance, it is a standard fact already observed in Kobayashi [Kob70] that the negativity of T_X (or the ampleness of T^*_X) implies the hyperbolicity of X. There are many ways of improving or generalizing this result. We present here a few simple examples of such generalizations.

§4.A. Exploiting curvature via potential theory

If (V, h) is a holomorphic vector bundle equipped with a smooth hermitian metric, we denote by $\nabla_h = \nabla'_h + \nabla''_h$ the associated Chern connection and by $\Theta_{V,h} = \frac{i}{2\pi}\nabla_h^2$ its Chern curvature tensor.

4.1. Proposition. *Let (X, V) be a compact directed manifold. Assume that V is non singular and that V^* is ample. Then (X, V) is hyperbolic.*

Proof (from an original idea of [Kob75]). Recall that a vector bundle E is said to be ample if $S^m E$ has enough global sections $\sigma_1, \ldots, \sigma_N$ so as to generate 1-jets of sections at any point, when m is large. One obtains a Finsler metric N on E^* by putting

$$N(\xi) = \Big(\sum_{1 \leqslant j \leqslant N} |\sigma_j(x) \cdot \xi^m|^2 \Big)^{1/2m}, \qquad \xi \in E^*_x,$$

and N is then a strictly plurisubharmonic function on the total space of E^* minus the zero section (in other words, the line bundle $\mathcal{O}_{P(E^*)}(1)$ has a metric of positive curvature). By the ampleness assumption on V^*, we thus have a Finsler metric N on V which is strictly plurisubharmonic outside the zero section. By the Brody lemma, if (X, V) is not hyperbolic, there is a non constant entire curve $g : \mathbb{C} \to X$ tangent to V such that $\sup_{\mathbb{C}} \|g'\|_\omega \leqslant 1$ for some given hermitian metric ω on X. Then $N(g')$ is a bounded subharmonic function on $\mathbb{C}$ which is strictly subharmonic on $\{g' \neq 0\}$. This is a contradiction, for any bounded subharmonic function on $\mathbb{C}$ must be constant. □

§4.B. Ahlfors-Schwarz lemma

Proposition 4.1 can be generalized a little bit further by means of the Ahlfors-Schwarz lemma (see e.g. [Lang87]; we refer to [Dem85] for the generalized version presented here; the proof is merely an application of the maximum principle plus a regularization argument).

4.2. Ahlfors-Schwarz lemma. *Let $\gamma(t) = \gamma_0(t)\, i\, dt \wedge d\overline{t}$ be a hermitian metric on Δ_R where $\log \gamma_0$ is a subharmonic function such that $i\,\partial\overline{\partial} \log \gamma_0(t) \geqslant A\,\gamma(t)$ in the sense of currents, for some positive constant A. Then γ can be compared with the Poincaré metric of Δ_R as follows:*

$$\gamma(t) \leqslant \frac{2}{A} \frac{R^{-2}|dt|^2}{(1 - |t|^2/R^2)^2}.$$

More generally, let $\gamma = i \sum \gamma_{jk} dt_j \wedge d\overline{t}_k$ be an almost everywhere positive hermitian form on the ball $B(0, R) \subset \mathbb{C}^p$, such that $-\operatorname{Ricci}(\gamma) := i\,\partial\overline{\partial} \log \det \gamma \geqslant A\gamma$ in the sense of currents, for some constant $A > 0$ (this means in particular that $\det \gamma = \det(\gamma_{jk})$ is such that $\log \det \gamma$ is plurisubharmonic). Then the γ-volume form is controlled by the Poincaré volume form :

$$\det(\gamma) \leqslant \Big(\frac{p+1}{AR^2} \Big)^p \frac{1}{(1 - |t|^2/R^2)^{p+1}}.$$

4.C. Applications of the Ahlfors-Schwarz lemma to hyperbolicity

Let (X, V) be a *compact* directed manifold. We assume throughout this subsection that V is *non singular.*

4.3. Proposition. *Assume V^* is "very big" in the following sense: there exists an ample line bundle L and a sufficiently large integer m such that the global sections in $H^0(X, S^m V^* \otimes L^{-1})$ generate all fibers over $X \smallsetminus Y$, for some analytic subset $Y \subsetneq X$. Then all entire curves $f : \mathbb{C} \to X$ tangent to V satisfy $f(\mathbb{C}) \subset Y$ [under our assumptions, X is a projective algebraic manifold and Y is an algebraic subvariety, thus it is legitimate to say that the entire curves are "algebraically degenerate"].*

Proof. Let $\sigma_1, \ldots, \sigma_N \in H^0(X, S^m V^* \otimes L^{-1})$ be a basis of sections generating $S^m V^* \otimes L^{-1}$ over $X \smallsetminus Y$. If $f : \mathbb{C} \to X$ is tangent to V, we define a semipositive hermitian form $\gamma(t) = \gamma_0(t)\, |dt|^2$ on $\mathbb{C}$ by putting

$$\gamma_0(t) = \sum \|\sigma_j(f(t)) \cdot f'(t)^m\|^{2/m}_{L^{-1}}$$

where $\| \ \|_L$ denotes a hermitian metric with positive curvature on L. If $f(\mathbb{C}) \not\subset Y$, the form γ is not identically 0 and we then find

$$i\,\partial\overline{\partial}\log\gamma_0 \geqslant \frac{2\pi}{m} f^*\Theta_L$$

where Θ_L is the curvature form. The positivity assumption combined with an obvious homogeneity argument yield

$$\frac{2\pi}{m} f^*\Theta_L \geqslant \varepsilon \|f'(t)\|_\omega^2\,|dt|^2 \geqslant \varepsilon'\,\gamma(t)$$

for any given hermitian metric ω on X. Now, for any t_0 with $\gamma_0(t_0) > 0$, the Ahlfors-Schwarz lemma shows that f can only exist on a disk $D(t_0, R)$ such that $\gamma_0(t_0) \leqslant \frac{2}{\varepsilon'} R^{-2}$, contradiction. □

There are similar results for p-measure hyperbolicity, e.g.

4.4. Proposition. *Assume that $\Lambda^p V^*$ is ample. Then (X, V) is infinitesimally p-measure hyperbolic. More generally, assume that $\Lambda^p V^*$ is very big with base locus contained in $Y \subsetneq X$ (see 3.3). Then $\mathbf{e}^p$ is non degenerate over $X \smallsetminus Y$.*

Proof. By the ampleness assumption, there is a smooth Finsler metric N on $\Lambda^p V$ which is strictly plurisubharmonic outside the zero section. We select also a hermitian metric ω on X. For any holomorphic map $f : \mathbb{B}_p \to X$ we define a semipositive hermitian metric $\widetilde{\gamma}$ on $\mathbb{B}_p$ by putting $\widetilde{\gamma} = f^*\omega$. Since ω need not have any good curvature estimate, we introduce the function $\delta(t) = N_{f(t)}(\Lambda^p f'(t) \cdot \tau_0)$, where $\tau_0 = \partial/\partial t_1 \wedge \cdots \wedge \partial/\partial t_p$, and select a metric $\gamma = \lambda\widetilde{\gamma}$ conformal to $\widetilde{\gamma}$ such that $\det\gamma = \delta$. Then λ^p is equal to the ratio $N/\Lambda^p\omega$ on the element $\Lambda^p f'(t) \cdot \tau_0 \in \Lambda^p V_{f(t)}$. Since X is compact, it is clear that the conformal factor λ is bounded by an absolute constant independent of f. From the curvature assumption we then get

$$i\,\partial\overline{\partial}\log\det\gamma = i\,\partial\overline{\partial}\log\delta \geqslant (f, \Lambda^p f')^*(i\,\partial\overline{\partial}\log N) \geqslant \varepsilon f^*\omega \geqslant \varepsilon'\,\gamma.$$

By the Ahlfors-Schwarz lemma we infer that $\det\gamma(0) \leqslant C$ for some constant C, i.e., $N_{f(0)}(\Lambda^p f'(0) \cdot \tau_0) \leqslant C'$. This means that the Kobayashi-Eisenman pseudometric $\mathbf{e}^p_{(X,V)}$ is positive definite everywhere and uniformly bounded from below. In the case $\Lambda^p V^*$ is very big with base locus Y, we use essentially the same arguments, but we then only have N being positive definite on $X \smallsetminus Y$. □

4.5. Corollary ([Gri71], KobO71]). *If X is a projective variety of general type, the Kobayashi-Eisenmann volume form $\mathbf{e}^n$, $n = \dim X$, can degenerate only along a proper algebraic set $Y \subsetneq X$.*

§4.C. Main conjectures concerning hyperbolicity

One of the earliest conjectures in hyperbolicity theory is the following statement due to Kobayashi ([Kob70], [Kob76]).

4.6. Conjecture (Kobayashi).

(a) *A (very) generic hypersurface $X \subset \mathbb{P}^{n+1}$ of degree $d \geqslant d_n$ large enough is hyperbolic.*

(b) *The complement $\mathbb{P}^n \smallsetminus H$ of a (very) generic hypersurface $H \subset \mathbb{P}^n$ of degree $d \geqslant d'_n$ large enough is hyperbolic.*

In its original form, Kobayashi conjecture did not give the lower bounds d_n and d'_n. Zaidenberg proposed the bounds $d_n = 2n + 1$ (for $n \geqslant 2$) and $d'_n = 2n + 1$ (for $n \geqslant 1$), based on the results of Clemens, Xu, Ein and Voisin already mentioned, and the following observation (cf. [Zai87], [Zai93]).

4.7. Theorem (Zaidenberg). *The complement of a general hypersurface of degree* $2n$ *in* $\mathbb{P}^n$ *is not hyperbolic.*

The converse of Corollary 4.5 is also expected to be true, namely, the generic non degeneracy of $\mathbf{e}^n$ should imply that X is of general type, but this is only known for surfaces (see [GrGr80] and [MoMu82]):

4.8. Conjecture (Green-Griffiths [GrGr80]). *A projective algebraic variety* X *is measure hyperbolic* (*i.e.* $\mathbf{e}^n$ *degenerates only along a proper algebraic subvariety*) *if and only if* X *is of general type.*

An essential step in the proof of the necessity of having general type subvarieties would be to show that manifolds of Kodaira dimension 0 (say, Calabi-Yau manifolds and holomorphic symplectic manifolds, all of which have $c_1(X) = 0$) are not measure hyperbolic, e.g. by exhibiting enough families of curves $C_{s,\ell}$ covering X such that $(2g(\overline{C}_{s,\ell})-2)/\deg(C_{s,\ell}) \to 0$. Another (even stronger) conjecture which we will investigate at the end of these notes is

4.9. Conjecture (Green-Griffiths [GrGr80]). *If* X *is a variety of general type, there exists a proper algebraic set* $Y \subsetneq X$ *such that every entire holomorphic curve* $f : \mathbb{C} \to X$ *is contained in* Y.

One of the early important result in the direction of Conjecture 4.9 is the proof of the Bloch theorem, as proposed by Bloch [Blo26a] and Ochiai [Och77]. The Bloch theorem is the special case of 4.9 when the irregularity of X satisfies $q = h^0(X, \Omega^1_X) > \dim X$. Various solutions have then been obtained in fundamental papers of Noguchi [Nog77, 81, 84], Kawamata [Kaw80] and Green-Griffiths [GrGr80], by means of different techniques. See section § 10 for a proof based on jet bundle techniques. A much more recent result is the striking statement due to Diverio, Merker and Rousseau [DMR10], confirming 4.9 when $X \subset \mathbb{P}^{n+1}$ is a generic non singular hypersurface of sufficiently large degree $d \geqslant 2^{n^5}$ (cf. §16). Conjecture 4.9 was also considered by S. Lang [Lang86, Lang87] in view of arithmetic counterparts of the above geometric statements.

4.10. Conjecture (Lang). *A projective algebraic variety* X *is hyperbolic if and only if all its algebraic subvarieties* (*including* X *itself*) *are of general type.*

4.11. Conjecture (Lang). *Let* X *be a projective variety defined over a number field* K.

(a) *If* X *is hyperbolic, then the set of* K*-rational points is finite.*

(a′) *Conversely, if the set of* K'*-rational points is finite for every finite extension* $K' \supset K$, *then* X *is hyperbolic.*

(b) *If* X *is of general type, then the set of* K*-rational points is not Zariski dense.*

(b′) *Conversely, if the set of* K'*-rational points is not Zariski dense for any extension* $K' \supset K$, *then* X *is of general type.*

In fact, in 4.11 (b), if $Y \subsetneq X$ is the “Green-Griffiths locus” of X, it is expected that $X \smallsetminus Y$ contains only finitely many rational K-points. Even when dealing only with the geometric statements, there are several interesting connections between these conjectures.

4.12. Proposition. *Conjecture* 4.9 *implies the "if" part of conjecture* 4.10*, and Conjecture* 4.8 *implies the "only if" part of Conjecture* 4.10*, hence* (4.8 *and* 4.9) $\Rightarrow$ (4.10)*.*

Proof. In fact if Conjecture 4.9 holds and every subariety Y of X is of general type, then it is easy to infer that every entire curve $f : \mathbb{C} \to X$ has to be constant by induction on $\dim X$, because in fact f maps $\mathbb{C}$ to a certain subvariety $Y \subsetneq X$. Therefore X is hyperbolic.

Conversely, if Conjecture 4.8 holds and X has a certain subvariety Y which is not of general type, then Y is not measure hyperbolic. However Proposition 2.4 shows that hyperbolicity implies measure hyperbolicity. Therefore Y is not hyperbolic and so X itself is not hyperbolic either. $\square$

4.13. Proposition. *Assume that the Green-Griffiths conjecture* 4.9 *holds. Then the Kobayashi conjecture* 4.6 (a) *holds with* $d_n = 2n + 1$*.*

Proof. We know by Ein [Ein88, Ein91] and Voisin [Voi96] that a very generic hypersurface $X \subset \mathbb{P}^{n+1}$ of degree $d \geqslant 2n+1$, $n \geqslant 2$, has all its subvarieties that are of general type. We have seen that the Green-Griffiths conjecture 4.9 implies the hyperbolicity of X in this circumstance. $\square$

§5. Projectivization of a directed manifold

§5.A. The 1-jet fonctor

The basic idea is to introduce a fonctorial process which produces a new complex directed manifold $(\widetilde{X}, \widetilde{V})$ from a given one (X, V). The new structure $(\widetilde{X}, \widetilde{V})$ plays the role of a space of 1-jets over X. We let

$$\widetilde{X} = P(V), \qquad \widetilde{V} \subset T_{\widetilde{X}}$$

be the projectivized bundle of lines of V, together with a subbundle $\widetilde{V}$ of $T_{\widetilde{X}}$ defined as follows: for every point $(x, [v]) \in \widetilde{X}$ associated with a vector $v \in V_x \smallsetminus \{0\}$,

$$\widetilde{V}_{(x,[v])} = \big\{\xi \in T_{\widetilde{X},\,(x,[v])}\,;\ \pi_*\xi \in \mathbb{C}v\big\}, \qquad \mathbb{C}v \subset V_x \subset T_{X,x}, \tag{5.1}$$

where $\pi : \widetilde{X} = P(V) \to X$ is the natural projection and $\pi_* : T_{\widetilde{X}} \to \pi^* T_X$ is its differential. On $\widetilde{X} = P(V)$ we have a tautological line bundle $\mathcal{O}_{\widetilde{X}}(-1) \subset \pi^* V$ such that $\mathcal{O}_{\widetilde{X}}(-1)_{(x,[v])} = \mathbb{C}v$. The bundle $\widetilde{V}$ is characterized by the two exact sequences

$$0 \longrightarrow T_{\widetilde{X}/X} \longrightarrow \widetilde{V} \xrightarrow{\ \pi_*\ } \mathcal{O}_{\widetilde{X}}(-1) \longrightarrow 0, \tag{5.2}$$

$$0 \longrightarrow \mathcal{O}_{\widetilde{X}} \longrightarrow \pi^* V \otimes \mathcal{O}_{\widetilde{X}}(1) \longrightarrow T_{\widetilde{X}/X} \longrightarrow 0, \tag{5.2$'$}$$

where $T_{\widetilde{X}/X}$ denotes the relative tangent bundle of the fibration $\pi : \widetilde{X} \to X$. The first sequence is a direct consequence of the definition of $\widetilde{V}$, whereas the second is a relative version of the Euler exact sequence describing the tangent bundle of the fibers $P(V_x)$. From these exact sequences we infer

$$\dim \widetilde{X} = n + r - 1, \qquad \operatorname{rank} \widetilde{V} = \operatorname{rank} V = r, \tag{5.3}$$

and by taking determinants we find $\det(T_{\widetilde{X}/X}) = \pi^* \det V \otimes \mathcal{O}_{\widetilde{X}}(r)$, thus

$$\det \widetilde{V} = \pi^* \det V \otimes \mathcal{O}_{\widetilde{X}}(r-1). \tag{5.4}$$

By definition, $\pi : (\widetilde{X}, \widetilde{V}) \to (X, V)$ is a morphism of complex directed manifolds. Clearly, our construction is fonctorial, i.e., for every morphism of directed manifolds $\Phi : (X, V) \to (Y, W)$, there is a commutative diagram

$$\begin{array}{ccc} (\widetilde{X}, \widetilde{V}) & \xrightarrow{\pi} & (X, V) \\ \widetilde{\Phi} \downarrow & & \downarrow \Phi \\ (\widetilde{Y}, \widetilde{W}) & \xrightarrow{\pi} & (Y, W) \end{array} \tag{5.5}$$

where the left vertical arrow is the meromorphic map $P(V) \dashrightarrow P(W)$ induced by the differential $\Phi_* : V \to \Phi^* W$ ($\widetilde{\Phi}$ is actually holomorphic if $\Phi_* : V \to \Phi^* W$ is injective).

§5.B. Lifting of curves to the 1-jet bundle

Suppose that we are given a holomorphic curve $f : \Delta_R \to X$ parametrized by the disk Δ_R of centre 0 and radius R in the complex plane, and that f is a tangent curve of the directed manifold, i.e., $f'(t) \in V_{f(t)}$ for every $t \in \Delta_R$. If f is non constant, there is a well defined and unique tangent line $[f'(t)]$ for every t, even at stationary points, and the map

$$\widetilde{f} : \Delta_R \to \widetilde{X}, \qquad t \mapsto \widetilde{f}(t) := (f(t), [f'(t)]) \tag{5.6}$$

is holomorphic (at a stationary point t_0, we just write $f'(t) = (t - t_0)^s u(t)$ with $s \in \mathbb{N}^*$ and $u(t_0) \neq 0$, and we define the tangent line at t_0 to be $[u(t_0)]$, hence $\widetilde{f}(t) = (f(t), [u(t)])$ near t_0 ; even for $t = t_0$, we still denote $[f'(t_0)] = [u(t_0)]$ for simplicity of notation). By definition $f'(t) \in \mathcal{O}_{\widetilde{X}}(-1)_{\widetilde{f}(t)} = \mathbb{C}\, u(t)$, hence the derivative f' defines a section

$$f' : T_{\Delta_R} \to \widetilde{f}^* \mathcal{O}_{\widetilde{X}}(-1). \tag{5.7}$$

Moreover $\pi \circ \widetilde{f} = f$, therefore

$$\pi_* \widetilde{f}'(t) = f'(t) \in \mathbb{C}u(t) \Longrightarrow \widetilde{f}'(t) \in \widetilde{V}_{(f(t), u(t))} = \widetilde{V}_{\widetilde{f}(t)}$$

and we see that $\widetilde{f}$ is a tangent trajectory of $(\widetilde{X}, \widetilde{V})$. We say that $\widetilde{f}$ is the *canonical lifting* of f to $\widetilde{X}$. Conversely, if $g : \Delta_R \to \widetilde{X}$ is a tangent trajectory of $(\widetilde{X}, \widetilde{V})$, then by definition of $\widetilde{V}$ we see that $f = \pi \circ g$ is a tangent trajectory of (X, V) and that $g = \widetilde{f}$ (unless g is contained in a vertical fiber $P(V_x)$, in which case f is constant).

For any point $x_0 \in X$, there are local coordinates $(z_1, \ldots, z_n)$ on a neighborhood Ω of x_0 such that the fibers $(V_z)_{z \in \Omega}$ can be defined by linear equations

$$V_z = \Big\{ \xi = \sum_{1 \leqslant j \leqslant n} \xi_j \frac{\partial}{\partial z_j} \, ; \, \xi_j = \sum_{1 \leqslant k \leqslant r} a_{jk}(z) \xi_k \text{ for } j = r+1, \ldots, n \Big\}, \tag{5.8}$$

where (a_{jk}) is a holomorphic $(n-r) \times r$ matrix. It follows that a vector $\xi \in V_z$ is completely determined by its first r components $(\xi_1, \ldots, \xi_r)$, and the affine chart $\xi_j \neq 0$ of $P(V)_{\restriction \Omega}$ can be described by the coordinate system

$$\Big(z_1, \ldots, z_n ; \frac{\xi_1}{\xi_j}, \ldots, \frac{\xi_{j-1}}{\xi_j}, \frac{\xi_{j+1}}{\xi_j}, \ldots, \frac{\xi_r}{\xi_j} \Big). \tag{5.9}$$

Let $f \simeq (f_1, \ldots, f_n)$ be the components of f in the coordinates $(z_1, \ldots, z_n)$ (we suppose here R so small that $f(\Delta_R) \subset \Omega$). It should be observed that f is uniquely determined by its initial value x and by the first r components $(f_1,\ldots,f_r)$. Indeed, as $f'(t)\in V_{f(t)}$, we can recover the other components by integrating the system of ordinary differential equations

$$f'_j(t)=\sum_{1\leqslant k\leqslant r} a_{jk}(f(t))f'_k(t),\qquad j>r, \tag{5.10}$$

on a neighborhood of 0, with initial data $f(0)=x$. We denote by $m=m(f,t_0)$ the *multiplicity* of f at any point $t_0\in\Delta_R$, that is, $m(f,t_0)$ is the smallest integer $m\in\mathbb{N}^*$ such that $f_j^{(m)}(t_0)\neq 0$ for some j. By (5.10), we can always suppose $j\in\{1,\ldots,r\}$, for example $f_r^{(m)}(t_0)\neq 0$. Then $f'(t)=(t-t_0)^{m-1}u(t)$ with $u_r(t_0)\neq 0$, and the lifting $\widetilde{f}$ is described in the coordinates of the affine chart $\xi_r\neq 0$ of $P(V)_{\restriction\Omega}$ by

$$\widetilde{f}\simeq\Big(f_1,\ldots,f_n;\frac{f'_1}{f'_r},\ldots,\frac{f'_{r-1}}{f'_r}\Big). \tag{5.11}$$

§5.C. Curvature properties of the 1-jet bundle

We end this section with a few curvature computations. Assume that V is equipped with a smooth hermitian metric h. Denote by $\nabla_h=\nabla'_h+\nabla''_h$ the associated Chern connection and by $\Theta_{V,h}=\frac{i}{2\pi}\nabla_h^2$ its Chern curvature tensor. For every point $x_0\in X$, there exists a "normalized" holomorphic frame $(e_\lambda)_{1\leqslant\lambda\leqslant r}$ on a neighborhood of x_0, such that

$$\langle e_\lambda,e_\mu\rangle_h=\delta_{\lambda\mu}-\sum_{1\leqslant j,k\leqslant n} c_{jk\lambda\mu}z_j\overline{z}_k+O(|z|^3), \tag{5.12}$$

with respect to any holomorphic coordinate system $(z_1,\ldots,z_n)$ centered at x_0. A computation of $d'\langle e_\lambda,e_\mu\rangle_h=\langle\nabla'_h e_\lambda,e_\mu\rangle_h$ and $\nabla_h^2 e_\lambda=d''\nabla'_h e_\lambda$ then gives

$$\nabla'_h e_\lambda=-\sum_{j,k,\mu} c_{jk\lambda\mu}\overline{z}_k\,dz_j\otimes e_\mu+O(|z|^2),$$

$$\Theta_{V,h}(x_0)=\frac{i}{2\pi}\sum_{j,k,\lambda,\mu} c_{jk\lambda\mu}dz_j\wedge d\overline{z}_k\otimes e_\lambda^*\otimes e_\mu. \tag{5.13}$$

The above curvature tensor can also be viewed as a hermitian form on $T_X\otimes V$. In fact, one associates with $\Theta_{V,h}$ the hermitian form $\langle\Theta_{V,h}\rangle$ on $T_X\otimes V$ defined for all $(\zeta,v)\in T_X\times_X V$ by

$$\langle\Theta_{V,h}\rangle(\zeta\otimes v)=\sum_{1\leqslant j,k\leqslant n,\,1\leqslant\lambda,\mu\leqslant r} c_{jk\lambda\mu}\zeta_j\overline{\zeta}_k v_\lambda\overline{v}_\mu. \tag{5.14}$$

Let h_1 be the hermitian metric on the tautological line bundle $\mathcal{O}_{P(V)}(-1)\subset\pi^*V$ induced by the metric h of V. We compute the curvature $(1,1)$-form $\Theta_{h_1}(\mathcal{O}_{P(V)}(-1))$ at an arbitrary point $(x_0,[v_0])\in P(V)$, in terms of $\Theta_{V,h}$. For simplicity, we suppose that the frame $(e_\lambda)_{1\leqslant\lambda\leqslant r}$ has been chosen in such a way that $[e_r(x_0)]=[v_0]\in P(V)$ and $|v_0|_h=1$. We get holomorphic local coordinates $(z_1,\ldots,z_n\,;\,\xi_1,\ldots,\xi_{r-1})$ on a neighborhood of $(x_0,[v_0])$ in $P(V)$ by assigning

$$(z_1,\ldots,z_n\,;\,\xi_1,\ldots,\xi_{r-1})\longmapsto(z,[\xi_1e_1(z)+\cdots+\xi_{r-1}e_{r-1}(z)+e_r(z)])\in P(V).$$

Then the function

$$\eta(z,\xi)=\xi_1e_1(z)+\cdots+\xi_{r-1}e_{r-1}(z)+e_r(z)$$

defines a holomorphic section of $\mathcal{O}_{P(V)}(-1)$ in a neighborhood of $(x_0, [v_0])$. By using the expansion (5.12) for h, we find

$$
\begin{aligned}
|\eta|^2_{h_1} = |\eta|^2_h = 1 + |\xi|^2 - \sum_{1 \leqslant j,k \leqslant n} c_{jkrr} z_j \overline{z}_k + O((|z| + |\xi|)^3),\\
\Theta_{h_1}(\mathcal{O}_{P(V)}(-1))_{(x_0,[v_0])} &= -\frac{i}{2\pi} \partial\overline{\partial} \log |\eta|^2_{h_1}\\
&= \frac{i}{2\pi}\Big(\sum_{1 \leqslant j,k \leqslant n} c_{jkrr} dz_j \wedge d\overline{z}_k - \sum_{1 \leqslant \lambda \leqslant r-1} d\xi_\lambda \wedge d\overline{\xi}_\lambda \Big). \qquad (5.15)
\end{aligned}
$$

§6. Jets of curves and Semple jet bundles

Let X be a complex n-dimensional manifold. Following ideas of Green-Griffiths [GrGr80], we let $J_k \to X$ be the bundle of k-jets of germs of parametrized curves in X, that is, the set of equivalence classes of holomorphic maps $f : (\mathbb{C}, 0) \to (X, x)$, with the equivalence relation $f \sim g$ if and only if all derivatives $f^{(j)}(0) = g^{(j)}(0)$ coincide for $0 \leqslant j \leqslant k$, when computed in some local coordinate system of X near x. The projection map $J_k \to X$ is simply $f \mapsto f(0)$. If $(z_1, \ldots, z_n)$ are local holomorphic coordinates on an open set $\Omega \subset X$, the elements f of any fiber $J_{k,x}$, $x \in \Omega$, can be seen as $\mathbb{C}^n$-valued maps

$$f = (f_1, \ldots, f_n) : (\mathbb{C}, 0) \to \Omega \subset \mathbb{C}^n,$$

and they are completetely determined by their Taylor expansion of order k at $t = 0$

$$f(t) = x + t\, f'(0) + \frac{t^2}{2!} f''(0) + \cdots + \frac{t^k}{k!} f^{(k)}(0) + O(t^{k+1}).$$

In these coordinates, the fiber $J_{k,x}$ can thus be identified with the set of k-tuples of vectors $(\xi_1, \ldots, \xi_k) = (f'(0), \ldots, f^{(k)}(0)) \in (\mathbb{C}^n)^k$. It follows that J_k is a holomorphic fiber bundle with typical fiber $(\mathbb{C}^n)^k$ over X (however, J_k is not a vector bundle for $k \geqslant 2$, because of the nonlinearity of coordinate changes; see formula (7.2) in § 7).

According to the philosophy developed throughout this paper, we describe the concept of jet bundle in the general situation of complex directed manifolds. If X is equipped with a holomorphic subbundle $V \subset T_X$, we associate to V a k-jet bundle $J_k V$ as follows.

6.1. Definition. *Let (X, V) be a complex directed manifold. We define $J_k V \to X$ to be the bundle of k-jets of curves $f : (\mathbb{C}, 0) \to X$ which are tangent to V, i.e., such that $f'(t) \in V_{f(t)}$ for all t in a neighborhood of 0, together with the projection map $f \mapsto f(0)$ onto X.*

It is easy to check that $J_k V$ is actually a subbundle of J_k. In fact, by using (5.8) and (5.10), we see that the fibers $J_k V_x$ are parametrized by

$$\big((f_1'(0), \ldots, f_r'(0)); (f_1''(0), \ldots, f_r''(0)); \ldots; (f_1^{(k)}(0), \ldots, f_r^{(k)}(0))\big) \in (\mathbb{C}^r)^k$$

for all $x \in \Omega$, hence $J_k V$ is a locally trivial $(\mathbb{C}^r)^k$-subbundle of J_k. Alternatively, we can pick a local holomorphic connection ∇ on V, defined on some open set $\Omega \subset X$, and compute inductively the successive derivatives

$$\nabla f = f', \qquad \nabla^j f = \nabla_{f'}(\nabla^{j-1} f)$$

with respect to ∇ along the cure $t \mapsto f(t)$. Then

$$(\xi_1, \xi_2, \ldots, \xi_k) = (\nabla f(0), \nabla^2 f(0), \ldots, \nabla^k f(0)) \in V_x^{\oplus k}$$

provides a "trivialization" $J^k V_{|\Omega} \simeq V_{|\Omega}^{\oplus k}$. This identification depends of course on the choice of ∇ and cannot be defined globally in general (unless we are in the rare situation where V has a global holomorphic connection). □

We now describe a convenient process for constructing "projectivized jet bundles", which will later appear as natural quotients of our jet bundles $J_k V$ (or rather, as suitable desingularized compactifications of the quotients). Such spaces have already been considered since a long time, at least in the special case $X = \mathbb{P}^2$, $V = T_{\mathbb{P}^2}$ (see Gherardelli [Ghe41], Semple [Sem54]), and they have been mostly used as a tool for establishing enumerative formulas dealing with the order of contact of plane curves (see [Coll88], [CoKe94]); the article [ASS92] is also concerned with such generalizations of jet bundles, as well as [LaTh96] by Laksov and Thorup.

We define inductively the *projectivized k-jet bundle* $P_k V = X_k$ (or *Semple k-jet bundle*) and the associated subbundle $V_k \subset T_{X_k}$ by

$$(X_0, V_0) = (X, V), \qquad (X_k, V_k) = (\widetilde{X}_{k-1}, \widetilde{V}_{k-1}). \tag{6.2}$$

In other words, $(P_k V, V_k) = (X_k, V_k)$ is obtained from (X, V) by iterating k-times the lifting construction $(X, V) \mapsto (\widetilde{X}, \widetilde{V})$ described in § 5. By (5.2–5.7), we find

$$\dim P_k V = n + k(r-1), \qquad \operatorname{rank} V_k = r, \tag{6.3}$$

together with exact sequences

$$0 \longrightarrow T_{P_k V/P_{k-1}V} \longrightarrow V_k \xrightarrow{(\pi_k)_*} \mathcal{O}_{P_k V}(-1) \longrightarrow 0, \tag{6.4}$$

$$0 \longrightarrow \mathcal{O}_{P_k V} \longrightarrow \pi_k^* V_{k-1} \otimes \mathcal{O}_{P_k V}(1) \longrightarrow T_{P_k V/P_{k-1}V} \longrightarrow 0. \tag{6.4'}$$

where π_k is the natural projection $\pi_k : P_k V \to P_{k-1} V$ and $(\pi_k)_*$ its differential. Formula (5.4) yields

$$\det V_k = \pi_k^* \det V_{k-1} \otimes \mathcal{O}_{P_k V}(r-1). \tag{6.5}$$

Every non constant tangent trajectory $f : \Delta_R \to X$ of (X, V) lifts to a well defined and unique tangent trajectory $f_{[k]} : \Delta_R \to P_k V$ of $(P_k V, V_k)$. Moreover, the derivative $f'_{[k-1]}$ gives rise to a section

$$f'_{[k-1]} : T_{\Delta_R} \to f_{[k]}^* \mathcal{O}_{P_k V}(-1). \tag{6.6}$$

In coordinates, one can compute $f_{[k]}$ in terms of its components in the various affine charts (5.9) occurring at each step: we get inductively

$$f_{[k]} = (F_1, \ldots, F_N), \qquad f_{[k+1]} = \Big(F_1, \ldots, F_N, \frac{F'_{s_1}}{F'_{s_r}}, \ldots, \frac{F'_{s_{r-1}}}{F'_{s_r}}\Big) \tag{6.7}$$

where $N = n + k(r-1)$ and $\{s_1, \ldots, s_r\} \subset \{1, \ldots, N\}$. If $k \geqslant 1$, $\{s_1, \ldots, s_r\}$ contains the last $r-1$ indices of $\{1, \ldots, N\}$ corresponding to the "vertical" components of the projection

$P_kV \to P_{k-1}V$, and in general, s_r is an index such that $m(F_{s_r},0) = m(f_{[k]},0)$, that is, F_{s_r} has the smallest vanishing order among all components F_s (s_r may be vertical or not, and the choice of $\{s_1,\ldots,s_r\}$ need not be unique).

By definition, there is a canonical injection $\mathcal{O}_{P_kV}(-1) \hookrightarrow \pi_k^*V_{k-1}$, and a composition with the projection $(\pi_{k-1})_*$ (analogue for order $k-1$ of the arrow $(\pi_k)_*$ in sequence (6.4)) yields for all $k \geqslant 2$ a canonical line bundle morphism

$$(6.8)\qquad \mathcal{O}_{P_kV}(-1) \hookrightarrow \pi_k^*V_{k-1} \xrightarrow{(\pi_k)^*(\pi_{k-1})_*} \pi_k^*\mathcal{O}_{P_{k-1}V}(-1),$$

which admits precisely $D_k = P(T_{P_{k-1}V/P_{k-2}V}) \subset P(V_{k-1}) = P_kV$ as its zero divisor (clearly, D_k is a hyperplane subbundle of P_kV). Hence we find

$$(6.9)\qquad \mathcal{O}_{P_kV}(1) = \pi_k^*\mathcal{O}_{P_{k-1}V}(1) \otimes \mathcal{O}(D_k).$$

Now, we consider the composition of projections

$$(6.10)\qquad \pi_{j,k} = \pi_{j+1} \circ \cdots \circ \pi_{k-1} \circ \pi_k : P_kV \longrightarrow P_jV.$$

Then $\pi_{0,k} : P_kV \to X = P_0V$ is a locally trivial holomorphic fiber bundle over X, and the fibers $P_kV_x = \pi_{0,k}^{-1}(x)$ are k-stage towers of $\mathbb{P}^{r-1}$-bundles. Since we have (in both directions) morphisms $(\mathbb{C}^r, T_{\mathbb{C}^r}) \leftrightarrow (X,V)$ of directed manifolds which are bijective on the level of bundle morphisms, the fibers are all isomorphic to a "universal" nonsingular projective algebraic variety of dimension $k(r-1)$ which we will denote by $\mathbb{R}_{r,k}$; it is not hard to see that $\mathbb{R}_{r,k}$ is rational (as will indeed follow from the proof of Theorem 7.11 below). The following Proposition will help us to understand a little bit more about the geometric structure of P_kV. As usual, we define the *multiplicity* $m(f,t_0)$ of a curve $f : \Delta_R \to X$ at a point $t \in \Delta_R$ to be the smallest integer $s \in \mathbb{N}^*$ such that $f^{(s)}(t_0) \neq 0$, i.e., the largest s such that $\delta(f(t), f(t_0)) = O(|t-t_0|^s)$ for any hermitian or riemannian geodesic distance δ on X. As $f_{[k-1]} = \pi_k \circ f_{[k]}$, it is clear that the sequence $m(f_{[k]},t)$ is non increasing with k.

6.11. Proposition. *Let $f : (\mathbb{C},0) \to X$ be a non constant germ of curve tangent to V. Then for all $j \geqslant 2$ we have $m(f_{[j-2]},0) \geqslant m(f_{[j-1]},0)$ and the inequality is strict if and only if $f_{[j]}(0) \in D_j$. Conversely, if $w \in P_kV$ is an arbitrary element and $m_0 \geqslant m_1 \geqslant \cdots \geqslant m_{k-1} \geqslant 1$ is a sequence of integers with the property that*

$$\forall j \in \{2,\ldots,k\}, \qquad m_{j-2} > m_{j-1} \quad \textit{if and only if}\ \pi_{j,k}(w) \in D_j,$$

there exists a germ of curve $f : (\mathbb{C},0) \to X$ tangent to V such that $f_{[k]}(0) = w$ and $m(f_{[j]},0) = m_j$ for all $j \in \{0,\ldots,k-1\}$.

Proof. i) Suppose first that f is given and put $m_j = m(f_{[j]},0)$. By definition, we have $f_{[j]} = (f_{[j-1]},[u_{j-1}])$ where $f'_{[j-1]}(t) = t^{m_{j-1}-1}u_{j-1}(t) \in V_{j-1}$, $u_{j-1}(0) \neq 0$. By composing with the differential of the projection $\pi_{j-1} : P_{j-1}V \to P_{j-2}V$, we find $f'_{[j-2]}(t) = t^{m_{j-1}-1}(\pi_{j-1})_*u_{j-1}(t)$. Therefore

$$m_{j-2} = m_{j-1} + \mathrm{ord}_{t=0}(\pi_{j-1})_*u_{j-1}(t),$$

and so $m_{j-2} > m_{j-1}$ if and only if $(\pi_{j-1})_*u_{j-1}(0) = 0$, that is, if and only if $u_{j-1}(0) \in T_{P_{j-1}V/P_{j-2}V}$, or equivalently $f_{[j]}(0) = (f_{[j-1]}(0),[u_{j-1}(0)]) \in D_j$.

ii) Suppose now that $w \in P_kV$ and $m_0, \ldots, m_{k-1}$ are given. We denote by $w_{j+1} = (w_j, [\eta_j])$, $w_j \in P_jV$, $\eta_j \in V_j$, the projection of w to $P_{j+1}V$. Fix coordinates $(z_1, \ldots, z_n)$ on X centered at w_0 such that the r-th component $\eta_{0,r}$ of η_0 is non zero. We prove the existence of the germ f by induction on k, in the form of a Taylor expansion

$$f(t) = a_0 + t\,a_1 + \cdots + t^{d_k} a_{d_k} + O(t^{d_k+1}), \qquad d_k = m_0 + m_1 + \cdots + m_{k-1}.$$

If $k = 1$ and $w = (w_0, [\eta_0]) \in P_1V_x$, we simply take $f(t) = w_0 + t^{m_0}\eta_0 + O(t^{m_0+1})$. In general, the induction hypothesis applied to $P_kV = P_{k-1}(V_1)$ over $X_1 = P_1V$ yields a curve $g : (\mathbb{C}, 0) \to X_1$ such that $g_{[k-1]} = w$ and $m(g_{[j]}, 0) = m_{j+1}$ for $0 \leqslant j \leqslant k-2$. If $w_2 \notin D_2$, then $[g'_{[1]}(0)] = [\eta_1]$ is not vertical, thus $f = \pi_1 \circ g$ satisfies $m(f, 0) = m(g, 0) = m_1 = m_0$ and we are done.

If $w_2 \in D_2$, we express $g = (G_1, \ldots, G_n; G_{n+1}, \ldots, G_{n+r-1})$ as a Taylor expansion of order $m_1 + \cdots + m_{k-1}$ in the coordinates (5.9) of the affine chart $\xi_r \neq 0$. As $\eta_1 = \lim_{t\to 0} g'(t)/t^{m_1-1}$ is vertical, we must have $m(G_s, 0) > m_1$ for $1 \leqslant j \leqslant n$. It follows from (6.7) that $G_1, \ldots, G_n$ are never involved in the calculation of the liftings $g_{[j]}$. We can therefore replace g by $f \simeq (f_1, \ldots, f_n)$ where $f_r(t) = t^{m_0}$ and $f_1, \ldots, f_{r-1}$ are obtained by integrating the equations $f'_j(t)/f'_r(t) = G_{n+j}(t)$, i.e., $f'_j(t) = m_0 t^{m_0-1} G_{n+j}(t)$, while $f_{r+1}, \ldots, f_n$ are obtained by integrating (5.10). We then get the desired Taylor expansion of order d_k for f. □

Since we can always take $m_{k-1} = 1$ without restriction, we get in particular:

6.12. Corollary. *Let $w \in P_kV$ be an arbitrary element. Then there is a germ of curve $f : (\mathbb{C}, 0) \to X$ such that $f_{[k]}(0) = w$ and $f'_{[k-1]}(0) \neq 0$ (thus the liftings $f_{[k-1]}$ and $f_{[k]}$ are regular germs of curve). Moreover, if $w_0 \in P_kV$ and w is taken in a sufficiently small neighborhood of w_0, then the germ $f = f_w$ can be taken to depend holomorphically on w.*

Proof. Only the holomorphic dependence of f_w with respect to w has to be guaranteed. If f_{w_0} is a solution for $w = w_0$, we observe that $(f_{w_0})'_{[k]}$ is a non vanishing section of V_k along the regular curve defined by $(f_{w_0})_{[k]}$ in P_kV. We can thus find a non vanishing section ξ of V_k on a neighborhood of w_0 in P_kV such that $\xi = (f_{w_0})'_{[k]}$ along that curve. We define $t \mapsto F_w(t)$ to be the trajectory of ξ with initial point w, and we put $f_w = \pi_{0,k} \circ F_w$. Then f_w is the required family of germs. □

Now, we can take $f : (\mathbb{C}, 0) \to X$ to be regular at the origin (by this, we mean $f'(0) \neq 0$) if and only if $m_0 = m_1 = \cdots = m_{k-1} = 1$, which is possible by Proposition 6.11 if and only if $w \in P_kV$ is such that $\pi_{j,k}(w) \notin D_j$ for all $j \in \{2, \ldots, k\}$. For this reason, we define

$$\begin{aligned} P_kV^{\mathrm{reg}} &= \bigcap_{2\leqslant j\leqslant k} \pi_{j,k}^{-1}(P_jV \smallsetminus D_j), \\ P_kV^{\mathrm{sing}} &= \bigcup_{2\leqslant j\leqslant k} \pi_{j,k}^{-1}(D_j) = P_kV \smallsetminus P_kV^{\mathrm{reg}}, \end{aligned} \tag{6.13}$$

in other words, P_kV^{reg} is the set of values $f_{[k]}(0)$ reached by all regular germs of curves f. One should take care however that there are singular germs which reach the same points $f_{[k]}(0) \in P_kV^{\mathrm{reg}}$, e.g., any s-sheeted covering $t \mapsto f(t^s)$. On the other hand, if $w \in P_kV^{\mathrm{sing}}$, we can reach w by a germ f with $m_0 = m(f, 0)$ as large as we want.

6.14. Corollary. *Let $w \in P_kV^{\mathrm{sing}}$ be given, and let $m_0 \in \mathbb{N}$ be an arbitrary integer larger than the number of components D_j such that $\pi_{j,k}(w) \in D_j$. Then there is a germ of curve*

$f : (\mathbb{C}, 0) \to X$ with multiplicity $m(f, 0) = m_0$ at the origin, such that $f_{[k]}(0) = w$ and $f'_{[k-1]}(0) \neq 0$.

§7. Jet differentials

§7.A. Green-Griffiths jet differentials

We first introduce the concept of jet differentials in the sense of Green-Griffiths [GrGr80]. The goal is to provide an intrinsic geometric description of holomorphic differential equations that a germ of curve $f : (\mathbb{C}, 0) \to X$ may satisfy. In the sequel, we fix a directed manifold (X, V) and suppose implicitly that all germs of curves f are tangent to V.

Let $\mathbb{G}_k$ be the group of germs of k-jets of biholomorphisms of $(\mathbb{C}, 0)$, that is, the group of germs of biholomorphic maps

$$t \mapsto \varphi(t) = a_1 t + a_2 t^2 + \cdots + a_k t^k, \qquad a_1 \in \mathbb{C}^*,\ a_j \in \mathbb{C},\ j \geqslant 2,$$

in which the composition law is taken modulo terms t^j of degree $j > k$. Then $\mathbb{G}_k$ is a k-dimensional nilpotent complex Lie group, which admits a natural fiberwise right action on $J_k V$. The action consists of reparametrizing k-jets of maps $f : (\mathbb{C}, 0) \to X$ by a biholomorphic change of parameter $\varphi : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$, that is, $(f, \varphi) \mapsto f \circ \varphi$. There is an exact sequence of groups

$$1 \to \mathbb{G}'_k \to \mathbb{G}_k \to \mathbb{C}^* \to 1$$

where $\mathbb{G}_k \to \mathbb{C}^*$ is the obvious morphism $\varphi \mapsto \varphi'(0)$, and $\mathbb{G}'_k = [\mathbb{G}_k, \mathbb{G}_k]$ is the group of k-jets of biholomorphisms tangent to the identity. Moreover, the subgroup $\mathbb{H} \simeq \mathbb{C}^*$ of homotheties $\varphi(t) = \lambda t$ is a (non normal) subgroup of $\mathbb{G}_k$, and we have a semidirect decomposition $\mathbb{G}_k = \mathbb{G}'_k \ltimes \mathbb{H}$. The corresponding action on k-jets is described in coordinates by

$$\lambda \cdot (f', f'', \ldots, f^{(k)}) = (\lambda f', \lambda^2 f'', \ldots, \lambda^k f^{(k)}).$$

Following [GrGr80], we introduce the vector bundle $E^{\mathrm{GG}}_{k,m} V^* \to X$ whose fibers are complex valued polynomials $Q(f', f'', \ldots, f^{(k)})$ on the fibers of $J_k V$, of weighted degree m with respect to the $\mathbb{C}^*$ action defined by H, that is, such that

$$Q(\lambda f', \lambda^2 f'', \ldots, \lambda^k f^{(k)}) = \lambda^m Q(f', f'', \ldots, f^{(k)}) \tag{7.1}$$

for all $\lambda \in \mathbb{C}^*$ and $(f', f'', \ldots, f^{(k)}) \in J_k V$. Here we view $(f', f'', \ldots, f^{(k)})$ as indeterminates with components

$$\big((f'_1, \ldots, f'_r); (f''_1, \ldots, f''_r); \ldots; (f^{(k)}_1, \ldots, f^{(k)}_r)\big) \in (\mathbb{C}^r)^k.$$

Notice that the concept of polynomial on the fibers of $J_k V$ makes sense, for all coordinate changes $z \mapsto w = \Psi(z)$ on X induce polynomial transition automorphisms on the fibers of $J_k V$, given by a formula

$$(\Psi \circ f)^{(j)} = \Psi'(f) \cdot f^{(j)} + \sum_{s=2}^{s=j} \sum_{j_1 + j_2 + \cdots + j_s = j} c_{j_1 \ldots j_s} \Psi^{(s)}(f) \cdot (f^{(j_1)}, \ldots, f^{(j_s)}) \tag{7.2}$$

with suitable integer constants $c_{j_1\ldots j_s}$ (this is easily checked by induction on s). In the "absolute case" $V = T_X$, we simply write $E^{\rm GG}_{k,m}T^*_X = E^{\rm GG}_{k,m}$. If $V \subset W \subset T_X$ are holomorphic subbundles, there are natural inclusions

$$J_kV \subset J_kW \subset J_k, \qquad P_kV \subset P_kW \subset P_k.$$

The restriction morphisms induce surjective arrows

$$E^{\rm GG}_{k,m} \to E^{\rm GG}_{k,m}W^* \to E^{\rm GG}_{k,m}V^*,$$

in particular $E^{\rm GG}_{k,m}V^*$ can be seen as a quotient of $E^{\rm GG}_{k,m}$. (The notation V^* is used here to make the contravariance property implicit from the notation). Another useful consequence of these inclusions is that one can extend the definition of J_kV and P_kV to the case where V is an arbitrary linear space, simply by taking the closure of $J_kV_{X\smallsetminus{\rm Sing}(V)}$ and $P_kV_{X\smallsetminus{\rm Sing}(V)}$ in the smooth bundles J_k and P_k, respectively.

If $Q \in E^{\rm GG}_{k,m}V^*$ is decomposed into multihomogeneous components of multidegree $(\ell_1, \ell_2, \ldots, \ell_k)$ in $f', f'', \ldots, f^{(k)}$ (the decomposition is of course coordinate dependent), these multidegrees must satisfy the relation

$$\ell_1 + 2\ell_2 + \cdots + k\ell_k = m.$$

The bundle $E^{\rm GG}_{k,m}V^*$ will be called the *bundle of jet differentials of order k and weighted degree m*. It is clear from (7.2) that a coordinate change $f \mapsto \Psi\circ f$ transforms every monomial $(f^{(\bullet)})^\ell = (f')^{\ell_1}(f'')^{\ell_2}\cdots(f^{(k)})^{\ell_k}$ of partial weighted degree $|\ell|_s := \ell_1 + 2\ell_2 + \cdots + s\ell_s$, $1 \leqslant s \leqslant k$, into a polynomial $((\Psi\circ f)^{(\bullet)})^\ell$ in $(f', f'', \ldots, f^{(k)})$ which has the same partial weighted degree of order s if $\ell_{s+1} = \cdots = \ell_k = 0$, and a larger or equal partial degree of order s otherwise. Hence, for each $s = 1, \ldots, k$, we get a well defined (i.e., coordinate invariant) decreasing filtration $F^\bullet_s$ on $E^{\rm GG}_{k,m}V^*$ as follows:

$$(7.3)\qquad F^p_s(E^{\rm GG}_{k,m}V^*) = \left\{\begin{matrix} Q(f', f'', \ldots, f^{(k)}) \in E^{\rm GG}_{k,m}V^* \text{ involving} \\ \text{only monomials } (f^{(\bullet)})^\ell \text{ with } |\ell|_s \geqslant p \end{matrix}\right\}, \qquad \forall p \in \mathbb{N}.$$

The graded terms ${\rm Gr}^p_{k-1}(E^{\rm GG}_{k,m}V^*)$ associated with the filtration $F^p_{k-1}(E^{\rm GG}_{k,m}V^*)$ are precisely the homogeneous polynomials $Q(f', \ldots, f^{(k)})$ whose monomials $(f^\bullet)^\ell$ all have partial weighted degree $|\ell|_{k-1} = p$ (hence their degree ℓ_k in $f^{(k)}$ is such that $m - p = k\ell_k$, and ${\rm Gr}^p_{k-1}(E^{\rm GG}_{k,m}V^*) = 0$ unless $k|m-p$). The transition automorphisms of the graded bundle are induced by coordinate changes $f \mapsto \Psi\circ f$, and they are described by substituting the arguments of $Q(f', \ldots, f^{(k)})$ according to formula (7.2), namely $f^{(j)} \mapsto (\Psi\circ f)^{(j)}$ for $j < k$, and $f^{(k)} \mapsto \Psi'(f)\circ f^{(k)}$ for $j = k$ (when $j = k$, the other terms fall in the next stage F^{p+1}_{k-1} of the filtration). Therefore $f^{(k)}$ behaves as an element of $V \subset T_X$ under coordinate changes. We thus find

$$(7.4)\qquad G^{m-k\ell_k}_{k-1}(E^{\rm GG}_{k,m}V^*) = E^{\rm GG}_{k-1,m-k\ell_k}V^* \otimes S^{\ell_k}V^*.$$

Combining all filtrations $F^\bullet_s$ together, we find inductively a filtration $F^\bullet$ on $E^{\rm GG}_{k,m}V^*$ such that the graded terms are

$$(7.5)\qquad {\rm Gr}^\ell(E^{\rm GG}_{k,m}V^*) = S^{\ell_1}V^* \otimes S^{\ell_2}V^* \otimes \cdots \otimes S^{\ell_k}V^*, \qquad \ell \in \mathbb{N}^k, \quad |\ell|_k = m.$$

The bundles $E_{k,m}^{\mathrm{GG}}V^*$ have other interesting properties. In fact,

$$E_{k,\bullet}^{\mathrm{GG}}V^* := \bigoplus_{m\geqslant 0} E_{k,m}^{\mathrm{GG}}V^*$$

is in a natural way a bundle of graded algebras (the product is obtained simply by taking the product of polynomials). There are natural inclusions $E_{k,\bullet}^{\mathrm{GG}}V^* \subset E_{k+1,\bullet}^{\mathrm{GG}}V^*$ of algebras, hence $E_{\infty,\bullet}^{\mathrm{GG}}V^* = \bigcup_{k\geqslant 0} E_{k,\bullet}^{\mathrm{GG}}V^*$ is also an algebra. Moreover, the sheaf of holomorphic sections $\mathcal{O}(E_{\infty,\bullet}^{\mathrm{GG}}V^*)$ admits a canonical derivation ∇^{GG} given by a collection of $\mathbb{C}$-linear maps

$$\nabla^{\mathrm{GG}} : \mathcal{O}(E_{k,m}^{\mathrm{GG}}V^*) \to \mathcal{O}(E_{k+1,m+1}^{\mathrm{GG}}V^*),$$

constructed in the following way. A holomorphic section of $E_{k,m}^{\mathrm{GG}}V^*$ on a coordinate open set $\Omega \subset X$ can be seen as a differential operator on the space of germs $f : (\mathbb{C}, 0) \to \Omega$ of the form

$$Q(f) = \sum_{|\alpha_1|+2|\alpha_2|+\cdots+k|\alpha_k|=m} a_{\alpha_1\ldots\alpha_k}(f)\,(f')^{\alpha_1}(f'')^{\alpha_2}\cdots(f^{(k)})^{\alpha_k} \tag{7.6}$$

in which the coefficients $a_{\alpha_1\ldots\alpha_k}$ are holomorphic functions on Ω. Then ∇Q is given by the formal derivative $(\nabla Q)(f)(t) = d(Q(f))/dt$ with respect to the 1-dimensional parameter t in $f(t)$. For example, in dimension 2, if $Q \in H^0(\Omega, \mathcal{O}(E_{2,4}^{\mathrm{GG}}))$ is the section of weighted degree 4

$$Q(f) = a(f_1, f_2)\, f_1'^3 f_2' + b(f_1, f_2)\, f_1''^2,$$

we find that $\nabla Q \in H^0(\Omega, \mathcal{O}(E_{3,5}^{\mathrm{GG}}))$ is given by

$$\begin{aligned}(\nabla Q)(f) = {}& \frac{\partial a}{\partial z_1}(f_1, f_2)\, f_1'^4 f_2' + \frac{\partial a}{\partial z_2}(f_1, f_2)\, f_1'^3 f_2'^2 + \frac{\partial b}{\partial z_1}(f_1, f_2)\, f_1' f_1''^2 \\ &+ \frac{\partial b}{\partial z_2}(f_1, f_2)\, f_2' f_1''^2 + a(f_1, f_2)\,\big(3 f_1'^2 f_1'' f_2' + f_1'^3 f_2''\big) + b(f_1, f_2)\; 2 f_1'' f_1'''.\end{aligned}$$

Associated with the graded algebra bundle $E_{k,\bullet}^{\mathrm{GG}}V^*$, we have an analytic fiber bundle

$$X_k^{\mathrm{GG}} := \mathrm{Proj}(E_{k,\bullet}^{\mathrm{GG}}V^*) = (J_kV \smallsetminus \{0\})/\mathbb{C}^* \tag{7.7}$$

over X, which has weighted projective spaces $\mathbb{P}(1^{[r]}, 2^{[r]}, \ldots, k^{[r]})$ as fibers (these weighted projective spaces are singular for $k > 1$, but they only have quotient singularities, see [Dol81] ; here $J_kV \smallsetminus \{0\}$ is the set of non constant jets of order k ; we refer e.g. to Hartshorne's book [Har77] for a definition of the Proj fonctor). As such, it possesses a canonical sheaf $\mathcal{O}_{X_k^{\mathrm{GG}}}(1)$ such that $\mathcal{O}_{X_k^{\mathrm{GG}}}(m)$ is invertible when m is a multiple of $\mathrm{lcm}(1, 2, \ldots, k)$. Under the natural projection $\pi_k : X_k^{\mathrm{GG}} \to X$, the direct image $(\pi_k)_*\mathcal{O}_{X_k^{\mathrm{GG}}}(m)$ coincides with polynomials

$$P(z\,;\,\xi_1, \ldots, \xi_k) = \sum_{\alpha_\ell\in\mathbb{N}^r,\,1\leqslant\ell\leqslant k} a_{\alpha_1\ldots\alpha_k}(z)\,\xi_1^{\alpha_1}\ldots\xi_k^{\alpha_k} \tag{7.8}$$

of weighted degree $|\alpha_1| + 2|\alpha_2| + \ldots + k|\alpha_k| = m$ on J^kV with holomorphic coefficients; in other words, we obtain precisely the sheaf of sections of the bundle $E_{k,m}^{\mathrm{GG}}V^*$ of jet differentials of order k and degree m.

7.9. Proposition. *By construction, if $\pi_k : X_k^{\mathrm{GG}} \to X$ is the natural projection, we have the direct image formula*

$$(\pi_k)_*\mathcal{O}_{X_k^{\mathrm{GG}}}(m) = \mathcal{O}(E_{k,m}^{\mathrm{GG}}V^*)$$

for all k and m.

§7.B. Invariant jet differentials

In the geometric context, we are not really interested in the bundles $(J_kV \smallsetminus \{0\})/\mathbb{C}^*$ themselves, but rather on their quotients $(J_kV \smallsetminus \{0\})/\mathbb{G}_k$ (would such nice complex space quotients exist!). We will see that the Semple bundle P_kV constructed in § 6 plays the role of such a quotient. First we introduce a canonical bundle subalgebra of $E^{\rm GG}_{k,\bullet}V^*$.

7.10. Definition. *We introduce a subbundle $E_{k,m}V^* \subset E^{\rm GG}_{k,m}V^*$, called the bundle of invariant jet differentials of order k and degree m, defined as follows: $E_{k,m}V^*$ is the set of polynomial differential operators $Q(f', f'', \ldots, f^{(k)})$ which are invariant under arbitrary changes of parametrization, i.e., for every $\varphi \in \mathbb{G}_k$*

$$Q\big((f\circ\varphi)', (f\circ\varphi)'', \ldots, (f\circ\varphi)^{(k)}\big) = \varphi'(0)^m Q(f', f'', \ldots, f^{(k)}).$$

Alternatively, $E_{k,m}V^* = (E^{\rm GG}_{k,m}V^*)^{\mathbb{G}'_k}$ is the set of invariants of $E^{\rm GG}_{k,m}V^*$ under the action of $\mathbb{G}'_k$. Clearly, $E_{\infty,\bullet}V^* = \bigcup_{k\geqslant 0}\bigoplus_{m\geqslant 0} E_{k,m}V^*$ is a subalgebra of $E^{\rm GG}_{k,m}V^*$ (observe however that this algebra is not invariant under the derivation $\nabla^{\rm GG}$, since e.g. $f''_j = \nabla^{\rm GG} f_j$ is not an invariant polynomial). In addition to this, there are natural induced filtrations $F^p_s(E_{k,m}V^*) = E_{k,m}V^* \cap F^p_s(E^{\rm GG}_{k,m}V^*)$ (all locally trivial over X). These induced filtrations will play an important role later on.

7.11. Theorem. *Suppose that V has rank $r \geqslant 2$. Let $\pi_{0,k} : P_kV \longrightarrow X$ be the Semple jet bundles constructed in section 6, and let $J_kV^{\rm reg}$ be the bundle of regular k-jets of maps $f : (\mathbb{C}, 0) \to X$, that is, jets f such that $f'(0) \neq 0$.*

i) *The quotient $J_kV^{\rm reg}/\mathbb{G}_k$ has the structure of a locally trivial bundle over X, and there is a holomorphic embedding $J_kV^{\rm reg}/\mathbb{G}_k \hookrightarrow P_kV$ over X, which identifies $J_kV^{\rm reg}/\mathbb{G}_k$ with $P_kV^{\rm reg}$ (thus P_kV is a relative compactification of $J_kV^{\rm reg}/\mathbb{G}_k$ over X).*

ii) *The direct image sheaf*
$$(\pi_{0,k})_*\mathcal{O}_{P_kV}(m) \simeq \mathcal{O}(E_{k,m}V^*)$$
can be identified with the sheaf of holomorphic sections of $E_{k,m}V^*$.

iii) *For every $m > 0$, the relative base locus of the linear system $|\mathcal{O}_{P_kV}(m)|$ is equal to the set $P_kV^{\rm sing}$ of singular k-jets. Moreover, $\mathcal{O}_{P_kV}(1)$ is relatively big over X.*

Proof. i) For $f \in J_kV^{\rm reg}$, the lifting $\widetilde f$ is obtained by taking the derivative $(f, [f'])$ without any cancellation of zeroes in f', hence we get a uniquely defined $(k-1)$-jet $\widetilde f : (\mathbb{C}, 0) \to \widetilde X$. Inductively, we get a well defined $(k-j)$-jet $f_{[j]}$ in P_jV, and the value $f_{[k]}(0)$ is independent of the choice of the representative f for the k-jet. As the lifting process commutes with reparametrization, i.e., $(f\circ\varphi)^\sim = \widetilde f\circ\varphi$ and more generally $(f\circ\varphi)_{[k]} = f_{[k]}\circ\varphi$, we conclude that there is a well defined set-theoretic map

$$J_kV^{\rm reg}/\mathbb{G}_k \to P_kV^{\rm reg}, \qquad f \bmod \mathbb{G}_k \mapsto f_{[k]}(0).$$

This map is better understood in coordinates as follows. Fix coordinates $(z_1, \ldots, z_n)$ near a point $x_0 \in X$, such that $V_{x_0} = \mathrm{Vect}(\partial/\partial z_1, \ldots, \partial/\partial z_r)$. Let $f = (f_1, \ldots, f_n)$ be a regular k-jet tangent to V. Then there exists $i \in \{1, 2, \ldots, r\}$ such that $f'_i(0) \neq 0$, and there is a unique reparametrization $t = \varphi(\tau)$ such that $f\circ\varphi = g = (g_1, g_2, \ldots, g_n)$ with $g_i(\tau) = \tau$ (we just express the curve as a graph over the z_i-axis, by means of a change of parameter $\tau = f_i(t)$, i.e. $t = \varphi(\tau) = f_i^{-1}(\tau)$). Suppose $i = r$ for the simplicity of notation. The space

P_kV is a k-stage tower of $\mathbb{P}^{r-1}$-bundles. In the corresponding inhomogeneous coordinates on these $\mathbb{P}^{r-1}$'s, the point $f_{[k]}(0)$ is given by the collection of derivatives

$$\big((g_1'(0),\ldots,g_{r-1}'(0));(g_1''(0),\ldots,g_{r-1}''(0));\ldots;(g_1^{(k)}(0),\ldots,g_{r-1}^{(k)}(0))\big).$$

[Recall that the other components $(g_{r+1},\ldots,g_n)$ can be recovered from $(g_1,\ldots,g_r)$ by integrating the differential system (5.10)]. Thus the map $J_kV^{\mathrm{reg}}/\mathbb{G}_k\to P_kV$ is a bijection onto P_kV^{reg}, and the fibers of these isomorphic bundles can be seen as unions of r affine charts $\simeq(\mathbb{C}^{r-1})^k$, associated with each choice of the axis z_i used to describe the curve as a graph. The change of parameter formula $\frac{d}{d\tau}=\frac{1}{f_r'(t)}\frac{d}{dt}$ expresses all derivatives $g_i^{(j)}(\tau)=d^jg_i/d\tau^j$ in terms of the derivatives $f_i^{(j)}(t)=d^jf_i/dt^j$

$$(7.12)\qquad \begin{aligned}
&(g_1',\ldots,g_{r-1}')=\Big(\frac{f_1'}{f_r'},\ldots,\frac{f_{r-1}'}{f_r'}\Big);\\
&(g_1'',\ldots,g_{r-1}'')=\Big(\frac{f_1''f_r'-f_r''f_1'}{f_r'^3},\ldots,\frac{f_{r-1}''f_r'-f_r''f_{r-1}'}{f_r'^3}\Big);\ \ldots\ ;\\
&(g_1^{(k)},\ldots,g_{r-1}^{(k)})=\Big(\frac{f_1^{(k)}f_r'-f_r^{(k)}f_1'}{f_r'^{k+1}},\ldots,\frac{f_{r-1}^{(k)}f_r'-f_r^{(k)}f_{r-1}'}{f_r'^{k+1}}\Big)+(\text{order}<k).
\end{aligned}$$

Also, it is easy to check that $f_r'^{2k-1}g_i^{(k)}$ is an invariant polynomial in f', f'', $\ldots$, $f^{(k)}$ of total degree $2k-1$, i.e., a section of $E_{k,2k-1}$.

ii) Since the bundles P_kV and $E_{k,m}V^*$ are both locally trivial over X, it is sufficient to identify sections σ of $\mathcal{O}_{P_kV}(m)$ over a fiber $P_kV_x=\pi_{0,k}^{-1}(x)$ with the fiber $E_{k,m}V_x^*$, at any point $x\in X$. Let $f\in J_kV_x^{\mathrm{reg}}$ be a regular k-jet at x. By (6.6), the derivative $f_{[k-1]}'(0)$ defines an element of the fiber of $\mathcal{O}_{P_kV}(-1)$ at $f_{[k]}(0)\in P_kV$. Hence we get a well defined complex valued operator

$$(7.13)\qquad Q(f',f'',\ldots,f^{(k)})=\sigma(f_{[k]}(0))\cdot(f_{[k-1]}'(0))^m.$$

Clearly, Q is holomorphic on $J_kV_x^{\mathrm{reg}}$ (by the holomorphicity of σ), and the $\mathbb{G}_k$-invariance condition of Def. 7.10 is satisfied since $f_{[k]}(0)$ does not depend on reparametrization and $(f\circ\varphi)_{[k-1]}'(0)=f_{[k-1]}'(0)\varphi'(0)$. Now, $J_kV_x^{\mathrm{reg}}$ is the complement of a linear subspace of codimension n in J_kV_x, hence Q extends holomorphically to all of $J_kV_x\simeq(\mathbb{C}^r)^k$ by Riemann's extension theorem (here we use the hypothesis $r\geqslant 2$; if $r=1$, the situation is anyway not interesting since $P_kV=X$ for all k). Thus Q admits an everywhere convergent power series

$$Q(f',f'',\ldots,f^{(k)})=\sum_{\alpha_1,\alpha_2,\ldots,\alpha_k\in\mathbb{N}^r}a_{\alpha_1\ldots\alpha_k}\,(f')^{\alpha_1}(f'')^{\alpha_2}\cdots(f^{(k)})^{\alpha_k}.$$

The $\mathbb{G}_k$-invariance (7.10) implies in particular that Q must be multihomogeneous in the sense of (7.1), and thus Q must be a polynomial. We conclude that $Q\in E_{k,m}V_x^*$, as desired.

Conversely, Corollary 6.12 implies that there is a holomorphic family of germs $f_w:(\mathbb{C},0)\to X$ such that $(f_w)_{[k]}(0)=w$ and $(f_w)_{[k-1]}'(0)\neq 0$, for all w in a neighborhood of any given point $w_0\in P_kV_x$. Then every $Q\in E_{k,m}V_x^*$ yields a holomorphic section σ of $\mathcal{O}_{P_kV}(m)$ over the fiber P_kV_x by putting

$$(7.14)\qquad \sigma(w)=Q(f_w',f_w'',\ldots,f_w^{(k)})(0)\,\big((f_w)_{[k-1]}'(0)\big)^{-m}.$$

iii) By what we saw in i-ii), every section σ of $\mathcal{O}_{P_kV}(m)$ over the fiber P_kV_x is given by a polynomial $Q \in E_{k,m}V_x^*$, and this polynomial can be expressed on the Zariski open chart $f'_r \neq 0$ of $P_kV_x^{\rm reg}$ as

$$Q(f', f'', \ldots, f^{(k)}) = f_r'^m \widehat{Q}(g', g'', \ldots, g^{(k)}), \tag{7.15}$$

where $\widehat{Q}$ is a polynomial and g is the reparametrization of f such that $g_r(\tau) = \tau$. In fact $\widehat{Q}$ is obtained from Q by substituting $f'_r = 1$ and $f_r^{(j)} = 0$ for $j \geqslant 2$, and conversely Q can be recovered easily from $\widehat{Q}$ by using the substitutions (7.12).

In this context, the jet differentials $f \mapsto f'_1, \ldots, f \mapsto f'_r$ can be viewed as sections of $\mathcal{O}_{P_kV}(1)$ on a neighborhood of the fiber P_kV_x. Since these sections vanish exactly on $P_kV^{\rm sing}$, the relative base locus of $\mathcal{O}_{P_kV}(m)$ is contained in $P_kV^{\rm sing}$ for every $m > 0$. We see that $\mathcal{O}_{P_kV}(1)$ is big by considering the sections of $\mathcal{O}_{P_kV}(2k-1)$ associated with the polynomials $Q(f', \ldots, f^{(k)}) = f_r'^{2k-1} g_i^{(j)}$, $1 \leqslant i \leqslant r-1$, $1 \leqslant j \leqslant k$; indeed, these sections separate all points in the open chart $f'_r \neq 0$ of $P_kV_x^{\rm reg}$.

Now, we check that every section σ of $\mathcal{O}_{P_kV}(m)$ over P_kV_x must vanish on $P_kV_x^{\rm sing}$. Pick an arbitrary element $w \in P_kV^{\rm sing}$ and a germ of curve $f : (\mathbb{C}, 0) \to X$ such that $f_{[k]}(0) = w$, $f'_{[k-1]}(0) \neq 0$ and $s = m(f, 0) \gg 0$ (such an f exists by Corollary 6.14). There are local coordinates $(z_1, \ldots, z_n)$ on X such that $f(t) = (f_1(t), \ldots, f_n(t))$ where $f_r(t) = t^s$. Let Q, $\widehat{Q}$ be the polynomials associated with σ in these coordinates and let $(f')^{\alpha_1}(f'')^{\alpha_2} \cdots (f^{(k)})^{\alpha_k}$ be a monomial occurring in Q, with $\alpha_j \in \mathbb{N}^r$, $|\alpha_j| = \ell_j$, $\ell_1 + 2\ell_2 + \cdots + k\ell_k = m$. Putting $\tau = t^s$, the curve $t \mapsto f(t)$ becomes a Puiseux expansion $\tau \mapsto g(\tau) = (g_1(\tau), \ldots, g_{r-1}(\tau), \tau)$ in which g_i is a power series in $\tau^{1/s}$, starting with exponents of τ at least equal to 1. The derivative $g^{(j)}(\tau)$ may involve negative powers of τ, but the exponent is always $\geqslant 1 + \frac{1}{s} - j$. Hence the Puiseux expansion of $\widehat{Q}(g', g'', \ldots, g^{(k)})$ can only involve powers of τ of exponent $\geqslant -\max_\ell((1 - \frac{1}{s})\ell_2 + \cdots + (k - 1 - \frac{1}{s})\ell_k)$. Finally $f'_r(t) = st^{s-1} = s\tau^{1-1/s}$, thus the lowest exponent of τ in $Q(f', \ldots, f^{(k)})$ is at least equal to

$$\begin{aligned}\Big(1 - \frac{1}{s}\Big)m - \max_\ell \Big(\Big(1 - \frac{1}{s}\Big)\ell_2 + \cdots + \Big(k - 1 - \frac{1}{s}\Big)\ell_k\Big)&\\ \geqslant \min_\ell \Big(1 - \frac{1}{s}\Big)\ell_1 + \Big(1 - \frac{1}{s}\Big)\ell_2 + \cdots + \Big(1 - \frac{k-1}{s}\Big)\ell_k&\end{aligned}$$

where the minimum is taken over all monomials $(f')^{\alpha_1}(f'')^{\alpha_2} \cdots (f^{(k)})^{\alpha_k}$, $|\alpha_j| = \ell_j$, occurring in Q. Choosing $s \geqslant k$, we already find that the minimal exponent is positive, hence $Q(f', \ldots, f^{(k)})(0) = 0$ and $\sigma(w) = 0$ by (7.14). □

Theorem (7.11 iii) shows that $\mathcal{O}_{P_kV}(1)$ is never relatively ample over X for $k \geqslant 2$. In order to overcome this difficulty, we define for every $\boldsymbol{a} = (a_1, \ldots, a_k) \in \mathbb{Z}^k$ a line bundle $\mathcal{O}_{P_kV}(\boldsymbol{a})$ on P_kV such that

$$\mathcal{O}_{P_kV}(\boldsymbol{a}) = \pi_{1,k}^*\mathcal{O}_{P_1V}(a_1) \otimes \pi_{2,k}^*\mathcal{O}_{P_2V}(a_2) \otimes \cdots \otimes \mathcal{O}_{P_kV}(a_k). \tag{7.16}$$

By (6.9), we have $\pi_{j,k}^*\mathcal{O}_{P_jV}(1) = \mathcal{O}_{P_kV}(1) \otimes \mathcal{O}_{P_kV}(-\pi_{j+1,k}^*D_{j+1} - \cdots - D_k)$, thus by putting $D_j^* = \pi_{j+1,k}^*D_{j+1}$ for $1 \leqslant j \leqslant k-1$ and $D_k^* = 0$, we find an identity

$$\begin{aligned}&\mathcal{O}_{P_kV}(\boldsymbol{a}) = \mathcal{O}_{P_kV}(b_k) \otimes \mathcal{O}_{P_kV}(-\boldsymbol{b} \cdot D^*), \qquad \text{where}\\ &\boldsymbol{b} = (b_1, \ldots, b_k) \in \mathbb{Z}^k, \quad b_j = a_1 + \cdots + a_j,\\ &\boldsymbol{b} \cdot D^* = \sum_{1 \leqslant j \leqslant k-1} b_j\, \pi_{j+1,k}^* D_{j+1}.\end{aligned} \tag{7.17}$$

In particular, if $\boldsymbol{b} \in \mathbb{N}^k$, i.e., $a_1 + \cdots + a_j \geqslant 0$, we get a morphism

$$\mathcal{O}_{P_kV}(\boldsymbol{a}) = \mathcal{O}_{P_kV}(b_k) \otimes \mathcal{O}_{P_kV}(-\boldsymbol{b} \cdot D^*) \to \mathcal{O}_{P_kV}(b_k). \tag{7.18}$$

7.19. Remark. As in Green-Griffiths [GrGr80], Riemann's extension theorem shows that for every meromorphic map $\Phi : X \dashrightarrow Y$ there are well-defined pullback morphisms

$$\Phi^* : H^0(Y, E_{k,m}^{\mathrm{GG}}) \to H^0(X, E_{k,m}^{\mathrm{GG}}), \qquad \Phi^* : H^0(Y, E_{k,m}) \to H^0(X, E_{k,m}).$$

In particular the dimensions $h^0(X, E_{k,m}^{\mathrm{GG}})$ and $h^0(X, E_{k,m}^{\mathrm{GG}})$ are bimeromorphic invariants of X. The same is true for spaces of sections of any subbundle of $E_{k,m}^{\mathrm{GG}}$ or $E_{k,m}$ constructed by means of the canonical filtrations $F_s^\bullet$.

7.20. Remark. As $\mathbb{G}_k$ is a non reductive group, it is not a priori clear that the graded ring $\mathcal{A}_{n,k,r} = \bigoplus_{m\in\mathbb{Z}} E_{k,m}V^\star$ is finitely generated (pointwise). This can be checked by hand ([Dem07a], [Dem07b]) for $n = 2$ and $k \geqslant 4$. Rousseau [Rou06b] also checked the case $n = 3$, $k = 3$, and then Merker [Mer08] proved the finiteness for $n = 2$, $k = 5$. Recently, Bérczi and Kirwan [BeKi10] found a nice geometric argument proving the finiteness in full generality.

§8. k-jet metrics with negative curvature

The goal of this section is to show that hyperbolicity is closely related to the existence of k-jet metrics with suitable negativity properties of the curvature. The connection between these properties is in fact a simple consequence of the Ahlfors-Schwarz lemma. Such ideas have been already developed long ago by Grauert-Reckziegel [GRec65], Kobayashi [Kob75] for 1-jet metrics (i.e., Finsler metrics on T_X) and by Cowen-Griffiths [CoGr76], Green-Griffiths [GrGr80] and Grauert [Gra89] for higher order jet metrics.

§8.A. Definition of k-jet metrics

Even in the standard case $V = T_X$, the definition given below differs from that of [GrGr80], in which the k-jet metrics are not supposed to be $\mathbb{G}'_k$-invariant. We prefer to deal here with $\mathbb{G}'_k$-invariant objects, because they reflect better the intrinsic geometry. Grauert [Gra89] actually deals with $\mathbb{G}'_k$-invariant metrics, but he apparently does not take care of the way the quotient space $J_k^{\mathrm{reg}}V/\mathbb{G}_k$ can be compactified; also, his metrics are always induced by the Poincaré metric, and it is not at all clear whether these metrics have the expected curvature properties (see 8.14 below). In the present situation, it is important to allow also hermitian metrics possessing some singularities ("singular hermitian metrics" in the sense of [Dem90b]).

8.1. Definition. *Let $L \to X$ be a holomorphic line bundle over a complex manifold X. We say that h is a singular metric on L if for any trivialization $L_{\restriction U} \simeq U \times \mathbb{C}$ of L, the metric is given by $|\xi|_h^2 = |\xi|^2 e^{-\varphi}$ for some real valued weight function $\varphi \in L^1_{\mathrm{loc}}(U)$. The curvature current of L is then defined to be the closed $(1,1)$-current $\Theta_{L,h} = \frac{i}{2\pi}\partial\overline{\partial}\varphi$, computed in the sense of distributions. We say that h admits a closed subset $\Sigma \subset X$ as its degeneration set if φ is locally bounded on $X \smallsetminus \Sigma$ and is unbounded on a neighborhood of any point of Σ.*

An especially useful situation is the case when the curvature of h is positive definite. By this, we mean that there exists a smooth positive definite hermitian metric ω and a continuous positive function ε on X such that $\Theta_{L,h} \geqslant \varepsilon\omega$ in the sense of currents, and we write in this case $\Theta_{L,h} \gg 0$. We need the following basic fact (quite standard when X is projective algebraic; however we want to avoid any algebraicity assumption here, so as to be able to cover the case of general complex tori in § 10).

8.2. Proposition. *Let L be a holomorphic line bundle on a compact complex manifold X.*

i) *L admits a singular hermitian metric h with positive definite curvature current $\Theta_{L,h} \gg 0$ if and only if L is big.*

Now, define B_m to be the base locus of the linear system $|H^0(X, L^{\otimes m})|$ and let

$$\Phi_m : X \smallsetminus B_m \to \mathbb{P}^N$$

be the corresponding meromorphic map. Let Σ_m be the closed analytic set equal to the union of B_m and of the set of points $x \in X \smallsetminus B_m$ such that the fiber $\Phi_m^{-1}(\Phi_m(x))$ is positive dimensional.

ii) *If $\Sigma_m \neq X$ and G is any line bundle, the base locus of $L^{\otimes k} \otimes G^{-1}$ is contained in Σ_m for k large. As a consequence, L admits a singular hermitian metric h with degeneration set Σ_m and with $\Theta_{L,h}$ positive definite on X.*

iii) *Conversely, if L admits a hermitian metric h with degeneration set Σ and positive definite curvature current $\Theta_{L,h}$, there exists an integer $m > 0$ such that the base locus B_m is contained in Σ and $\Phi_m : X \smallsetminus \Sigma \to \mathbb{P}_m$ is an embedding.*

Proof. i) is proved e.g. in [Dem90b, 92], and ii) and iii) are well-known results in the basic theory of linear systems. □

We now come to the main definitions. By (6.6), every regular k-jet $f \in J_kV$ gives rise to an element $f'_{[k-1]}(0) \in \mathcal{O}_{P_kV}(-1)$. Thus, measuring the "norm of k-jets" is the same as taking a hermitian metric on $\mathcal{O}_{P_kV}(-1)$.

8.3. Definition. *A smooth, (resp. continuous, resp. singular) k-jet metric on a complex directed manifold (X, V) is a hermitian metric h_k on the line bundle $\mathcal{O}_{P_kV}(-1)$ over P_kV (i.e. a Finsler metric on the vector bundle V_{k-1} over $P_{k-1}V$), such that the weight functions φ representing the metric are smooth (resp. continuous, L^1_{loc}). We let $\Sigma_{h_k} \subset P_kV$ be the singularity set of the metric, i.e., the closed subset of points in a neighborhood of which the weight φ is not locally bounded.*

We will always assume here that the weight function φ is quasi psh. Recall that a function φ is said to be quasi psh if φ is locally the sum of a plurisubharmonic function and of a smooth function (so that in particular $\varphi \in L^1_{\mathrm{loc}}$). Then the curvature current

$$\Theta_{h_k^{-1}}(\mathcal{O}_{P_kV}(1)) = \frac{i}{2\pi} \partial\overline{\partial}\varphi.$$

is well defined as a current and is locally bounded from below by a negative $(1,1)$-form with constant coefficients.

8.4. Definition. *Let h_k be a k-jet metric on (X, V). We say that h_k has negative jet curvature (resp. negative total jet curvature) if $\Theta_{h_k}(\mathcal{O}_{P_kV}(-1))$ is negative definite along the subbundle $V_k \subset T_{P_kV}$ (resp. on all of T_{P_kV}), i.e., if there is $\varepsilon > 0$ and a smooth hermitian metric ω_k on T_{P_kV} such that*

$$\langle \Theta_{h_k^{-1}}(\mathcal{O}_{P_kV}(1)) \rangle(\xi) \geqslant \varepsilon |\xi|^2_{\omega_k}, \qquad \forall \xi \in V_k \subset T_{P_kV} \quad (\textit{resp.} \quad \forall \xi \in T_{P_kV}).$$

(*If the metric h_k is not smooth, we suppose that its weights φ are quasi psh, and the curvature inequality is taken in the sense of distributions.*)

It is important to observe that for $k \geqslant 2$ there cannot exist any smooth hermitian metric h_k on $\mathcal{O}_{P_kV}(1)$ with positive definite curvature along $T_{X_k/X}$, since $\mathcal{O}_{P_kV}(1)$ is not relatively ample over X. However, it is relatively big, and Prop. 8.2 i) shows that $\mathcal{O}_{P_kV}(-1)$ admits a singular hermitian metric with negative total jet curvature (whatever the singularities of the metric are) if and only if $\mathcal{O}_{P_kV}(1)$ is big over P_kV. It is therefore crucial to allow singularities in the metrics in Def. 8.4.

§8.B. Special case of 1-jet metrics

A 1-jet metric h_1 on $\mathcal{O}_{P_1V}(-1)$ is the same as a Finsler metric $N = \sqrt{h_1}$ on $V \subset T_X$. Assume until the end of this paragraph that h_1 is smooth. By the well known Kodaira embedding theorem, the existence of a smooth metric h_1 such that $\Theta_{h_1^{-1}}(\mathcal{O}_{P_1V}(1))$ is positive on all of T_{P_1V} is equivalent to $\mathcal{O}_{P_1V}(1)$ being ample, that is, V^* ample.

8.5 Remark. In the absolute case $V = T_X$, there are only few examples of varieties X such that T_X^* is ample, mainly quotients of the ball $\mathbb{B}_n \subset \mathbb{C}^n$ by a discrete cocompact group of automorphisms.

The 1-jet negativity condition considered in Definition 8.4 is much weaker. For example, if the hermitian metric h_1 comes from a (smooth) hermitian metric h on V, then formula (5.15) implies that h_1 has negative total jet curvature (i.e. $\Theta_{h_1^{-1}}(\mathcal{O}_{P_1V}(1))$ is positive) if and only if $\langle\Theta_{V,h}\rangle(\zeta\otimes v) < 0$ for all $\zeta \in T_X \smallsetminus \{0\}$, $v \in V \smallsetminus \{0\}$, that is, if (V,h) is *negative in the sense of Griffiths*. On the other hand, $V_1 \subset T_{P_1V}$ consists by definition of tangent vectors $\tau \in T_{P_1V,(x,[v])}$ whose horizontal projection ${}^H\tau$ is proportional to v, thus $\Theta_{h_1}(\mathcal{O}_{P_1V}(-1))$ is negative definite on V_1 if and only if $\Theta_{V,h}$ satisfies the much weaker condition that the *holomorphic sectional curvature* $\langle\Theta_{V,h}\rangle(v\otimes v)$ is negative on every complex line. □

§8.C. Vanishing theorem for invariant jet differentials

We now come back to the general situation of jets of arbitrary order k. Our first observation is the fact that the k-jet negativity property of the curvature becomes actually weaker and weaker as k increases.

8.6. Lemma. *Let (X,V) be a compact complex directed manifold. If (X,V) has a $(k-1)$-jet metric h_{k-1} with negative jet curvature, then there is a k-jet metric h_k with negative jet curvature such that $\Sigma_{h_k} \subset \pi_k^{-1}(\Sigma_{h_{k-1}}) \cup D_k$. (The same holds true for negative total jet curvature).*

Proof. Let ω_{k-1}, ω_k be given smooth hermitian metrics on $T_{P_{k-1}V}$ and T_{P_kV}. The hypothesis implies

$$\langle\Theta_{h_{k-1}^{-1}}(\mathcal{O}_{P_{k-1}V}(1))\rangle(\xi) \geqslant \varepsilon|\xi|^2_{\omega_{k-1}}, \qquad \forall\xi \in V_{k-1}$$

for some constant $\varepsilon > 0$. On the other hand, as $\mathcal{O}_{P_kV}(D_k)$ is relatively ample over $P_{k-1}V$ (D_k is a hyperplane section bundle), there exists a smooth metric $\widetilde{h}$ on $\mathcal{O}_{P_kV}(D_k)$ such that

$$\langle\Theta_{\widetilde{h}}(\mathcal{O}_{P_kV}(D_k))\rangle(\xi) \geqslant \delta|\xi|^2_{\omega_k} - C|(\pi_k)_*\xi|^2_{\omega_{k-1}}, \qquad \forall\xi \in T_{P_kV}$$

for some constants $\delta, C > 0$. Combining both inequalities (the second one being applied to $\xi \in V_k$ and the first one to $(\pi_k)_*\xi \in V_{k-1}$), we get

$$\begin{aligned}\langle\Theta_{(\pi_k^*h_{k-1})^{-p}\widetilde{h}}(\pi_k^*\mathcal{O}_{P_{k-1}V}(p)\otimes\mathcal{O}_{P_kV}(D_k))\rangle(\xi) \geqslant\\ \geqslant \delta|\xi|^2_{\omega_k} + (p\varepsilon - C)|(\pi_k)_*\xi|^2_{\omega_{k-1}}, \qquad \forall\xi \in V_k.\end{aligned}$$

Hence, for p large enough, $(\pi_k^* h_{k-1})^{-p}\widetilde{h}$ has positive definite curvature along V_k. Now, by (6.9), there is a sheaf injection

$$\mathcal{O}_{P_kV}(-p) = \pi_k^*\mathcal{O}_{P_{k-1}V}(-p)\otimes\mathcal{O}_{P_kV}(-pD_k)\hookrightarrow\big(\pi_k^*\mathcal{O}_{P_{k-1}V}(p)\otimes\mathcal{O}_{P_kV}(D_k)\big)^{-1}$$

obtained by twisting with $\mathcal{O}_{P_kV}((p-1)D_k)$. Therefore $h_k := ((\pi_k^*h_{k-1})^{-p}\widetilde{h})^{-1/p} = (\pi_k^*h_{k-1})\widetilde{h}^{-1/p}$ induces a singular metric on $\mathcal{O}_{P_kV}(1)$ in which an additional degeneration divisor $p^{-1}(p-1)D_k$ appears. Hence we get $\Sigma_{h_k} = \pi_k^{-1}\Sigma_{h_{k-1}}\cup D_k$ and

$$\Theta_{h_k^{-1}}(\mathcal{O}_{P_kV}(1)) = \frac{1}{p}\Theta_{(\pi_k^*h_{k-1})^{-p}\widetilde{h}} + \frac{p-1}{p}[D_k]$$

is positive definite along V_k. The same proof works in the case of negative total jet curvature. □

One of the main motivations for the introduction of k-jets metrics is the following list of algebraic sufficient conditions.

8.7. Algebraic sufficient conditions. We suppose here that X is projective algebraic, and we make one of the additional assumptions i), ii) or iii) below.

i) Assume that there exist integers $k, m > 0$ and $\boldsymbol{b}\in\mathbb{N}^k$ such that the line bundle $\mathcal{O}_{P_kV}(m)\otimes\mathcal{O}_{P_kV}(-\boldsymbol{b}\cdot D^*)$ is ample over P_kV. Set $A = \mathcal{O}_{P_kV}(m)\otimes\mathcal{O}_{P_kV}(-\boldsymbol{b}\cdot D^*)$. Then there is a smooth hermitian metric h_A on A with positive definite curvature on P_kV. By means of the morphism $\mu : \mathcal{O}_{P_kV}(-m)\to A^{-1}$, we get an induced metric $h_k = (\mu^*h_A^{-1})^{1/m}$ on $\mathcal{O}_{P_kV}(-1)$ which is degenerate on the support of the zero divisor $\mathrm{div}(\mu) = \boldsymbol{b}\cdot D^*$. Hence $\Sigma_{h_k} = \mathrm{Supp}(\boldsymbol{b}\cdot D^*)\subset P_kV^{\mathrm{sing}}$ and

$$\Theta_{h_k^{-1}}(\mathcal{O}_{P_kV}(1)) = \frac{1}{m}\Theta_{h_A}(A) + \frac{1}{m}[\boldsymbol{b}\cdot D^*] \geqslant \frac{1}{m}\Theta_{h_A}(A) > 0.$$

In particular h_k has negative total jet curvature.

ii) Assume more generally that there exist integers $k, m > 0$ and an ample line bundle L on X such that $H^0(P_kV, \mathcal{O}_{P_kV}(m)\otimes\pi_{0,k}^*L^{-1})$ has non zero sections $\sigma_1,\ldots,\sigma_N$. Let $Z\subset P_kV$ be the base locus of these sections; necessarily $Z\supset P_kV^{\mathrm{sing}}$ by 7.11 iii). By taking a smooth metric h_L with positive curvature on L, we get a singular metric h'_k on $\mathcal{O}_{P_kV}(-1)$ such that

$$h'_k(\xi) = \Big(\sum_{1\leqslant j\leqslant N}|\sigma_j(w)\cdot\xi^m|^2_{h_L^{-1}}\Big)^{1/m},\qquad w\in P_kV,\quad \xi\in\mathcal{O}_{P_kV}(-1)_w.$$

Then $\Sigma_{h'_k} = Z$, and by computing $\frac{i}{2\pi}\partial\overline{\partial}\log h'_k(\xi)$ we obtain

$$\Theta_{h_k'^{-1}}(\mathcal{O}_{P_kV}(1)) \geqslant \frac{1}{m}\pi_{0,k}^*\Theta_L.$$

By (7.17) and an induction on k, there exists $\boldsymbol{b}\in\mathbb{Q}_+^k$ such that $\mathcal{O}_{P_kV}(1)\otimes\mathcal{O}_{P_kV}(-\boldsymbol{b}\cdot D^*)$ is relatively ample over X. Hence $A = \mathcal{O}_{P_kV}(1)\otimes\mathcal{O}_{P_kV}(-\boldsymbol{b}\cdot D^*)\otimes\pi_{0,k}^*L^{\otimes p}$ is ample on X for $p\gg 0$. The arguments used in i) show that there is a k-jet metric h''_k on $\mathcal{O}_{P_kV}(-1)$ with $\Sigma_{h''_k} = \mathrm{Supp}(\boldsymbol{b}\cdot D^*) = P_kV^{\mathrm{sing}}$ and

$$\Theta_{h_k''^{-1}}(\mathcal{O}_{P_kV}(1)) = \Theta_A + [\boldsymbol{b}\cdot D^*] - p\,\pi_{0,k}^*\Theta_L,$$

where Θ_A is positive definite on P_kV. The metric $h_k = (h_k'^{mp}h_k'')^{1/(mp+1)}$ then satisfies $\Sigma_{h_k} = \Sigma_{h_k'} = Z$ and

$$\Theta_{h_k^{-1}}(\mathcal{O}_{P_kV}(1)) \geqslant \frac{1}{mp+1}\Theta_A > 0.$$

iii) If $E_{k,m}V^*$ is ample, there is an ample line bundle L and a sufficiently high symmetric power such that $S^p(E_{k,m}V^*) \otimes L^{-1}$ is generated by sections. These sections can be viewed as sections of $\mathcal{O}_{P_kV}(mp) \otimes \pi_{0,k}^* L^{-1}$ over P_kV, and their base locus is exactly $Z = P_kV^{\rm sing}$ by 7.11 iii). Hence the k-jet metric h_k constructed in ii) has negative total jet curvature and satisfies $\Sigma_{h_k} = P_kV^{\rm sing}$. □

An important fact, first observed by [GRe65] for 1-jet metrics and by [GrGr80] in the higher order case, is that k-jet negativity implies hyperbolicity. In particular, the existence of enough global jet differentials implies hyperbolicity.

8.8. Theorem. *Let (X,V) be a compact complex directed manifold. If (X,V) has a k-jet metric h_k with negative jet curvature, then every entire curve $f : \mathbb{C} \to X$ tangent to V is such that $f_{[k]}(\mathbb{C}) \subset \Sigma_{h_k}$. In particular, if $\Sigma_{h_k} \subset P_kV^{\rm sing}$, then (X,V) is hyperbolic.*

Proof. The main idea is to use the Ahlfors-Schwarz lemma, following the approach of [GrGr80]. However we will give here all necessary details because our setting is slightly different. Assume that there is a k-jet metric h_k as in the hypotheses of Theorem 8.8. Let ω_k be a smooth hermitian metric on T_{P_kV}. By hypothesis, there exists $\varepsilon > 0$ such that

$$\langle \Theta_{h_k^{-1}}(\mathcal{O}_{P_kV}(1))\rangle(\xi) \geqslant \varepsilon|\xi|^2_{\omega_k} \qquad \forall \xi \in V_k.$$

Moreover, by (6.4), $(\pi_k)_*$ maps V_k continuously to $\mathcal{O}_{P_kV}(-1)$ and the weight e^φ of h_k is locally bounded from above. Hence there is a constant $C > 0$ such that

$$|(\pi_k)_*\xi|^2_{h_k} \leqslant C|\xi|^2_{\omega_k}, \qquad \forall \xi \in V_k.$$

Combining these inequalities, we find

$$\langle \Theta_{h_k^{-1}}(\mathcal{O}_{P_kV}(1))\rangle(\xi) \geqslant \frac{\varepsilon}{C}|(\pi_k)_*\xi|^2_{h_k}, \qquad \forall \xi \in V_k.$$

Now, let $f : \Delta_R \to X$ be a non constant holomorphic map tangent to V on the disk Δ_R. We use the line bundle morphism (6.6)

$$F = f'_{[k-1]} : T_{\Delta_R} \to f_{[k]}^*\mathcal{O}_{P_kV}(-1)$$

to obtain a pullback metric

$$\gamma = \gamma_0(t)\,dt \otimes d\overline{t} = F^*h_k \qquad \text{on } T_{\Delta_R}.$$

If $f_{[k]}(\Delta_R) \subset \Sigma_{h_k}$ then $\gamma \equiv 0$. Otherwise, $F(t)$ has isolated zeroes at all singular points of $f_{[k-1]}$ and so $\gamma(t)$ vanishes only at these points and at points of the degeneration set $(f_{[k]})^{-1}(\Sigma_{h_k})$ which is a polar set in Δ_R. At other points, the Gaussian curvature of γ satisfies

$$\frac{i\,\partial\overline{\partial}\log\gamma_0(t)}{\gamma(t)} = \frac{-2\pi\,(f_{[k]})^*\Theta_{h_k}(\mathcal{O}_{P_kV}(-1))}{F^*h_k} = \frac{\langle\Theta_{h_k^{-1}}(\mathcal{O}_{P_kV}(1))\rangle(f'_{[k]}(t))}{|f'_{[k-1]}(t)|^2_{h_k}} \geqslant \frac{\varepsilon}{C},$$

since $f'_{[k-1]}(t) = (\pi_k)_* f'_{[k]}(t)$. The Ahlfors-Schwarz lemma 4.2 implies that γ can be compared with the Poincaré metric as follows:

$$\gamma(t) \leqslant \frac{2C}{\varepsilon}\frac{R^{-2}|dt|^2}{(1-|t|^2/R^2)^2} \quad\Longrightarrow\quad |f'_{[k-1]}(t)|^2_{h_k} \leqslant \frac{2C}{\varepsilon}\frac{R^{-2}}{(1-|t|^2/R^2)^2}.$$

If $f : \mathbb{C} \to X$ is an entire curve tangent to V such that $f_{[k]}(\mathbb{C}) \not\subset \Sigma_{h_k}$, the above estimate implies as $R \to +\infty$ that $f_{[k-1]}$ must be a constant, hence also f. Now, if $\Sigma_{h_k} \subset P_k V^{\rm sing}$, the inclusion $f_{[k]}(\mathbb{C}) \subset \Sigma_{h_k}$ implies $f'(t) = 0$ at every point, hence f is a constant and (X, V) is hyperbolic. □

Combining Theorem 8.8 with 8.7 ii) and iii), we get the following consequences.

8.9. Corollary. *Assume that there exist integers $k, m > 0$ and an ample line bundle L on X such that $H^0(P_kV, \mathcal{O}_{P_kV}(m) \otimes \pi^*_{0,k}L^{-1}) \simeq H^0(X, E_{k,m}(V^*) \otimes L^{-1})$ has non zero sections $\sigma_1, \ldots, \sigma_N$. Let $Z \subset P_kV$ be the base locus of these sections. Then every entire curve $f : \mathbb{C} \to X$ tangent to V is such that $f_{[k]}(\mathbb{C}) \subset Z$. In other words, for every global $\mathbb{G}_k$-invariant polynomial differential operator P with values in L^{-1}, every entire curve f must satisfy the algebraic differential equation $P(f) = 0$.* □

8.10. Corollary. *Let (X, V) be a compact complex directed manifold. If $E_{k,m}V^*$ is ample for some positive integers k, m, then (X, V) is hyperbolic.* □

8.11. Remark. Green and Griffiths [GrGr80] stated that Corollary 8.9 is even true with sections $\sigma_j \in H^0(X, E^{\rm GG}_{k,m}(V^*) \otimes L^{-1})$, in the special case $V = T_X$ they consider. We refer to [SiYe97] by Siu and Yeung for a detailed proof of this fact, based on a use of the well-known logarithmic derivative lemma in Nevanlinna theory (the original proof given in [GrGr80] does not seem to be complete, as it relies on an unsettled pointwise version of the Ahlfors-Schwarz lemma for general jet differentials); other proofs seem to have been circulating in the literature in the last years. We give here a very short proof for the case when f is supposed to have a bounded derivative (thanks to the Brody criterion, this is enough if one is merely interested in proving hyperbolicity, thus Corollary 8.10 will be valid with $E^{\rm GG}_{k,m}V^*$ in place of $E_{k,m}V^*$). In fact, if f' is bounded, one can apply the Cauchy inequalities to all components f_j of f with respect to a finite collection of coordinate patches covering X. As f' is bounded, we can do this on sufficiently small discs $D(t, \delta) \subset \mathbb{C}$ of constant radius $\delta > 0$. Therefore all derivatives f', f'', … $f^{(k)}$ are bounded. From this we conclude that $\sigma_j(f)$ is a bounded section of f^*L^{-1}. Its norm $|\sigma_j(f)|_{L^{-1}}$ (with respect to any positively curved metric $|\ \ |_L$ on L) is a bounded subharmonic function, which is moreover strictly subharmonic at all points where $f' \neq 0$ and $\sigma_j(f) \neq 0$. This is a contradiction unless f is constant or $\sigma_j(f) \equiv 0$. □

The above results justify the following definition and problems.

8.12. Definition. *We say that X, resp. (X, V), has non degenerate negative k-jet curvature if there exists a k-jet metric h_k on $\mathcal{O}_{P_kV}(-1)$ with negative jet curvature such that $\Sigma_{h_k} \subset P_kV^{\rm sing}$.*

8.13. Conjecture. *Let (X, V) be a compact directed manifold. Then (X, V) is hyperbolic if and only if (X, V) has nondegenerate negative k-jet curvature for k large enough.*

This is probably a hard problem. In fact, we will see in the next section that the smallest admissible integer k must depend on the geometry of X and need not be uniformly bounded as soon as $\dim X \geqslant 2$ (even in the absolute case $V = T_X$). On the other hand, if (X, V) is hyperbolic, we get for each integer $k \geqslant 1$ a generalized Kobayashi-Royden metric $\mathbf{k}_{(P_{k-1}V, V_{k-1})}$ on V_{k-1} (see Definition 2.1), which can be also viewed as a k-jet metric h_k on $\mathcal{O}_{P_kV}(-1)$; we will call it the *Grauert k-jet metric* of (X, V), although it formally differs from the jet metric considered in [Gra89] (see also [DGr91]). By looking at the projection $\pi_k : (P_kV, V_k) \to (P_{k-1}V, V_{k-1})$, we see that the sequence h_k is monotonic, namely $\pi_k^* h_k \leqslant h_{k+1}$ for every k. If (X, V) is hyperbolic, then h_1 is nondegenerate and therefore by monotonicity $\Sigma_{h_k} \subset P_k V^{\rm sing}$ for $k \geqslant 1$. Conversely, if the Grauert metric satisfies $\Sigma_{h_k} \subset P_k V^{\rm sing}$, it is easy to see that (X, V) is hyperbolic. The following problem is thus especially meaningful.

8.14. Problem. *Estimate the k-jet curvature $\Theta_{h_k^{-1}}(\mathcal{O}_{P_kV}(1))$ of the Grauert metric h_k on (P_kV, V_k) as k tends to $+\infty$.*

§8.D. Vanishing theorem for non invariant k-jet differentials

We prove here a more general vanishing theorem which strengthens Theorem 8.8 and Corollary 8.9. In this form, the result is due to Siu and Yeung ([SiYe96a, SiYe97], [Siu97], cf. also [Dem97] for a more detailed account (in French)).

8.15. Fundamental vanishing theorem. *Let (X, V) be a directed projective variety and $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$ an entire curve tangent to V. Then for every global section $P \in H^0(X, E^{\rm GG}_{k,m} V^* \otimes \mathcal{O}(-A))$ where A is an ample divisor of X, one has $P(f\,;\, f', f'', \ldots, f^{(k)}) = 0$.*

Proof. After raising P to a power P^s and replacing $\mathcal{O}(-A)$ with $\mathcal{O}(-sA)$, one can always assume that A is very ample divisor. We interpret $E^{\rm GG}_{k,m} V^* \otimes \mathcal{O}(-A)$ as the bundle of complex valued differential operators whose coefficients $a_\alpha(z)$ vanish along A.

Let us first give the proof of (8.15) in the special case where f is a brody curve, i.e. $\sup_{t\in\mathbb{C}} \|f'(t)\|_\omega < +\infty$ with respect to a given Hermitian metric ω on X. Fix a finite open covering of X by coordinate balls $B(p_j, R_j)$ such that the balls $B_j(p_j, R_j/4)$ still cover X. As f' is bounded, there exists $\delta > 0$ such that for $f(t_0) \in B(p_j, R_j/4)$ we have $f(t) \in B(p_j, R_j/2)$ whenever $|t - t_0| < \delta$, uniformly for every $t_0 \in \mathbb{C}$. The Cauchy inequalities applied to the components of f in each of the balls imply that the derivatives $f^{(j)}(t)$ are bounded on $\mathbb{C}$, and therefore, since the coefficients $a_\alpha(z)$ of P are also uniformly bounded on each of the balls $B(p_j, R_j/2)$ we conclude that $g := P(f\,;\, f', f'', \ldots, f^{(k)})$ is a bounded holomorphic function on $\mathbb{C}$. After moving A in the linear system $|A|$, we may further assume that $\operatorname{Supp} A$ intersects $f(\mathbb{C})$. Then g vanishes somewhere, hence $g \equiv 0$ by Liouville's theorem, as expected.

The proof for the general case where f' is unbounded is slightly more subtle (cf. [Siu87]), and makes use of Nevanlinna theory, especially the logarithmic derivative lemma. Assume that $g = P(f', \ldots, f^{(k)})$ does not vanish identically. Fix a hermitian metric h on $\mathcal{O}(-A)$ such that $\omega := \Theta_{\mathcal{O}(A),h^{-1}} > 0$ is a Kähler metric. The starting point is the inequality

$$\frac{i}{2\pi} \partial\overline{\partial} \log \|g\|_h^2 = \frac{i}{2\pi} \partial\overline{\partial} \log \|P(f', \ldots, f^{(k)})\|_h^2 \geqslant f^*\omega.$$

In fact, as we are on $\mathbb{C}$, the Lelong-Poincaré equation shows that the left hand side is equal to the right hand side plus a certain linear combination of Dirac measures at points where $P(f', \ldots, f^{(k)})$ vanishes. Let us consider the growth and proximity functions

$$T_{f,\omega}(r) := \int_{r_0}^{r} \frac{d\rho}{\rho} \int_{D(0,\rho)} f^*\omega, \tag{8.16}$$

$$m_g(r) := \frac{1}{2\pi}\int_0^{2\pi} \log_+ \|g(r\,e^{i\theta})\|_h^2\,d\theta. \tag{8.17}$$

We get

$$T_{f,\omega}(r) \leqslant \int_{r_0}^r \frac{d\rho}{\rho}\int_{D(0,\rho)} \frac{i}{2\pi}\partial\overline{\partial}\log\|g\|_h^2 = m_g(r) + \mathrm{Const} \tag{8.18}$$

thanks to the Jensen formula. Now, consider a (finite) family of rational functions (u_j) on X such that one can extract local coordinates from local determinations of the logarithms $\log u_j$ at any point of X (if X is embedded in some projective space, it is sufficient to take rational functions of the form $u_j(z) = \ell_j(z)/\ell'_j(z)$ where ℓ_j, ℓ'_j are linear forms; we also view the u_j's as rational maps $u_j : X \dashrightarrow \mathbb{P}^1$). One can then express locally $P(f', \ldots, f^{(k)})$ as a polynomial Q in the logarithmic derivatives $D^p(\log u_j \circ f)$, with holomorphic coefficients in f, i.e.,

$$g = P(f', \ldots, f^{(k)}) = Q\big(f, D^p(\log u_j \circ f)_{p,j}\big), \qquad Q(z, v_{p,j}) = \sum a_\alpha(z) v^\alpha.$$

By compactness of X, we infer

$$m_g(r) = \frac{1}{2\pi}\int_0^{2\pi} \log_+ \|g(r\,e^{i\theta})\|_h^2\,d\theta \leqslant C_1 \sum_{j,\,1\leqslant p\leqslant k} m_{D^p(\log u_j\circ f)}(r) + C_2 \tag{8.19}$$

with suitable constants C_1, C_2. The logarithmic derivative lemma states that for every meromorphic function $h : \mathbb{C} \to \mathbb{P}^1$ we have

$$m_{D^p \log h}(r) \leqslant \log r + (1+\varepsilon)\log_+ T_{h,\omega_{\rm FS}}(r) + O(1) \quad /\!/,$$

where the notation // indicates as usual that the inequality holds true outside a set of finite Lebesgue measure in $[0, +\infty[$. We apply this to $h = u_j \circ f$ and use the standard fact that $T_{u_j\circ f,\omega_{\rm FS}}(r) \leqslant C_j T_{f,\omega}(r)$. We find in this way

$$m_{D^p(\log u_j\circ f)}(r) \leqslant C_3\big(\log r + \log_+ T_{f,\omega}(r)\big) \quad /\!/. \tag{8.20}$$

By putting (8.18–8.20) together, one obtains

$$T_{f,\omega}(r) \leqslant C\big(\log r + \log_+ T_{f,\omega}(r)\big) \quad /\!/.$$

We infer from here that $T_{f,\omega}(r) = O(\log r)$, hence $f(\mathbb{C})$ has a finite total area. By well known facts of Nevanlinna theory, we conclude that $C = \overline{f(\mathbb{C})}$ is a rational curve and that f extends as a rational map $\mathbb{P}^1 \to X$. In particular the derivative f' is bounded, but then the first case of the proof can be applied to conclude that $g = P(f', \ldots, f^{(k)}) \equiv 0$. □

§8.E. Bloch theorem

The core of the result can be expressed as a characterization of the Zariski closure of an entire curve drawn on a complex torus. The proof is a simple consequence of the Ahlfors-Schwarz lemma (more specifically Theorem 8.8), combined with a jet bundle argument. We refer to [Och], [GrG80] (also [Dem95]) for a detailed proof.

8.21. Theorem. *Let Z be a complex torus and let $f : \mathbb{C} \to Z$ be a holomorphic map. Then the (analytic) Zariski closure $\overline{f(\mathbb{C})}^{\rm Zar}$ is a translate of a subtorus, i.e. of the form $a + Z'$, $a \in Z$, where $Z' \subset Z$ is a subtorus.*

The converse is of course also true: for any subtorus $Z' \subset Z$, we can choose a dense line $L \subset Z'$, and the corresponding map $f : \mathbb{C} \simeq a + L \hookrightarrow Z$ has Zariski closure $\overline{f(\mathbb{C})}^{\rm Zar} = a + Z'$.

§9. Morse inequalities and the Green-Griffiths-Lang conjecture

The goal of this section is to study the existence and properties of entire curves $f : \mathbb{C} \to X$ drawn in a complex irreducible n-dimensional variety X, and more specifically to show that they must satisfy certain global algebraic or differential equations as soon as X is projective of general type. By means of holomorphic Morse inequalities and a probabilistic analysis of the cohomology of jet spaces, we are able to prove a significant step of a generalized version of the Green-Griffiths-Lang conjecture on the algebraic degeneracy of entire curves. The use of holomorphic Morse inequalities was first suggested in [Dem07a], and then carried out in an algebraic context by S. Diverio in his PhD work ([Div08, Div09]). The general more analytic and more powerful results presented here first appeared in [Dem11]. We refer to [Dem12] for a more detailed exposition.

§9.A. Introduction

Our main target is the following deep conjecture concerning the algebraic degeneracy of entire curves, which generalizes the similar absolute statements given in § 4 (see also [GrGr79], [Lang86, Lang87]).

9.1. Generalized Green-Griffiths-Lang conjecture. *Let (X, V) be a projective directed manifold such that the canonical sheaf K_V is big (in the absolute case $V = T_X$, this means that X is a variety of general type, and in the relative case we will say that (X, V) is of general type). Then there should exist an algebraic subvariety $Y \subsetneq X$ such that every non constant entire curve $f : \mathbb{C} \to X$ tangent to V is contained in Y.*

The precise meaning of K_V and of its bigness will be explained below – our definition *does not coincide* with other frequently used definitions and is in our view better suited to the study of entire curves of (X, V). One says that (X, V) is Brody-hyperbolic when there are no entire curves tangent to V. According to (generalized versions of) conjectures of Kobayashi [Kob70, Kob76] the hyperbolicity of (X, V) should imply that K_V is big, and even possibly ample, in a suitable sense. It would then follow from conjecture (9.1) that (X, V) is hyperbolic if and only if for every irreducible variety $Y \subset X$, the linear subspace

$$V_{\widetilde{Y}} = \overline{T_{\widetilde{Y} \smallsetminus E} \cap \mu_*^{-1} V} \subset T_{\widetilde{Y}} \tag{9.2}$$

has a big canonical sheaf whenever $\mu : \widetilde{Y} \to Y$ is a desingularization and E is the exceptional locus.

By definition, proving the algebraic degeneracy means finding a non zero polynomial P on X such that all entire curves $f : \mathbb{C} \to X$ satisfy $P(f) = 0$. As already explained in § 8, all known methods of proof are based on establishing first the existence of certain algebraic differential equations $P(f\,;\, f', f'', \ldots, f^{(k)}) = 0$ of some order k, and then trying to find enough such equations so that they cut out a proper algebraic locus $Y \subsetneq X$. We use for this global sections of $H^0(X, E^{\rm GG}_{k,m} V^* \otimes \mathcal{O}(-A))$ where A is ample, and apply the fundamental vanishing theorem 8.15. It is expected that the global sections of $H^0(X, E^{\rm GG}_{k,m} V^* \otimes \mathcal{O}(-A))$ are precisely those which ultimately define the algebraic locus $Y \subsetneq X$ where the curve f should lie. The problem is then reduced to (i) showing that there are many non zero sections of $H^0(X, E^{\rm GG}_{k,m} V^* \otimes \mathcal{O}(-A))$ and (ii) understanding what is their joint base locus. The first part of this program is the main result of this section.

9.3. Theorem. *Let (X, V) be a directed projective variety such that K_V is big and let A be an ample divisor. Then for $k \gg 1$ and $\delta \in \mathbb{Q}_+$ small enough, $\delta \leqslant c(\log k)/k$, the number of sections $h^0(X, E^{\rm GG}_{k,m} V^* \otimes \mathcal{O}(-m\delta A))$ has maximal growth, i.e. is larger that $c_k m^{n+kr-1}$*

for some $m \geqslant m_k$*, where* $c, c_k > 0$*,* $n = \dim X$ *and* $r = \operatorname{rank} V$*. In particular, entire curves* $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$ *satisfy (many) algebraic differential equations.*

The statement is very elementary to check when $r = \operatorname{rank} V = 1$, and therefore when $n = \dim X = 1$. In higher dimensions $n \geqslant 2$, only very partial results were known at this point, concerning merely the absolute case $V = T_X$. In dimension 2, Theorem 9.3 is a consequence of the Riemann-Roch calculation of Green-Griffiths [GrGr79], combined with a vanishing theorem due to Bogomolov [Bog79] – the latter actually only applies to the top cohomology group H^n, and things become much more delicate when extimates of intermediate cohomology groups are needed. In higher dimensions, Diverio [Div08, Div09] proved the existence of sections of $H^0(X, E^{\rm GG}_{k,m}V^* \otimes \mathcal{O}(-1))$ whenever X is a hypersurface of $\mathbb{P}^{n+1}_{\mathbb{C}}$ of high degree $d \geqslant d_n$, assuming $k \geqslant n$ and $m \geqslant m_n$. More recently, Merker [Mer10] was able to treat the case of arbitrary hypersurfaces of general type, i.e. $d \geqslant n+3$, assuming this time k to be very large. The latter result is obtained through explicit algebraic calculations of the spaces of sections, and the proof is computationally very intensive. Bérczi [Ber10] also obtained related results with a different approach based on residue formulas, assuming $d \geqslant 2^{7n\log n}$.

All these approaches are algebraic in nature. Here, however, our techniques are based on more elaborate curvature estimates in the spirit of Cowen-Griffiths [CoGr76]. They require holomorphic Morse inequalities (see 9.10 below) – and we do not know how to translate our method in an algebraic setting. Notice that holomorphic Morse inequalities are essentially insensitive to singularities, as we can pass to non singular models and blow-up X as much as we want: if $\mu : \widetilde{X} \to X$ is a modification then $\mu_*\mathcal{O}_{\widetilde{X}} = \mathcal{O}_X$ and $R^q\mu_*\mathcal{O}_{\widetilde{X}}$ is supported on a codimension 1 analytic subset (even codimension 2 if X is smooth). It follows from the Leray spectral sequence that the cohomology estimates for L on X or for $\widetilde{L} = \mu^*L$ on $\widetilde{X}$ differ by negligible terms, i.e.

$$h^q(\widetilde{X}, \widetilde{L}^{\otimes m}) - h^q(X, L^{\otimes m}) = O(m^{n-1}). \tag{9.4}$$

Finally, singular holomorphic Morse inequalities (in the form obatined by L. Bonavero [Bon93]) allow us to work with singular Hermitian metrics h; this is the reason why we will only require to have big line bundles rather than ample line bundles. In the case of linear subspaces $V \subset T_X$, we introduce singular Hermitian metrics as follows.

9.5. Definition. *A singular Hermitian metric on a linear subspace* $V \subset T_X$ *is a metric* h *on the fibers of* V *such that the function* $\log h : \xi \mapsto \log |\xi|^2_h$ *is locally integrable on the total space of* V*.*

Such a metric can also be viewed as a singular Hermitian metric on the tautological line bundle $\mathcal{O}_{P(V)}(-1)$ on the projectivized bundle $P(V) = V \smallsetminus \{0\}/\mathbb{C}^*$, and therefore its dual metric h^* defines a curvature current $\Theta_{\mathcal{O}_{P(V)}(1),h^*}$ of type $(1,1)$ on $P(V) \subset P(T_X)$, such that

$$p^*\Theta_{\mathcal{O}_{P(V)}(1),h^*} = \frac{i}{2\pi}\partial\overline{\partial}\log h, \qquad \text{where } p : V \smallsetminus \{0\} \to P(V).$$

If $\log h$ is quasi-plurisubharmonic (or quasi-psh, which means psh modulo addition of a smooth function) on V, then $\log h$ is indeed locally integrable, and we have moreover

$$\Theta_{\mathcal{O}_{P(V)}(1),h^*} \geqslant -C\omega \tag{9.6}$$

for some smooth positive $(1,1)$-form on $P(V)$ and some constant $C > 0$; conversely, if (9.6) holds, then $\log h$ is quasi-psh.

9.7. Definition. *We will say that a singular Hermitian metric h on V is* admissible *if h can be written as $h = e^{\varphi} h_{0|V}$ where h_0 is a smooth positive definite Hermitian on T_X and φ is a quasi-psh weight with analytic singularities on X, as in Definition* 9.5. *Then h can be seen as a singular Hermitian metric on $\mathcal{O}_{P(V)}(1)$, with the property that it induces a smooth positive definite metric on a Zariski open set $X' \subset X \smallsetminus \mathrm{Sing}(V)$; we will denote by* $\mathrm{Sing}(h) \supset \mathrm{Sing}(V)$ *the complement of the largest such Zariski open set X'.*

If h is an admissible metric, we define $\mathcal{O}_h(V^*)$ to be the sheaf of germs of holomorphic sections sections of $V^*_{|X\smallsetminus\mathrm{Sing}(h)}$ which are h^*-bounded near $\mathrm{Sing}(h)$; by the assumption on the analytic singularities, this is a coherent sheaf (as the direct image of some coherent sheaf on $P(V)$), and actually, since $h^* = e^{-\varphi}h_0^*$, it is a subsheaf of the sheaf $\mathcal{O}(V^*) := \mathcal{O}_{h_0}(V^*)$ associated with a smooth positive definite metric h_0 on T_X. If r is the generic rank of V and m a positive integer, we define similarly $K^m_{V,h}$ to be sheaf of germs of holomorphic sections of $(\det V^*_{|X'})^{\otimes m} = (\Lambda^r V^*_{|X'})^{\otimes m}$ which are $\det h^*$-bounded, and $K^m_V := K^m_{V,h_0}$.

If V is defined by $\alpha : X \dashrightarrow G_r(T_X)$, there always exists a modification $\mu : \widetilde{X} \to X$ such that the composition $\alpha\circ\mu : \widetilde{X} \to G_r(\mu^*T_X)$ becomes holomorphic, and then $\mu^*V_{|\mu^{-1}(X\smallsetminus\mathrm{Sing}(V))}$ extends as a locally trivial subbundle of μ^*T_X which we will simply denote by μ^*V. If h is an admissible metric on V, then μ^*V can be equipped with the metric $\mu^*h = e^{\varphi\circ\mu}\mu^*h_0$ where μ^*h_0 is smooth and positive definite. We may assume that $\varphi\circ\mu$ has divisorial singularities (otherwise just perform further blow-ups of $\widetilde{X}$ to achieve this). We then see that there is an integer m_0 such that for all multiples $m = pm_0$ the pull-back $\mu^*K^m_{V,h}$ is an invertible sheaf on $\widetilde{X}$, and $\det h^*$ induces a smooth non singular metric on it (when $h = h_0$, we can even take $m_0 = 1$). By definition we always have $K^m_{V,h} = \mu_*(\mu^*K^m_{V,h})$ for any $m \geqslant 0$. In the sequel, however, we think of $K_{V,h}$ not really as a coherent sheaf, but rather as the "virtual" $\mathbb{Q}$-line bundle $\mu_*(\mu^*K^{m_0}_{V,h})^{1/m_0}$, and we say that $K_{V,h}$ is big if $h^0(X, K^m_{V,h}) \geqslant cm^n$ for $m \geqslant m_1$, with $c > 0$, i.e. if the invertible sheaf $\mu^*K^{m_0}_{V,h}$ is big in the usual sense.

At this point, it is important to observe that "our" canonical sheaf K_V differs from the sheaf $\mathcal{K}_V := i_*\mathcal{O}(K_V)$ associated with the injection $i : X \smallsetminus \mathrm{Sing}(V) \hookrightarrow X$, which is usually referred to as being the "canonical sheaf", at least when V is the space of tangents to a foliation. In fact, $\mathcal{K}_V$ is always an invertible sheaf and there is an obvious inclusion $K_V \subset \mathcal{K}_V$. More precisely, the image of $\mathcal{O}(\Lambda^r T^*_X) \to \mathcal{K}_V$ is equal to $\mathcal{K}_V \otimes_{\mathcal{O}_X} \mathcal{J}$ for a certain coherent ideal $\mathcal{J} \subset \mathcal{O}_X$, and the condition to have h_0-bounded sections on $X \smallsetminus \mathrm{Sing}(V)$ precisely means that our sections are bounded by $\mathrm{Const}\sum|g_j|$ in terms of the generators (g_j) of $\mathcal{K}_V \otimes_{\mathcal{O}_X} \mathcal{J}$, i.e. $K_V = \mathcal{K}_V \otimes_{\mathcal{O}_X} \overline{\mathcal{J}}$ where $\overline{\mathcal{J}}$ is the integral closure of $\mathcal{J}$. More generally,

$$K^m_{V,h} = \mathcal{K}^m_V \otimes_{\mathcal{O}_X} \overline{\mathcal{J}}^{m/m_0}_{h,m_0} \tag{9.8}$$

where $\overline{\mathcal{J}}^{m/m_0}_{h,m_0} \subset \mathcal{O}_X$ is the (m/m_0)-integral closure of a certain ideal sheaf $\mathcal{J}_{h,m_0} \subset \mathcal{O}_X$, which can itself be assumed to be integrally closed; in our previous discussion, μ is chosen so that $\mu^*\mathcal{J}_{h,m_0}$ is invertible on $\widetilde{X}$.

The discrepancy already occurs e.g. with the rank 1 linear space $V \subset T_{\mathbb{P}^n_{\mathbb{C}}}$ consisting at each point $z \neq 0$ of the tangent to the line $(0z)$ (so that necessarily $V_0 = T_{\mathbb{P}^n_{\mathbb{C}},0}$). As a sheaf (and not as a linear space), $i_*\mathcal{O}(V)$ is the invertible sheaf generated by the vector field $\xi = \sum z_j\partial/\partial z_j$ on the affine open set $\mathbb{C}^n \subset \mathbb{P}^n_{\mathbb{C}}$, and therefore $\mathcal{K}_V := i_*\mathcal{O}(V^*)$ is generated over $\mathbb{C}^n$ by the unique 1-form u such that $u(\xi) = 1$. Since ξ vanishes at 0, the generator u is *unbounded* with respect to a smooth metric h_0 on $T_{\mathbb{P}^n_{\mathbb{C}}}$, and it is easily seen that K_V is the non invertible sheaf $K_V = \mathcal{K}_V \otimes \mathfrak{m}_{\mathbb{P}^n_{\mathbb{C}},0}$. We can make it invertible by considering the blow-up $\mu : \widetilde{X} \to X$ of $X = \mathbb{P}^n_{\mathbb{C}}$ at 0, so that μ^*K_V is isomorphic to $\mu^*\mathcal{K}_V \otimes \mathcal{O}_{\widetilde{X}}(-E)$

where E is the exceptional divisor. The integral curves C of V are of course lines through 0, and when a standard parametrization is used, their derivatives do not vanish at 0, while the sections of $i_*\mathcal{O}(V)$ do – another sign that $i_*\mathcal{O}(V)$ and $i_*\mathcal{O}(V^*)$ are the *wrong objects* to consider. Another standard example is obtained by taking a generic pencil of elliptic curves $\lambda P(z) + \mu Q(z) = 0$ of degree 3 in $\mathbb{P}^2_{\mathbb{C}}$, and the linear space V consisting of the tangents to the fibers of the rational map $\mathbb{P}^2_{\mathbb{C}} \dashrightarrow \mathbb{P}^1_{\mathbb{C}}$ defined by $z \mapsto Q(z)/P(z)$. Then V is given by

$$0 \longrightarrow i_*\mathcal{O}(V) \longrightarrow \mathcal{O}(T_{\mathbb{P}^2_{\mathbb{C}}}) \xrightarrow{PdQ-QdP} \mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(6) \otimes \mathcal{J}_S \longrightarrow 0$$

where $S = \mathrm{Sing}(V)$ consists of the 9 points $\{P(z) = 0\} \cap \{Q(z) = 0\}$, and $\mathcal{J}_S$ is the corresponding ideal sheaf of S. Since $\det \mathcal{O}(T_{\mathbb{P}^2}) = \mathcal{O}(3)$, we see that $\mathcal{K}_V = \mathcal{O}(3)$ is ample, which seems to contradict 9.1 since all leaves are elliptic curves. There is however no such contradiction, because $K_V = \mathcal{K}_V \otimes \mathcal{J}_S$ is not big in our sense (it has degree 0 on all members of the elliptic pencil). A similar example is obtained with a generic pencil of conics, in which case $\mathcal{K}_V = \mathcal{O}(1)$ and $\mathrm{card}\, S = 4$.

For a given admissible Hermitian structure (V, h), we define similarly the sheaf $E^{\mathrm{GG}}_{k,m}V^*_h$ to be the sheaf of polynomials defined over $X \smallsetminus \mathrm{Sing}(h)$ which are “h-bounded”. This means that when they are viewed as polynomials $P(z\,;\,\xi_1, \ldots, \xi_k)$ in terms of $\xi_j = (\nabla^{1,0}_{h_0})^j f(0)$ where $\nabla^{1,0}_{h_0}$ is the $(1,0)$-component of the induced Chern connection on (V, h_0), there is a uniform bound

$$\big|P(z\,;\,\xi_1, \ldots, \xi_k)\big| \leqslant C\Big(\sum \|\xi_j\|_h^{1/j}\Big)^m \tag{9.9}$$

near points of $X \smallsetminus X'$ (see section 2 for more details on this). Again, by a direct image argument, one sees that $E^{\mathrm{GG}}_{k,m}V^*_h$ is always a coherent sheaf. The sheaf $E^{\mathrm{GG}}_{k,m}V^*$ is defined to be $E^{\mathrm{GG}}_{k,m}V^*_h$ when $h = h_0$ (it is actually independent of the choice of h_0, as follows from arguments similar to those given in section 2). Notice that this is exactly what is needed to extend the proof of the vanishing theorem 8.15 to the case of a singular linear space V ; the value distribution theory argument can only work when the functions $P(f\,;\, f', \ldots, f^{(k)})(t)$ do not exhibit poles, and this is guaranteed here by the boundedness assumption.

Our strategy can be described as follows. We consider the Green-Griffiths bundle of k-jets $X^{\mathrm{GG}}_k = J^kV \smallsetminus \{0\}/\mathbb{C}^*$, which by (9.3) consists of a fibration in *weighted projective spaces*, and its associated tautological sheaf

$$L = \mathcal{O}_{X^{\mathrm{GG}}_k}(1),$$

viewed rather as a virtual $\mathbb{Q}$-line bundle $\mathcal{O}_{X^{\mathrm{GG}}_k}(m_0)^{1/m_0}$ with $m_0 = \mathrm{lcm}(1, 2, \ldots, k)$. Then, if $\pi_k : X^{\mathrm{GG}}_k \to X$ is the natural projection, we have

$$E^{\mathrm{GG}}_{k,m} = (\pi_k)_*\mathcal{O}_{X^{\mathrm{GG}}_k}(m) \quad \text{and} \quad R^q(\pi_k)_*\mathcal{O}_{X^{\mathrm{GG}}_k}(m) = 0 \text{ for } q \geqslant 1.$$

Hence, by the Leray spectral sequence we get for every invertible sheaf F on X the isomorphism

$$H^q(X, E^{\mathrm{GG}}_{k,m}V^* \otimes F) \simeq H^q(X^{\mathrm{GG}}_k, \mathcal{O}_{X^{\mathrm{GG}}_k}(m) \otimes \pi_k^*F).$$

The latter group can be evaluated thanks to holomorphic Morse inequalities. Let us recall the main statement.

9.10. Holomorphic Morse inequalities ([Dem85]). *Let X be a compact complex manifolds, $E \to X$ a holomorphic vector bundle of rank r, and (L, h) a hermitian line* *bundle. The dimensions $h^q(X, E\otimes L^k)$ of cohomology groups of the tensor powers $E\otimes L^k$ satisfy the following asymptotic estimates as $k\to+\infty$:*

(WM) *Weak Morse inequalities*:

$$h^q(X, E\otimes L^k) \leqslant r\frac{k^n}{n!}\int_{X(L,h,q)} (-1)^q\Theta^n_{L,h} + o(k^n)\ .$$

(SM) *Strong Morse inequalities*:

$$\sum_{0\leqslant j\leqslant q} (-1)^{q-j}h^j(X, E\otimes L^k) \leqslant r\frac{k^n}{n!}\int_{X(L,h,\leqslant q)} (-1)^q\Theta^n_{L,h} + o(k^n)\ .$$

(RR) *Asymptotic Riemann-Roch formula*:

$$\chi(X, E\otimes L^k) := \sum_{0\leqslant j\leqslant n} (-1)^j h^j(X, E\otimes L^k) = r\frac{k^n}{n!}\int_X \Theta^n_{L,h} + o(k^n)\ .$$

Moreover (cf. Bonavero's PhD thesis [Bon93]), if $h = e^{-\varphi}$ is a singular hermitian metric with analytic singularities, the estimates are still true provided all cohomology groups are replaced by cohomology groups $H^q(X, E\otimes L^k\otimes \mathcal{I}(h^k))$ twisted with the multiplier ideal sheaves

$$\mathcal{I}(h^k) = \mathcal{I}(k\varphi) = \big\{f\in \mathcal{O}_{X,x},\ \ \exists V\ni x,\ \int_V |f(z)|^2e^{-k\varphi(z)}d\lambda(z) < +\infty\big\}.$$

The special case of 9.10 (SM) when $q=1$ yields a very useful criterion for the existence of sections of large multiples of L.

9.11. Corollary. *Under the above hypotheses, we have*

$$h^0(X, E\otimes L^k) \geqslant h^0(X, E\otimes L^k) - h^1(X, E\otimes L^k) \geqslant r\frac{k^n}{n!}\int_{X(L,h,\leqslant 1)} \Theta^n_{L,h} - o(k^n)\ .$$

Especially L is big as soon as $\int_{X(L,h,\leqslant 1)}\Theta^n_{L,h} > 0$ for some hermitian metric h on L.

Now, given a directed manifold (X, V), we can associate with any admissible metric h on V a metric (or rather a natural family) of metrics on $L = \mathcal{O}_{X_k^{\rm GG}}(1)$. The space $X_k^{\rm GG}$ always possesses quotient singularities if $k\geqslant 2$ (and even some more if V is singular), but we do not really care since Morse inequalities still work in this setting thanks to Bonavero's generalization. As we will see, it is then possible to get nice asymptotic formulas as $k\to+\infty$. They appear to be of a *probabilistic nature* if we take the components of the k-jet (i.e. the successive derivatives $\xi_j = f^{(j)}(0)$, $1\leqslant j\leqslant k$) as random variables. This probabilistic behaviour was somehow already visible in the Riemann-Roch calculation of [GrGr79]. In this way, assuming K_V big, we produce a lot of sections $\sigma_j = H^0(X_k^{\rm GG}, \mathcal{O}_{X_k^{\rm GG}}(m)\otimes\pi_k^*F)$, corresponding to certain divisors $Z_j\subset X_k^{\rm GG}$. The hard problem which is left in order to complete a proof of the generalized Green-Griffiths-Lang conjecture is to compute the base locus $Z = \bigcap Z_j$ and to show that $Y = \pi_k(Z)\subset X$ must be a proper algebraic variety.

§9.B. Hermitian geometry of weighted projective spaces

The goal of this section is to introduce natural Kähler metrics on weighted projective spaces, and to evaluate the corresponding volume forms. Here we put $d^c = \frac{i}{4\pi}(\overline{\partial}-\partial)$ so that $dd^c = \frac{i}{2\pi}\partial\overline{\partial}$. The normalization of the d^c operator is chosen such that we have precisely $(dd^c \log|z|^2)^n = \delta_0$ for the Monge-Ampère operator in $\mathbb{C}^n$. Given a k-tuple of "weights" $a = (a_1, \ldots, a_k)$, i.e. of integers $a_s > 0$ with $\gcd(a_1, \ldots, a_k) = 1$, we introduce the weighted projective space $P(a_1, \ldots, a_k)$ to be the quotient of $\mathbb{C}^k \smallsetminus \{0\}$ by the corresponding weighted $\mathbb{C}^*$ action:

$$P(a_1, \ldots, a_k) = \mathbb{C}^k \smallsetminus \{0\}/\mathbb{C}^*, \qquad \lambda \cdot z = (\lambda^{a_1} z_1, \ldots, \lambda^{a_k} z_k). \tag{9.12}$$

As is well known, this defines a toric $(k-1)$-dimensional algebraic variety with quotient singularities. On this variety, we introduce the possibly singular (but almost everywhere smooth and non degenerate) Kähler form $\omega_{a,p}$ defined by

$$\pi_a^*\omega_{a,p} = dd^c\varphi_{a,p}, \qquad \varphi_{a,p}(z) = \frac{1}{p}\log \sum_{1\leqslant s\leqslant k} |z_s|^{2p/a_s}, \tag{9.13}$$

where $\pi_a : \mathbb{C}^k \smallsetminus \{0\} \to P(a_1, \ldots, a_k)$ is the canonical projection and $p > 0$ is a positive constant. It is clear that $\varphi_{p,a}$ is real analytic on $\mathbb{C}^k \smallsetminus \{0\}$ if p is an integer and a common multiple of all weights a_s, and we will implicitly pick such a p later on to avoid any difficulty. Elementary calculations give the following well-known formula for the volume

$$\int_{P(a_1,\ldots,a_k)} \omega_{a,p}^{k-1} = \frac{1}{a_1 \ldots a_k} \tag{9.14}$$

(notice that this is independent of p, as it is obvious by Stokes theorem, since the cohomology class of $\omega_{a,p}$ does not depend on p).

Our later calculations will require a slightly more general setting. Instead of looking at $\mathbb{C}^k$, we consider the weighted $\mathbb{C}^*$ action defined by

$$\mathbb{C}^{|r|} = \mathbb{C}^{r_1} \times \ldots \times \mathbb{C}^{r_k}, \qquad \lambda \cdot z = (\lambda^{a_1} z_1, \ldots, \lambda^{a_k} z_k). \tag{9.15}$$

Here $z_s \in \mathbb{C}^{r_s}$ for some k-tuple $r = (r_1, \ldots, r_k)$ and $|r| = r_1 + \ldots + r_k$. This gives rise to a weighted projective space

$$\begin{aligned} &P(a_1^{[r_1]}, \ldots, a_k^{[r_k]}) = P(a_1, \ldots, a_1, \ldots, a_k, \ldots, a_k), \\ &\pi_{a,r} : \mathbb{C}^{r_1} \times \ldots \times \mathbb{C}^{r_k} \smallsetminus \{0\} \longrightarrow P(a_1^{[r_1]}, \ldots, a_k^{[r_k]}) \end{aligned} \tag{9.16}$$

obtained by repeating r_s times each weight a_s. On this space, we introduce the degenerate Kähler metric $\omega_{a,r,p}$ such that

$$\pi_{a,r}^*\omega_{a,r,p} = dd^c\varphi_{a,r,p}, \qquad \varphi_{a,r,p}(z) = \frac{1}{p}\log \sum_{1\leqslant s\leqslant k} |z_s|^{2p/a_s} \tag{9.17}$$

where $|z_s|$ stands now for the standard Hermitian norm $(\sum_{1\leqslant j\leqslant r_s} |z_{s,j}|^2)^{1/2}$ on $\mathbb{C}^{r_s}$. This metric is cohomologous to the corresponding "polydisc-like" metric $\omega_{a,p}$ already defined, and therefore Stokes theorem implies

$$\int_{P(a_1^{[r_1]},\ldots,a_k^{[r_k]})} \omega_{a,r,p}^{|r|-1} = \frac{1}{a_1^{r_1} \ldots a_k^{r_k}}. \tag{9.18}$$

Using standard results of integration theory (Fubini, change of variable formula...), one obtains:

9.19. Proposition. *Let $f(z)$ be a bounded function on $P(a_1^{[r_1]},\ldots,a_k^{[r_k]})$ which is continuous outside of the hyperplane sections $z_s=0$. We also view f as a $\mathbb{C}^*$-invariant continuous function on $\prod(\mathbb{C}^{r_s}\smallsetminus\{0\})$. Then*

$$\begin{aligned}&\int_{P(a_1^{[r_1]},\ldots,a_k^{[r_k]})} f(z)\,\omega_{a,r,p}^{|r|-1}\\ &=\frac{(|r|-1)!}{\prod_s a_s^{r_s}}\int_{(x,u)\in\Delta_{k-1}\times\prod S^{2r_s-1}} f(x_1^{a_1/2p}u_1,\ldots,x_k^{a_k/2p}u_k)\prod_{1\leqslant s\leqslant k}\frac{x_s^{r_s-1}}{(r_s-1)!}\,dx\,d\mu(u)\end{aligned}$$

where Δ_{k-1} is the $(k-1)$-simplex $\{x_s\geqslant 0,\ \sum x_s=1\}$, $dx=dx_1\wedge\ldots\wedge dx_{k-1}$ its standard measure, and where $d\mu(u)=d\mu_1(u_1)\ldots d\mu_k(u_k)$ is the rotation invariant probability measure on the product $\prod_s S^{2r_s-1}$ of unit spheres in $\mathbb{C}^{r_1}\times\ldots\times\mathbb{C}^{r_k}$. As a consequence

$$\lim_{p\to+\infty}\int_{P(a_1^{[r_1]},\ldots,a_k^{[r_k]})} f(z)\,\omega_{a,r,p}^{|r|-1}=\frac{1}{\prod_s a_s^{r_s}}\int_{\prod S^{2r_s-1}} f(u)\,d\mu(u).$$

Also, by elementary integrations by parts and induction on $k, r_1,\ldots,r_k$, it can be checked that

$$\int_{x\in\Delta_{k-1}}\prod_{1\leqslant s\leqslant k} x_s^{r_s-1}dx_1\ldots dx_{k-1}=\frac{1}{(|r|-1)!}\prod_{1\leqslant s\leqslant k}(r_s-1)!\ . \tag{9.20}$$

This implies that $(|r|-1)!\prod_{1\leqslant s\leqslant k}\frac{x_s^{r_s-1}}{(r_s-1)!}\,dx$ is a probability measure on Δ_{k-1}.

§9.C. Probabilistic estimate of the curvature of k-jet bundles

Let (X,V) be a compact complex directed non singular variety. To avoid any technical difficulty at this point, we first assume that V is a holomorphic vector subbundle of T_X, equipped with a smooth Hermitian metric h.

According to the notation already specified in § 7, we denote by J^kV the bundle of k-jets of holomorphic curves $f:(\mathbb{C},0)\to X$ tangent to V at each point. Let us set $n=\dim_{\mathbb{C}}X$ and $r=\operatorname{rank}_{\mathbb{C}}V$. Then $J^kV\to X$ is an algebraic fiber bundle with typical fiber $\mathbb{C}^{rk}$, and we get a projectivized k-jet bundle

$$X_k^{\rm GG}:=(J^kV\smallsetminus\{0\})/\mathbb{C}^*,\qquad \pi_k: X_k^{\rm GG}\to X \tag{9.21}$$

which is a $P(1^{[r]},2^{[r]},\ldots,k^{[r]})$ weighted projective bundle over X, and we have the direct image formula $(\pi_k)_*\mathcal{O}_{X_k^{\rm GG}}(m)=\mathcal{O}(E_{k,m}^{\rm GG}V^*)$ (cf. Proposition 7.9). In the sequel, we do not make a direct use of coordinates, because they need not be related in any way to the Hermitian metric h of V. Instead, we choose a local holomorphic coordinate frame $(e_\alpha(z))_{1\leqslant\alpha\leqslant r}$ of V on a neighborhood U of x_0, such that

$$\langle e_\alpha(z),e_\beta(z)\rangle=\delta_{\alpha\beta}+\sum_{1\leqslant i,j\leqslant n,\,1\leqslant\alpha,\beta\leqslant r}c_{ij\alpha\beta}z_i\overline{z}_j+O(|z|^3) \tag{9.22}$$

for suitable complex coefficients $(c_{ij\alpha\beta})$. It is a standard fact that such a normalized coordinate system always exists, and that the Chern curvature tensor $\frac{i}{2\pi}D^2_{V,h}$ of (V,h) at x_0 is then given by

$$\Theta_{V,h}(x_0) = -\frac{i}{2\pi}\sum_{i,j,\alpha,\beta} c_{ij\alpha\beta}\,dz_i\wedge d\overline{z}_j\otimes e^*_\alpha\otimes e_\beta. \tag{9.23}$$

Consider a local holomorphic connection ∇ on $V_{|U}$ (e.g. the one which turns (e_α) into a parallel frame), and take $\xi_k = \nabla^k f(0)\in V_x$ defined inductively by $\nabla^1 f = f'$ and $\nabla^s f = \nabla_{f'}(\nabla^{s-1}f)$. This gives a local identification

$$J_kV_{|U}\to V^{\oplus k}_{|U},\qquad f\mapsto(\xi_1,\ldots,\xi_k) = (\nabla f(0),\ldots,\nabla f^k(0)),$$

and the weighted $\mathbb{C}^*$ action on J_kV is expressed in this setting by

$$\lambda\cdot(\xi_1,\xi_2,\ldots,\xi_k) = (\lambda\xi_1,\lambda^2\xi_2,\ldots,\lambda^k\xi_k).$$

Now, we fix a finite open covering $(U_\alpha)_{\alpha\in I}$ of X by open coordinate charts such that $V_{|U_\alpha}$ is trivial, along with holomorphic connections ∇_α on $V_{|U_\alpha}$. Let θ_α be a partition of unity of X subordinate to the covering (U_α). Let us fix $p>0$ and small parameters $1=\varepsilon_1\gg\varepsilon_2\gg\ldots\gg\varepsilon_k>0$. Then we define a global weighted Finsler metric on J^kV by putting for any k-jet $f\in J^k_xV$

$$\Psi_{h,p,\varepsilon}(f) := \Big(\sum_{\alpha\in I}\theta_\alpha(x)\sum_{1\leqslant s\leqslant k}\varepsilon_s^{2p}\|\nabla^s_\alpha f(0)\|^{2p/s}_{h(x)}\Big)^{1/p} \tag{9.24}$$

where $\|\ \|_{h(x)}$ is the Hermitian metric h of V evaluated on the fiber V_x, $x=f(0)$. The function $\Psi_{h,p,\varepsilon}$ satisfies the fundamental homogeneity property

$$\Psi_{h,p,\varepsilon}(\lambda\cdot f) = \Psi_{h,p,\varepsilon}(f)\,|\lambda|^2 \tag{9.25}$$

with respect to the $\mathbb{C}^*$ action on J^kV, in other words, it induces a Hermitian metric on the dual L^* of the tautological $\mathbb{Q}$-line bundle $L_k = \mathcal{O}_{X_k^{\mathrm{GG}}}(1)$ over X_k^{GG}. The curvature of L_k is given by

$$\pi_k^*\Theta_{L_k,\Psi^*_{h,p,\varepsilon}} = dd^c\log\Psi_{h,p,\varepsilon} \tag{9.26}$$

Our next goal is to compute precisely the curvature and to apply holomorphic Morse inequalities to $L\to X_k^{\mathrm{GG}}$ with the above metric. It might look a priori like an untractable problem, since the definition of $\Psi_{h,p,\varepsilon}$ is a rather unnatural one. However, the "miracle" is that the asymptotic behavior of $\Psi_{h,p,\varepsilon}$ as $\varepsilon_s/\varepsilon_{s-1}\to 0$ is in some sense uniquely defined and very natural. It will lead to a computable asymptotic formula, which is moreover simple enough to produce useful results.

9.27. Lemma. *On each coordinate chart U equipped with a holomorphic connection ∇ of $V_{|U}$, let us define the components of a k-jet $f\in J^kV$ by $\xi_s=\nabla^s f(0)$, and consider the rescaling transformation*

$$\rho_{\nabla,\varepsilon}(\xi_1,\xi_2,\ldots,\xi_k) = (\varepsilon_1^1\xi_1,\varepsilon_2^2\xi_2,\ldots,\varepsilon_k^k\xi_k)\quad \textit{on } J^k_xV,\ x\in U$$

(*it commutes with the* $\mathbb{C}^*$*-action but is otherwise unrelated and not canonically defined over* X *as it depends on the choice of* ∇*). Then, if* p *is a multiple of* $\mathrm{lcm}(1,2,\ldots,k)$ *and* $\varepsilon_s/\varepsilon_{s-1}\to 0$ *for all* $s=2,\ldots,k$*, the rescaled function* $\Psi_{h,p,\varepsilon}\circ\rho^{-1}_{\nabla,\varepsilon}(\xi_1,\ldots,\xi_k)$ *converges towards*

$$\Big(\sum_{1\leqslant s\leqslant k}\|\xi_s\|_h^{2p/s}\Big)^{1/p}$$

on every compact subset of $J^kV_{|U}\smallsetminus\{0\}$*, uniformly in* C^∞ *topology.*

Proof. Let $U\subset X$ be an open set on which $V_{|U}$ is trivial and equipped with some holomorphic connection ∇. Let us pick another holomorphic connection $\widetilde\nabla=\nabla+\Gamma$ where $\Gamma\in H^0(U,\Omega^1_X\otimes\mathrm{Hom}(V,V)$. Then $\widetilde\nabla^2 f=\nabla^2 f+\Gamma(f)(f')\cdot f'$, and inductively we get

$$\widetilde\nabla^s f=\nabla^s f+P_s(f\,;\,\nabla^1 f,\ldots,\nabla^{s-1}f)$$

where $P(x\,;\,\xi_1,\ldots,\xi_{s-1})$ is a polynomial with holomorphic coefficients in $x\in U$ which is of weighted homogeneous degree s in $(\xi_1,\ldots,\xi_{s-1})$. In other words, the corresponding change in the parametrization of $J^kV_{|U}$ is given by a $\mathbb{C}^*$-homogeneous transformation

$$\widetilde\xi_s=\xi_s+P_s(x\,;\,\xi_1,\ldots,\xi_{s-1}).$$

Let us introduce the corresponding rescaled components

$$(\xi_{1,\varepsilon},\ldots,\xi_{k,\varepsilon})=(\varepsilon_1^1\xi_1,\ldots,\varepsilon_k^k\xi_k),\qquad(\widetilde\xi_{1,\varepsilon},\ldots,\widetilde\xi_{k,\varepsilon})=(\varepsilon_1^1\widetilde\xi_1,\ldots,\varepsilon_k^k\widetilde\xi_k).$$

Then

$$\begin{aligned}\widetilde\xi_{s,\varepsilon}&=\xi_{s,\varepsilon}+\varepsilon_s^s\,P_s(x\,;\,\varepsilon_1^{-1}\xi_{1,\varepsilon},\ldots,\varepsilon_{s-1}^{-(s-1)}\xi_{s-1,\varepsilon})\\&=\xi_{s,\varepsilon}+O(\varepsilon_s/\varepsilon_{s-1})^s\,O(\|\xi_{1,\varepsilon}\|+\ldots+\|\xi_{s-1,\varepsilon}\|^{1/(s-1)})^s\end{aligned}$$

and the error terms are thus polynomials of fixed degree with arbitrarily small coefficients as $\varepsilon_s/\varepsilon_{s-1}\to 0$. Now, the definition of $\Psi_{h,p,\varepsilon}$ consists of glueing the sums

$$\sum_{1\leqslant s\leqslant k}\varepsilon_s^{2p}\|\xi_k\|_h^{2p/s}=\sum_{1\leqslant s\leqslant k}\|\xi_{k,\varepsilon}\|_h^{2p/s}$$

corresponding to $\xi_k=\nabla^s_\alpha f(0)$ by means of the partition of unity $\sum\theta_\alpha(x)=1$. We see that by using the rescaled variables $\xi_{s,\varepsilon}$ the changes occurring when replacing a connection ∇_α by an alternative one ∇_β are arbitrary small in C^∞ topology, with error terms uniformly controlled in terms of the ratios $\varepsilon_s/\varepsilon_{s-1}$ on all compact subsets of $V^k\smallsetminus\{0\}$. This shows that in C^∞ topology, $\Psi_{h,p,\varepsilon}\circ\rho^{-1}_{\nabla,\varepsilon}(\xi_1,\ldots,\xi_k)$ converges uniformly towards $(\sum_{1\leqslant s\leqslant k}\|\xi_k\|_h^{2p/s})^{1/p}$, whatever the trivializing open set U and the holomorphic connection ∇ used to evaluate the components and perform the rescaling are. □

Now, we fix a point $x_0\in X$ and a local holomorphic frame $(e_\alpha(z))_{1\leqslant\alpha\leqslant r}$ satisfying (9.22) on a neighborhood U of x_0. We introduce the rescaled components $\xi_s=\varepsilon_s^s\nabla^s f(0)$ on $J^kV_{|U}$ and compute the curvature of

$$\Psi_{h,p,\varepsilon}\circ\rho^{-1}_{\nabla,\varepsilon}(z\,;\,\xi_1,\ldots,\xi_k)\simeq\Big(\sum_{1\leqslant s\leqslant k}\|\xi_s\|_h^{2p/s}\Big)^{1/p}$$

(by Lemma 9.27, the errors can be taken arbitrary small in C^∞ topology). We write $\xi_s = \sum_{1\leqslant\alpha\leqslant r} \xi_{s\alpha} e_\alpha$. By (9.22) we have

$$\|\xi_s\|_h^2 = \sum_\alpha |\xi_{s\alpha}|^2 + \sum_{i,j,\alpha,\beta} c_{ij\alpha\beta} z_i \overline{z}_j \xi_{s\alpha} \overline{\xi}_{s\beta} + O(|z|^3|\xi|^2).$$

The question is to evaluate the curvature of the weighted metric defined by

$$\begin{aligned}\Psi(z\,;\,\xi_1,\ldots,\xi_k) &= \Big(\sum_{1\leqslant s\leqslant k} \|\xi_s\|_h^{2p/s}\Big)^{1/p}\\ &= \Big(\sum_{1\leqslant s\leqslant k} \Big(\sum_\alpha |\xi_{s\alpha}|^2 + \sum_{i,j,\alpha,\beta} c_{ij\alpha\beta} z_i \overline{z}_j \xi_{s\alpha}\overline{\xi}_{s\beta}\Big)^{p/s}\Big)^{1/p} + O(|z|^3).\end{aligned}$$

We set $|\xi_s|^2 = \sum_\alpha |\xi_{s\alpha}|^2$. A straightforward calculation yields

$$\begin{aligned}&\log \Psi(z\,;\,\xi_1,\ldots,\xi_k) =\\ &= \frac{1}{p}\log \sum_{1\leqslant s\leqslant k} |\xi_s|^{2p/s} + \sum_{1\leqslant s\leqslant k} \frac{1}{s}\frac{|\xi_s|^{2p/s}}{\sum_t |\xi_t|^{2p/t}} \sum_{i,j,\alpha,\beta} c_{ij\alpha\beta} z_i\overline{z}_j \frac{\xi_{s\alpha}\overline{\xi}_{s\beta}}{|\xi_s|^2} + O(|z|^3).\end{aligned}$$

By (9.26), the curvature form of $L_k = \mathcal{O}_{X_k^{\mathrm{GG}}}(1)$ is given at the central point x_0 by the following formula.

9.28. Proposition. *With the above choice of coordinates and with respect to the rescaled components $\xi_s = \varepsilon_s^s \nabla^s f(0)$ at $x_0 \in X$, we have the approximate expression*

$$\Theta_{L_k,\Psi^*_{h,p,\varepsilon}}(x_0,[\xi]) \simeq \omega_{a,r,p}(\xi) + \frac{i}{2\pi} \sum_{1\leqslant s\leqslant k} \frac{1}{s} \frac{|\xi_s|^{2p/s}}{\sum_t |\xi_t|^{2p/t}} \sum_{i,j,\alpha,\beta} c_{ij\alpha\beta} \frac{\xi_{s\alpha}\overline{\xi}_{s\beta}}{|\xi_s|^2}\, dz_i\wedge d\overline{z}_j$$

where the error terms are $O(\max_{2\leqslant s\leqslant k}(\varepsilon_s/\varepsilon_{s-1})^s)$ uniformly on the compact variety X_k^{GG}. Here $\omega_{a,r,p}$ is the (degenerate) Kähler metric associated with the weight $a = (1^{[r]}, 2^{[r]}, \ldots, k^{[r]})$ of the canonical $\mathbb{C}^*$ action on J^kV.

Thanks to the uniform approximation, we can (and will) neglect the error terms in the calculations below. Since $\omega_{a,r,p}$ is positive definite on the fibers of $X_k^{\mathrm{GG}} \to X$ (at least outside of the axes $\xi_s = 0$), the index of the $(1,1)$ curvature form $\Theta_{L_k,\Psi^*_{h,p,\varepsilon}}(z,[\xi])$ is equal to the index of the $(1,1)$-form

$$\gamma_k(z,\xi) := \frac{i}{2\pi} \sum_{1\leqslant s\leqslant k} \frac{1}{s} \frac{|\xi_s|^{2p/s}}{\sum_t |\xi_t|^{2p/t}} \sum_{i,j,\alpha,\beta} c_{ij\alpha\beta}(z) \frac{\xi_{s\alpha}\overline{\xi}_{s\beta}}{|\xi_s|^2}\, dz_i\wedge d\overline{z}_j \tag{9.29}$$

depending only on the differentials $(dz_j)_{1\leqslant j\leqslant n}$ on X. The q-index integral of $(L_k, \Psi^*_{h,p,\varepsilon})$ on X_k^{GG} is therefore equal to

$$\begin{aligned}&\int_{X_k^{\mathrm{GG}}(L_k,q)} \Theta^{n+kr-1}_{L_k,\Psi^*_{h,p,\varepsilon}} =\\ &= \frac{(n+kr-1)!}{n!(kr-1)!} \int_{z\in X} \int_{\xi\in P(1^{[r]},\ldots,k^{[r]})} \omega^{kr-1}_{a,r,p}(\xi) \mathbb{1}_{\gamma_k,q}(z,\xi)\gamma_k(z,\xi)^n\end{aligned}$$

where $\mathbb{1}_{\gamma_k,q}(z,\xi)$ is the characteristic function of the open set of points where $\gamma_k(z,\xi)$ has signature $(n-q,q)$ in terms of the dz_j's. Notice that since $\gamma_k(z,\xi)^n$ is a determinant, the product $\mathbb{1}_{\gamma_k,q}(z,\xi)\gamma_k(z,\xi)^n$ gives rise to a continuous function on X_k^{GG}. Formula 9.20 with $r_1=\ldots=r_k=r$ and $a_s=s$ yields the slightly more explicit integral

$$\int_{X_k^{\mathrm{GG}}(L_k,q)}\Theta^{n+kr-1}_{L_k,\Psi^*_{h,p,\varepsilon}}=\frac{(n+kr-1)!}{n!(k!)^r}\times$$
$$\int_{z\in X}\int_{(x,u)\in\Delta_{k-1}\times(S^{2r-1})^k}\mathbb{1}_{g_k,q}(z,x,u)g_k(z,x,u)^n\,\frac{(x_1\ldots x_k)^{r-1}}{(r-1)!^k}\,dx\,d\mu(u),$$

where $g_k(z,x,u)=\gamma_k(z,x_1^{1/2p}u_1,\ldots,x_k^{k/2p}u_k)$ is given by

$$g_k(z,x,u)=\frac{i}{2\pi}\sum_{1\leqslant s\leqslant k}\frac{1}{s}x_s\sum_{i,j,\alpha,\beta}c_{ij\alpha\beta}(z)\,u_{s\alpha}\overline{u}_{s\beta}\,dz_i\wedge d\overline{z}_j \tag{9.30}$$

and $\mathbb{1}_{g_k,q}(z,x,u)$ is the characteristic function of its q-index set. Here

$$d\nu_{k,r}(x)=(kr-1)!\,\frac{(x_1\ldots x_k)^{r-1}}{(r-1)!^k}\,dx \tag{9.31}$$

is a probability measure on Δ_{k-1}, and we can rewrite

$$\begin{aligned}&\int_{X_k^{\mathrm{GG}}(L_k,q)}\Theta^{n+kr-1}_{L_k,\Psi^*_{h,p,\varepsilon}}=\frac{(n+kr-1)!}{n!(k!)^r(kr-1)!}\times\\ &\int_{z\in X}\int_{(x,u)\in\Delta_{k-1}\times(S^{2r-1})^k}\mathbb{1}_{g_k,q}(z,x,u)g_k(z,x,u)^n\,d\nu_{k,r}(x)\,d\mu(u).\end{aligned} \tag{9.32}$$

Now, formula (9.30) shows that $g_k(z,x,u)$ is a "Monte Carlo" evaluation of the curvature tensor, obtained by averaging the curvature at random points $u_s\in S^{2r-1}$ with certain positive weights x_s/s; we should then think of the k-jet f as some sort of random variable such that the derivatives $\nabla^k f(0)$ are uniformly distributed in all directions. Let us compute the expected value of $(x,u)\mapsto g_k(z,x,u)$ with respect to the probability measure $d\nu_{k,r}(x)\,d\mu(u)$. Since $\int_{S^{2r-1}}u_{s\alpha}\overline{u}_{s\beta}d\mu(u_s)=\frac{1}{r}\delta_{\alpha\beta}$ and $\int_{\Delta_{k-1}}x_s\,d\nu_{k,r}(x)=\frac{1}{k}$, we find

$$\mathbf{E}(g_k(z,\bullet,\bullet))=\frac{1}{kr}\sum_{1\leqslant s\leqslant k}\frac{1}{s}\cdot\frac{i}{2\pi}\sum_{i,j,\alpha}c_{ij\alpha\alpha}(z)\,dz_i\wedge d\overline{z}_j.$$

In other words, we get the normalized trace of the curvature, i.e.

$$\mathbf{E}(g_k(z,\bullet,\bullet))=\frac{1}{kr}\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)\Theta_{\det(V^*),\det h^*}, \tag{9.33}$$

where $\Theta_{\det(V^*),\det h^*}$ is the $(1,1)$-curvature form of $\det(V^*)$ with the metric induced by h. It is natural to guess that $g_k(z,x,u)$ behaves asymptotically as its expected value $\mathbf{E}(g_k(z,\bullet,\bullet))$ when k tends to infinity. If we replace brutally g_k by its expected value in (9.32), we get the integral

$$\frac{(n+kr-1)!}{n!(k!)^r(kr-1)!}\frac{1}{(kr)^n}\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)^n\int_X\mathbb{1}_{\eta,q}\eta^n,$$

where $\eta := \Theta_{\det(V^*),\det h^*}$ and $\mathbb{1}_{\eta,q}$ is the characteristic function of its q-index set in X. The leading constant is equivalent to $(\log k)^n/n!(k!)^r$ modulo a multiplicative factor $1+O(1/\log k)$. By working out a more precise analysis of the deviation, the following result has been proved in [Dem11] and [Dem12].

9.34. Probabilistic estimate. *Fix smooth Hermitian metrics h on V and $\omega = \frac{i}{2\pi}\sum \omega_{ij}dz_i\wedge d\overline{z}_j$ on X. Denote by $\Theta_{V,h} = -\frac{i}{2\pi}\sum c_{ij\alpha\beta}dz_i\wedge d\overline{z}_j\otimes e_\alpha^*\otimes e_\beta$ the curvature tensor of V with respect to an h-orthonormal frame (e_α), and put*

$$\eta(z) = \Theta_{\det(V^*),\det h^*} = \frac{i}{2\pi}\sum_{1\leqslant i,j\leqslant n}\eta_{ij}dz_i\wedge d\overline{z}_j, \qquad \eta_{ij} = \sum_{1\leqslant\alpha\leqslant r} c_{ij\alpha\alpha}.$$

Finally consider the k-jet line bundle $L_k = \mathcal{O}_{X_k^{\rm GG}}(1)\to X_k^{\rm GG}$ equipped with the induced metric $\Psi^*_{h,p,\varepsilon}$ (as defined above, with $1=\varepsilon_1\gg\varepsilon_2\gg\ldots\gg\varepsilon_k>0$). When k tends to infinity, the integral of the top power of the curvature of L_k on its q-index set $X_k^{\rm GG}(L_k,q)$ is given by

$$\int_{X_k^{\rm GG}(L_k,q)}\Theta^{n+kr-1}_{L_k,\Psi^*_{h,p,\varepsilon}} = \frac{(\log k)^n}{n!\,(k!)^r}\bigg(\int_X \mathbb{1}_{\eta,q}\eta^n + O((\log k)^{-1})\bigg)$$

for all $q=0,1,\ldots,n$, and the error term $O((\log k)^{-1})$ can be bounded explicitly in terms of Θ_V, η and ω. Moreover, the left hand side is identically zero for $q>n$.

The final statement follows from the observation that the curvature of L_k is positive along the fibers of $X_k^{\rm GG}\to X$, by the plurisubharmonicity of the weight (this is true even when the partition of unity terms are taken into account, since they depend only on the base); therefore the q-index sets are empty for $q>n$. It will be useful to extend the above estimates to the case of sections of

$$L_k = \mathcal{O}_{X_k^{\rm GG}}(1)\otimes\pi_k^*\mathcal{O}\Big(\frac{1}{kr}\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)F\Big) \tag{9.35}$$

where $F\in\mathrm{Pic}_{\mathbb{Q}}(X)$ is an arbitrary $\mathbb{Q}$-line bundle on X and $\pi_k : X_k^{\rm GG}\to X$ is the natural projection. We assume here that F is also equipped with a smooth Hermitian metric h_F. In formulas (9.32–9.34), the renormalized curvature $\eta_k(z,x,u)$ of L_k takes the form

$$\eta_k(z,x,u) = \frac{1}{\frac{1}{kr}(1+\frac{1}{2}+\ldots+\frac{1}{k})}g_k(z,x,u) + \Theta_{F,h_F}(z), \tag{9.36}$$

and by the same calculations its expected value is

$$\eta(z) := \mathbf{E}(\eta_k(z,\bullet,\bullet)) = \Theta_{\det V^*,\det h^*}(z) + \Theta_{F,h_F}(z). \tag{9.37}$$

Then the variance estimate for $\eta_k-\eta$ is unchanged, and the L^p bounds for η_k are still valid, since our forms are just shifted by adding the constant smooth term $\Theta_{F,h_F}(z)$. The probabilistic estimate 9.34 is therefore still true in exactly the same form, provided we use (9.35 – 9.37) instead of the previously defined L_k, η_k and η. An application of holomorphic Morse inequalities gives the desired cohomology estimates for

$$\begin{aligned} &h^q\Big(X, E_{k,m}^{\rm GG}V^*\otimes\mathcal{O}\Big(\frac{m}{kr}\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)F\Big)\Big)\\ &\qquad = h^q(X_k^{\rm GG},\mathcal{O}_{X_k^{\rm GG}}(m)\otimes\pi_k^*\mathcal{O}\Big(\frac{m}{kr}\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)F\Big)\Big),\end{aligned}$$

provided m is sufficiently divisible to give a multiple of F which is a $\mathbb{Z}$-line bundle.

9.38. Theorem. *Let (X,V) be a directed manifold, $F\to X$ a $\mathbb{Q}$-line bundle, (V,h) and (F,h_F) smooth Hermitian structure on V and F respectively. We define*

$$L_k = \mathcal{O}_{X_k^{\rm GG}}(1)\otimes \pi_k^*\mathcal{O}\Big(\frac{1}{kr}\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)F\Big),$$
$$\eta = \Theta_{\det V^*,\det h^*}+\Theta_{F,h_F}.$$

Then for all $q\geqslant 0$ and all $m\gg k\gg 1$ such that m is sufficiently divisible, we have

$$\text{(a)}\qquad h^q(X_k^{\rm GG},\mathcal{O}(L_k^{\otimes m}))\leqslant \frac{m^{n+kr-1}}{(n+kr-1)!}\frac{(\log k)^n}{n!\,(k!)^r}\Big(\int_{X(\eta,q)}(-1)^q\eta^n+O((\log k)^{-1})\Big),$$

$$\text{(b)}\qquad h^0(X_k^{\rm GG},\mathcal{O}(L_k^{\otimes m}))\geqslant \frac{m^{n+kr-1}}{(n+kr-1)!}\frac{(\log k)^n}{n!\,(k!)^r}\Big(\int_{X(\eta,\leqslant 1)}\eta^n-O((\log k)^{-1})\Big),$$

$$\text{(c)}\qquad \chi(X_k^{\rm GG},\mathcal{O}(L_k^{\otimes m}))= \frac{m^{n+kr-1}}{(n+kr-1)!}\frac{(\log k)^n}{n!\,(k!)^r}\big(c_1(V^*\otimes F)^n+O((\log k)^{-1})\big).$$

Green and Griffiths [GrGr79] already checked the Riemann-Roch calculation (9.38 c) in the special case $V=T_X^*$ and $F=\mathcal{O}_X$. Their proof is much simpler since it relies only on Chern class calculations, but it cannot provide any information on the individual cohomology groups, except in very special cases where vanishing theorems can be applied; in fact in dimension 2, the Euler characteristic satisfies $\chi=h^0-h^1+h^2\leqslant h^0+h^2$, hence it is enough to get the vanishing of the top cohomology group H^2 to infer $h^0\geqslant\chi$; this works for surfaces by means of a well-known vanishing theorem of Bogomolov which implies in general

$$H^n\Big(X,E_{k,m}^{\rm GG}T_X^*\otimes\mathcal{O}\Big(\frac{m}{kr}\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)F\Big)\Big)=0$$

as soon as $K_X\otimes F$ is big and $m\gg 1$.

In fact, thanks to Bonavero's singular holomorphic Morse inequalities [Bon93], everything works almost unchanged in the case where $V\subset T_X$ has singularities and h is an admissible metric on V (see Definition 9.7). We only have to find a blow-up $\mu:\widetilde{X}_k\to X_k$ so that the resulting pull-backs μ^*L_k and μ^*V are locally free, and $\mu^*\det h^*$, $\mu^*\Psi_{h,p,\varepsilon}$ only have divisorial singularities. Then η is a $(1,1)$-current with logarithmic poles, and we have to deal with smooth metrics on $\mu^*L_k^{\otimes m}\otimes\mathcal{O}(-mE_k)$ where E_k is a certain effective divisor on X_k (which, by our assumption in 9.7, does not project onto X). The cohomology groups involved are then the twisted cohomology groups

$$H^q(X_k^{\rm GG},\mathcal{O}(L_k^{\otimes m})\otimes\mathcal{J}_{k,m})$$

where $\mathcal{J}_{k,m}=\mu_*(\mathcal{O}(-mE_k))$ is the corresponding multiplier ideal sheaf, and the Morse integrals need only be evaluated in the complement of the poles, that is on $X(\eta,q)\smallsetminus S$ where $S=\mathrm{Sing}(V)\cup\mathrm{Sing}(h)$. Since

$$(\pi_k)_*\big(\mathcal{O}(L_k^{\otimes m})\otimes\mathcal{J}_{k,m}\big)\subset E_{k,m}^{\rm GG}V^*\otimes\mathcal{O}\Big(\frac{m}{kr}\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)F\Big)\Big)$$

we still get a lower bound for the H^0 of the latter sheaf (or for the H^0 of the un-twisted line bundle $\mathcal{O}(L_k^{\otimes m})$ on $X_k^{\rm GG}$). If we assume that $K_V\otimes F$ is big, these considerations also allow us to obtain a strong estimate in terms of the volume, by using an approximate Zariski decomposition on a suitable blow-up of (X, V). The following corollary implies in particular Theorem 9.3.

9.39. Corollary. *If F is an arbitrary $\mathbb{Q}$-line bundle over X, one has*

$$h^0\Big(X_k^{\mathrm{GG}}, \mathcal{O}_{X_k^{\mathrm{GG}}}(m) \otimes \pi_k^*\mathcal{O}\Big(\frac{m}{kr}\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)F\Big)\Big) \geqslant \frac{m^{n+kr-1}}{(n+kr-1)!}\frac{(\log k)^n}{n!\,(k!)^r}\Big(\mathrm{Vol}(K_V \otimes F) - O((\log k)^{-1})\Big) - o(m^{n+kr-1}),$$

when $m \gg k \gg 1$, in particular there are many sections of the k-jet differentials of degree m twisted by the appropriate power of F if $K_V \otimes F$ is big.

Proof. The volume is computed here as usual, i.e. after performing a suitable modification $\mu : \widetilde{X} \to X$ which converts K_V into an invertible sheaf. There is of course nothing to prove if $K_V \otimes F$ is not big, so we can assume $\mathrm{Vol}(K_V \otimes F) > 0$. Let us fix smooth Hermitian metrics h_0 on T_X and h_F on F. They induce a metric $\mu^*(\det h_0^{-1} \otimes h_F)$ on $\mu^*(K_V \otimes F)$ which, by our definition of K_V, is a smooth metric. By the result of Fujita [Fuj94] on approximate Zariski decomposition, for every $\delta > 0$, one can find a modification $\mu_\delta : \widetilde{X}_\delta \to X$ dominating μ such that

$$\mu_\delta^*(K_V \otimes F) = \mathcal{O}_{\widetilde{X}_\delta}(A + E)$$

where A and E are $\mathbb{Q}$-divisors, A ample and E effective, with

$$\mathrm{Vol}(A) = A^n \geqslant \mathrm{Vol}(K_V \otimes F) - \delta.$$

If we take a smooth metric h_A with positive definite curvature form Θ_{A,h_A}, then we get a singular Hermitian metric $h_A h_E$ on $\mu_\delta^*(K_V \otimes F)$ with poles along E, i.e. the quotient $h_A h_E/\mu^*(\det h_0^{-1} \otimes h_F)$ is of the form $e^{-\varphi}$ where φ is quasi-psh with log poles $\log|\sigma_E|^2$ (mod $C^\infty(\widetilde{X}_\delta)$) precisely given by the divisor E. We then only need to take the singular metric h on T_X defined by

$$h = h_0 e^{\frac{1}{r}(\mu_\delta)^*\varphi}$$

(the choice of the factor $\frac{1}{r}$ is there to correct adequately the metric on $\det V$). By construction h induces an admissible metric on V and the resulting curvature current $\eta = \Theta_{K_V,\det h^*} + \Theta_{F,h_F}$ is such that

$$\mu_\delta^*\eta = \Theta_{A,h_A} + [E], \qquad [E] = \text{current of integration on } E.$$

Then the 0-index Morse integral in the complement of the poles is given by

$$\int_{X(\eta,0)\smallsetminus S} \eta^n = \int_{\widetilde{X}_\delta} \Theta_{A,h_A}^n = A^n \geqslant \mathrm{Vol}(K_V \otimes F) - \delta$$

and (9.39) follows from the fact that δ can be taken arbitrary small. □

9.40. Example. In some simple cases, the above estimates can lead to very explicit results. Take for instance X to be a smooth complete intersection of multidegree $(d_1, d_2, \ldots, d_s)$ in $\mathbb{P}_{\mathbb{C}}^{n+s}$ and consider the absolute case $V = T_X$. Then $K_X = \mathcal{O}_X(d_1 + \ldots + d_s - n - s - 1)$ and one can check via explicit bounds of the error terms (cf. [Dem11], [Dem12]) that a sufficient condition for the existence of sections is

$$k \geqslant \exp\Big(7.38\, n^{n+1/2}\Big(\frac{\sum d_j + 1}{\sum d_j - n - s - a - 1}\Big)^n\Big).$$

This is good in view of the fact that we can cover arbitrary smooth complete intersections of general type. On the other hand, even when the degrees d_j tend to $+\infty$, we still get a large lower bound $k \sim \exp(7.38\, n^{n+1/2})$ on the order of jets, and this is far from being optimal: Diverio [Div08, Div09] has shown e.g. that one can take $k = n$ for smooth hypersurfaces of high degree, using the algebraic Morse inequalities of Trapani [Tra95]. The next paragraph uses essentially the same idea, in our more analytic setting. □

§9.D. Non probabilistic estimate of the Morse integrals

We assume here that the curvature tensor $(c_{ij\alpha\beta})$ satisfies a lower bound

$$\sum_{i,j,\alpha,\beta} c_{ij\alpha\beta}\xi_i\overline{\xi}_j u_\alpha \overline{u}_\beta \geqslant -\sum \gamma_{ij}\xi_i\overline{\xi}_j\,|u|^2, \qquad \forall \xi \in T_X,\ u \in X \tag{9.41}$$

for some semipositive $(1,1)$-form $\gamma = \frac{i}{2\pi}\sum \gamma_{ij}(z)\,dz_i \wedge d\overline{z}_j$ on X. This is the same as assuming that the curvature tensor of (V^*, h^*) satisfies the semipositivity condition

$$\Theta_{V^*,h^*} + \gamma \otimes \mathrm{Id}_{V^*} \geqslant 0 \tag{9.41′}$$

in the sense of Griffiths, or equivalently $\Theta_{V,h} - \gamma \otimes \mathrm{Id}_V \leqslant 0$. Thanks to the compactness of X, such a form γ always exists if h is an admissible metric on V. Now, instead of replacing Θ_V with its trace free part $\widetilde{\Theta}_V$ and exploiting a Monte Carlo convergence process, we replace Θ_V with $\Theta_V^\gamma = \Theta_V - \gamma \otimes \mathrm{Id}_V \leqslant 0$, i.e. $c_{ij\alpha\beta}$ by $c^\gamma_{ij\alpha\beta} = c_{ij\alpha\beta} + \gamma_{ij}\delta_{\alpha\beta}$. Also, we take a line bundle $F = A^{-1}$ with $\Theta_{A,h_A} \geqslant 0$, i.e. F seminegative. Then our earlier formulas (9.28), (9.35), (9.36) become instead

$$g_k^\gamma(z,x,u) = \frac{i}{2\pi}\sum_{1\leqslant s\leqslant k}\frac{1}{s}x_s \sum_{i,j,\alpha,\beta} c^\gamma_{ij\alpha\beta}(z)\,u_{s\alpha}\overline{u}_{s\beta}\,dz_i \wedge d\overline{z}_j \geqslant 0, \tag{9.42}$$

$$L_k = \mathcal{O}_{X_k^{\mathrm{GG}}}(1) \otimes \pi_k^*\mathcal{O}\Big(-\frac{1}{kr}\Big(1 + \frac{1}{2} + \ldots + \frac{1}{k}\Big)A\Big), \tag{9.43}$$

$$\Theta_{L_k} = \eta_k(z,x,u) = \frac{1}{\frac{1}{kr}(1 + \frac{1}{2} + \ldots + \frac{1}{k})} g_k^\gamma(z,x,u) - (\Theta_{A,h_A}(z) + r\gamma(z)). \tag{9.44}$$

In fact, replacing Θ_V by $\Theta_V - \gamma \otimes \mathrm{Id}_V$ has the effect of replacing $\Theta_{\det V^*} = \mathrm{Tr}\,\Theta_{V^*}$ by $\Theta_{\det V^*} + r\gamma$. The major gain that we have is that $\eta_k = \Theta_{L_k}$ is now expressed as a difference of semipositive $(1,1)$-forms, and we can exploit the following simple lemma, which is the key to derive algebraic Morse inequalities from their analytic form (cf. [Dem94], Theorem 12.3).

9.45. Lemma. *Let $\eta = \alpha - \beta$ be a difference of semipositive $(1,1)$-forms on an n-dimensional complex manifold X, and let $\mathbb{1}_{\eta,\leqslant q}$ be the characteristic function of the open set where η is non degenerate with a number of negative eigenvalues at most equal to q. Then*

$$(-1)^q \mathbb{1}_{\eta,\leqslant q}\,\eta^n \leqslant \sum_{0\leqslant j\leqslant q}(-1)^{q-j}\alpha^{n-j}\beta^j,$$

in particular

$$\mathbb{1}_{\eta,\leqslant 1}\,\eta^n \geqslant \alpha^n - n\alpha^{n-1}\wedge\beta \qquad \textit{for } q=1.$$

Proof. Without loss of generality, we can assume $\alpha > 0$ positive definite, so that α can be taken as the base hermitian metric on X. Let us denote by

$$\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n \geqslant 0$$

the eigenvalues of β with respect to α. The eigenvalues of $\eta = \alpha - \beta$ are then given by

$$1-\lambda_1 \leqslant \ldots \leqslant 1-\lambda_q \leqslant 1-\lambda_{q+1} \leqslant \ldots \leqslant 1-\lambda_n,$$

hence the open set $\{\lambda_{q+1} < 1\}$ coincides with the support of $\mathbb{1}_{\eta,\leqslant q}$, except that it may also contain a part of the degeneration set $\eta^n = 0$. On the other hand we have

$$\binom{n}{j}\alpha^{n-j}\wedge\beta^j = \sigma_n^j(\lambda)\,\alpha^n,$$

where $\sigma_n^j(\lambda)$ is the j-th elementary symmetric function in the λ_j's. Thus, to prove the lemma, we only have to check that

$$\sum_{0\leqslant j\leqslant q}(-1)^{q-j}\sigma_n^j(\lambda) - \mathbb{1}_{\{\lambda_{q+1}<1\}}(-1)^q\prod_{1\leqslant j\leqslant n}(1-\lambda_j) \geqslant 0.$$

This is easily done by induction on n (just split apart the parameter λ_n and write $\sigma_n^j(\lambda) = \sigma_{n-1}^j(\lambda) + \sigma_{n-1}^{j-1}(\lambda)\,\lambda_n$). □

We apply here Lemma 9.45 with

$$\alpha = g_k^\gamma(z,x,u), \qquad \beta = \beta_k = \frac{1}{kr}\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)(\Theta_{A,h_A}+r\gamma),$$

which are both semipositive by our assumption. The analogue of (9.32) leads to

$$\begin{aligned}
&\int_{X_k^{\mathrm{GG}}(L_k,\leqslant 1)} \Theta^{n+kr-1}_{L_k,\Psi^*_{h,p,\varepsilon}}\\
&\quad= \frac{(n+kr-1)!}{n!(k!)^r(kr-1)!}\int_{z\in X}\int_{(x,u)\in\Delta_{k-1}\times(S^{2r-1})^k}\mathbb{1}_{g_k^\gamma-\beta_k,\leqslant 1}\,(g_k^\gamma-\beta_k)^n\,d\nu_{k,r}(x)\,d\mu(u)\\
&\quad\geqslant \frac{(n+kr-1)!}{n!(k!)^r(kr-1)!}\int_{z\in X}\int_{(x,u)\in\Delta_{k-1}\times(S^{2r-1})^k}((g_k^\gamma)^n - n(g_k^\gamma)^{n-1}\wedge\beta_k)\,d\nu_{k,r}(x)\,d\mu(u).
\end{aligned}$$

The resulting integral now produces a "closed formula" which can be expressed solely in terms of Chern classes (at least if we assume that γ is the Chern form of some semipositive line bundle). It is just a matter of routine to find a sufficient condition for the positivity of the integral. One can first observe that g_k^γ is bounded from above by taking the trace of $(c_{ij\alpha\beta})$, in this way we get

$$0 \leqslant g_k^\gamma \leqslant \Big(\sum_{1\leqslant s\leqslant k}\frac{x_s}{s}\Big)(\Theta_{\det V^*}+r\gamma)$$

where the right hand side no longer depends on $u \in (S^{2r-1})^k$. Also, g_k^γ can be written as a sum of semipositive $(1,1)$-forms

$$g_k^\gamma = \sum_{1\leqslant s\leqslant k} \frac{x_s}{s}\theta^\gamma(u_s), \qquad \theta^\gamma(u) = \sum_{i,j,\alpha,\beta} c_{ij\alpha\beta}^\gamma u_\alpha \overline{u}_\beta \, dz_i \wedge d\overline{z}_j,$$

hence for $k \geqslant n$ we have

$$(g_k^\gamma)^n \geqslant n! \sum_{1\leqslant s_1<\ldots<s_n\leqslant k} \frac{x_{s_1}\ldots x_{s_n}}{s_1\ldots s_n}\,\theta^\gamma(u_{s_1})\wedge\theta^\gamma(u_{s_2})\wedge\ldots\wedge\theta^\gamma(u_{s_n}).$$

Since $\int_{S^{2r-1}} \theta^\gamma(u)\,d\mu(u) = \frac{1}{r}\operatorname{Tr}(\Theta_{V^*}+\gamma) = \frac{1}{r}\Theta_{\det V^*}+\gamma$, we infer from this

$$\begin{aligned}&\int_{(x,u)\in\Delta_{k-1}\times(S^{2r-1})^k} (g_k^\gamma)^n \, d\nu_{k,r}(x)\,d\mu(u)\\ &\qquad\geqslant n! \sum_{1\leqslant s_1<\ldots<s_n\leqslant k} \frac{1}{s_1\ldots s_n}\Big(\int_{\Delta_{k-1}} x_1\ldots x_n\,d\nu_{k,r}(x)\Big)\Big(\frac{1}{r}\Theta_{\det V^*}+\gamma\Big)^n.\end{aligned}$$

By putting everything together, we conclude:

9.46. Theorem. *Assume that $\Theta_{V^*} \geqslant -\gamma\otimes \mathrm{Id}_{V^*}$ with a semipositive $(1,1)$-form γ on X. Then the Morse integral of the line bundle*

$$L_k = \mathcal{O}_{X_k^{\mathrm{GG}}}(1)\otimes\pi_k^*\mathcal{O}\Big(-\frac{1}{kr}\Big(1+\frac12+\ldots+\frac1k\Big)A\Big), \qquad A\geqslant 0$$

satisfies for $k\geqslant n$ the inequality

$$\begin{aligned}&\frac{1}{(n+kr-1)!}\int_{X_k^{\mathrm{GG}}(L_k,\leqslant 1)} \Theta_{L_k,\Psi_{h,p,\varepsilon}^*}^{n+kr-1}\\ (*)\quad &\geqslant \frac{1}{n!(k!)^r(kr-1)!}\int_X c_{n,r,k}\big(\Theta_{\det V^*}+r\gamma\big)^n - c'_{n,r,k}\big(\Theta_{\det V^*}+r\gamma\big)^{n-1}\wedge\big(\Theta_{A,h_A}+r\gamma\big)\end{aligned}$$

where

$$\begin{aligned} c_{n,r,k} &= \frac{n!}{r^n}\Big(\sum_{1\leqslant s_1<\ldots<s_n\leqslant k}\frac{1}{s_1\ldots s_n}\Big)\int_{\Delta_{k-1}} x_1\ldots x_n\,d\nu_{k,r}(x),\\ c'_{n,r,k} &= \frac{n}{kr}\Big(1+\frac12+\ldots+\frac1k\Big)\int_{\Delta_{k-1}}\Big(\sum_{1\leqslant s\leqslant k}\frac{x_s}{s}\Big)^{n-1}d\nu_{k,r}(x).\end{aligned}$$

Especially we have a lot of sections in $H^0(X_k^{\mathrm{GG}}, mL_k)$, $m\gg 1$, as soon as the difference occurring in $(*)$ is positive.

The statement is also true for $k<n$, but then $c_{n,r,k}=0$ and the lower bound $(*)$ cannot be positive. By Corollary 9.11, it still provides a non trivial lower bound for $h^0(X_k^{\mathrm{GG}}, mL_k) - h^1(X_k^{\mathrm{GG}}, mL_k)$, though. For $k\geqslant n$ we have $c_{n,r,k}>0$ and $(*)$ will be positive if $\Theta_{\det V^*}$ is large enough. By Formula 9.20 we have

$$(9.47)\qquad c_{n,r,k} = \frac{n!\,(kr-1)!}{(n+kr-1)!}\sum_{1\leqslant s_1<\ldots<s_n\leqslant k}\frac{1}{s_1\ldots s_n} \geqslant \frac{(kr-1)!}{(n+kr-1)!},$$

(with equality for $k = n$), and by ([Dem11], Lemma 2.20 (b)) we get the upper bound

$$\frac{c'_{n,k,r}}{c_{n,k,r}} \leqslant \frac{(kr+n-1)r^{n-2}}{k/n}\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)^n\Big[1+\frac{1}{3}\sum_{m=2}^{n-1}\frac{2^m(n-1)!}{(n-1-m)!}\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)^{-m}\Big].$$

The case $k = n$ is especially interesting. For $k = n \geqslant 2$ one can show (with $r \leqslant n$ and H_n denoting the harmonic sequence) that

$$\frac{c'_{n,k,r}}{c_{n,k,r}} \leqslant \frac{n^2+n-1}{3}n^{n-2}\exp\Big(\frac{2(n-1)}{H_n}+n\log H_n\Big) \leqslant \frac{1}{3}\big(n\log(n\log 24n)\big)^n. \tag{9.48}$$

We will later need the particular values that can be obtained by direct calculations (cf. Formula (9.20 and [Dem11, Lemma 2.20]).

$$c_{2,2,2} = \frac{1}{20},\qquad c'_{2,2,2} = \frac{9}{16},\qquad \frac{c'_{2,2,2}}{c_{2,2,2}} = \frac{45}{4}, \tag{9.49_2}$$

$$c_{3,3,3} = \frac{1}{990},\qquad c'_{3,3,3} = \frac{451}{4860},\qquad \frac{c'_{3,3,3}}{c_{3,3,3}} = \frac{4961}{54}. \tag{9.49_3}$$

§10. Hyperbolicity properties of hypersurfaces of high degree

§10.A. Global generation of the twisted tangent space of the universal family

In [Siu02, Siu04], Y.T. Siu developed a new stategy to produce jet differentials, involving meromorphic vector fields on the total space of jet bundles – these vector fields are used to differentiate the sections of $E^{\mathrm{GG}}_{k,m}$ so as to produce new ones with less zeroes. The approach works especially well on universal families of hypersurfaces in projective space, thanks to the good positivity properties of the relative tangent bundle, as shown by L. Ein [Ein88, Ein91] and C. Voisin [Voi96]. This allows at least to prove the hyperbolicity of generic surfaces and generic 3-dimensional hypersurfaces of sufficiently high degree. We reproduce here the improved approach given by [Pau08] for the twisted global generation of the tangent space of the space of vertical two jets. The situation of k-jets in arbitrary dimension n is substantially more involved, details can be found in [Mer09].

Consider the universal hypersurface $\mathcal{X} \subset \mathbb{P}^{n+1} \times \mathbb{P}^{N_d}$ of degree d given by the equation

$$\sum_{|\alpha|=d} A_\alpha\, Z^\alpha = 0,$$

where $[Z] \in \mathbb{P}^{n+1}$, $[A] \in \mathbb{P}^{N_d}$, $\alpha = (\alpha_0, \ldots, \alpha_{n+1}) \in \mathbb{N}^{n+2}$ and

$$N_d = \binom{n+d+1}{d} - 1.$$

Finally, we denote by $\mathcal{V} \subset \mathcal{X}$ the vertical tangent space, i.e. the kernel of the projection

$$\pi : \mathcal{X} \to U \subset \mathbb{P}^{N_d}$$

where U is the Zariski open set parametrizing smooth hypersurfaces, and by $J_k\mathcal{V}$ the bundle of k-jets of curves tangent to $\mathcal{V}$, i.e. curves contained in the fibers $X_s = \pi^{-1}(s)$. The goal is to describe certain meromorphic vector fields on the total space of $J_k\mathcal{V}$. In the special case $n = 2$, $k = 2$ considered by Păun [Pau08], one fixes the affine open set

$$\mathcal{U}_0 = \{Z_0 \neq 0\} \times \{A_{0d00} \neq 0\} \simeq \mathbb{C}^3 \times \mathbb{C}^{N_d}$$

in $\mathbb{P}^3 \times \mathbb{P}^{N_d}$ with the corresponding inhomogeneous coordinates $(z_j = Z_j/Z_0)_{j=1,2,3}$ and $(a_\alpha = A_\alpha/A_{0d00})_{|\alpha|=d,\alpha_1<d}$. Since α_0 is determined by $\alpha_0 = d - (\alpha_1 + \alpha_2 + \alpha_3)$, with a slight abuse of notation in the sequel, α will be seen as a multiindex $(\alpha_1, \alpha_2, \alpha_3)$ in $\mathbb{N}^3$, with moreover the convention that $a_{d00} = 1$. On this affine open set we have

$$\mathcal{X}_0 := \mathcal{X} \cap \mathcal{U}_0 = \Big\{ z_1^d + \sum_{|\alpha|\leqslant d,\alpha_1<d} a_\alpha\, z^\alpha = 0 \Big\}.$$

We now write down equations for the open variety $J_2\mathcal{V}_0$, where we indicated with $\mathcal{V}_0$ the restriction of $\mathcal{V} \subset T_{\mathcal{X}}$, the kernel of the differential of the second projection, to $\mathcal{X}_0$: elements in $J_2\mathcal{V}_0$ are therefore 2-jets of germs of "vertical" holomorphic curves in $\mathcal{X}_0$, that is curves tangent to vertical fibers. The equations, which live in a natural way in $\mathbb{C}^3_{z_j} \times \mathbb{C}^{N_d}_{a_\alpha} \times \mathbb{C}^3_{z'_j} \times \mathbb{C}^3_{z''_j}$, stand as follows.

$$\begin{aligned}
&\sum_{|\alpha|\leqslant d} a_\alpha\, z^\alpha = 0,\\
&\sum_{1\leqslant j\leqslant 3}\ \sum_{|\alpha|\leqslant d} a_\alpha \frac{\partial z^\alpha}{\partial z_j}\, z'_j = 0,\\
&\sum_{1\leqslant j\leqslant 3}\ \sum_{|\alpha|\leqslant d} a_\alpha \frac{\partial z^\alpha}{\partial z_j}\, z''_j + \sum_{1\leqslant j,k\leqslant 3}\ \sum_{|\alpha|\leqslant d} a_\alpha \frac{\partial^2 z^\alpha}{\partial z_j \partial z_k}\, z'_j z'_k = 0.
\end{aligned}$$

Let $\mathcal{W}_0$ to be the closed algebraic subvariety of $J_2\mathcal{V}_0$ defined by

$$\mathcal{W}_0 = \{(z, a, z', z'') \in J_2\mathcal{V}_0 \mid z' \wedge z'' = 0\}$$

and let $\mathcal{W}$ be the Zariski closure of $\mathcal{W}_0$ in $J_2\mathcal{V}$: we call this set the *Wronskian locus* of $J_2\mathcal{V}$. Explicit calculations (cf. [Pau08]) then produce the following vector fields:

First family of tangent vector fields. For any multiindex α such that $\alpha_1 \geqslant 3$, consider the vector field

$$\theta^{300}_\alpha = \frac{\partial}{\partial a_\alpha} - 3z_1 \frac{\partial}{\partial a_{\alpha-\delta_1}} + 3z_1^2 \frac{\partial}{\partial a_{\alpha-2\delta_1}} - z_1^3 \frac{\partial}{\partial a_{\alpha-3\delta_1}},$$

where $\delta_j \in \mathbb{N}^4$ is the multiindex whose j-th component is equal to 1 and the others are zero. For the multiindexes α which verify $\alpha_1 \geqslant 2$ and $\alpha_2 \geqslant 1$, define

$$\begin{aligned}
\theta^{210}_\alpha = \frac{\partial}{\partial a_\alpha} - 2z_1 \frac{\partial}{\partial a_{\alpha-\delta_1}} - z_2 \frac{\partial}{\partial a_{\alpha-\delta_2}} + z_1^2 \frac{\partial}{\partial a_{\alpha-2\delta_1}}\\
+ 2z_1 z_2 \frac{\partial}{\partial a_{\alpha-\delta_1-\delta_2}} - z_1^2 z_2 \frac{\partial}{\partial a_{\alpha-2\delta_1-\delta_2}}.
\end{aligned}$$

Finally, for those α for which $\alpha_1, \alpha_2, \alpha_3 \geqslant 1$, set

$$\begin{aligned}
\theta^{111}_\alpha = \frac{\partial}{\partial a_\alpha} - z_1 \frac{\partial}{\partial a_{\alpha-\delta_1}} - z_2 \frac{\partial}{\partial a_{\alpha-\delta_2}} - z_3 \frac{\partial}{\partial a_{\alpha-\delta_3}}\\
+ z_1 z_2 \frac{\partial}{\partial a_{\alpha-\delta_1-\delta_2}} + z_1 z_3 \frac{\partial}{\partial a_{\alpha-\delta_1-\delta_3}} + z_2 z_3 \frac{\partial}{\partial a_{\alpha-\delta_2-\delta_3}}\\
- z_1 z_2 z_3 \frac{\partial}{\partial a_{\alpha-\delta_1-\delta_2-\delta_3}}.
\end{aligned}$$

Second family of tangent vector fields. We construct here the holomorphic vector fields in order to span the $\partial/\partial z_j$-directions. For $j = 1, 2, 3$, consider the vector field

$$\frac{\partial}{\partial z_j} - \sum_{|\alpha+\delta_j|\leqslant d} (\alpha_j + 1) a_{\alpha+\delta_j} \frac{\partial}{\partial a_\alpha}.$$

Third family of tangent vector fields. In order to span the jet directions, consider a vector field of the following form:

$$\theta_B = \sum_{|\alpha|\leqslant d, \alpha_1<d} p_\alpha(z, a, b) \frac{\partial}{\partial a_\alpha} + \sum_{1\leqslant j\leqslant 3} \sum_{k=1}^{2} \xi_j^{(k)} \frac{\partial}{\partial z_j^{(k)}},$$

where $\xi^{(k)} = B \cdot z^{(k)}$, $k = 1, 2$, and $B = (b_{jk})$ varies among 3×3 invertible matrices with complex entries. By studying more carefully these three families of vector fields, one obtains:

10.1. Theorem. *The twisted tangent space $T_{J_2\mathcal{V}} \otimes \mathcal{O}_{\mathbb{P}^3}(7) \otimes \mathcal{O}_{\mathbb{P}^{N_d}}(1)$ is generated over by its global sections over the complement $J_2\mathcal{V} \smallsetminus \mathcal{W}$ of the Wronskian locus $\mathcal{W}$. Moreover, one can choose generating global sections that are invariant with respect to the action of $\mathbb{G}_2$ on $J_2\mathcal{V}$.*

By similar, but more computationally intensive arguments [Mer09], one can investigate the higher dimensional case. The following result strengthens the initial announcement of [Siu04].

10.2. Theorem. *Let $J_k^{\mathrm{vert}}(\mathcal{X})$ be the space of vertical k-jets of the universal hypersurface*

$$\mathcal{X} \subset \mathbb{P}^{n+1} \times \mathbb{P}^{N_d}$$

parametrizing all projective hypersurfaces $X \subset \mathbb{P}^{n+1}$ of degree d. Then for $k = n$, there exist constants c_n and c'_n such that the twisted tangent bundle

$$T_{J_k^{\mathrm{vert}}(\mathcal{X})} \otimes \mathcal{O}_{\mathbb{P}^{n+1}}(c_n) \otimes \mathcal{O}_{\mathbb{P}^{N_d}}(c'_n)$$

is generated by its global $\mathbb{G}_k$-invariant sections outside a certain exceptional algebraic subset $\Sigma \subset J_k^{\mathrm{vert}}(\mathcal{X})$. One can take either $c_n = \frac{1}{2}(n^2 + 5n)$, $c'_n = 1$ and Σ defined by the vanishing of certain Wronskians, or $c_n = n^2 + 2n$ and a smaller set $\widetilde{\Sigma} \subset \Sigma$ defined by the vanishing of the 1-jet part.

10.B. General strategy of proof

Let again $\mathcal{X} \subset \mathbb{P}^{n+1} \times \mathbb{P}^{N_d}$ be the universal hypersurface of degree d in $\mathbb{P}^{n+1}$.

(10.3) *Assume that we can prove the existence of a non zero polynomial differential operator*

$$P \in H^0(\mathcal{X}, E_{k,m}^{\mathrm{GG}} T_{\mathcal{X}}^* \otimes \mathcal{O}(-A)),$$

where A is an ample divisor on $\mathcal{X}$, at least over some Zariski open set U in the base of the projection $\pi : \mathcal{X} \to U \subset \mathbb{P}^{N_d}$.

Observe that we now have a lot of techniques to do this; the existence of P over the family follows from lower semicontinuity in the Zariski topology, once we know that such a section P exists on a generic fiber $X_s = \pi^{-1}(s)$. Let $\mathcal{Y} \subset \mathcal{X}$ be the set of points $x \in \mathcal{X}$ where $P(x) = 0$, as an element in the fiber of the vector bundle $E_{k,m}^{\mathrm{GG}} T_{\mathcal{X}}^* \otimes \mathcal{O}(-A))$ at x. Then $\mathcal{Y}$

is a proper algebraic subset of $\mathcal{X}$, and after shrinking U we may assume that $Y_s = \mathcal{Y} \cap X_s$ is a proper algebraic subset of X_s for every $s \in U$.

(10.4) *Assume also, according to Theorems* 10.1 *and* 10.2*, that we have enough global holomorphic* $\mathbb{G}_k$*-invariant vector fields* θ_i *on* $J_k\mathcal{V}$ *with values in the pull-back of some ample divisor* B *on* $\mathcal{X}$*, in such a way that they generate* $T_{J_k\mathcal{V}} \otimes p_k^*B$ *over the dense open set* $(J_k\mathcal{V})^{\mathrm{reg}}$ *of regular* k*-jets, i.e.* k*-jets with non zero first derivative* (*here* $p_k : J_k\mathcal{V} \to \mathcal{X}$ *is the natural projection*).

Considering jet differentials P as functions on $J_k\mathcal{V}$, the idea is to produce new ones by taking differentiations

$$Q_j := \theta_{j_1} \ldots \theta_{j_\ell} P, \qquad 0 \leqslant \ell \leqslant m,\ j = (j_1, \ldots, j_\ell).$$

Since the θ_j's are $\mathbb{G}_k$-invariant, they are in particular $\mathbb{C}^*$-invariant, thus

$$Q_j \in H^0(\mathcal{X}, E^{\mathrm{GG}}_{k,m} T^*_{\mathcal{X}} \otimes \mathcal{O}(-A + \ell B))$$

(and Q is in fact $\mathbb{G}'_k$ invariant as soon as P is). In order to be able to apply the vanishing theorems of § 8, we need $A - mB$ to be ample, so A has to be large compared to B. If $f : \mathbb{C} \to X_s$ is an entire curve contained in some fiber $X_s \subset \mathcal{X}$, its lifting $j_k(f) : \mathbb{C} \to J_k\mathcal{V}$ has to lie in the zero divisors of all sections Q_j. However, every non zero polynomial of degree m has at any point some non zero derivative of order $\ell \leqslant m$. Therefore, at any point where the θ_i generate the tangent space to $J_k\mathcal{V}$, we can find some non vanishing section Q_j. By the assumptions on the θ_i, the base locus of the Q_j's is contained in the union of $p_k^{-1}(\mathcal{Y}) \cup (J_k\mathcal{V})^{\mathrm{sing}}$; there is of course no way of getting a non zero polynomial at points of $\mathcal{Y}$ where P vanishes. Finally, we observe that $j_k(f)(\mathbb{C}) \not\subset (J_k\mathcal{V}^{\mathrm{sing}}$ (otherwise f is constant). Therefore $j_k(f)(\mathbb{C}) \subset p_k^{-1}(\mathcal{Y})$ and thus $f(\mathbb{C}) \subset \mathcal{Y}$, i.e. $f(\mathbb{C}) \subset Y_s = \mathcal{Y} \cap X_s$.

10.5. Corollary. *Let* $\mathcal{X} \subset \mathbb{P}^{n+1} \times \mathbb{P}^{N_d}$ *be the universal hypersurface of degree* d *in* $\mathbb{P}^{n+1}$*. If* $d \geqslant d_n$ *is taken so large that conditions* (10.3) *and* (10.4) *are met with* $A - mB$ *ample, then the generic fiber* X_s *of the universal family* $\mathcal{X} \to U$ *satisfies the Green-Griffiths conjecture, namely all entire curves* $f : \mathbb{C} \to X_s$ *are contained in a proper algebraic subvariety* $Y_s \subset X_s$*, and the* Y_s *can be taken to form an algebraic subset* $\mathcal{Y} \subset \mathcal{X}$*.*

This is unfortunately not enough to get the hyperbolicity of X_s, because we would have to know that Y_s itself is hyperbolic. However, one can use the following simple observation due to Diverio and Trapani [DT10]. The starting point is the following general, straightforward remark. Let $\mathcal{E} \to \mathcal{X}$ be a holomorphic vector bundle let $\sigma \in H^0(\mathcal{X}, \mathcal{E}) \neq 0$; then, up to factorizing by an effective divisor D contained in the common zeroes of the components of σ, one can view σ as a section

$$\sigma \in H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{O}_{\mathcal{X}}(-D)),$$

and this section now has a zero locus without divisorial components. Here, when $n \geqslant 2$, the very generic fiber X_s has Picard number one by the Noether-Lefschetz theorem, and so, after shrinking U if necessary, we can assume that $\mathcal{O}_{\mathcal{X}}(-D)$ is the restriction of $\mathcal{O}_{\mathbb{P}^{n+1}}(-p)$, $p \geqslant 0$ by the effectivity of D. Hence D can be assumed to be nef. After performing this simplification, $A - mB$ is replaced by $A - mB + D$, which is still ample if $A - mB$ is ample. As a consequence, we may assume $\operatorname{codim} \mathcal{Y} \geqslant 2$, and after shrinking U again, that all Y_s have $\operatorname{codim} Y_s \geqslant 2$.

10.6. Additional statement. *In corollary* 10.5*, under the same hypotheses* (10.3) *and* (10.4)*, one can take all fibers* Y_s *to have* $\operatorname{codim} Y_s \geqslant 2$*.*

This is enough to conclude that X_s is hyperbolic if $n = \dim X_s \leqslant 3$. In fact, this is clear if $n = 2$ since the Y_s are then reduced to points. If $n = 3$, the Y_s are at most curves, but we know by Ein and Voisin that a generic hypersurface $X_s \subset \mathbb{P}^4$ of degree $d \geqslant 7$ does not possess any rational or elliptic curve. Hence Y_s is hyperbolic and so is X_s, for s generic. □

10.7. Corollary. *Assume that $n = 2$ or $n = 3$, and that $\mathcal{X} \subset \mathbb{P}^{n+1} \times \mathbb{P}^{N_d}$ is the universal hypersurface of degree $d \geqslant d_n \geqslant 2n + 1$ so large that conditions* (10.3) *and* (10.4) *are met with $A - mB$ ample. Then the very generic hypersurface $X_s \subset \mathbb{P}^{n+1}$ of degree d is hyperbolic.*

§10.C. Proof of the Green-griffiths conjecture for generic hypersurfaces in $\mathbb{P}^{n+1}$

The most striking progress made at this date on the Green-Griffiths conjecture itself is a recent result of Diverio, Merker and Rousseau [DMR10], confirming the statement when $X \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ is a generic hypersurface of large degree d, with a (non optimal) sufficient lower bound $d \geqslant 2^{n^5}$. Their proof is based in an essential way on Siu's strategy as developed in § 10.B, combined with the earlier techniques of [Dem95]. Using our improved bounds from § 9.D, we obtain here a better estimate (actually of exponential order one $O(\exp(n^{1+\varepsilon})$ rather than order 5).

10.8. Theorem. *A generic hypersurface $X \subset \mathbb{P}^{n+1}$ of degree $d \geqslant d_n$ with*

$$d_2 = 286, \quad d_3 = 7316, \quad d_n = \left\lfloor \frac{n^4}{3}\big(n\log(n\log(24n))\big)^n \right\rfloor \quad \textit{for } n \geqslant 4,$$

satisfies the Green-Griffiths conjecture.

Proof. Let us apply Theorem 9.46 with $V = T_X$, $r = n$ and $k = n$. The main starting point is the well known fact that $T^*_{\mathbb{P}^{n+1}} \otimes \mathcal{O}_{\mathbb{P}^{n+1}}(2)$ is semipositive (in fact, generated by its sections). Hence the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^{n+1}}(-d) \to T^*_{\mathbb{P}^{n+1}|X} \to T^*_X \to 0$$

implies that $T^*_X \otimes \mathcal{O}_X(2) \geqslant 0$. We can therefore take $\gamma = \Theta_{\mathcal{O}(2)} = 2\omega$ where ω is the Fubini-Study metric. Moreover $\det V^* = K_X = \mathcal{O}_X(d - n - 2)$ has curvature $(d - n - 2)\omega$, hence $\Theta_{\det V^*} + r\gamma = (d + n - 2)\omega$. The Morse integral to be computed when $A = \mathcal{O}_X(p)$ is

$$\int_X \Big(c_{n,n,n}(d + n - 2)^n - c'_{n,n,n}(d + n - 2)^{n-1}(p + 2n)\Big)\omega^n,$$

so the critical condition we need is

$$d + n - 2 > \frac{c'_{n,n,n}}{c_{n,n,n}}(p + 2n).$$

On the other hand, Siu's differentiation technique requires $\frac{m}{n^2}(1 + \frac{1}{2} + \ldots + \frac{1}{n})A - mB$ to be ample, where $B = \mathcal{O}_X(n^2 + 2n)$ by Merker's result 10.2. This ampleness condition yields

$$\frac{1}{n^2}\Big(1 + \frac{1}{2} + \ldots + \frac{1}{n}\Big)p - (n^2 + 2n) > 0,$$

so one easily sees that it is enough to take $p = n^4 - 2n$ for $n \geqslant 3$. Our estimates (9.48) and (9.49) give the expected bound d_n. □

Thanks to 10.6, one also obtains the generic hyperbolicity of 2 and 3-dimensional hypersurfaces of large degree.

10.9. Theorem. *For $n = 2$ or $n = 3$, a generic hypersurface $X \subset \mathbb{P}^{n+1}$ of degree $d \geqslant d_n$ is Kobayashi hyperbolic.*

By using more explicit calculations of Chern classes (and invariant jets rather than Green-Griffiths jets) Diverio-Trapani [DT10] obtained the better lower bound $d \geqslant d_3 = 593$ in dimension 3. In the case of surfaces, Paun [Pau08] obtained $d \geqslant d_2 = 18$, using deep results of McQuillan [McQ98].

One may wonder whether it is possible to use jets of order $k < n$ in the proof of 10.8 and 10.9. Diverio [Div08] showed that the answer is negative (his proof is based on elementary facts of representation theory and a vanishing theorem of Brückmann-Rackwitz [BR90]):

10.10. Proposition ([Div08])**.** *Let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface. Then*

$$H^0(X, E^{\mathrm{GG}}_{k,m} T^*_X) = 0$$

for $m \geqslant 1$ and $1 \leqslant k < n$. More generally, if $X \subset \mathbb{P}^{n+s}$ is a smooth complete intersection of codimension s, there are no global jet differentials for $m \geqslant 1$ and $k < n/s$.

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Université Joseph Fourier Grenoble I
Institut Fourier (Mathématiques)
UMR 5582 du C.N.R.S., BP 74
38402 Saint-Martin d'Hères, France
e-mail: demailly@fourier.ujf-grenoble.fr